'VISCOSITY SOLUTIONS'- AN INTRODUCTION TO THE BASICS OF THE THEORY

by

Banu Baydil

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
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in
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ABSTRACT

'Viscosity Solutions'- An Introduction To The Basics Of The Theory

In this work, concepts that appear in the basic theory of viscosity solutions theory is surveyed. Structures of sub and super differentials and sub and super semijets, and concepts of generalized second derivative, generalized 'maximum principle' and generalized 'comparison principle' are studied. Basic properties of semiconvex functions and sup (Jensen's) convolutions are presented. Existence and uniqueness of solutions of the Dirichlet Problem for first and second order nonlinear elliptic partial differential equations is studied.

Key words: viscosity solutions, nonlinear elliptic partial differential equations, maximum principles, comparison theorems, Perron's method.

ÖZET

'Viskozite Çözümleri'-Teorinin Temellerine Bir Giriş

Bu çalışmada viskozite çözümleri teorisinin temelini oluşturan kavramlar ele alınmıştır. Alt ve üst birinci ve ikinci türev kümelerinin yapıları, genelleştirilmiş ikinci türev, genelleştirilmiş 'maksimum prensibi' ve genelleştirilmiş 'karşılaştırma prensibi' kavramları incelenmiştir. Yarı konveks fonksiyonlar ve sup (Jensen) konvülasyonlarına ait temel özellikler verilmiştir. Birinci ve ikinci derece doğrusal olmayan elliptik kısmi diferansiyel denklemler için Dirichlet problemi ele alınarak bu problemin çözümlerinin varlık ve tekliği incelenmiştir.

Anahtar kelimler: viskozite çözümleri, doğrusal olmayan elliptik kısmi diferansiyel denklemler, maksimum prensipleri, karşılaştırma teoremleri, Perron yöntemi.

Anneme,
Tüm kalbimle...

ACKNOWLEDGMENTS

I would like to thank my advisor and supervisor Prof. Dr. Albert Erkip for his continuous encouragement, support and guidance during my studies at Sabancı University; Prof. Dr. H. Mete Soner for introducing us to viscosity solutions during 'Research Semester on Qualitative Theory of Non-Linear Partial Differential Equations' at TUBITAK (Turkish National Council of Scientific and Technical Research)-Feza Gürsey Basic Sciences Research Institute; and Prof. Dr. Alp Eden for his endless efforts to motivate and teach his students.

I would also like to express my thanks to Sabancı University family, and especially to Prof. Dr. Tosun Terzioğlu, to my advisor Prof. Dr. Albert Erkip once again, to Prof. Dr. Alev Topuzoğlu, and to Dr. Huriye Arıkan for providing me with a unique experience during my Masters' studies.

TABLE OF CONTENTS

LIST OF FIGURES

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TABLE OF CONTENTS

LIST OF FIGURES

1

INTRODUCTION

The first time I was introduced to 'viscosity solutions' was in Prof. H. Mete Soner's lecture during the research semester on 'Qualitative Behavior of Nonlinear Partial Differential Equations' that took place at TUBITAK (Turkish National Council of Scientific and Technical Research)-Feza Gürsey Basic Sciences Research Institute, Istanbul, Turkey. The idea then fascinated me for two reasons, one was that it was a complete different way of looking at the things, with a different pair of glasses, in a different perspective, and the other was that it was a rather new development in mathematics which proved to be enormously promising in a very short period of time. Afterwards, I have decided to write my MSc. thesis in this area in order to be able to learn more on the subject along the way.

'Viscosity Solutions' was first introduced by M. G. Crandall and P.-L. Lions in 1983. Since then over a thousand papers appeared in well known mathematical journals. Scope of these papers ranged from the theory to applications and to numerical computations and they spanned a spectrum of subjects ranging from control theory to image processing, from phase transitions to mathematical finance. This was an evidence of the importance of the theory in applied mathematics; and in fact 'viscosity solutions' turned out to be the right class of weak solutions of certain fully nonlinear first and second order elliptic and parabolic partial differential equations (pde's).

The major breakthrough in the theory after 1983, came in 1988 with Jensen when he was able to show uniqueness for second order equations. Jensen's observation was that even if $Du^2(\hat{x})$ might not exist at a local maximum $\hat{x}$ of a semiconvex function (See Section 2.2 for a definition), near $\hat{x}$ one could find a sequence of x_n's converging to $\hat{x}$ such that $Du^2(x_n) \leq 0$. Hence this was actually a generalized second derivative test for semiconvex functions. Most of the above mentioned papers were written after this breakthrough.

Later on, in the second half of 1990's, P-L. Lions and P. E. Souganidis introduced 'viscosity solutions' to the area of nonlinear stochastic pde's.

Recently, a four year (1998-2002) TMR (Training and Mobility of Researches)-network project has been organized under the European Union TMR program bringing together researches from 10 different institutions in Europe working on different aspects of the theory; and preprints of the latest results from this project can be obtained from their web page http://www.ceremade.dauphine.fr/reseaux/TMR-viscosite/.

This survey thesis is mainly based on two major papers in the field. The first one is 'Some Properties of Viscosity Solutions of Hamilton-Jacobi Equations' published by M. G. Crandall, L. C. Evans and P.-L. Lions in 1984 and the second one is the famous survey article 'User's Guide to Viscosity Solutions of Second Order Partial Differential equations" by M. G. Crandall, H. Ishii and P.-L. Lions, published in 1992. Also, the books 'Controlled Markov Processes and Viscosity solutions' by W. H. Fleming and H. M. Soner, 'Fully Nonlinear Elliptic Equations' by X. Cabré and L .A. Caffarelli, and 'Viscosity Solutions and Applications, C.I.M.E. Lecture Series (1660)' by Bardi et.al. are extensively used. (See references for details of the sources.)

The name 'viscosity' comes from a traditional engineering application where a nonlinear first order pde is approximated by quasilinear first order equations which are obtained from the initial pde by adding a regularizing $\varepsilon \triangle u^\varepsilon$ term, which is called a 'viscosity term', and these approximate equations can be solved by classical or numerical methods and the limit of their solution hopefully solves the initial equation.

This classical method was called method of 'vanishing viscosity'; and it was observed at the very beginning of the research in this area that 'vanishing viscosity' method yielded viscosity solutions (See Section 4.2.3). However, this is only a historical connection and viscosity solutions do not have more to do with this method or the viscosity term. The definition of viscosity solutions as will be seen in this survey is an intrinsic one.

'Viscosity Solutions' theory is a highly nonlinear approach, for it does not make use of differentiation as is the case in other weak solution approaches. It is a "maximum principle", "generalized second derivative" approach, and it is a "real analysis" approach using facts from calculus rather then making use of results from functional analysis. Throughout this thesis we will try to emphasize these points as much as possible.

Within this theory, several concepts of classical theory can be relaxed, generalized and replaced by their correspondents. We can name some of them as follows:

1) Continuity to upper and lower semicontinuity (See Section 2.4 for definition),

2) Differentiation to sub- and super-differentials (See Section 2.2 for definition),

3) Second derivative to second order sub- and super semijets (See Section 2.2 for definition),

4) Differential equation to pair of differential inequalities.

Throughout this thesis, nonlinear scaler second order pde's will be considered, and first order cases and analogues of certain concepts will be introduced along the way. The presentation is preferred to be a ahistorical one in order to avoid repetitions of the same ideas.

In Chapter 2, basic definitions and motivation will be given, in particular structures and properties of semijets and subdifferentials will be emphasized. Later on, the link between linearization of a nonlinear mapping at a function u_0 and the 'properness' property of this nonlinear mapping, and the maximum principle that this nonlinear mapping is to satisfy will be discussed. Links with linear elliptic theory will be pointed out by considering several simple examples regarding applications of maximum principles. Finally, viscosity solution concept will be introduced via two perspectives and two equivalent definitions will be stated.

In Chapter 3, since in viscosity solutions theory one inevitably works with upper and lower semicontinuous functions (See Section 2.4 for a definition), it is important to know how to work with their regularizations, therefore, the basic tools, namely semiconvex functions and sup convolutions and their properties and links with semi-jets, that will be needed in the analysis will be introduced first. Later on Jensen's lemma will be proved, and generalization of the second derivative concept, in other words a 'maximum principle' for upper semicontinuous solutions will be presented. In the literature this last result is referred to as 'theorem on sums'.

In Chapter 4, the Dirichlet Problem on a bounded domain is considered. First, the approach to be able to obtain a comparison result is discussed, then the conditions under which a comparison result can be obtained are derived, and as a trivial consequence of the comparison result, uniqueness is presented. The method and the necessary conditions for comparison in first order case is presented shortly and why the method for first order cases does not work in second order cases is illustrated by a simple example. In the second part of Chapter 4, existence of solutions is considered for the same Dirichlet Problem. There are several ways existence schemes can be shown, and among them Perron's method, which presupposes comparison, is chosen to be presented in this work. We note that this is an existence scheme rather then an existence result, since the existence of solutions in this method depends further on existence of a subsolution and a supersolution that agrees on the boundary of the domain. The conditions under which such a sub and super solution exist is very problem specific and in different problem types it is dealt with different results from classical analysis. Hence, Perron's method can be considered more as an existence scheme. Finally an existence scheme for first order case is presented, and this is the historical connection we have mentioned above, the method of 'vanishing viscosity'.

This thesis is written with a view of providing the basics of the theory in order to save time and effort for future students who would want to work on the subject, and is thought of as a concise guide with basic tools for the beginner with almost no knowledge on the subject and hence as a guide to the present introductory guides and books for the theory. Therefore, we have tried to answer the questions of why's as much as possible, and tried to state what is in between the lines of usual proofs and goes unstated. We have tried to visualize certain material along the sequel, and hence used $n = 1$ illustrations and in some cases stated the proofs for $n = 2$. Also, some of the exercises present in some of the introductory books to the theory are solved and included as examples. In the notes sections to each chapter, content specific references are given.

Throughout this thesis, the fact that one is trying to generalize a theory for nonlinear equations, and that one is trying to generalize a 'weak solution concept' and that since one will be working with nondifferentiable functions, one needs a generalization of 'differentiability' is simultaneously kept in mind.

2

PRELIMINARIES AND MOTIVATION

2.1. Introduction

We will first start with directly presenting the below definition for a viscosity subsolution, viscosity supersolution and viscosity solution of a certain type of nonlinear elliptic PDE. As we try to understand what this definition means by going over its constituent terms, we will find ourselves introduced to viscosity solution theory.

Definition 2.1 *Let F be a continuous proper second order nonlinear elliptic partial differential operator, and $\Omega \subset R^n$. Then, a function $u \in USC(\Omega)$ is a viscosity subsolution of $F = 0$ on Ω if*

$$F(x, u(x), p, X) \leq 0 \text{ for all } x \in \Omega \text{ and } (p, X) \in J_{\Omega}^{2,+}u(x),$$

A function $u \in LSC(\Omega)$ is a viscosity supersolution of $F = 0$ on Ω if

$$F(x, u(x), p, X) \geq 0 \text{ for all } x \in \Omega \text{ and } (p, X) \in J_{\Omega}^{2,-}u(x),$$

and a function $u \in C(\Omega)$ is a viscosity solution of $F = 0$ on Ω if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ on Ω.

Our first aim will be to investigate this definition and try to understand what it means. In order to be able to do so, we will begin with exploring its components; for example, when first presented with such a definition one immediately asks what a $J_\Omega^{2,+}u(x)$, or a $J_\Omega^{2,-}u(x)$ is, or how F is defined and what 'proper' is for a second order nonlinear elliptic operator.

Next, we will ask the questions of "why do we require F to be proper, or u to be upper semicontinuous for a subsolution and lower semicontinuous for a supersolution", and "what could be the motivation behind this definition", "how possibly could its equivalents be stated", and "finally, what could its merits be?".

Along our way, we will also be defining viscosity subsolutions/supersolutions (and hence viscosity solutions) of first order nonlinear elliptic partial differential operators, and first order analogues of $J_\Omega^{2,+}u(x)$ and $J_\Omega^{2,-}u(x)$.

2.2. Second Order Semijets & First Order Differentials

Definition 2.2 *Let $(p, X) \in R^n \times S(N)$, $u : \Omega \to R$, and $\hat{x} \in \Omega$. Then $(p, X) \in J_\Omega^{2,+}u(\hat{x})$, if*

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), (x - \hat{x}) \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega.$$

$J_\Omega^{2,+}u(\hat{x})$ is then called the second order superjet of u at $\hat{x}$.

Similarly $(p, X) \in J_\Omega^{2,-}u(\hat{x})$, if

$$u(x) \geq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), (x - \hat{x}) \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega.$$

$J_\Omega^{2,-}u(\hat{x})$ is then called the second order subjet of u at $\hat{x}$.

Before proceeding any further in working with these sets, let us try to understand their first order analogues.

2.2.1. First order case

Let us start by presenting our motivation behind the definitions that will follow.

We call a function $u : \Omega \to R$ differentiable at a point $\hat{x} \in \Omega$, and let $Du(\hat{x}) = p \in R^n$, if

$$u(x) = u(\hat{x}) + \langle p, x - \hat{x}\rangle + o(|x - \hat{x}|) \text{ as } x \to \hat{x} \text{ in } \Omega$$

holds. In fact, we can view this equality as the conjunction of two other inequalities

$$\begin{aligned} \limsup_{x\to\hat{x}} \frac{(u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle)}{|x - \hat{x}|} &\leq 0 \\ \text{and } \liminf_{x\to\hat{x}} \frac{(u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle)}{|x - \hat{x}|} &\geq 0 \end{aligned}$$

since

$$\begin{aligned} u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle &= o(|x - \hat{x}|) \text{ as } x \to \hat{x} \text{ in } \Omega \text{ implies that} \\ \lim_{x\to\hat{x}} \frac{|u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle|}{|x - \hat{x}|} &= 0. \end{aligned}$$

If u is not differentiable at $\hat{x}$, and however, if it is continuous at this point (and even if it is not continuous but upper of lower semicontinuous) then one of the inequalities might still hold at $\hat{x}$. Therefore, we define the following:

Definition 2.3 *Let $u : \Omega \to R$, and $\hat{x} \in \Omega$. The superdifferential of u at $\hat{x}$ is the set of $p \in R^n$ such that*

$$\limsup_{x\to\hat{x}} \frac{(u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle)}{|x - \hat{x}|} \leq 0 \textit{ holds.} \tag{2.1}$$

and is denoted by $D^+u(\hat{x})$.

Similarly, the subdifferential of u at $\hat{x}$ is the set of $p \in R^n$ such that

$$\liminf_{x\to\hat{x}} \frac{(u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle)}{|x - \hat{x}|} \geq 0 \textit{ holds.} \tag{2.2}$$

and is denoted by $D^-u(\hat{x})$.

Let us take $n = 1$, and try to get a rough geometrical picture of the above definition.

Let

$$u(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 2 \\ \frac{1}{2}x + 1 & \text{if } x \leq 2 \end{cases},$$

clearly u is continuous, but not differentiable at $\hat{x} = 2$.

Associate to each $p \in R$, the line with slope p that is touching the graph of u at $x = 2$. Let l_1 be the line with slope $p_1 = \lim_{x \to 2^+} \frac{(u(x)-u(2))}{|x-2|} = 2$ and l_2 be the line with slope $p_2 = \lim_{x \to 2-} \frac{(u(2)-u(x))}{|x-2|} = \frac{1}{2}$. See Figure 2.1 at the end of the chapter. Let $x_n \to 2^+$. Then slope of any line whose half graph left to $x = 2$ lies in the region S1 satisfies (2.2) as $x_n \to 2^+$, and is a candidate to be in $D^-u(2)$. Let $y_n \to 2^-$. Then slope of any line whose half graph right to $x = 2$ lies in the region S2 satisfies (2.2) as $y_n \to 2^-$, and is a candidate to be in $D^-u(2)$. Since we require (2.2) to hold as $x \to 2$, this requires that both of these cases hold simultaneously. Hence, slope of any line whose graph lies in the shaded region is actually in $D^-u(2)$. Note that this shaded region is controlled by the lines l_1 and l_2, and that $D^-u(2) = \left[\frac{1}{2}, 2\right] \subset R$. Through a similar geometrical analysis we see that $D^+u(2) = \emptyset$, since this time there can be no line whose right half graph is in the corresponding region S3, and whose left half graph is in the corresponding region S4 simultaneously. We note at this point that at $x = 2$ graph of u is concave up.

Now let $v(x) = -u(x) = \begin{cases} -\frac{1}{2}x^2 & \text{if } x \geq 2 \\ -\frac{1}{2}x - 1 & \text{if } x < 2 \end{cases}$

Following the same geometrical approach we see that this time $D^-v(2) = \emptyset$ and $D^+v(2) = \left[-\frac{1}{2}, -2\right] = -D^-u(2)$. See Figure 2.2 at the end of the chapter. We also note that this time at $x = 2$ graph of u is concave down.

Finally, it is also important to note that when u is differentiable at $\hat{x} \in \Omega$, then $l_1 = l_2$ and the corresponding shaded regions for both D^+u and D^-u become just the graph of this unique line and $D^+u(\hat{x}) = D^-u(\hat{x}) = \{Du(\hat{x})\}$.

Hence when dealing with continuous functions that are not differentiable at certain points, at the points of nondifferentiability we can in a way replace the concept of differentiability with the weaker concept of subdifferentials and superdifferentials. Furthermore, as hinted by above geometrical approach, we can characterize these sets as follows:

Lemma 2.4 *Let $u \in C(\Omega)$ be differentiable at $\hat{x} \in \Omega$. Then, there exists $\varphi_1, \varphi_2 \in C^1(\Omega)$ such that $D\varphi_1(\hat{x}) = D\varphi_2(\hat{x}) = Du(\hat{x})$ and $u - \varphi_1$ has a strict local maximum value of zero at $\hat{x}$, and $u - \varphi_2$ has a strict local minimum value of zero at $\hat{x}$.*

By strictness of the maximum we mean that there is a nondecreasing function $h : (0, \infty) \to (0, \infty)$ and $s, r > 0$ such that

$$u(x) - \varphi_1(x) \leq u(\hat{x}) - \varphi(\hat{x}) - h(s) \text{ for } s \leq |x - \hat{x}| \leq r.$$

Similarly, by strictness of the minimum we mean that there is a nondecreasing function $h : (0, \infty) \to (0, \infty)$ and $s, r > 0$ such that

$$u(x) - \varphi_1(x) \geq u(\hat{x}) - \varphi(\hat{x}) + h(s) \text{ for } s \leq |x - \hat{x}| \leq r.$$

See Figure 2.3 at the end of the chapter to have an idea in $n = 1$ for a differentiable (locally linearizable) u at $\hat{x}$.

Lemma 2.5 is a special case of Proposition 2.6, hence we will not give a separate proof for it.

Proposition 2.5 *Let $u \in C(\Omega)$, $\hat{x} \in \Omega$, $p \in R^n$. Then the following are equivalent:*

i) $p \in D^+u(\hat{x})$ (respectively $D^-u(\hat{x})$) and

ii) there exists $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local maximum (respectively minimum) at $\hat{x}$ and $D\varphi(\hat{x}) = p$.

Proof. We will give the proof for the $D^+u(\hat{x})$ and the local maximum case.

Let $p \in D^+u(\hat{x})$. Then near $\hat{x}$, $u(x) \leq u(\hat{x}) + \langle p, x - \hat{x}\rangle + o(|x - \hat{x}|)$. Let $\alpha(x) = \{u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle\}^+$, where $\{h\}^+ = \max\{h, 0\}$. Then, since $\alpha(x) = o(|x - \hat{x}|)$ and $\alpha(\hat{x}) = 0$,

$$\alpha(x) = \alpha(\hat{x}) + \langle 0, x - \hat{x}\rangle + o(|x - \hat{x}|)$$

holds and hence $\alpha(x)$ is differentiable at $\hat{x}$ and $D\alpha(\hat{x}) = 0$. Let $\beta_1 \in C^1(\Omega)$ be given for this α by the previous lemma. Then

$$\beta_1(\hat{x}) = \alpha(\hat{x}) = 0,\ D\beta_1(\hat{x}) = D\alpha(\hat{x}) = 0$$

and near $\hat{x}$

$$\begin{aligned} \alpha(x) - \beta_1(x) &\leq \alpha(\hat{x}) - \beta_1(\hat{x}) = 0, \text{ so that we have} \\ \{u(x) - (u(\hat{x}) + \langle p, x - \hat{x} \rangle)\}^+ - \beta_1(x) &\leq 0. \end{aligned}$$

Let

$$\varphi(x) = u(\hat{x}) + \langle p, x - \hat{x} \rangle + \beta_1(x).$$

Then

$$\begin{aligned} \varphi(\hat{x}) &= u(\hat{x}) \text{ since } \beta_1(\hat{x}) = 0, \\ D\varphi(\hat{x}) &= p \text{ since } D\beta_1(\hat{x}) = 0; \end{aligned}$$

and near $\hat{x}$ we have

$$\begin{aligned} u(x) - \varphi(x) &= u(x) - u(\hat{x}) - \langle p, x - \hat{x} \rangle - \beta_1(x) \\ &\leq \{u(x) - (u(\hat{x}) + \langle p, x - \hat{x} \rangle)\}^+ - \beta_1(x) \\ &\leq 0 = u(\hat{x}) - \varphi(\hat{x}). \end{aligned}$$

Hence $u - \varphi$ has a local maximum at $\hat{x}$ and $D\varphi(\hat{x}) = p$.

Now, if $u - \varphi$ has a local maximum at $\hat{x}$, then near $\hat{x}$ we have

$$\begin{aligned} u(x) - \varphi(x) &\leq u(\hat{x}) - \varphi(\hat{x}) \text{ so that} \\ u(x) &\leq u(\hat{x}) - \varphi(\hat{x}) + \varphi(x) \text{ by Taylor expansion of } \varphi, \\ \text{we have } u(x) &\leq u(\hat{x}) - \varphi(\hat{x}) + \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|) \\ \text{which gives us that } u(x) &\leq u(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|). \end{aligned}$$

Hence

$$\limsup_{x - \hat{x}} \frac{(u(x) - u(\hat{x}) - \langle D\varphi(\hat{x}), x - \hat{x} \rangle)}{|x - \hat{x}|} \leq 0$$

and $D\varphi(\hat{x}) \in D^+u(\hat{x})$.

See Figure 2.4 for an illustration for $n = 1$. ∎

Proposition 2.6 *Let $u \in C(\Omega)$, $\hat{x} \in \Omega$, $p \in R^n$. Then, the following are equivalent:*

i) $p \in D^+u(\hat{x})$ (respectively $D^-u(\hat{x})$) and

ii) there exists $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a strict maximum (respectively minimum) at $\hat{x}$ and $D\varphi(\hat{x}) = p$.

Proof. This time we will construct such a function φ.

Let $p \in D^+u(\hat{x})$. Then near $\hat{x}$, $u(x) \leq u(\hat{x}) + \langle p, x - \hat{x}\rangle + o(|x - \hat{x}|)$.

Let

$$\gamma(s) = \sup\left\{(u(x) - u(\hat{x}) - \langle p, x - \hat{x}\rangle)^+ : x \in \Omega, \text{ and } |x - \hat{x}| \leq s\right\}.$$

Then $\gamma(s)$ is nondecreasing, $0 \leq \gamma(s)$and as $s \to 0$, $\gamma(s) = o(s)$. Let $\tau(s) \in C(\Omega)$ be such that $\gamma(s) \leq \tau(s)$, and $\tau(s)$ is nondecreasing and also $\tau(s) = o(s)$.

We will assume that $\hat{x} = 0$ to ease the notation.

Let

$$T(s) = \frac{1}{s}\int_s^{2s} \tau(z)dz \text{ for } s > 0, \text{ and } T(s) = 0 \text{ for } s = 0, \text{ then}$$

for $s > 0$, $T(s)$ is continuous, we have to check at $s = 0$.

$$\begin{aligned} 0 &\leq T(s) \leq \frac{1}{s}\int_s^{2s} \tau(2s)dz \text{ since } \tau(x) \leq \tau(2s) \text{ for } s \leq |x| \leq 2s, \text{ then} \\ &\leq \frac{1}{s}\tau(2s)\int_s^{2s} dz = \frac{1}{s}\tau(2s)(2s - s) = \tau(2s) \text{ hence} \\ 0 &\leq T(s) \leq \tau(2s). \end{aligned}$$

Then as $s \to 0$, $T(s) \to 0 = T(0)$, hence $T(s)$ is continuous at $s = 0$. Furthermore,

$$\begin{aligned} sT(s) &= \int_s^{2s} \tau(z)dz \\ \frac{d}{ds}(sT(s)) &= \frac{d}{ds}(\int_s^{2s} \tau(z)dz) \\ T(s) + s\frac{d}{ds}(T(s)) &= 2\tau(2s) - \tau(s) \\ \frac{d}{ds}(T(s)) &= \frac{1}{s}(2\tau(2s) - \tau(s) - T(s)). \end{aligned}$$

Hence for $s > 0$, $\frac{d}{ds}(T(s))$ is continuous and we have to check at $s = 0$.

$$\begin{aligned}
\left|\frac{d}{ds}(T(s))\right| &\leq \frac{1}{s}(\tau(2s) + \tau(2s) + \tau(2s)) = \frac{3}{s}\tau(2s) \\
\left|\frac{d}{ds}(T(s))\right| &\leq 0 \text{ as } s \to 0.
\end{aligned}$$

So, $\frac{d}{ds}(T(s))$ is continuous at $s = 0$.

Hence $T(s)$ and $\frac{d}{ds}(T(s))$ are continuous. Furthermore, $T(0) = \frac{d}{ds}(T(0)) = 0$.

Now we go back using $\hat{x}$. Let

$$\varphi(x) = u(\hat{x}) + T(|x - \hat{x}|) + \langle p, x - \hat{x}\rangle + |x - \hat{x}|^4,$$

then $\varphi(\hat{x}) = u(\hat{x})$, and $D\varphi(\hat{x}) = p$.

Since we have

$$\begin{aligned}
T(s) &= \frac{1}{s}\int_s^{2s} \tau(z)dz \geq \frac{1}{s}\int_s^{2s} \tau(s)dz \\
&= \frac{1}{s}\tau(s)(2s - s) = \tau(s) \text{ since } \tau(s) \leq \tau(x) \text{ for } s \leq |x| \leq 2s \qquad (2.3) \\
\text{and } u(x) - \langle p, x - \hat{x}\rangle &\leq \gamma(s) \leq \tau(s) \qquad (2.4)
\end{aligned}$$

then, we have

$$\begin{aligned}
\varphi(x) &= T(|x - \hat{x}|) + \langle p, x - \hat{x}\rangle + |x - \hat{x}|^4 \text{ by (2.3)}, \\
&\geq \tau(s) + \langle p, x - \hat{x}\rangle + |x - \hat{x}|^4 \text{ by (2.4)}, \\
&\geq u(x) + |x - \hat{x}|^4 \text{ for all } x \in \Omega.
\end{aligned}$$

Then we have

$$\begin{aligned}
u(x) - \varphi(x) &\leq 0 - |x - \hat{x}|^4 = u(\hat{x}) - \varphi(\hat{x}) - |x - \hat{x}|^4, \text{ hence} \\
u(x) - \varphi(x) &\leq u(\hat{x}) - \varphi(\hat{x}) - |x - \hat{x}|^4 \text{ for all } x \in \Omega.
\end{aligned}$$

Now, let $h(t) : (0, \infty) \to (0, \infty)$ be $h(t) = t^4$ and let $r > 0$. Then for $s \leq |x - \hat{x}| \leq r$ $s^4 = h(s) \leq h(|x - \hat{x}|) = |x - \hat{x}|^4$ since h is nondecreasing, hence we have

$$u(x) - \varphi(x) \leq u(\hat{x}) - \varphi(\hat{x}) - h(s) \text{ for } s \leq |x - \hat{x}| \leq r.$$

Hence $u - \varphi$ has a strict maximum at $\hat{x}$, with $h(s) = s^4$ as strictness.

The proof of $ii) \to i)$ is same as it is in the previous proposition. ■

One can view Lemma 2.4 as a special case of Proposition 2.6, where u is differentiable at $\hat{x}$ and therefore $D^+u(\hat{x}) = \{Du(\hat{x})\}$ and $Du(\hat{x}) = p = D\varphi(\hat{x})$.

Having this insight now, we can go back to the second order case.

