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Monotonicity and envyfree assignments*

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Summary. Given any problem involving assignment of indivisible objects and a sum of money among individuals, there is an efficient envyfree allocation (namely the minmax money allocation) which can be extended monotonically to a new efficient envyfree allocation for any object added or individual removed, and another (the maximin value allocation) extendable similarly for any object removed or person added. Still, the efficient envyfree solution is largely incompatible with the resource and population monotonicity axioms: The minmax money and maxmin value allocations are unique in being extendable.

1. Introduction

Resource monotonicity and *population monotonicity* have been put forward as recommendable norms in the division of commonly claimed resources (Roemer (1986), Thomson (1983)). Resource monotonicity requires for instance that no member of a fixed population should be worse off in case resources expand, and population monotonicity requires the opposite in case the population expands but resources stay fixed. Compelling as they appear as equity standards, these criteria turn out to be rather strong, in ways not entirely evident at first sight. Various characterizations have shown, in fact, that monotonicity axioms lead to solutions of the welfare egalitarian type (Kalai (1977), Kalai and Samet (1985), Thomson (1983), Dutta and Vohra (1991), Sprumont (1992)), suggesting in particular that they are likely to be incompatible with other standards. For instance, Moulin and Thomson (1985) have exposed how such an incompatibility exists between resource monotonicity and the *no-envy* criterion (Foley (1967), Kolm (1972)), perhaps the most prominent notion of equity in the literature (Thomson (1990)). Their demon-

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stration holds on the domain of resources which are divisible and between which there are complementarities. Our purpose in this paper is to uncover the extent of (in)compatibility between the resource/population monotonicity criteria and envyfreeness on the domain of assignment problems.

An *assignment problem* consists of n individuals having common claims on m indivisible objects and a sum of money X. Each individual is to receive a *bundle* containing (at most) one object and some money from X. Alkan, Demange and Gale (1991) have extensively studied this problem, and in particular shown that the set of efficient envyfree allocations is always nonempty, when valuations are continuously increasing in money and no object is of infinite value.[1]

Our investigation here continues a line of query initiated by ADG (1991) and hinges on a criterion which incorporates monotonicity requirements on all *one-member* variations in the set of individuals or objects. The criterion asks whether there always exist *extendable* efficient envyfree allocations for an assignment problem, in the sense that, following any one-member variation in the problem, a new efficient envyfree allocation can be found which makes everyone better off or everyone worse off, as the context would require.

We present two sets of results. Together they chart out the extent of compatibility between the monotonicity criteria and (efficient) envyfreeness on the present domain. The first set shows that two particular selections from the efficient envyfree solution are extendable in one of two directions respectively, while the second set shows that *all* other allocations fail in this regard. The efficient envyfree solution is thus seen to be "largely" nonmonotonic. Simple examples show, in fact, that there may exist *no* envyfree allocations which fulfil the monotonicity criteria under two-member variations.

To describe our results in further detail, let us first point out that envyfreeness implies efficiency when $m \geq n^2$, but not otherwise. As ADG (1991) have shown, though, envyfreeness can be strengthened by a condition, which holds vacuously when $m \geq n$, and then efficiency is a consequence. Efficient envyfree allocations may exist, on the other hand, which are not *strongly* envyfree in this sense. Let the *value* of a bundle to an individual be the pure-money equivalent of that bundle for the individual and call V the set of all value vectors (in the space of individuals) which are realizable by strongly envyfree allocations. Again as shown in ADG (1991), any social welfare function of the minmax/maxmin variety defines a *single-valued* selection from V, and the same holds if one replaces V by the set of all money vectors (in the space of objects) which pertain to strongly envyfree allocations. Two among these selections, namely the *maxmin value* allocation and the *minmax money* allocation, turn out to have the special status in comparative statics mentioned above.

In fact, ADG (1991) showed for the case $m = n$ that, if one starts out with the minmax money allocation, then for any object to be added to the initial set, a strongly envyfree allocation can be found in which no one is worse off. Our first

[1] The efficient envyfree set in fact is generically multivalued and has a connectedness property that can be seen as generalized convexity (ADG (1991).

2 This was shown by Svensson (1983)

theorem here extends this fact to *all* one-member variations and with no restriction on m, n. It says that the minmax money allocation is *upper extendable*, i.e., one may accommodate a unidirectional change in welfare for any addition of an object or for any removal of an individual (Theorem 1a), and that the maxmin value allocation is *lower extendable*, i.e., the same holds for any removal of an object or for any addition of an individual (Theorem 1b).[3]

Our second set of results establishes counterparts to Theorem 1. Theorem 2a states that, given any assignment problem (with $m \geq n$), for *any* envyfree allocation other than the minmax money allocation, there exists an object whose addition makes at least one individual worse off in any new efficient envyfree allocation.[4] The minmax money allocation is thus unique in being upper extendable. Interestingly, the situation is different for extendability in the other direction. We present an assignment problem where *all* envyfree allocations are lower extendable. As analogue to Theorem 2a, nevertheless, a weak uniqueness result holds for the maxmin value allocation, which we state as Theorem 2b: There exist assignment problems where all lower extendable allocations are confined to an arbitrarily small subset of the envyfree set.

