On Algorithmic Solutions to Simple Allocation Problems*

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Abstract

We interpret solution rules to a class of simple allocation problems as data on the choices of a policy-maker. We study the properties of *rational* rules. We show that every *rational* rule falls into a class of *algorithmic rules* that we describe. The *Equal Gains* rule is a member of this class and it uniquely satisfies *rationality, continuity,* and *equal treatment of equals.* Its dual, the Equal Losses rule, uniquely satisfies *continuity, equal treatment of equals,* and two properties that constitute the dual of rationality: *translation down* and *translation up.*

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1 Introduction

Revealed preference theory studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Choice rules for which this is possible are called *rational.* Most of the earlier work on rationality analyzes consumers' demand choices from budget sets (e.g. see Samuelson, 1938, 1948). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be "rationalized" as maximizing the underlying preferences of an impartial arbitrator (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 1999; Sánchez, 2000).

In this paper, we study the implications of *rationality* on a class of solutions to *simple allocation problems.*[1] A simple allocation problem for a society N is an $|N|+1$ dimensional nonnegative real vector $\left(c_1, ...c_{|N|}, E\right) \in \mathbb{R}^N_+$ satisfying $\sum_N c_i \geqq E$ where E, the **endowment** has to be allocated among agents in N who are characterized by c, the **characteristic vector**. Simple allocation problems have a wide range of applications. We discuss them in detail in *Subsection 1.1.*

Our results are as follows. In **Section 3**, we describe a class of *algorithmic rules.* Such a rule chooses an allocation for each problem (c, E) by following an algorithm which gradually reduces each agent i's characteristic value from E to c_i. At each Step $k \in \{0, ..., n\}$, the algorithm takes into account only the characteristics of k agents (that is, for each $|S| = k$, defining c^S as $c^S = \left(c_S, E_{N \setminus S}\right)$, the algorithm calculates $x^S = F\left(c^S, E\right)$).[2] If there is a feasible allocation x^S (*i.e.* satisfying $x^S \leqq c$), the algorithm chooses it. Otherwise, it moves to the next step.

Proposition 2 states that every *rational* rule is *algorithmic.* The converse, however, is not true, even for *continuous* rules. *Proposition 4* states that every *rational* rule which is

[1] Thus, as discussed in *Subsection 1.1*, we interpret an allocation rule on simple allocation problems as representing the choices of a decision-maker (*e.g.* a public-policy maker, a tax codifier or a bankruptcy judge).

[2] With an abuse of notation, we use $E_{N \setminus S}$ to denote an $|N \setminus S|$ dimensional vector whose every coordinate is E.

continuous with respect to the characteristic vector is an *algorithmic* rule that additionally satisfies the following properties: first, for problems (c, E) with $\sum_N c_i > E$, the algorithm stops at a Step $k < n$; second, if the algorithm stops at Step k with coalition S, then each member of S receives his characteristic value.

Section 4 contains the main result of the paper: *Theorem 1* uses the above results to show that the *Equal Gains rule* uniquely satisfies *rationality, continuity*, and *equal treatment of equals* (a fairness property which requires that agents with identical characteristics should receive identical shares).

The literature contains other characterizations of the *Equal Gains rule.* Dagan (1996) shows that this rule uniquely satisfies *equal treatment of equals, truncation invariance*, and *composition up.*[3] Schummer and Thomson (1997) show that the allocation chosen by the *Equal Gains rule* minimizes (*i*) the difference between the largest and the smallest share and (*ii*) the variance of the shares. In a related result, Bosmans and Lauwers (2006) show that the allocation chosen by the *Equal Gains rule* Lorenz dominates every other allocation. Herrero and Villar (2002) and Yeh (2004) show that the *Equal Gains rule* uniquely satisfies *conditional full compensation* and *composition down.*[4] Finally, Yeh (2006) shows that the *Equal Gains rule* uniquely satisfies *conditional full compensation* and *own-claim monotonicity.*

Our characterization is logically independent from these previous results. Furthermore, the main principles employed in these characterizations (such as "composition", *full compensation*, or *Lorenz domination*) are quite different than our main axiom: *rationality.* Also, with the exception of Schummer and Thomson (1997) and Bosmans and Louwers (2006), the above characterizations use properties that relate the rule's behavior at different social endowment levels. This is not the case for *Theorem 1.*

[3] *Composition up* requires that dividing the social endowment in two, first allocating one part, revising the characteristic vector accordingly, and then allocating the rest produces the same final allocation as allocating all the social endowment at once.

[4] *Conditional full compensation* roughly requires agents with sufficiently small characteristic values to receive their characteristic values. *Composition down* deals with the following scenario: after the social endowment is allocated, we discover that the actual social endowment is smaller; then, it requires that using the original characteristic vector or the initially chosen allocation should produce the same final outcome.

Our final results are in **Section 5** where we first introduce two properties: *translation down* and *translation up*. Both are concerned with the implications of translating a problem by simultaneously changing, at the same amount, the characteristic value of an agent and the endowment. For such translations, these properties require that the initial allocation be translated the same way. We show, in *Lemma 5*, that a rule satisfies *translation down* and *translation up* if and only if its dual rule is *rational*. In *Theorem 2*, we then use this lemma and *Theorem 1* to show that the *Equal Losses rule* uniquely satisfies *translation down, translation up, continuity*, and *equal treatment of equals.*

In a companion paper (Kıbrıs, 2008), we carry out a revealed preference analysis on simple allocation problems and study *rational, transitive-rational*, and *representable* rules. There, we show that an allocation rule is *rational* if and only if it satisfies a standard property called *contraction independence* (also called *independence of irrelevant alternatives* in the context of bargaining by Nash (1950) and *Property* α in the context of consumer choice by Sen (1971)). In this paper, we make extensive use of this equivalence.

In the next subsection, we discuss the various applications of our analysis. In *Section 2*, we present our model and further discuss rational rules. In the following sections, we present our results as summarized above.

