APPLICATION OF A GENERAL RISK MANAGEMENT MODEL TO PORTFOLIO PROBLEMS WITH ELLIPTICAL DISTRIBUTIONS

by

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APPLICATION OF A GENERAL RISK MANAGEMENT MODEL TO PORTFOLIO PROBLEMS WITH ELLIPTICAL DISTRIBUTIONS

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to my family

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Keywords: elliptical distributions, risk management, linear loss functions, Value-at-Risk, Conditional Value-at-Risk, portfolio optimization, disutility.

Abstract

Focus is directed to a class of risk measures for portfolio optimization with two types of disutility functions, where the random return variables of financial instruments are assumed to be distributed by multivariate elliptical distributions. Recent risk measures, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are considered under this setting. If the joint distribution function of the financial instruments is elliptical and disutility is taken as linear reflecting the behavior of risk neutral investors, then the optimal solution of the mathematical models with objective functions formed by VaR and CVaR measures, is equivalent to the solution of the corresponding Markowitz model. To solve the Markowitz model, a very fast, finite step algorithm proposed in the literature, has been modified and implemented . Finally, the CVaR model with convex increasing disutility functions reflecting the behavior of risk averse investors has been introduced. Although, a convex objective function exists in this case, an analytic form cannot be obtained. However, unlike generating scenarios from multivariate distributions as suggested in the literature, the objective function can be closely estimated by simulating realizations only from univariate distributions. The thesis concludes with a thorough computational study on a sample data collected from the Istanbul Stock Exchange for each different class introduced.

GENEL BİR RİSK YÖNETİMİ MODELİNİN ELLİPTİK DAĞILIMLI PORTFÖY PROBLEMLERİNE UYGULAMASI

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Anahtar Kelimeler: eliptik dağılımlar, risk yönetimi, lineer kayıp fonksiyonları, Riske Maruz Değer, Koşullu Riske Maruz Değer, portföy optimizasyonu, negatif fayda.

Özet

Finansal yatırım araçlarının rasgele getiri değişkenlerinin çok değişkenli eliptik dağılımlarla dağıldığının kabul edildiği iki tip negatif faydaya sahip portföy optimizasyonu için bir dizi risk ölçütüne odaklanılmıştır. Bu bağlamda güncel risk ölçütleri olan Riske Maruz Değer (VaR) ve Koşullu Riske Maruz Değer (CVaR) göz önünde bulundurulmuştur. Finansal yatırım araçlarının ortak dağılım fonksiyonunun eliptik olması ve negatif faydanın riske karşı kayıtsız yatırımcıların davranışını yansıtacak şekilde lineer kabul edilmesi halinde, VaR ve CVaR ölçütleriyle oluşturulan objektif fonksiyonlara sahip matematiksel modellerin optimum çözümü ilgili Markowitz modelinin çözümüne eşdeğerdir. Markowitz modelini çözebilmek için literatürde önerilen oldukça hızlı bir sonlu adım algoritması değiştirilerek uygulanmıştır. Son olarak riskten kaçınan yatırımcıların davranışını yansıtan konveks artış sergileyen negatif faydaya sahip CVaR modeli sunulmuştur. Bu modelde konveks objektif fonksiyon mevcut olmasına rağmen analitik bir form elde edilememektedir. Ancak objektif fonksiyon, literatürde geçtiği üzere çok değişkenli dağılımlardan senaryolar üretmeden farklı olarak gerçeklemelerin yalnızca tek değişkenli dağılımlardan simüle edilmesiyle yakınen tahmin edilebilir. Bu tez sunulan farklı her sınıf için İstanbul Menkul Kıymetler Borsası kaynaklı örneklem verileri baz alınarak gerçekleştirilen kapsamlı bir bilişimsel çalışma ile sonlandırılmaktadır.

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CHAPTER 1

INTRODUCTION

In the world of finance and engineering, making decisions that minimize "risk" and, at the same time, meet desired objectives. However, quantifying risk differs according to the measure used. There are two central approaches for perceiving risk and quantifying it. Risk can be identified as a function of deviation from expectation or a function of absolute loss. The first approach, that has triggered the idea of using function of standard deviation as the risk measure, is Markowitz's modern portfolio theory, [9]. The second introduces the recent risk measures Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). This chapter, first concentrates on Markowitz approach and then studies intermediate risk measures between variance and VaR.

Before detailing risk measures, portfolio should first be defined: A collection of stocks or bonds, a book of derivatives, a collection of risky loans, or a financial institution's overall position in risky assets. The definition of the portfolio changes with the type of risk. There mainly are three types of risk: market, credit, and operational. For example, if the credit risk of a bank is to be evaluated, then the portfolio of the bank would be the credits given to institutions. In operational risk, portfolio is assessed as internal or external frauds, failed processes, people, or computer systems as given in the Basel II amendment. After deciding on the portfolio, the uncertainty of each component has to be defined over a probability space, since we would like to decide according to future states of each component. Following the identification of random events, the definition of the loss function must to be given since the risk of loss is the desired quantity. Lastly, the selection of the risk measure is handled.

As Markowitz mentions in [9], the process of selecting a portfolio involves two stages. The first stage begins with observations and ends with decisions about the future performances of the components of portfolio, the behaviors of the random variables. The second stage, on the other hand begins with interpretation of these future performances and ends with the choice of portfolio. This thesis is concerned with the second stage as the decision maker would like to maximize the portfolio return as well as minimize the risk of portfolio loss. Balancing the portfolio is therefore desired with respect to return and loss according to the decision maker. There is a property that the risk measure should satisfy; for the same amount of return, a diversified portfolio is preferred to all non-diversified portfolios.

In Markowitz's approach, the portfolio problem does not deal with loss function, the presumption is accepted that the law of large numbers will insure that actual return of the portfolio will be almost same as the expected return of the portfolio. Consider a portfolio of n assets over a certain period of time. x_i denotes the capital invested in the asset i, by $\mathbf{x} \in \mathbb{R}^n$, the portfolio vector; and by $\mathbf{Y} \in \mathbb{R}^n$, the random vectors of asset returns, yielding $\mathbf{x}_i\mathbf{Y}_i$ capitals for the financial instrument i. Suppose $\mathbf{Y}$ is given by a joint probability distribution function with the expectation $\mu = \mathbb{E}(\mathbf{Y})$, and the covariance matrix:

$$\Sigma = \mathbb{E}\left[(\mathbf{Y}-\mu)(\mathbf{Y}-\mu)^\top\right].$$

Hence, the total return of the portfolio is $\mathbf{x}^\top\mathbf{Y}$ and the risk of the portfolio is taken as the variance of the return, $\mathbf{x}^\top\Sigma\mathbf{x}$. The risk is taken as deviation of the portfolio's return from its expected return. The investor's goal of splitting between risk and return can be formulated as

$$\begin{aligned} \min \quad & \tfrac{1}{2}\mathbf{x}^\top\Sigma\mathbf{x} \\ \text{s.t} \quad & \mathbf{e}^\top\mathbf{x} = 1 \\ & \mu^\top\mathbf{x} = r \\ & \mathbf{x} \geq 0. \end{aligned}$$

where $\mathbf{e} \in \mathbb{R}^n$ denotes the vector of 1's. $\mathbf{e}^\top\mathbf{x} = 1$ is normalized budget equation. Covariance matrix, Σ is assumed to be positive definite and μ is not a multiple of $\mathbf{e}$. The inability to use short selling brings up the nonnegativity constraints. Short selling is the act of selling without actually owning an asset. To differentiate the problem, a risk-free asset also can be added to the portfolio, where return of the asset is known from the beginning.

The optimal portfolio for the above problem changes as the value of r. Markowitz has used this problem to derive an efficient frontier where every portfolio on the frontier maximizes the expected return for a given variance or minimizes the variance for a given expected return. Thus, the investor must make a trade off between risk and return. Criticisms the mean variance approach can be listed as:

- ◇ The model relies on the mean and the variance of the distribution of the returns where the higher moments could give different kinds of interpretations and informations.

- ◇ The model is sufficient for normally distributed returns and quadratic utility functions however, stock prices are better fitted to non-normal distributions.
- ◇ The actual utility of the investors can be approximated by a quadratic curve only for some relevant range of returns.
- ◇ A risk measure does not have to be symmetric.
- ◇ The returns of the portfolio higher than the expected return of the portfolio is not taken as risk by the decision maker.

The obvious disadvantage of the Markowitz model is taking greater values of return of portfolio as risk, emphasized by Markowitz himself, too. So, he proposes the idea of lower semi variance as the risk measure. The lower semi variance of portfolio can be defined as

$$\mathbb{E}(|\mathbb{E}(\mathbf{Y}) - \mathbf{x}^\top \mathbf{Y}|_+^2)$$

where $|u|_+$ refers to $\max\{0, u\}$. This risk measures handles the problem of taking positive returns as risk however, we still use the first two moments of the portfolio, and this risk measure is same as variance when $\mathbf{x}^\top \mathbf{Y}$ follows a normal distribution. Hence, the optimization problem would be

$$\begin{aligned} \min \quad & \mathbb{E}(|\mathbb{E}(\mathbf{Y}) - \mathbf{x}^\top \mathbf{Y}|_+^2) \\ \text{s.t} \quad & \mathbf{e}^\top \mathbf{x} = 1 \\ & \mu^\top \mathbf{x} = r \\ & \mathbf{x} \geq 0. \end{aligned}$$

With the introduction of lower semi variance as risk measure, a period of downside risk has started. Similar to the lower semi variance, the lower semi absolute deviation can also be given as the risk measure;

$$\mathbb{E}(|\mathbb{E}(\mathbf{Y}) - \mathbf{x}^\top \mathbf{Y}|_+). \tag{1.1}$$

Roy [15], as mentioned in the article written by David Nawrocki [11], believes that a mathematical utility function would not be sufficient for an investor to maximize the portfolio return. He has stated that an investor will prefer the safety of principal first and will set a minimum acceptable return that will conserve his principal. He has called the minimum acceptable return "a disaster level" and the resulting method, a " Roy's safety first method". His belief is the investor prefers the investment with the smallest probability of going below either the disaster level or the target return, $\min_{\mathbf{x}\in X}\mathbb{P}\{\mathbf{x}^\top\mathbf{Y} \leq \tau\}$. Actually this risk measure combines utility theory with downside risk. If the disaster level is fixed by the decision vector as τ, then below target risk measure can be given as

$$\mathrm{BT}_p(\tau, \mathbf{x}) = \mathbb{E}(|\tau - \mathbf{x}^\top\mathbf{Y}|_+^p)^{1/p}. \tag{1.2}$$

where the following optimization problem is formed,

$$\begin{aligned}
\min \quad & \mathbb{E}(|\tau - \mathbf{x}^\top\mathbf{Y}|_+^p)^{1/p} \\
\text{s.t} \quad & \mathbf{e}^\top\mathbf{x} = 1 \\
& \mu^\top\mathbf{x} = r \\
& \mathbf{x} \geq 0.
\end{aligned}$$

The level of p defines the characteristic of the decision maker. If $p \leq 1$, then the investor is risk seeking, if $p = 1$ risk neutral and if $p \geq 1$, the decision maker is risk averse. Below target risk measure, a convex function with respect to $\mathbf{x}$, for all values of p, has also been called as Lower Partial Moments in the literature. Setting p to the correct valuebis very important, since p affects the result directly [11]. If risk measure is taken as,

$$\mathbb{E}(|\tau - \mathbf{x}^\top\mathbf{Y}|_+^2),$$

it is called, target semi-variance, the combination of Equations 1.1 and 1.2.

Following the lower partial moments, the idea of VaR measure is introduced. In order to evaluate VaR, the decision maker should decide on a probability level, β. VaR is the lowest amount of loss such that, with the predefined probability level, loss will not exceed the lowest amount. We denote that lowest amount as VaR, since the VaR value depends on the chosen probability level. If function $f(\mathbf{x}, \mathbf{Y})$ defines loss, VaR can be given as

$$\mathbb{P}\{f(\mathbf{x}, \mathbf{Y}) \geq \mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))\} = 1 - \beta.$$

The definition of the $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$ measure will be made clear in the following chapters of this thesis. $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$ is not a convex function of $\mathbf{x}$; therefore, minimizing $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$ with respect to $\mathbf{x}$ is difficult. Financial institutions mainly use VaR to measure their risk since VaR is a very popular risk measure, rather than deciding upon the portfolio by minimizing VaR. There are many models established to measure VaR of the portfolio, such as time series models; ARCH and GARCH. However, optimization through use is problematic.