In our model, presented next in Section 2, objects may be desirable or undesirable and neither the total amount of money nor individual moneys are restricted in sign. It is evident that in this broad framework, the *proper direction* to postulate for changes in welfare would depend on the context, in particular the sign of $m - n$. The extendability criterion we formulate, in Section 3, brings together what is pertinent in this regard. In Section 4 we establish some properties of the minmax money and maxmin value allocations, which we use in proving the main results in Section 5. Section 6 contains an example of incompatibility with two-member variations and closing remarks.

2. The assignment problem and the efficient envyfree solution

An *assignment problem* is a triplet (P, O, X) where P is a finite set of *individuals*, O is a finite set of *objects*, and $X \in R$ (the real line) is an amount of *money*. We denote (α, x) a *bundle* consisting of object α and x units of money. We use the notation (ϕ, x) for a bundle containing no object and call ϕ a *null object*. We assume that, for every individual A and bundle (α, x), there exists an amount of money $\varphi_A(\alpha, \mathrm{x})$, called the *value* of (α, x) to A, such that A is indifferent between (α, x) and $(\phi, \varphi_A(\alpha, \mathrm{x}))$. We assume that the functions $\varphi_A(\alpha, \mathrm{x})$ are continuous and strictly increasing in x. A *dummy* is an individual for whom all objects are null objects..

Let $\mathrm{A} = (P, O, X)$ be an assignment problem. Let $h = \max\{|O| - |P|, 0\}$ and $k = \max\{|P| - |O|, 0\}$. Denote P^* the union of P with h dummies, and denote O^* the union of O with k null objects. An *assignment* for A is a triple (μ, v, x) where x is a *money vector* associating x_α units of money to each $\alpha \in O^*$, μ is a bijection between P^* and O^*, and v is the *value vector* given by $v_A = \varphi_A(\mu(A), x_{\mu(A)})$ for each $A \in P^*$.

[3] Tadenuma and Thomson (1989) obtained Theorem 1b for the case $n = 1$.

[4] For the *quasilinear* domain, i.e., where individuals' valuations of bundles are separable in money homogeneously for all objects, ADG (1991) have obtained this fact by an argument special to the domain.

An assignment (μ, v, x) is *feasible* if $\sum_{A\in P} x_{\mu(A)} = X$ and *efficient* if, in addition, there is no feasible assignment (μ', v', x') such that $v' > v$.[5] A feasible assignment (μ, v, x) is *envyfree* (resp., *strongly envyfree*) if

$$v_A \geq \varphi_A(\alpha, x_\alpha) \quad \text{for all } \alpha \in O^*,$$

for every $A \in P$ (resp., for every $A \in P^*$). (Note that an envyfree assignment (μ, v, x) is strongly envyfree if and only if $x_\omega = \max\{x_\alpha | \alpha \in O\}$ for all $\omega \in O - \mu(P)$, a condition which is vacuously true in case $|P| \geq |O|$.) We call (v, x) a feasible, efficient, envyfree or strongly envyfree *allocation*, if there is an assignment (μ, v, x) with the same property.

Let $\mathscr{A}$ be the set of all assignment problems. The *efficient envyfree solution* is a correspondence on $\mathscr{A}$, which we will denote Φ, where $\Phi(\mathsf{A})$ is the set of all efficient envyfree allocations for every A. We define the *strongly envyfree solution* $\Phi^* \subseteq \Phi$ similarly. ADG (1991) have shown that $\Phi^*(\mathsf{A}) \neq \emptyset$ for every $\mathsf{A} = (P, O, X) \in \mathscr{A}$. We point out for emphasis that, while $\Phi(\mathsf{A}) = \Phi^*(\mathsf{A})$ when $|P| \geq |O|$, there may exist efficient envyfree allocations which are not strongly envyfree as well as envyfree allocations which are not efficient when $|P| < |O|$.

Our purpose is to examine the monotonicity of Φ with respect to variations in the set of individuals or objects. We shall make frequent use of the following fact which states Φ^* is monotonic with respect to variations in money in a strong manner.

Money Monotonicity Theorem (to be abbreviated **MMT**): Let $\mathsf{A} = (P, O, X) \in \mathscr{A}$. Consider the family of assignment problems $\{\mathsf{A}(\delta) | \delta \in R\}$ where $\mathsf{A}(\delta) = (P, O, X + \delta)$. For any $\hat{\mathbf{a}} \in \Phi^*(\mathsf{A})$, there exists a path of allocations $\{\hat{\mathbf{a}}(\delta) | \delta \in R\}$ continuous in δ where $\hat{\mathbf{a}}(0) = \hat{\mathbf{a}}$, $\hat{\mathbf{a}}(\delta) \in \Phi^*(\mathsf{A}(\delta))$, and $\hat{\mathbf{a}}(\delta) \gg \hat{\mathbf{a}}(\delta')$ whenever $\delta > \delta'$.

A milder version of this theorem had been given in ADG (1991). Essentially the same argument applies here; we omit the proof.

3. A monotonicity criterion: extendability

Our focus will be on a monotonicity criterion that involves *one*-individual or one-object variations in $\mathsf{A} = (P, O, X) \in \mathscr{A}$. Consider any solution Ψ on the domain $\mathscr{A}$. The criterion demands that there always be an allocation $\hat{\mathbf{a}} \in \Psi(\mathsf{A})$ which permits a *unidirectional* change in welfare for *any* one-member variation in A, in the sense that a Ψ – allocation can be found after the variation making everyone better off *or* everyone worse off relative to $\hat{\mathbf{a}}$. The proper direction to postulate here naturally depends on the type of variation, such as whether it is the addition or removal of an object, then whether the object is desirable or nondesirable, then also on the sign of $|P| - |O|$, similarly on whether the variation is the addition or removal of an individual, then on whether the individuals are taking part in benefit or burden. Our definition below brings together all such particularities.