1.1 Examples and Applications

A *simple allocation problem* for a society N is an $|N|+1$ dimensional nonnegative real vector $\left(c_1, ...c_{|N|}, E\right)$ which, with the exception of the last application below, is interpreted as follows. A social endowment E of a perfectly divisible commodity is to be allocated among members of N. Each agent $i \in N$ is characterized by an amount c_i of the commodity. Next, we discuss the alternative interpretations of c and E at various applications.

1. **Taxation:** A public authority is to collect an amount E of *tax* from a society N. Each agent i has *income* c_i. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of "parametric solutions" to this problem.

2. **Bankruptcy:** A bankruptcy judge is to allocate the remaining *assets* E of a bankrupt

firm among its creditors, N. Each agent i has *credited* c_i to the bankrupt firm and now, claims this amount. For example, see O'Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

3. **Permit Allocation**: The Environmental Protection Agency is to allocate an amount E of *pollution permits* among firms in N (such as CO_2 emission permits allocated among energy producers). Each firm i, depending on its location, is imposed by the local authority an *emission constraint* c_i on its pollution level. For more on this application, see Kıbrıs (2003) and the literature cited therein.

4. **Single-peaked or Saturated Preferences**: A social planner is to allocate E units of a perfectly divisible commodity among members of N. Each agent i is known to have preferences with *peak (saturation point)* c_i. The rest of the preference information is disregarded as typical in several well-known solutions to this problem, such as the Uniform rule or the Proportional rule. For example, see Sprumont (1991) and the following literature.

5. **Demand Rationing**: A supplier is to allocate its *production* E among demanders in N. Each demander i *demands* c_i units of the commodity. The supply-chain management literature contains detailed analysis of this problem.[5] For example, see Cachon and Lariviere (1999) and the literature cited therein.

6. **Bargaining with Quasilinear Preferences and Claims:** An arbitrator is to allocate E units of a *numeriare good* among agents who have quasilinear preferences with respect to it. Each agent holds a *claim* c_i on what he should receive. For examples of bargaining problems with claims, see Chun and Thomson (1992) and the following literature. For bargaining problems with quasilinear preferences, see Moulin (1985) and the following literature.

7. **Surplus Sharing**: A social planner is to allocate the *return* E of a project among its investors in N. Each investor i has invested s_i. The project is profitable, that is,

[5] We would like to thank Rakesh Vohra for bringing this application to our attention.

$\sum_N s_i \leqq E$. Using the principal that no agent should receive less than his investment, define the *maximal share of an agent i* as $c_i = E - \sum_{N \setminus \{i\}} s_j$. Note that $\sum_N c_i \geqq E$. The surplus sharing problem can now be analyzed as a simple allocation problem. For more on surplus-sharing problems, see Moulin (1985 and 1987) and the following literature.

8. **Consumer Choice under fixed prices and rationing:** A consumer has to allocate his *income* E among a set N of commodities. The prices of the commodities are fixed and thus, do not change from one problem to another. (With appropriate choice of consumption units, normalize the price vector so that all commodities have the same price.) As typical in the fixed-price literature, the consumer also faces "rationing constraints" on how much he can consume of each commodity. Let c_i be the agent's *consumption constraint* on commodity i. See Benassy (1993) or Kıbrıs and Küçükşenel (2008) for more on rationing rules.

2 Model

Let $N = \{1, ..., n\}$ be the set of agents. For $i \in N$, let e_i be the i^{th} unit vector in $\mathbb{R}^N_+$. Let $e = \sum_N e_i$. We use the vector inequalities $\leqq, \leq, <$. For $c \in \mathbb{R}^N_+$, $\alpha \in \mathbb{R}_+$, and $S \subseteq N$, with an abuse of notation, we write $\left(c_S, \alpha_{N \setminus S}\right)$ to denote the vector which coincides with c on S and which chooses α for every coordinate in $N \setminus S$.

A **simple allocation problem** for N is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geqq E$ (please see Figures 1 and 2). We call E the **endowment** and c the **characteristic vector**. As discussed in Subsection 1.1, depending on the application, E can be an asset or a liability and c can be a vector or incomes, claims, demands, preference peaks, or consumption constraints. Let $\mathcal{C}$ be the set of all simple allocation problems for N. Given a simple allocation problem $(c, E) \in \mathcal{C}$, let $X(c, E) = \left\{x \in \mathbb{R}^N_+ \mid x \leqq c \text{ and } \sum x_i \leqq E\right\}$ be the **choice set of** (c, E).

An allocation **rule** $F : \mathcal{C} \rightarrow \mathbb{R}^N_+$ assigns each simple allocation problem (c, E) to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c, E) = E$. Note that each rule F satisfies

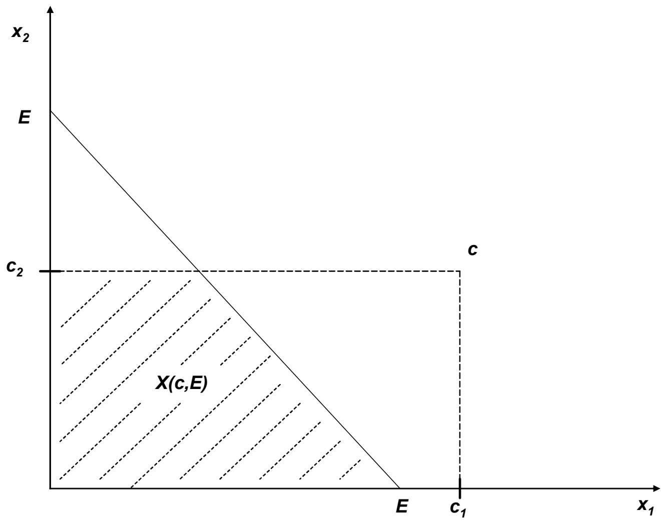

Figure 1: A two-agent simple allocation problem.

$F(c,E) \leqq c$. Depending on the application, this might be interpreted as satisfying the consumption constraints or as an efficiency requirement (as in the case of single-peaked preferences) or that no agent be taxed more than his income. Also, $\sum_N F_i(c,E) = E$ can be interpreted as an *efficiency* property or, as in taxation, a feasibility requirement or as in consumer choice, the Walras law.

