Kolmogorov problem on widths asymptotics and pluripotential theory

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Abstract

Given a compact set K in an open set D on a Stein manifold Ω, $\dim \Omega = n$, the set A_K^D of all restrictions of functions, analytic in D with absolute value bounded by 1, is a compact subset of $C(K)$. The problem on the strict asymptotics for Kolmogorov diameters (widths) :

$$-\ln d_s \left(A_K^D \right) \sim \sigma \ s^{1/n}, \ s \to \infty.$$

was stated by Kolmogorov in an equivalent formulation for ε-entropy of that set [13, 14, 16]. It was conjectured in [46, 47] that for "good" pairs (K, D) such an asymptotics holds with the constant $\sigma = 2\pi \left(\frac{n!}{C(K,D)} \right)^{1/n}$, where $C(K, D)$ is the pluricapacity of the "pluricondenser" (K, D), introduced by Bedford-Taylor [6]. In the one-dimensional case it is equivalent to Kolmogorov's conjecture about the ε-entropy of the set A_K^D, which has been confirmed by efforts of many authors (Erokhin, Babenko, Zahariuta, Levin-Tikhomirov, Widom, Nguyen, Skiba - Zahariuta, Fisher - Miccheli, et al).

In [46, 47] the above problem had been reduced (the proof was only sketched there) to the certain problem of pluripotential theory about approximating of the relative Green pluripotential of the "pluricondenser" (K, D) by pluripotentials with finite set of logarithmical singulatities. The latter problem has been solved recently by Poletsky [25] and Nivoche [23, 24]. Here we give a detailed proof of the above-mentioned reduction, which provides, together with the Nivoche-Poletsky result, a positive solution of our conjecture about asymptotics of Kolmogorov diameters.

1   Introduction

Let K be a compact set in an open set D on a Stein manifold Ω, $H^\infty (D)$ the Banach space of all bounded and analytic in D functions with the uniform

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norm, and A_K^D be a compact subset in the space of continuous functions $C(K)$ consisted of all restrictions of functions from the unit ball $\mathbb{B}_{H^\infty(D)}$; since *it will be always assumed that the restriction operator is injective*, one may infer that $A_K^D = \mathbb{B}_{H^\infty(D)}$. Kolmogorov raised the problem about strict asymptotics for ε-entropy of this set ([13, 14, 31, 36, 16])

$$H_\varepsilon\left(A_K^D\right) \sim \tau \left(\ln \frac{1}{\varepsilon}\right)^{n+1}, \ \varepsilon \to 0, \tag{1}$$

with some constant τ (the weak asymptotics for $H_\varepsilon\left(A_K^D\right)$ had been already proved by him under some natural restrictions on K and D).

For a set A in a Banach space X the *Kolmogorov diameters (or widths) of A with respect to the unit ball B_X* of the space X are the numbers (see,e.g., [32]):

$$d_i\left(A, \mathbb{B}_X\right) := \inf_{L \in \mathcal{L}_s} \ \sup_{x \in A} \ \inf_{y \in L} \ \|x - y\|_X, \tag{2}$$

where $\mathcal{L}_i$ is the set of all i-dimensional subspaces of X. For a pair of normed spaces $Y \hookrightarrow X$ with a linear continuous imbedding we shall write simply $d_i(Y, X)$ instead of $d_i(\mathbb{B}_Y, \mathbb{B}_X)$; in particular, $d_i\left(A_K^D\right) := d_i(H^\infty(D), AC(K))$, where $AC(K)$ is the completion of the set of all traces of functions, analytic on K in the space $C(K)$.

From Levin-Tikhomirov [18] it follows that the asymptotics (1) is equivalent to the following asymptotics for Kolmogorov diameters of the set A_K^D:

$$\ln d_i\left(A_K^D\right) \sim -\sigma \ i^{1/n}, \ i \to \infty. \tag{3}$$

with the constant $\sigma = \left(\frac{2}{(n+1)\tau}\right)^{1/n}$.

Kolmogorov conjectured that, in the case $n = 1$, the constant τ coincides with the Green capacity $\tau(K, D)$ for proper pairs (K, D). Recall that the Green capacity of a condenser (K, D) on a Riemann surface Ω is the number $\tau(K, D) := \frac{1}{2\pi} \int \Delta\omega$, where $\omega(z) = \omega(D, K; z)$ is the generalized Green potential, defined by the formula

$$\omega(z) := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \in Sh(D), \ u|_K \leq 0, \ u(\zeta) < 1 \text{ in } D\}, \tag{4}$$

here $Sh(D)$ stands for the class of all subharmonic functions in D and $\Delta\omega$ is understood as a positive Borel measure (supported by K).

