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No Equilibrium in Infinite Economies: Two Examples*

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Two examples of the nonexistence of equilibrium are given in a production economy with preferences over an infinite horizon. The relation of the examples to various existence theorems is discussed. *Journal of Economic Literature* Classification Number: 021.

I. INTRODUCTION

The purpose of this paper is to present two examples of an economy possessing an optimum but no competitive equilibrium. Their essential features are having production and preferences over an infinite horizon; in fact, it seems similar examples may not occur in overlapping generations and/or pure exchange models. There is a growing literature of equilibrium existence theorems in this field. We choose to mention here the ones due to Bewley [3] and Toussaint [7] on production economies, and those of Peleg and Yaari [6] and Wilson [8] on exchange economies. In relation to these results, our first example shows (i) that the assumption in both [3] and [7] on "strongly adequate" endowments cannot be relaxed by itself and (ii) that [6] on [8] do not extend to production. Our second example also satisfies all but one of the assumptions in [3] and [7]. In this case it is continuity of preferences—amounting, roughly, to impatience—which is shown to be necessary for an equilibrium. We elaborate on these points in the section following the examples.

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II. The Examples

Consider an economy with a consumption good C_t, a production good M_t and the following two activities $\mathscr{C}^t$ and $\mathscr{M}^t$ in every period $t = 0, 1,\ldots$. The activity $\mathscr{C}^t$ transforms one unit of C_t into one-half unit of C_{t+1}, while $\mathscr{M}^t$ transforms one unit of M_t into one-half units of C_{t+1} and M_{t+1}. Production consists of all nonnegative linear combinations of these two activities, and it is possible to freely dispose of any good at any period.

Let c_t and m_t denote, respectively, quantities of C_t and M_t. We now describe the consumer:

Example 1. The consumption set consists of all sequences $c = (c_t)$ with $c_t \geqslant 1$ for all $t \geqslant 0$. Preferences are represented by the utility function $U(c) = \sum_0^\infty 2^{-t} u(c_t) = \sum_0^\infty 2^{-t} \sqrt{c_t - 1}$. Endowments are $\mathbf{c}_0 = 3$, $\mathbf{m}_0 = 1$, $\mathbf{c}_t = 2^{-1} - 2^{-2t} + 2^{-4t}$ and $\mathbf{m}_t = \frac{1}{2}$ for $t \geqslant 1$.

Example 2. Any nonnegative sequence $c = (c_t)$ can be consumed, and a consumption sequence c is at least as good as another $\bar{c}$ if and only if $\liminf_{T\to\infty} \sum_0^T u_t(c_t) - u_t(\bar{c}_t) \geqslant 0$, where $u_t(\cdot) = 2^t(1 + 2^{-2t}) \log(\cdot)$. Endowments are $\mathbf{c}_0 = 3$, $\mathbf{m}_0 = 1$, and $\mathbf{c}_t = \mathbf{m}_t = \frac{1}{2}$.

Proposition 1. *There is a unique optimal allocation in each example.*

Proof (Example 1). We will show that the consumption sequence $c^* = (c_t^*) = (1 + 2^{-4t})$ is the only optimal consumption in this case:

Looking at the activities $\mathscr{M}^t$ and the endowments $\mathbf{m}_t$, it is clear that there can be at most one unit of M_t available for production at any period $t \geqslant 0$. This implies there can be at most one-half unit of C_t which can be produced and added to the endowments $\mathbf{c}_t$ at any $t \geqslant 1$. Thus there are at most $\mathbf{c}_0 = \mathbf{c}_0 = 3$ and $\mathbf{c}_t = \mathbf{c}_t + \frac{1}{2} = 1 - 2^{-2t} + 2^{-4t}$ plus $\frac{1}{2}s_{t-1}$ units of C_t available for consumption at any period t, where s_t denotes the amount of C_t used as input in the activity $\mathscr{C}_t$. It follows therefore that *attainable* consumption sequences $c = (c_t)$ are those satisfying the inequalities

$$c_t + s_t \leqslant \mathbf{c}_t + 2^{-1} s_{t-1}, \qquad t \geqslant 1.$$

From $c_0 + s_0 \leqslant \mathbf{c}_0$ and the nonnegativity of s_t's, we easily get

$$0 \leqslant \sum_0^\infty 2^t(\mathbf{c}_t - c_t), \tag{1}$$

and upon checking $\sum_0^\infty 2^t(c_t^* - \mathbf{c}_t) = 0$, we obtain from (1) that

$$0 \leqslant \sum_0^\infty 2^t(c_t^* - c_t) \tag{2}$$

for any attainable consumption sequence $c = (c_t)$.