The following are some well-known examples of rules. The **Proportional rule** allocates the endowment proportional to the characteristic values: for each $i \in N$, $PRO_i(c,E) = \frac{c_i}{\sum_N c_j}E$. In the taxation literature, this rule is called a *Linear Tax.* The **Equal Gains rule** allocates the endowment equally, subject to no agent receiving more than his characteristic value: for each $i \in N$, $EG_i(c,E) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \min\{c_i, \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Uniform rule,* in the bankruptcy literature it is called the *Constrained Equal Awards rule*, and in the taxation literature, it is called the *Leveling Tax.* The **Equal Losses rule** equalizes the losses agents incur, subject to no agent receiving a negative share: for each $i \in N$, $EL_i(c,E) = \max\{0, c_i - \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \max\{0, c_i - \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Equal Distance rule,* in the bankruptcy literature it is called

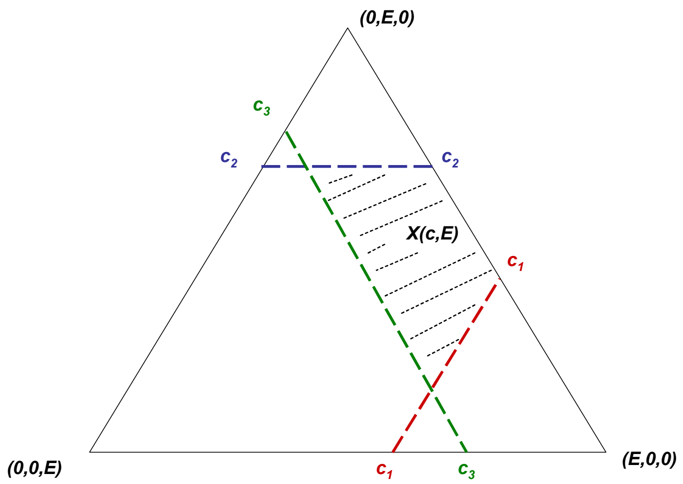

Figure 2: A three-agent simple allocation problem.

the *Constrained Equal Losses rule*, and in the taxation literature, it is called the *Head Tax*. The **Talmud rule** (Aumann and Maschler, 1985) assigns equal gains until each agent receives half his characteristic value and then uses the equal losses idea: $TAL(c,E) = EG\left(\frac{1}{2}c, \min\left\{E, \frac{1}{2}\sum_N c_i\right\}\right) + EL\left(\frac{1}{2}c, \max\left\{0, E - \sum_N c_i\right\}\right).$

A rule F is **rational** if there is a binary relation $B \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c,E) \in \mathcal{C},\ F(c,E) = \{x \in X(c,E) \mid \text{for each } y \in X(c,E),\ xBy\}$. Kıbrıs (2008) shows that *rationality* is equivalent to the following property. A rule F satisfies **contraction independence** if a chosen alternative from a set is still chosen from subsets (contractions) that contain it: for each pair $(c,E), (c',E) \in \mathcal{C},\ F(c,E) \in X(c',E) \subseteq X(c,E)$ implies $F(c',E) = F(c,E)$. In the literature, this property is also referred to as *independence of irrelevant alternatives* (Nash, 1950) or *Sen's property* α (Sen, 1971).

Theorem A. (Kıbrıs, 2008) A rule F is *rational* if and only if it is *contraction independent.*

The following lemma is from Kıbrıs (2008). For completeness, we include the simple proof.

Lemma 1 *A rule F is* rational *if and only if for each $(c, E), (c', E) \in \mathcal{C}$ it satisfies the following properties*

Property (i). *if for each $i \in N$, $\min\{c_i, E\} = \min\{c'_i, E\}$, then $F(c, E) = F(c', E)$,*

Property (ii). *if $F(c, E) \leqq c' \leqq c$, then $F(c', E) = F(c, E)$.*

Proof. $(\Rightarrow)$ Assume that F is *rational.* Then, by Theorem A, it satisfies *contraction independence.* Let $(c, E), (c', E) \in \mathcal{C}$. First, assume that for each $i \in N$, $\min\{c_i, E\} = \min\{c'_i, E\}$. Let $x \in \mathbb{R}^N_+$ satisfy $\sum_N x_i \leqq E$. Then, $x \leqq c$ if and only if $x \leqq c'$. Thus $X(c, E) = X(c', E)$. This implies $F(c, E) = F(c', E)$. Next, assume that $F(c, E) \leqq c' \leqq c$. Then, $F(c, E) \in X(c', E) \subseteq X(c, E)$, which by *contraction independence*, implies $F(c, E) = F(c', E)$.

$(\Leftarrow)$ Assume that (i) and (ii) are satisfied. Let $(c, E), (c', E) \in \mathcal{C}$ be such that $F(c, E) \in X(c', E) \subseteq X(c, E)$. Then for each $i \in N$, either $c'_i \leqq c_i$ or $\min\{c'_i, E\} = \min\{c_i, E\}$. Let $S = \{i \in N \mid c'_i \leqq c_i\}$. Let $c'' = (c'_S, c_{N \setminus S})$. Then $F(c, E) \leqq c'' \leqq c$ and by (ii), $F(c, E) = F(c'', E)$. Now, for each $i \in N$, $\min\{c'_i, E\} = \min\{c''_i, E\}$. Thus, by (i), $F(c'', E) = F(c', E)$. Altogether, we obtain $F(c, E) = F(c', E)$. Thus, F satisfies *contraction independence.* Then, by Theorem A, F is *rational.* ■

Property (i) of *Lemma 1* is called *truncation-invariance* for rules on bankruptcy and taxation problems (Thomson, 2003 and 2007). Property (ii) says that a decrease in characteristic values does not change the initially chosen allocation as long as it remains feasible.