2.2.2. Second order case

Let us recall the definition of second order superjet of u at $\hat{x} \in \Omega$ once again: $(p, X) \in J^{2,+}_{\Omega} u(\hat{x})$ if

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), (x - \hat{x}) \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega.$$

Paralleling our discussion for the first order case, we this time note that if a function $u : \Omega \to R$ is such that $u \in C^2(\Omega)$, and at at some $\hat{x}$, $Du(\hat{x}) = p \in R^n$, $D^2u(\hat{x}) = X \in S(N)$, then by its Taylor expansion around $\hat{x}$, we know that

$$u(x) = u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), (x - \hat{x}) \rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega.$$

Rearranging the terms, we arrive at, as $x \to \hat{x}$ in Ω

$$\begin{aligned}
u(x) &= u(\hat{x}) - \langle p, \hat{x} \rangle + \frac{1}{2} \langle X\hat{x}, \hat{x} \rangle + \langle p - X\hat{x}, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x - \hat{x}|^2) \\
&= l_0 + l(x) + \frac{1}{2} \langle Ax, x \rangle + o(|x - \hat{x}|^2) \\
\text{where } l_0 &= u(\hat{x}) - \langle p, \hat{x} \rangle + \frac{1}{2} \langle X\hat{x}, \hat{x} \rangle \text{ is a constant,} \\
l(x) &= \langle p - X\hat{x}, x \rangle \text{ is a linear function, and} \\
A &= X \text{ is a symmetric matrix.}
\end{aligned}$$

Now, we note that a paraboloid is a polynomial in x of degree 2, and any paraboloid P can be written as

$$P(x) = l_0 + l(x) + \frac{1}{2} \langle Ax, x \rangle$$

where l_0 is a constant, $l(x)$ is a linear function, and A is a symmetric matrix. Hence in the case that $u \in C^2(\Omega)$, we have

$$u(x) = P(x) + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega$$

for some paraboloid $P(x)$. Moreover, we will make the following definitions:

Definition 2.7 *A paraboloid P will be called of opening M, whenever*

$$P(x) = l_0 + l(x) \pm \frac{M}{2} |x|^2,$$

where M is a positive constant, l_0 is a constant and l is a linear function. Then, P is convex when we have $+\frac{M}{2}|x|^2$, and concave when we have $-\frac{M}{2}|x|^2$ as the third term.

Definition 2.8 *Let $u, v \in C(\Omega)$. Ω be open, and $\hat{x} \in \Omega$. If*

$$\begin{aligned} u(x) &\leq v(x) \textit{ for all } x \in \Omega \textit{ and} \\ u(\hat{x}) &= v(\hat{x}), \textit{ then} \end{aligned}$$

we will say that v touches u by above at $\hat{x}$. Similarly, if

$$\begin{aligned} u(x) &\geq v(x) \textit{ for all } x \in \Omega \textit{ and} \\ u(\hat{x}) &= v(\hat{x}), \textit{ then} \end{aligned}$$

we will say that v touches u by below at $\hat{x}$.

In the above case when $u \in C^2(\Omega)$, then by letting $P_\varepsilon(x) = P(x) + \frac{\varepsilon}{2}|x - \hat{x}|^2$ where $\varepsilon > 0$, we have

$$u(x) = P(x) + o(|x - \hat{x}|^2) \leq P(x) + \frac{\varepsilon}{2}|x - \hat{x}|^2 = P_\varepsilon(x) \text{ in a neighborhood of } \hat{x}.$$

Hence, $P_\varepsilon(x)$ is a paraboloid that touches u by above at $\hat{x}$; and similarly, $P_{(-\varepsilon)}(x)$ is a paraboloid that touches u by below at $\hat{x}$.

Within this perspective, we can take as our generalized pointwise definition for second order differentiability at $\hat{x} \in \Omega$, when $u \in C(\Omega)$ and fails to be $C^2(\Omega)$, as follows:

Definition 2.9 *$u \in C(\Omega)$ will be called punctually second order differentiable at $\hat{x} \in \Omega$, if there is a paraboloid P such that*

$$u(x) = P(x) + o(|x-\hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega \text{ holds,}$$

and we will define, $D^2u(\hat{x}) = D^2P(\hat{x})$.

In the case that this fails to hold then we can expect either

$$u(x) \leq P(x) + o(|x-\hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega$$

or

$$u(x) \geq P(x) + o(|x-\hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega \text{ to hold.}$$

In the first case then

$$u(x) \leq P(x) + o(|x-\hat{x}|^2) \leq P(x) + \frac{\varepsilon}{2}|x-\hat{x}|^2 = P_\varepsilon(x) \text{ in a neighborhood of } \hat{x},$$

and $P_\varepsilon(x)$ will be touching u by above at $\hat{x}$, and in the second case

$$u(x) \geq P(x) + o(|x-\hat{x}|^2) \geq P(x) - \frac{\varepsilon}{2}|x-\hat{x}|^2 = P_{(-\varepsilon)}(x) \text{ in a neighborhood of } \hat{x}$$

and $P_{(-\varepsilon)}(x)$ will be touching u by below at $\hat{x}$.

Then, whenever $(p, X) \in J_\Omega^{2,+}u(\hat{x})$ is given, since upon rearrangement, and as $x \to \hat{x}$ in Ω,

$$\begin{aligned} u(x) &\leq u(\hat{x}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), (x-\hat{x})\rangle + o(|x-\hat{x}|^2) \text{ will imply} \\ &\leq P(x) + o(|x-\hat{x}|^2) \text{ as } x \to \hat{x} \text{ in } \Omega, \end{aligned}$$

we can say that there is a paraboloid

$$P_\varepsilon(x) = u(\hat{x}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), (x-\hat{x})\rangle + \frac{\varepsilon}{2}|x-\hat{x}|^2$$

that touches u by above at $\hat{x}$.

Similarly, whenever $(p, X) \in J_{\Omega}^{2,-}u(\hat{x})$ is given, we can say that there is a paraboloid

$$P_{(-\varepsilon)}(x) = u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), (x-\hat{x})\rangle - \frac{\varepsilon}{2}|x-\hat{x}|^2$$

that touches u by below at $\hat{x}$.

Furthermore, if φ is $C^2(\Omega)$ and $\hat{x}$ is a local maximum of $u - \varphi$, then

$$u(x) - \varphi(x) \leq u(\hat{x}) - \varphi(\hat{x}) \text{ for } x \text{ near } \hat{x},$$

and by Taylor expansion of φ, we have

$$\begin{aligned} u(x) - \varphi(x) &\leq u(\hat{x}) - \varphi(\hat{x}) + \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle \\ &\quad + \frac{1}{2}\left\langle D^2\varphi(\hat{x})(x-\hat{x}), (x-\hat{x})\right\rangle + o(|x-\hat{x}|^2) \\ &\leq u(\hat{x}) + \langle D\varphi(\hat{x}), x-\hat{x}\rangle + \frac{1}{2}\left\langle D^2\varphi(\hat{x})(x-\hat{x}), (x-\hat{x})\right\rangle + o(|x-\hat{x}|^2) \end{aligned}$$

so that $(D\varphi(\hat{x}), D^2\varphi(\hat{x}))$ will be in $J_{\Omega}^{2,+}u(\hat{x})$, and

$$P_{\varepsilon}(x) = u(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle + \frac{1}{2}\left\langle D^2\varphi(\hat{x})(x-\hat{x}), (x-\hat{x})\right\rangle + \frac{\varepsilon}{2}|x-\hat{x}|^2$$

will be touching u by above at $\hat{x}$.

Following a similar manner as in Proposition 2.6 in first order case, given $(p, X) \in J_{\Omega}^{2,+}u(\hat{x})$, by taking

$$T(s) = \frac{2}{3s^2}\int_s^{2s}\int_k^{2k}\tau(z)dzdk,$$

we can construct a function φ such that $\varphi \in C^2(\Omega)$, and $u - \varphi$ attains its maximum at $\hat{x}$.

At this point, we will give two rather detailed examples which will assist us in having a picture of these sets.

Example 2.10 *On R let us define the function*

$$u(x) = \left\{ \begin{array}{cc} 0 & \text{for } x \leq 0, \\ ax + \frac{b}{2}x^2 & \text{for } x \geq 0. \end{array} \right\}.$$

We will see that

$$J_R^{2,+}u(0) = \left\{ \begin{array}{cc} \emptyset & \text{if } a > 0, \\ \{0\} \times [\max\{0,b\}, \infty) & \text{if } a = 0, \\ ((a,0) \times R) \cup (\{0\} \times [0,\infty)) \cup (\{a\} \times [b,\infty)) & \text{if } a < 0. \end{array} \right\}$$

Solution: We are looking for pairs of $(p, X) \in R \times S(1)$ for which the inequality

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2)$$

holds as $x \longrightarrow \hat{x}$. Since we will be computing the second order superjet of u (in the set $\Omega = R$) at $\hat{x} = 0$ this inequality becomes:

$$u(x) \leq u(0) + \langle p, x - 0 \rangle + \frac{1}{2} \langle X(x - 0), x - 0 \rangle + o(|x - 0|^2)$$

as $x \longrightarrow 0$.

Moreover, since $u(x)$ is piecewise defined around $\hat{x} = 0$ we actually have two inequalities to hold simultaneously:

1) $0 \leq 0 + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x|^2)$ as $x \longrightarrow 0^-$ and

2) $ax + \frac{b}{2}x^2 \leq 0 + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x|^2)$ as $x \longrightarrow 0^+$.

At this point, we note that the inequality (1) is independent of the constants a and b; and that the second inequality leads us to three main cases, namely, $a < 0$, $a = 0$ and $a > 0$; and that $S(1) = R$, so that $(p, X) \in R \times R$; and also that the scaler product is usual multiplication in R.

Case 1: $a = 0$, $\quad u(x) = \left\{ \begin{array}{cc} 0 & \text{for } x \leq 0, \\ \frac{b}{2}x^2 & \text{for } x \geq 0. \end{array} \right\}$

In this case, we can graph $u(x)$ as in Figure 2.5 (for $b > 0$), see end of the chapter for the figure. On the left of $x = 0$, the graph is a straight line and u has slope 0. On the right of $x = 0$, the graph is a quadratic and u has slope bx and second derivative (bending) b. The function u is differentiable at the point $x = 0$ with $u'(0) = 0$ however not twice differentiable at $x = 0$ (unless $b = 0$).

Then the inequalities (1) and (2) become:

1.1) $0 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^-$ and

1.2) $\frac{b}{2}x^2 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^+$.

For this specific $u(x)$, we are looking for $(p, X) \in R \times R$ such that (1.1) and(1.2) will hold simultaneously.

If $p = 0$, then we have

$$\begin{aligned}
\text{from (1.1) } \frac{0}{x^2} &\leq \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence } 0 \leq \frac{1}{2}X, \text{ so that} \\
X &\geq 0 \text{ as } x \to 0^- \text{ has to hold,} \\
\text{from (1.2) } \frac{0}{x^2} &\leq \frac{1}{2}\frac{(X-b)x^2}{x^2} + 0, \text{ hence } 0 \leq \frac{1}{2}(X-b), \text{ so that} \\
X &\geq b \text{ as } x \to 0^+ \text{ has to hold.}
\end{aligned}$$

Hence for $p = 0$ if we have $X \geq \max\{0, b\}$, then the desired inequalities will hold as $x \to 0$ in R.

If $p < 0$ then

$$\begin{aligned}
\text{from (1.2) } \frac{-px}{x^2} &\leq \frac{1}{2}\frac{(X-b)x^2}{x^2} + 0, \text{ hence} \\
\frac{-p}{x} &\leq \frac{1}{2}(X-b) \text{ as } x \to 0^+ \text{ has to hold,}
\end{aligned}$$

however since left hand-side (LHS) of this last inequality$\to \infty$ as $x \to 0^+$, for any fixed $p < 0$, $b \in R$, there does not exist any $(X - b)$ (and hence any X) that will make (1.2) hold.

If $p > 0$, then px is the line with slope p going through the origin, see Figure 2.6 at the end of the chapter,

$$\begin{aligned}
\text{from (1.1) } \frac{-px}{x^2} &\leq \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence} \\
\frac{-p}{x} &\leq \frac{1}{2}X \text{ as } x \to 0^- \text{ has to hold,}
\end{aligned}$$

however since LHS of this last inequality $\to \infty$ as $x \to 0^-$, for any fixed $p > 0$, there does not exist any X that will make (1.1) hold.

So, if $a = 0$, we have

$$(p, X) \in \{0\} \times [\max\{0, b\}, \infty), \text{ i.e. } J_R^{2,+}u(0) = \{0\} \times [\max\{0, b\}, \infty)$$

Case 2: $a > 0$, $\qquad u(x) = \left\{ \begin{array}{cc} 0 & \text{for } x \leq 0, \\ ax + \frac{b}{2}x^2 & \text{for } x \geq 0. \end{array} \right\}.$

In this case, we can graph $u(x)$ as in Figure 2.7 (for $b > 0$), see end of the chapter for the figure. On the left of $x = 0$, the graph is a straight line and u has slope 0. On the right of $x = 0$, the graph is that of a line ax plus a quadratic this time and u has slope $a + bx$ and second derivative (bending) b. This time, the function u is not differentiable at the point $x = 0$ since $L_1 = \lim_{h \to 0^+} \frac{u(0+h)-u(0)}{h} = \lim_{h \to 0^+} \frac{ah+\frac{b}{2}h^2-0}{h} = a$ and $L_2 = \lim_{h \to 0^-} \frac{u(0+h)-u(0)}{h} = 0$ and $L_1 \neq L_2$ since $a > 0$.

Now, the inequalities (1) and (2) become:

2.1) $0 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^-$ and

2.2) $ax + \frac{b}{2}x^2 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^+$.

If $p > 0$, through (2.1), we have the same result given by (1.1) as above in Case 1, since this equation has not changed.

If $p < 0$, then

$$\begin{aligned} \text{from (2.2)} \ \frac{(a-p)x}{x^2} &\leq \frac{1}{2}\frac{(X-b)x^2}{x^2} + 0, \text{ hence} \\ \frac{(a-p)}{x} &\leq \frac{1}{2}(X-b) \text{ as } x \to 0^+ \text{ has to hold,} \end{aligned}$$

however since LHS of this last inequality $\to \infty$ as $x \to 0^+$, for any fixed $p < 0$, $a > 0$, $b \in R$, there does not exist any $(X - b)$ (and hence any X) that will make (2.2) hold.

If $p = 0$, then

$$\begin{aligned} \text{from (2.2)} \ \frac{ax}{x^2} &\leq \frac{1}{2}\frac{(X-b)x^2}{x^2} + 0, \text{ hence} \\ \frac{a}{x} &\leq \frac{1}{2}(X-b) \text{ as } x \to 0^+ \text{ has to hold,} \end{aligned}$$

however, since LHS of this last inequality $\to \infty$ as $x \to 0^+$, for any fixed $a > 0$, $b \in R$, there does not exist any $(X - b)$ (and hence any X) that will make (2.2) hold.

So, if $a > 0$, we have

$$(p, X) \in \emptyset, ie J_R^{2,+}u(0) = \emptyset.$$

Case 3: $a < 0$, $\qquad u(x) = \left\{ \begin{array}{cc} 0 & \text{for } x \leq 0, \\ ax + \frac{b}{2}x^2 & \text{for } x \geq 0. \end{array} \right\}.$

In this case, we can graph $u(x)$ as in Figure 2.8 (for $b > 0$) at the end of the chapter. On the left of $x = 0$, the graph is again a straight line and u has slope 0. On the right of $x = 0$, the graph is that of a line ax plus a quadratic and u has slope $a + bx$ and second derivative (bending) b. Again the function u is not differentiable at the point $x = 0$ since $L_1 = \lim_{h \to 0^+} \frac{u(0+h)-u(0)}{h} = \lim_{h \to 0^+} \frac{ah+\frac{b}{2}h^2-0}{h} = a$ and $L_2 = \lim_{h \to 0^-} \frac{u(0+h)-u(0)}{h} = 0$ and $L_1 \neq L_2$ since $a < 0$.

This case looks quite similar to the previous case, however, let us see that it is not so.

For this function $u(x)$, the inequalities (1) and (2) become:

3.1) $0 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^-$ and

3.2) $ax + \frac{b}{2}x^2 \leq px + \frac{1}{2}Xx^2 + o(|x|^2)$ as $x \to 0^+$.

If $p > 0$, then through (3.1), we have the same result given by (1.1) as above in Case 1, since this equation has not changed.

If $p = 0$, then we have

$$\begin{aligned} \text{from (3.1) } \frac{0}{x^2} &\leq \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence } 0 \leq \frac{1}{2}X, \text{ so that} \\ X &\geq 0 \text{ as } x \to 0^- \text{ has to hold,} \end{aligned}$$

$$\begin{aligned} \text{from (3.2) } 0 &\leq \frac{-ax}{x^2} + \frac{1}{2}\frac{(X-b)x^2}{x^2} + 0, \text{ hence} \\ 0 &\leq \frac{|a|}{x} + \frac{1}{2}(X-b), \text{ so that } \frac{1}{2}(X-b) \geq -\frac{|a|}{x}, \text{ and hence} \\ X &\geq b - \frac{2|a|}{x} \text{ as } x \to 0^+ \text{ has to hold, and since} \end{aligned}$$

the right hand-side(RHS) of this last inequality $\to -\infty$ as $x \to 0^+$, for any fixed $a < 0$, $b \in R$; any $X \in R$ would make (3.2) hold.

Hence for $p = 0$ we need to have $X \geq 0$ for the two inequalities to hold simultaneously.

Hence if $(p, X) \in \{0\} \times [0, \infty)$ then $(p, X) \in J_R^{2,+}u(0)$.

If $p<0$, then

$$\begin{aligned} \text{from (3.1) } 0 &\leq \frac{px}{x^2}+\frac{1}{2}\frac{Xx^2}{x^2}+0, \text{ hence } 0 \leq \frac{p}{x}+\frac{1}{2}X, \text{ so that} \\ X &\geq \frac{-2p}{x} \text{ as } x \to 0^-, \text{ and since} \end{aligned}$$

RHS of this last inequality $\to -\infty$ as $x \to 0^-$, for any fixed $p<0$, any $X \in R$ would make (3.1) hold,

$$\begin{aligned} \text{from (3.2) } \frac{0}{x^2} &\leq \frac{(p-a)x}{x^2}+\frac{1}{2}\frac{(X-b)x^2}{x^2}+0, \text{ hence} \\ 0 &\leq \frac{(p-a)}{x}+\frac{1}{2}(X-b) \text{ as } x \to 0^+ \text{ has to hold, but then} \end{aligned}$$

if $p<a$, this gives $\frac{(X-b)}{2} \geq \frac{a-p}{x}$ and since $a-p$ is positive in this case, RHS of this last inequality $\to \infty$ as $x \to 0^+$, for any fixed $p<a$, $a<0$, $b \in R$, and there does not exist any $(X-b)$ (and hence any X) that will make (3.2) hold;

if $p>a$, this gives $\frac{(X-b)}{2} \geq \frac{a-p}{x}$ and since $a-p$ is negative in this case, RHS of this last inequality $\to -\infty$ as $x \to 0^+$, for any fixed $0>p>a$, $b \in R$, and any $X \in R$ will make (3.2) hold; and

if $p=a$, this gives $\frac{(X-b)}{2} \geq 0$ and $X \geq b$ will make (3.2) hold.

Hence for $p<0$ we need to have $X \in R$ if $p>a$, and we need to have $X \geq b$ if $p=a$, in order for (3.1) and (3.2) to hold simultaneously.

So, for $a<0$, we have

$$\begin{aligned} (p,X) &\in \{0\}\times[0,\infty)\cup(a,0)\times R)\cup\{a\}\times[b,\infty)\text{, i.e.} \\ J_R^{2,+}u(0) &= \{0\}\times[0,\infty)\cup(a,0)\times R)\cup\{a\}\times[b,\infty) \end{aligned}$$

Example 2.11 *This time, we will look at the second order superjet of the same above function $u(x)$ at $\hat{x}=0$ on the domain $\Omega=[-1,0]$, i.e.. $J_{[-1,0]}^{2,+}u(0)$.*

Note that in this case $\hat{x}=0$ is a boundary point of the domain.

Solution: Again we are looking for pairs of $(p,X) \in R\times R$ for which the inequality

$$u(x) \leq u(\hat{x})+\langle p, x-\hat{x}\rangle+\frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle+o(|x-\hat{x}|^2)$$

holds as $x \to \hat{x}$. Then, this gives us the following simultaneous inequalities:

1) $0 \leq 0 + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + o(|x|^2)$ as $x \to 0^-$ and

2) $ax + \frac{b}{2}x^2 \leq 0 + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + o(|x|^2)$ as $x \to 0^+$.

However since in this case $\hat{x} = 0$ is a boundary point of the domain, the second inequality does not apply (or else we can say that it holds by voidness for this domain), and the first inequality is the only governing inequality that we need to satisfy. Therefore, our result will not depend on a which appears in the second inequality.

Again continuing case by case:

If $p = 0$, then

$$\begin{aligned} \text{from (1) } \frac{0}{x^2} &\leq \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence } 0 \leq \frac{1}{2}X, \text{ so that} \\ X &\geq 0 \text{ as } x \to 0^- \text{ has to hold;} \end{aligned}$$

if $p > 0$, then

$$\begin{aligned} \text{from (1) } \frac{-px}{x^2} &\leq \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence} \\ \frac{-p}{x} &\leq \frac{1}{2}X \text{ as } x \to 0^- \text{ has to hold,} \end{aligned}$$

however since LHS of this last inequality $\to \infty$ as $x \to 0^-$, for any fixed $p > 0$, there does not exist any X that will make (1) hold.

if $p < 0$, then

$$\begin{aligned} \text{from (1) } \frac{0}{x^2} &\leq \frac{px}{x^2} + \frac{1}{2}\frac{Xx^2}{x^2} + 0, \text{ hence } 0 \leq \frac{p}{x} + \frac{1}{2}X, \\ \text{so that } X &\geq \frac{-2p}{x} \text{ as } x \to 0^- \text{ has to hold,} \end{aligned}$$

however since RHS of this inequality $\to -\infty$ as $x \to 0^-$, for any fixed $p < 0$, any $X \in R$ would make (1) hold.

Hence if $(p, X) \in \{0\} \times [0, \infty)$ or if $(p, X) \in (-\infty, 0) \times R$, the inequality (1) will hold.

Thus,

$$J^{2,+}_{[-1,0]}u(0) = (-\infty, 0) \times R \cup \{0\} \times [0, \infty).$$

Remark: After these two examples, let us first note that, as seen in the above examples $J_{\Omega}^{2,+}u(x)$ need not be a closed set; and second that we can define the following mapping

$$\begin{aligned} J_{\Omega}^{2,+}u : \Omega &\to 2^{R^n \times S(N)} \\ x &\to J_{\Omega}^{2,+}u(x) \end{aligned}$$

where $J_{\Omega}^{2,+}u(x) \subset R^n \times S(N)$. Hence, $J_{\Omega}^{2,+}u$ is a set-valued mapping. (Similarly, we can define a corresponding set-valued mapping $J_{\Omega}^{2,-}u$ in the case of second order subjets.) Moreover, as we have seen by the previous two examples, $J_{\Omega}^{2,+}u(x)$ (respectively $J_{\Omega}^{2,-}u(x)$) depends on Ω; however, once $\hat{x}$ is an interior point of the domain, as also seen from the two examples, both inequalities (1) and (2) are effective and once $\hat{x}$ is on the boundary only one of them is effective. Hence, we can say that for all the sets Ω for which $\hat{x}$ is an interior point we will have the same $J_{\Omega}^{2,+}u(\hat{x})$ (respectively $J_{\Omega}^{2,-}u(x)$) value for the same function independent of the domain Ω. We will denote this common value by $J^{2,+}u(\hat{x})$ (respectively by $J^{2,-}u(x)$).

Finally, in this subsection, we will state three properties of semijets, first two of which we will be using in the following chapters, and next define closures of semijets, which we also be using in the following chapters.

Proposition 2.12 *Let $u : \Omega \to R$, and $\hat{x} \in \Omega$. Then,*

$$J_{\Omega}^{2,-}u(\hat{x}) = -J_{\Omega}^{2,+}(-u)(\hat{x}).$$

Proof. Let $(p, X) \in J_{\Omega}^{2,-}u(\hat{x})$. Then as $x \to \hat{x}$

$$\begin{aligned} u(x) &\geq u(\hat{x}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \text{ if and only if} \\ -u(x) &\leq -u(\hat{x}) - \langle p, x-\hat{x}\rangle - \frac{1}{2}\langle X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \text{ if and only if} \\ (-u)(x) &\leq (-u)(\hat{x}) + \langle -p, x-\hat{x}\rangle + \frac{1}{2}\langle -X(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2) \end{aligned}$$

if and only if

$$\begin{aligned}(-p,-X) &\in J_{\Omega}^{2,+}(-u)(\hat{x}) \text{ if and only if}\\ -(p,X) &\in J_{\Omega}^{2,+}(-u)(\hat{x}) \text{ if and only if}\\ (p,X) &\in -J_{\Omega}^{2,+}(-u)(\hat{x}).\end{aligned}$$

Hence, the desired set equality follows. ■

As a result of Proposition 2.18, the following Proposition 2.19 will also hold when $J_{\Omega}^{2,+}$ is replaced by $J_{\Omega}^{2,-}$ everywhere.

Proposition 2.13 *Let $u : \Omega \to R$, and $\varphi : \Omega \to R$ be $C^2(\Omega)$. Then,*

$$J_{\Omega}^{2,+}(u-\varphi)(x) = \left\{(p - D\varphi(x), X - D^2\varphi(x)) : (p,X) \in J_{\Omega}^{2,+}u(x)\right\}.$$

Proof. Fix $\hat{x} \in \Omega$. Then we have the set equality

$$J_{\Omega}^{2,+}(u-\varphi)(\hat{x}) = \left\{(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})) : (p,X) \in J_{\Omega}^{2,+}u(\hat{x})\right\}.$$

So, we will proceed as follows:

Let $(q,Y) \in J_{\Omega}^{2,+}(u-\varphi)(\hat{x})$, then as $x \to \hat{x}$,

$$\begin{aligned}(u-\varphi)(x) &= u(x) - \varphi(x) \leq (u-\varphi)(\hat{x}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle Y(x-\hat{x}), x-\hat{x}\rangle\\ &\quad + o(|x-\hat{x}|^2)\\ &= u(\hat{x}) - \varphi(\hat{x}) + \langle q, x-\hat{x}\rangle + \frac{1}{2}\langle Y(x-\hat{x}), x-\hat{x}\rangle + o(|x-\hat{x}|^2).\end{aligned}$$

Furthermore, by Taylor expansion of φ, we have:

$$\varphi(x) = \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x-\hat{x}\rangle + \frac{1}{2}\left\langle D^2\varphi(\hat{x})(x-\hat{x}), x-\hat{x}\right\rangle + o(|x-\hat{x}|^2) \text{ as } x \to \hat{x}.$$

Hence as $x \to \hat{x}$,

$$u(x) \leq u(\hat{x}) + \langle D\varphi(\hat{x}) + q, x-\hat{x}\rangle + \frac{1}{2}\left\langle (D^2\varphi(\hat{x}) + Y)(x-\hat{x}), x-\hat{x}\right\rangle + o(|x-\hat{x}|^2),$$

so that

$$(D\varphi(\hat{x}) + q, D^2\varphi(\hat{x}) + Y) \in J_{\Omega}^{2,+}u(\hat{x}). \text{ Then}$$

$$q = p_1 - D\varphi(\hat{x}) \text{ and } Y = X_1 - D^2\varphi(\hat{x}) \text{ for some } (p_1, X_1) \in J_\Omega^{2,+}u(\hat{x}), \text{ hence}$$

$$\begin{aligned} (q, Y) &\in \left\{(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})) : (p, X) \in J_\Omega^{2,+}u(\hat{x})\right\}, \text{ and} \\ J_\Omega^{2,+}(u - \varphi)(\hat{x}) &\subset \left\{(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})) : (p, X) \in J_\Omega^{2,+}u(\hat{x})\right\}. \end{aligned}$$

This time, let $(q, Y) \in \left\{(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})) : (p, X) \in J_\Omega^{2,+}u(\hat{x})\right\}$, then

$$q = p_1 - D\varphi(\hat{x}) \text{ and } Y = X_1 - D^2\varphi(\hat{x}) \text{ for some } (p_1, X_1) \in J_\Omega^{2,+}u(\hat{x}), \text{ but then}$$

$$u(x) \leq u(\hat{x}) + \langle p_1, x - \hat{x}\rangle + \frac{1}{2}\langle X_1(x - \hat{x}), x - \hat{x}\rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x}, \text{ and}$$

$$\varphi(x) = \varphi(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle + \frac{1}{2}\left\langle D^2\varphi(\hat{x})(x - \hat{x}), x - \hat{x}\right\rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x},$$

so that

$$\begin{aligned} u(x) - \varphi(x) \leq\ & u(\hat{x}) - \varphi(\hat{x}) + \langle p_1 - D\varphi(\hat{x}), x - \hat{x}\rangle \\ & + \frac{1}{2}\left\langle (X_1 - D^2\varphi(\hat{x}))(x - \hat{x}), x - \hat{x}\right\rangle + o(|x - \hat{x}|^2) \end{aligned}$$

as $x \to \hat{x}$, hence

$$(u - \varphi)(x) \leq (u - \varphi)(\hat{x}) + \langle q, x - \hat{x}\rangle + \frac{1}{2}\langle Y(x - \hat{x}), x - \hat{x}\rangle + o(|x - \hat{x}|^2) \text{ as } x \to \hat{x},$$

so that

$$(q, Y) \in J_\Omega^{2,+}(u - \varphi)(\hat{x}), \text{ hence}$$

$$\left\{(p - D\varphi(\hat{x}), X - D^2\varphi(\hat{x})) : (p, X) \in J_\Omega^{2,+}u(\hat{x})\right\} \subset J_\Omega^{2,+}(u - \varphi)(\hat{x}).$$

Thus, the desired equality follows from the two inclusions at $\hat{x}$, furthermore since $\hat{x}$ was arbitrary, it also holds for any x in Ω. ▬

Proposition 2.14 *For $u, v : \Omega \to R$, we have*

$$J_\Omega^{2,+}u(x) + J_\Omega^{2,+}v(x) \subset J_\Omega^{2,+}(u + v)(x).$$

Proof. Fix $\hat{x} \in \Omega$. Let $(q,Y) \in J_\Omega^{2,+}u(\hat{x}) + J_\Omega^{2,+}v(\hat{x})$. Since

$$J_\Omega^{2,+}u(x) + J_\Omega^{2,+}v(x) = \left\{ \begin{array}{c} (p,X) : (p,X) = (p_1,X_1) + (p_2 + X_2) \\ \text{for some } (p_1,X_1) \in J_\Omega^{2,+}u(x) \text{ and } (p_2 + X_2) \in J_\Omega^{2,+}v(x) \end{array} \right\}$$

$(q,Y) = (p_1,X_1) + (p_2 + X_2)$ for some $(p_1,X_1) \in J_\Omega^{2,+}u(\hat{x})$ and $(p_2 + X_2) \in J_\Omega^{2,+}v(\hat{x})$.