Kolmogorov's hypothesis has been confirmed by efforts of many authors ([4, 8, 9, 41, 18, 22, 39, 48, 10]). The following statement gives an idea of those one-dimensional results.

Proposition 1 *Let K be a non-polar compact subset of an open set D of on open one-dimensional Riemann surface Ω, $K = \widehat{K}_D$, and D is a relatively compact open set in Ω with boundary ∂D consisted of a countable set of compact connected components at least one of which has more that one point. Then the asymptotics (3) holds with $n = 1$ and $\sigma = \frac{1}{\tau(K,D)}$.*

An important tool in the proof of such results (see, e.g., [48]) is the classical fact of potential theory about the approximation of the potential (4) by finite combinations $\sum_{k=1}^{m} \alpha_k \, g_D(\zeta_k, z)$, where $g_D(\zeta, z)$ is the Green function of D with the unit logarithmic singularity at ζ; it is for the lack of a proper multidimensional analogue of this fact that the problem on asymptotics (3) for $n \geq 2$ was known for a long time only in some particular cases (see, e.g.,[16, 36, 43])

In [46] (see, also [47]) it was conjectured that for a good enough pair $K \subset D$ on a Stein manifold Ω, $\dim \Omega = n$, the asymptotics (3) holds with $\sigma = 2\pi \left(\frac{n!}{C(K,D)} \right)^{1/n}$, where $C(K, D)$ is the pluricapacity of the "pluricondenser" (K, D), introduced by Bedford-Taylor [6]. It was shown there how to reduce the problem about the asymptotics (3) for $n \geq 2$ to the certain problem of pluripotential theory (suggested as an analogue of the above one-dimensional fact). We state this problem below after some necessary definitions.

The *Green pluripotential* $\omega(z) = \omega(D, K; z)$ of a *pluricondenser* (K, D) on a Stein manifold Ω is defined by the same formula (4) with the class $Psh(D)$ of all *plurisubharmonic functions* in D instead of $Sh(D)$ in it. We say that (K, D) is a *pluriregular pair* on Ω provided the conditions: (a) K is a compact subset of an open set $D \subset \Omega$ such that $K =. \widehat{K}_D$ and D has no component disjoint with K; (b) $\omega(D, K; z) \equiv 0$ on K and $\lim \omega(D, K; z_j) = 1$ for any discrete sequence $\{z_j\}$ in D. Given $F = \{\zeta_1, \ldots, \zeta_\mu, \ldots, \zeta_m\} \subset D$ and $\alpha = (\alpha_\mu) \in \mathbb{R}_+^n$ the *Green multipole plurisubharmonic function* $g_D(F, \alpha; z)$ is defined ([46, 12, 17, 47]) as a regularized upper envelope of the family of all functions $u \in Psh(D)$, negative in D and satisfying the estimate $u(z) \leq \alpha_\mu \ln |t(\zeta_\mu) - t(z)| + const$ in some neighborhood U_μ of each point ζ_μ (in any local coordinates $t : U_\mu \to \mathbb{C}^n$); see below in section 5 about this function more in detail.

Problem 2 *([46, 47]) Given a pluriregular pair (K, D) on a Stein manifold does there exist a sequence of multipole Green functions $g_D\left(F^{(j)}, \alpha^{(j)}; z\right)$ converging to $\omega(D, K; z) - 1$ uniformly on any compact subset of $D \setminus K$?*

This problem has been solved recently by Poletsky [25] and Nivoche [23, 24] (see below Proposition 38), which covers an important part in the final positive proof of our conjecture on Kolmogorov problem.

In this paper we represent a detailed proof of the reduction part which was only sketched in the survey [47] (somewhat more comprehensive proof from [46] has not been published).

It is natural to modify the Kolmogorov problem in the following more general way. Denote by $A(D)$ the Fréchet space of all functions analytic in D with the topology of uniform convergence on compact subsets and by $A(K)$ the locally convex space of all germs of analytic functions on K with the usual inductive limit topology. We are concerned with the strict asymptotics of the sequence of Kolmogorov diameters $d_i(X_1, X_0)$ of the unit ball of a Banach space X_1 with respect to the unit ball of a Banach space X_0 for couples of Banach spaces X_0, X_1 satisfying the linear continuous imbeddings:

$$X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0 \tag{5}$$

and closely related with the spaces $A(D)$ and $A(K)$ in the following sense.

Definition 3 *We say that a couple of Banach spaces X_0, X_1 satisfying the imbeddings (5) is admissible for a pair (K, D) if for any other couple of Banach spaces Y_0, Y_1 satisfying the linear continuous imbeddings:*

$$X_1 \hookrightarrow Y_1 \hookrightarrow A(D); \; A(K) \hookrightarrow Y_0 \hookrightarrow X_0,$$

we have $\ln d_i (Y_1, Y_0) \sim \ln d_i (X_1, X_0)$ as $s \to \infty$.