In what follows, we will make extensive use of the equivalence stated in *Lemma 1.*

3 Rationality vs Algorithmic Rules

Using *Lemma 1*, it is straightforward to check that the *Equal Gains rule* is *rational* while the *Proportional rule, Equal Losses rule*, and the *Talmudic rule* are not. One important difference of the *Equal Gains rule* from the others is that it can be alternatively defined as choosing the outcome of the following algorithm: let $(c, E) \in \mathcal{C}$,

Step 1. Propose equal division of the endowment among all agents, that is, let $x^1 =$

$EG(E_N, E)$.[6] If no agent receives more than his characteristic value, that is, if $x^1 \leqq c$, stop and let $EG(c, E) = x^1$. Otherwise, let $S_1 = \{i \in N \mid x_i^1 \geqq c_i\}$ and move to Step 2.

Step k. (for $k = 2, ..., |N|$) Propose to each $i \in S_{k-1}$ his characteristic value. (These are agents who, in Step $k-1$, were proposed a share at least as large as their characteristic values). Propose equal division of the remaining endowment among the remaining agents, that is, let $x^k = EG\left(c_{S_{k-1}}, E_{N \setminus S_{k-1}}, E\right)$. If no agent receives more than his characteristic value, that is, if $x^k \leqq c$, stop and let $EG(c, E) = x^k$. Otherwise, let $S_k = \left\{i \in N \mid x_i^k \geqq c_i\right\}$ and move to Step $k+1$.

In what follows, we present a property that generalizes this idea.

Definition 1 *A rule F is* ***algorithmic*** *if for each $(c, E) \in \mathcal{C}$, $F(c, E)$ is the outcome of the following algorithm:*

Step 0. *If $F(E_N, E) \leqq c$, let $F(c, E) = F(E_N, E)$. Otherwise, move to Step 1.*

Step 1. *If there is $S \subseteq N$ such that $|S| = 1$ and $F\left(c_S, E_{N \setminus S}, E\right) \leqq c$, let $F(c, E) = F\left(c_S, E_{N \setminus S}, E\right)$. Otherwise, move to Step 2.*

For $k = 2, ..., |N|$:

Step k. *If there is $S \subseteq N$ such that $|S| = k$ and $F\left(c_S, E_{N \setminus S}, E\right) \leqq c$, let $F(c, E) = F\left(c_S, E_{N \setminus S}, E\right)$. Otherwise, move to Step $k+1$.*

If F is not an *algorithmic* rule, two things can happen: either (i) the algorithm finds a feasible allocation at some step k, but another allocation is chosen for the original problem, or (ii) the algorithm finds multiple feasible allocations at some step k and it is not clear which one to choose.

The *Equal Gains rule* is not the only *algorithmic rule.* Any *rational* rule satisfies the property.

Proposition 2 If a rule F is *rational* then it is an *algorithmic rule.*

Proof. Let F be *rational.* We first show that for each pair $S, T \subseteq N$ and each $(c, E) \in \mathcal{C}$ such that $F_T\left(c_S, E_{N \setminus S}, E\right) \leqq c_T$ and $F_S\left(c_T, E_{N \setminus T}, E\right) \leqq c_S$, we have $F\left(c_S, E_{N \setminus S}, E\right) =$

[6] Note that E_N denotes a claims vector whose every coordinate is E.

$F\left(c_T, E_{N\setminus T}, E\right)$. To see this, assume $F_T\left(c_S, E_{N\setminus S}, E\right) \leqq c_T$ and $F_S\left(c_T, E_{N\setminus T}, E\right) \leqq c_S$. Then

$$F\left(c_S, E_{N\setminus S}, E\right) \leqq \left(c_{S\cup T}, E_{(S\cup T)^c}\right) \leqq \left(c_S, E_{N\setminus S}\right).$$

By *Lemma 1,* $F\left(c_{S\cup T}, E_{(S\cup T)^c}, E\right) = F\left(c_S, E_{N\setminus S}, E\right)$. Similarly,

$$F\left(c_T, E_{N\setminus T}, E\right) \leqq \left(c_{S\cup T}, E_{(S\cup T)^c}\right) \leqq \left(c_T, E_{N\setminus T}\right)$$

implies $F\left(c_{S\cup T}, E_{(S\cup T)^c}, E\right) = F\left(c_T, E_{N\setminus T}, E\right)$. Thus, $F\left(c_S, E_{N\setminus S}, E\right) = F\left(c_T, E_{N\setminus T}, E\right)$.

We now show that F is an *algorithmic rule.* Let $(c, E) \in \mathcal{C}$. First assume that $F(E_N, E) \leqq c$. Then by *Lemma 1*, $F(c, E) = F(E_N, E)$. Thus, the algorithm yields the allocation at Step 0. Now suppose $F(E_N, E) \nleqq c$.

Since $F(c_N, E) \leqq c$, there is some $k \in \{1, ..., n\}$ such that (i) for each $T \subset N$ such that $|T| < k$, we have $F\left(c_T, E_{N\setminus T}, E\right) \nleqq c$ and (ii) for some $S \subseteq N$ such that $|S| = k$, we have $F\left(c_S, E_{N\setminus S}, E\right) \leqq c$. We will next show that the algorithm then stops at Step k and yields $F\left(c_S, E_{N\setminus S}, E\right)$.

Because of (i), the algorithm does not stop at any Step $l < k$. Note that for each $S' \subseteq N$ such that $|S'| = k$, if $F\left(c_{S'}, E_{N\setminus S'}, E\right) \leqq c$, then by the first paragraph, $F\left(c_{S'}, E_{N\setminus S'}, E\right) = F\left(c_S, E_{N\setminus S}, E\right)$. Applying *Lemma 1* to this unique allocation, $F(c, E) = F\left(c_S, E_{N\setminus S}, E\right)$, the desired conclusion. Thus, F is *algorithmic.* ■

Surprisingly, not every *algorithmic rule* is *rational.* The following example demonstrates this point.