We call an object α *desirable* (resp., *undesirable*) if $\varphi_A(\alpha, \mathsf{x})$ is strictly greater (resp., smaller) than x for all $\mathsf{x} \in R$ and for all individuals A.

[5] We use the vector inequalities $\geq$, $>$, $\gg$.

Notation: For any vector z, denote z_M, z_m the maximum and minimum components of z respectively. Given any $\mathsf{A} = (P, O, X) \in \mathscr{A}$, denote $\mathsf{A} \cup A$, $\mathsf{A} \cup \alpha$, $\mathsf{A} - A$, $\mathsf{A} - \alpha$ the assignment problems $(P \cup \{A\}, O, X), (P, O \cup \{\alpha\}, X), (P - \{A\}, O, X), (P, O - \{\alpha\}, X)$ respectively. In the definition below, we will somewhat abuse notation and write $v \geq v'$ for value vectors v, v' of two assigment problems A, A' to mean $v_A \geq v'_A$ for every individual A who is present in both A and A'.

Definition: Let Ψ be a solution on the domain $\mathscr{A}$. Let $\mathsf{A} = (P, O, X) \in \mathscr{A}$.

An allocation $(v, x) \in \Psi(\mathsf{A})$ is *upper extendable* if

(i) for any desirable (resp., undesirable) $\alpha \notin O$, there exists $(v', x') \in \Psi(\mathsf{A} \cup \alpha)$ such that $v' \geq v$ if $|P| \leq |O|$ and $v' \gg v$ (resp., $v' \ll v$) if $|P| > |O|$, and
(ii) for any $A \in P$, there exists $(v', x') \in \Psi(\mathsf{A} - A)$ such that $v' \gg v$ if $x_M > 0$, $v' = v$ if $x_M = 0$, and $v' \ll v$ if $x_M < 0$.

An allocation $(v, x) \in \Psi(\mathsf{A})$ is *lower extendable* if

(i) for any desirable (resp., undesirable) $\alpha \in O$, there exists $(v', x') \in \Psi(\mathsf{A} - \alpha)$ such that $v' \leq v$ if $|P| < |O|$ and $v' \ll v$ (resp., $v' \gg v$) if $|P| \geq |O|$, and
(ii) for any $A \notin P$, there exists $(v', x') \in \Psi(\mathsf{A} \cup A)$ such that $v' \ll v$ if $v_m > 0$, $v' = v$ if $v_m = 0$, and $v' \gg v$ if $v_m < 0$.

We say that Ψ is *extendable* if for every $\mathsf{A} \in \mathscr{A}$ there exists in $\Psi(\mathsf{A})$ an allocation which is upper extendable and an allocation which is lower extendable.

We will show in Section 5 that the efficient envyfree solution Φ is extendable.

4. The minmax money and maxmin value allocations

The following two selections from Φ^* turn out to have a pivotal role in our investigation. Let $\mathsf{A} = (P, O, X) \in \mathscr{A}$.

Definition: Call an allocation $(v, x) \in \Phi^*(\mathsf{A})$,

(i) *minmax money* allocation if $x_M \leq x'_M$ for all $(v', x') \in \Phi^*(\mathsf{A})$,
(ii) *maxmin value* allocation if $v_m \geq v'_m$ for all $(v', x') \in \Phi^*(\mathsf{A})$.

Let (μ, v, x) be a strongly envyfree assignment for A. A *μ-path from $A \in P^*$ to $\alpha \in O^*$* is a sequence of k distinct pairs $(A_i, \alpha_i) \in P^* \times O^*$ for some $k \geqslant 1$, where

$$A = A_1, \quad \alpha = \alpha_k, \quad \alpha_i = \mu(A_i) \quad \text{for} \quad 1 \leq i \leq k,$$

and $v_{A_i} = \varphi_{A_i}(\alpha_{i-1}, x_{\alpha_{i-1}})$ for $1 < i \leq k$.

We call any $\alpha \in O^*$ with $x_\alpha = x_M$ a *max object* and any $A \in P^*$ with $v_A = v_m$ a *min individual.*

Lemma 1: (i) If (μ, v, x) is a minmax money assignment, then there exists from every $A \in P^*$ a μ-path to some max object. (ii) Assume all $\alpha \in O$ are desirable. If (μ, v, x) is a maxmin value assignment, then there exists to every $\alpha \in O^*$ a μ-path from some min individual.[6]

[6] The converse statements are also true (Alkan (1989)).

Proof: We shall only prove (ii) here. The argument for (i) is analogous and has been given in ADG (1991) for the case $|P|=|O|$.

Let (μ, v, x) be a maxmin value assignment for A. Let O' be the set of all $\alpha \in O^*$ to which there exists a μ-path from some min individual. In particular, all min individuals are in $P' \equiv \mu(O')$. Suppose the lemma is false, i.e., $O^* - O'$ is nonempty. Note that $v_A < \varphi_A(\alpha, x_\alpha)$ for any $A \in P^* - P'$ and any $\alpha \in O'$.