On the other hand, we have from the concavity of $u(\cdot)$ that $u(c_t^*) - u(c_t) \geqslant u'(c_t^*)(c_t^* - c_t)$ for any $c_t \geqslant 1$. Computing $u'(c_t^*) = 2^{2t-1}$, we obtain

$$2^{t-1}(c_t^* - c_t) \leqslant 2^{-t}(u(c_t^*) - u(c_t))$$

for each t, with strict inequality if $c_t \neq c_t^*$. Adding over all $t \geqslant 0$ and using (2) gives

$$0 < U(c^*) - U(c)$$

for any $c \neq c^*$, which says c^* is the unique optimal consumption.

Proof (Example 2). Follow the above steps to see that the consumption sequence $c^* = (1 + 2^{-2t})$ gives the only optimal allocation in this case. ∎

Now in search of a competitive equilibrium, let nonnegative numbers π_t and μ_t denote, respectively, the price of C_t and M_t. Since there are constant returns in production, profits in equilibrium can only be zero. The consumer's wealth or budget, therefore, is simply the value of his endowments, i.e., $\sum_0^\infty \pi_t \mathbf{c}_t + \mu_t \mathbf{m}_t$. It may of course be that this wealth is infinite. With no ambiguity, however, we define as in (8), that a consumption sequence is *budget-feasible* if $\liminf_{T\to\infty} D^T \leqslant 0$, where $D^T = \sum_0^T \pi_t c_t - (\pi_t \mathbf{c}_t + \mu_t \mathbf{m}_t)$ measures the consumer's debt in the first T periods. In standard fashion, we say a *competitive equilibrium* exists if there is a nontrivial sequence of prices, a sequence of production decisions, and a consumption sequence c such that (i) the operated activities yield zero profits while all others have nonpositive profitability, (ii) c maximizes the consumer's preferences among all budget-feasible consumption sequences, and (iii) supply meets demand, i.e., c is attainable.

Our demonstration that no competitive equilibrium exists in the above examples utilizes the following fact:

LEMMA. *A competitive equilibrium allocation is optimal.*

Note that, if there were several consumers, the budget-feasibility condition might be violated for some or all and yet be satisfied in the aggregate. The standard argument for the lemma might fail therefore, and it is not clear why it would hold in general. There is, however, the following simple proof for the case of our single-consumer model.

Proof. We show that any attainable consumption sequence is budget-feasible with respect to prices generating nonpositive profitability in the production sector:

Let $c = (c_t)$ be any attainable consumption sequence. Then there is a sequence of net outputs $Y = (c^t, m^t)$ such that $C \leqslant E + Y$, where $C = (c_t, 0)$ and $E = (\mathbf{c}_t, \mathbf{m}_t)$. Let $P = (\pi_t, \mu_t)$ be any sequence of prices with respect to which each activity yields nonpositive profits. Note that for any period T, the value of net outputs in the first T periods, $P^T Y^T$, can only be nonpositive, since it amounts to the sum of the first T period profits minus the value of inputs used in period T. The consumer's debt, therefore, is $D^T = P^T(C^T - E^T) \leqslant P^T Y^T \leqslant 0$. Thus, c is budget-feasible.

The lemma now follows directly from the definition of competitive equilibrium. ∎

PROPOSITION 2. *There exists no competitive equilibrium in either example.*

Proof. Since competitive equilibrium allocations are optimal, in either example the consumer would have to be demanding the consumption sequence c^* given in Proposition 1. But both of these allocations require all the activities $\mathscr{C}^t$ and $\mathscr{M}^t$ to be operated at positive levels. Every activity therefore must have zero profitability; that is, equilibrium prices (π_t) and (μ_t), if they exist, must satisfy the equations

$$\begin{aligned} \pi_t &= \tfrac{1}{2}\pi_{t+1}, \\ \mu_t &= \tfrac{1}{2}\pi_{t+1} + \tfrac{1}{2}\mu_{t+1}, \qquad t \geqslant 0. \end{aligned} \tag{3}$$

Solving (3) we get $\mu_0 = t\pi_0 + (\tfrac{1}{2})^{t+1}\mu_{t+1}$ for any $t \geqslant 0$, which implies $\pi_0 = \pi_t = 0$ for all t. But then the consumer has infinite demand, and therefore no equilibrium exists. ∎

We make two observations at this point on this failure of equilibrium:

First, any finite truncation of each economy obviously satisfies all the assumptions behind the classical existence theorem for finite models, so that what is being displayed here is purely a failure of existence in the limit. Moreover, note that prices decentralizing the productive sector do exist: let all consumption goods be free and all production goods have the same positive price. It follows directly from the above proof, in fact, that these are the only prices supporting the production decisions with which optimality may be attained. We thus have the following picture: the "optimal" production prices specify a world where all consumption goods are free even though they are available in limited quantities, and there is an unsatisfied consumer with positive wealth. It seems such a situation could arise in finite economies only in connection to very peculiar production sets, and even then only as a boundary phenomenon.

Second, we should mention that our examples possess *quasi*-equilibria, by which is understood a price-and-attainable allocation pair where demand may not be satisfied but there exists no cheaper consumption than the present one. In fact, any attainable allocation and the prices quoted in the previous paragraph constitute a quasi-equilibrium. As we shall indicate, however, our examples can be extended without altering any of their relevant features, so as not to have any quasi-equilibrium which is optimal.