Example 1 *(An* algorithmic rule *that is not* rational*) Let* $N = \{1, 2\}$ *and let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be defined as* $f(x) = \max\{0, 2x - 10\}$

$$F(c, E) = \begin{cases} (0, 10) & \textit{if} \quad E = 10,\ c_2 \geqq 10 \\ (10 - f(c_2), f(c_2)) & \textit{if} \quad E = 10,\ c_2 < 10,\ c_1 \geqq 10 - f(c_2) \\ (c_1, 10 - c_1) & \textit{if} \quad E = 10,\ c_2 < 10,\ c_1 < 10 - f(c_2) \\ EG(c, E) & \textit{if} \quad E \neq 10. \end{cases}$$

The rule F in the example is *algorithmic.* The reason is, first, $F(E_N, E) = (0, 10)$ is chosen in every problem for which it is feasible. All other problems have $c_2 < 10$ and for them, $F(E_1, c_2, E) = (10 - f(c_2), f(c_2))$ is chosen whenever it is feasible. All remaining problems have $c_1 < 10 - f(c_2)$ and $c_2 < 10$ and the definition of an algorithmic rule does not determine how they should be solved. The rule F is not *rational* since $F(10, 9, 10) = (2, 8)$ and $F(3, 8, 10) \neq (2, 8)$ violates Property (ii) of *Lemma 1*. Finally note that F is also *continuous*.

We next introduce a subclass of algorithmic rules.

Definition 2 *A rule F is an* ***algorithmic rule with boundary condition*** *if (i) F is an algorithmic rule, (ii) for each $(c, E) \in \mathcal{C}$, if $F(c, E)$ is first obtained at Step k with the coalition $S \subseteq N$ such that $|S| = k$, then $F_S(c, E) = c_S$, (iii) for each $(c, E) \in \mathcal{C}$ such that $\sum_N c_i > E$, there is $k < n$ such that $F(c, E)$ is first obtained at a Step k of the algorithm.*

It turns out that any *contraction independent* and *c-continuous* rule is of this form. To prove this result, we use the following lemma.

Lemma 3 Assume that F is *rational* and *c-continuous.* Then, for each $(c, E) \in \mathcal{C}$, $i \in N$, and $\delta \in \mathbb{R}_+$, $F_i(c + \delta e_i, E) > c_i$ implies $F_i(c, E) = c_i$.

Proof. Let $(c, E) \in \mathcal{C}$, $i \in N$, and $\delta \in \mathbb{R}_+$ satisfy $F_i(c + \delta e_i, E) > c_i$. Suppose $F_i(c, E) < c_i$. Then by *claims-continuity*, there is $\varepsilon < \delta$ such that $F_i(c + \varepsilon e_i, E) = c_i$. But then, $F(c + \varepsilon e_i, E) \leqq c \leqq c + \varepsilon e_i$, by *Lemma 1*, implies $F(c, E) = F(c + \varepsilon e_i, E)$, a contradiction. ■

We next present the proposition.

Proposition 4 If a rule F is *rational* and *c-continuous*, then it is an *algorithmic rule with boundary condition.*

Proof. Let F satisfy the given properties. Then, by *Proposition 2*, F is an algorithmic rule.

Next, let $(c, E) \in \mathcal{C}$ and assume that $F(c, E)$ is first obtained at Step k with the coalition $S \subseteq N$ such that $|S| = k$. Then, $F(c, E) = F\left(c_S, E_{N \setminus S}, E\right)$. If $S = \emptyset$, $F_S(c, E) = c_S$ is trivially satisfied. So assume $S \neq \emptyset$ and let $i \in S$. Since $F(c, E)$ is not obtained at Step $k-1$,

$F\left(c_{S\backslash i}, E_{N\backslash(S\backslash i)}, E\right) \nleqq c$. Since $F\left(c_S, E_{N\backslash S}, E\right) \leqq c$, this implies $F_i\left(c_{S\backslash i}, E_{N\backslash(S\backslash i)}, E\right) > c_i$.[7] Then, by *Lemma 3*, $F_i\left(c_S, E_{N\backslash S}, E\right) = c_i$. Since this conclusion holds for all $i \in S$, we have $F_S(c, E) = c_S$.

Finally, let $(c, E) \in \mathcal{C}$ be such that $\sum_N c_i > E$. We show that $F(c, E)$ is first obtained at a Step $k < n$. For contradiction, suppose $F(c, E)$ is first obtained at Step n. Then, by the previous paragraph, $F(c, E) = c$. But then, $\sum_N F_i(c, E) = \sum_N c_i = E < \sum_N c_i$, a contradiction. ■

For two-agent problems, any *algorithmic rule with boundary condition* is *c-continuous.* The following example demonstrates that for larger societies, this relationship does not hold anymore.

Example 2 *(An* algorithmic rule with boundary condition *which is not* c-continuous*) Let* $N = \{1, 2, 3\}$, *let* $\omega = (3, 1, 1)$, *and let* EG^ω *be the weighted Equal Gains rule with weight vector* ω, *defined as follows: for each* $(c, E) \in \mathcal{C}$, $EG_1^\omega(c, E) = \min\{c_1, 3\rho\}$ *and for* $i \in \{2, 3\}$, $EG_i^\omega(c, E) = \min\{c_i, \rho\}$ *where* $\rho \in \mathbb{R}_+$ *satisfies* $\sum_N EG_i^\omega(c, E) = E$. *Finally, let* F *be defined as*

$$F(c, E) = \begin{cases} EG(c, E) & if \quad c_3 > 0, \\ EG^\omega(c, E) & if \quad c_3 = 0. \end{cases}$$

4 A Characterization of the Equal Gains Rule

The following is an important fairness notion. A rule F satisfies **equal treatment of equals** if two agents with identical characteristics are always awarded equal shares: for each $(c, E) \in \mathcal{C}$ and $i, j \in N$, $c_i = c_j$ implies $F_i(c, E) = F_j(c, E)$. A large class of rules, including the four central ones introduced in Section 2, satisfy this property (*e.g.* see Young, 1987). Among them, however, the Equal Gains rule is the only *rational* rule.

Theorem 1 A rule F satisfies *rationality, c-continuity,* and *equal treatment of equals* if and only if it is the *Equal Gains rule.*

[7] Because, by *Lemma 1*, $F\left(c_{S\backslash i}, E_{N\backslash(S\backslash i)}, E\right) \neq F\left(c_S, E_{N\backslash S}, E\right)$ implies $F\left(c_{S\backslash i}, E_{N\backslash(S\backslash i)}, E\right) \nleqq \left(c_S, E_{N\backslash S}\right)$ and this implies $F_i\left(c_{S\backslash i}, E_{N\backslash(S\backslash i)}, E\right) > c_i$.