Since all $\alpha \in O$ are desirable, any dummy in P^* is a min individual and so belongs to P', in particular $P^* - P' \subseteq P$. If P' consisted exclusively of dummies, then we could slightly increase x_α for $\alpha \in O'$ without violating feasibility or strong envyfreeness, which would then increase v_m contrary to the fact that (v, x) is a maxmin value allocation. So $P' \cap P$ is nonempty.

By MMT now, we can slightly increase x_α for every $\alpha \in O'$ and slightly decrease x_α for every $\alpha \in O^* - O'$, while maintaining feasibility and strong envyfreeness, again contrary to (v, x) being maxmin value. So $O' = O^*$. ◆

Lemma 2: If (μ, v, x), (μ', v', x') are strongly envyfree assignments for A and $P' = \{A \in P^* | v_A > v'_A\}$, $O' = \{\alpha \in O^* | x_\alpha > x'_\alpha\}$, then $\mu(P') = \mu'(P') = O'$.

Lemma 2 is a version of the so-called Decomposition Lemma; for proof, see ADG (1991). We use it in the proof below as well as in the next section.

Lemma 3: If (v^*, x^*) is a minmax money allocation and (v, x) is any other strongly envyfree allocation for A, then there exists an $\alpha \in O^*$ such that $x_\alpha > x^*_\alpha = x^*_M$.

Proof: Let $\hat{\mathbf{a}}^* = (\mu^*, v^*, x^*)$ be a minmax money assignment and $\hat{\mathbf{a}} = (\mu, v, x)$ be any strongly envyfree assignment for A such that $x \neq x^*$. Suppose the lemma is false. Then $O' \equiv \{\alpha \in O^* | x_\alpha \leq x^*_\alpha\}$ contains all $\hat{\mathbf{a}}^*$-max objects, in particular any object in $\mu^*(P^* - P)$. So, by feasibility, $O^* - O'$ is nonempty.

By Lemma 1(i), there exists an individual $A \in \mu^*(O')$ such that $v^*_A = \varphi_A(\alpha, x^*_\alpha)$ for some $\alpha \in O^* - O'$. By Lemma 2, $\mu(A) \equiv \alpha' \in O'$. Thus, $v_A = \varphi_A(\alpha', x_{\alpha'}) \leq \varphi_A(\alpha', x^*_{\alpha'}) \leq v^*_A$ since $x_{\alpha'} \leq x^*_{\alpha'}$ and $\hat{\mathbf{a}}^*$ is envyfree. On the other hand, $v^*_A = \varphi_A(\alpha, x^*_\alpha) < \varphi_A(\alpha, x_\alpha)$ since $x^*_\alpha < x_\alpha$. So $v_A < \varphi_A(\alpha, x_\alpha)$, contradicting $\hat{\mathbf{a}}$ is envyfree. ◆

One deduces directly from Lemma 3 that there exists a *unique* minmax money allocation for each $\mathsf{A} \in \mathscr{A}$. (Note that $x = x'$ if and only if $v = v'$ for any pair of allocations (v, x), $(v', x') \in \Phi(\mathsf{A})$.) One would likewise see that maxmin value allocations are unique when all $\alpha \in O$ are desirable. This single-valuedness property has been established in ADG (1991) for a broader class of selections from Φ on the domain where $|P| = |O|$.

5. Extendability and the efficient envyfree solution

For simplicity, we present our first result below under the assumption that *all objects are desirable.* Following the proof, we shall remark how it carries over to the domain where undesirable objects are allowed as well.

Theorem 1: The efficient envyfree solution is extendable:

a) The minmax money allocation is upper extendable.
b) The maxmin value allocation is lower extendable.

Proof: Let $A = (P, O, X) \in \mathscr{A}$.

a) Let $\hat{\mathbf{a}} = (\mu, v, x)$ be a minmax money assignment. To see that $\hat{\mathbf{a}}$ fulfils criterion (i) of upper extendability, take any object $\beta \notin O$. Let x_β be the smallest money such that $v_B = \varphi_B(\beta, x_\beta)$ for at least one individual $B \in P$. By Lemma 1(i), $\hat{\mathbf{a}}$ has a μ-path $(A_1, \alpha_1), \ldots, (A_K, \alpha_k)$ where $A_1 = B$ and $x\alpha_k = x_M$. *Case* 1: $|P| > |O|$. Since all objects are desirable, it must be that α_k is a null object and $x_M > x_\beta$. Alter $\hat{\mathbf{a}}$ to $\hat{\mathbf{a}}'$ by giving B the bundle (β, x_β), A_i the bundle $(\alpha_{i-1}, x_{\alpha_{i-1}})$ for $i > 1$, and all remaining $A \in P$ the same bundles as in $\hat{\mathbf{a}}$. Note that $\hat{\mathbf{a}}'$ is an assignment with the same value vector as $\hat{\mathbf{a}}$ and is strongly envyfree for $(P, O \cup \beta, X - x_M + x_\beta)$. The desired conclusion follows from MMT. *Case* 2: $|P| \leq |O|$. If $x_M > x_\beta$, apply the same reassignment and reasoning as above. If on the other hand $x_M \leq x_\beta$, the assignment $\hat{\mathbf{a}}$ (plus giving (β, x_M) to the new dummy) is still strongly envyfree for $A \cup \beta$.