Then, as $x \to \hat{x}$,

$$\begin{aligned} u(x) &\leq u(\hat{x}) + \langle p_1, x - \hat{x} \rangle + \frac{1}{2} \langle X_1(x-\hat{x}), x - \hat{x} \rangle + o(|x-\hat{x}|^2) \text{ and} \\ v(x) &\leq v(\hat{x}) + \langle p_2, x - \hat{x} \rangle + \frac{1}{2} \langle X_2(x-\hat{x}), x - \hat{x} \rangle + o(|x-\hat{x}|^2) \text{ so that} \\ (u+v)(x) &\leq (u+v)(\hat{x}) + \langle p_1 + p_2, x - \hat{x} \rangle + \frac{1}{2} \langle (X_1 + X_2)(x-\hat{x}), x - \hat{x} \rangle \\ &\quad + o(|x-\hat{x}|^2). \end{aligned}$$

Hence

$$\begin{aligned} (p_1 + p_2, X_1 + X_2) &\in J_\Omega^{2,+}(u+v)(\hat{x}), \text{ so that} \\ (p_1,X_1) + (p_2 + X_2) &\in J_\Omega^{2,+}(u+v)(\hat{x}), \text{ so that} \\ (q,Y) &\in J_\Omega^{2,+}(u+v)(\hat{x}). \end{aligned}$$

Thus

$$J_\Omega^{2,+}u(\hat{x}) + J_\Omega^{2,+}v(\hat{x}) \in J_\Omega^{2,+}(u+v)(\hat{x}).$$

Since $\hat{x}$ was arbitrary, it also holds for any x in Ω. ∎

Definition 2.15 *Let $x \in \Omega$, by the closure of set-valued mapping $J_\Omega^{2,+}u$, we mean*

$$\begin{aligned} \bar{J}_\Omega^{2,+}u : \Omega &\to 2^{R^n \times S(N)} \\ x &\to \bar{J}_\Omega^{2,+}u(x) \end{aligned}$$

where

$$\bar{J}_\Omega^{2,+}u(x) = \left\{ \begin{array}{c} (p,X) \in R^n \times S(N) : \textit{there is } (x_n,p_n,X_n) \in \Omega \times R^n \times S(N) \\ \textit{such that } (p_n,X_n) \in J_\Omega^{2,+}u(x_n) \textit{ and} \\ (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X). \end{array} \right\}$$

is the closure of the second order superjet of u *at* x. *Similarly, by the closure of set-valued mapping* $J_{\Omega}^{2,-}u$, *we mean*

$$\begin{aligned} \bar{J}_{\Omega}^{2,-}u : \Omega &\to 2^{R^n \times S(N)} \\ x &\to \bar{J}_{\Omega}^{2,-}u(x) \end{aligned}$$

where

$$\bar{J}_{\Omega}^{2,-}u(x) = \left\{ \begin{array}{c} (p, X) \in R^n \times S(N) : \textit{there is } (x_n, p_n, X_n) \in \Omega \times R^n \times S(N) \\ \textit{such that } (p_n, X_n) \in J_{\Omega}^{2,-}u(x_n) \textit{ and} \\ (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X). \end{array} \right\}$$

is the closure of the second order subjet of u *at* x.

2.3. Ellipticity, Linearization, "Properness" and "Maximum Principle"

Before going any further, we will make the following observations:

1) In linear equations the type (namely, ellipticity, parabolicity or hyperbolicity) of the equation is determined by the differential equation itself; however, in nonlinear equations "type" depends on the individual solutions. We will elaborate on this assertion first. Let us for the moment accept $u : \Omega \to R$ to be twice differentiable on $\Omega \subset R^n$ and (after leaving aside the lower order terms) consider the second order nonlinear partial differential equation

$$\mathsf{z}(u) = F(D^2u) = 0.$$

Here,

$$D^2u = \begin{bmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ . & . & . \\ u_{x_nx_1} & . & u_{x_nx_n} \end{bmatrix}$$

is the Hessian matrix of second derivatives of u, F is a mapping such that $F : S(N) \to R$, and $S(N)$ is the set of real symmetric $N \times N$ matrices; and we will assume F to be smooth. In this case, we can view F as a function of N^2 variables such that $F(p_{11,}p_{12}, ..., p_{1n}, p_{21}, ..., p_{nn})$ where $p_{ij} = u_{x_i x_j}$. Then z is defined to be "elliptic" at some "solution" C^2 function $u^0(x)$ if

$$p(\xi) = -\sum_{i,j} \frac{\partial F}{\partial p_{ij}}(u^0(x))\xi_i \xi_j > 0 \text{ for } \xi \neq 0.$$

Furthermore, "linearization" of z at some u^0 is a linear map $D\mathsf{z}(u^0) : C^\infty(\Omega) \to C^\infty(\Omega)$ defined as follows: for $\phi \in C^\infty(\Omega)$,

$$\begin{aligned} D\mathsf{z}(u^0)(\phi) &= \lim_{t\to 0} \frac{\mathsf{z}(u^0 + t\phi) - \mathsf{z}(u^0)}{t} \\ &= \lim_{t\to 0} \frac{F(D^2u^0 + tD^2\phi) - F(D^2u^0)}{t} \\ &= \sum_{i,j} \frac{\partial F}{\partial p_{ij}}(u^0)\phi_{x_i x_j} \end{aligned}$$

and moreover, in this case $D\mathsf{z}(u^0)(\phi) = \sum \left(\frac{\partial F}{\partial p_{ij}}(u^0)\right)\phi_{x_i x_j}$. Hence z being "elliptic" will correspond to its linearization about any fixed u^0 being an "elliptic" operator.

2) Now, we will consider some examples of scaler coefficient, linear elliptic partial differential equations and simple applications of maximum principle. Throughout, u will be $C^2(\Omega)$:

a) Let $n = 1$, and consider the linear elliptic partial differential mapping L as being the Laplacian, i.e. let $L(D^2u) = -\Delta u = -u''$. Let $-\Delta u = 1$, then any $C^2(\Omega)$ function of the form $u(x) = a + bx - \frac{1}{2}x^2$ solves this equation on Ω, hence is a classical solution. In this case, if $p(x)$ is a paraboloid (parabola in $n = 1$) and $u - p$ has a local maximum at some $\hat{x} \in \Omega$, then $p''(\hat{x}) \geq -1$, i.e. $L(D^2p(\hat{x})) \leq 1$; and if $p(x)$ is a parabola and $u - p$ has a local minimum at some $\hat{x} \in \Omega$, then $p''(\hat{x}) \leq 1$; i.e. $L(D^2p(\hat{x})) \geq 1$.

b) Let $n = 2$, and $L(D^2u) = -\Delta u$. Suppose $-\Delta u(\hat{x}, \hat{y}) < 0$, then "maximum principle" says that u cannot have a local maximum at $(\hat{x}, \hat{y}) \in \Omega \subset R^2$. Proof:

Suppose $(\hat{x}, \hat{y})$ is a local maximum of u, then $\bigtriangledown u(\hat{x}, \hat{y}) = 0$, and $u_{xx}(\hat{x}, \hat{y}) \leq 0$ and $u_{yy}(\hat{x}, \hat{y}) \leq 0$, but then $-\Delta u(\hat{x}, \hat{y}) = -u_{xx}(\hat{x}, \hat{y}) - u_{yy}(\hat{x}, \hat{y}) \geq 0$ hence we arrive at a contradiction, so $(\hat{x}, \hat{y})$ cannot be a local maximum of u. We can restate the same statement as: If u has a local maximum at $(\hat{x}, \hat{y})$, then $-\Delta u(\hat{x}, \hat{y}) \geq 0$ has to hold.

c) Let $\hat{x} \in R^n$. This time let L also depend on u. Let $L(u, D^2u) = -\Delta u + \gamma u$. Let $-\Delta u + \gamma u \leq 0$, and $u = 0$ on $\partial\Omega$. Suppose u has a local maximum at $\hat{x} \in \Omega$. Then $\Delta u(\hat{x}) \leq 0$, and $\gamma u(\hat{x}) \leq \Delta u(\hat{x}) \leq 0$. In order for the classical maximum principle to hold we need to have $\gamma > 0$, since only then the assertion of the classical maximum principle for this case (which is $u(\hat{x}) \leq 0$ and hence $u(x) \leq 0$ on Ω) holds. In this case since $L(u, D^2u) = -\Delta u + \gamma u = -tr(D^2u) + \gamma u$, the condition of $\gamma > 0$ corresponds to L being strictly increasing in u.

d) This time, let L depend on Du as well and be defined as $L(u, Du, D^2u) = -\Delta u + \alpha Du + \gamma u$. Let $w = u - v$, and $\gamma > 0$. Suppose $L(w, Dw, D^2w) = -\Delta w + \alpha Dw + \gamma w \leq 0$ (then $-\Delta u + \alpha Du + \gamma u \leq -\Delta v + \alpha Dv + \gamma v$) and w has a maximum at $\hat{x}$. Then, $Du(\hat{x}) - Dv(\hat{x}) = Dw(\hat{x}) = 0$, hence $Du(\hat{x}) = Dv(\hat{x})$ and $\Delta u(\hat{x}) - \Delta v(\hat{x}) = \Delta w(\hat{x}) \leq 0$, so that $\Delta u(\hat{x}) \leq \Delta v(\hat{x})$. Hence $\gamma u(\hat{x}) \leq \gamma v(\hat{x})$, and since $\gamma > 0$, we have $u(\hat{x}) \leq v(\hat{x})$, i.e. $w(\hat{x}) \leq 0$. Note also in this case that,

$$\begin{aligned} L(u(\hat{x}), Du(\hat{x}), D^2v(\hat{x})) &= -\Delta v(\hat{x}) + \alpha Du(\hat{x}) + \gamma u(\hat{x}) \\ &\leq -\Delta u(\hat{x}) + \alpha Du(\hat{x}) + \gamma u(\hat{x}) \\ &= L(u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})). \end{aligned}$$

After these observations, we would like to state that we have two main issues at hand. One is that generalizing a similar "maximum principle" approach to nonlinear equations, and the other is that generalizing the class of solutions to a larger class than that of classical solutions. In the latter, one when we make such a generalization, we would like to have consistency in order to have it as an acceptable generalization. In other words, we would like the classical solutions still be solutions within the new generalized concept of solution. Within this perspective we are now ready to proceed in defining the properties of the mapping F that will allow it to be considered under the theory of viscosity solutions.

Let F be a mapping from $\Omega \times R \times R^n \times S(N)$ into R. We will consider nonlinear partial differential equations of the form $F(x, u, Du, D^2u) = 0$ and in the case that u is C^2, $Du = (u_{x_1}, ..., u_{x_n})$ denotes the gradient matrix of first order partial derivatives of u, and D^2u denotes the Hessian matrix described above. Since later on we will require u only to be continuous and not necessarily differentiable (but still can solve the equation within the new solution concept) Du and D^2u will not have their classical meanings and we will write instead $F(x, r, p, X)$ to indicate the value of F at $(x, r, p, X) \in \Omega \times R \times R^n \times S(N)$. Having made these clarifications we can now proceed as follows:

Definition 2.16 *We will say that F satisfies the restricted "maximum principle", if for any $\varphi, \psi \in C^2$ such that $\psi - \varphi$ has a local maximum at $\hat{x}$ and $\varphi(\hat{x}) = \psi(\hat{x})$ holds the following inequality*

$$F(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x}))$$

is satisfied.

At this point, if we ask the question of "under what condition imposed on F we can guarantee that F satisfies this 'maximum principle"' we arrive at the following condition:

Proposition 2.17 *Above defined F satisfies the restricted "maximum principle" if and only if the following antimonotonicity condition*

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ for } Y \leq X$$

holds. Here, $X, Y \in S(N)$ and $Y \leq X$ is the ordering in $S(N)$ that is given by:

$$Y \leq X \text{ if and only if } \langle X\xi, \xi\rangle \leq \langle Y\xi, \xi\rangle \text{ for } \xi \in R^n.$$

Before proving this proposition, we will interpret it first. Let F be as in our observation 1) above. Let us fix a matrix $Y \in S(N)$ and a vector $\xi \in R^n$. Letting $X = Y + t(\xi \otimes \xi)$ where $t > 0$ and

$$(\xi \otimes \xi) = \begin{bmatrix} \xi_1\xi_1 & \cdots & \xi_1\xi_n \\ . & . & . \\ \xi_n\xi_1 & \cdots & \xi_n\xi_n \end{bmatrix} \in S(N),$$

we have by the antimonotonicity condition that

$$\frac{1}{t}(F(Y + t(\xi \otimes \xi)) - F(Y)) \leq 0.$$

When we let $t \to 0^+$, since we assume F to be smooth, we conclude that

$$D\mathsf{z}(Y)(\xi \otimes \xi) = \lim_{t \to 0^+} \frac{(F(Y + t(\xi \otimes \xi)) - F(Y))}{t} \leq 0$$

Since, we have

$$\left[\frac{\partial F}{\partial r_{ij}}(Y)\right] \cdot (\xi \otimes \xi) = D\mathsf{z}(Y)(\xi \otimes \xi) \leq 0,$$

where $\cdot$ is not the matrix multiplication, but the dot product of the elements in R^{n^2}, then we have

$$p(\xi) = -\sum_{i,j} \frac{\partial^2 F}{\partial p_{ij}}(u^0(x))\xi_i\xi_j \geq 0.$$

Hence, we can interpret the condition of antimonotonicity as meaning that the "linearization" of z about any fixed u^0 being an "elliptic" operator, and furthermore since it allows for the value of zero, then possibly being a "degenerate elliptic" operator. Therefore, this antimonotonicity condition will be named as z being "degenerate elliptic". Now, we will prove the proposition:

Proof. Let $\varphi, \psi \in C^2$ be such that $\varphi - \psi$ has a minimum at $\hat{x}$, and $\varphi(\hat{x}) = \psi(\hat{x})$. Then by calculus we have $D\varphi(\hat{x}) = D\psi(\hat{x})$, and $D^2\varphi(\hat{x}) \geq D^2\psi(\hat{x})$. Hence, if antimonotonicity holds, we have

$$F(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x})).$$

so that "maximum principle" is satisfied. For the converse, assume antimonotonicity does not hold at some $\hat{x}$. Then for $x \in \Omega$, let $\varphi(x) = r + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x - \hat{x}\rangle$ and $\psi(x) = r + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle Y(x-\hat{x}), x-\hat{x}\rangle$. Then $\varphi - \psi$ has a minimum at $\hat{x}$ such that $\varphi(\hat{x}) = \psi(\hat{x})$, and $\varphi, \psi \in C^2$. Then, since monotonicity does not hold at $\hat{x}$, F does not satisfy "maximum principle". ■

Now, let us consider an example where F is first order. Let $F(x, r, p, X) = H(x, r, p)$ for some function H. Then F is clearly degenerate elliptic. However, in this case restricted "maximum principle" does not say much for if $\psi, \varphi \in C^2$, $\psi - \varphi$ has a maximum at $\hat{x}$ and $\varphi(\hat{x}) = \psi(\hat{x})$ holds, since then by calculus we have $D\varphi(\hat{x}) = D\psi(\hat{x})$ and the inequality $H(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x})) = H(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}))$ holds automatically. However, instead of having the requirement that $\varphi(\hat{x}) = \psi(\hat{x})$ holds, we can require that at a maximum $\hat{x}$ of $\psi - \varphi$, the inequality $\varphi(\hat{x}) \leq \psi(\hat{x})$ to hold, in other words, we can require $\psi - \varphi$ to have a nonnegative maximum at $\hat{x}$; and additionally require F to be strictly increasing in r, (i.e. $r \leq s$ implying $F(x, r, p, X) \leq F(x, s, p, X)$), to guarantee that

$$F(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x}))$$

will still be satisfied. Hence, by modifying this requirement of $\varphi(\hat{x}) = \psi(\hat{x})$ in the definition of restricted "maximum principle" we are imposing on F a second structural condition, namely monotonicity in r, so that the inequality of the maximum principle will still hold.

Hence as a result of this modification, we have the following:

Definition 2.18 *We will say that F satisfies the maximum principle, if for any $\varphi, \psi \in C^2$ such that $\psi - \varphi$ has a nonnegative maximum at $\hat{x}$ the following inequality*

$$F(\hat{x}, \varphi(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, \psi(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x}))$$

is satisfied.

Proposition 2.19 *In this case, F satisfies the maximum principle if and only if the following conditions*

$$\begin{aligned} &\textit{(i)} \quad F(x,r,p,X) \leq F(x,r,p,Y) \textit{ for } Y \leq X, \textit{ and} \\ &\textit{(ii)} \quad F(x,r,p,X) \leq F(x,s,p,X) \textit{ for } r \leq s \textit{ hold.} \end{aligned}$$

In the case that F satisfies (i), F will be called degenerate elliptic, if in addition F satisfies (ii), F will then be called proper.

Hence, we are able to provide an answer to another one of our promised questions at the beginning of this chapter.

In the next section, we will see that if F satisfies the maximum principle, in other words if F is proper, within the context of the new solution concept, classical solutions will still continue to be a solution and that maximum principle, or in other words F being proper will guarantee us the consistency. Also, in the next section, we will see how we define viscosity solutions by taking off from maximum principle.

2.4. Viscosity Solutions

In this section we will define a generalized solution concept for the equation

$$F(x,u,Du,D^2u) = 0. \tag{2.5}$$

Throughout this work we will assume F to be proper and continuous as indicated by the previous section and try to make us of the maximum principle in our generalizations. Hence, taking off from maximum principle, let us assume $u, v \in C^2(\Omega)$, and see what type of information we would have in our hands in this case. Let us start by also assuming that u is a subsolution (classical since $u \in C^2(\Omega)$) of this equation. Then, we know that

$$F(x,u(x),Du(x),D^2u(x)) \leq 0 \text{ for all } x \in \Omega.$$

If also $\hat{x}$ is a local maximum of $u - v$, we would have $Du(\hat{x}) = Dv(\hat{x})$, and $D^2u(\hat{x}) \leq D^2v(\hat{x})$ from calculus. Hence we can use the fact that F is proper (in particular the degenerate ellipticity part) to obtain

$$F(\hat{x}, u(\hat{x}), Dv(\hat{x}), D^2v(\hat{x})) \leq F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0$$

at the maximum $\hat{x}$. This would hold true for any $v \in C^2(\Omega)$, in the case that u is also $C^2(\Omega)$. Now, we are aiming at defining a solution concept that would allow functions u that are not necessarily differentiable to be considered as candidates for solutions. If we look at the above derived inequality once more closely, we see that we have actually obtained the following result that is independent of the derivatives of u,

$$F(\hat{x}, u(\hat{x}), Dv(\hat{x}), D^2v(\hat{x})) \leq 0.$$

Hence, in the case that u were not differentiable, we could take this inequality to hold for $v \in C^2(\Omega)$ whenever $u - v$ has a maximum point, to be the definition of a subsolution. If we compare this last inequality to the one we obtained from u being a solution, in other words to the following inequality

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0$$

we then see that in the case that u is not differentiable, we have as a matter of fact at $\hat{x}$ 'transferred' the derivative onto a smooth test function v at the expense of $u - v$ having a local maximum at $\hat{x}$. Within this perspective let us define viscosity subsolutions, supersolutions and solutions for (2.5).

Definition 2.20 (1) *Let F be proper, Ω open subset of R^n, and $u \in USC(\Omega)$, $v \in LSC(\Omega)$. Then u is a viscosity subsolution of $F = 0$ in Ω, if*

for every $\varphi \in C^2(\Omega)$ and local maximum point $\hat{x} \in \Omega$ of $u - \varphi$

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0 \text{ holds.}$$

Similarly, v is a viscosity supersolution of $F = 0$ in Ω, if

$$\text{for every } \varphi \in C^2(\Omega) \text{ and local minimum point } \hat{x} \in \Omega \text{ of } v - \varphi$$

$$F(\hat{x}, v(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0 \text{ holds.}$$

A function w is a viscosity solution of $F = 0$ in Ω, if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$.

In the definition we have required a subsolution to be upper semicontinuous and a super solution to be lower semicontinuous. One of the reasons for this is that upper semicontinuous functions and lower semicontinuous functions assume their maximums and respectively minimums on compact sets and we will want to produce maxima related with these functions. The other reason is that later on we would like to produce continuous solutions with Perron's process, in which we obtain continuous solutions in the limit of a sequence of some functions, and this can be done in more generality in the classes of upper and lower semicontinuous functions, since these classes are larger then the class of continuous functions and can still yield continuous functions in the limit. Hence, the theory will inevitably require us to work with upper and lower semicontinuous functions consistently. Therefore at the end of this section we will give shortly the definitions, some properties and examples of upper and lower semicontinuous functions.

Now, recalling the results we have obtained in Section 2.1 for semijets, we can immediately give the following equivalent definition for subsolutions, supersolutions, and solutions.

Definition 2.21 (2) *Let F be a continuous proper second order nonlinear elliptic partial differential operator, and $\Omega \subset R^n$. Then, a function $u \in USC(\Omega)$ is a viscosity subsolution of $F = 0$ in Ω if*

$$F(x, u(x), p, X) \leq 0 \text{ for all } x \in \Omega \text{ and } (p, X) \in J_{\Omega}^{2,+}u(x),$$

A function $u \in LSC(\Omega)$ is a viscosity supersolution of $F = 0$ in Ω if

$$F(x, u(x), p, X) \geq 0 \text{ for all } x \in \Omega \text{ and } (p, X) \in J_{\Omega}^{2,-}u(x),$$

and a function $u \in C(\Omega)$ is a viscosity solution of $F = 0$ in Ω if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω.

Actually, this was the first definition we have presented at the beginning of this chapter to motivate the whole discussion.

Now, it is easy to see that these two definitions are equivalent since, if u is a viscosity solution in the sense of Definition (1) , then for every $(p, X) \in J_{\Omega}^{2,+}u(\hat{x})$, we can construct, as indicated in Section 2.1, a $\varphi \in C^2(\Omega)$ with $D\varphi(\hat{x}) = p$, $D^2\varphi(\hat{x}) = X$, such that $u - \varphi$ will have a maximum at $\hat{x}$, then the result follows automatically from Definition (1) ; conversely, if u is a solution in the sense of Definition (2), then for $\varphi \in C^2(\Omega)$, if $u - \varphi$ has a local maximum at $\hat{x}$, then $(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in J_{\Omega}^{2,+}u(\hat{x})$, and the result will follow from Definition (2) automatically.

Throughout this work we will work with both definitions interchangeably.

Once having generalized the solution concept for $F(x, u, Du, D^2u) = 0$, next we need to check that it is consistent with the classical solution concept. In other words, classical solutions need still continue to be solutions under the new concept.

Proposition 2.22 *Let $u \in C^2(\Omega)$ be a solution of $F(x, u, Du, D^2u) = 0$ in the classical sense. Then u is also a viscosity solution of $F(x, u, Du, D^2u) = 0$.*

Proof. Since $u \in C^2(\Omega)$ and a classical solution then at every $\hat{x} \in \Omega$ we have

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) = 0,$$

also since $J_{\Omega}^{2,+}u(\hat{x}) = J_{\Omega}^{2,-}u(\hat{x}) = \{(Du(\hat{x}), D^2u(\hat{x}))\}$ then we have $F(x, u(x), p, X) = 0$ for all $x \in \Omega$ and $(p, X) \in J_{\Omega}^{2,+}u(x)$. ∎

Proposition 2.23 *If u is a viscosity solution of $F(x, u, Du, D^2u) = 0$, and u is twice differentiable at some $\hat{x}$, then u solves $F(x, u, Du, D^2u) = 0$ in the classical sense at $\hat{x}$,*

i.e. $F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) = 0.$

Proof. If u is twice differentiable at $\hat{x}$, then

$$J_\Omega^{2,+}u(\hat{x}) = J_\Omega^{2,-}u(\hat{x}) = \left\{(Du(\hat{x}), D^2u(\hat{x}))\right\}.$$

And since u is a viscosity solution from Definition (2), we obtain that

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) = 0.$$

$\blacksquare$

From this point on, we will omit the term viscosity, since we will be dealing with viscosity subsolutions, supersolutions and solutions consistently.

Next, we will have the promised definitions and properties concerning upper and lower semicontinuous functions.

Definition 2.24 *A function $u : \Omega \to R$ is called upper semicontinuous (USC) at $x_0 \in \Omega$, if given any $\varepsilon > 0$, there exists a neighborhood of x_0 in which $u(x) < u(x_0) + \varepsilon$. Similarly, u is called lower semicontinuous (LSC) at a point $x_0 \in \Omega$ if given any $\varepsilon > 0$, there exists a neighborhood of x_0 in which $u(x) > u(x_0) - \varepsilon$.*

Remark: Equivalently, $u : \Omega \to R$ is called upper semicontinuous if $u^{-1}(\lambda, \infty)$ is open for every λ. Similarly, u is called lower semicontinuous if $u^{-1}(-\infty, \lambda)$ is open for every λ.

Example 2.25 *Let* $u_b(x) = \begin{cases} 0 & x < a \\ b & x = a \\ 1 & x > a \end{cases}$ *, then depending on b, u_b is upper or lower semicontinuous.*

If $b = 0$, then u_b is lower semicontinuous, $b = \frac{1}{2}$, then u_b is neither lower nor upper semicontinuous, $b = 1$, then u_b is upper semicontinuous.