For any pluriregular pair (K, D) there exists an admissible couple X_0, X_1 (see below, Corollary 18) and the asymptotic class of the sequence $\ln d_i (X_1, X_0)$ is rather a characteristic of the pair (K, D), than of any individual couple X_0, X_1 admissible for this pair. Moreover, since the spaces $A(K)$ and $A(D)$ are nuclear, there are admissible couples of Hilbert spaces, which allows to apply the Hilbert scales technics.

Problem 4 *([46, 47]) Let (K, D) be a pluriregular pair "compact set-open set" on a Stein manifold Ω. Does the strict asymptotics*

$$\ln d_i (X_1, X_0) \sim -2\pi \left(\frac{n! \, i}{C(K, D)} \right)^{1/n}, \; s \to \infty \tag{6}$$

hold for some (hence, for any) couple of Banach spaces X_0, X_1, admissible for (K, D)?

Developing our approach from [46, 47] and applying the above-mentioned result of Nivoche-Poletsky we shall give the positive solution of this problem, namely the following theorem will be proved in Section 9 after substantial preparatory considerations in sections 2-8.

Theorem 5 *The asymptotics (6) holds for any couple of Banach spaces X_0, X_1, admissible for (K, D) if (K, D) is a pluriregular pair on a Stein manifold Ω.*

Remark 6 *The statement of this theorem remains true assuming that the pair (K, D) satisfies all conditions in the definition of a pluriregular pair besides that K is suposed to be only non-pluripolar (instead of the condition $\omega(D, K; z) \equiv 0$ on K). But in what follows, for the simplicity sake, we will consider only pluriregular pairs.*

As a consequence we obtain an answer to the question about the asymptotics (3) specifying the fuzzy terms "good enough" or "proper" in the above conjectures.

Corollary 7 *Given a pluriregular pair (K, D) the asymptotics*

$$\ln d_i \left(A_K^D \right) \sim -2\pi \left(\frac{n! \, i}{C(K, D)} \right)^{1/n}, \; i \to \infty. \tag{7}$$

holds if and only if the couple $(AC(K), H^\infty(D))$ is admissible for (K, D).

The last statement, though being final, is too general and calls for some concrete description of admissibility. We discuss some necessary and sufficient conditions of admissibility of a couple $(AC(K), H^\infty(D))$ in the sections 4,9. On the other hand, we consider there certain classes of *pluriregular* pairs (K, D), for which the asymptotics (3) does not hold with the constant $\sigma = 2\pi \left(\frac{n!}{C(K,D)}\right)^{1/n}$ but may be true with some smaller constant.

2 Preliminaries

Notation. For a pair of positive sequences we write $a_i \prec b_i$ if there is a constant C such that $a_i \leq C\, b_i$. If X and Y are locally convex spaces, then $X \hookrightarrow Y$ stands for a linear continuous imbedding. We use the notation: $|x|_F := \sup\{|x(z)| : z \in F\}$. Given a Banach space X the notation $\mathbb{B}_X$ is for its closed unit ball.

Some facts of Pluripotential Theory. In what follows Ω is a Stein manifold, $\dim \Omega = n$. The set of all *plurisubharmonic* functions in an open set $D \subset \Omega$ is denoted by $Psh(D)$; $MP(D)$ stands for the set of all *maximal plurisubharmonic* functions in D. Let us remind that $u \in Psh(D)$ is *maximal* in D if for any subdomain $G \Subset D$ and for each function $v \in Psh(D)$ it follows that the inequality $v(z) \leq u(z)$ takes place on G provided that it is valid on the boundary ∂G. A Stein manifold Ω is called *pluriregular* (or *hyperconvex*) if there is a negative function $u \in Psh(\Omega)$ such that $\lim u(z_j) = 0$ for any sequence $\{z_j\}$ having no limit point in Ω. An open set $D \Subset \Omega$ is *strongly pluriregular* if there is an open set $G \Supset D$ and a function $u \in C(G) \cap Psh(G)$ such that $D = \{z \in G : u(z) < 0\}$.

The *Green pluripotential of a condenser* (K, D) on Ω (that is K is a compact set in an open set $D \subset \Omega$) is the function

$$\omega(z) = \omega(D, K; z) := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \in P(K, D)\}, \qquad (8)$$

where $P(K, D)$ is the set of all $u \in Psh(D)$ such that $u|_K \leq 0$ and $u(\zeta) < 1$ in D. The following two families of sublevel sets are important for further considerations:

$$D_\alpha := \{z \in D : \omega(z) < \alpha\}, \quad K_\alpha := \{z \in D : \omega(z) \leq \alpha\},\ 0 < \alpha < 1. \qquad (9)$$

A compact set $K \subset \Omega$ is *pluriregular* if $\omega(D, K; z) \equiv 0$ on K for any open set $D \supset K$. We say that *a pair (condenser)* (K, D) is *pluriregular* if (a) both K and D are pluriregular; (b) $\widehat{K}_D = K$; (c) D has no components disjoint with K. It is known that $\omega(D, K; z)$ is continuous in D for a pluriregular condenser (see, e.g. [43, 40]).