To see that $\hat{\mathbf{a}}$ fulfils criterion (ii) as well, take any $B \in P$. By Lemma 1(i), $\hat{\mathbf{a}}$ has a μ-path as above. After $\hat{\mathbf{a}}$ to $\hat{\mathbf{a}}'$ by giving A_i the bundle $(\alpha_{i-1}, x_{\alpha_{i-1}})$ for every $i > 1$ and all remaining $A \in P$ the same bundles as in $\hat{\mathbf{a}}$. Again $\hat{\mathbf{a}}'$ is an assignment with the same value vector as $\hat{\mathbf{a}}$ and is strongly envyfree for $(P - B,\ O,\ X - x_M)$. The conclusion follows from MMT.

b) Let $\hat{\mathbf{a}} = (\mu, v, x)$ be a maxmin value assignment. (i): Take any $\beta \in O$. By Lemma 1(ii), $\hat{\mathbf{a}}$ has a μ-path $(A_1, \alpha_1), \ldots, (A_k, \alpha_k)$ where $vA_1 = v_m$ and $\alpha_k = \beta$. *Case* 1: $|P| \geq |O|$. Giving A_1 the bundle (ϕ, v_m), and reassigning along this path as in the above paragraph, one gets a strongly envyfree assignment for $(P, O - \beta,\ X - x_\beta + v_m)$. Since β is desirable, $x_\beta < \varphi_{A_1}(\beta, x_\beta) \leq v_{A_1} = v_m$, and the conclusion follows from MMT again. *Case* 2: $|P| < |O|$. Since all $\alpha \in O$ are desirable, A_1 is a dummy, i.e., $v_m = v_{A_1} = x_{\alpha_1} = x_M$. Drop β, A_1 and reassign along the μ-path. One has a strongly envyfree assignment for $(P, O - \beta,\ X - x_\beta + x_M)$. Apply MMT.

(ii): Take any $B \notin P$. Let β be an object in O^* such that $\varphi_B(\beta, x_\beta) \geq \varphi_B(\alpha, x_\alpha)$ for all $\alpha \in O^*$. By Lemma 1(ii) again, $\hat{\mathbf{a}}$ has a μ-path as above. Give B the bundle (β, x_β), reassign along the μ-path, and in case $|P| \geq |O|$ give A_1 the bundle (ϕ, v_m), in case $|P| < |O|$ drop (the dummy) A_1. In each case, one has a strongly envyfree assignment for $(P \cup B,\ O,\ X + v_m)$. The proof follows from MMT. ◆

Remark 1: On the general domain where undesirable as well as desirable objects are allowed, one may check the proof above that Theorem 1a and 1b continue to hold in the cases $|P| \leq |O|$ and $|P| \geq |O|$ respectively. For the remaining cases $|P| > |O|$ and $|P| < |O|$, respectively, let us call $(v, x) \in \Phi^*(A)$ a *minimum pure money* allocation if $x_\omega \leq x'_\omega$ for any $\omega \in O^* - O$ and a *maximum dummy value* allocation if $v_A \geq v'_A$ for any $A \in P^* - P$, for all $(v', x') \in \Phi^*(A)$. Then, Theorem 1a and 1b hold via these (single-valued) selections respectively; for details, one may see Alkan (1989). Note that, the minimum pure money allocation is the minmax money allocation when all $\alpha \in O$ are desirable and the minmin money allocation when all $\alpha \in O$ are undesirable. Likewise, the maximum dummy value allocation is the maxmin value allocation when all $\alpha \in O$ are desirable and the maxmax money allocation when all $\alpha \in O$ are undesirable.

We now turn to our second set of results which establish that the minmax money allocation is unique in being upper extendable and that, in a weaker sense, the maxmin value allocation is unique in being lower extendable.

Theorem 2 a): Let $\mathsf{A} = (P, O, X) \in \mathscr{A}$ where $|P| \geq |O|$ and all $\alpha \in O$ are desirable. If (v, x) is an envyfree allocation distinct from the minmax money allocation for A, then there exists a desirable object $\omega \notin O$ such that $(v', x') \in \Phi(\mathsf{A} \cup \omega)$ implies $v'_A < v_A$ for some $A \in P$, i.e., (v, x) is not upper extendable.

Proof: Let $\hat{\mathbf{a}}^* = (\mu^*, v^*, x^*)$ be a minmax money assignment and $\hat{\mathbf{a}} = (\mu, v, x)$ be any envyfree assignment for A such that $v \neq v^*$ (and so $x \neq x^*$). Since all $\alpha \in O$ are desirable, $v^*_A > x^*_M$ for all $A \in P$. Pick a $\delta > 0$ such that $v^*_A > x^*_M + \delta$ for all $A \in P$ and $x_\alpha + \delta < x^*_M$ for all $\alpha \in O^*$ with $x_\alpha < x^*_M$. Next let k be a positive scalar greater than $(v_A - v^*_A)/(x_\alpha - x^*_M)$ for all $A \in P$ and for all $\alpha \in O^*$ such that $x_\alpha \neq x^*_M$. We shall prove the theorem by introducing a new object ω, defined for all $A \in P$ by

$$\varphi_A(\omega, \mathrm{x}) = \max\{v^*_A + k(\mathrm{x} - x^*_M), \mathrm{x} + \delta\} \tag{1}$$

and showing that there exists no allocation in $\Phi(\mathsf{A} \cup \omega)$ in which everyone is at least as well off as in $\hat{\mathbf{a}}$. Note that ω is a desirable object.