Proposition 2.26 *Let $u \in USC(\Omega)$. If $x_n, x_0 \in \Omega$ and $\lim_{n\to\infty} x_n = x_0$, then*

$$\limsup u(x_n) \leq u(x_0).$$

Proof. Since u is upper semicontinuous, we know that given any $\delta > 0$, there is a neighborhood of x_0 such that for all x in this neighborhood of x_0, $u(x) \leq u(x_0) + \delta$. Hence if we have a sequence x_n converging to x_0, the sequence $u(x_n)$ cannot have an accumulation point which is strictly greater then $u(x_0)$. In other words, $\limsup_{n\to\infty} u(x_n) \leq u(x_0)$. ∎

Theorem 2.27 *Let $u \in USC(\Omega)$ be bounded from above, and Ω be compact. Then u attains its supremum on Ω.*

Proof. Let $M = \sup_\Omega u(x)$. Then, there exists a sequence of x_n such that $u(x_n) \to M$. Since Ω is compact x_n has a convergent subsequence say x_{n_k} say converging to some $x_0 \in \Omega$. Then by semicontinuity we have $u(x_{n_k}) \leq u(x_0)$. But then since $u(x_n) \to M$ and $u(x_{n_k})$ is a subsequence of $u(x_n)$, we have $u(x_{n_k}) \to M$ also. Hence we have $u(x_{n_k}) \leq u(x_0) \leq M$, and in the limit we achieve $M \leq u(x_0) \leq M$, giving us $u(x_0) = M$, hence supremum is achieved at $x_0 \in \Omega$. ∎

2.5. Figures

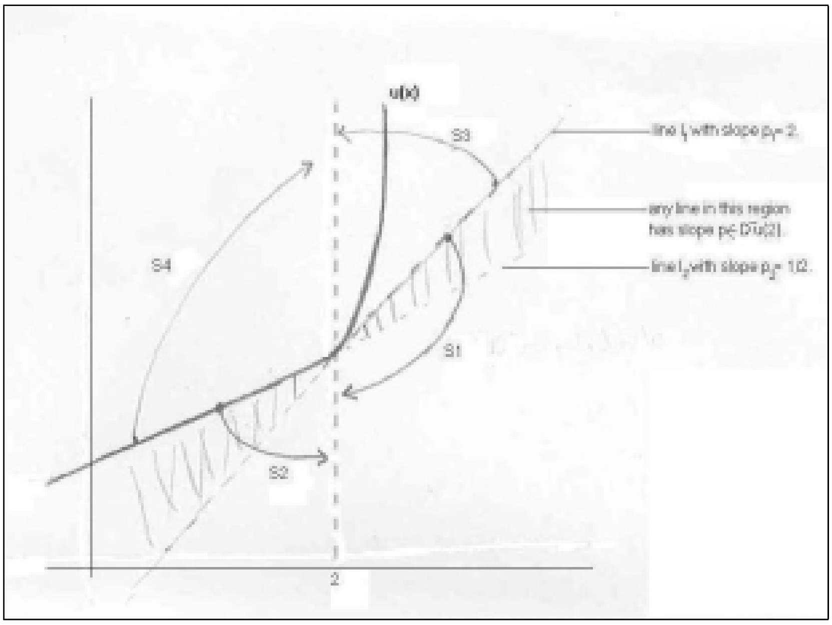

Figure 2.1

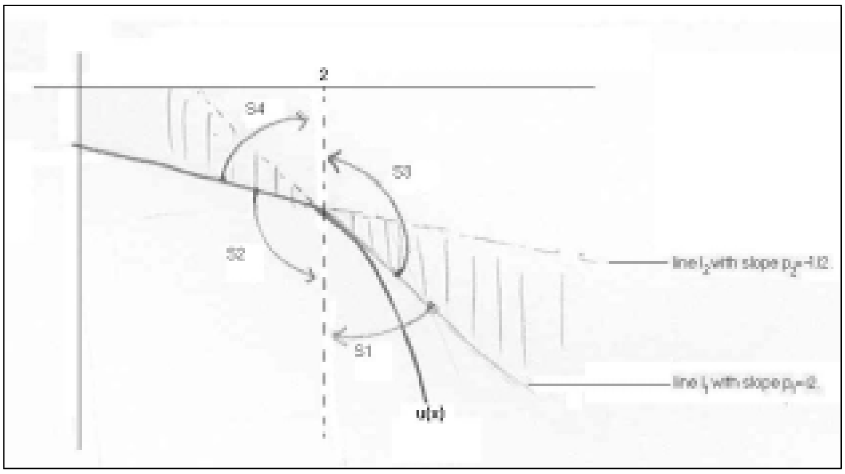

Figure 2.2

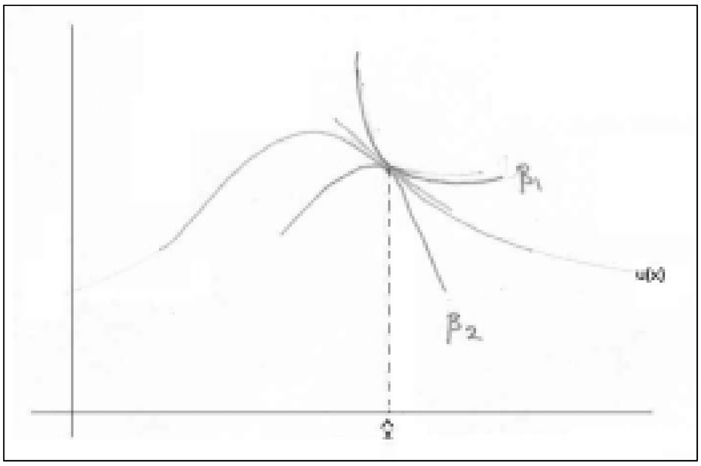

Figure 2.3

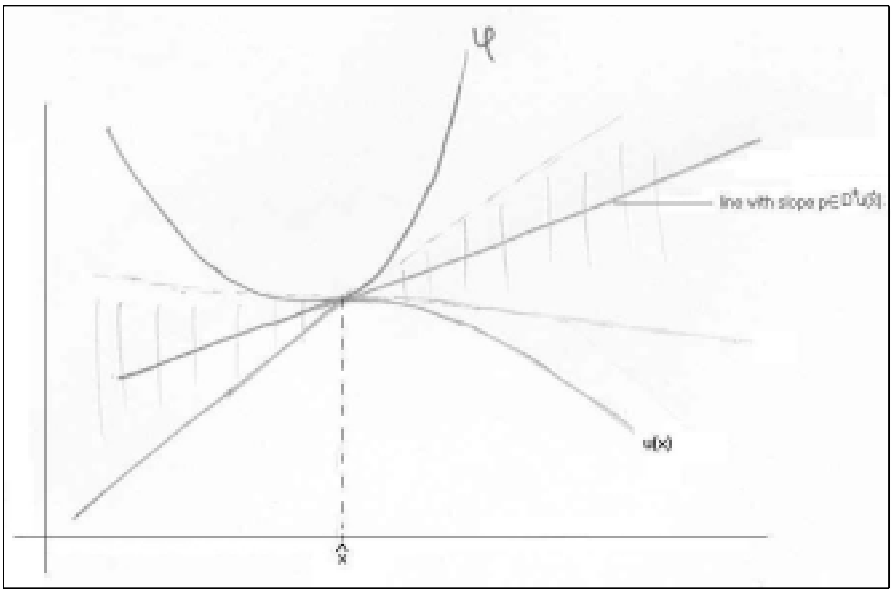

Figure 2.4

In Figures 2.3 and 2.4, we are neglecting the $o(|x - \hat{x}|)$ term. It is possible to have $u(x) \geq u(\hat{x}) + \langle p, x - \hat{x} \rangle$ but still $u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + o(|x - \hat{x}|)$ to hold. In fact, in the proof of Proposition 2.5, the function $\alpha(x)$ is used to record the intervals and the differences when the case that $u(x) \geq u(\hat{x}) + \langle p, x - \hat{x} \rangle$ holds, so that $u - \varphi$ can have a local maximum.

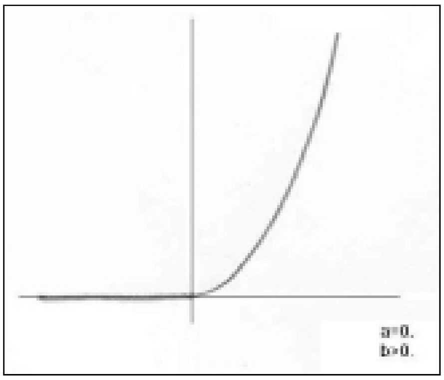

Figure 2.5

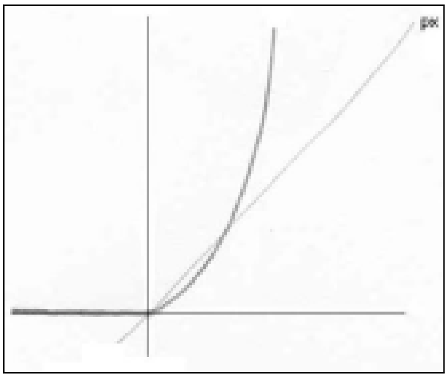

Figure 2.6

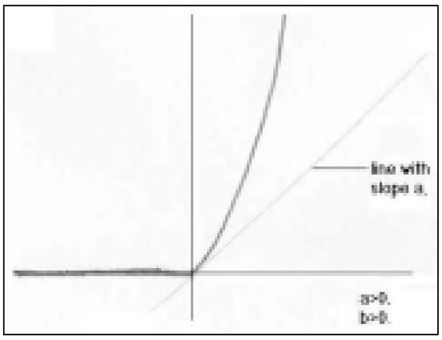

Figure 2.7

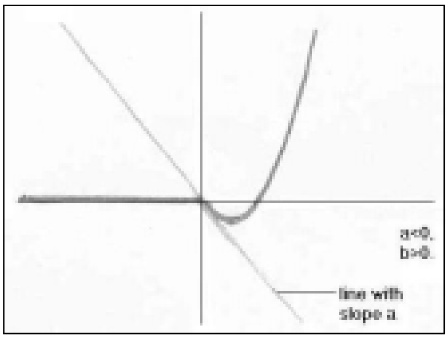

Figure 2.8

2.6. Notes

Discussion for first order case follows the discussion in [**C-E-L**]. Proof of Proposition 2.6 is parallel to the proof of second order parabolic version of the same proposition in [**F-S**]. The idea of paraboloids and punctual second order differentiability presented in Section 2.2 occurs in [**Cab-Caf**]. Definition 2.22 and Proposition 2.23 are from [**F-S**], and interpretation of antimonotonicity appears in L. C. Evans's lecture note 'Regularity for Fully Nonlinear Elliptic Equations and Motion by Mean Curvature' in [**B-et.al.**].

3

GENERALIZATIONS OF SECOND DERIVATIVE TESTS - "MAXIMUM & COMPARISON PRINCIPLES"

3.1. Introduction

In this chapter, our first aim will be to prone a generalized second derivative test for upper semicontinuous functions and we will call it maximum principle for upper semicontinuous functions. Once having done that, using this maximum principle, we will then aim at deriving the conditions under which comparison would hold for the Dirichlet Problem

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega. \tag{DP}$$

where Ω will be a bounded subset of R^n.

Let us try to identify the problem we have at hand in this process.

In the classical case, if we want to derive a comparison result using maximum principle, we would use the fact that at a maximum point $\hat{x}$, for a C^2 function w we would have

$$Dw(\hat{x}) = 0 \text{ and } D^2w(\hat{x}) \leq 0. \tag{3.1}$$

It is also important to note that in this case, i.e. when w is C^2, we also have that

$$J^2w(\hat{x}) = J^{2,+}w(\hat{x}) \cap J^{2,-}w(\hat{x}) = \left\{Dw(\hat{x}), D^2w(\hat{x})\right\}.$$

We first need to see how this preceding information would work: If u and v are C^2 subsolution and supersolution of the (DP) and if $w = u - v$ has an interior maximum $\hat{x} \in \Omega$, then by (3.1) we would have

$$Du(\hat{x}) = Dv(\hat{x}) \text{ and } D^2u(\hat{x}) \leq D^2v(\hat{x}).$$

On the other hand, the other piece of information we have would come from the fact that u and v are subsolution and supersolution respectively and that also F is proper. These pieces of information would lead us respectively to

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0 \leq F(\hat{x}, v(\hat{x}), Dv(\hat{x}), D^2v(\hat{x}))$$

$$F(\hat{x}, v(\hat{x}), Dv(\hat{x}), D^2v(\hat{x})) \leq F(\hat{x}, v(\hat{x}), Du(\hat{x}), D^2u(\hat{x})).$$

Hence we would have

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq F(\hat{x}, v(\hat{x}), Du(\hat{x}), D^2u(\hat{x})).$$

If F also satisfies the structure condition of being strictly nondecreasing in r this would then lead us to conclude that

$$u(\hat{x}) \leq v(\hat{x})$$

and since $\hat{x}$ was a local maximum of $u - v$ would have

$$u(x) - v(x) \leq u(\hat{x}) - v(\hat{x}) \leq 0$$

and hence we would obtain the result that $u \leq v$ on Ω.

As we have seen in the preceding chapter, viscosity solutions need not be differentiable, the only regularity we assume for them is continuity. Moreover, subsolutions and supersolutions are allowed to be even semicontinuous. Therefore, in order to be able to make a similar deduction as above, we need to define a corresponding maximum principle for semicontinuous functions.

Also in the preceding chapter, we have defined an alternative way of dealing with differentiability at a nondifferentiable point x of a semicontinuous function u, which was considering the elements of the sets $J^{2,+}u(x)$, $J^{2,-}u(x)$ in the place of a possible derivative value. We can use the same approach here as well. At a maximum $\hat{x}$ of w, we can consider $J^{2,+}w(\hat{x})$, $J^{2,-}w(\hat{x})$, hence actually use the sets $J^{2,+}u(\hat{x})$, $J^{2,-}u(\hat{x})$, $J^{2,+}v(\hat{x})$, $J^{2,-}v(\hat{x})$. Since the information

$$Du(\hat{x}) = Dv(\hat{x}) \text{ and } D^2u(\hat{x}) \leq D^2v(\hat{x})$$

is actually a way of comparing the values of Du with Dv, and, D^2u with D^2v at $\hat{x}$, i.e. comparing the values of some elements present in the set values of $J^{2,+}u$, $J^{2,-}u$, $J^{2,+}v$, $J^{2,-}v$ at $\hat{x}$, (noting that in the case of u and v being C^2, we have

$$\begin{aligned} J^{2,+}u(\hat{x}) &= J^{2,-}u(\hat{x}) = \left\{(Du(\hat{x}), D^2u(\hat{x}))\right\} \\ J^{2,+}v(\hat{x}) &= J^{2,-}v(\hat{x}) = \left\{(Dv(\hat{x}), D^2v(\hat{x}))\right\} \text{);} \end{aligned}$$

then, in this case that u and v might not be differentiable at $\hat{x}$, we can try comparing some elements present in the set values of $J^{2,+}u$ and $J^{2,-}u$, and ,$J^{2,+}v$ and $J^{2,-}v$ at $\hat{x}$ in order to be able to deduce a result paralleling

$$Du(\hat{x}) = Dv(\hat{x}) \text{ and } D^2u(\hat{x}) \leq D^2v(\hat{x}).$$

However, at this point we have a problem. The sets $J^{2,+}u(\hat{x})$, $J^{2,-}u(\hat{x})$, $J^{2,+}v(\hat{x})$, $J^{2,-}v(\hat{x})$ could very well be empty and prevent us from deducing any kind of information that would have been obtained via comparing their elements in the case that they were not nonempty. Hence, we have to overcome this obstacle. One way of doing this would be approximating $\hat{x}$ via a sequence of points $\hat{x}_\alpha$ of which we would like to have the following first set of information: $\hat{x}_\alpha$ are maximums of some functions w_α; as $\alpha \to \infty$, the process of maximization of w_α approximates the process of maximization of w; and as $\alpha \to \infty$, $\hat{x}_\alpha \to \hat{x}$. The process of 'doubling the variables', which we will introduce in the sequel, will provide us with such an approximation process. The price of overcoming this obstacle would be however changing the usual setting in which we were normally comparing the values of Du and Dv,and, D^2u and D^2v at $\hat{x}$. Hence, we have to interpret what the information of $Dw(\hat{x}) = 0$ and $D^2w(\hat{x}) \leq 0$ would correspond to under this new setting.

In doubling the variables technique, the functions w_α would be of the form

$$w_\alpha(z) = u(x) - v(y) - \varphi_\alpha(x, y)$$

where φ is a C^2 function and z represents the doubled variable (x, y). If we assume for the moment that u and v are also C^2, then w_α would be C^2, and assuming further that $D_x\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha) = -D_y\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)$, then at a maximum $\hat{z}_\alpha$ of w_α, by using the classical maximum principle $Dw_\alpha(\hat{z}_\alpha) = 0$ and $D^2w_\alpha(\hat{z}_\alpha) \leq 0$ we would obtain that

$$\begin{aligned}
0 &= Dw_\alpha(\hat{z}_\alpha) = Du(\hat{x}_\alpha) - Dv(\hat{y}_\alpha) - D(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \\
Du(\hat{x}_\alpha) &= D_x\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha) \text{ and } Dv(\hat{y}_\alpha) = -D_y\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha) \\
Du(\hat{x}_\alpha) &= Dv(\hat{y}_\alpha),
\end{aligned}$$

and

$$
\begin{aligned}
0 \;\geq\; D^2 w_\alpha(\hat{z}_\alpha) &= \begin{bmatrix} D_{xx} w_\alpha(\hat{z}_\alpha) & D_{xy} w_\alpha(\hat{z}_\alpha) \\ D_{yx} w_\alpha(\hat{z}_\alpha) & D_{yy} w_\alpha(\hat{z}_\alpha) \end{bmatrix} \\
&= \begin{bmatrix} D_{xx} u(\hat{x}_\alpha) - D_{xx}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) & -D_{xy}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \\ -D_{yx}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) & -D_{yy} v(\hat{y}_\alpha) - D_{yy}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \end{bmatrix} \\
&= \begin{bmatrix} D^2 u(\hat{x}_\alpha) & 0 \\ 0 & -D^2 v(\hat{y}_\alpha) \end{bmatrix} - \begin{bmatrix} D_{xx}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) & D_{xy}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \\ D_{yx}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) & D_{yy}(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \end{bmatrix} \\
&= \begin{bmatrix} D^2 u(\hat{x}_\alpha) & 0 \\ 0 & -D^2 v(\hat{y}_\alpha) \end{bmatrix} - D^2(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \text{ which would give}
\end{aligned}
$$

$$
\begin{bmatrix} D^2 u(\hat{x}_\alpha) & 0 \\ 0 & -D^2 v(\hat{y}_\alpha) \end{bmatrix} \leq D^2(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)).
$$

Since these two pieces of information due to maximum principle for C^2 functions is again actually a way of comparing the values of Du and Dv,and, D^2u and D^2v; in our quest for defining a maximum principle for semicontinuous functions, the second set of information we would like to have at these maximums $\hat{z}_\alpha$ would be the existence of some elements (p, X), (q, Y) in $J^{2,+}u(\hat{x}_\alpha)$ and $J^{2,+}v(\hat{y}_\alpha)$ respectively or in $J^{2,-}u(\hat{x}_\alpha)$ and $J^{2,-}v(\hat{y}_\alpha)$ respectively such that the following type of information

$$
\begin{aligned}
p &= q \\
\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} &\leq D^2(\varphi_\alpha(\hat{x}_\alpha, \hat{y}_\alpha)) \text{ holds.}
\end{aligned}
$$

This time, 'Theorem of Sums' will provide us with this kind of information; however, in a slightly modified manner. Hence 'Theorem of Sums' can be seen as the maximum principle for semicontinuous functions. The fact that it will provide us with a slightly modified version of the above argument will be due to the fact that the function w_α we will be considering will be semicontinuous, and hence that we have to work with its regularizations. As a consequence of this, the theorem will provide us with a result concerning closures of semijets, rather then semijets themselves. However, this will not lead to a problem, since the function F which we will evaluate using this information will be continuous, and existence of an element in the closure of a semijet would amount to existence of a corresponding convergent sequence of quadruples $(x_n, u(x_n), p_n, X_n)$ on which we can evaluate F, and pass to the limit under continuity.

Since along our way, we would have to work with regularizations of semicontinuous functions, and in particular the ones we will be employing would be sup convolutions, and that these particular regularizations are semiconvex, we will begin our presentation with introducing semiconvex functions and some of their related properties, then we will introduce sup convolutions and some of their related properties. Afterwards having equipped with this information, we will prove 'theorem of sums', in other words maximum principle for semicontinuous functions. Then, we will introduce 'doubling variables' technique and its justification, and we would be ready for the next chapter where we will investigate the conditions under which comparison holds for the above stated (DP); and once having determined them, assuming that they hold, we will almost automatically have the uniqueness result for the above stated (DP).

3.2. Semiconvex Functions

Definition 3.1 *Let G be a compact subset of R^n. A function $u(x) \in C(G)$ is called* semiconvex *if for every bounded $B \subset G$ there is a constant $\kappa_B \geq 0$ such that the function $u_B(x) = u(x) + \kappa_B |x|^2$ is convex on every convex subset of B. Then κ_B is called a semiconvexity constant for u_B.*

Lemma 3.2 *Let u be a semiconvex function in G, and $\hat{x}$ be an interior maximum of u. Then u is differentiable at $\hat{x}$ with $Du(\hat{x}) = 0$.*

Proof. Without loss of generality assume $\hat{x} = 0$.(Otherwise, we can shift $\hat{x}$ and the proof would still work.) Let $B \subset G$ be a convex, bounded neighborhood of $\hat{x}$.

Since u is semiconvex, there is a $\kappa_B \geq 0$ such that

$$u_B(x) = u(x) + \kappa_B |x|^2 \tag{3.2}$$

is convex on B. Then, the separation theorem for convex functions says that there is a $p \in R^n$ such that

$$u_B(x) \geq u_B(0) + p.x \text{ for } x \in B.$$

Since $u_B(0) = u(0) + \kappa_B |0|^2 = u(0)$, we then have

$$u_B(x) \geq u(0) + p.x. \tag{3.3}$$

Then, for all $x \in B$, by (3.2) we have $u(x) = u_B(x) - \kappa_B |x|^2$, and then by (3.3) we have

$$u(x) \geq u(0) + p.x - \kappa_B |x|^2 \tag{3.4}$$

and since $\hat{x} = 0$ is a maximum of u ($u(0) \geq u(x)$ for all $x \in B$), we have,

$$u(x) \geq u(x) + p.x - \kappa_B |x|^2 .$$

This gives us that $0 \geq p.x - \kappa_B |x|^2$, which implies

$$\kappa_B |x|^2 \geq p.x \text{ for all } x \in B. \tag{3.5}$$

If we let $\varepsilon > 0$ be sufficiently small so that $x_\varepsilon = \varepsilon p$ is in B (this is possible since B is a convex bounded neighborhood of 0), then by (3.5),

$$\begin{aligned}
\kappa_B |x_\varepsilon|^2 &\geq p.x_\varepsilon \\
\kappa_B |\varepsilon p|^2 &\geq p.\varepsilon p \text{ for } x_\varepsilon = \varepsilon p \\
\varepsilon^2 \kappa_B |p|^2 &\geq \varepsilon |p|^2 \\
\varepsilon \kappa_B |p|^2 &\geq |p|^2 \text{ upon letting } \varepsilon \to 0 \\
0 &\geq |p|^2 \text{ since } p \text{ was a fixed element of } R^n, \\
\text{Hence, } p &= 0.
\end{aligned}$$

Since $\hat{x} = 0$ is a maximum of u, $u(0) - u(x) \geq 0$ for all $x \in B$, and by (3.4) we have

$$\begin{aligned} \kappa_B |x|^2 - p.x &\geq u(0) - u(x) \text{ since } p = 0, \\ \kappa_B |x|^2 &\geq u(0) - u(x) \\ \kappa_B |x|^2 &\geq 0 \text{ since } \hat{x} = 0 \text{ is a maximum of } u. \end{aligned} \tag{3.6}$$

Letting $x = h$ and dividing by $|h|$ and taking limits as $|h| \to 0$ in (3.6) gives us that u is differentiable at $\hat{x} = 0$ and that $Du(\hat{x}) = 0$. ∎

Theorem 3.3 *(Jensen's Lemma) Let $u(x) \in C(G)$ be semiconvex. Let $G \subset R^n$ be bounded and let u have a strict local maximum in G. i.e. let*

$$\mu = \sup_G u - \sup_{\partial G} u > 0.$$

Then, there are constants $c_0 > 0$, and $\delta_0 > 0$ such that $m(M_\delta) \geq c_0\delta^2$ for all $\delta \leq \delta_0$, where m denotes the Lebesque measure and the set M_δ is defined as follows:

For $\delta > 0$, let

$$M_\delta = \left\{ \begin{array}{c} x_\alpha \in int(G) : \text{there is } p \in R^n \text{ such that } |p| \leq \delta \\ \text{and } u(x) \leq u(x_\alpha) + p.(x - x_\alpha) \text{ for all } x \in G \end{array} \right\}$$

Remark 1: If we observe the set M_δ closely we notice that the condition $u(x) \leq u(x_\alpha) + p.(x - x_\alpha)$ for all $x \in G$ implies that $u(x) - p.x \leq u(x_\alpha) - p.x_\alpha$ for all $x \in G$ which then implies that x_α is a local maximum for a function $u_s(x)$ defined as $u_s(x) = u(x) + s.x$ where $s = -p$ giving us $|s| \leq \delta$.

Remark 2: Classical maximum principle states that if a C^2 function u has a maximum at some point a interior its domain then $Du(a) = 0$ and $D^2u(a) \leq 0$. We would like to be able to have a similar information concerning the interior maximum of semiconvex functions. For this we would like to make use of some points near this interior maximum and the related information we will have about these nearby points via some limit process. However, in order to be able to do that we need to make sure that we have 'enough' of these points. Actually this theorem and the next theorem we will be stating will provide us a way of knowing this. The remark at the end of the proof of the next theorem will make this point more clear.

Proof. We will assume that $u(x) \in C^2(G)$ and $n = 2$. Let $\hat{x} \in G$ be a maximum of u. For $p \in R^2$ define

$$u_p(x) = u(x) - p.(x - \hat{x}) \text{ for } x \in G.$$

Then,

$$\sup_{x \in G} u_p(x) \geq u_p(x) \text{ for all } x \in G$$

and in particular for $x = \hat{x}$, therefore

$$\sup_{x \in G} u_p(x) \geq u_p(\hat{x}) = u(\hat{x}) - p.(\hat{x} - \hat{x}) = u(\hat{x}) = \sup_{x \in G} u(x) = \mu + \sup_{x \in \partial G} u \tag{3.7}$$

and since

$$\begin{aligned} \sup_{x \in \partial G} u_p(x) &= \sup_{x \in \partial G} (u(x) - p.(x - \hat{x})) \leq \sup_{x \in \partial G} u(x) + \sup_{x \in \partial G} (-p.(x - \hat{x})) \\ &= \sup_{x \in \partial G} u(x) - \inf_{x \in \partial G} (p.(x - \hat{x})) \end{aligned}$$

gives

$$\sup_{x \in \partial G} u_p(x) + \inf_{x \in \partial G} (p.(x - \hat{x})) \leq \sup_{x \in \partial G} u(x) \tag{3.8}$$

then by (3.7) and (3.8) we have

$$\sup_{x \in G} u_p(x) \geq \mu + \sup_{x \in \partial G} u_p(x) + \inf_{x \in \partial G} (p.(x - \hat{x})) \tag{3.9}$$

Since G is bounded let $r = \sup_{x,y \in \partial G} |x - y|$. Let $x_0 = \frac{x+y}{2}$. Then

$$G \subset \bar{B}(x_0, \frac{r}{2}) \text{ gives } \hat{x} \in \bar{B}(x_0, \frac{r}{2}) \text{ which gives } \sup_{x \in \partial G} |x - \hat{x}| \leq r,$$

and we have

$$\begin{aligned} \sup_{x \in \partial G} (-p.(x - \hat{x})) &\leq \sup_{x \in \partial G} |p|\,|x - \hat{x}| \leq |p| \sup_{x \in \partial G} |x - \hat{x}| \leq |p|\, r \\ &= |p| \sup_{x,y \in \partial G} |x - y| \end{aligned}$$

so that

$$\begin{aligned}\inf_{x\in\partial G}(p.(x-\hat{x})) &= -\sup_{x\in\partial G}(-p.(x-\hat{x}))\\ &\geq -|p|\sup_{x,y\in\partial G}|x-y|\end{aligned}$$

Hence (3.9) becomes

$$\sup_{x\in G}u_p(x) \geq \mu - |p|\sup_{x,y\in\partial G}|x-y| + \sup_{x\in\partial G}u_p(x) \tag{3.10}$$

But then

$$\sup_{x\in G}u_p(x) - \sup_{x\in\partial G}u_p(x) \geq \mu - |p|\sup_{x,y\in\partial G}|x-y|$$

Now if

$$\mu - |p|\sup_{x,y\in\partial G}|x-y| > 0$$

then u_p has an interior maximum, in other words, if

$$|p| < \frac{\mu}{\sup_{x,y\in\partial G}|x-y|}$$

then u_p has an interior maximum. Let us call

$$\delta_0 = \frac{\mu}{\sup_{x,y\in\partial G}|x-y|}.$$

Letting $\tilde{x}$ be a maximum of u_p, then

$$\begin{aligned}u_p(\tilde{x}) &\geq u_p(x) \text{ for all } x\in G\\ u_p(\tilde{x}) &= u(\tilde{x}) - p.(\tilde{x}-\hat{x}) \geq u(x) - p.(x-\hat{x}) = u_p(x)\\ u(\tilde{x}) - p.\tilde{x} + p.\hat{x} &\geq u(x) - p.x + p.\hat{x}\\ u(\tilde{x}) + p.(x-\tilde{x}) &\geq u(x) \text{ for all } x\in G \text{ gives us} \qquad (3.11)\\ \tilde{x} &\in M_\delta \text{ where } |p|\leq\delta, \text{ in particular } \tilde{x}\in M_{|p|}.\end{aligned}$$

If we define $u_r(x) = u(\tilde{x}) + p.(x-\tilde{x})$, then by (3.11) we have for all $x\in G$

$$u_r(x) \geq u(x) \text{ and } u_r(\tilde{x}) = u(\tilde{x}).$$

However, $u_r(x)$ is a linear function (since $u(\tilde{x})$ is a fixed number) and at $\tilde{x}$ the graph of u touches the graph of u_r, furthermore it is also below the graph of u_r for all $x \in G$. Since we have assumed u to be twice differentiable, we then see that at $\tilde{x}$,

$$\begin{aligned} Du(\tilde{x}) &= Du_r(\tilde{x}) = p \\ D^2u(\tilde{x}) &\leq Du_r(\tilde{x}) = 0. \end{aligned}$$

Hence, letting $p \in \bar{B}_\delta$ such that $\delta < \delta_0$, then $|p| < \delta_0$, so the above defined function u_p has an interior maximum $\tilde{x}$ in the set M_δ, and also $p = Du(\tilde{x})$, then this gives us that $p \in Du(M_\delta)$, and hence we have

$$\bar{B}_\delta \subset Du(M_\delta). \tag{3.12}$$

On the other hand, if $x_0 \in M_\delta$ then there exists a p such that $|p| \leq \delta$, and when u is twice differentiable, which is the case we have assumed at the very beginning, via the construction of the above function $u_r(x)$ we can see that $Du(x_0) = p$. i.e. $Du(x_0) \in \bar{B}_\delta$ and hence

$$Du(M_\delta) \subset \bar{B}_\delta. \tag{3.13}$$

So that, from (3.12) and (3.13), we have $Du(M_\delta) = \bar{B}_\delta$ for all $\delta < \delta_0$.