Due to Bedford and Taylor [5, 6] (inspired by [7]), the *Monge-Ampére operator* $u \to (dd^c u)^n$ is well-defined as an operator from the space $L^\infty(\Omega, loc) \cap Psh(\Omega)$ to the space $\mathcal{M}(\Omega)$ of non-negative Borel measures with the weak convergence topology; it is "continuous" with respect to monotone sequences

of functions; therewith this operator is continuous as an operator from the space $C\left(\Omega\right) \cap Psh\left(D\right)$ to the space $\mathcal{M}\left(\Omega\right)$. Maximality of a function $u \in Psh\left(\Omega\right) \cap L^{\infty}\left(\Omega, loc\right)$ in $D \subset \Omega$ is equivalent to $\left(dd^c u\right)^n = 0$ in D; in particular, $\left(dd^c \omega\left(D, K; z\right)\right)^n \equiv 0$ in $D \smallsetminus K$ for a pluriregular pair $\left(K, D\right)$ ([5, 26]).

Of prime importance for our considerations is the notion of *pluricapacity of a condenser*, which in the case of a pluriregular condenser can be written in the form ([6]):

$$C\left(K, D\right) := \int_K \left(dd^c \omega\left(D, K; z\right)\right)^n. \tag{10}$$

For facts from Pluripotential Theory which are not explained here (or below) we send the reader to the book [12] warning only that our notation may differ from used there.

Spaces. Let D be an open set on a Stein manifold Ω. Denote by $A(D)$ the Fréchet space of all functions analytic in D with the topology of *locally uniform (or compact) convergence* on D, determined by the sequence of seminorms

$$\|x\|_s := |x|_{K_s} = \max\left\{|x\left(z\right)| : z \in K_s\right\}, \quad x \in A(D), \ s \in \mathbb{N}, \tag{11}$$

where K_s is any non-decreasing sequence of compact subsets exhausting D. By $A(K)$ we denote the locally convex space of all germs of analytic functions on K with the usual inductive limit topology.

Given a compact set K on an open set $D \subset \Omega$ the *restriction operator* $J = J_{D,K} : A\left(D\right) \to A(K)$ is an operator which maps any function $f \in A\left(D\right)$ to the germ $\varphi = Jf \in A(K)$, generated by f.

If X, Y are locally convex spaces and $i : X \to Y$ is a linear continuous injection we say that X is *imbedded* (linearly and continuously) into Y, identifying often X with its image $i\left(X\right)$ and writing $X \hookrightarrow Y$. If this imbedding is dense (i.e. $i\left(X\right)$ is a dense set in Y) then the conjugate mapping $i^* := Y^* \to X^*$ is also a linear continuous injection, so we can identify any linear functional $y^* \in Y^*$ with its image $y' := i^*\left(y^*\right) = y^*|_X$ and write in this case that $Y^* = Y' := i^*\left(Y^*\right) \hookrightarrow X^*$; this imbedding is also dense if the space X is reflexive.

In particular, for a pluriregular pair $\left(K, D\right)$ we shall write

$$A\left(D\right) = J_{D,K}\left(A\left(D\right)\right) \hookrightarrow A\left(K\right); \tag{12}$$
$$A\left(K\right)^* = A\left(K\right)' := J_{D,K}^*\left(A\left(K\right)^*\right) \hookrightarrow A\left(D\right)^*.$$

Given an open set $D \subset \Omega$ the elements of the space $A\left(D\right)^*$ are called usually *analytic functionals* on D (see, e.g., [11]); so, any functional from $A\left(K\right)^*$ is identified in (12) with the corresponding analytic functional on D. Given $F \subset D$ the non-bounded seminorm is introduced

$$|x'|_F^* := \sup\left\{|x'\left(x\right)| : x \in A\left(D\right), \ |x|_F \leq 1\right\} \tag{13}$$

on $A\left(D\right)^*$, which is definitely a norm if $\left(K, D\right)$ with $K = \overline{F}$ is a pluriregular condenser.