We shall first treat the case $|P| = |O|$. Suppose to the contrary that $\hat{\mathbf{a}}' = (\mu', v', x')$ is an efficient envyfree assignment for $\mathsf{A} \cup \omega$ such that

$$v'_A \geq v_A \text{ for all } A \in P \tag{2}$$

The proof is in three steps. Step (i) shows that μ' must assign ω to some individual in P (i.e., not to a dummy). Step (ii) shows that this individual must not be worse off in $\hat{\mathbf{a}}'$ than in $\hat{\mathbf{a}}^*$, while Step (iii) shows the opposite. From this contradiction, the theorem follows for $|P| = |O|$. (As we shall point out in the end, the case $|P| > |O|$ is less involved. For instance, Step (i) then holds vacuously.)

Step (i) $\mu'(\omega) \equiv B \in P$.

If not, $\hat{\mathbf{a}}'$ (ignoring (ω, x'_ω) and the dummy) is an efficient envyfree assignment for A. But then since $\hat{\mathbf{a}}$ is efficient, (2) implies $v'_P = v$, and consequently, $x'_O = x$. By Lemma 3, there exists an object $\beta \in O$ with

$$x_\beta > x^*_\beta = x^*_M. \tag{3}$$

Then, using (1) and (3), $\varphi_A(\omega, x'_\beta) = \varphi_A(\omega, x_\beta) \geq v^*_A + k(x_\beta - x^*_M) > v^*_A + ((v_A - v^*_A)/(x_\beta - x^*_M))\,(x_\beta - x^*_M) = v_A = v'_A$ for every $A \in P$. In particular, $A = \mu'(\beta)$ would be better off receiving (ω, x'_β) instead of (β, x'_β), contrary to efficiency of $\hat{\mathbf{a}}'$.

Step (ii) $v'_B \geq v^*_B$.

By Step (i), there is an object $\eta \in O$ such that $\mu'(\eta) \notin P$. We *claim*

$$x_\eta \geq x^*_M. \tag{4}$$

Consider any $\alpha \in O - \eta$ and let $\mu'(\alpha) = A$. Then, from (2) and envyfreeness, $\varphi_A(\alpha, x'_\alpha) = v'_A \geq v_A \geq \varphi_A(\alpha, x_\alpha)$ so $x'_\alpha \geq x_\alpha$. On the other hand, by feasibility, $x'_\omega + \sum_{O-\eta} x'_\alpha = x_\eta + \sum_{O-\eta} x_\alpha$. Hence $x'_\omega \leq x_\eta$. Therefore, if (4) were false, using (1), we would get $v'_B = \varphi_B(\omega, x'_\omega) \leq \varphi_B(\omega, x_\eta) = \max\{v^*_B + k(x_\eta - x^*_M),\ x_\eta + \delta\} < \max\{v^*_B + ((v_B - v^*_B)/(x_\eta - x^*_M))\,(x_\eta - x^*_M), x^*_M\} = \max\{v_B, x^*_M\}$. But, $\max\{v_B, x^*_M\} = v_B$, since by envyfreeness, (3), and the desirability of β, $v_B \geq \varphi_B(\beta, x_\beta) > \varphi_B(\beta, x^*_M) \geq x^*_M$. So $v'_B < v_B$ contradicting (2). End of claim.

Let $P' = \{A \in P \mid v'_A < v^*_A\}$. Take any $A \in P'$ and let $\mu^*(A) = \alpha$. Using envyfreeness and (2), $\varphi_A(\alpha, x_\alpha) \leq v_A \leq v'_A < v^*_A = \varphi_A(\alpha, x^*_\alpha)$ hence $x_\alpha < x^*_\alpha$ so from (4) $\alpha \neq \eta$ in particular $\mu'(\alpha) \equiv A' \in P$. Then, using envyfreeness, $\varphi_A(\alpha, x'_\alpha) \le v'_A < v^*_A = \varphi_A(\alpha, x^*_\alpha)$ so $x'_\alpha < x^*_\alpha$ which implies $v'_{A'} = \varphi_{A'}(\alpha, x'_\alpha) < \varphi_{A'}(\alpha, x^*_\alpha) \le v^*_{A'}$, that is $A' \in P'$.

Call $\mu^*(P') = O'$. We have just shown that $\mu'(O') \subseteq P'$. Of course $|O'| = |P'|$ so in fact $\mu'(O') = P'$. Therefore, if B were in P', $\mu'(B)$ would be an object in O', contradicting $\mu'(B) = \omega$. Hence $B \notin P'$.

Step (iii) $v'_B < v^*_B$.

This will follow from our *claim*

(5) $$x'_\omega < x^*_M,$$

for then, using (1), $v'_B = \varphi_B(\omega, x'_\omega) < \varphi_B(\omega, x^*_M) = \max\{v^*_B, x^*_M + \delta\} = v^*_B$.

If $x'_\omega > x^*_M$, by feasibility, $\hat{\mathbf{a}}'$ assigns at least one individual A an α with $x'_\alpha < x^*_\alpha$, but then using envyfreeness and (1) $v'_A = \varphi_A(\alpha, x'_\alpha) < \varphi_A(\alpha, x^*_\alpha) \le v^*_A = \varphi_A(\omega, x^*_M) < \varphi_A(\omega, x'_\omega)$, contrary to envyfreeness, therefore

(6) $$x'_\omega \le x^*_M.$$

Now let $\mu^*(\beta) = C$. From (3) and Lemma 2, $v_C > v^*_C$, so from (2)

(7) $$v'_C > v^*_C.$$

This implies $\mu'(C) \ne \omega$ for if not from (6) and (1) $v'_C = \varphi_C(\omega, x'_\omega) \le \varphi_C(\omega, x^*_M) = v^*_C$ contradicting (7). Call $\mu'(C) = v$. From (7) and envyfreeness, $\varphi_C(v, x'_v) = v'_C > v^*_C \ge \varphi_C(v, x^*_v)$ so $x'_v > x^*_v$. With this inequality at hand now, we get (5) simply by repeating the argument in the first sentence of this paragraph. This establishes our claim and the proof is over for the case $|P| = |O|$.