Proof. It is straightforward to show that EG satisfies the given properties. Conversely, let F be a rule that satisfies them. We next show $F = EG$. First note that, since F is *rational*, it satisfies the two properties of *Lemma 1*.

Let $(c, E) \in \mathcal{C}$. If $\sum_N c_i = E$, $F(c, E) = EG(c, E) = c$. Alternatively, assume $\sum_N c_i > E$. Note that, by *Proposition 4*, F and EG are both *algorithmic rules with boundary condition*.

Step 1: For each Step $k \in \{0, ..., n-1\}$ of the algorithm and for each $S \subsetneq N$ such that $|S| = k$, we have $F\left(c_S, E_{N\setminus S}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$. To prove this, we use induction. Initially, let $k = 0$. Thus, $S = \emptyset$ and by *equal treatment of equals*, $F\left(c_S, E_{N\setminus S}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$. Now let $k \in \{1, ..., n-1\}$ and assume that the statement holds for each $l < k$. Let $S \subsetneq N$ be such that $|S| = k$.

Case 1: There is $l < k$ and $T \subsetneq S$ such that $|T| = l$ and $F\left(c_S, E_{N\setminus S}, E\right) = F\left(c_T, E_{N\setminus T}, E\right)$. Then, by our assumption, $F\left(c_T, E_{N\setminus T}, E\right) = EG\left(c_T, E_{N\setminus T}, E\right)$. Thus, $EG\left(c_T, E_{N\setminus T}, E\right) \leqq \left(c_S, E_{N\setminus S}\right)$. By *Lemma 1* applied to EG, $EG\left(c_T, E_{N\setminus T}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$. Combining the equalities, we then have $F\left(c_S, E_{N\setminus S}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$.

Case 2: For each $l < k$ and $T \subsetneq S$ with $|T| = l$, $F\left(c_S, E_{N\setminus S}, E\right) \neq F\left(c_T, E_{N\setminus T}, E\right)$. Thus, $F\left(c_S, E_{N\setminus S}, E\right)$ is first obtained at Step k. Since, F is *algorithmic with boundary condition*, then $F_S\left(c_S, E_{N\setminus S}, E\right) = c_S$. Since F satisfies *equal treatment of equals*, for each $i \in N \setminus S$, $F_i\left(c_S, E_{N\setminus S}, E\right) = \frac{E - \sum_{j\in S} c_j}{n - |S|}$. By the induction hypothesis, for each $l < k$ and $T \subsetneq N$ with $|T| = l$, $F\left(c_T, E_{N\setminus T}, E\right) = EG\left(c_T, E_{N\setminus T}, E\right)$. By assumption of Case 2, for each $l < k$ and $T \subsetneq S$ with $|T| = l$, $F\left(c_S, E_{N\setminus S}, E\right) \neq F\left(c_T, E_{N\setminus T}, E\right)$, and thus, by *Lemma 1* applied to F, $F\left(c_T, E_{N\setminus T}, E\right) \nleqq \left(c_S, E_{N\setminus S}\right)$. Thus, for each $l < k$ and $T \subsetneq S$ with $|T| = l$, $EG\left(c_T, E_{N\setminus T}, E\right) \nleqq \left(c_S, E_{N\setminus S}, E\right)$. Therefore, $EG\left(c_S, E_{N\setminus S}, E\right)$ is first obtained at Step k. Since EG is *algorithmic with boundary condition*, then $EG_S\left(c_S, E_{N\setminus S}, E\right) = c_S$. Since EG satisfies *equal treatment of equals*, for each $i \in N \setminus S$, $EG_i\left(c_S, E_{N\setminus S}, E\right) = \frac{E - \sum_{j\in S} c_j}{n - |S|}$. Combining this with the observations on F, we have $F\left(c_S, E_{N\setminus S}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$.

Step 2: $F(c, E) = EG(c, E)$. First, since F is *algorithmic with boundary condition*, there is $k < n$ and $S \subsetneq N$ with $|S| = k$ such that $F(c, E) = F\left(c_S, E_{N\setminus S}\right)$. Then, by *Step 1*, $F\left(c_S, E_{N\setminus S}, E\right) = EG\left(c_S, E_{N\setminus S}, E\right)$. Therefore, $EG\left(c_S, E_{N\setminus S}, E\right) \leqq c$. Then, by *Lemma 1* applied to EG, we have $EG(c, E) = EG\left(c_S, E_{N\setminus S}, E\right)$. Combining the equalities, $F(c, E) = EG(c, E)$. ■

The above characterization is tight. Without *rationality*, *Proportional rule* becomes admissible. Without *equal treatment of equals*, all *algorithmic rules with boundary condition*, such as the weighted versions of the Equal Gains rule, become admissible. Finally, the following example presents a rule that violates *c-continuity* but satisfies the other properties.

Example 3 *(A rule that satisfies* rationality *and* equal treatment of equals *but not* c-continuity*) Let* $N = \{1, 2\}$ *. Let* F *be defined as*

$$F(c, E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \textit{if } c_1 \geqq \frac{E}{2} \textit{ and } c_2 \geqq \frac{E}{2}, \\ (E, 0) & \textit{if } c_1 \geqq E \textit{ and } c_2 < \frac{E}{2}, \\ (c_1, E - c_1) & \textit{if } E - c_2 < c_1 < E \textit{ and } c_2 < \frac{E}{2}, \\ (0, E) & \textit{if } c_1 < \frac{E}{2} \textit{ and } c_2 \geqq E, \\ (E - c_2, c_2) & \textit{if } c_1 < \frac{E}{2} \textit{ and } E - c_1 < c_2 < E. \end{cases}$$

5 Translation Properties and Duality

We start this section by introducing two properties that require that the solution to a problem be covariant under certain translations of the problem. Similar properties have been analyzed in bargaining theory (*e.g.* see Thomson, 1981).