Minimizing portfolio by using VaR also brought the notion of coherent risk measures which measure, according to Artzner *et.al.* [1], as real valued risk measure ρ, on the space of real-valued random variables depending only on the distribution. Coherent risk measures fulfill the following properties as defined in Embrechts [1, 3]:

1. **Monotonicity:** For any two random variables $\mathbf{X}$ and $\mathbf{Y}$ we have

$$\mathbf{X} \geq \mathbf{Y} \Rightarrow \rho(\mathbf{X}) \geq \rho(\mathbf{Y}).$$

2. **Subadditivity:** For any two random variables $\mathbf{X}$ and $\mathbf{Y}$ we have

$$\rho(\mathbf{X}+\mathbf{Y}) \leq \rho(\mathbf{X}) + \rho(\mathbf{Y}).$$

3. **Positive Homogeneity:** For $\lambda > 0$ and a random variable $\mathbf{X}$ we have

$$\rho(\lambda \mathbf{X}) = \lambda \rho(\mathbf{X}).$$

4. **Translation Invariance:** For any $a \in \mathbb{R}^n$ and a random variable $\mathbf{Y}$, we have

$$\rho(\mathbf{X}+a) = \rho(\mathbf{X}) + a.$$

VaR satisfies coherency properties under conditions discussed in the following chapters. However, coherency is not a general property for VaR. It has some undesirable mathematical properties as a lack of subadditivity and convexity. In the case of a finite number of scenarios, which means a measure is taken with respect to realizations, VaR also becomes nonsmooth, nonconvex, and multiextremum function.

CVaR, a related measure to VaR, is defined as the conditional expectation of losses exceeding the corresponding VaR amount, that can be formulated as

$$\mathrm{CVaR}_\beta(f(\mathbf{x},\mathbf{Y})) = \frac{1}{1-\beta}\,\mathbb{E}(f(\mathbf{x},\mathbf{Y}) \mid f(\mathbf{x},\mathbf{Y}) \geq \mathrm{VaR}_\beta(f(\mathbf{x},\mathbf{Y}))).$$

CVaR is also defined as Expected Shortfall in the literature. CVaR not only maintains the advantages of VaR, but also eliminates the computational disadvantages of VaR, since CVaR is a coherent and a convex function over $\mathbf{x}$, independent from the joint distribution of random vector $\mathbf{Y}$. Although VaR and CVaR do not use the higher moments of the probability density function of the loss, $f(\mathbf{x}, \mathbf{Y})$, they give more information about the skewness and kurtosis of the distribution. If skewness of the distribution is high, then these two values will be rather far away from each other, whereas they will be closer if the kurtosis of distribution is high, as compared to the relations between VaR and CVaR, when the underlying joint distribution function is normal.

Besides risk measures, Stochastic Dominance, one other approach that decides upon the efficient frontier of a portfolio is pointed out by Porter, Wart and Ferguson in (1973), [12]. Stochastic Dominance converts the probability distribution of an investment into a cumulative probability curve. Then the mathematical analysis of the cumulative probability curve is used to determine if one investment is superior to another. The advantages of stochastic dominance are: applicability for all probability distributions and inclusion of all risk averse utility assumptions. The disadvantage of the method is that operation research algorithms are not applicable to this method. The tests have shown that below target semi-variance portfolios are members of stochastic dominance efficient sets while the below mean semi-variance portfolios were not.

This thesis uses VaR and CVaR as the risk measure to decide upon the optimal portfolio. Thus, the scope of this work is wider in terms of VaR and CVaR. Risk measures are used in portfolio optimization problems. In the following section, the disutility functions are introduced along with risk measures, VaR and CVaR, and the details of their roles in portfolio management are discussed.

1.1 Portfolio Optimization Problem

In portfolio optimization, the decision maker tries to allocate his or her capital to n financial instruments, so that considered risk is minimized. The loss of the decision maker is given by the real-valued function $f(\mathbf{x}, \mathbf{Y})$, where $\mathbf{x} \subseteq \mathbb{R}^n$ is the decision vector denoting the allocations, X is a closed convex set; and $\mathbf{Y} \in \mathbb{R}^n$ is a random vector denoting the uncertain returns of the financial instruments. Hence, $\mathbf{x}$ denotes the portfolio. X can simply be defined as;

$$x_i \geq 0 \text{ for } i = 1, ..., n, \text{ with } \sum_{i=1}^{n} x_i = 1.$$

This thesis assumes that short-selling is not allowed: hence, allocations are nonnegative. It should be reiterated that not allowing short selling exacerbates the problem as compared to the allowed case, since if short selling is allowed, then the problem immediately has an analytic solution. Since $\mathbf{Y}$ is a random vector, loss function $f(\mathbf{x}, \mathbf{Y})$ is also a random variable. The cumulative distribution function of random loss is given by

$$\Psi_{\mathbf{x}}(\alpha) := \mathbb{P}\{f(\mathbf{x}, \mathbf{Y}) \leq \alpha\}.$$

We assume that this cumulative distribution function of random loss is everywhere continuous with respect to α. The corresponding inverse cumulative distribution function, or the quantile function, then becomes

$$\alpha_\beta(\mathbf{x}) := \mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y})) := \min\{\alpha \in \mathbb{R} : \Psi_{\mathbf{x}}(\alpha) \geq \beta\} = \Psi_{\mathbf{x}}^{\leftarrow}(\beta), \tag{1.3}$$

where "$\leftarrow$" denotes the inverse of a function. Within risk management, the value $\alpha_\beta(\mathbf{x})$ is known as the Value-at-Risk (VaR) of the loss $f(\mathbf{x}, \mathbf{Y})$ at the probability level β with $\beta \in (0, 1)$. Since function $\alpha \longmapsto \Psi_{\mathbf{x}}(\alpha)$ is continuous , it is well known for every $u \in (0, 1)$ that [17]

$$\mathbb{P}\{\Psi_{\mathbf{x}}(f(\mathbf{x}, \mathbf{Y})) \leq u\} = \mathbb{P}\{f(\mathbf{x}, \mathbf{Y}) \leq \Psi_{\mathbf{x}}^{\leftarrow}(u)\} = \Psi_{\mathbf{x}}(\Psi_{\mathbf{x}}^{\leftarrow}(u)) = u. \tag{1.4}$$

This property is given to emphasize the fact that $\Psi_{\mathbf{x}}(f(\mathbf{x}, \mathbf{Y}))$ is uniformly distributed. Therefore $\Psi_{\mathbf{x}}(f(\mathbf{x}, \mathbf{Y}))$ and $\mathbf{U}$, where $\mathbf{U}$ stands for a standard uniformly distributed random variable has the same distribution [3, 14]. Another recent risk measure is given by

$$\phi_\beta(\mathbf{x}) := \mathrm{CVaR}_\beta(f(\mathbf{x}, \mathbf{Y})) := (1 - \beta)^{-1}\, \mathbb{E}(f(\mathbf{x}, \mathbf{Y}) 1_{\{f(\mathbf{x}, \mathbf{Y}) \geq \alpha_\beta(\mathbf{x})\}}),$$

with probability level β. This risk measure is called the Conditional Value-at-Risk (CVaR) of the loss $f(\mathbf{x}, \mathbf{Y})$ at level β [13]. One main reason for introducing this new risk measure is that as a function of $\mathbf{x}$, CVaR is convex when loss function $f(\mathbf{x}, \mathbf{Y})$ is also convex in $\mathbf{x}$. Generally VaR does not have this property unless random vector $\mathbf{Y}$ has an elliptical distribution, and a certain class of loss functions and decision makers exist.

To show that $\phi_\beta(\mathbf{x})$ is convex in $\mathbf{x}$, Function $F_\beta(\mathbf{x}, \alpha)$, first introduced in [13] is used. For $0 < \beta < 1$, function $F_\beta(\mathbf{x}, \alpha) : X \longmapsto \mathbb{R}$ is given by,

$$F_\beta(\mathbf{x}, \alpha) := \alpha + (1-\beta)^{-1}\ \mathbb{E}(\max\{f(\mathbf{x}, \mathbf{Y}) - \alpha, 0\}).$$

Convexity of $\phi_\beta(\mathbf{x})$ will be shown by the help of the following lemma introduced in [13]:

Lemma 1.1.1 It follows for every $\mathbf{x} \in X$ that

$$\phi_\beta(\mathbf{x}) = \min_{\alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha).$$

Moreover, if the closed interval $S_\beta(\mathbf{x})$ denotes the set of optimal solutions of the above optimization problem, then the left end point of this interval is equal to $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$.

Now, if $f(\mathbf{x}, \mathbf{Y})$ is convex in $\mathbf{x}$, then clearly the function $(\mathbf{x}, \alpha) \longmapsto \max\{f(\mathbf{x}, \mathbf{Y}) - \alpha, 0\}$ is also convex in $(\mathbf{x}, \alpha)$. Since expectation is a linear operator, the function $F_\beta(\mathbf{x}, \alpha)$ is convex. By Lemma 1.1.1, the function $\phi_\beta(\mathbf{x})$ is also convex.

VaR is a very popular risk measure and in risk management, originally the following optimization problem is desired to be considered:

$$\min_{\mathbf{x} \in X}\ \alpha_\beta(\mathbf{x}) = \min_{\mathbf{x} \in X}\ \mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y})) = \min_{\mathbf{x} \in X}\ \Psi_{\mathbf{x}}^{\leftarrow}(\beta). \tag{VP}$$

Since this function depends on the cumulative distribution function of loss $f(\mathbf{x}, \mathbf{Y})$, it may happen that even for convex loss function $f(\mathbf{x}, \mathbf{Y})$, $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$ can be a nonconvex function. Thus, the above problem belongs to the field of global optimization and might be difficult to solve. However, minimizing the following may be considered

$$\min_{\mathbf{x} \in X} \phi_\beta(\mathbf{x}) = \min_{\mathbf{x} \in X}\ \mathrm{CVaR}_\beta(f(\mathbf{x}, \mathbf{Y})). \tag{CVP}$$

Since $\phi_\beta(\mathbf{x})$ is a convex function in $\mathbf{x}$, the above problem is always a convex program. Moreover, if $\mathbf{x}^*$ is an optimal solution of (CVP), using Lemma 1.1.1, the corresponding $\mathrm{VaR}_\beta(f(\mathbf{x}, \mathbf{Y}))$ can be identified quite quickly by implementing bisection methods.

Following [13], $\phi_\beta(\mathbf{x})$ is employed since this definition is more suitable for a probabilistic interpretation of CVaR. Furthermore, applying Relation (1.3) and (1.4) with

$\beta \in (0,1)$, the following set equivalences are obvious

$$\begin{aligned}\mathbb{P}\{f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x})\} &= \mathbb{P}\{f(\mathbf{x},\mathbf{Y}) \geq \Psi_{\mathbf{x}}^{\leftarrow}(\beta)\} \\ &= 1 - \mathbb{P}\{f(\mathbf{x},\mathbf{Y}) < \Psi_{\mathbf{x}}^{\leftarrow}(\beta)\} \\ &= 1 - \mathbb{P}\{\Psi_{\mathbf{x}}(f(\mathbf{x},\mathbf{Y})) < \beta\} \\ &= 1-\beta.\end{aligned} \tag{1.5}$$

Relation 1.5 shows that,

$$\begin{aligned}\phi_\beta(\mathbf{x}) &= (1-\beta)^{-1}\,\mathbb{E}(f(\mathbf{x},\mathbf{Y})1_{\{f(\mathbf{x},\mathbf{Y})\geq\alpha_\beta(\mathbf{x})\}}) \\ &= (1-\beta)^{-1}\,\mathbb{E}(f(\mathbf{x},\mathbf{Y}) \mid f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x}))\;\mathbb{P}(f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x})) \\ &= (1-\beta)^{-1}\,\mathbb{E}(f(\mathbf{x},\mathbf{Y}) \mid f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x}))\;(1-\beta) \\ &= \mathbb{E}(f(\mathbf{x},\mathbf{Y}) \mid f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x})).\end{aligned} \tag{1.6}$$

The indicator function, $1_{\{f(\mathbf{x},\mathbf{Y})\geq\alpha_\beta(\mathbf{x})\}}$, equals zero, unless the condition is satisfied. Thus, in the second equation, only the probability that $f(\mathbf{x},\mathbf{Y}) \geq \alpha_\beta(\mathbf{x})$ is under consideration. In the final step, Definition 1.3 is utilized. Observe that this property justifies the name Conditional Value-at-Risk and now this expression fits definition offered in the Introduction: CVaR measure is the conditional expectation of the losses exceeding the corresponding VaR amount.