Scales. A family of Banach spaces X_α, $\alpha_0 \leq \alpha \leq \alpha_1$, is called a *scale of Banach spaces* (or simply a *scale*) if for arbitrary $\alpha_0 \leq \alpha < \beta \leq \alpha$ two conditions hold: 1) $X_\alpha \hookrightarrow X_\beta$ and 2) $\|x\|_\gamma \leq C(\alpha, \beta, \gamma) \left(\|x\|_{X_\alpha}\right)^{1-\tau(\gamma)} \left(\|x\|_{X_\beta}\right)^{\tau(\gamma)}$ with $\tau(\gamma) = \frac{\gamma - \alpha}{\beta - \alpha}$, $\alpha < \gamma < \beta$. In what follows we send the reader to the monograph [15] for further notions and results about scales.

Here we turn our attention to a particular case of *Hilbert scale* $H_\alpha = H_0^{1-\alpha} H_1^\alpha$, $\alpha \in (-\infty, \infty)$, spanned on a couple of Hilbert spaces with a *dense compact imbedding* $H_1 \hookrightarrow H_0$. Since under such assumptions there is a *common orthogonal basis* $\{\varphi_i\}$ for H_0 and H_1, normalized in H_0 and enumerated by non-decreasing of norms in the space H_1:

$$\|\varphi_i\|_{H_0} = 1, \ i \in \mathbb{N}; \ \mu_i = \mu_i(H_0, H_1) := \|\varphi_i\|_{H_1} \nearrow \infty. \tag{14}$$

this scale is determined by the norms

$$\|x\|_{H_\alpha} := \left(\sum_{i \in \mathbb{N}} |\xi_i|^2 \ \mu_i^{2\alpha}\right)^{1/2}, \ x = \sum_{i \in \mathbb{N}} \xi_i \ \varphi_i, \tag{15}$$

so that the space H_α consists of $x \in H_0$ with a finite norm (15) if $\alpha \geq 0$, while H_α is the completion of H_0 by the norm (15)) if $\alpha < 0$.

Diameters. We shall use the following equivalent definition of the Kolmogorov diameters (2):

$$d_i(X_1, X_0) = \inf\{\inf\{\lambda > 0 : \mathbb{B}_{X_1} \subset \lambda \mathbb{B}_{X_0} + L\} : L \in \mathcal{L}_i\}, \tag{16}$$

where $\mathcal{L}_i$ is the set of all i-dimensional subspaces of X_1.

In the conditions concerned with (14) the following simple expression for the diameters holds (see, e.g., [21], Corollary 3):

$$d_i(H_1, H_0) = \frac{1}{\mu_{i+1}(H_0, H_1)}, \ i \in \mathbb{N}. \tag{17}$$

Hence for the Hilbert scale $H_\alpha = H_0^{1-\alpha} H_1^\alpha$, due to representation (15), we have the equality

$$d_i(H_{\alpha_1}, H_{\alpha_0}) = \frac{1}{\mu_{i+1}(H_0, H_1)^{\alpha_1 - \alpha_0}} = (d_i(H_1, H_0))^{\alpha_1 - \alpha_0}, \ \alpha_1 < \alpha_0. \tag{18}$$

Proposition 8 *Let $X_1 \hookrightarrow Y_1 \hookrightarrow Y_0 \hookrightarrow X_0$ be a quadruple of Banach spaces with dense imbeddings, then there is a constant M such that*

$$d_i(X_1, X_0) \leq M \ d_i(Y_1, Y_0), \ i \in \mathbb{N}. \tag{19}$$

3 Hadamard type inequalities for analytic functions and functionals

Analogously to the one-dimensional case, one of the main applications of Green pluripotential is Two Constants Theorem for analytic functions, in particular,

Hadamard type interpolational estimates ([28], see also [43]), which may be written, for a pluriregular pair (K, D), in the form

$$|f|_{D_\alpha} \leq (|f|_K)^{1-\alpha} \ (|f|_D)^{\alpha}, \ 0 < \alpha < 1, \ f \in H^\infty(D), \tag{20}$$

where the intermediate sets D_α are defined in (9). Those estimates are very useful for constructing of common bases for the spaces $A(K)$ and $A(D)$ ([41, 43, 46, 47], see also [22, 48, 49]), since they provide good estimates for the system $\{f_i(z)\}$, examined for being a basis.

Hadamard type interpolational estimates for analytic functionals are of no less importance. They are needed to provide good estimates for the biorthogonal system of analytic functionals $\left\{f_i'\right\}$. In the one-dimensional case, due to Grothendieck-Köthe-Silva duality, analytic functionals can be represented as analytic functions in the complement of K, so one can use the same inequalities (20) to estimate functionals (see, e.g., [41, 22, 48]). For $n \geq 2$, though this direct way fails, the following analogue of Two Constant Theorem for analytic functionals holds.