In order to be able to derive a conclusion about Lebesque measure of M_δ we will use change of variables formula. To be able to do that we need to define a diffeomorphism form M_δ onto some set. We can do this through using the set $Du(M_\delta) = \bar{B}_\delta$ since from this equation we see that Du maps M_δ onto $\bar{B}_\delta$.

Now, for $\varepsilon > 0$ define

$$\xi_\varepsilon(x) = Du(x) - \varepsilon x \text{ for } x \in G.$$

If $x \in M_\delta$ then

$$u(y) \leq u(x) + p.(y - x) \text{ for all } y \in G.$$

This implies that

$$u(y) \leq u(x) + Du(x).(y - x) \tag{3.14}$$

since $p = Du(x)$ when $x \in M_\delta$ and u is smooth.

On the other hand if also $y \in M_\delta$ then

$$u(x) \leq u(y) + Du(y).(x - y) \tag{3.15}$$

Summing up (3.14) and (3.15) we get

$$\begin{aligned} u(y) + u(x) &\leq u(x) + u(y) + Du(y).(x - y) + Du(x).(y - x) \\ 0 &\geq (Du(x) - Du(y))(x - y) \text{ for all } x, y \in M_\delta. \end{aligned} \tag{3.16}$$

Hence

$$\begin{aligned} (\xi_\varepsilon(x) - \xi_\varepsilon(y))(x - y) &= (Du(x) - \varepsilon x - Du(y) - \varepsilon y)(x - y) \\ &= (Du(x) - Du(y))(x - y) - \varepsilon(x - y)(x - y) \\ &\leq -\varepsilon \left|x - y\right|^2 \text{ for all } x, y \in M_\delta \text{ by (3.16)}. \end{aligned}$$

Hence

$$(\xi_\varepsilon(x) - \xi_\varepsilon(y))(x - y) \leq -\varepsilon \left|x - y\right|^2 \text{ for all } x, y \in M_\delta. \tag{3.17}$$

This implies that ξ_ε is a one-to-one mapping of M_δ (since otherwise assume we have $x, y \in M_\delta$ such that $x \neq y$ but $\xi_\varepsilon(x) = \xi_\varepsilon(y)$. Then we would have a contradiction $0 \leq -\varepsilon \left|x - y\right|^2 > 0$ by (3.17).) Moreover, Jacobian $J_{\xi_\varepsilon} = Det(D\xi_\varepsilon) = Det(D^2u - \varepsilon) < 0$ since$D^2u \leq 0$. Hence J_{ξ_ε} is nonzero. Hence ξ_ε is a diffeomorphism from M_δ onto$\xi_\varepsilon(M_\delta)$ and we can use change of variables formula which states that

$$\int_{\xi_\varepsilon(M_\delta)} d\xi_\varepsilon(x) = \int_{M_\delta} \left|Det(D\xi_\varepsilon)\right| dm(x) \tag{3.18}$$

so that we have

$$\int_{\xi_\varepsilon(M_\delta)} d\xi_\varepsilon(x) = \int_{M_\delta} \left|Det(D^2u - \varepsilon)\right| dm(x) \tag{3.19}$$

Letting $\varepsilon \to 0$ we have

$$\int_{Du(M_\delta)} d\xi(x) = \int_{M_\delta} \left|Det(D^2u(x))\right| dm(x)$$

since boundary of $Du(M_\delta)$ has measure 0.

Letting

$$I = \int_{M_\delta} \left| Det(D^2 u(x)) \right| dm(x)$$

then

$$I = \int_{B_\delta} d\xi(x) = m(\bar{B}_1)\delta^2 \tag{3.20}$$

where $m(\bar{B}_1)$ is the measure of the unit ball in $R^2$and $\delta < \delta_0$.

On the other hand, if we let

$$\lambda = \sup \left\{ Det(-D^2 u(x)) : x \in G,\, D^2 u(x) \leq 0 \right\}$$

(this supremum exists since u is semiconvex), then we have

$$I = \int_{M_\delta} \left| Det(D^2 u(x)) \right| dm(x) = \int_{M_\delta} Det(-D^2 u(x)) dm(x) \leq \lambda m(M_\delta) \tag{3.21}$$

since $D^2 u(x) \leq 0$ for all $x \in M_\delta$. From (3.20) and (3.21) we obtain $m(\bar{B}_1)\delta^2 \leq \lambda m(M_\delta)$. Letting $c_0 = \frac{m(\bar{B}_1)}{\lambda}$, we have the desired result that $m(M_\delta) \geq c_0 \delta^2$ for all $\delta \leq \delta_0$.

We have assumed that u was twice differentiable. When u is not twice differentiable an approximation via mollification with smooth functions u_m that have the same semiconvexity constant with u and that converge uniformly to u on G results in corresponding sets K_m to obey the above results for large m and then since $\limsup_{m\to\infty} M_\delta^m = \cap_{M=1}^{\infty} \cup_{m=M}^{\infty} M_\delta^m \subset M_\delta$ holds we have the desired result. ∎

Theorem 3.4 *(Alexandrov's Theorem) Let $u : R^n \to R$ be a semiconvex function. Then u is twice differentiable almost everywhere (i.e. except possibly on a set of measure 0) on R^n.*

We will accept this classical result without proof.

Remark: Jensen's lemma can be viewed as generalization of maximum principle for semiconvex functions. Let us try to explain this by considering a semiconvex function u and assuming that $\hat{x}$ is a strict interior maximum of u. In this case, $D^2u(\hat{x})$ might not exist. However, by letting $\delta = \frac{1}{m}$ for positive integers m, we know by Jensen's lemma that the set M_δ, i.e. $M_{\frac{1}{m}}$ is of positive measure for each m. Alexandrov's theorem states that u is twice differentiable almost everywhere. Therefore, for each m the set $M_{\frac{1}{m}}$ contains points that are twice differentiable. Letting x_m be a twice differentiable point of u in $M_{\frac{1}{m}}$, then as $m \to \infty$, $x_m \to \hat{x}$. Repeating an argument in the proof above by defining the function $u_r(y) = u(x_m) + p(y - x_m)$, for each m, then by (3.11) we have for all $y \in G$

$$u_r(y) \geq u(y) \text{ and } u_r(x_m) = u(x_m).$$

Since $u_r(y)$ is a linear function (because $u(x_m)$ is a fixed number) and at x_m the graph of u touches the graph of u_r, and that it is also below the graph of u_r for all $y \in G$, in addition also because u is twice differentiable at each x_m, we then have at each x_m,

$$\begin{aligned} Du(x_m) &= Du_r(x_m) = p \text{ and} \\ D^2u(x_m) &\leq Du_r(x_m) = 0. \end{aligned}$$

Since $|p| \leq \frac{1}{m}$, we then have $|Du(x_m)| \leq \frac{1}{m}$. Hence even if we do not know whether u is twice differentiable at $\hat{x}$, we at least know that there is a sequence $x_m \to \hat{x}$, for which

$$|Du(x_m)| \leq \frac{1}{m} \text{ and } D^2u(x_m) \leq 0 \text{ holds,}$$

in other words for which

$$\lim_{m\to\infty} |Du(x_m)| \leq 0 \text{ and } D^2u(x_m) \leq 0.$$

Within this perspective we can see Jensen's lemma as a generalized maximum principle for semiconvex functions.

3.3. Sup Convolution

Sup convolutions will allow us to be able to regularize merely semicontinuous functions. Throughout this section we will assume $u : \Omega \to R^n$ to be bounded from above and we will extend u to R^n by letting u take the value of $-\infty$ on unimportant sets, in other words in our case we will let $u(x) = -\infty$ for $x \notin \Omega$. This way, we will be considering upper semicontinuous functions $u : R^n \to R \cup \{-\infty\}$.

Definition 3.5 *Let $\Omega \subset R^n$ be closed, $\varepsilon > 0$, and $u : \Omega \to R^n$ be such that $u \in USC(\Omega)$. For $y \in R^n$, let*

$$\hat{u}_\varepsilon(y) = \sup_{x\in\Omega}(u(x) - \frac{1}{2\varepsilon}|x-y|^2).$$

This process of constructing $\hat{u}_\varepsilon$'s is called 'sup convolution'.

It provides us with an approximation of u in the sense that $lim_{\varepsilon\to 0}\hat{u}_\varepsilon(y) = u(y)$ for $y \in R^n$. In the case that u is continuous this convergence is uniform. Furthermore, it is also a regularization of u since we will see in a moment that $\hat{u}_\varepsilon$ is semiconvex.

We will start by giving some technical lemmas:

Lemma 3.6 *$\hat{u}_\varepsilon(y) \geq u(y)$.*

Proof. Clearly,

$$\hat{u}_\varepsilon(y) = \sup_{x\in\Omega}(u(x) - \frac{1}{2\varepsilon}|x-y|^2) \geq u(x) - \frac{1}{2\varepsilon}|x-y|^2 \text{ for all } x \in \Omega,$$

in particular for $x = y$, therefore

$$\hat{u}_\varepsilon(y) \geq u(x) - \frac{1}{2\varepsilon}|x-y|^2 = u(y).$$

$\blacksquare$

Lemma 3.7

$$\hat{u}_\varepsilon(y) = \sup\left\{u(x) - \frac{1}{2\varepsilon}|x-y|^2 : |x-y| \leq 2\sqrt{\varepsilon M}\right\}$$

where $M = \|u\| = \sup_\Omega u(x)$. Therefore it is attained at some $y^* \in \Omega$ with $|y^* - y| \leq 2\sqrt{\varepsilon M}$, and thus supremum is finite.

Proof. Fix $y \in \Omega$. Let $|x-y| > 2\sqrt{\varepsilon M}$, then

$$u(x) - \frac{1}{2\varepsilon}|x-y|^2 < u(x) - 2M \le -M \le u(y) \le \hat{u}_\varepsilon(y)$$

hence supremum cannot occur on the set $\left\{x \in \Omega : |x-y| > 2\sqrt{\varepsilon M}\right\}$. Since the set

$$\left\{x \in \Omega : |x-y| \le 2\sqrt{\varepsilon M}\right\}$$

is compact and $u \in USC(\Omega)$, u attains its supremum on this set. ■

Lemma 3.8 *$\hat{u}_\varepsilon$ is continuous.*

Proof. Let $|y-z| < h \le 1$.

$$\begin{aligned}
\hat{u}_\varepsilon(y) &= u(y^*) - \frac{1}{2\varepsilon}|y^*-y|^2 \\
&= u(y^*) - \frac{1}{2\varepsilon}|y^*-z|^2 + \frac{1}{2\varepsilon}\left(|y^*-z|^2 - |y^*-y|^2\right) \\
&\le \hat{u}_\varepsilon(z) + \frac{1}{2\varepsilon}\left(|y^*-z|^2 - |y^*-y|^2\right) \\
&= \hat{u}_\varepsilon(z) + \frac{1}{2\varepsilon}\left(|y^*-z| + |y^*-y|\right)\left(|y^*-z| - |y^*-y|\right) \\
&\le \hat{u}_\varepsilon(z) + \frac{1}{2\varepsilon}\left(|y^*-y| + |y^*-y| + |y-z|\right)|y-z| \\
&< \hat{u}_\varepsilon(z) + \frac{1}{2\varepsilon}\left(4\sqrt{\varepsilon M} + h\right)|y-z| \\
\hat{u}_\varepsilon(y) - \hat{u}_\varepsilon(z) &< \frac{1}{2\varepsilon}\left(4\sqrt{\varepsilon M} + 1\right)|y-z| .
\end{aligned}$$

But by symmetry the opposite also holds, none of the constants depend on y. So

$$|\hat{u}_\varepsilon(y) - \hat{u}_\varepsilon(z)| < \frac{1}{2\varepsilon}\left(4\sqrt{\varepsilon M} + 1\right)|y-z|$$

Given $\varepsilon > 0$, let $h \le \frac{2\varepsilon\sqrt[2]{\varepsilon}}{4M+1}$ then $|y-z| < \frac{2\varepsilon\sqrt[2]{\varepsilon}}{4M+1}$, and $|\hat{u}_\varepsilon(y) - \hat{u}_\varepsilon(z)| < \varepsilon$. ■

Lemma 3.9 *If $s \le r$ then $\hat{s}_\varepsilon \le \hat{r}_\varepsilon$.*

Proof. $s \le r$ implies that $s(x) \le r(x)$, then fixing $y \in \Omega$ we have

$$\begin{aligned}
s(x) - \frac{1}{2\varepsilon}|x-y|^2 &\le r(x) - \frac{1}{2\varepsilon}|x-y|^2 \\
\sup_{x\in\Omega}(s(x) - \frac{1}{2\varepsilon}|x-y|^2) &\le \sup_{x\in\Omega}(r(x) - \frac{1}{2\varepsilon}|x-y|^2)
\end{aligned}$$

so that, $\hat{s}_\varepsilon(y) \leq \hat{r}_\varepsilon(y)$. Since y was arbitrary this holds for all $y \in \Omega$. Hence $\hat{s}_\varepsilon \leq \hat{r}_\varepsilon$. $\blacksquare$

Proposition 3.10 *Let $\Omega \subset R^n$ be compact and $u \in C(\Omega)$, then $\hat{u}_\varepsilon(y) \to u(y)$ uniformly on Ω as $\varepsilon \to 0$.*

Proof. By the first one of the above technical lemmas, $\hat{u}_\varepsilon(y) \geq u(y)$. Conversely, let y^* be a point at which supremum is attained. Since u is continuos, given $h > 0$, there exists $d > 0$ such that $u(x) \leq u(y) + h$. Let $\varepsilon < \frac{d^2}{4M}$. Then, since $|y^* - y| \leq 2\sqrt{\varepsilon M}$, we will have $|y^* - y| < d$, then $u(y^*) \leq u(y) + h$ holds. But then,

$$\hat{u}_\varepsilon(y) = u(y^*) - \frac{1}{2\varepsilon}|y^* - y|^2 \leq u(y) + h - \frac{1}{2\varepsilon}|y^* - y|^2 \leq u(y) + h.$$

Hence for $\varepsilon < \frac{d^2}{4M}$ we will have $0 \leq \hat{u}_\varepsilon(y) - u(y) \leq h$. $\blacksquare$

Proposition 3.11 *Let $\Omega \subset R^n$ be closed. Then, $\hat{u}_\varepsilon(y)$ is semiconvex on Ω .*

Proof. We will give the proof using definition of semiconvexity. In other words we want to find a $\kappa_\Omega > 0$ such that the function defined by

$$\check{u}_\varepsilon(y) = \hat{u}_\varepsilon(y) + \kappa_\Omega |y|^2$$

is convex on every convex subset of Ω.

Claim 3.12 *$\kappa_\Omega = \frac{1}{2\varepsilon}$ i.e. $\check{u}_\varepsilon(y) = \hat{u}_\varepsilon(y) + \frac{1}{2\varepsilon}|y|^2$ is convex on every convex subset of Ω.*

We will show that for every $y + h$, $y - h$, and $y \in \Omega$, we have

$$\check{u}_\varepsilon(y + h) + \check{u}_\varepsilon(y - h) - 2\check{u}_\varepsilon(y) \geq 0.$$

Fix $y \in \Omega$. Let y^* be the point supremum is achieved then, we have

$$\begin{aligned} \hat{u}_\varepsilon(y + h) &\geq u(y^*) - \frac{1}{2\varepsilon}|y^* - (y + h)|^2 \text{ and} \\ \hat{u}_\varepsilon(y - h) &\geq u(y^*) - \frac{1}{2\varepsilon}|y^* - (y - h)|^2 \end{aligned}$$

Hence,

$$
\begin{aligned}
& \check{u}_\varepsilon(y+h) + \check{u}_\varepsilon(y-h) - 2\check{u}_\varepsilon(y) \\
= \;& \hat{u}_\varepsilon(y+h) + \frac{1}{2\varepsilon}|y+h|^2 + \hat{u}_\varepsilon(y-h) + \frac{1}{2\varepsilon}|y-h|^2 - 2\hat{u}_\varepsilon(y) - 2\frac{1}{2\varepsilon}|y|^2 \\
\geq \;& u(y^*) - \frac{1}{2\varepsilon}|y^* - (y+h)|^2 + \frac{1}{2\varepsilon}|y+h|^2 + u(y^*) - \frac{1}{2\varepsilon}|y^* - (y-h)|^2 \\
& + \frac{1}{2\varepsilon}|y-h|^2 - 2u(y^*) + 2\frac{1}{2\varepsilon}|y^* - y|^2 - 2\frac{1}{2\varepsilon}|y|^2 \\
= \;& \frac{1}{2\varepsilon}\left\{ \begin{array}{c} (|y+h|^2 + |y-h|^2 - 2|y|^2) \\ -(|y^* - (y+h)|^2 + |y^* - (y-h)|^2 - 2|y^* - y|^2 \end{array} \right\} \\
\geq \;& \frac{1}{2\varepsilon}\left\{ \begin{array}{c} |y+h|^2 + |y-h|^2 - |y|^2 - |y^*|^2 \\ -|y+h|^2 - |y^*|^2 - |y-h|^2 + 2|y^*| + 2|y|^2 \end{array} \right\} \\
= \;& 0
\end{aligned}
$$

Hence $\check{u}_\varepsilon(y+h) + \check{u}_\varepsilon(y-h) - 2\check{u}_\varepsilon(y) \geq 0$. Since y was arbitrary this holds true on Ω.

$\blacksquare$

At this point, we will see the important role sup convolution of a subsolution plays. However, before that we need another theorem which is important for sup convolutions and which gives us a relation between second order semijets of a sup convolution of a function u and second order semijets of the function u itself. More specifically, it will tell us that if (p, X) is in the second order semijet of $\hat{u}_\varepsilon$ at x_0 then (p, X) will also be in a second order semijet of u but this time at $x_0 + \varepsilon p$. For convenience of notation from now on we will drop the lower index ε of sup convolution $\hat{u}_\varepsilon$ of u, and hence write $\hat{u}$ instead of $\hat{u}_\varepsilon$, however we will keep in mind that $\hat{u}$ depends on ε.

In the literature, the theorem below is referred to as "magic properties of sup convolution".

Theorem 3.13 *Let $u : R^n \to R$ be $USC(R^n)$. If $(p, X) \in J^{2,+}\hat{u}(y_0)$, and T is any real $n \times n$ matrix, then*

1) $(p, \frac{1}{\varepsilon}(I - T^*)(I - T) + T^*XT) \in J^{2,+}u(y_0 + \varepsilon p)$, (here T^* denotes adjoint of T).

2) $x_0 = y_0 + \varepsilon p$ *is the unique point such that* $\hat{u}(y_0) = u(x_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2$.

3) If we choose $T = I$*, then* $(p, X) \in J^{2,+}u(y_0 + \varepsilon p)$.

Proof. We will give the proof in $n = 2$, and will prove 2) first.

Let $(p, X) \in J^{2,+}\hat{u}(y_0)$. Then as we know from Chapter 2, there exits a $\varphi \in C^2(R^2)$ such that $\hat{u} - \varphi$ assumes its maximum at y_0, i.e. $\hat{u}(y) - \varphi(y) \leq \hat{u}(y_0) - \varphi(y_0)$ for all $y \in R^2$, and $D\varphi(y_0) = p$ and $D^2\varphi(y_0) = X$.

Let y^* be the point supremum is achieved, then we have

$$\begin{aligned}
u(x) - \frac{1}{2\varepsilon}|x - y|^2 - \varphi(y) &\leq \sup_{x\in\Omega}(u(x) - \frac{1}{2\varepsilon}|x - y|^2) - \varphi(y) \\
&= \hat{u}(y) - \varphi(y) \leq \hat{u}(y_0) - \varphi(y_0) \\
&= \sup_{x\in\Omega}(u(x) - \frac{1}{2\varepsilon}|x - y_0|^2) - \varphi(y_0) \\
&\leq u(y^*) - \frac{1}{2\varepsilon}|y^* - y_0|^2 - \varphi(y_0) \text{ for all } x \in R^2.
\end{aligned}$$

Hence we have

$$u(x) - \frac{1}{2\varepsilon}|x - y|^2 - \varphi(y) \leq u(y^*) - \frac{1}{2\varepsilon}|y^* - y_0|^2 - \varphi(y_0) \text{ for all } x \in R^2. \tag{3.22}$$

Then by letting $x = y^*$ in (3.22), we see that

$$\begin{aligned}
u(y^*) - \frac{1}{2\varepsilon}|y^* - y|^2 - \varphi(y) &\leq u(y^*) - \frac{1}{2\varepsilon}|y^* - y_0|^2 - \varphi(y_0) \text{ for all } x \in R^2 \\
\frac{1}{2\varepsilon}|y^* - y|^2 + \varphi(y) &\geq \frac{1}{2\varepsilon}|y^* - y_0|^2 + \varphi(y_0).
\end{aligned}$$

Since y is arbitrary this means that the function $\alpha(y) = \frac{1}{2\varepsilon}|y^* - y|^2 + \varphi(y)$ has a minimum at y_0. But this function is C^2, hence we can apply first and second derivative tests to see that

$$\begin{aligned}
D\alpha(y_0) &= 0 \text{ hence } -\frac{1}{\varepsilon}(y^* - y_0) + D\varphi(y_0) = 0 \text{ so that} \\
y^* &= \varepsilon D\varphi(y_0) + y_0 = \varepsilon p + y_0 \text{ uniquely, and that}
\end{aligned}$$

$$\begin{aligned}
D^2\alpha(y_0) &\geq 0 \text{ hence } \frac{1}{\varepsilon}I + D^2\varphi(y_0) \geq 0 \text{ hence} \\
D^2\varphi(y_0) &\geq -\frac{1}{\varepsilon}I \text{ so that } X \geq -\frac{1}{\varepsilon}.
\end{aligned}$$

(**Remark:** This last inequality, $X \geq -\frac{1}{\varepsilon}I$ will turn out to be important later on).

Since $y^* = \varepsilon p + y_0$ uniquely, and we have taken y^* as a maximum of $\hat{u}$, then y^* is the unique point for which $\hat{u}(y_0) = u(y^*) - \frac{1}{2\varepsilon}|y^* - y_0|^2$ holds. Next we will prove 1).

In (3.22) if we let $y = T(x - y^*) + y_0$ we have for all $x \in R^2$

$$\begin{aligned} & u(x) - \frac{1}{2\varepsilon}|(I-T)x + Ty^* - y_0|^2 - \varphi(Tx - Ty^* + y_0) \\ \leq\ & u(y^*) - \frac{1}{2\varepsilon}|y^* - y_0|^2 - \varphi(y_0) \end{aligned}$$

But then letting

$$\beta(x) = \frac{1}{2\varepsilon}|(I-T)x + Ty^* - y_0|^2 + \varphi(Tx - Ty^* + y_0)$$

we have

$$u(x) - \beta(x) \leq u(y^*) - \beta(y^*) \text{ for all } x \in R^2.$$

In other words y^* is a maximum point of $u - \beta$, and since β is C^2, we have

$$(D\beta(y^*), D^2\beta(y^*)) \in J^{2,+}u(y^*)$$

But then since

$$D\beta(x) = \frac{1}{\varepsilon}((I-T)x + Ty^* - y_0)(I-T) + D\varphi(Tx - Ty^* + y_0),$$

we have

$$\begin{aligned} D\beta(y^*) &= \frac{1}{\varepsilon}((I-T)y^* + Ty^* - y_0)(I-T) + D\varphi(Ty^* - Ty^* + y_0)T \\ &= \frac{1}{\varepsilon}(y^* - y_0)(I-T) + D\varphi(y_0)T \\ &= \frac{1}{\varepsilon}(\varepsilon p)(I-T) + pT = p - pT + pT = p, \end{aligned}$$

and since

$$D^2\beta(x) = \frac{1}{\varepsilon}(I-T)^*(I-T) + T^* D^2\varphi(Tx - Ty^* + y_0)T,$$

we have

$$\begin{aligned} D^2\beta(y^*) &= \frac{1}{\varepsilon}(I-T)^*(I-T) + T^* D^2\varphi(Ty^* - Ty^* + y_0)T \\ &= \frac{1}{\varepsilon}(I-T)^*(I-T) + T^*XT. \end{aligned}$$

Hence

$$(p, \frac{1}{\varepsilon}(I - T^*)(I - T) + T^* X T) \in J^{2,+}u(y^*). \tag{3.23}$$

To prove 3) we then let $T = I$ in (3.23), and have $(p, X) \in J^{2,+}u(y^*)$ as desired. ∎

Corollary 3.14 *If* $(0, X) \in \bar{J}^{2,+}\hat{u}(0)$, *then* $(0, X) \in \bar{J}^{2,+}u(0)$.

Proof. Let $(0, X) \in \bar{J}^{2,+}\hat{u}(0)$. But then this means that

$$\begin{aligned}
\text{there exists } (y_n, p_n, X_n) &\in R^2 \times R^2 \times S(N) \text{ such that} \\
(p_n, X_n) &\in J^{2,+}\hat{u}(y_n) \text{ and} \\
(y_n, \hat{u}(y_n), p_n, X_n) &\to (0, \hat{u}(0), 0, X) \text{ as } n \to \infty.
\end{aligned}$$

But then we know from the previous theorem that $(p_n, X_n) \in J^{2,+}u(x_n)$ where $x_n = y_n + \varepsilon p_n$ and

$$\hat{u}(y_n) = u(x_n) - \frac{1}{2\varepsilon} |x_n - y_n|^2 \tag{3.24}$$

In order to be able to show that $(0, X) \in \bar{J}^{2,+}u(0)$ holds we will claim that

$$(x_n, u(x_n), p_n, X_n) \to (0, u(0), 0, X) \text{ as } n \to \infty.$$

Since as $n \to \infty$, $y_n \to 0$, $p_n \to 0$, $X_n \to 0$ is given, we can easily deduce that $x_n \to 0$, hence we need to show that $u(x_n) \to u(0)$.

Since as $n \to \infty$, $\frac{1}{2\varepsilon} |x_n - y_n|^2 \to 0$, and also since it is given that $\hat{u}(y_n) \to \hat{u}(0)$, we know by (3.24) that

$$u(x_n) \to \hat{u}(0) \tag{3.25}$$

So we will be done if we can show that $\hat{u}(0) = u(0)$. But by the first one of the above technical lemmas we know that $\hat{u}(0) \geq u(0)$, hence we are left to show that $\hat{u}(0) \leq u(0)$. Since u is upper semicontinuous, as $x_n \to 0$, we have

$$u(0) \geq \limsup_{n \to \infty} u(x_n).$$

But then by (3.25) we know that $u(x_n) \to \hat{u}(0)$ which implies that

$$\limsup_{n\to\infty} u(x_n) = \hat{u}(0),$$

thus, we get $u(0) \geq \hat{u}(0)$ as desired. As a result, there exits

$$\begin{aligned}
(x_n, p_n, X_n) &\in R^2 \times R^2 \times S(N) \text{ such that} \\
(p_n, X_n) &\in J^{2,+}u(x_n) \text{ and} \\
(x_n, u(x_n), p_n, X_n) &\to (0, u(0), 0, X).
\end{aligned}$$

This means that $(0, X) \in \bar{J}^{2,+}u(0)$. ∎

Corollary 3.15 *Let $F(u, Du, D^2u)$ be proper and let u be a subsolution of*

$$F(u, Du, D^2u) = 0.$$

Then $\hat{u}$ is also a subsolution of $F(u, Du, D^2u) = 0$.