Hence, now all of the definitions are given for the risk management problem. The portfolio $\mathbf{x}$, consisting of n financial instruments, has the loss function $f(\mathbf{x},\mathbf{Y})$, where $\mathbf{Y}$ is a random vector. The risk measures taken are VaR and CVaR.

1.2 Outline

The outline of the thesis is as follows. Chapter 2, gives a brief literature review on the Markowitz problem, VaR, and CVaR measures. Literature Review is followed in Chapter 3 by the properties of VaR and CVaR in the elliptic world with risk neutral and risk averse investors. The finite step algorithm is given in Chapter 4 together with an extensive numerical study on a sample data collected from the Istanbul Stock Exchange. The conclusion is consigned in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

The concept of risk is actually part of our everyday lives. Risk can be thought as the probability of the materialization of undesirable future events , which cannot be forecast at the the time we actually make our decisions. We need to somehow rank choices with respect to the probability of materialization in order to maximize our utilization. That principle is the same with financial risks. When the investor decides to allocate capital on several assets, the return of this investment is uncertain. However, in order for the investor make a decision, a portfolio ranking measure is necessary. The first measure that comes to mind is that the return of the investment should be high, along with risk to differentiate, in case the returns of the two choices are the same. These thoughts bring up the idea of "Modern Portfolio Theory", introduced by Harry Markowitz in 1950s [9]. As Steinbach puts , "The classical mean-variance approach offered the first systematic treatment of a dilemma that each investor faces: the conflicting objectives of *high profit* versus *low risk*" [18].

Steinbach supplies all ideas behind the Markowitz Model, together with portfolio problems consisting of risky, risk-free financial instruments and guaranteed loss along with the same portfolios studied with downside risk measures [18]. Risky assets are the financial instruments that have uncertain returns whereas risk-free assets have known yield. Guaranteed loss offers the choice of not allocating all owned capital. The most important difference between the feasible set examined in [18] and the feasible set of the models introduced in this thesis is nonnegativity constraints. Since Short Selling is allowed in [18], that means $\mathbf{x} \geq \mathbf{0}$, the Markowitz problem, becomes a quadratic problem that has an analytical solution. Additionally, multistage models are introduced in [18]. Multistage is the case when a portfolio cannot be liquidated until after the passage of a certain amount of time. The investor decides to invest how much to which asset at time 0, and the decision maker can change this portfolio allocation through the time horizon without turning the portfolio into cash. Since there are many decision stages in between, this model is called the multistage model. The approach to multistage models contains forming a tree, where time 0 stage forms the root node of the tree. Mainly, this work not only provided reference for this research, but also offered a compact introduction to modern portfolio theory.

The introduction, visualizes the improvement of the risk measures ending with VaR and CVaR upon which this study also concentrates. We have classified and analyzed a general risk management model applied to portfolio problems where the returns of the assets are elliptically distributed. We discuss linear loss functions as well as two different characteristics of decision makers. The term *elliptical world* actually refers to the returns of the financial instruments being elliptically distributed. For instance, normal and student-t are two typical elliptical distributions. Most VaR models used in finance are based on multivariate normal distribution due to normal distribution's analytical expression in portfolio calculation and simulation. The wider class of elliptical distributions, however, examines the kurtosis of return distributions in a relatively more realistic way. Embrechts *et.al* thoroughly defines the elliptical world and also discusses the properties of VaR and CVaR in the elliptical world. This work assisted in classifying the elliptical world with respect to risk measures. On the other hand, most significantly the discussion about characterization of decision makers is not entailed in the work of Embrechts *et.al.*, [3], which has drawn attention to in this work.

Jules Kamdem distinguishes portfolios with respect to dependence on portfolio loss and joint distribution of the random returns of the assets [6]. The original RiskMetric methodology for estimating VaR works well for linear portfolios where the loss is a linear function of the returns, and the joint distribution of the financial instruments is multivariate normal. Actually, under this setting, VaR reduces to variance introduced in the Markowitz problem. Two adjustments exist: either a non-linear instead of linear loss function will be introduced or portfolios of non-normally distributed financial instruments will be scrutinized. Although Jules Kamdem provides the idea of changing the linear loss function, his study involves the analysis of portfolios with non-normally distributed returns. The most significant difference between this thesis and the work by Jules Kamdem is implementation of the similar ideas. Jules Kamdem observes the properties of VaR in the elliptic world as well, however he mainly uses calculus as a tool, rather than probability theory, which is the main tool employed in this thesis to examine disutility functions together with elliptical distributions.

In the literature, the minimization of VaR is not a common approach. Researchers focus mainly on estimating the joint distribution of the assets and then by using this estimation, VaR is determined. In the calculation of VaR, the choice of risk horizons and confidence level, which is also called by probability level, has an important effect. Alternative risk horizons can be a day, a week, a month, or a quarter. Risk horizons are established as the observation period of the random vector $\mathbf{Y}$. In order to calculate VaR, the distribution of rate of returns has to be known which is the complicated part. If the distribution can be found or estimated, VaR can be calculated accordingly. Generally there are three approaches for the estimation of the distribution:

1. Historical simulation

2. Estimation of portfolio mean and variance

3. Monte carlo simulation

This topic alone is another main issue in risk management. Regarding the estimation of VaR, importance of minimizing risk needs to be emphasized.

As defined by Rockafellar and Uryasev [13], the VaR of the portfolio with respect to a specified probability level β is the lowest amount α such that with probability β, the loss will not exceed α. The CVaR is the conditional expectation of the losses above the amount α. VaR risk performance measure is widely standardized because of its acceptance as the common risk measure by the banks and also various estimation techniques have been proposed, as mentioned above.

Apart from the two risk measures mentioned above, this study also focuses on two disutility functions. At first discussing risk neutral decision makers, the return of the portfolio is taken as the a linear function of the product of two independent random variables, as the linear portfolio defined in [6]. The equivalence between the portfolio problems with VaR and CVaR measures, and the corresponding Markowitz model is also pointed out. A finite step algorithm proposed in the literature [10] is revised, and applied to the Markowitz model. The core idea of Michelots algorithm, based on projection of points on canonical simplices and elementary cones is preserved. Instead of directly assigning variables to constants though, once again projections on canonical simplices are used. The algorithm introduced in [10] has never been programmed. Although the core idea has been kept in the thesis, a new rule is given for the simplification of the application of the algorithm. Since the algorithm is finite step, the optimal portfolio is found quite quickly. The Michelot algorithm, though, is not introduced for solving Markowitz problems beforehand. In the elliptical world with linear loss functions, CVaR measure can be estimated by generating univariate realizations.

The idea of defining and minimizing VaR and CVaR by using operations research techniques has been extensively studied by Rockafellar and Uryasev in [13]. They have come up with three theorems about the relation between the optimization problems enclosing VaR and CVaR measures. The approach in [13] not only estimates VaR, but also brings up the idea of minimizing the portfolio problem with respect to CVaR. In the elliptical world, VaR is estimated as well as the portfolio problem with respect to VaR is minimized. Rockafellar and Uryasev discretisize the problem over scenarios; a scenario stands for the different rates of returns of the assets. Even if only a single stage exists, infinite scenarios can be constructed. While discretisizing the problem in order for the law of large numbers to become applicable, as many scenarios as possible as possible should be generated. It should be kept in mind that these scenarios are rare event generations, since the larger value of probability level is under consideration. Although the main classifications are the same, the exposition in this thesis uses probability theory rather than integrals.

A special attention is paid to place the model introduced by Rockafellar and Uryasev in the elliptical world. The properties and the forms of the risk measures change according to the loss function taken. If the investor is risk averse, the loss function is therefore nonlinear. Rockafellar and Uryasev have used multidimensional scenario generation for estimating the general CVaR objective in the portfolio problem. Their generalized results that can be applied to all of the sectors of risk are however hard to implement in practice. In the elliptical world, this work emphasizes most importantly that scenario generation can simply be done using univariate realizations. This extension allows significant reduction in simulation time. Rockafellar and Uryasev [13] have also dealt with financial concepts such as hedging, jumps and discontinuity, which are beyond the scope of this thesis. CVaR is also discussed by Embrechts *et.al.* under the name expected shortfall or mean excess loss.

CHAPTER 3

PROPERTIES OF VaR AND CVaR IN THE ELLIPTIC WORLD

Elliptic world is a key issue in this work. First of all, since normal distribution is a member, it is a rather important class. In addition to this, even if non-normal random vectors are considered, if the distribution of the portfolio is conditioned on a certain time, portfolio return behaves normally [5]. To analyze general risk model for portfolio management first the following class of multivariate distributions are introduced [3, 4]. In order to understand the definitions, we need to remember orthogonality. A matrix U is called an orthogonal matrix if $U^\top U = UU^\top = I$. We also adopt the following notation: $\mathbf{X} =_d \mathbf{Y}$ means that random variable $\mathbf{X}$ has the same distribution as $\mathbf{Y}$.

Definition 3.0.1 A random vector $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)^\top$ has a spherical distribution if for any $n \times n$ orthogonal matrix U, it holds that,

$$U\mathbf{X} =_d \mathbf{X}.$$

The spherical distributions are actually the probability density functions of random variables which are closed under rotational transformations. Namely, we are moving on a sphere's surface and have the same class of distributions. If a probability density function (pdf) $g(.)$ is spherical, the same function can be characterized in terms of the radius of the sphere. Showing that the expectation of elliptically distributed random vector $\mathbf{X}$ is always $\mathbf{0}$, is easy.

$$\begin{aligned} U\mathbf{X} =_d \mathbf{X} \; &\Longleftrightarrow \mathbb{E}(U\mathbf{X}) = \mathbb{E}(\mathbf{X}) \\ &\Longleftrightarrow U\mathbb{E}(\mathbf{X}) = \mathbb{E}(\mathbf{X}) \\ &\Longleftrightarrow \mathbb{E}(\mathbf{X}) = U^{-1}\mathbb{E}(\mathbf{X}) \\ &\Longleftrightarrow U\mathbb{E}(\mathbf{X}) = U^{-1}\mathbb{E}(\mathbf{X}) \\ &\Longleftrightarrow \mathbb{E}(\mathbf{X}) = 0. \end{aligned}$$

An important observation is that any spherically distributed random variable $\mathbf{X}$ can be represented as the product $S\mathbf{Z}$, of independent random variables S and $\mathbf{Z}$, where $\mathbf{Z}$ is a multivariate normally distributed random vector with mean $\mathbf{0}$ and covariance matrix I, and S is a univariate nonnegative random variable with corresponding cumulative distribution function $G(\cdot)$ that satisfies $G(0) = 0$ [3];

$$\mathbf{X} =_d S\mathbf{Z}. \tag{3.1}$$

For example the multivariate t-distribution with ν degrees of freedom can be constructed by taking $S \sim \sqrt{\nu}/\sqrt{\chi^2_\nu}$ as shown in [3]. $\mathbf{X}$ is related to known notions in literature such as the characteristic function or Fourier transform of a random variable. A related important class of multivariate distribution is given by the following, [3, 4];

Definition 3.0.2 A random vector $\mathbf{Y} = (\mathbf{Y}_1, ..., \mathbf{Y}_n)^\top$ has an elliptical distribution if there exists an affine mapping $\mathbf{x} \longmapsto A\mathbf{x} + \mu$ and a random vector $\mathbf{X}$ having a spherical distribution such that $\mathbf{Y} = A\mathbf{X} + \mu$.