Theorem 9 (*[43, 46, 47]) Let (K, D) be a pluriregular pair on a Stein manifold Ω and D be strongly pluriregular. Then for each $\varepsilon > 0$ and $\alpha \in (0, 1)$ there is a constant $M = M(\alpha, \varepsilon)$ such that for any $x' \in AC(K)' \hookrightarrow A(D)^*$ the estimates hold:*

$$|x'|_{D_\alpha}^* \leq M \ \left(|x'|_K^*\right)^{1-\alpha+\varepsilon} \ \left(|x'|_D^*\right)^{\alpha-\varepsilon} . \tag{21}$$

4   Adherent spaces

Let E be a Fréchet space, $\left\{\|x\|_p, \ p \in \mathbb{N}\right\}$ a system of seminorms defined its topology,

$$\|x^*\|_p^* := \sup\left\{|x^*(x)| : x \in E, \ \|x\|_p \leq 1\right\}, \quad x^* \in E^*, \ p \in \mathbb{N}, \tag{22}$$

be the system of polar (non-bounded, in general) norms, and

$$U_p := \left\{x \in E : \|x\|_p \leq 1\right\}, \quad U_p^\circ := \left\{x^* \in E^* : \|x^*\|_p^* \leq 1\right\}, \quad p \in \mathbb{N}. \tag{23}$$

The following interpolation property proved to be useful in studying of structural properties of Fréchet spaces (see, e.g., [42, 45, 37, 38, 20]).

Definition 10 *A Fréchet space E satisfies the property $\mathcal{D}_2$ (we write also $E \in (\mathcal{D}_2)$) if for every $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ there is a constant C provided the estimate:*

$$\left(\|x^*\|_q^*\right)^2 \leq C \ \|x^*\|_p^* \ \|x^*\|_r^*, \quad x^* \in E^* \tag{24}$$

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Definition 11 *A Banach space* $X \hookrightarrow E$ *is said to be adherent to* E *if for each* $p \in \mathbb{N}$ *and any* $\delta > 0$ *there is* $q \in \mathbb{N}$ *and a constant* $C > 0$ *such that*

$$\|x^*\|_q^* \leq C \left(\|x^*\|_{X^*}^* \right)^{1-\delta} \left(\|x^*\|_p^* \right)^{\delta}, \quad x^* \in E^*.$$

It is easy to check that X is adherent to E if and only if one of the following conditions holds (see, e.g.,[37, 20]):

(i) *for any neighborhood* V *of zero in* E *and each* $\delta > 0$ *there is* $p \in N$ *and a constant* $C > 0$ *such that*

$$U_p \subset t^{\delta} \; \mathbb{B}_X \; + \frac{C}{t^{1-\delta}} \; V, \quad t > 0; \tag{25}$$

(ii) *for any neighborhood* V *of zero in* E *and each* $\delta > 0$ *there is* $p \in N$ *and a constant* $C > 0$ *such that*

$$t^{1-\delta} \; V^{\circ} \cap \frac{1}{t^{\delta}} \mathbb{B}_{X^*} \; \subset C \; U_p^{\circ}, \quad t > 0 \tag{26}$$

where $V^{\circ} := \{x^* \in E^* : |x^*(x)| \leq 1, \; x \in V\}$.

Proposition 12 *(D. Vogt [37], Lemma 4) Let a Schwartz Fréchet space* E *satisfy the property* $\mathcal{D}_2$. *Then there is a Banach space* $X \hookrightarrow E$ *adherent to* E.

Proposition 13 *Let* Ω *be a Stein manifold, having finite set of connected components. Then the following statements are equivalent:* *(i)* Ω *is pluriregular;* *(ii)* $A(\Omega) \in \mathcal{D}_2$; *(iii) there exists a Hilbert space* $H \hookrightarrow E$ *adherent to the space* $A(\Omega)$.

Proof. The relations $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$ are due to [43, 45] (see also, [46, 47]), therewith the proof of $(i) \Rightarrow (ii)$ is based on Hadamard type inequalities for analytic functionals (see, Theorem 9 above); $(ii) \Leftrightarrow (iii)$ follows from Vogt's result (see, Proposition 12 above), taking into account that $A(\Omega)$ is nuclear. ∎
It should be mentioned that Aytuna [2] constructed, under the assumption (i), an adherent Hilbert space for $A(\Omega)$ as a weighted L_2-space, applying Hörmander's $\bar{\partial}$-problem technics. For good enough domains we have the following easy description of adherent spaces for $A(\Omega)$ (see, e.g., [46, 47]).

Proposition 14 *Let* D *be a strongly pluriregular domain on a Stein manifold. Then any Banach space* X *satisfying the dense imbeddings* $A(\overline{D}) \hookrightarrow X \hookrightarrow A(D)$ *is adherent to* $A(D)$; *in particular, the space* $H^{\infty}(D)$ *is adherent to* $A(D)$.