To conclude, let us state how our proof above also applies for the case $|P| > |O|$. As already mentioned, Step (i) holds vacuously. Step (ii) is shorter because the first paragraph is now unnecessary for the argument in the second paragraph and can be skipped altogether. Step (iii) holds identically. ◆

Remark 2: We mention that Theorem 2a admits a relatively simple proof with respect to strongly envyfree allocations, that is, if one replaces the set Φ in the statement with Φ^*. For emphasis, let us repeat that $\Phi^*(\mathsf{A})$ is often a proper subset of $\Phi(\mathsf{A})$ when $|P| < |O|$. It is therefore not surprising that our proof is more complicated for the case $|P| = |O|$ than for $|P| > |O|$. We shall conjecture here that, the minimum pure money allocation is the unique upper extendable allocation, when $|P| > |O|$ and undesirable as well as desirable objects are allowed. Let us also remark that the remaining case $|P| < |O|$ poses some difficulties not confronted in our proof above and it might be that the minmax money allocation is not the unique upper extendable allocation when $|P| < |O|$. (As already mentioned, ADG (1991) have proved Theorem 2a on the quasilinear domain. The proof in that case is much less involved but appears not to lend itself to a generalization onto the nonlinear domain.)

Note that the characterization obtained in Theorem 2a uses only criterion (i) of upper extendability. As powerful a characterization does not obtain via criterion (ii): Consider the problem $(\{A, B\}, \{\alpha\}, 0)$ where $\varphi_A(\alpha, \mathsf{x}) = 2 + \mathsf{x}$ and $\varphi_B(\alpha, \mathsf{x}) = \mathsf{x}$. An assignment here is (efficient) envyfree if and only if A gets $(\alpha, -\mathsf{x})$, B gets (ϕ, x), and $\mathsf{x} \in [0, 1]$. The minmax money allocation, where $\mathsf{x} = 0$, is the unique allocation which fulfils criterion (ii) in case A leaves, since B would be worse off with respect to any other initial allocation. On the other hand, *all* allocations fulfil criterion (ii) in case B leaves.

Interestingly, an analogue of Theorem 2a does not hold for lower extendability. Here in fact is an assignment problem *all* of whose efficient envyfree allocations are lower extendable.

Example 1: Consider the problem $\mathsf{A} = (\{A, B\}, \{\alpha, \beta\}, 0)$ where $\varphi_A(\alpha, \mathsf{x}) = \varphi_B(\beta, \mathsf{x}) = 3 + \mathsf{x}$ and $\varphi_A(\beta, \mathsf{x}) = \varphi_B(\alpha, \mathsf{x}) = 1 + \mathsf{x}$. Note that an assignment $\hat{\mathbf{a}}$ for A is envyfree if and only if A gets (α, x), B gets $(\beta, -\mathsf{x})$, and $\mathsf{x} \in [-1, 1]$. Take any such assignment $\hat{\mathbf{a}}(\mathsf{x})$. It is straightforward to check that $\hat{\mathbf{a}}(\mathsf{x})$ fulfils criterion (i) of lower extendability. We show below that $\hat{\mathbf{a}}(\mathsf{x})$ also fulfils criterion (ii) and so is lower extendable.

Let C be any individual and consider $\mathsf{A} \cup C$. Restrict $\mathsf{x} \in [0, 1]$ and consider the money vector $(x_\alpha, x_\beta, x_\phi) = (\mathsf{x}, -\mathsf{x}, 3 - \mathsf{x})$ where the moneys add up to $Y = 3 - \mathsf{x}$. Note that, A prefers α while B is indifferent between β and ϕ. Therefore, if C does not strictly prefer α, then one has an envyfree assignment for $Y > 0$, and so $\hat{\mathbf{a}}(\mathsf{x})$ fulfils criterion (ii) by MMT. So suppose C strictly prefers α. Now decrease x_α in the money vector $(\mathsf{x}, -\mathsf{x}, 3 - \mathsf{x})$ by $\delta > 0$ until either A or C prefers one of β, ϕ, and note that at this point an envyfree assignment exists for $Y = 3 - \mathsf{x} - \delta$. Check that $\delta \leq 2\mathsf{x}$, for A prefers ϕ at $\delta = 2\mathsf{x}$. Thus $Y \geq 0$ and so $\hat{\mathbf{a}}(\mathsf{x})$ meets criterion (ii) by MMT again. (The case $\mathsf{x} \in [-1, 0]$ is treated identically.) ◆

There obtains for lower extendability, nevertheless, a weak uniqueness result which we next state. It will suffice to restrict attention to the class $\mathscr{C}$ of all assignment problems $\mathsf{A} = (\{A, B\}, \{\alpha\}, 0) \in \mathscr{A}$ where the set of envyfree value vectors, $V(\mathsf{A}) \equiv \{v \mid (v, x) \in \Phi(\mathsf{A})\}$, is a linear segment in R^2_+ of at least unit length. (Let $d(v, v') = |v_A - v'_A| + |v_B - v'_B|$.)