For convenience, an elliptically distributed random vector $\mathbf{Y}$ that is defined as above, is denoted by $(A, \mu, \mathbf{X})$. Elliptical distributions are the elongated and translated versions of the spherical distributions.

The portfolio $\mathbf{x}$, consists of n financial instruments, has the loss function $f(\mathbf{x}, \mathbf{Y})$, where $\mathbf{Y}$ is elliptically distributed. The risk measures taken are VaR and CVaR. The properties of VaR and CVaR changes according to the loss function, $f(\mathbf{x}, \mathbf{Y})$, and disutility function taken while defining the portfolio problem in Chapter 1. In this study, only linear loss function is considered, where $f(\mathbf{x}, \mathbf{Y}) = -\mathbf{x}^\top\mathbf{Y}$ for the rest of the study. Two types of disutility functions are brought in. Linear disutility functions correspond to decision makers being risk-neutral in utility theory and convex disutility functions correspond to decision makers being risk averse. Risk-seeking decision makers do not make a logical sense in credit's risk or operational risk. Hence, these two classes seem to be sufficient. The first subsection contains the analysis with risk neutral decision makers, where as the second one entails the risk averse ones.

3.1 Risk Neutral Decision Maker

Loss is established as $f(\mathbf{x}, \mathbf{Y}) = -\mathbf{x}^\top\mathbf{Y}$ and disutility function is linear. This kind of portfolios are called linear portfolios. The interpretation of this form is quite intuitive;

the return of a portfolio is the sum of the returns of the individual instruments weighted by corresponding allocations. Then the loss is taken as the negative of the return. Thus, in the elliptic world with linear portfolios, actually the linear combinations of elliptically distributed random variables are considered. One of the properties of elliptically distributed random variables is that the marginal distributions of these random variables have the same characteristic function as the joint distribution function; i.e., both marginal distributions and the linear convolutions are elliptically distributed. Showing the following result is possible now.

Lemma 3.1.1 If $\mathbf{Y}$ has an elliptical distribution with representation $(A, \mu, \mathbf{X})$ and $det(A) \neq 0$, then it follows for every $\mathbf{x} \in \mathbb{R}^n$ that

$$-\mathbf{x}^\top \mathbf{Y} \;=_d\; \|A\mathbf{x}\| SV - \mathbf{x}^\top \mu,$$

with S having a cumulative distribution function G,where $G(0) = 0$. S is independent of the univariate random variable V, which has a standard normal distribution with mean 0 and variance 1.

Proof: If $\mathbf{Y}$ has an elliptical distribution with the above representation, then it can be written as

$$\mathbf{Y} = A\mathbf{X} + \mu.$$

It is given that $det(A) \neq 0$, then it follows by relation (3.1) that there exists a non-negative random variable S, with $S \sim G$ satisfying $G(0) = 0$, and a multivariate standard normally distributed random vector $\mathbf{Z}$ (with mean $\mathbf{0}$ and covariance matrix I) independent of S, such that;

$$\mathbf{Y} =_d AS\mathbf{Z} + \mu =_d SA\mathbf{Z} + \mu. \tag{3.2}$$

This shows for any $\mathbf{x} \in X \subseteq \mathbb{R}^n$ that,

$$-\mathbf{x}^\top \mathbf{Y} =_d -S\mathbf{x}^\top A\mathbf{Z} - \mathbf{x}^\top \mu. \tag{3.3}$$

Since $-\mathbf{x}^\top A\mathbf{Z}$ has a univariate normal distribution with mean 0 and positive variance, $\mathbf{x}^\top AA^\top \mathbf{x} = \|A\mathbf{x}\|^2$. We obtain by relation (3.3) that

$$-\mathbf{x}^\top \mathbf{Y} \;=_d\; \|A\mathbf{x}\|\, S\frac{-\mathbf{x}^\top A\mathbf{Z}}{\|A\mathbf{x}\|} - \mathbf{x}^\top \mu \;=_d\; \|A\mathbf{x}\|\, SV - \mathbf{x}^\top \mu$$

where $V = \frac{-\mathbf{x}^\top A\mathbf{Z}}{\|A\mathbf{x}\|}$ is a univariate standard normal random variable, independent of S.
$\square$

The above lemma allows us to represent the loss of the portfolio, which is also a random variable, as a linear function of the product of two independent random variables, where one of these random variables is multivariate standard normally distributed random vector and the other one is a univariate nonnegative random variable. Now, this representation will be used to define VaR and CVaR analytically.

Lemma 3.1.2 If $\mathbf{Y}$ has an elliptical distribution with $det(A) \neq 0$ and $f(\mathbf{x}, \mathbf{Y}) = -\mathbf{x}^\top \mathbf{Y}$, then the Value-at-Risk, $\alpha_\beta(\mathbf{x})$ or $\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y})$, is given by

$$\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y}) \;=\; \|A\mathbf{x}\|\, F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu \;=\; \Psi_{\mathbf{x}}^{\leftarrow}(\beta),$$

where F is the cumulative distribution function of the random variable SV considered in Lemma 3.1.1.

Proof: $\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y})$ is defined as $\min\{\alpha \in \mathbb{R} : \Psi_{\mathbf{x}}(\alpha) \geq \beta\}$ in Relation 1.3. If $\Psi_{\mathbf{x}}(\alpha)$ is continuous with respect to α, then this reduces to $\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y}) = \{\alpha \in \mathbb{R} : \Psi_{\mathbf{x}}(\alpha) = \beta\}$.To compute $\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y})$ we observe $\Psi_{\mathbf{x}}$ by Lemma 3.1.1:

$$\begin{aligned}
\Psi_{\mathbf{x}}(\alpha) &= \mathbb{P}\{-\mathbf{x}^\top \mathbf{Y} \leq \alpha\} = \mathbb{P}\{\|A\mathbf{x}\|\mathbf{SV} - \mathbf{x}^\top \mu \leq \alpha\} \\
&= \mathbb{P}\{\mathbf{SV} \leq \|A\mathbf{x}\|^{-1}(\alpha + \mathbf{x}^\top \mu)\} \\
&= F(\|A\mathbf{x}\|^{-1}(\alpha + \mathbf{x}^\top \mu)).
\end{aligned}$$

Now α is obtained out of the relation $\Psi_{\mathbf{x}}(\alpha) = \beta$.

$$\begin{aligned}
& \Psi_{\mathbf{x}}(\alpha) = \beta \\
\iff\; & F(\|A\mathbf{x}\|^{-1}(\alpha + \mathbf{x}^\top \mu)) = \beta \\
\iff\; & \|A\mathbf{x}\|^{-1}(\alpha + \mathbf{x}^\top \mu) = F^{\leftarrow}(\beta) \\
\iff\; & \alpha = \|A\mathbf{x}\| F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu
\end{aligned}$$

Hence $\alpha_\beta(\mathbf{x}) = \Psi_{\mathbf{x}}^{\leftarrow}(\beta) = \|A\mathbf{x}\|\, F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu$ and the desired result is verified. $\square$

Consequently, we have the analytical form for the VaR if $\mathbf{Y}$ has multivariate elliptical distribution and $f(\mathbf{x}, \mathbf{Y}) = -\mathbf{x}^\top \mathbf{Y}$. Moreover, $F^{\leftarrow}(\beta)$ can be simply calculated using standard inversion. An analytical form also exists for CVaR whenever the same conditions are satisfied. The result is given by the following lemma.

Lemma 3.1.3 If $\mathbf{Y}$ has an elliptical distribution with $det(A) \neq 0$ and $f(\mathbf{x}, \mathbf{Y}) = -\mathbf{x}^\top \mathbf{Y}$, then

$$\mathrm{CVaR}_\beta(-\mathbf{x}^\top \mathbf{Y}) = \phi_\beta(\mathbf{x}) = \|A\mathbf{x}\|\, \theta - \mathbf{x}^\top \mu,$$

with $\theta := \mathbb{E}(SV \mid SV \geq F^{\leftarrow}(\beta))$ and F which is the cumulative distribution function of the random variable SV.

Proof: We know by Lemma 3.1.1 that

$$-\mathbf{x}^\top \mathbf{Y} \ =_d \ \|A\mathbf{x}\| SV - \mathbf{x}^\top \mu,$$

and by relation (1.6) that

$$\mathrm{CVaR}_\beta(-\mathbf{x}^\top \mathbf{Y}) = \mathbb{E}(f(\mathbf{x}, \mathbf{Y}) \mid f(\mathbf{x}, \mathbf{Y}) \geq \mathrm{VaR}_\beta f(\mathbf{x}, \mathbf{Y})).$$

If we plug in the result obtained by Lemma 3.1.2, then

$$\begin{aligned}
\mathrm{CVaR}_\beta(-\mathbf{x}^\top \mathbf{Y}) &= \mathbb{E}(-\mathbf{x}^\top \mathbf{Y} \mid -\mathbf{x}^\top \mathbf{Y} \geq \alpha_\beta(\mathbf{x})) \\
&= \mathbb{E}(\|A\mathbf{x}\|\, SV - \mathbf{x}^\top \mu \mid \|A\mathbf{x}\|\, SV - \mathbf{x}^\top \mu \geq \Psi_{\mathbf{x}}^{\leftarrow}(\beta)) \\
&= \mathbb{E}(\|A\mathbf{x}\|\, SV \mid \|A\mathbf{x}\|\, SV - \mathbf{x}^\top \mu \geq \|A\mathbf{x}\|\, F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu) - \mathbf{x}^\top \mu \\
&= \mathbb{E}(\|A\mathbf{x}\|\, SV \mid SV \geq F^{\leftarrow}(\beta)) - \mathbf{x}^\top \mu \\
&= \|A\mathbf{x}\|\, \mathbb{E}(SV \mid SV \geq F^{\leftarrow}(\beta)) - \mathbf{x}^\top \mu
\end{aligned}$$

So, if we label $\theta := \mathbb{E}(SV \mid SV \geq F^{\leftarrow}(\beta))$, the desired result follows. □

Since VaR and CVaR are defined analytically by Lemmas 3.1.2 and 3.1.3, respectively, now the optimization problems (VP) and (CVP) defined in Section 1.1, can be rewritten.

$$\begin{aligned}
\min \quad & \|A\mathbf{x}\|\, F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu \\
\text{s.t} \quad & \textstyle\sum_{i=1}^n x_i = 1 \\
& x_i \geq 0 \qquad\qquad \forall i = 1, ..., n.
\end{aligned} \tag{VP}$$

and

$$\begin{aligned}
\min \quad & \|A\mathbf{x}\|\, (\mathbb{E}(SV \mid SV \geq F^{\leftarrow}(\beta))) - \mathbf{x}^\top \mu \\
\text{s.t} \quad & \textstyle\sum_{i=1}^n x_i = 1 \\
& x_i \geq 0 \qquad\qquad\qquad\qquad \forall i = 1, ..., n.
\end{aligned} \tag{CVP}$$

VaR and CVaR may be used for ranking the risks or determining the optimal risk-minimizing portfolios. Within the elliptic world, using VaR or CVaR is equivalent to using the Markowitz approach under the condition that a certain return is attained. These measures may give different objective function values but they all yield the same optimal decision vector $\mathbf{x}$. We next formalize this discussion.

Lemma 3.1.4 If $\mathbf{Y}$ has an elliptical distribution with the presentation $(A, \mu, \mathbf{X})$ where $det(A) \neq 0$, $f(\mathbf{x}, \mathbf{Y}) = \lambda(-\mathbf{x}^\top \mathbf{Y}) + a$ for $\lambda \geq 0$ and $a \in \mathbb{R}$ and the risk measure $\rho : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies the following two properties:

1. $\rho((-\mathbf{x}^{T\top}\mathbf{Y}) + a) = a + \rho(-\mathbf{x}^\top \mathbf{Y})$,

2. $\rho(\lambda(-\mathbf{x}^\top \mathbf{Y})) = \lambda \rho(-\mathbf{x}^\top \mathbf{Y})$.