Definition 15 *Let* K *be a compact set on a Stein manifold* Ω *and a Banach space* X *satisfy the dense imbedding* $A(K) \hookrightarrow X$. *We say that* X *is adherent to* $A(K)$ *if its dual* $X^* \hookrightarrow A(K)^*$ *is adherent to* $A(K)^*$ *(in the sense of Definition 11).*

The following fact cannot be obtained from Proposition 13 for $\dim \Omega \geq 2$, because there is no proper multidimensional analogue of the Grothendieck–Köthe-Silva duality.

Proposition 16 *([43, 44]) Let K be a compact set on a Stein manifold Ω such that $A(\Omega)$ is dense in $A(K)$ (i.e. K is a Runge set on Ω) and Ω has no connected component disjoint with K. The following statements are equivalent: (i) K is pluriregular; (ii) $A(K)^* \in \mathcal{D}_2$; (iii) there is an adherent to $A(K)$ Hilbert space $H \hookleftarrow A(K)$; (iv) the space $AC(K)$ is adherent to $A(K)$.*

It follows from $(i) \Rightarrow (ii)$ any Hilbert space H, satisfying the dense imbeddings $A(K) \hookrightarrow H \hookrightarrow AC(K)$, is adherent to $A(K)$. A more explicit example of a Hilbert space adherent to $A(K)$ is the space $H = AL_2(K, \mu)$ obtained as a completion of $A(K)$ by the norm

$$\|x\| := \left(\int_K |x(z)|^2 \; d\mu \right)^{1/2} ,$$

where $\mu := (dd^c \omega)^n$ with $\omega(z) = \omega(D, K; z)$ and D is any open set composing a pluriregular pair (K, D) with K ([46, 49, 47], see also [50] for a characterization of Borel measures μ providing that $H = AL_2(K, \mu)$ is adherent to $A(K)$).

Definition 17 *Given a pluriregular pair (K, Ω) a couple of Banach spaces (X_0, X_1) is said to be adherent to a couple $(A(K), A(\Omega))$ if*

$$X_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow X_0 \tag{27}$$

and X_1 is adherent to $A(\Omega)$, X_0 is adherent to $A(K)$.

Propositions 13 and 16 yield the following

Corollary 18 *For any pluriregular pair (K, Ω) there exists a couple of Banach (Hilbert) spaces (X_0, X_1) adherent to $(A(K), A(\Omega))$.*

The following statement was proved in [43] for the particular case of Hilbert couples $(X_0, \ X_1)$; here we derive the general case from the Hilbert version using standard technics of Banach scales (see, e.g., [15]).

Theorem 19 *Suppose that $(K, \ D)$ is a pluriregular pair, D is a Stein manifold. Let $(X_0, \ X_1)$ be a couple of Banach spaces adherent to the couple $(A(K), A(D))$ such that X_1 is imbedded normally into X_0 and $\mathbb{B}_{X_1}$ is closed in X_0. Let X_α, $0 \leq \alpha \leq 1$, be any regular normal scale of Banach spaces connecting the spaces $X_0, \ X_1$. Then the following linear continuous imbeddings are true:*

$$A(K_\alpha) \hookrightarrow X_\alpha \hookrightarrow A(\ D_\alpha), \quad 0 < \alpha < 1, \tag{28}$$

where $K_\alpha, \ D_\alpha$ have been defined in (9).

Proof. Consider any regular normal scale X_α, $0 < \alpha < 1$, connecting X_0 and X_1. First we are going to show that the imbeddings

$$A(D) \hookleftarrow X_\alpha \hookleftarrow A(K) \tag{29}$$

hold for each $\alpha \in (0,1)$. By the adherence of X_1 to $A(D)$, for each $\alpha < 1$ there is $\beta < 1$ such that

$$|x'|_{D_\beta}^* \leq C \left(\|x'\|_{X_0^*} \right)^{1-\alpha} \left(\|x'\|_{X_1^*} \right)^\alpha, \quad x' \in A(D)^*.$$

Therefore for the minimal scale $X_\alpha^{\min}$ connecting the spaces X_0 and X_1 ([15]) we obtain the estimate

$$\|x\|_{X_\alpha^{\min}} := \sup_{x' \in X_0^*} \left\{ \frac{|x'(x)|}{\left(\|x'\|_{X_0^*} \right)^{1-\alpha} \left(\|x'\|_{X_1^*} \right)^\alpha} \right\} \leq C \sup_{x \backslash \in X_0^\backslash} \left\{ \frac{|x'(x)|}{|x'|_{D_\beta}^*} \right\} = C |x|_{D_\beta}.$$

Hence $A(D) \hookleftarrow X_\alpha^{\min}$, $0 < \alpha < 1$. Then, due to Lions-Peetre [19] (see, also [15], Chapter IV, Theorem 2.20), the left imbedding in (29) holds for any scale X_α, $0 < \alpha < 1$, connecting X_0 and X_1.