A rule F satisfies **translation down** *(Figure 3, left)* if for each $(c, E) \in \mathcal{C}$, each $i \in N$, and each $\delta \in (0, F_i(c, E)]$, we have $F(c - \delta e_i, E - \delta) = F(c, E) - \delta e_i$. It satisfies **translation up** *(Figure 3, right)* if for each $(c, E) \in \mathcal{C}$, each $i \in N$ such that $c_i \geqq \sum_N c_j - E$, and each $\delta \in (0, \infty)$, we have $F(c + \delta e_i, E + \delta) = F(c, E) + \delta e_i$. Both properties are concerned with the implications of translating a problem by simultaneously changing, at the same amount, the characteristic value of an agent and the endowment. For such translations, these properties require that the initial allocation be translated the same way.

It turns out that these two properties are very closely related to *rationality*. The relationship is through a concept called duality. The **dual of a rule F**, $\mathbf{F^d}$, allocates what is available in the same way as F allocates what is missing, that is, for each $(c, E) \in \mathcal{C}$, $F^d(c, E) = c - F(c, \sum_N c_i - E)$. Aumann and Maschler (1985) quote several passages from the Talmud where the notion of duality is implicitly discussed and self-duality of a rule (that is, the rule coinciding with its dual) is promoted.

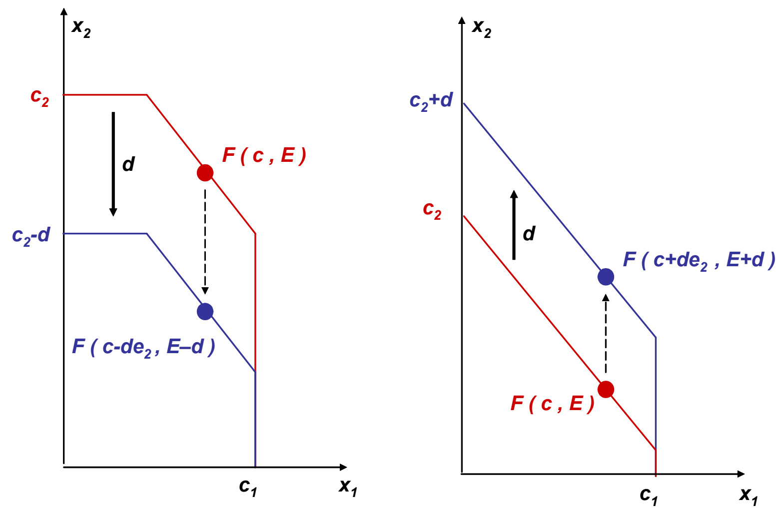

Figure 3: *Translation down* (**left**) and *translation up* (**right**).

The duality of rules can also be used to define a notion of duality for properties. A **property Π is the dual of another property $\Pi^{\mathbf{d}}$** if whenever a rule F satisfies Π, its dual rule F^d satisfies Π^d. Some properties, such as *c-continuity* or *equal treatment of equals*, are self-dual. That is, a rule F satisfies *c-continuity* (or *equal treatment of equals*) if and only if its dual F^d satisfies the same property.

The following result shows that *translation down* and *translation up* are, together, the dual of *rationality* (or equivalently, *contraction independence*).

Lemma 5 A rule F satisfies *rationality* if and only if its dual F^d satisfies *translation up* and *translation down*.

Proof. First assume that F satisfies *rationality.* Then it satisfies the two properties of *Lemma 1*.Let $(c, E) \in \mathcal{C}$, $i \in N$, and $\delta \in (0, \infty)$.

Claim 1. F^d satisfies *translation down.* To see this, assume $\delta \leqq F_i^d(c, E)$. Let $\overline{E} = \sum_N c_i - E$. Then, $\delta \leqq c_i - F_i\left(c, \overline{E}\right)$ implies $F\left(c, \overline{E}\right) \leqq c - \delta e_i \leqq c$. Then, by Property (ii) of *Lemma 1*, $F\left(c - \delta e_i, \overline{E}\right) = F\left(c, \overline{E}\right)$. This implies $F^d(c - \delta e_i, E - \delta) = F^d(c, E) - \delta e_i$.

Claim 2. F^d satisfies *translation up.* To see this, assume $c_i \geqq \sum_N c_j - E$. Let $\overline{E} = \sum_N c_i - E$. Then, $\min\left\{c_i, \overline{E}\right\} = \overline{E} = \min\left\{c_i + \delta, \overline{E}\right\}$. Then, by Property (i) of *Lemma 1*, $F\left(c + \delta e_i, \overline{E}\right) = F\left(c, \overline{E}\right)$. This implies $F^d\left(c + \delta e_i, E + \delta\right) = F^d\left(c, E\right) + \delta e_i$.

Next, assume that F^d satisfies *translation down* and *translation up.* Let $(c, E), (c', E) \in \mathcal{C}$.

Claim 3. F satisfies Property (ii) of *Lemma 1.* Assume $F(c, E) \leqq c' \leqq c$. Let $A = \left\{j \in N \mid c'_j < c_j\right\}$. If $A = \emptyset$, $F(c', E) = F(c, E)$ trivially holds. Alternatively, let $i \in A$. Let $\delta = c_i - c'_i$ and $\overline{E} = \sum_N c_i - E$. Then $F_i(c, E) \leqq c_i - \delta$ implies $\delta \in (0, F_i^d\left(c, \overline{E}\right)]$. Thus, by *translation down,* $F^d\left(c - \delta e_i, \overline{E} - \delta\right) = F^d\left(c, \overline{E}\right) - \delta e_i$. This implies $F(c - \delta e_i, E) = F(c, E)$. Applying the same argument to each $i \in A$, we obtain $F(c', E) = F(c, E)$.