Then for any two portfolios ${\mathbf{x}_1}^\top \mathbf{Y}$ and ${\mathbf{x}_2}^\top \mathbf{Y}$ having the same expectation, it follows that

$$\rho({\mathbf{x}_1}^\top \mathbf{Y}) \leq \rho({\mathbf{x}_2}^\top \mathbf{Y}) \Longleftrightarrow \sigma^2({\mathbf{x}_1}^\top \mathbf{Y}) \leq \sigma^2({\mathbf{x}_2}^\top \mathbf{Y}).$$

Proof: $\mathbb{E}({\mathbf{x}_1}^\top \mathbf{Y}) = \mathbb{E}({\mathbf{x}_2}^\top \mathbf{Y})$ is given.

$$\rho({\mathbf{x}_1}^\top \mathbf{Y}) \leq \rho({\mathbf{x}_2}^\top \mathbf{Y}) \text{ if and only if } \rho({\mathbf{x}_1}^\top \mathbf{Y} - \mathbb{E}({\mathbf{x}_1}^\top \mathbf{Y})) \leq \rho({\mathbf{x}_2}^\top \mathbf{Y} - \mathbb{E}({\mathbf{x}_2}^\top \mathbf{Y}))$$

by the first property of the risk measure given above. Since $\mathbf{Y}$ has an elliptical distribution, we know by Lemma 3.1.1 that $\mathbf{x}^\top \mathbf{Y} - \mathbb{E}(\mathbf{x}^\top \mathbf{Y}) =_d \|A\mathbf{x}\| SV$, therefore;

$$\rho(\|A\mathbf{x}_1\| SV) = \|A\mathbf{x}_1\| \rho(SV) \ \leq \ \|A\mathbf{x}_2\| \rho(SV) = \rho(\|A\mathbf{x}_2\| SV)$$

which is equivalent with $\|A\mathbf{x}_1\| \leq \|A\mathbf{x}_2\|$ by the second property of the risk measure given above. Thus, we have the desired result under the condition that $\mathbb{E}({\mathbf{x}_1}^\top \mathbf{Y}) = \mathbb{E}({\mathbf{x}_2}^\top \mathbf{Y})$. □

By Lemma 3.1.4 and [3] we have

$$\arg \min_{\mathbf{x}_1 \in X} \rho({\mathbf{x}_1}^\top \mathbf{Y}) = \arg \min_{\mathbf{x}_1 \in X} \sigma^2({\mathbf{x}_1}^\top \mathbf{Y})$$

with the feasible region

$$X = \{\mathbf{x} \ : \ \mathbf{e}^\top \mathbf{x} = 1, \mu^\top \mathbf{x} = r, \mathbf{x} \geq \mathbf{0}\},$$

where r is the predetermined expected return of the portfolio. We now easily show that VaR and CVaR also belong to the special class of risk measures that are given in Lemma 3.1.4.

Lemma 3.1.5 It follows for $\rho(\mathbf{Z}) := \mathbb{E}(\mathbf{Z} \mid \mathbf{Z} \geq \mathrm{VaR}_\beta(\mathbf{Z}))$ with $\mathbf{Z}$, a univariate random variable, and for every random vector $\mathbf{Y}$ not necessarily elliptically distributed,that we have $\rho(-\lambda\mathbf{x}^\top\mathbf{Y}) = \lambda\rho(-\mathbf{x}^\top\mathbf{Y})$ for every $\lambda \geq 0$ and $\rho(-\mathbf{x}^\top\mathbf{Y} + a) = a + \rho(-\mathbf{x}^\top\mathbf{Y})$ for every $a \in \mathbb{R}$.

Proof: For every $\lambda \geq 0$ and $a \in \mathbb{R}$, to compute $\mathrm{VaR}_\beta(-\lambda\mathbf{x}^\top\mathbf{Y} + a)$, we observe by Lemma 3.1.1 and Lemma 3.1.2 that $\Psi_\mathbf{x}(\alpha) = \mathbb{P}\{-\lambda\mathbf{x}^\top\mathbf{Y} + a \leq \alpha\} = \mathbb{P}\{-\mathbf{x}^\top\mathbf{Y} \leq \frac{\alpha-a}{\lambda}\} = F_\mathbf{x}(u^{\leftarrow}(\alpha))$. This implies;

$$\Psi_\mathbf{x}(\alpha) = \beta \Longleftrightarrow F_\mathbf{x}(\frac{\alpha - a}{\lambda}) = \beta \Longleftrightarrow \frac{\alpha - a}{\lambda} = F_\mathbf{x}^{\leftarrow}(\beta) \Longleftrightarrow \alpha = \lambda F_\mathbf{x}^{\leftarrow}(\beta) + a$$

Therefore, $\mathrm{VaR}_\beta(-\lambda\mathbf{x}^\top\mathbf{Y} + a) = \lambda\mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y}) + a$ is proven. This property of VaR will be used to show that CVaR has the above properties.

$$\begin{aligned}
\rho(-\lambda\mathbf{x}^\top\mathbf{Y}) &= \mathbb{E}(-\lambda\mathbf{x}^\top\mathbf{Y} \mid -\lambda\mathbf{x}^\top\mathbf{Y} \geq \mathrm{VaR}_\beta(-\lambda\mathbf{x}^\top\mathbf{Y})) \\
&= \mathbb{E}(-\lambda\mathbf{x}^\top\mathbf{Y} \mid -\mathbf{x}^\top\mathbf{Y} \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y})) \\
&= \lambda\mathbb{E}(-\lambda\mathbf{x}^\top\mathbf{Y} \mid -\mathbf{x}^\top\mathbf{Y} \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y})) \\
&= \lambda\rho(-\mathbf{x}^\top\mathbf{Y})
\end{aligned}$$

we have shown the first property. Moreover, $\forall a \in \mathbb{R}$ it follows that ,

$$\begin{aligned}
\rho(-\mathbf{x}^\top\mathbf{Y} + a) &= \mathbb{E}(-\mathbf{x}^\top\mathbf{Y} + a| - \mathbf{x}^\top\mathbf{Y} + a \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y} + a)) \\
&= \mathbb{E}(-\mathbf{x}^\top\mathbf{Y} + a| - \mathbf{x}^\top\mathbf{Y} + a \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y}) + a) \\
&= \mathbb{E}(-\mathbf{x}^\top\mathbf{Y} + a| - \mathbf{x}^\top\mathbf{Y} \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y})) \\
&= a + \mathbb{E}(-\mathbf{x}^\top\mathbf{Y}| - \mathbf{x}^\top\mathbf{Y} \geq \mathrm{VaR}_\beta(-\mathbf{x}^\top\mathbf{Y})) \\
&= a + \rho(-x^\top\mathbf{Y})
\end{aligned}$$

and the second property is also shown. □

The first property is called positive homogeneity and the second one is translation invariance [3]. Actually, these two properties are the same as the following property: if the function u is an increasing affine function given by $u(t) = \lambda t + a$, where $\lambda \geq 0$, then

$$\rho(u(-\mathbf{x}^\top\mathbf{Y})) = u(\rho(-\mathbf{x}^\top\mathbf{Y}))$$

where ρ is the risk measure that is defined in Lemma 3.1.4. This section concludes the fact that, if we are in the elliptic world, $\mathbf{Y}$ has multivariate elliptical distribution, and $f(\mathbf{x}, \mathbf{Y})$ is an increasing affine function, then minimizing VaR and CVaR measure is equivalent to solving the corresponding Markowitz optimization problem given by

$$\min\{\mathbf{x}^\top \Sigma \mathbf{x} : \mathbf{x} \geq \mathbf{0}, \sum_{i=1}^{n} x_i = 1, \mu^\top \mathbf{x} = r\},$$

where r is the expected return. To solve the resulting Markowitz problem, a finite stepped algorithm proposed in [10] is modified and implemented. The algorithm and the implementation is given in the computational approach section.

3.2 Risk Averse Decision Maker

In the last subsection we have simply taken linear disutility function. However the investor can have an increasing, right continuous and convex disutility function $u : \mathbb{R} \longmapsto \mathbb{R}$. In this case it follows that $f(\mathbf{x}, \mathbf{Y}) = u(-\mathbf{x}^\top \mathbf{Y})$.

Definition 3.2.1 An investor is called risk-averse if its disutility is given by $u(-\mathbf{x}^\top \mathbf{Y})$ with $u(.)$ an increasing convex function. The investor is called risk-neutral if its loss function is given by $-\mathbf{x}^\top \mathbf{Y}$.

The investor is not supposed to be risk neutral. As an example of a risk-averse measure we mention the so called downside risk given by $f(-\mathbf{x}^\top \mathbf{Y})$ with $f(t) = (\max\{t-\tau, 0\})^p$ with $p > 1$. In this case $\tau > 0$ represent a fixed positive number representing the acceptable loss for an investor (cf. [3]). Clearly for $p > 1$ the function is convex. By lemma 3.1.1, it follows immediately, for the disutility function $u(.)$ and elliptically distributed $\mathbf{Y}$, that

$$u(-\mathbf{x}^\top \mathbf{Y}) \;=_d\; u(\|A\mathbf{x}\|\, SV - \mathbf{x}^\top \mu). \tag{3.4}$$

By the above relation we fathom the distribution of the random variable $u(-\mathbf{x}^\top \mathbf{Y})$ formed by disutility function $u(.)$, if $u(.)$ is an increasing, right continuous function.

Lemma 3.2.2 If $u(.)$ is a strictly increasing continuous function and $\mathbf{Y}$ has an elliptical distribution with $\det(A) \neq 0$, then the Value-at-Risk $\mathrm{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y}))$ is given by

$$\mathrm{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y})) = u(\|A\mathbf{x}\| F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu) = \Psi_{\mathbf{x}}^{\leftarrow}(\beta)$$

with F being the cumulative distribution function of the random variable SV considered in Lemma 3.1.1.

Proof: To compute $\text{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y}))$ we observe by Lemma 3.1.1 that

$$\begin{aligned}\Psi_{\mathbf{x}}(\alpha) &= \mathbb{P}\{u(-\mathbf{x}^\top \mathbf{Y}) \le \alpha\} \\ &= \mathbb{P}\{\|A\mathbf{x}\| SV - \mathbf{x}^\top \mu \le u^{\leftarrow}(\alpha)\} \\ &= \mathbb{P}\{SV \le \|A\mathbf{x}\|^{-1}(u^{\leftarrow}(\alpha) + \mathbf{x}^\top \mu)\} \\ &= F(\|A\mathbf{x}\|^{-1}(u^{\leftarrow}(\alpha) + \mathbf{x}^\top \mu)).\end{aligned}$$

This shows

$$\begin{aligned} & \Psi_{\mathbf{x}}(\alpha) = \beta \\ \iff & F(\|A\mathbf{x}\|^{-1}(u^{\leftarrow}(\alpha) + \mathbf{x}^\top \mu)) = \beta \\ \iff & \|A\mathbf{x}\|^{-1}(u^{\leftarrow}(\alpha) + \mathbf{x}^\top \mu) = F^{\leftarrow}(\beta) \\ \iff & \alpha = u(\|A\mathbf{x}\| F^{\leftarrow}(\beta) - \mathbf{x}^\top \mu)\end{aligned}$$

and so the desired result follows. □

By the above lemma, we obtain the following important result;

$$\text{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y})) = u(\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y})) \tag{3.5}$$

Actually, this relation does not hold only for elliptically distributed $\mathbf{Y}$. Observation 3.5 is a rather general result for VaR measure. Consider a random vector $\mathbf{Y}$ with an arbitrary multivariate distribution and the distribution of the random variable $-\mathbf{x}^\top \mathbf{Y}$ is denoted by $F_{\mathbf{x}}$. Since $u(.)$ is an increasing and continuous function, the cumulative distribution function can be illustrated as $\Psi_{\mathbf{x}}(\alpha) = \mathbb{P}\{u(-\mathbf{x}^\top \mathbf{Y}) \le \alpha\} = \mathbb{P}\{-\mathbf{x}^\top \mathbf{Y} \le u^{\leftarrow}(\alpha)\} = F_{\mathbf{x}}(u^{\leftarrow}(\alpha))$. Therefore, the following equivalence can be observed:

$$\Psi_{\mathbf{x}}(\alpha) = \beta \iff F_{\mathbf{x}}(u^{\leftarrow}(\alpha)) = \beta \iff u^{\leftarrow}(\alpha) = F_{\mathbf{x}}^{\leftarrow}(\beta) \iff \alpha = u(F_{\mathbf{x}}^{\leftarrow}(\beta)).$$

Hence the relation (3.5) holds. Therefore, with VaR measure the decision maker choses the same portfolio whether he is risk averse or risk-neutral. Only the objective function value, the quantity of risk changes.

$$\min_{\mathbf{x} \in X} \text{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y})) = \min_{\mathbf{x} \in X} u(\text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y}))$$

For the risk seeking decision makers the disutility function is not a strictly increasing function. If $u(.)$ has this kind of property, $u(\min_{\mathbf{x} \in X} \text{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y}))$ becomes a concave function and the Relation (3.5) does not hold. Nonetheless, it seems more logical for the risk-averse decision makers to minimize their risk measure. Irrelevancy of VaR with respect to the types of disutility functions can be considered as an argument to use CVaR instead of VaR. We have shown in Lemma 3.1.3 that, in case of a linear disutility function and elliptically distributed random vector $\mathbf{Y}$, CVaR has an analytical solution. The following lemma illustrates that CVaR does not have an analytical solution for general disutility functions, even if the random vector $\mathbf{Y}$ is elliptically distributed.