Proof. We need to show that for every $y \in R^n$ and $(p, X) \in J^{2,+}\hat{u}(y)$,

$$F(\hat{u}(y), p, X) \leq 0.$$

Let $y_0 \in R^n$, and $(p_0, X_0) \in J^{2,+}\hat{u}(y_0)$. Then by the above theorem we know that

$$(p_0, X_0) \in J^{2,+}u(y_0 + \varepsilon p_0) = J^{2,+}u(x_0) \text{ and} \tag{3.26}$$

$$\hat{u}(y_0) = u(x_0) - \frac{1}{2\varepsilon}|x_0 - y_0|^2 \text{ upon denoting } x_0 = y_0 + \varepsilon p_0. \tag{3.27}$$

Since u is a subsolution of $F(u, Du, D^2u) = 0$ we have

$$F(u(x_0), p_0, X_0) \leq 0.$$

But then this implies that

$$F(\hat{u}(y_0) + \frac{1}{2\varepsilon}|x_0 - y_0|^2, p_0, X_0) \leq 0$$

since by (3.26)

$$u(x_0) = \hat{u}(y_0) + \frac{1}{2\varepsilon} |x_0 - y_0|^2 .$$

Furthermore, letting

$$\alpha(y) = \hat{u}(y) + \frac{1}{2\varepsilon} |x_0 - y|^2 ,$$

we have for every $y \in R^n$,

$$\hat{u}(y) \leq \alpha(y)$$

which implies that

$$\hat{u} \leq \alpha$$

and if we combine this with the fact that F is proper, we get

$$F(u, p, X) \leq F(\alpha, p, X)$$

which gives us

$$\begin{aligned} F(\hat{u}(y_0), p_0, X_0) &\leq F(\alpha(y_0), p_0, X_0) \\ &= F(\hat{u}(y_0) + \frac{1}{2\varepsilon} |x_0 - y_0|^2 , p_0, X_0) \\ &\leq 0 \end{aligned}$$

hence

$$F(\hat{u}(y_0), p_0, X_0) \leq 0.$$

Since y_0 was arbitrary we have,

$$F(\hat{u}(y), p, X) \leq 0 \text{ for every } y \in R^n \text{ and } (p, X) \in J^{2,+}\hat{u}(y)$$

hence $\hat{u}$ is also a subsolution of

$$F(u, Du, D^2u) = 0.$$

$\blacksquare$

Example 3.16 *Let $B \in S(2)$ such that $B < \frac{1}{2\varepsilon} I$, and let $u(x) = \langle Bx, x \rangle$. Then $\hat{u}(y) = \langle B(I - 2\varepsilon B)^{-1} y, y \rangle$.*

Proof. Let in

$$\hat{u}(y) = \sup(u(x) - \frac{1}{2\varepsilon}|x-y|^2)$$

supremum be achieved at y^*, then we have

$$\begin{aligned} \hat{u}(y) &= u(y^*) - \frac{1}{2\varepsilon}|y^* - y|^2 \\ &= \langle By^*, y^* \rangle - \frac{1}{2\varepsilon}|y^* - y|^2. \end{aligned}$$

Let

$$\alpha(x) = \langle Bx, x \rangle - \frac{1}{2\varepsilon}|x-y|^2,$$

then $\alpha(x)$ is twice differentiable with maximum at y^*. Hence

$$D\alpha(y^*) = 0 \text{ gives } 2By^* - \frac{1}{\varepsilon}(y^* - y) = 0 \text{ which gives } y^* = (I - 2\varepsilon B)^{-1}y,$$

then

$$\begin{aligned} \frac{1}{2\varepsilon}|y^* - y|^2 &= \frac{1}{2\varepsilon}|y^* - (I - 2\varepsilon B)y^*|^2 \\ &= \frac{1}{2\varepsilon}|(I - I + 2\varepsilon B)y^*|^2 \\ &= \frac{1}{2\varepsilon}|2\varepsilon By^*|^2 = \frac{(2\varepsilon)(2\varepsilon)}{2\varepsilon}|By^*|^2 = (2\varepsilon)\langle By^*, By^* \rangle \end{aligned}$$

and then

$$\begin{aligned} \hat{u}(y) &= \langle By^*, y^* \rangle - \frac{1}{2\varepsilon}|y^* - y|^2 \\ &= \langle By^*, y^* \rangle - (2\varepsilon)\langle By^*, By^* \rangle \\ &= \langle By^*, y^* - 2\varepsilon By^* \rangle = \left\langle B(I - 2\varepsilon B)^{-1}y, y \right\rangle. \end{aligned}$$

∎

The theorem above has given us a relation between second order semijets of a sup convolution of a function u and second order semijets of the function u itself. However, we would like to know also when an element that we can control exists in the closure of second order semijet of a sup convolution of a function u, so that via this element we can pass to closure of second order semijet of the function u itself. The following theorem will give us this result.

Theorem 3.17 *Let $u(y) \in C(R^n)$, $B \in S(N)$, and let $u(y) + \frac{1}{2\varepsilon}|y|^2$ be convex. Then if*

$$u(y) - \frac{1}{2}\langle By, y\rangle$$

has a maximum at $y = 0$, in other words if

$$\max_{y \in R^n}(u(y) - \frac{1}{2}\langle By, y\rangle) = u(0),$$

then there is an $X \in S(N)$ such that

$$(0, X) \in \bar{J}^2 u(0) = \bar{J}^{2,+}u(0) \cap \bar{J}^{2,-}u(0) \text{ and } -\frac{1}{\varepsilon}I \leq X \leq B.$$

Proof. Our aim in this proof is to be able to find a sequence $(p_n, X_n) \in J^{2,+}u(y_n)$ such that

$$y_n \to 0,\ p_n \to 0,\ X_n \to X \text{ and } u(y_n) \to u(0) \text{ holds,}$$

(since u is continuous, the latter will hold automatically once $y_n \to 0$ holds), and a sequence $(\tilde{p}_n, \tilde{X}_n) \in J^{2,-}u(\tilde{y}_n)$ such that

$$\tilde{y}_n \to 0,\ \tilde{p}_n \to 0,\ \tilde{X}_n \to X \text{ and } u(\tilde{y}_n) \to u(0) \text{ holds.}$$

We will localize our attention around $y = 0$. Hence let $G = \bar{B}(0, r)$. Since u is semiconvex on G (with semiconvexity constant $\kappa_G = \frac{1}{2\varepsilon}$) by the convexity assumption given, by Alexandrov's Theorem we know that u is twice differentiable almost everywhere on G. Let Γ be the set of points of G where u is twice differentiable. Consider the function

$$\alpha(y) = u(y) - \frac{1}{2}\langle By, y\rangle - |y|^4.$$

Our first claim is that $\alpha(y)$ has a strict maximum at $y = 0$. Assume there exists $y_1 \neq 0$ such that $\alpha(y_1) = \alpha(0)$,then we have

$$u(y_1) - \frac{1}{2}\langle By_1, y_1\rangle - |y_1|^4 = u(0) - \frac{1}{2}\langle B0, 0\rangle - |0|^4$$

implying

$$u(y_1) - \frac{1}{2}\langle By_1, y_1\rangle - |y_1|^4 = u(0)$$

implying

$$u(y_1) - \frac{1}{2}\langle By_1, y_1\rangle = u(0) + |y_1|^4 > u(0),$$

but then this contradicts to

$$\max_{y \in R^2}(u(y) - \frac{1}{2}\langle By, y\rangle) = u(0).$$

Hence $\alpha(y)$ has a strict maximum at $y = 0$.

Our second claim is that $\alpha(y)$ is semiconvex on G with semiconvexity constant $\kappa_G = \frac{\gamma + \frac{1}{\varepsilon} + 14r^2}{2}$ where γ is an eigenvalue of B. In other words, we claim that the function

$$\beta(y) = \alpha(y) + \frac{\gamma + \frac{1}{\varepsilon} + 14r^2}{2}|y|^2$$

is convex on every convex subset of G. Now,

$$\beta(y) = u(y) - \frac{1}{2}\langle By, y\rangle - |y|^4 + \frac{\gamma}{2}|y|^2 + \frac{1}{2\varepsilon}|y|^2 + \frac{14r^2}{2}|y|^2.$$

Since $u(y) + \frac{1}{2\varepsilon}|y|^2$ is convex we need only to show that

$$\varphi(y) = -\frac{1}{2}\langle By, y\rangle - |y|^4 + \frac{\gamma}{2}|y|^2 + \frac{14r^2}{2}|y|^2 \text{ is convex.}$$

$$\begin{aligned}
\varphi(y) &= -\frac{1}{2}\langle By, y\rangle - |y|^4 + \frac{\gamma}{2}|y|^2 + \frac{14r^2}{2}|y|^2 \\
&= -\frac{1}{2}\langle By, y\rangle + \frac{\gamma}{2}\langle y, y\rangle - |y|^4 + \frac{14r^2}{2}|y|^2 \\
&= -\frac{1}{2}\langle (B - \gamma I)y, y\rangle - |y|^4 + \frac{14r^2}{2}|y|^2 \\
&= -|y|^4 + \frac{14r^2}{2}|y|^2
\end{aligned}$$

and since this function is twice differentiable on G it suffices to check its Hessian.

$$\begin{aligned}
D^2\varphi(y) &= -12\begin{bmatrix} y_1^2 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & y_n^2 \end{bmatrix} + \\
& r^2 I
\end{aligned}$$

and since $|y| \leq r$ in G, we have $\langle D^2u\,(x)\,v, v\rangle \geq 0$ for every $v \in R^n$, hence we have $D^2\varphi(y) \geq 0$ on G, and hence $\varphi(y)$ is convex which implies that $\beta(y)$ is convex on G. Thus, $\alpha(y)$ is semiconvex on G. Then we can use Jensen's Theorem for $\alpha(y)$ and deduce that the set

$$M_\delta = \left\{ \begin{array}{c} y \in G : \text{there is a } p \in R^2 \text{ with } |p| < \delta \text{ for which} \\ \alpha_p(y) = \alpha(y) + \langle p, y\rangle \text{ has a local maximum at } y \end{array} \right\}$$

has positive measure. But then $closure(M_\delta \cap \Gamma) = closure(M_\delta)$ in other words $M_\delta \cap \Gamma$, the set of points in M_δ where u is also twice differentiable, is dense in M_δ, so one can converge to any point in M_δ by a sequence in $M_\delta \cap \Gamma$. Furthermore $y = 0$ is in each M_δ. (It is also the unique such point since it is a strict maximum of $\alpha(y)$.) Hence there is a sequence in each $M_\delta \cap \Gamma$ such that this sequence converges to $y = 0$. Let us consider from now on only $\delta = \frac{1}{m}$, $m = 1, \ldots.$ Then we can form a sequence of $y_m$such that each

$$\text{each } y_m \in M_{\frac{1}{m}} \cap \Gamma,\ |y_m| < \frac{1}{m}, \text{ and } y_m \to 0.$$

Then, at each y_m, there is a $p_m \in R^n$ with $|p| < \frac{1}{m}$ for which

$$\alpha_{p_m}(y) = \alpha(y) + \langle p_m, y\rangle$$

has a local maximum, and furthermore at each y_m, u is twice differentiable. Then we have

$$\begin{aligned}
D\alpha_{p_m}(y_m) &= Du(y_m) - By_m - 4\,|y_m|^2\,y_m + p_m = 0 \text{ implying} \\
Du(y_m) &= By_m + 4\,|y_m|^2\,y_m - p_m \text{ implying} \\
|Du(y_m)| &= \left|By_m + 4\,|y_m|^2\,y_m - p_m\right| \\
&\leq |By_m| + 4\,|y_m|^3 + |p_m| \\
&\leq \frac{1}{m}(|B| + 4(\frac{1}{m})^3 + 1) \\
&= c(\frac{1}{m}) \text{ where } c \text{ is a constant, implying} \\
|Du(y_m)| &= O(\frac{1}{m})
\end{aligned}$$

and we also have

$$
\begin{aligned}
D^2\alpha_{p_m}(y_m) &= D^2u(y_m) - B - 8\begin{bmatrix} y_1^2 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & y_n^2 \end{bmatrix} \leq 0 \text{ implying} \\
D^2u(y_m) &\leq B + 8\begin{bmatrix} y_1^2 & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & y_n^2 \end{bmatrix} \leq B + 8(\frac{1}{m})I \text{ implying} \\
D^2u(y_m) &= B + O(\frac{1}{m}).
\end{aligned}
$$

Furthermore since also $u(y) + \frac{1}{2\varepsilon}|y|^2$ is convex and u is twice differentiable at each y_m, we have

$$
\begin{aligned}
D^2(u(y_m) + \frac{1}{2\varepsilon}|y_m|^2) &\geq 0 \text{ implying } D^2u(y_m) + \frac{1}{\varepsilon}I \geq 0 \text{ so that} \\
D^2u(y_m) &\geq -\frac{1}{\varepsilon}I.
\end{aligned}
$$

Then, since $|Du(y_m)| = O(\frac{1}{m})$, as $m \to \infty$ we have $Du(y_m) \to 0$; and since $D^2u(y_m)$ is bounded, it has a convergent subsequence $D^2u(y_{m_l})$ that converge to some $X \in S(N)$. We also have $Du(y_{m_l}) \to 0$. Furthermore since u is twice differentiable at each y_m, we have

$$
(Du(y_m), D^2u(y_m)) \in J^2u(y_m) = J^{2,+}u(y_m) \cap J^{2,-}u(y_m).
$$

Hence, we conclude that $(0, X) \in \bar{J}^2u(0)$, and as $m \to \infty$, $-\frac{1}{\varepsilon}I \leq X \leq B$. ■

Example 3.18 *Let us go back to our previous example and note that the condition* $B < \frac{1}{2\varepsilon}I$ *holds was implicitly imposed by the fact that we were considering* $\hat{u}_\varepsilon(y) = \sup(u(x) - \frac{1}{2\varepsilon}|x-y|^2)$ *and that it was necessary in order for the supremum to be achieved at* $\hat{x}$. *Now let us ask the following question:*

If we were not given the fact that $B < \frac{1}{2\varepsilon}I$ *holds then under what conditions could we derive a similar result for* $u(x) = \frac{1}{2}\langle Bx, x\rangle$ *and its sup convolution* $\hat{u}(y)$*?*

Proof. Let us assume for the moment that $\hat{u}_\varepsilon(y) = \sup(u(x) - \frac{1}{2\varepsilon}|x-y|^2)$achieves its supremum at y^*. Then we would have

$$\hat{u}_\varepsilon(y) = u(y^*) - \frac{1}{2\varepsilon}|y^* - y|^2 = \frac{1}{2}\langle By^*, y^*\rangle - \frac{1}{2\varepsilon}|y^* - y|^2.$$

Let again

$$\alpha(x) = \frac{1}{2}\langle Bx, x\rangle - \frac{1}{2\varepsilon}|x-y|^2,$$

then $\alpha(x)$ would be twice differentiable and in order for the maximum to be at y^*, we would need to have $D\alpha(y^*) = 0$ and this would require

$$By^* - \frac{1}{\varepsilon}(y^* - y) = 0,$$

i.e. $y^*(I - \varepsilon B) = y$. But then this would require $I - \varepsilon B$ to be invertible. Furthermore, we would also need $D^2\alpha(y^*) \le 0$, and this would require $B \le \frac{1}{\varepsilon}I$. Combining these two it becomes obvious that we need to have $B < \frac{1}{\varepsilon}I$.

Now our next question is which choice of ε would guarantee us that this latter condition holds. We note that

$$B < \frac{1}{\varepsilon}I \text{ iff } \varepsilon B - I < 0 \text{ iff } \varepsilon(B - \frac{1}{\varepsilon}I) < 0 \text{ iff } \frac{1}{\varepsilon} > \|B\|$$

since

$$B < \frac{1}{\varepsilon}I \text{ implies } \langle Bx, x\rangle \le \left\langle \frac{1}{\varepsilon}Ix, x\right\rangle \text{ for all } x \in R^n,$$

but then

$$\begin{aligned} \sup\{\langle Bx, x\rangle : \|x\| = 1\} &\le \sup\left\{\left\langle \frac{1}{\varepsilon}Ix, x\right\rangle : \|x\| = 1\right\} \text{ hence} \\ \|B\| &\le \left\|\frac{1}{\varepsilon}I\right\| = \frac{1}{\varepsilon} \end{aligned}$$

where

$$\begin{aligned} \|B\| &= \max\{|\nu| : \text{where } \nu \text{ is an eigenvalue of } B\} \\ &= \sup\{\langle Bx, x\rangle : \|x\| = 1\}. \end{aligned}$$

Hence if $\frac{1}{\varepsilon} = \frac{1}{\gamma} + \|B\|$ then the desired condition will hold when $\gamma > 0$. Also, the reason we have added to $\|B\|$ a term of the form $\frac{1}{\gamma}$ is that as $\gamma \to 0$ we want to penalize the term $|x-y|^2$ in $\hat{u}_\varepsilon(y)$ more and more, so that as $\gamma \to 0$, $\hat{u}_\varepsilon(y) \to u(y)$.

$\blacksquare$

Example 3.19 *Hence we can restate our previous example as follows:*

Let $B \in S(2)$, and $u(x) = \frac{1}{2}\langle Bx, x\rangle$. Then for $\gamma > 0$, and $\frac{1}{\varepsilon} = \frac{1}{\gamma} + \|B\|$, where $\|B\|$ is as above,

$$\hat{u}(y) = \left\langle B(I - \varepsilon B)^{-1} y, y \right\rangle .$$

Furthermore, since $(I - \varepsilon B)^{-1} = I + \gamma B$ we have $\hat{u}(y) = \langle B(I + \gamma B)y, y\rangle$.

Now we are ready to prove a preliminary version of our long promised theorem.

3.4. Theorem on Sums - A Comparison Principle for Semicontinuous Functions

The merits of the 'Theorem on Sums' is mentioned in the introduction to this chapter, hence we will directly go on proving a preliminary version of this theorem. Later on, we will extend it to a more general version.

Theorem 3.20 *Let u_1, $u_2 \in USC(R^2)$, $u_1(0) = u_2(0) = 0$, $A \in S(4)$, and let*

$$w(x) = u_1(x_1) + u_2(x_2) \leq \frac{1}{2}\langle Ax, x\rangle \ \ \text{for } x = (x_1, x_2) \in R^4.$$

Then, for every $\varepsilon > 0$, there exists X_1, $X_2 \in S(2)$, such that

$$(0, X_1) \in \bar{J}^{2,+}u_1(0), (0, X_2) \in \bar{J}^{2,+}u_2(0)$$

and the block diagonal matrix with entries satisfies

$$-(\frac{1}{\gamma} + \|A\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \gamma A^2.$$

Proof. Let $\beta(x) = \frac{1}{2}\langle Ax, x\rangle$. Since $w \leq \beta$ we have $\hat{w}_\varepsilon \leq \hat{\beta}_\varepsilon$ for same $\varepsilon > 0$. Then, since we know by the last example of the previous section $\hat{\beta}_\varepsilon(y) = \frac{1}{2}\langle A(I+\gamma A)y, y\rangle$, where $\frac{1}{\varepsilon} = \frac{1}{\gamma} + \|A\|$, we have

$$\hat{w}_\varepsilon(y) \leq \frac{1}{2}\langle A(I+\gamma A)y, y\rangle . \tag{3.28}$$

We will drop the subscript ε for the sup convolution for the moment. Now also

$$\begin{aligned}
\hat{w}(y) &= \sup_{x\in R^4}(w(x) - \frac{1}{2\varepsilon}|x-y|^2) \\
&= \sup_{x\in R^4}(u_1(x_1) + u_2(x_2) - \frac{1}{2\varepsilon}(|x_1-y_1|^2 + |x_2-y_2|^2) \\
&= \sup_{x=(x_1,x_2)\in R^4}(u_1(x_1) - \frac{1}{2\varepsilon}|x_1-y_1|^2 + u_2(x_2) - \frac{1}{2\varepsilon}|x_2-y_2|^2) \\
&= \sup_{x\in R^2}(u_1(x_1) - \frac{1}{2\varepsilon}|x_1-y_1|^2) + \sup_{x\in R^2}(u_2(x_2) - \frac{1}{2\varepsilon}|x_2-y_2|^2) \\
&= \hat{u}_1(y) + \hat{u}_2(y).
\end{aligned}$$

Since

$$u \leq \hat{u} \text{ implies } 0 = u_1(0) \leq \hat{u}_1(0) \text{ and } 0 = u_2(0) \leq \hat{u}_2(0)$$

we also have

$$\hat{u}_1(0) + \hat{u}_2(0) = \hat{w}(0) \leq \frac{1}{2}\langle A(I+\gamma A)0, 0\rangle = 0$$

and this implies that

$$\hat{u}_1(0) = 0 \text{ and } \hat{u}_2(0) = 0.$$

But then

$$(\hat{w}-\beta)(y) = \hat{w}(y) - \frac{1}{2}\left\langle (A+\gamma A^2)y, y\right\rangle \leq 0 \text{ by (3.28) for all } y$$

and $(\hat{w}-\beta)(0) = \hat{w}(0) - 0 = 0$ implies that $\hat{w}-\beta$ has a maximum at $y=0$.

Furthermore since $\hat{w}$ is semiconvex with semiconvexity constant $\kappa_G = \frac{1}{2\varepsilon}$ on a compact neighborhood of $y=0$, we can proceed as in the proof of last theorem of previous section to obtain

$$-(\frac{1}{\gamma} + \|A\|) \leq D^2\hat{w}(y_{\frac{1}{m}}) \leq A + \gamma A^2 + O((\frac{1}{m})^2).$$

But then since

$$D^2\hat{w}(y_{\frac{1}{m}}) = \begin{pmatrix} D^2\hat{u}_1(x^1_{\frac{1}{m}}) & 0 \\ 0 & D^2\hat{u}_2(x^2_{\frac{1}{m}}) \end{pmatrix}$$

letting

$$X^1_{\frac{1}{m}} = D^2\hat{u}_1(x^1_{\frac{1}{m}}) \text{ and } X^2_{\frac{1}{m}} = D^2\hat{u}_2(x^2_{\frac{1}{m}})$$

as $\frac{1}{m} \to 0$ we have

$$X^1_{\frac{1}{m}} \to X_1 \text{ and } X^2_{\frac{1}{m}} \to X_2$$

and hence

$$-(\frac{1}{\gamma} + \|A\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \gamma A^2.$$

Furthermore, in the same way we will also obtain $D\hat{w}(y_{\frac{1}{m}}) = O(\frac{1}{m})$. Then, we will have

$$D\hat{w}(y_{\frac{1}{m}}) = D\hat{w}(x^1_{\frac{1}{m}}, x^2_{\frac{1}{m}}) = (D\hat{u}_1(x^1_{\frac{1}{m}}), D\hat{u}_2(x^2_{\frac{1}{m}}))$$

letting

$$p^1_{\frac{1}{m}} = D\hat{u}_1(x^1_{\frac{1}{m}}) \text{ and } p^2_{\frac{1}{m}} = D\hat{u}_2(x^2_{\frac{1}{m}})$$

we have $(p^1_{\frac{1}{m}}, p^2_{\frac{1}{m}}) = O(\frac{1}{m})$. As $\frac{1}{m} \to 0$ we have $p^1_{\frac{1}{m}} \to 0$, and $p^2_{\frac{1}{m}} \to 0$. Since

$$\begin{aligned} (p^1_{\frac{1}{m}}, X^1_{\frac{1}{m}}) &= (D\hat{u}_1(x^1_{\frac{1}{m}}), D^2\hat{u}_1(x^1_{\frac{1}{m}})) \text{ and} \\ (D\hat{u}_1(x^1_{\frac{1}{m}}), D^2\hat{u}_1(x^1_{\frac{1}{m}})) &\in J^2\hat{u}_1(x^1_{\frac{1}{m}}) \end{aligned}$$

we have as in the proof of Theorem 3.17,

$$(0, X_1) \in \bar{J}^2\hat{u}_1(0) \text{ and similarly } (0, X_2) \in \bar{J}^2\hat{u}_2(0). \tag{3.29}$$

Furthermore, noting that u_1 and u_2 are bounded from above (since they are upper semicontinuous, on a compact neighborhood of $y = 0$ they will be bounded from above and outside a neighborhood of zero they can be modified to be bounded from above and this will not affect the analysis) and also that (3.29) implies

$$(0, X_1) \in \bar{J}^{2,+}\hat{u}_1(0) \text{ and } (0, X_2) \in \bar{J}^{2,+}\hat{u}_2(0)$$

we have by Corollary 3.14,

$$(0, X_1) \in \bar{J}^{2,+}u_1(0) \text{ and } (0, X_2) \in \bar{J}^{2,+}u_2(0).$$

$\blacksquare$

In the above proof, the fact that we had $u_1(0) = u_2(0) = 0$, and w bounded by a pure quadratic $\beta(x) = \langle Ax, x \rangle$ provided us with the fact that 0 is a maximum point $\hat{w}(y) - \frac{1}{2}\langle Ay, y \rangle$, and hence we could carry out the analysis around $y = 0$. Furthermore, note that $D\beta(0)$ was equal to zero, we had $D\beta(y_{\frac{1}{m}}) = O(\frac{1}{m})$, and as a consequence $D\hat{w}(y_{\frac{1}{m}}) \to 0$ as $\frac{1}{m} \to 0$.

Hence if we can reduce a general problem to this case, then we would be able to carry out the same analysis. So, let us consider this case. Let $\hat{x}$ be a maximum of $w - \varphi$, where φ is a twice differentiable function. First we will translate $\hat{x}$ to 0.

Let $\tilde{\varphi}(x) = \varphi(\hat{x} + x)$, then $\tilde{\varphi}(0) = \varphi(\hat{x})$. From now on for simplicity of notation we will call $\tilde{\varphi}$ as φ, and keep in mind that the new φ is a shifted version of the former φ and has carried out the local properties of former φ around $\hat{x}$ to around 0. Now since φ is C^2, by its Taylor expansion near $x = 0$ we have

$$\varphi(x) = \varphi(0) + D\varphi(0)x + \frac{1}{2}\left\langle D^2\varphi(0)x, x\right\rangle + o(|x|^2).$$

Now let

$$\begin{aligned} \check{\varphi}(x) &= \varphi(x) - \varphi(0) - D\varphi(0)x \text{ then } \check{\varphi}(x) = \frac{1}{2}\left\langle D^2\varphi(0)x, x\right\rangle + o(|x|^2) \text{ and} \\ D\check{\varphi}(x) &= D\varphi(x) - D\varphi(0) \text{ and hence } D\check{\varphi}(0) = D\varphi(0) - D\varphi(0) = 0 \text{ and} \\ D^2\check{\varphi}(x) &= D^2\varphi(x) \text{ and hence } D^2\check{\varphi}(0) = D^2\varphi(0) \text{ and also } \check{\varphi}(0) = 0. \end{aligned}$$

Similarly, we can translate w so that $\tilde{w}(x) = w(\hat{x} + x)$, then $\tilde{w}(0) = w(\hat{x})$, and again for simplicity of notation we will call $\tilde{w}$ as w, and keep in mind that the new w is a shifted version of the former w and has carried out the local properties of former w around $\hat{x}$ to around 0.

Moreover, we also have to keep the previous local relation now between our new w and $\check{\varphi}$. Note that the values of $\check{\varphi}$ is obtained by first shifting the values of by

$\varphi(x)$ by $\varphi(0)$ so that $\check{\varphi}(0) = 0$ now and second by modifying $\varphi(x)$ around $x = 0$ by subtracting a factor of $D\varphi(0)x$. Therefore in order to be able to keep the previous local relation we need to define a new $\check{w}$ whose values are shifted by $w(0)$, so that now $\check{w}(0) = 0$. Furthermore we need to modify w by subtracting a factor of $D\varphi(0)x$ so that the previous relation between w and φ around $\hat{x}$ is now preserved between $\check{w}$ and $\check{\varphi}$ around $x = 0$. Hence we define

$$\check{w}(x) = w(x) - w(0) - D\varphi(0)x.$$

But then we have around $x = 0$,

$$\begin{aligned}
\check{w}(x) - \check{\varphi}(x) &= w(x) - w(0) - D\varphi(0)x - \varphi(x) + \varphi(0) + D\varphi(0)x \\
&= (w - \varphi)(x) - w(0) - D\varphi(0)x + \varphi(0) + D\varphi(0)x \\
&\leq (w - \varphi)(0) - w(0) + \varphi(0) \\
&= w(0) - \varphi(0) - w(0) + \varphi(0) - D\varphi(0)0 + D\varphi(0)0 \\
&= w(0) - w(0) - D\varphi(0)0 - \varphi(0) + \varphi(0) + D\varphi(0)0 \\
&= \check{w}(0) - \check{\varphi}(0) = 0
\end{aligned}$$

and hence 0 is a maximum point of $\check{w} - \check{\varphi}$. Since $\check{w}(x) - \check{\varphi}(x) \leq 0$, we have

$$\check{w}(x) - \frac{1}{2}\langle Ax, x\rangle - o(|x|^2) \leq 0 \text{ where } A = D^2\varphi(0).$$

Now the problem at this point is if $o(|x|^2) > 0$, then considering $\check{w}(x) \leq \frac{1}{2}\langle Ax, x\rangle$ would not suffice as an upper bound for $\check{w}(x)$, since we have $\check{w}(x) \leq \frac{1}{2}\langle Ax, x\rangle + o(|x|^2)$ and $o(|x|^2) > 0$. However, if instead of $\frac{1}{2}\langle Ax, x\rangle$, we consider $\frac{1}{2}\langle (A + \eta I)x, x\rangle$, then this would suffice as an upper bound since we will have

$$\check{w}(x) \leq \frac{1}{2}\langle Ax, x\rangle + o(|x|^2) < \frac{1}{2}\langle (A + \eta I)x, x\rangle \text{ if } \eta > 0$$

Hence we arrive at the following generalization:

Theorem 3.21 *Let u_1, $u_2 \in USC(R^2)$, $\varphi \in C^2(R^2)$, and let*

$$w(x) = u_1(x_1) + u_2(x_2) \textit{ for } x = (x_1, x_2) \in R^4.$$

If $\hat{x} = (\hat{x}_1, \hat{x}_2) \in R^4$ *is a local maximum of* $w - \varphi$ *relative to* R^4*, then for every* $\varepsilon > 0$*, there exists* $X_1,\ X_2 \in S(2)$*, such that*

$$(D_{x_1}\varphi(\hat{x}), X_1) \in \bar{J}^{2,+}u_1(\hat{x}_1) \text{ and } (D_{x_2}\varphi(\hat{x}), X_2) \in \bar{J}^{2,+}u_2(\hat{x}_2)$$

and the block diagonal matrix with entries satisfies

$$-(\frac{1}{\gamma} + \|A\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \gamma A^2,$$

where $A = D^2\varphi(\hat{x}) \in S(4)$*.*

Proof. We apply the previous theorem to the above derived $\breve{w}$ as w, $\breve{\varphi}$ asφ with upper bound $\frac{1}{2}\langle (A + \eta I)x, x\rangle$ for $\breve{w}$, and note that

$$\begin{aligned} \breve{w}(x) &= w(x) - w(0) - D\varphi(0)x \\ &= u_1(x_1) + u_2(x_2) - u_1(0) - u_2(0) - D_{x_1}\varphi(0)x_1 - D_{x_2}\varphi(0)x_2 \\ &= \breve{u}_1(x_1) + \breve{u}_2(x_2) \end{aligned}$$

upon letting

$$\begin{aligned} \breve{u}_1(x_1) &= u_1(x_1) - u_1(0) - D_{x_1}\varphi(0)x_1 \text{ and} \\ \breve{u}_2(x_2) &= u_2(x_2) - u_2(0) - D_{x_2}\varphi(0)x_2 \end{aligned}$$

and noting that $\breve{u}_1,\ \breve{u}_2 \in USC(R^2)$, we obtain

$$(0, X_1) \in \bar{J}^{2,+}\breve{u}_1(0) \text{ and } (0, X_2) \in \bar{J}^{2,+}\breve{u}_2(0) \text{ and}$$

$$-(\frac{1}{\gamma} + \|(A + \eta I)\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq (A + \eta I) + \gamma(A + \eta I)^2.$$

Now, since for $\upsilon \in C^2$, we have

$$\bar{J}^{2,+}(\tau - \upsilon)(x) = \left\{ \left(p - D\upsilon(x), X - D^2\upsilon(x)\right) : (p, X) \in \bar{J}^{2,+}\tau(x) \right\}$$

from Chapter 2, letting $\tau = \check{u}_1(x_1)$ and $\upsilon = -u_1(0) - D_{x_1}\varphi(0)x_1$, $p = 0$, and $X = X_1$, we have

$$D\upsilon(0) = -D_{x_1}\varphi(0) \text{ and } D^2\upsilon(0) = 0,$$

and hence

$$(D_{x_1}\varphi(0), X_1) \in \bar{J}^{2,+}(\tau - \upsilon)(x) = \bar{J}^{2,+}u_1(0)$$

and similarly we have

$$(D_{x_2}\varphi(0), X_2) \in \bar{J}^{2,+}u_2(0).$$

Since we had a shifted version of the former w and have carried out the local properties of former w around $\hat{x}$ to around 0, we have:

$$(D_{x_1}\varphi(\hat{x}), X_1) \in \bar{J}^{2,+}u_1(\hat{x}_1) \text{ and } (D_{x_2}\varphi(\hat{x}), X_2) \in \bar{J}^{2,+}u_2(\hat{x}_2).$$

Upon letting $\eta \to 0$, we furthermore obtain:

$$-(\frac{1}{\gamma} + \|A\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \gamma A^2.$$

$\blacksquare$

We can make one more generalization of this theorem:

Let us assume instead of $u_1,\ u_2 \in USC(R^2)$, we had $u_1,\ u_2 \in USC(\Omega)$, where Ω was a locally compact subset of R^n. Then, we could restrict u_i to a compact neighborhood K_i of $\hat{x}_i$ in Ω, and extend it to R^n by $u_i(x_i) = -\infty$ for $x_i \notin K_i$. Then, the new $u_i \in USC(R^n)$ since each K_i is compact. Given $u_i(\hat{x}_i) > -\infty$, we would also have $\bar{J}^{2,+}_{\Omega}u_i(\hat{x}_i) = \bar{J}^{2,+}_{R^n}u_i(\hat{x}_i)$, and $\hat{x}$ would still be a local maximum of $w - \varphi$ relative to R^{2n}.