The economy has a consumption good C_t and $t+1$ production goods $M_{t,j}$, $0 \leqslant j \leqslant t$, in every period $t \geqslant 0$. Production is generated by $\mathscr{C}^t$ as before and the following $t+1$ activities $\mathscr{M}^{t,j}$, $0 \leqslant j \leqslant t$, $t \geqslant 0$. Activity $\mathscr{M}^{t,0}$ is the same as $\mathscr{M}^t$ given originally, and for each $j = 1, \ldots, t$, activity $\mathscr{M}^{t,j}$ transforms one unit of $M_{t,j}$ into one-half unit each of $M_{t+1,j-1}$ and $M_{t+1,j}$. The consumers are the same as before in all regards except that their endowment of the production goods are given by

$$
\begin{aligned}
\mathbf{m}_{t,j} &= 1, & & t = j, \\
&= \tfrac{1}{2}, & & t = j+1, \\
&= 0, & & t \neq j, j+1 \quad \text{for } j = 0, \ldots, t,\ t \geqslant 0.
\end{aligned}
$$

It is easily seen that the set of attainable consumption sequences in these modified examples is the same as previously. Proposition 1 therefore holds identically. Letting $\mu_{t,j}$ denote the prices of the production goods and following the same argument as in Proposition 2, we now get that prices supporting the optimal production decisions must satisfy

$$
\begin{aligned}
\pi_t &= \tfrac{1}{2}\pi_{t+1}, \\
\mu_{t,0} &= \tfrac{1}{2}\pi_{t+1} + \tfrac{1}{2}\mu_{t+1,0}, \\
\mu_{t,j} &= \tfrac{1}{2}\mu_{t+1,j-1} + \tfrac{1}{2}\mu_{t+1,j}, \qquad j = 1, \ldots, t.
\end{aligned}
$$

These equations have no nonnegative solution other than the trivial one with all unknowns being equal to zero. (See Alkan [1] for a version of this example in connection to existence of production prices.) The conclusion is that no optimal quasi-equilibrium exists in the modified examples.

III. Remarks

We start with the following definitions:

(i) Production is *bounded* if from uniformly bounded inputs only uniformly bounded outputs can be obtained.

(ii) Endowments are *weakly* (*strongly*) *adequate* if it is possible to have a consumption sequence with some (at least some fixed) positive amount of every consumption good.

(iii) Preferences are *myopic* if a little more consumption in the near future is preferred to any large constant increase in consumption at all periods after some date in the distant future. "Myopicness," for our purposes, stands for continuity of preferences. See [4] for a precise definition and its relation to continuity with respect to various topologies on the commodity space).

(1) The above three properties are the important ones to be checked in support of the points we have made in the introduction with respect to [3, 6–8]. It is straightforward to check that production is bounded in both of our examples, and that the endowments in Example 1 are weakly while the ones in Example 2 are strongly adequate. As for preferences, we can show directly that those in Example 1 are myopic. Actually, the behaviour of $u(x) = \sqrt{x-1}$ in the utility $U(\cdot)$, say for $x \geqslant 2$, is immaterial for what this example bears. We could alternatively have worked with a $\tilde{U}(\cdot)$ such that $\tilde{U}(c) < \infty$ for all c, which would imply continuity in all related work in the literature.

(2) Preferences in Example 2 are not myopic. It is not quite clear, however, why economically "sensible preferences" are the myopic ones, as seems to be suggested in [3] for instance. It is true that optima may fail to exist an obvious way when the consumer "values" future consumption and there is sufficient productivity—when, of course, an equilibrium in the normal sense would not obtain. This is not the case, however, in Example 2, where in fact utility has a finite value at the optimum.

(3) Weakly adequate endowments can be made into strongly adequate ones by suitably changing units of measurement. It should be clear, however, that this might destroy the boundedness of production and/or the continuity of preferences. Such a transformation in fact does both to Example 1, so that it displays a different phenomenon than Example 2.

(4) The "adequacy of endowments" assumption plays a powerful role in the proof of the classical theorem for finite economies, and by itself prevents a nonexistence of equilibrium as described in the previous section. Under this assumption, furthermore, in finite economies quasi-equilibria are the same as competitive equilibria. Example 2 shows that more than this interiority assumption is necessary for such implications to hold in infinite economies with production.

(5) Finally, we refer to Florenzano [5] for a theorem on the existence of quasi-equilibria in exchange economies under a weaker adequacy assumption. The modified version of Example 1 differs from the set-up of this work only in having production. Unless it is conceivable that all "equilibria" may turn out to be nonoptimal, this seems also to suggest that having production may make a difference in infinite economy models. We should add, however, that the difficulties shown here are due to the presence of an infinitely living consumer and that they may not occur in the finite-life agent models (e.g., [2]).

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