Lemma 3.2.3 If $u(.)$ is a strictly increasing, continuous function and $\mathbf{Y}$ has an elliptical distribution with $det(A) \neq 0$, then;

$$\text{CVaR}_\beta(u(-\mathbf{x}^\top\mathbf{Y})) \neq u(\text{CVaR}_\beta(-\mathbf{x}^\top\mathbf{Y}))$$

Proof: We observe by Lemma 3.1.1 that,

$$\begin{aligned}\text{CVaR}_\beta(u(-\mathbf{x}^\top\mathbf{Y})) &= \mathbb{E}(u(-\mathbf{x}^\top\mathbf{Y}) \mid u(-\mathbf{x}^\top\mathbf{Y}) \geq \alpha_\beta(\mathbf{x})) \\ &= \mathbb{E}(u(\|A\mathbf{x}\|SV - \mathbf{x}^\top\mu) \mid u(\|A\mathbf{x}\|SV - \mathbf{x}^\top\mu) \geq u(\|A\mathbf{x}\|F^{\leftarrow}(\beta) - \mathbf{x}^\top\mu)) \\ &= \mathbb{E}(u(\|A\mathbf{x}\|SV - \mathbf{x}^\top\mu) \mid SV \geq F^{\leftarrow}(\beta))\end{aligned}$$

since we cannot change the places of $u(.)$ and $\mathbb{E}(.)$, the desired result is obtained. □

Although the analytical form of the solution does not exist, the objective function can be closely estimated by simulations from two independent univariate distributions, the distributions of S and V. The CVaR measure is estimated in [13] by generating scenarios from multivariate distributions. Thus, in the elliptic world, simulation from multivariate distributions is reduced to univariate simulation.

$$\text{CVaR}_\beta(u(-\mathbf{x}^\top\mathbf{Y})) = \mathbb{E}(u(\|A\mathbf{x}\|SV - \mathbf{x}^\top\mu) \mid SV \geq F^{\leftarrow}(\beta)). \tag{3.6}$$

is used to estimate the objective function of CVP when the decision maker is risk averse, by generating realizations for S and V. Illustration and the comparison of the technique is given in the Computational Results, Chapter 4.

CHAPTER 4

COMPUTATIONAL RESULTS

We devote this section to introduce the modified algorithm as well as to illustrate our analysis with the proposed solution approach. We first give a finite step algorithm to solve problem (MP). Then, we present a thorough numerical study on a set of data compiled from the Istanbul Stock Exchange.

4.1 Modified Michelot Algorithm

At the end of Section 3.1, we have emphasized that when $\mathbf{Y}$ has a multivariate elliptical distribution and $f(\mathbf{x}, \mathbf{Y})$ is a bilinear, then minimizing VaR and CVaR measures are equivalent to solving the corresponding Markowitz problem (MP) with the predetermined expected return r.

The algorithm introduced by Michelot finds in finite steps the projection of a given vector onto a special polytope [10]. The main idea of this algorithm is to use the analytic solutions of a sequence of projections onto canonical simplices and elementary cones. To apply Michelot's algorithm, we use a transformation $\mathbf{y} = \Sigma^{\frac{1}{2}}\mathbf{x}$. Then problem (MP) becomes

$$\min\{\mathbf{y}^\top\mathbf{y} \mid \mathbf{d}^\top\mathbf{y} = 1,\ \eta^\top\mathbf{y} = r,\ A\mathbf{y} \geq \mathbf{0}\}, \tag{4.1}$$

where $\mathbf{d}^\top = \mathbf{e}^\top\Sigma^{-1/2}$, $\eta^\top = \mu^\top\Sigma^{-1/2}$ and $A = \Sigma^{-1/2}$. Note that the matrix A is the same as the matrix used for elliptical distributions in Chapter 3. To modify Michelot's algorithm according to our problem, we need to introduce several sets. Let

$$\begin{aligned} \mathcal{V} &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{d}^\top\mathbf{y} = 1,\ \eta^\top\mathbf{y} = r\}, \\ \mathcal{Y}_\mathcal{I} &= \{\mathbf{y} \in \mathbb{R}^n \mid (A\mathbf{y})_i = 0,\ i \in \mathcal{I}\}, \\ \mathcal{V}_\mathcal{I} &= \mathcal{V} \cap \mathcal{Y}_\mathcal{I}, \end{aligned} \tag{4.2}$$

where $\mathcal{I} \subseteq \{1, 2, \cdots, n\}$ denotes an index set and $(A\mathbf{y})_i$ denotes the ith component of vector $A\mathbf{y}$. Algorithm 4.1.1 gives the steps of the Modified Michelot Algorithm. The algorithm starts with solving the quadratic programming problem $\min\{\mathbf{y}^\top \mathbf{y} \mid \mathbf{y} \in \mathcal{V}\}$. It is easy to show that the analytic solution for this quadratic program [2] is given by

$$\bar{\mathbf{y}} = \begin{bmatrix} \mathbf{d}^\top \\ \eta^\top \end{bmatrix}^\top \begin{bmatrix} \mathbf{d}^\top \mathbf{d} & \mathbf{d}^\top \eta \\ \eta^\top \mathbf{d} & \eta^\top \eta \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix}. \tag{4.3}$$

Notice that some of the components $(A\mathbf{y})_i$ may be negative. After identifying the most negative component and initializing the index set $\mathcal{I}$, the algorithm iterates between projections of the incumbent solution $\bar{\mathbf{x}}$ onto subspace $\mathcal{Y}_\mathcal{I}$, and then onto subspace $\mathcal{V}_\mathcal{I}$ until none of the components are negative; i.e., the solution is optimal. The first projection [2] is given by

$$\begin{aligned} P_{\mathcal{Y}_\mathcal{I}}(\bar{\mathbf{y}}) &:= \arg\min\{\tfrac{1}{2}\|\mathbf{y} - \bar{\mathbf{y}}\| \mid \mathbf{y} \in \mathcal{Y}_\mathcal{I}\} \\ &= \left(I - [A]_{i\in\mathcal{I}}^\top \left([A]_{i\in\mathcal{I}} [A]_{i\in\mathcal{I}}^\top\right)^{-1} [A]_{i\in\mathcal{I}}^\top\right) \bar{\mathbf{y}}, \end{aligned} \tag{4.4}$$

where $[A]_{i\in\mathcal{I}}$ denotes the submatrix formed by the rows $i \in \mathcal{I}$ of A. Similarly, the second projection [2] yields

$$\begin{aligned} P_{\mathcal{V}_\mathcal{I}}(\bar{\mathbf{y}}) &:= \arg\min\{\tfrac{1}{2}\|\mathbf{y} - \bar{\mathbf{y}}\| \mid \mathbf{y} \in \mathcal{V}_\mathcal{I}\}, \\ &= \bar{\mathbf{y}} - \begin{bmatrix} \mathbf{d}^\top \\ \eta^\top \\ [A]_{i\in\mathcal{I}} \end{bmatrix}^\top \left(\begin{bmatrix} \mathbf{d}^\top \\ \eta^\top \\ [A]_{i\in\mathcal{I}} \end{bmatrix} \begin{bmatrix} \mathbf{d}^\top \\ \eta^\top \\ [A]_{i\in\mathcal{I}} \end{bmatrix}^\top \right)^{-1} \left(\begin{bmatrix} \mathbf{d}^\top \\ \eta^\top \\ [A]_{i\in\mathcal{I}} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} 1 \\ r \\ 0 \end{bmatrix} \right). \end{aligned} \tag{4.5}$$

Since we have a finite number of assets, the dimension of the index set $\mathcal{I}$ is also finite. This shows that the modified algorithm terminates in at most n iterations (see also; [10]).

Algorithm 1 Modified Michelot Algorithm

1. Set $\bar{\mathbf{y}}$ as in (4.3). If $A\bar{\mathbf{y}} \geq \mathbf{0}$ then stop; $\bar{\mathbf{y}}$ is optimal. Otherwise, select i with most negative $(A\bar{\mathbf{y}})_i$, set $\mathcal{I} \leftarrow \{i\}$, and then go to Step 2.
2. Set $\bar{\mathbf{y}} \leftarrow P_{\mathcal{Y}_\mathcal{I}}(\bar{\mathbf{y}})$ as in (4.4), and then go to Step 3.
3. Set $\bar{\mathbf{y}} \leftarrow P_{\mathcal{V}_\mathcal{I}}(\bar{\mathbf{y}})$ as in (4.5). If $A\bar{\mathbf{y}} \geq \mathbf{0}$ then stop; $\bar{\mathbf{y}}$ is optimal. Otherwise, select i with most negative $(A\bar{\mathbf{y}})_i$, update $\mathcal{I} \leftarrow \mathcal{I} \cup \{i\}$, and then go to Step 2.

4.2 Sample Data

To conduct the experiments, a portfolio with 26 financial instruments from seven sectors is compiled from Istanbul Stock Exchange. The data is collected over 48 months starting from July 2001 until June 2005. Table 4.1 shows the data of the instruments. The first column gives the individual instruments that are grouped into their respective sectors. The last two columns correspond to average monthly returns and variance of monthly returns, respectively, which are given as percentages. Monthly returns are calculated by proportioning the difference between the initial and the final prices of the corresponding stocks with the initial prices for the respective months. The interpretation of negative monthly return is given as the decline of stock price for that month. The exact formulation for monthly return can be found in the Istanbul Stock Exchange website. The second column refers to mainly the sample variance of monthly returns for the corresponding stock. All the figures in the table are calculated by using TL currency.

Before conducting experiments, we need to test that the returns are indeed elliptically distributed. Moreover we should also use the point estimator of the covariance matrix. However, the test and estimation assume that the data is not time dependent. Therefore, we have first shown by Von-Neumann test that our data is time independent. The details of this test are given in Appendix A. After establishing that, we have used a macro named `%MULTINORM` in `SAS` to show that the data is coming from a multivariate normal distribution [19]. Appendix B includes the details about multivariate normality test. In the subsequent part of this section, Σ denotes the point estimator of the covariance matrix, and $A := \Sigma^{1/2}$ is found by the Cholesky decomposition of the covariance matrix. To implement the solution procedure we have used `MATLAB` 14 on a Xeon-2 GHz personal computer.