Hence we have the 'theorem on sums' in the following generality:

Theorem 3.22 *Let $u_1,\ u_2 \in USC(\Omega)$, where Ω is a locally compact subset of R^n, φ be C^2 in a neighborhood of Ω, Set*

$$w(x) = u_1(x_1) + u_2(x_2) \textit{ for } x = (x_1, x_2) \in \Omega \times \Omega.$$

If $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \Omega \times \Omega$ is a local maximum of $w - \varphi$ relative to $\Omega \times \Omega$, then for every $\varepsilon > 0$, there exists X_1, $X_2 \in S(2)$, such that

$$(D_{x_1}\varphi(\hat{x}), X_1) \in \bar{J}^{2,+}_{\Omega} u_1(\hat{x}_1) \text{ and } (D_{x_2}\varphi(\hat{x}), X_2) \in \bar{J}^{2,+}_{\Omega} u_2(\hat{x}_2)$$

and the block diagonal matrix with entries X_i satisfies

$$-(\frac{1}{\gamma} + \|A\|) \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \gamma A^2,$$

where $A = D^2\varphi(\hat{x}) \in S(4)$.

3.5. Notes

Sections 3.2 and 3.3 parallel a similar presentation in Chapter V of [**F-S**], and Lemma 3.2 and Theorem 3.3 are also from the same chapter as. Change of variables formula and related information can be found in [**A-B**]. Theorem 3.13 is presented in the form it is given in M.G. Crandall's lecture note 'Viscosity Solutions: A primer' in [**B-et.al.**]. Example 3.16 was given as an exercise in the same lecture notes.

4

EXISTENCE AND UNIQUENESS OF SOLUTIONS

4.1. Comparison and Uniqueness (Second Order Case)

In this section, we would like to prove a comparison result for viscosity solutions of (DP). In other words, for a $u \in USC(\bar{\Omega})$ subsolution and $v \in LSC(\bar{\Omega})$ supersolution of $F = 0$ in a bounded open subset Ω of R^n, if we know that $u \leq v$ on the boundary of this subset Ω, we want to be able to deduce from this information that $u \leq v$ on this subset Ω.

In order to be able to do that, we need to be able to compare values of u and v inside Ω, and deduce that $u - v \leq 0$ on this subset. However, it would suffice to show that if $\hat{x}$ were an interior maximum of $u - v$ then that $(u - v)(\hat{x}) \leq 0$ holds, since then $(u - v)(x) \leq (u - v)(\hat{x}) \leq 0$ would hold.

We also know that since $F(x, r, p, X)$ is proper it has a relation with u and v in a way that if $u \leq v$ then $F(x, u, p, X) \leq F(x, v, p, X)$, so we would like to deduce at a maximum $\hat{x}$ that

$$F(\hat{x}, u(\hat{x}), p, X) \leq F(\hat{x}, v(\hat{x}), p, X) \tag{4.1}$$

which would imply that $u - v \leq 0$ if F is strictly nondecreasing in r. Moreover, we know that u is a subsolution and v is a supersolution of $F = 0$, so that we have

$$F(\hat{x}, u(\hat{x}), p, X) \leq 0 \leq F(\hat{x}, v(\hat{x}), q, Y)$$

for every p, q, X, Y; now, if furthermore $X \leq Y$ holds then we have

$$F(\hat{x}, v(\hat{x}), q, Y) \leq F(\hat{x}, v(\hat{x}), q, X).$$

Hence, these last two inequalities would provide us with

$$F(\hat{x}, u(\hat{x}), p, X) \leq F(\hat{x}, v(\hat{x}), q, X) \tag{4.2}$$

which could in return provide us with the inequality (4.1) we would like to have, if we could make a bridge between them by knowing that some $p = q$ and $X \leq Y$ such that $(p, X) \in J^{2,+}u(\hat{x})$, and $(q, Y) \in J^{2,-}v(\hat{x})$ exists.

Now, if we suppose for the moment that $u, v \in C^2$ then we would have for $w = (u - v) \in C^2$and at a local maximum of w in Ω,

$$Dw(\hat{x}) = 0 \text{ and } D^2w(\hat{x}) \leq 0$$

by first and second order tests for a maximum, and this would give us that

$$Du(\hat{x}) = Dv(\hat{x}) \text{ and } D^2u(\hat{x}) \leq D^2v(\hat{x}),$$

and since we also know that

$$J^{2,+}u(\hat{x}) \cap J^{2,-}u(\hat{x}) = J^2u(\hat{x}) = \left\{(Du(\hat{x}), D^2u(\hat{x}))\right\},$$

and similarly for $J^2v(\hat{x})$, and we could have as our bridge

$$p = q = Du(\hat{x}) = Dv(\hat{x}) \text{ and } X = D^2u(\hat{x}) \text{ and } Y = D^2v(\hat{x})$$

and plug in (4.2) to obtain (4.1) i.e.

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq F(\hat{x}, v(\hat{x}), Du(\hat{x}), D^2u(\hat{x}))$$

as desired.

However, since we do not have $u, v \in C^2$, we do not know whether we have Du, Dv, D^2u, D^2v etc. Hence we cannot directly make such a substitution.

Instead we have $J^{2,+}u$, $J^{2,-}u$, $J^{2,+}v$, $J^{2,-}v$ etc. which can play a similar role, and we can make use of these sets. Yet this time, at a maximum $\hat{x}$ of $u-v$, it is possible that these sets can be empty. Therefore at this point, we have two pieces of information at hand:

One is that we need to approximate this maximum $\hat{x}$ with points x_α such that we know that certain elements exist in their $J^{2,+}u, J^{2,-}v(x_\alpha)$ etc. that we can use in place of Du, Dv, D^2u, D^2v above for each x_α. Then, any possible conclusion/control we may derive about behavior of F at $\hat{x}$, we need to derive it through its behavior at these x_α's.

The other is that by the 'theorem on sum', we know that $\bar{J}^{2,+}u, \bar{J}^{2,+}(-v)$ etc. is not empty at a local maximum $\hat{x}$ of $w-\varphi$, (where $w = u+(-v)$, noting $-v \in USC(\bar{\Omega})$, and φ is some C^2 function), and contains an element that we can control via $D\varphi$, and $D^2\varphi$ of this C^2 function φ.

Hence we would prefer an approximation of $\hat{x}$ by some points x_α, such that x_α is a maximum of some $(w-\varphi)_\alpha$ function. Furthermore, in view of the 'theorem on sum', we would also desire φ to be such that $D_{x_1}\varphi(x_\alpha) = -D_{x_2}\varphi(x_\alpha)$, (since then we would have

$$\begin{aligned}
(p, X) &= (D_{x_1}\varphi(x_\alpha), X_1) \in \bar{J}^{2,+}u(x_\alpha), \\
(D_{x_2}\varphi(x_\alpha), X_2) &\in J^{2,+}(-v)(x_\alpha) = -J^{2,-}v(x_\alpha) \text{ and hence} \\
(q, Y) &= (-D_{x_2}\varphi(x_\alpha), -X_2) \in J^{2,-}v(x_\alpha) \text{ so that} \\
p &= q \text{ would hold for this } x_\alpha.)
\end{aligned}$$

A good candidate then for such a function φ would be $\varphi(x, y) = \frac{1}{2}\alpha\,|x-y|^2$, where $\alpha > 0$ is a parameter, and so we would have $D_x\varphi = \alpha(x-y)$ and $D_y\varphi = -\alpha(x-y)$.

Moreover, since as $\alpha \to \infty$ the quantity $-\frac{1}{2}\alpha\,|x-y|^2$ would penalize the difference between any fixed x and y more and more, and by maximizing the function

$$u(x) - v(y) - \frac{1}{2}\alpha |x - y|^2$$

over $\bar{\Omega} \times \bar{\Omega}$ and letting $\alpha \to \infty$, we would approximate maximizing the function $u(x) - v(x)$ over $\bar{\Omega}$. In other words, if x_α were maximum points of

$$u(x) - v(y) - \frac{1}{2}\alpha |x - y|^2$$

for each α, and if $x_\alpha \to \hat{x}$ were to hold then $\hat{x}$ would in return be maximum of $u(x) - v(x)$.

More formally speaking, this approximation process would be described mathematically as follows:

Proposition 4.1 *(Doubling the Variables) Let Ω be a subset of R^n, $w \in USC(\Omega)$, $\varphi \in LSC(\Omega)$, $\varphi \geq 0$, and let*

$$M_\alpha = \sup_\Omega (w(x) - \alpha\varphi(x)) \text{ for } \alpha > 0.$$

Let M_α be finite for large α, and

$$\{N = x \in \Omega : \varphi(x) = 0\} \neq \emptyset.$$

Let $x_\alpha \in \Omega$ be such that

$$\lim_{\alpha \to \infty} (M_\alpha - (w(x_\alpha) - \alpha\varphi(x_\alpha)) = 0.$$

Then the following holds:

$$\text{1) } \lim_{\alpha \to \infty} \alpha\varphi(x_\alpha) = 0 \text{ and}$$

$$\begin{aligned} \text{2) If } x_\alpha &\to \hat{x} \in \Omega \text{ as } \alpha \to \infty \text{ then } \varphi(\hat{x}) = 0 \text{ and} \\ \lim_{\alpha \to \infty} M_\alpha &= w(\hat{x}) = \sup_N w(x). \end{aligned}$$

Proof. Now,

$$\sup_N w = \sup_N (w - \alpha\varphi) \leq \sup_\Omega (w - \alpha\varphi) = M_\alpha \leq M_1,$$

and M_α decreases as $\alpha \to \infty$ since $\varphi \geq 0$. Hence $\lim_{\alpha\to\infty} M_\alpha$ exists, and is finite by assumption. We will first prove 1): Let

$$\delta_\alpha = M_\alpha - (w(x_\alpha) - \alpha\varphi(x_\alpha)$$

and $\lim_{\alpha\to\infty} \delta_\alpha = 0$ since given. Now since,

$$\begin{aligned} M_{\frac{\alpha}{2}} &= \sup_\Omega (w(x) - \frac{\alpha}{2}\varphi(x)) \\ &\geq w(x_\alpha) - \frac{\alpha}{2}\varphi(x_\alpha) = w(x_\alpha) - \alpha\varphi(x_\alpha) + \frac{\alpha}{2}\varphi(x_\alpha) \\ &= M_\alpha - \delta_\alpha + \frac{\alpha}{2}\varphi(x_\alpha) \text{ this would imply} \end{aligned}$$

$$2(M_{\frac{\alpha}{2}} - M_\alpha + \delta_\alpha) \geq \alpha\varphi(x_\alpha),$$

and as $\alpha \to \infty$, left hand side (LHS)$\to 0$, hence

$$\lim_{\alpha\to\infty} \alpha\varphi(x_\alpha) = 0.$$

Next we will prove 2): Let α_n be a sequence of α. Then, $\lim_{n\to\infty} \varphi(x_{\alpha_n}) = 0$, and assume $x_{\alpha_n} \to \hat{x}$. Since $\varphi \in LSC(\Omega)$, we have

$$0 = \limsup_{n\to\infty} \varphi(x_{\alpha_n}) \geq \varphi(\hat{x}) \geq 0, \text{ hence } \varphi(\hat{x}) = 0.$$

Also,

$$\begin{aligned} w(\hat{x}) &\geq \lim_{n\to\infty} w(x_{\alpha_n}) - 0 = \lim_{n\to\infty} (w(x_{\alpha_n}) - \alpha_n\varphi(x_{\alpha_n})) \\ &= \lim_{n\to\infty} M_{a_n} \geq \sup_N w \geq w(\hat{x}) \end{aligned}$$

the first and the last inequalities follow from the fact that $w \in USC(\Omega)$, and $\hat{x} \in N$ respectively, hence we have the result

$$\lim_{\alpha\to\infty} M_\alpha = w(\hat{x}) = \sup_N w(x).$$

$\blacksquare$

On $\bar{\Omega} \times \bar{\Omega}$, we can apply this lemma to $w(x, y) = u(x) - v(y)$, since it is upper semicontinuous, and to $\varphi(x, y) = \frac{1}{2} |x - y|^2$, since φ is lower semicontinuous, $\varphi(x, y) \geq 0$, and $N = \{(x, y) : \varphi(x, y) = 0\} \neq \emptyset$. When we let

$$M_\alpha = \sup_{\bar{\Omega} \times \bar{\Omega}} (w(x, y) - \alpha\varphi(x, y))$$

since the supremum is taken over an upper semicontinuous function over a compact region, M_α is finite and is achieved on $\bar{\Omega} \times \bar{\Omega}$. In other words we have for each α, (x_α, y_α) such that

$$M_\alpha = w(x_\alpha, y_\alpha) - \alpha\varphi(x_\alpha, y_\alpha) \text{ holds.}$$

But then, since these pairs satisfy the condition

$$\lim_{\alpha \to \infty} (M_\alpha - (w(x_\alpha, y_\alpha) - \alpha\varphi(x_\alpha, y_\alpha)) = 0$$

immediately, we will choose and consider these (x_α, y_α). They will consist the sequence of points we want to approximate our $\hat{x}$ with.

At this point we need to note one more thing. Since we want to show that $u \leq v$ on Ω, we can assume on the contrary that there is a $z \in \Omega$ such that

$$u(z) > v(z)$$

and try to contradict this later on. If we make this assumption then this will imply that

$$M_\alpha \geq u(z) - v(z) = \delta > 0 \text{ for } \alpha > 0.$$

Now, if (x_α, y_α) has a limit point $\hat{x}$, then since as $\alpha \to \infty$,

$$\alpha\varphi(x_\alpha, y_\alpha) \to 0,$$

and this limit point has to be of the form $(\hat{x}, \hat{x})$. If $\hat{x} \in \partial\Omega$, then since $u \leq v$ on $\partial\Omega$, we would have

$$\begin{aligned} \limsup_{\alpha \to \infty} M_\alpha &= \limsup_{\alpha \to \infty} (w(x_\alpha, y_\alpha) - \alpha\varphi(x_\alpha, y_\alpha)) \\ &\leq u(\hat{x}) - v(\hat{x}) \leq 0. \end{aligned}$$

In view of our contrary assumption, we had $M_\alpha \geq \delta > 0$, so we cannot have such a limit point on $\partial(\Omega \times \Omega)$. Hence, for large α we need to have $(x_\alpha, y_\alpha) \in \Omega \times \Omega$.

Now, we can apply the 'theorem on sums' to each (x_α, y_α) with the corresponding functions they maximize. Let

$$u_1 = u,\ u_2 = -v,\ w(x,y) = u(x) - v(y),\ \varphi(x,y) = \frac{\alpha}{2}|x-y|^2.$$

Now, $(x_\alpha, y_\alpha) \in \Omega \times \Omega$ is a local maximum of $w - \varphi$ relative to $\Omega \times \Omega$. Then, we have

$$\begin{aligned} D_x\varphi(x_\alpha, y_\alpha) &= -D_y\varphi(x_\alpha, y_\alpha) = \alpha(x_\alpha - y_\alpha), \text{ and} \\ A &= \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},\ A^2 = 2\alpha A,\ \|A\| = 2\alpha, \end{aligned}$$

and we know that for every $\varepsilon > 0$ there exist $X_\alpha, -Y_\alpha \in S(N)$ such that

$$\begin{aligned} (\alpha(x_\alpha - y_\alpha), X_\alpha) &\in \bar{J}_\Omega^{2,+}u(x_\alpha) \text{ and} \\ (-\alpha(x_\alpha - y_\alpha), -Y_\alpha) &\in \bar{J}_\Omega^{2,+}(-v)(y_\alpha) = -\bar{J}_\Omega^{2,-}v(y_\alpha) \text{ implying} \\ (\alpha(x_\alpha - y_\alpha), Y_\alpha) &\in \bar{J}_\Omega^{2,-}v(y_\alpha), \text{ such that} \end{aligned}$$

$$-(\frac{1}{\gamma} + 2\alpha)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq \alpha(1 + 2\gamma\alpha)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \text{ holds.}$$

If we let $\gamma = \frac{1}{\alpha}$, then we obtain

$$-3\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} \leq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Since then

$$\begin{aligned} \left\langle \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ \xi \end{pmatrix} \right\rangle &\leq 3\alpha \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ \xi \end{pmatrix} \right\rangle \\ &= 0, \end{aligned}$$

we have $\langle X_\alpha\xi, \xi\rangle - \langle Y_\alpha\xi, \xi\rangle \leq 0$, which gives us that

$$X_\alpha \leq Y_\alpha.$$

Now, $(\alpha(x_\alpha - y_\alpha), X_\alpha) \in \bar{J}^{2,+}_\Omega u(x_\alpha)$ implies that there exists a sequence

$$\begin{aligned}(\alpha(x^n_\alpha - y^n_\alpha), X^n_\alpha) \quad &\in \quad J^{2,+}_\Omega u(x^n_\alpha) \text{ such that} \\ x^n_\alpha \quad &\to \quad x_\alpha,\ u(x^n_\alpha) \to u(x_\alpha),\ \alpha(x^n_\alpha - y^n_\alpha) \to \alpha(x_\alpha - y_\alpha),\ X^n_\alpha \to X,\end{aligned}$$

similarly for $(\alpha(x_\alpha - y_\alpha), Y_\alpha) \in \bar{J}^{2,-}_\Omega v(y_\alpha)$.

Since u is a subsolution, we have

$$F(x^n_\alpha, u(x^n_\alpha), \alpha(x^n_\alpha - y^n_\alpha), X^n_\alpha) \le 0 \text{ for ever } n$$

and since F is continuous,

$$F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X_\alpha) \le 0.$$

Similarly, since v is a supersolution we have

$$0 \le F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha).$$

Hence we arrive at

$$F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X_\alpha) \le F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha).$$

In order to be able to conclude more information from this inequality we need to impose on F certain structure conditions. If we can control F as x changes via a modulus of continuity function then this would lead us to the following:

$$F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X_\alpha) \le F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha) \text{ and}$$

$$\begin{aligned}& F(y_\alpha, u(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha) - \omega((\alpha\,|x-y| + 1)\,|x-y|) \\ \le\ & F(x_\alpha, u(x_\alpha), \alpha(x_\alpha - y_\alpha), X_\alpha) \\ \le\ & F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha),\end{aligned}$$

hence

$$\begin{aligned}& F(y_\alpha, u(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha) - F(y_\alpha, v(y_\alpha), \alpha(x_\alpha - y_\alpha), Y_\alpha) \\ \le\ & \omega((\alpha\,|x-y| + 1)\,|x-y|)\end{aligned}$$

Then if furthermore, F is strictly nondecreasing in r this would lead us to

$$\begin{aligned}\gamma\left(u(y_\alpha)-v(y_\alpha)\right) &\leq F(y_\alpha,u(y_\alpha),\alpha(x_\alpha-y_\alpha),Y_\alpha)-F(y_\alpha,v(y_\alpha),\alpha(x_\alpha-y_\alpha),Y_\alpha)\\ &\leq \omega((\alpha\left|x-y\right|+1)\left|x-y\right|)\end{aligned}$$

in the limit giving us $u(\hat{x})-v(\hat{y})\leq 0$. But this would contradict to the fact that

$$u(\hat{x})-v(\hat{y})\geq u(\hat{x})-v(\hat{y})-\frac{\alpha}{2}\left|\hat{x}-\hat{y}\right|^2=M_\alpha\geq u(z)-v(z)=\delta>0$$

Hence it would not be possible to have a $z\in\Omega$ such that $u(z)>v(z)$. Therefore, we have the following:

Theorem 4.2 *Consider the (DP) stated above. Assume, F is proper and satisfies the two structure conditions below. Let $u\in USC(\bar{\Omega})$, and $v\in LSC(\bar{\Omega})$ be subsolution and supersolution of (DP) respectively. Then $u\leq v$ in Ω.*

The two required structure conditions are as follows:

1) there is a function $\omega:[0,\infty]\to[0,\infty]$ such that $\omega(0+)=0$ and

$$\begin{aligned}&F(y_\alpha,u(y_\alpha),\alpha(x_\alpha-y_\alpha),Y_\alpha)-\omega((\alpha\left|x-y\right|+1)\left|x-y\right|)\\ \leq\ &F(x_\alpha,u(x_\alpha),\alpha(x_\alpha-y_\alpha),X_\alpha)\ \textit{holds and}\end{aligned}$$

2) there is a $\gamma>0$ such that

$$\begin{aligned}&\gamma\left(u(y_\alpha)-v(y_\alpha)\right)\\ \leq\ &F(y_\alpha,u(y_\alpha),\alpha(x_\alpha-y_\alpha),Y_\alpha)-F(y_\alpha,v(y_\alpha),\alpha(x_\alpha-y_\alpha),Y_\alpha)\ \textit{holds.}\end{aligned}$$

Once comparison holds, we would have uniqueness automatically.

Theorem 4.3 *Assume the conditions for comparison hold. Then if u and v are both viscosity solutions of the (DP), then $u=v$.*

Proof. Since comparison holds and u is a subsolution and v is a supersolution, we have $u\leq v$. Also since v is a subsolution and u is a supersolution, we have $u\leq v$. Hence $u=v$ on Ω ■

4.1.1. First order case

This time we would like to show that comparison holds for the Dirichlet Problem in the first order case. The problem is as follows:

$$H(x, u, Du) = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega. \tag{DP-2}$$

where Ω will be bounded subset of R^n. We want to show that if u is a subsolution of (DP-2), and v is a supersolution of (DP-2) then $u \leq v$. As before since u and v are $USC(\bar{\Omega})$, $LSC(\bar{\Omega})$ respectively, we are dealing with functions that might not have derivatives at certain points. Therefore, in the case that we want to make use of maximum principles at a maximum $\hat{x}$ of $u - v$ with the fact that u is a subsolution and v is a supersolution of $H(x, r, p)$ in particular at $\hat{x}$, in order to be able to conclude that $u(\hat{x}) - v(\hat{x}) \leq 0$, (and as a result to conclude that $u \leq v$ on Ω), we cannot directly employ Du or Dv since they might not exist at this maximum $\hat{x}$, just as we have seen in the second order case. Hence, we once again employ the technique of doubling the variables to be able to make use of smooth "test functions" on which we can transfer the derivatives; however, this time with a slightly different perspective. This technique will allow us to use the information of u being a subsolution and v being a supersolution to evaluate $H(x, r, p)$ at maxima of some approximating functions, from which we could derive a general result for u and v on all of Ω in the limit. Once again our main tool will be the technique of 'doubling variables'.

Let u be a subsolution, and v be a supersolution of (DP). Let us consider the functions given by

$$w_\varepsilon(x, y) = u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 .$$

As we have seen before w_ε attains its maximum at some $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$, and since $\bar{\Omega} \times \bar{\Omega}$ is compact these maxima converge to a point of the form $(\hat{x}, \hat{x})$. If this point is on the boundary of Ω, then we will automatically be done since this would imply that $\limsup_{\varepsilon \to 0} w_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq 0$. In the case that these maxima converge to a point in Ω, then for small ε, $(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ has to lie in Ω, hence we can evaluate $H(x, r, p)$ at these points upon the following observation.

Consider the map $x \to u(x) - \varphi(x, \hat{y}_\varepsilon)$, where $\varphi(x, y) = \frac{1}{2\varepsilon}|x - y|^2$ is a C^2 function, then $\hat{x}_\varepsilon$ is a maximum of this map. Then from the definition of u being a subsolution we have

$$H(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \leq 0.$$

Similarly, consider the map $y \to -v(y) - \varphi(\hat{x}_\varepsilon, y)$, then $\hat{y}_\varepsilon$ is a maximum of this map, hence is a minimum of $y \to v(y) - (-\varphi(\hat{x}_\varepsilon, y))$. Then from the definition of v being a supersolution we have

$$H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \geq 0.$$

Hence we arrive at the following,

$$H(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \leq 0 \text{ at } (\hat{x}_\varepsilon, \hat{y}_\varepsilon).$$

From this point on we would like to deduce that $u \leq v$. Our question will be as follows: Under what assumptions we can deduce this result.

Let $H(x, u, Du) = u + H(Du)$, then the above inequality would reduce to

$$\begin{aligned} u(\hat{x}_\varepsilon) + H(D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - v(\hat{y}_\varepsilon) - H(-D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) &\leq 0 \\ u(\hat{x}_\varepsilon) + H(D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - v(\hat{y}_\varepsilon) - H(D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) &\leq 0 \end{aligned}$$

since $D_x\varphi = -D_y\varphi$. Then we would have

$$u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon) \leq 0$$

and since

$$\begin{aligned} \limsup_{\varepsilon\to 0} w_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) &\leq \limsup_{\varepsilon\to 0}(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)) \leq 0 \text{ and} \\ u(x) - v(x) &= w_\varepsilon(x, x) \leq w_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \text{ holds} \\ \text{we have } u(x) - v(x) &\leq \limsup_{\varepsilon\to 0} w_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq 0, \text{ hence} \end{aligned}$$

in the limit this would tell us that $u \leq v$.