Claim 4. F satisfies Property (i) of *Lemma 1.* Assume for each $j \in N$, $\min\{c_j, E\} = \min\left\{c'_j, E\right\}$. Let $A = \left\{j \in N \mid c'_j < c_j\right\}$, $B = \left\{j \in N \mid c'_j > c_j\right\}$, and $c'' = \left(c'_A, c_{N\setminus A}\right)$. By Claim 3, $F(c'', E) = F(c, E)$. We will next show $F(c', E) = F(c'', E)$. If $B = \emptyset$, this trivially holds. Alternatively, let $i \in B$. Let $\delta = c'_i - c''_i$ and $\overline{E} = \sum_N c''_j - E$. Note that $c''_i = c_i$. Then $\min\{c''_i, E\} = \min\{c'_i, E\}$ implies $E \leqq c''_i$. Thus, $c''_i \geqq \sum_N c''_j - \overline{E}$. Then, by *translation up,* $F^d\left(c'' + \delta e_i, \overline{E} + \delta\right) = F^d\left(c'', \overline{E}\right) + \delta e_i$. This implies $F(c'' + \delta e_i, E) = F(c'', E)$. Applying the same argument to each $i \in B$, we obtain $F(c', E) = F(c'', E)$. Together, $F(c', E) = F(c, E)$.

Claims 3 and 4, by *Lemma 1*, imply that F is *rational.* ■

The following result is a corollary of *Theorem 1* and *Lemma 5.* It also uses the fact that the *Equal Gains* and *Equal Losses* rules are dual rules (*e.g.* see Thomson, 2003).

Theorem 2 A rule F satisfies *translation up, translation down, c-continuity,* and *equal treatment of equals* if and only if it is the *Equal Losses rule.*

Proof. It is straightforward to show that EL satisfies the given properties. Conversely, let F be a rule that satisfies them. We next show $F = EL$. By *Lemma 5*, F^d satisfies *rationality.* Since *c-continuity* and *equal treatment of equals* are self-dual properties, F^d also satisfies them. Thus, by *Theorem 1*, $F^d = EG$. Since EG and EL are dual rules, then, $F = EL$. ■

References

[1] Aumann, R.J. and Maschler, M., 1985, Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, *Journal of Economic Theory*, 36, 195-213.

[2] Bénassy, J.P., 1993, Nonclearing Markets: Microeconomic Concepts and Macroeconomic Applications. *Journal of Economic Literature*, 31, 732-761.

[3] Bosmans K. and Lauwers, L., 2006, Lorenz Dominance Relationships Between Rules for Solving Claims Problems, mimeo.

[4] Bossert, W., 1994, Rational Choice and Two Person Bargaining Solutions, *Journal of Mathematical Economics*, 23, 549-563.

[5] Cachon, G. and M. Lariviere, 1999, Capacity Choice and Allocation: Strategic Behavior and Supply Chain Performance, *Management Science,* 45, 1091-1108.

[6] Chun, Y. and Thomson, W., 1992, Bargaining Problems with Claims, *Mathematical Social Sciences*, 24, 19-33.

[7] Dagan N., 1996, New Characterization of Old Bankruptcy Rules, *Social Choice and Welfare*, 13, 51–59.

[8] Edgeworth, F. Y., 1898, The Pure Theory of Taxation, in R. A. Musgrave and A. T. Peacock (Eds.), (1958) *Classiscs in the Theory of Public Finance*. Macmillan, New York.

[9] Herrero, C. and Villar, A., 2002, Sustainability in Bankruptcy Problems, *TOP*, 10:2, 261-273.

[10] Kıbrıs, Ö., 2003, Constrained Allocation Problems with Single-Peaked Preferences: An Axiomatic Analysis, *Social Choice and Welfare*, 20:3, 353-362.

[11] Kıbrıs, Ö., 2008, A Revealed Preference Analysis of Solutions to Simple Allocation Problems, *Sabanci University Working Paper*.

[12] Kıbrıs, Ö. and Küçükşenel, S., 2008, Uniform Trade Rules for Uncleared Markets, *Social Choice and Welfare*, forthcoming.

[13] Moulin, H., 1985, Egalitarianism and utilitarianism in quasi-linear bargaining, *Econometrica*, 53, 49–67.

[14] Moulin, H., 1987, Equal or proportional division of a surplus, and other methods, *International Journal of Game Theory,* 16, 161–186.

[15] Nash, J.F., 1950, The Bargaining Problem, *Econometrica,* 18, 155-162.

[16] O'Neill, B., 1982, A Problem of Rights Arbitration from the Talmud, *Mathematical Social Sciences*, 2, 287-301.

[17] Peters, H. and P. Wakker, 1991, Independence of Irrelevant Alternatives and Revealed Group Preferences, *Econometrica,* 59, 1787-1801.

[18] Samuelson, P.A., 1938, A Note on the Pure Theory of Consumer's Behaviour, *Economica,* 5, 61-71.

[19] Samuelson, P.A., 1948, Consumption Theory in Terms of Revealed Preferences, *Economica*, 15, 243-253.

[20] Sánchez, M. C., 2000, Rationality of Bargaining Solutions, *Journal of Mathematical Economics*, 389-399.

[21] Schummer J. and Thomson, W., 1997, Two Derivations of the Uniform rule, *Economics Letters*, 55, 333-337.

[22] Sen, A.K., 1971, Choice Functions and Revealed Preferences, *Review of Economic Studies*, 38, 307-317.

[23] Sprumont, Y., 1991, The Division Problem With Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule, *Econometrica*, 49, 509-519.

[24] Thomson, W., 1981, Nash's Bargaining Solution and Utilitarian Choice Rules, *Econometrica*, 49:2, 535-538.

[25] Thomson, W., 2003, Axiomatic and Game-Theoretic Analysis of Bankruptcy and Taxation Problems: A Survey, *Mathematical Social Sciences*, 45, 249-297.

[26] Thomson, W., 2007, How to Divide When There Isn't Enough: From the Talmud to Game Theory, book manuscript.

[27] Yeh, C-H., 2004, Sustainability, Exemption, and the Constrained Equal Awards rule: A Note, *Mathematical Social Sciences*, 47, 103-110.

[28] Yeh, C-H., 2006, Protective Properties and the Constrained Equal Awards Rule for Claims Problems: A Note, *Social Choice and Welfare*, 27:2, 221-230.

[29] Young, P., 1987, On dividing an amount according to individual claims or liabilities. *Mathematics of Operations Research,* 12, 398–414.