4.3 Risk Neutral Decision Maker

As we have discussed in Section 3.1, for risk neutral decision makers and returns elliptical distributed, the optimal solution of problems (MP), (VP), and (CVP) coincide,

$$\mathbf{x}^* := \arg\min_{\mathbf{x}\in X} \mathbf{x}^\top \Sigma \mathbf{x} = \arg\min_{\mathbf{x}\in X} \|A\mathbf{x}\| \operatorname{VaR}_\beta(\mathbf{Z}) - \mathbf{x}^\top \mu = \arg\min_{\mathbf{x}\in X} \|A\mathbf{x}\| \operatorname{CVaR}_\beta(\mathbf{Z}) - \mathbf{x}^\top \mu.$$

Since $\mathbf{Y}$ has a multivariate normal distribution, $\mathbf{S} = 1$, $\mathbf{V}$ is a standard normal variable, and $F^{\leftarrow}$ is the inverse cumulative distribution function of $\mathbf{V}$.

Stocks Within Sectors	**Average Monthly Returns (%)**	**Variance of Monthly Returns (%)**
B1	3.06	286.97
B2	4.55	314.18
B3	2.60	397.98
B4	5.82	431.77
B5	3.45	352.04
B6	4.16	181.15
IC1	3.48	233.79
IC2	4.87	309.87
IC3	3.70	471.92
H1	2.75	274.13
H2	2.65	244.09
H3	2.07	222.58
H4	4.18	265.7
IT1	4.89	847.75
IT2	4.90	418.91
NM1	5.04	241.85
NM2	4.70	219.40
NM3	1.32	227.32
NM4	4.85	185.10
NM5	6.83	228.76
CP1	1.40	182.92
CP2	3.91	237.57
CP3	3.79	353.87
CP4	3.10	186.92
CP5	6.05	342.24
T1	1.48	335.80

B1,...,B6 : Banks and Special Finance Corporations

IC1,...,IC3 : Insurance Companies

H1,...,H4 : Holding and Investment Companies

IT1,IT2 : Investment Trusts

NM1,...,NM5 : Manufacture of Chemicals and of Petroleum, Rubber and Plastic Products

T1 : Telecommunication

Table 4.1: Main statistics of financial instruments.

Notice that problem (MP) is a convex optimization problem that can be solved by off-the-shelf nonlinear optimization solvers. To compare the performance of the modified Michelot Algorithm with a standard solver, we have also solved problem (MP) with `fmincon` function in `MATLAB`. We initially set the expected return, $r = \big(\max\{\mu_i : 1 \leq i \leq 26\} + \min\{\mu_i : 1 \leq i \leq 26\}\big)/2$. Both `fmincon` and Algorithm 4.1 return the same optimal solution. There are only six nonzero allocations to the financial instruments,

$$\text{B6} = 36.41\%, \text{NM3} = 0.29\%, \text{NM4} = 7.51\%, \text{NM5} = 9.57\%, \text{CP3} = 15.34\%, \text{CP4} = 30.87\%.$$

Solution of problem (MP) decides on the nonzero allocations by looking at the variance and covariance of the assets. We observe from Table 4.1 that the variances of the chosen assets are relatively low. Among the correlated assets, (MP) selects the one which has the minimum variance. Table 4.2 shows the comparison of the computation times. Clearly, the modified Michelot Algorithm outperforms `fmincon`. This is an expected result, since Algorithm 4.1 is customized to solve problem (MP). This improvement is especially important when the number of assets is huge. Notice also that it is interesting in finance to see the efficient frontier. In this case many problems need to be solved for different values of r. Therefore, the speed of the algorithm used to solve this sequence of problems becomes important.

Computation Times	
fmincon	**Modified Michelot**
3.078125 sec.	0.09375 sec.

Table 4.2: Comparison of Algorithm4.1 with `fmincon` function in terms of computation times.

The objective function values in problems (MP), (VP) and (CVP) are in percentages and as risk measures they illustrate at what percentage the optimal portfolio is going to lose. The problem (MP) deals with expected returns, where as (VP) and (CVP) investigate unexpected returns, where numerical objective function values can be helpful to visualize the unexpected situation and to compare portfolios. We have the analytic forms of the objective functions in problems (MP) and (VP). On the other hand, to compute the optimal objective function value of problem (CVP), generating realizations V_j such that $V_j \geq F^{\leftarrow}(\beta)$, is needed. Then θ can be estimated by

$$\theta \approx \frac{1}{N}\sum_{j=1}^{N} V_j,$$

where N is the sample size of the realizations that satisfy the condition $V_j \geq F^{\leftarrow}(\beta)$. This estimation becomes more precise for higher N values. Table 4.3 illustrates the improvement in the estimation as N increases. Here we set $\beta = 0.95$. The second column shows the change in θ as N increases. Similarly, the last column gives the convergence of the objective function value of problem (CVP). We note that the convergence can be observed after a large number of points is sampled.

N	θ	$\phi(\mathbf{x}^*)$
100	2.0426	17.9175
500	2.0938	18.4689
1000	2.0871	18.3964
5000	2.0664	18.1742
25000	2.0639	18.1494
100000	2.0636	18.1454

Table 4.3: Estimation of the optimal objective function value of problem (CVP); $\beta = 0.95$.

Naturally the time required to generate realizations for estimating θ becomes larger as N increases. In addition, the number of samples that pass the condition $V_j \geq F^{\leftarrow}(\beta)$ decreases as β increases. Therefore, we conduct another test to see the effect of N and β in computation times. Table 4.4 shows that as expected, the computation times rapidly increases when N and β are increased. A remedy for decreasing the large computation times may be using some rare event generation techniques [14].

β	N=100	N=500	N=1000	N=5000	N=25000	N=100000
0.90	0.12 sec.	0.28 sec.	0.53 sec.	2.77 sec.	25.81 sec.	361.52 sec.
0.95	0.08 sec.	0.42 sec.	0.86 sec.	4.83 sec.	37.17 sec.	398.41 sec.
0.99	0.34 sec.	2.09 sec.	4.31 sec.	22.12 sec.	123.62 sec.	749.20 sec.
0.999	4.01 sec.	21.95 sec.	44.41 sec.	220.50 sec.	1120.08 sec.	4677.50 sec.

Table 4.4: Time required to estimate θ for different N and β values.

The optimal objective function value of problem (CVP) is an upperbound for the optimal objective function value of problem (VP) [13]. One expects that the difference between these two optimal objective function values decrease as β increases, especially when the distribution function is not highly skewed. However, we should note that this is not always the case, since the difference is expected to increase as the skewness of the underlying joint distribution increases. Table 4.5 shows for our data that the optimal values for problems (VP) and (CVP) indeed decreases as β increases. Since the returns are multivariate normal, and hence elliptic, the joint distribution is not highly skewed.

β	$\alpha_\beta(\mathbf{x}^*)$	$\phi_\beta(\mathbf{x}^*)$
0.90	9.7226	14.8322
0.95	13.6347	18.1494
0.99	20.9730	24.6247
0.999	29.1986	32.1890

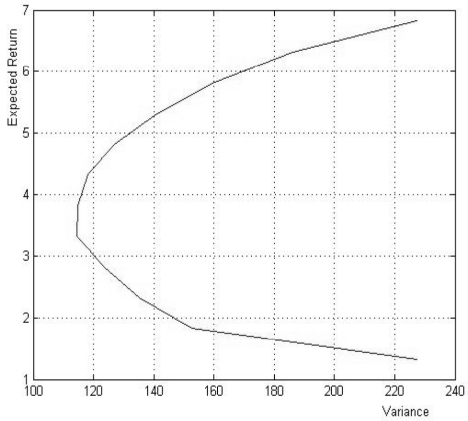
Table 4.5: Difference between optimal objective function values of problems (VP) and (CVP); $N = 25000$.

To this end we have used a fixed expected return value, r. In the analysis of portfolio optimization problems, it is quite common to construct the efficient frontier of r versus the considered risk measure. Figure 4.1 illustrates the minimum-variance sets for all three risk measures [8]. Far left point of the minimum-variance set is called the minimum-variance point. Most investors will prefer the portfolio with the smallest variance for the given mean. This type of investors is said to be risk averse. Additionally, most investors will prefer the portfolio with the largest mean for a given level of variance. This property of investors is termed nonsatiation. Thus, only the upper part from the minimum-variance point of the minimum-variance set will be of interest to investors who are risk averse and satisfy nonsatiation. This upper portion is named as efficient frontier.

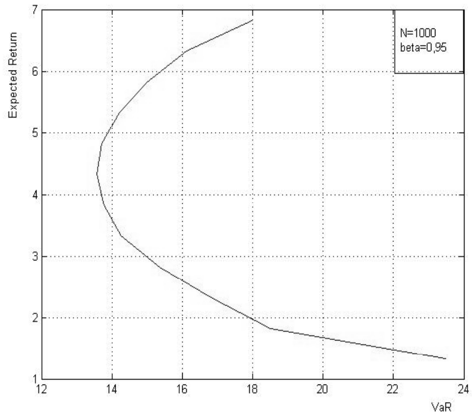
4.4 Risk Averse Decision Maker

As we have defined in Chapter 3.2, an investor is called risk averse if its disutility function is an increasing, convex function. Hence, the rate of change of disutility of a risk averse investor increases as the loss increases. In addition to this, loss is more significant than gain for a risk averse investor. Exponential utility function is one of the four types of utility functions that are mostly preferred, [8]. Therefore, $u(t) = e^{ct} - 1$ is chosen for the disutility function, where c is a scale parameter and 1 is subtracted in order to have zero disutility in case of zero loss. We initially set the scale parameter, $c = 1/(max\{\mu_i : 1 \geq i \geq 26\}$. It is logical in the sense that the maximum level of return is attained as all of the capital is allocated to the asset with the highest expected return.

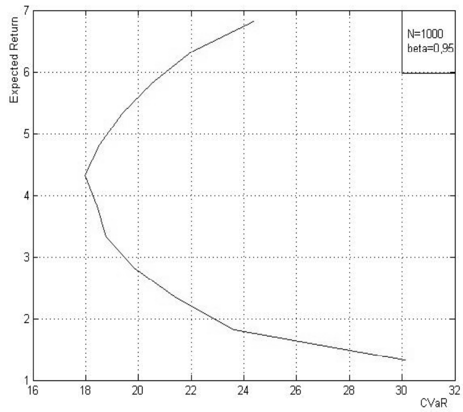
As discussed in Section 3.2, when the investor is risk averse and the returns are elliptically distributed, the optimal solutions of problems (VP) and (MP) with affine



(a) Problem (MP)



(b) Problem (VP)



(c) Problem (CVP)

Figure 4.1: Minimum-variance sets of expected return versus different risk measures.

loss function coincide.

$$\mathbf{x}^* := \arg\min_{\mathbf{x}\in X} \mathbf{x}^\top \Sigma \mathbf{x} = \arg\min_{\mathbf{x}\in X} \mathrm{VaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y})) = \arg\min_{\mathbf{x}\in X} u(\mathrm{VaR}_\beta(-\mathbf{x}^\top \mathbf{Y})).$$

However, the optimal solution of problem (CVP) for risk averse decision makers depends on N and β values taken. Thus, the optimal solution of problem (CVP) is denoted by $\mathbf{x}^*_C(N,\beta)$.

We have the analytic form of the objective function in problem (VP). On the other hand, the objective function of problem (CVP) needs to be estimated to solve the model. To obtain $\mathrm{CVaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y}))$, generating realizations S_j and V_j which satisfy $S_jV_j \geq F^{\leftarrow}(\beta)$, is necessary. Then, $\mathrm{CVaR}_\beta(u(-\mathbf{x}^T \mathbf{Y}))$ can be estimated by

$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{CVaR}_\beta(u(-\mathbf{x}^\top \mathbf{Y}_j)) = \frac{1}{N}\sum_{j=1}^{N} u(\|A\mathbf{x}\| S_j V_j - \mathbf{x}^\top \mu) \tag{4.6}$$

where N is the sample size of the realizations that satisfy $S_jV_j \geq F^{\leftarrow}(\beta)$. Accordingly, problem (CVP) is shown as

$$\min\{\frac{1}{N}\sum_{j=1}^{N} u(\|A\mathbf{x}\| S_j V_j - \mathbf{x}^\top \mu) \ : \ \mathbf{e}^\top \mathbf{x} = 1, \mathbf{x} \geq \mathbf{0}\}.$$

Since $\mathbf{Y}$ has a multivariate normal distribution, $S = 1$, V is a standard normal random variable and $F^{\leftarrow}$ is the inverse cumulative normal distribution function. This estimation, thus the optimal portfolio of problem (CVP), becomes more accurate for higher N values. Table 4.6 points up the improvement in the estimation of objective function and the optimal portfolio as N increases. Here we set $\beta = 0.95$. The second column illustrates the change in θ as N increases, similar to Table 4.3 in the previous section. Correspondingly, the third column provides the convergence of the objective function value of problem (CVP). The nonzero allocations, $\mathbf{x}^*_C(N,\beta)$ values, are given in the last column. Note that, $\mathbf{x}^*_C(N,\beta)$ also converges as N increases.