Suppose next $H(x, u, Du) = u + H(Du) - f(x)$, then similarly we would have

$$u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon) \leq f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon)$$

and in the case that f was uniformly continuous, this would give us

$$\begin{aligned} u(x) - v(x) &\leq \limsup_{\varepsilon \to 0} w_\varepsilon(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \leq \limsup_{\varepsilon \to 0}(u(\hat{x}_\varepsilon) - v(\hat{y}_\varepsilon)) \\ &\leq \limsup_{\varepsilon \to 0} f(\hat{x}_\varepsilon) - f(\hat{y}_\varepsilon) \leq 0. \end{aligned}$$

Suppose this time H were also to depend on x. Then, we would need a modulus of continuity estimate on H in order to be able to have control on it as x changes. Assume therefore that

$$|H(x, r, p) - H(y, r, p)| \leq \omega(|x - y| + p\,|x - y|)$$

where ω is a function such that $\omega(0+) = 0$, then we would have

$$H(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \leq 0 \text{ implying}$$

$$\begin{aligned} & H(\hat{x}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - H(\hat{y}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \\ & +H(\hat{y}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \\ \leq\ & 0 \text{ which would in turn imply} \end{aligned}$$

$$H(\hat{y}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) - H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_y\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \leq \omega(|x - y| + p\,|x - y|)$$

and since $D_x\varphi = -D_y\varphi$ this would give us

$$H(\hat{y}_\varepsilon, u(\hat{x}_\varepsilon), D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) \leq H(\hat{y}_\varepsilon, v(\hat{y}_\varepsilon), -D_x\varphi(\hat{x}_\varepsilon, \hat{y}_\varepsilon)) + \omega(|x - y| + p\,|x - y|).$$

Then we would require H to be 'strictly nondecreasing' in r, so that this last inequality would imply

$$u(\hat{x}_\varepsilon) \leq v(\hat{y}_\varepsilon)$$

and in the limit we would obtain $u \leq v$. We would formulate 'strictly nondecreasing' as follows: if $r \geq s$, then there exists $\gamma > 0$ such that $H(x,r,p) - H(x,s,p) \geq \gamma(r-s)$.

Since comparison would hold under these assumptions for the corresponding type of F, we would also have uniqueness as a result.

Theorem 4.4 *Assume the conditions for comparison hold. If u and v are both viscosity solutions of the (DP-2), then $u = v$.*

Proof. Since comparison holds and u is a subsolution and v is a supersolution we have $u \leq v$. Also since v is a subsolution and u is a supersolution we have $u \leq v$. Hence $u = v$ on Ω ∎

Before concluding this chapter we would like to show that the method for first order case falls short in the second order case.

Example 4.5 *Consider (DP-2) for $F(x,r,p,X) = r + G(p,X)$, assume G is continuous and degenerate elliptic. Let $u \in USC(\bar{\Omega})$ and $v \in LSC(\bar{\Omega})$ be subsolution and supersolution of (DP-2) respectively. We will apply the method of first order case. Let us consider maximum of*

$$u(x) - v(y) - \frac{1}{2\varepsilon}|x-y|^2 \text{ over } \Omega \times \Omega \text{ and we will let } \varepsilon \to 0.$$

Then, we again consider, when ε, is small an interior maximum $(\hat{x}, \hat{y})$ of this function, and following the method we arrive at

$$u(\hat{x}) + G(\frac{\hat{x}-\hat{y}}{\varepsilon}, \frac{1}{\varepsilon}I) \leq v(\hat{y}) + G(\frac{\hat{x}-\hat{y}}{\varepsilon}, -\frac{1}{\varepsilon}I).$$

and cannot go any further since $I \geq -I$.

4.2. Existence (Second Order Case)

In this section, we will consider the existence of a solution for the Dirichlet Problem (DP):

$$F(x,u,Du,D^2u) = 0 \text{ in } \Omega, \text{ and } u = 0 \text{ on } \partial\Omega. \tag{DP}$$

where Ω will be a bounded subset of R^n. Among several possible methods, the one we will be employing here will be a variant of the idea of Perron's method for the linear problem, hence will also be called Perron's method. Classical implementation of Perron's Method for linear problems can be found in [**J**].

Much of the work that would be needed to show existence via this way has been actually accomplished in the previous chapter and section for Perron's method presupposes the existence of comparison for the (DP) at hand. Let us try to see how this works.

In view of the comparison result of the previous section, we can say that if comparison holds for (DP), and if the (DP) has a solution, then this solution has to be the largest subsolution. For if w is the solution of (DP), then w is both a subsolution and a supersolution. Let $u \geq w$ be a subsolution of (DP), but then since w is a supersolution , by comparison we have, $w \geq u$. Hence we would have $w = u$, which then tells us that w has to be the largest subsolution.

Within this light, as a first step to proving this existence scheme we will show that a maximal subsolution exists; afterwards in our second step we will show that it is in fact a subsolution which is also the solution of (DP). As a matter of fact, this second step actually asserts that a maximal subsolution cannot afford to be not a solution. If w is the maximal subsolution that we obtain in step 1, and if it is not a solution, then at some point $\hat{x}$ it has to fail to be a supersolution. What we have known so far is that w is $USC(\bar{\Omega})$, however, in order to be able to consider it as a supersolution it needs to be $LSC(\bar{\Omega})$. Therefore, in our second step, to secure lower semicontinuity, we will look at w_*, lower semicontinuous envelope of w, i.e. the largest lower semicontinuous function that is less then or equal to w, and we will try to get a contradiction to maximality of w. The best way of doing this will be constructing around $\hat{x}$ a function larger then w, which will also be a subsolution of (DP). Hence the second step of our proof will be carrying out this construction. Now let us proceed with our first step.

4.2.1. Step 1: Construction of a maximal subsolution

Let us consider

$$w(x) = \sup \{u(x) : u \text{ is a subsolution of } F = 0 \text{ in } \Omega.\},$$

and denote $\{u(x) : u \text{ is a subsolution of } F = 0 \text{ in } \Omega.\}$ by K, and assume that K is nonempty.

For upper semicontinuity to hold, we will consider w^*, upper semicontinuous envelope of w, i.e. the smallest upper semicontinuous function such that $w \leq w^*$ holds. We will also assume that $w^*(x) < \infty$. We want to show that w^* is a subsolution of $F = 0$ in Ω. In other words we want to show that

$$\text{for } (p, X) \in J_\Omega^{2,+} w^*(z),\ F(z, w^*(z), p, X) \leq 0 \text{ for } z \text{ in } \Omega.$$

If we can find a sequence of $(y_n, u_n(y_n), p_n, X_n) \to (z, w^*(z), p, X)$ such that $(p_n, X_n) \in J_\Omega^{2,+} u_n(y_n)$, and if furthermore u_n are subsolutions of $F = 0$ in Ω, then we will know that $F(y_n, u_n(y_n), p_n, X_n) \leq 0$ and since F is continuous, we can pass to the limit to conclude that $F(z, w^*(z), p, X) \leq 0$. The following lemma will provide us with this sequence.

Lemma 4.6 *Let $\Omega \subset R^n$ be locally compact, $u \in USC(\Omega)$, $z \in \Omega$, and $(p, X) \in J_\Omega^{2,+} u(z)$. Suppose u_n is a sequence of USC functions on Ω such that*

i) there exists $x_n \in \Omega$ such that $(x_n, u_n(x_n)) \to (z, u(z))$ and

ii) if $z_n \in \Omega$ and $z_n \to x \in \Omega$, then $\limsup_{n\to\infty} u_n(z_n) \leq u(x)$.

Then, there exists

$$\begin{aligned} \hat{x}_n &\in \Omega, (p_n, X_n) \in J_\Omega^{2,+} u_n(\hat{x}_n) \textit{ such that} \\ (\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) &\to (z, u(z), p, X). \end{aligned}$$

Proof. Since the following analysis is local, without loss of generality we will take $z = 0$, and carry it around this point. First, we will find a candidate sequence of $\hat{x}_n$, and show that $\hat{x}_n \to 0$. Second, we will show that $u_n(\hat{x}_n) \to u(0)$. Third, we will find (p_n, X_n) such that $(p_n, X_n) \in J_\Omega^{2,+} u_n(\hat{x}_n)$, and finally we will show that $(p_n, X_n) \to (p, X)$

1) Since $(p, X) \in J_\Omega^{2,+} u(0)$, we have

$$u(x) \leq u(0) + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x|^2) \text{ for } x \in \Omega.$$

Let $c(x) = o(|x|^2)$ term. Then, given any $\delta > 0$, there is an $r > 0$ such that $|c(x)| \leq \delta |x|^2$ for $x \in \Omega$ and $|x| \leq r$. Let $N_r = \{x \in \Omega : |x| \leq r\}$. N_r is compact. Then, we have

$$u(x) \leq u(0) + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + \delta |x|^2 \text{ for } x \in N_r$$

Let

$$\varphi(x) = \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + 2\delta |x|^2.$$

Then, $\varphi \in C^2(\Omega)$,

$$\varphi(0) = 0,\ D\varphi(0) = p,\ D^2\varphi(0) = X + 4\delta I$$

and

$$u(x) - \varphi(x) \leq u(0) - \delta |x|^2 = u(0) - \varphi(0) - \delta |x|^2 \text{ for } x \in N_r.$$

Define $h : (0, \infty) \to (0, \infty)$ as $h(a) = \delta |a|^2$. Then, h is nondecreasing and $h(s) = \delta s^2 \leq \delta |x|^2$ for $s \leq |x| \leq r$. So we have

$$u(x) - \varphi(x) \leq u(0) - \varphi(0) - h(s) \text{ for } s \leq |x| \leq r$$

Hence $u - \varphi$ has a strict maximum at $x = 0$.

(Recall from Chapter 2 that h, within this respect, is called the strictness of the maximum.)

Now, assumption i) says that there exists a sequence of $x_n \in \Omega$ such that

$$(x_n, u_n(x_n)) \to (0, u(0)).$$

Now, since each u_n is upper semicontinuous, and $\varphi \in C^2$, $(u_n - \varphi) \in USC(\Omega)$, hence attains its maximum on N_r. Let $\hat{x}_n \in N_r$, be a maximum point of $u_n(x) - \varphi(x)$ over N_r. Then

$$\begin{aligned} u_n(x) - \varphi(x) &\leq u_n(\hat{x}_n) - \varphi(\hat{x}_n) \text{ and hence,} \\ u_n(x) &\leq u_n(\hat{x}_n) + \varphi(x) - \varphi(\hat{x}_n) \text{ for } x \in N_r. \end{aligned}$$

Since N_r is compact, $\hat{x}_n \in N_r$, it has a convergent subsequence, say $\hat{x}_{n_j}$, that converges to some $y \in N_r$ as $j \to \infty$. Then, in particular for $x = x_{n_j}$, from the last inequality we will have

$$u_{n_j}(x_{n_j}) \leq u_{n_j}(\hat{x}_{n_j}) + \varphi(x_{n_j}) - \varphi(\hat{x}_{n_j}).$$

Taking $\liminf$ as $j \to \infty$, we have

$$\begin{aligned} u(0) &\leq (\liminf u_{n_j}(\hat{x}_{n_j})) + \varphi(0) - \varphi(y) \\ &= (\liminf u_{n_j}(\hat{x}_{n_j})) - \varphi(y). \end{aligned} \tag{4.3}$$

By assumption ii), however we have

$$\liminf u_{n_j}(\hat{x}_{n_j}) \leq \limsup u_{n_j}(\hat{x}_{n_j}) \leq u(y).$$

Hence,

$$u(0) \leq u(y) - \varphi(y) \leq u(0) - \delta\, |y|^2$$

The last inequality holds since $y \in N_r$, and we had $u(x) - \varphi(x) \leq u(0) - \delta\, |x|^2$ for $x \in N_r$ before. Hence

$$u(0) \leq u(0) - \delta\, |y|^2 \text{ give us that } y = 0, \text{ and therefore } \hat{x}_{n_j} \to 0.$$

But then every subsequence of $\hat{x}_n$ has a convergent subsequence that converges to $y = 0$. Hence $\hat{x}_n$ converges to 0.

2) But then by (4.3) which gives us that

$$u(0) \leq \liminf u_n(\hat{x}_n)$$

and by ii) which gives us that

$$\limsup_{n\to\infty} u_n(\hat{x}_n) \le u(0),$$

we have

$$u(0) = \lim_{n\to\infty} u_n(\hat{x}_n).$$

3) Since $\hat{x}_n$ is a maximum of $u_n(x) - \varphi(x)$, we know that

$$(D\varphi(\hat{x}_n), D^2\varphi(\hat{x}_n)) \in J_\Omega^{2,+} u_n(\hat{x}_n).$$

Hence

$$(p + 4\delta\hat{x}_n + X\hat{x}_n, X + 4\delta I) \in J_\Omega^{2,+} u_n(\hat{x}_n).$$

4) As $\hat{x}_n \to 0$, $D\varphi(\hat{x}_n) \to D\varphi(0) = p$, $D^2\varphi(\hat{x}_n) \to D^2\varphi(0) = X + 4\delta I$.

As $\delta \to 0$, $X + 4\delta I \to X$. ■

We can also interpret this lemma as follows: If z is a strict maximum of $u - \varphi_\delta$, and u_n, (p, X) is as given, and $(D\varphi_\delta(z), D^2\varphi_\delta(z)) \to (p, X)$ as $\delta \to 0$, then we have a corresponding sequence of $\hat{x}_n$ of maxima of $u_n - \varphi_\delta$ such that $\hat{x}_n \to z$, $u_n(\hat{x}_n) \to u(z)$, $(D\varphi_\delta(\hat{x}_n), D^2\varphi_\delta(\hat{x}_n)) \to (p, X)$ as $n \to \infty$ and $\delta \to 0$.

Now that we have the desired sequence, we can proceed in achieving the aim of our first step:

Lemma 4.7 *Let $\Omega \subset R^n$ be locally compact, F continuous and proper. Let*

$$w(x) = \sup\{u(x) : u \textit{ is a viscosity subsolution of } F = 0 \textit{ in } \Omega.\},$$

and denote $\{u(x) : u$ is a viscosity subsolution of $F = 0$ in $\Omega.\}$ by K, and assume that this set is nonempty and also that $w^*(x) < \infty$ for $x \in \Omega$. Then w^* is a viscosity subsolution of $F = 0$ in Ω.

Proof. Let $z \in \Omega$ and $(p, X) \in J_\Omega^{2,+} w^*(z)$. Now, by construction of w^*, there exists $x_n \in \Omega$, and $w(x_n)$ such that $w(x_n) \to w^*(z)$ as $x_n \to z$. Furthermore, since w is supremum of

$$\{u(x) : u \text{ is a viscosity subsolution of } F = 0 \text{ in } \Omega.\},$$

we have for any $y \in \Omega$ a sequence of $u_\alpha(y)$ such that $u_\alpha(y) \to w(y)$.In particular for each x_n, we have a sequence of $u_\alpha(x_n) \to w(x_n)$ where choice of the sequence u_α depends on x_n. Thus, we can form a sequence $u_n(x_n)$ such that $u_n(x_n) \to w^*(z)$.

Then, for this u_n, if $z_n \in \Omega$ such that $z_n \to x$, we have

$$\limsup_{n\to\infty} u_n(z_n) \leq w^*(x)$$

by definition of w^*. Hence we satisfy the assumptions of the Lemma 4.6 with $u = w^*$. Then Lemma 4.6 guarantees us the existence of a sequence $\hat{x}_n \in \Omega$, and $(p_n, X_n) \in J^{2,+}_\Omega u_n(\hat{x}_n)$ such that

$$(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \to (z, w^*(z), p, X).$$

Now, since $(p_n, X_n) \in J^{2,+}_\Omega u_n(\hat{x}_n)$, and u_n are subsolutions we have for each u_n,

$$F(\hat{x}_n, u_n(\hat{x}_n), p_n, X_n) \leq 0 in \Omega.$$

Since F is continuous, as $n \to \infty$, in the limit we have

$$F(z, w^*(z), p, X) \leq 0.$$

Since this holds for any $z \in \Omega$, w^* is a subsolution of $F = 0$ in Ω. ∎

Next, we will proceed to our second step in providing an existence scheme to above (DP).

4.2.2. Step 2: Perron's method and existence

Lemma 4.8 *Let Ω be open, u be a subsolution of $F = 0$ in Ω. If u_* is not a supersolution at some $\hat{x}$, i.e. if there exists $(p, X) \in J^{2,-}_\Omega u_*(\hat{x})$, for which $F(\hat{x}, u_*(\hat{x}), p, X) < 0$, then for any small $\kappa > 0$, there is a subsolution U_κ of $F = 0$ in Ω, such that*

$$\begin{aligned} U_\kappa(x) &\geq u(x) \text{ and } \sup_\Omega(U_\kappa - u) > 0 \\ U_\kappa(x) &= u(x) \text{ for } x \in \Omega \text{ and } |x - \hat{x}| \geq \kappa. \end{aligned}$$

Proof. Without loss of generality we will assume that $0 \in \Omega$ and $\hat{x} = 0$.

Assume u_* is not a supersolution at $x = 0$, i.e. assume that there exists $(p, X) \in J_\Omega^{2,-} u_*(0)$, for which $F(0, u_*(0), p, X) < 0$. Then since $(p, X) \in J_\Omega^{2,-} u_*(0)$ we have

$$u_*(x) \geq u_*(0) + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle + o(|x|^2) \text{ for } x \in \Omega.$$

Let $c(x) = o(|x|^2)$ term. Then, given any $\gamma > 0$, there is an $r > 0$ such that $|c(x)| \leq \gamma |x|^2$ for $x \in \Omega$ and $|x| \leq r$. Then, we have

$$u_*(x) \geq u_*(0) + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle - \gamma |x|^2 \text{ for } |x| \leq r$$

Let $\varphi_\gamma(x) = u_*(0) + \langle p, x \rangle + \frac{1}{2} \langle Xx, x \rangle - 2\gamma |x|^2$. Then, $\varphi_\gamma \in C^2(\Omega)$,

$$\varphi_\gamma(0) = u_*(0),\ D\varphi_\gamma(x) = p + Xx + 4\gamma x,\ D^2\varphi_\gamma(x) = X + 4\gamma I$$

and

$$u_*(x) \geq \varphi_\gamma(x) + \gamma |x|^2 \text{ for } |x| \leq r. \tag{4.5}$$

Define $h : (0, \infty) \to (0, \infty)$ as $h(a) = \delta |a|^2$. Then, h is nondecreasing and $h(s) = \delta s^2 \leq \delta |x|^2$ for $s \leq |x| \leq r$. So we have

$$u_*(x) \geq \varphi_\gamma(x) + h(s) \text{ for } s \leq |x| \leq r.$$

Then

$$u_*(x) - \varphi_\gamma(x) \geq u_*(0) - \varphi_\gamma(0) + h(s) \text{ for } s \leq |x| \leq r.$$

Hence $u - \varphi_\gamma$ has a strict minimum at $x = 0$.

(Recall from Chapter 2 that h, within this respect, is called the strictness of the minimum.)

Let $\varphi_{\delta,\gamma}(x) = \varphi_\gamma(x) + \delta$. Then,

$$D\varphi_{\delta,\gamma}(x) = D\varphi_\gamma(x),\ D^2\varphi_{\delta,\gamma}(x) = D^2\varphi_\gamma(x).$$

Since $F(0, u_*(0), p, X) < 0$ and F is continuous there are neighborhoods N_1, N_2, N_3, of 0, p, and X respectively on which

$$F(x, u_*(x), p', X') < 0 \text{ for } x \in N_1, p' \in N_2, X' \in N_3.$$

Choosing γ small enough so that $\{x : |x| < r\} \subset N_1$, then by (4.5)

$$F(x, \varphi_\gamma(x), p, X) \leq F(x, u_*(x), p, X) < 0 \text{ for } |x| < r.$$

Choosing γ further smaller if necessary so that $D\varphi_\gamma(x) \in N_2$, $D^2\varphi_\gamma(x) \in N_3$, then again by continuity of F we have

$$F(x, \varphi_\gamma(x), D\varphi_\gamma(x), D^2\varphi_\gamma(x)) < 0 \text{ for } |x| < r.$$

Since F is continuous, this time there is a neighborhood N_4 of $\varphi_\gamma(x)$, such that

$$F(x, \beta, D\varphi_\gamma(x), D^2\varphi_\gamma(x)) \text{ for } \beta \in N_4 \text{ and } |x| < r.$$

Hence if we choose δ small enough so that $\varphi_{\delta,\gamma}(x) = (\varphi_\gamma(x) + \delta) \in N_4$ for $|x| < r$, then

$$F(x, \varphi_{\delta,\gamma}(x), D\varphi_\gamma(x), D^2\varphi_\gamma(x)) < 0 \text{ for } |x| < r.$$

Hence

$$F(x, \varphi_{\delta,\gamma}(x), D\varphi_{\delta,\gamma}(x), D^2\varphi_{\delta,\gamma}(x)) < 0 \text{ for } |x| < r.$$

This last inequality tells us that $\varphi_{\delta,\gamma}$ is a classical subsolution of $F \leq 0$ in the $B_r = \{x : |x| < r\}$ for small γ, r, $\delta > 0$.

Now, since we had

$$u_*(x) \geq \varphi_\gamma(x) + h(s) \text{ for } s \leq |x| \leq r$$

for $s = \frac{r}{2}$, we will have

$$u_*(x) \geq \varphi_\gamma(x) + h(\frac{r}{2}) \text{ for } \frac{r}{2} \leq |x| \leq r.$$

If $\delta < \frac{1}{2}h(\frac{r}{2}) = \frac{r^2}{8}\gamma$, then for $\frac{r}{2} \leq |x| \leq r$, we will have

$$u_*(x) \geq \varphi_\gamma(x) + h(\frac{r}{2}) \ > \varphi_\gamma(x) + \delta = \varphi_{\delta,\gamma}.$$

Hence,

$$u_*(x) > \varphi_{\delta,\gamma}(x) \text{ for } \frac{r}{2} \leq |x| \leq r.$$

Define

$$U(x) = \begin{cases} \max(u(x), \varphi_{\delta,\gamma}(x)) & \text{if } |x| < r \\ u(x) & \text{otherwise} \end{cases}.$$

By the previous lemma, then $U(x)$ is a subsolution of $F = 0$ in Ω.

By definition of u_*, we have a sequence $(x_n, u(x_n))$ that converges to $(0,\ u_*(0))$. Then,

$$\lim_{n\to\infty}(U(x_n) - u(x_n)) = \varphi_{\delta,\gamma}(0) - u_*(0) = u_*(0) + \delta - u_*(0) > 0.$$

Hence, in every neighborhood of 0 there are points such that $U(x) > u(x)$.

So, given any $\kappa > 0$, if we choose γ, r small enough so that $r \leq \kappa$, we have

$$\begin{aligned} U_\kappa(x) &\geq u(x) \text{ and } \sup_\Omega(U_\kappa - u) > 0, \\ U_\kappa(x) &= u(x) \text{ for } x \in \Omega \text{ and } |x| \geq \kappa. \end{aligned}$$

$\blacksquare$

Hence we have seen that if for a subsolution u, u_* fails to be a supersolution at some point, then u cannot be the maximal subsolution. Now we are ready to state and prove the existence scheme for the (DP).

Theorem 4.9 *Let comparison hold for (DP). Suppose also that there is a subsolution u and a supersolution v of (DP) that agree on the boundary, i.e. u and v satisfy the boundary condition $u_*(x) = v^*(x) = 0$ for $x \in \partial\Omega$. Then*

$$W(x) = \sup\{w(x) : u \leq w \leq v \textit{ and } w \textit{ is a subsolution of (DP)}\}$$

is a solution of (DP).

Proof. Let us note first that $u_* \leq W_* \leq W \leq W^* \leq v^*$. Hence, on the boundary we have $W_* = W = W^* = 0$. Second, by Lemma 4.7, W^* is a subsolution of (DP), and therefore by comparison we have $W^* \leq v$. Hence, since W is the supremum over a set that contains W^*, we have the the other inequality $W \geq W^*$, so $W = W^*$. Thus, W is a subsolution.

Third, let us assume that W_* fails to be a supersolution at some point $\hat{x} \in \Omega$. Then by Lemma 4.8, we have functions W_κ with the properties defined in the lemma. For sufficiently small κ, on the boundary, we have $W_\kappa = W = 0$. Hence W_κ is a subsolution of (DP). Then by comparison, $W_\kappa \leq v$. Furthermore, $u \leq W \leq W_\kappa$, hence $u \leq W_\kappa \leq v$. Since W is the maximal subsolution between u and v, we have $W_\kappa \leq W$, which contradicts to the fact that $\sup_\Omega(W_\kappa - W) > 0$. Hence W_* is a supersolution of the (DP). Then by comparison once again, we have $W \leq W_*$. but by definition $W_* \leq W$. Hence, we have $W = W_*$.

Thus, we have $W^* = W = W_*$. Hence by being both upper and lower semicontinuous W is continuous, and by being both a subsolution and a supersolution of (DP), W is a solution of (DP). ∎

4.2.3. First order case

In this section, we will give an existence scheme for the (DP-2) in the first order case. The method we will be employing will make use of addition of a regularizing term to the original equation and afterwards passing to limit via the solutions of these approximate equations and showing that the limiting equation of the solutions of the approximate equations is a solution of the original equation under certain assumptions. This method will also account for the origin of the term 'viscosity' in the theory, since the added regularizing term $\varepsilon \bigtriangleup u$ is called a viscosity term in physical applications, and hence that this method is called the method of 'vanishing viscosity'. Hence, we will be seeing that the method of 'vanishing viscosity' is one of the ways of producing viscosity solutions among several other methods.

Theorem 4.10 *Let $\varepsilon > 0$ and let $F_\varepsilon(x, r, p)$ be a family of continuous functions such that $F_\varepsilon(x, r, p)$ converges uniformly on compact subsets of $\Omega \times R \times R^n$ to some function $F(x, r, p)$ as $\varepsilon \to 0$; and suppose $u^\varepsilon \in C^2(\Omega)$ is a solution of*

$$-\varepsilon \bigtriangleup u + F_\varepsilon(x, u^\varepsilon, Du^\varepsilon) = 0 \text{ in } \Omega, \tag{4.6}$$

and u^ε converge uniformly on compact subsets of Ω to some $u \in C(\Omega)$. Then u is a viscosity solution of $F(x, u, Du) = 0$ in Ω.

Proof. We will first use test functions from $C^2(\Omega)$. Let $\phi \in C^2(\Omega)$,we will show first that u is a subsolution. Assume $u - \phi$ has a local maximum at $\hat{x} \in \Omega$. We then have to to show that

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}) \leq 0.$$

Then, $u - (\phi + \delta \left|x - \hat{x}\right|^2)$ has a strict local maximum at $\hat{x}$. Consider the closed ball $B = B(\hat{x}; r)$. Then for sufficiently small $r > 0$, $\max_{\partial B}(u - (\phi + \delta \left|x - \hat{x}\right|^2)) < (u - (\phi + \delta \left|x - \hat{x}\right|^2))(\hat{x})$. Then, since $u^\varepsilon \to u$ uniformly on B, $\max_{\partial B}(u^\varepsilon - (\phi + \delta \left|x - \hat{x}\right|^2)) < (u^\varepsilon - (\phi + \delta \left|x - \hat{x}\right|^2))(\hat{x})$ for ε small. As a result $u^\varepsilon - (\phi + \delta \left|x - \hat{x}\right|^2)$ has a local maximum at some point x_ε in the interior of B, and by choosing a sequence of r converging to 0, $x_\varepsilon \to \hat{x}$, as $\varepsilon \to 0$. Then at $x = x_\varepsilon$,

$$\begin{aligned} Du^\varepsilon(x_\varepsilon) &= D(\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon) \text{ and} \\ \triangle u^\varepsilon(x_\varepsilon) &\leq \triangle(\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon). \end{aligned} \tag{4.7}$$

Then from (4.6), we have

$$F_\varepsilon(x_\varepsilon, u^\varepsilon(x_\varepsilon), D(\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon)) \leq \varepsilon \triangle (\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon) \tag{4.8}$$

Since, as $\varepsilon \to 0$,

$$\begin{aligned} u^\varepsilon(x_\varepsilon) &\to u(\hat{x}), \\ D(\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon) &\to D(\phi + \delta \left|x - \hat{x}\right|^2)(\hat{x}) = D\phi(\hat{x}), \text{ and} \\ \varepsilon \triangle (\phi + \delta \left|x - \hat{x}\right|^2)(x_\varepsilon) &\to 0 \end{aligned}$$

and F is continuous, we have

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}) \leq 0. \tag{4.9}$$

However, we have to show this for test functions from $C^1(\Omega)$. Let $\phi \in C^1(\Omega)$, and assume $u - \phi$ has a local maximum at $\hat{x} \in \Omega$. We then have to to show that

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}) \leq 0.$$

Let $\phi_n \in C^2(\Omega)$ such that $\phi_n \to \phi$ in $C^1(\Omega)$. Consider $\phi_n + \delta |x - \hat{x}|^2$. For n large enough, $u - (\phi_n + \delta |x - \hat{x}|^2)$ has a local maximum at some $x_n \in \Omega$ and $x_n \to \hat{x}$. Then as shown above, for each n we have,

$$F(x_n, u(x_n), D + D(\phi_n + \delta |x - \hat{x}|^2)(x_n)) \leq 0. \tag{4.10}$$

Since, as $n \to \infty$,

$$\begin{aligned} u(x_n) &\to u(\hat{x}), \text{ and} \\ D(\phi_n + \delta |x - \hat{x}|^2)(x_n) &\to D\phi(\hat{x}) + D(\delta |x - \hat{x}|^2)(\hat{x}) = D\phi(\hat{x}) \end{aligned}$$

and F is continuous, we have,

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}) \leq 0. \tag{4.9}$$

Hence u is a subsolution.

In the case that $\phi \in C^1(\Omega)$, and $u - \phi$ has a local minimum at $\hat{x} \in \Omega$, we consider $\phi_n - \delta |x - \hat{x}|^2$, and we have the reversed inequalities in (4.7)-(4.10), resulting in a reversed inequality in (4.9). Hence we have that u is a supersolution.

Therefore u is a solution of $F(x, u, Du) = 0$ in Ω. ∎

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