Theorem 2 b): Denote (v^*, x^*) the maxmin value allocation. For any $\varepsilon > 0$, there exists an assignment problem $\mathsf{A} \in \mathscr{C}$ such that, if $(v, x) \in \Phi(\mathsf{A})$ and $d(v, v^*) > \varepsilon$, then $v'_A > v_A$ for all $(v', x') \in \Phi(\mathsf{A} \cup B')$ where B' is a replica of B, i.e., (v, x) is not lower extendable.

Proof: Consider $\mathsf{A} \in \mathscr{C}$ with $\varphi_A(\alpha, \mathsf{x}) = \mathsf{x}$, $\varphi_B(\alpha, \mathsf{x}) = (1/1 - 2\delta)(1 - \delta + \delta\mathsf{x})$ for $\mathsf{x} \leq 1$ (and some $\delta \in (0, 1/2)$). Note that an assignment for A is envyfree if and only if A gets (ϕ, x), B gets $(\alpha, -\mathsf{x})$, and $\mathsf{x} \in [0, 1]$. Thus, $V(\mathsf{A})$ is the linear segment with endpoints $((1 - \delta)/(1 - 2\delta), 0)$ and $v^* = (1, 1)$.

Check that $\mathsf{A} \cup B'$ has a unique envyfree allocation, in which B or B' gets the bundle $(\alpha, -2 + 2\delta)$, the other two individuals get $(\phi, 1 - \delta)$ each, and so everyone attains the value $1 - \delta$. Thus, if $(v, x) \in \Phi(\mathsf{A})$ is lower extendable, then $v_A \leq 1 - \delta$. That is, as one computes, v must lie in the segment whose endpoints are $(1 - \delta, (1 - \delta)^2/(1 - 2\delta))$ and $(1, 1)$, i.e., $d(v, v^*) = |v_A - 1| + |v_B - 1| \leq \delta(1 - \delta)/(1 - 2\delta) \equiv \varepsilon$. ◆

6. Concluding remarks

We have shown that, while there always exist two particular allocations in the efficient envyfree solution which permit monotonic extensions for all one-member variations in the set of individuals or objects, in one of two directions respectively, no other allocations have this property. These results thus identify the boundary of compatibility between envyfreeness and population/resource monotonicity in the assignment problem and show that the efficient envyfree solution is largely nonmonotonic. As illustrated below, compatibility in fact disappears entirely under two-member variations.

Example 2: Initially, there is one individual A holding an object α of zero value to him. Then, individuals B and B' join, each of whom attaches a value of 3 to α. The (efficient) envyfree solution is a singleton: Either B or B' gets α and pays the other two individuals 1 each. A is better off. No-envy is incompatible with population monotonicity. (An analogous example given in ADG (1991), with a two-object variation, shows that no-envy is incompatible with resource monotonicity.) ◆

In closing, we mention a reference and offer an observation regarding solutions to the assignment problem which do meet the monotonicity criteria. As Moulin (1992) and Mo and Gong (1990) have shown, on the quasilinear domain, there is an efficient solution to the assignment problem which is both resource and population monotonic, namely the Shapley Value solution, determined on the TU assignment game where the worth of a coalition is defined as the maximum total value its members can attain if they alone had access to all the resources. The underlying fact is that the game is concave. The monotonicities hold, in fact, for any weighted Shapley Value solution. Outside the quasilinear domain, however, the (NTU) assignment game is typically nonmonotonic, and not surprisingly, Shapley Value type solutions (e.g., the Kalai and Samet (1985) egalitarian solution) do not fulfil the monotonicity criteria. We record here the following example on how extreme a prescription the population monotonicity axiom, for instance, may here lead to.

Two individuals A and B have equal claims on one object α. The value of (α, x) to A is $100 + \mathrm{x}$ for all x, and to B is $100 + \mathrm{x}$ for $\mathrm{x} \geq 0$ while $100 + 10\mathrm{x}$ for $\mathrm{x} \leq 0$. Then, if one requires population monotonicity along with the axioms of efficiency and equal treatment of equals (i.e., identical individuals should fare identically), A should get the object (by the efficiency axiom) and transfer no more than 10 to B meaning that he himself achieve at least 90. (Proof: First, consider α with A absent and $k + 1$ copies of B present. By equal treatment of equals, each B should achieve the value $100/(10k + 1)$. Next consider α with A and $k + 1$ copies of B present. Then, A should get α, and by population monotonicity transfer no more than $100/(10k + 1)$ to each B, himself achieving at least $100 - 100\,(k + 1)/(10k + 1)$, which equals 90 in the limit. So, when $k = 0$ initially, A should achieve at least 90 hence transfer no more than 10 to B.) Note that the efficient value frontier in R^2 here consists of all splits of 100 and one may feel hard put to justify why B should end up getting (so much) less than A. For concreteness, consider the following instance of this assignment problem: A and B receive as inheritance an object that has a market value of 100 thousand dollars. Neither individual finds any use value in the object and so each would sell it if he/she were the recipient. The two individuals are in fact all alike except that A lives in a dollar country while B lives in a country the currency of which exchanges against the dollar at the official rate of 1 for inflows but at the black market rate of 10 for outflows.

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