N	θ	$\phi_\beta(\mathbf{x}^*_C(N,\beta))$	**Nonzero Allocations**[†]
100	9.7226	16.5148	35.52% , 7.80%, 12.84%, 14.58%, 29.27%
500	13.6347	18.5219	35.63%, 7.77%, 12.44%, 14.68%, 29.49%
1000	20.9730	18.2439	35.62%, 7.77%, 12.46%, 14.67%, 29.48%
5000	29.1986	17.1857	35.56%, 7.79%, 12.66%, 14.62%, 29.36%

[†]B6, NM4, NM5, CP3, CP4

Table 4.6: Estimation of the optimal objective function value of problem (CVP)and $\mathbf{x}^*_C(N,\beta)$ values; $\beta = 0.95$.

$\mathbf{x}^*_C(N,\beta)$ values differ from $\mathbf{x}^*$. Problem (CVP) does not allocate to the financial instrument NM3. Since, in the objective function of problem (CVP), $\mathbf{x}$ is multiplied by μ, the problem tries to balance mean and variance of the financial instruments. We observe from Table 4.1 in the previous section that, among the six nonzero allocations, the model eliminates the one with the lowest return and the highest variance; NM3. Subsequently, the model tries to allocate more to the asset with the highest return; NM5.

The numbers of samples that pass the condition $V_j \geq F^{\leftarrow}(\beta)$ decreases as β increases, as mentioned in the previous section. Naturally, the time required to generate realizations for estimating θ becomes larger as N increases. In addition, the model solves an optimization problem with the estimated objective function that lengthen the required time even more. Table 4.7 illustrates the effect of N and β on computation times to estimate θ and to solve for $\mathbf{x}^*_C(N,\beta)$. Table 4.7 also demonstrates that as expected the effective increment in the time requirements when the model estimates the objective function and minimize over it compared to the Table 4.4 in the previous section.

β	$N = 100$	N=500	$N = 1000$	$N = 5000$
0.90	24.8594 sec.	120.6563 sec.	215.9063 sec.	1181.2969 sec.
0.95	29.1875 sec.	162.7031 sec.	304.6719 sec.	1692.8281 sec.
0.99	30.7813 sec.	137.5313 sec.	272.2813 sec.	1755.5156 sec.
0.999	45.8125 sec.	218.2500 sec.	482.7188 sec.	1963.4531 sec.

Table 4.7: Time required to estimate $\phi_\beta(\mathbf{x}^*_C(N,\beta))$ and to minimize for different N and β values.

$\alpha_\beta(\mathbf{x}^*)$ and $\phi_\beta(\mathbf{x}^*_C(N,\beta))$ values are given in Table 4.8. Here we set $N = 5000$. Observe that $\alpha_\beta(\mathbf{x}^*)$ increases rapidly as β values increase, where as the increment of β does not effect the level of $\phi_\beta(\mathbf{x}^*_C(N,\beta))$ that much. Similarly, the difference between the two optimal objective function values decreases as β increases. There is a clear difference between the objective function values in Tables 4.4 and 4.8. Actually, this is a guessable observation, since disutility of risk averse investor increases exponentially as the loss increases.

β	$\alpha_\beta(\mathbf{x}*)$	$\phi_\beta(\mathbf{x}_C^*(N,\beta))$
0.90	3.1507	123.8318
0.95	6.3593	128.2890
0.99	20.5465	126.9068
0.999	70.8327	124.8263

Table 4.8: Difference between optimal objective function values of problems (VP) and (CVP) for risk averse investors; $N = 5000$.

CHAPTER 5

CONCLUSION

In this study we classify and analyze a general risk management model applied to portfolio problems. Two recent risk measures are considered; Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)and two types of disutility functions are considered; for risk neutral and risk averse decision makers. In this classification, the distributions of the asset returns are taken as elliptical, which allows us to represent the return of the portfolio as a linear function of the product of two independent random variables, where one of these random variables is multivariate standard normally distributed random vector and the other one is a univariate nonnegative random variable. Thus, if disutility function is linear in loss and the joint return distribution is elliptical, both VaR and CVaR measures are convex functions and have analytic forms. In utility theory, linear loss functions can be interpreted as the investors being risk neutral. However, in this case it can be shown that both optimization problems (objective functions formed by VaR and CVaR measures) are equivalent to classical Markowitz model. If the investor is risk averse and the risk measure is VaR, we show that investor's decision is not affected by the utility. Nonetheless, under the same setting this observation does not hold for CVaR measure. Although, a convex objective function exists in this case, an analytic form can not be defined with CVaR measure. However, unlike generating scenarios from multivariate distributions as suggested in the literature, we show that the objective function can be closely estimated by simulating realizations only from univariate distributions.

Computational experience is given with the mathematical analysis of the problem. For linear disutility functions where the random vector has elliptical distribution, a finite step algorithm in literature is modified and implemented to solve the corresponding Markowitz problem. The comparison of the computational times of the algorithm with `fmincon` function of `MATLAB` is shown. Simulation from multivariate distributions is reduced to generating univariate realizations for convex disutility functions. Numerical results for both sections are discussed.

There are several directions in which this research can be extended. We have stabilized the loss function as $-\mathbf{x}^\top \mathbf{Y}$. The loss function can be changed. The same type of classification can be made through non elliptical world. In the non elliptical world, if CVaR is chosen as the risk measure, it is still a convex function for linear disutility functions. VaR measure still entails irrelevancy property to characteristic of decision makers. Analytic forms of VaR and CVaR do not exist in non elliptical world. Therefore, the notion of copulas can be implemented to simulate realizations from n-dimensional multivariate distribution, [3]. Additionally, mathematical finance approaches, such as hedging, can be implemented in the model. Marginal contribution of each sector within the portfolio and each financial instrument can be evaluated. A multistage model can be given by using same classification. Lastly, since multivariate asset returns is a random vector, stochastic or time series models can be applied.

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Appendix A

Time Dependency Test

Each financial instrument's data must be tested for time dependency by using Von-Neumann ratio. The Von-Neumann ratio can be given as

$$d_i = \frac{\sum_{t=1}^{47}(Y_i^{t+1} - Y_i^t)}{\sum_{t=1}^{48}(Y_i^t - \bar{Y}_i)}$$

for each asset i, where t denotes the observations and $\bar{Y}_i$ denotes the average monthly returns for each financial instrument i. Then,

$$r_i = \frac{d_i - 2}{2} \text{ and } t_i = \frac{r_i\sqrt{49}}{\sqrt{1 - {r_i}^2}}$$

are calculated, where t_i is a t-statistic, that will be compared by $t(0.005, 49+3) = 2.68$. We will reject the null hypothesis that the data is independent if $|t_i| \geq t(0.005, 49+3)$. t_i figures are given in Table A.1. According to Table A.1, we do not reject the null hypothesis since all test statistics are less than the given amount $t(0.005, 49+3)$.

B1	**B2**	**B3**	**B4**	**B5**	**B6**	**IC1**	**IC2**	**IC3**	**H1**	**H2**	**H3**	**H4**
−0.25	−0.17	0.61	−0.87	−1.00	1.54	1.78	1.41	0.36	1.42	0.96	0.97	0.07
IT1	**IT2**	**NM1**	**NM2**	**NM3**	**NM4**	**NM5**	**CP1**	**CP2**	**CP3**	**CP4**	**CP5**	**T1**
−1.10	−0.08	1.15	2.21	1.59	1.59	0.13	0.57	1.52	0.44	0.91	1.44	-0.08

Table A.1: Von-Neumann statistics for each financial instrument.

Appendix B

Multivariate Normality Test

The process of testing multivariate normality involves assessment of marginal normality of each variable at a time and then evaluation of the multivariate skewness and kurtosis statistics. Romeu and Ozturk's simulation study illustrates that the multivariate tests of skewness and kurtosis proposed by Mardia are the most stable and reliable tests for evaluating multivariate normality [19]. The Mardia's skewness test statistic is

$$\hat{\beta}_{1,26} = \sum_{i=1}^{48} \sum_{j=1} 48 \left[(\mathbf{y}_i - \bar{\mathbf{y}})' \Sigma^{-1} (\mathbf{y}_j - \bar{\mathbf{y}}) \right]^3 / 48^2$$

for the data given, where $\mathbf{y}$ is a 26-dimensional vector and Σ is the estimated covariance matrix. Mardia proved that $48\hat{\beta}_{1,26}/6$ has a chi-square distribution with ν degrees of freedom, where $\nu = p(p+1)(p+2)/6$ and for this context $p = 26$. The Mardia's kurtosis test statistic is given as

$$\hat{\beta}_{2,26} = \sum_{i=1}^{48} \left[(\mathbf{y}_i - \bar{\mathbf{y}})' \Sigma^{-1} (\mathbf{y}_i - \bar{\mathbf{y}}) \right]^2 / 48$$

distributed normally with mean $\mu = 26(26+2)$ and variance $\sigma^2 = 8 * 26(26+2)/48$. The results of these tests can be found in Table B.1. The tests are rejected if they have small p-values, for example less than $0,05$. We do not reject the null hypothesis.

Q-Q plot is the most popular way of observing the distribution function of a random variable. Q-Q plots are usually used for observing normality. If the plot is linear on $x = y$ line, then the data is assumed to be normal, where as nonlinear plot indicates that the distribution is nonnormal. Obviously, the test based on the Q-Q plot is subjective. However, it gives intuitive ideas about the distribution, even if the distribution is non-normal. The Q-Q plot of the data can be found in Figure B.1. The linearity of the plot can be seen immediately.

Observations	Test	Skewness and Kurtosis	Test Statistic Value	p-value
48	Mardia Skewness	398.534	3402.97	0.06
48	Mardia Kurtosis	709.029	-1.72	0.09

Table B.1: Multivariate Normality Tests.

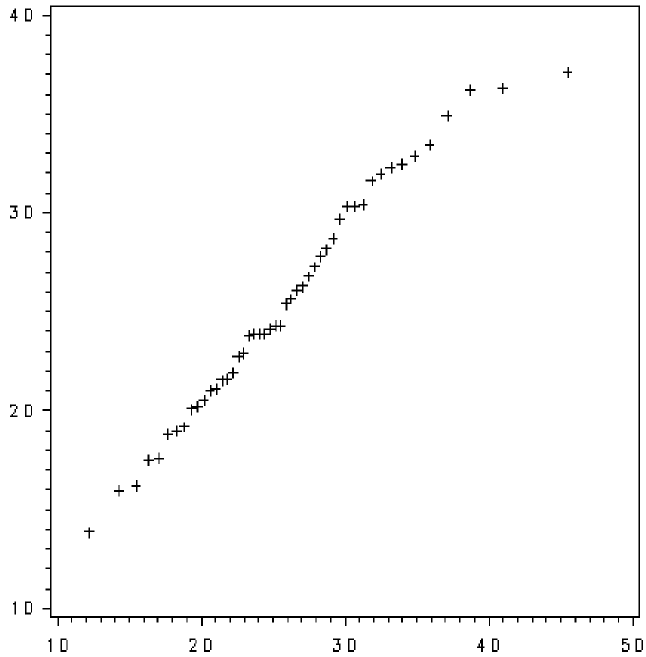

Figure B.1: Chi-Square Q-Q plot.