The imbeddings (27) imply the natural dense imbeddings $X_0^* \hookleftarrow A(K)^* \hookleftarrow A(D)^* \hookleftarrow Y_0$, where Y_0 is the closure of $A(\Omega)^*$ in X_1^*. Since $\mathbb{B}_{X_1}$ is closed in X_0 we have, by Aronszajn-Gagliardo [1], that Y_0 is a norming set for X_1. Therefore, taking into account the reflexivity of $A(K)$, the adherence of X_1 to $A(K)$ implies that for any $\gamma > 0$ there is $\delta > 0$ and $C > 0$ such that

$$|x|_{D_\delta} \leq C \left(\|x\|_{X_0} \right)^{1-\gamma} \left(\|x\|_{X_1} \right)^\gamma, \quad x \in X_1.$$

Then $\widehat{X}_\gamma \hookrightarrow AC(D_\delta) \hookrightarrow A(K)$, where $\widehat{X}_\theta := (X_0, X_1)_{\theta, L_{1,*}, L_{1,*}}$ is the *maximal scale of means* ([15], Chapter IV, Lemma 2.6). Therefore, applying this imbedding with $\gamma < \alpha/2$ and taking into account that any regular scale is almost imbedded into any scale ([15], Chapter IV, Corollary 3), we obtain the right imbedding in (28).

Now we take any pair of Hilbert spaces H_0, H_1 satisfying the imbeddings

$$X_1 \hookrightarrow H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0 \hookrightarrow X_0.$$

Then by (29) the imbeddings

$$X_1 \hookrightarrow H_1 \hookrightarrow X_{1-\varepsilon} \hookrightarrow X_\varepsilon \hookrightarrow H_0 \hookrightarrow X_0$$

hold for every $\varepsilon : 0 < \varepsilon < 1/2$. Applying now (29) to the Hilbert scale $H_\alpha = (H_0)^{1-\alpha} (H_1)^\alpha$, which is true due to [43, 46, 47], and using the interpolation property of scales [15], we obtain the imbeddings

$$X_{\alpha+\varepsilon(1-\alpha)} \hookrightarrow H_\alpha \hookrightarrow A(D_\alpha), \quad A(K_\alpha) \hookrightarrow H_\alpha \hookrightarrow X_{\alpha(1-\varepsilon)}, \quad 0 < \alpha < 1.$$

Since $\varepsilon > 0$ may be taken arbitrarily small here, we obtain (28) what ends the proof. ∎

The following result will be useful for investigating the problem about asymptotics (7).

Theorem 20 *Let a couple of Banach spaces* (X_0, X_1) *be adherent to the couple* $(A(K), A(\Omega))$. *Then it is admissible for* (K, Ω).

Proof. First we show that any couple of Hilbert spaces (H_0, H_1), which is adherent to $(A(K), A(\Omega))$, is admissible for (K, Ω). Indeed, let (Y_0, Y_1) be any couple of Banach spaces with the imbeddings

$$H_1 \hookrightarrow Y_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow Y_0 \hookrightarrow H_0.$$

Then, by Proposition 8, $d_i(H_1, H_0) \prec d_i(Y_1, Y_0)$. On the other hand, due to (18),(28),
$$d_i(Y_1, Y_0) \prec d_i(H_{1-\varepsilon}, H_\varepsilon) = (d_i(H_1, H_0))^{1-2\varepsilon}$$
for arbitrary $\varepsilon > 0$. Therefore $\ln d_i(Y_1, Y_0) \sim \ln d_i(H_1, H_0)$. So, (H_0, H_1) is admissible for (K, Ω).

Let (X_0, X_1) be an arbitrary couple of Banach spaces (X_0, X_1) adherent to $(A(K), A(\Omega))$. Then, due to Propositions 13 and 16 , we can find a couple of Hilbert spaces (H_0, H_1) admissible for (K, Ω) and such that

$$X_1 \hookrightarrow H_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow H_0 \hookrightarrow X_0.$$

Without loss of generality we may assume that the marginal imbeddings are normalized.

Fix any $\delta > 0$ and choose any sequence of positive numbers $\alpha_p \uparrow 1$. Then, by the imbeddings (28), the system of norms $\left\{ \|x\|_{H_{\alpha_p}}, \ p \in \mathbb{N} \right\}$ defines the original topology of the space $A(\Omega)$. Since X_1 is adherent to the space $E = A(\Omega)$, we obtain, applying (25) with $V = \mathbb{B}_{X_0} \cap E$, $U_p = \mathbb{B}_{H_{\alpha_p}} \cap E$, that there is $p = p(\delta)$ and $C = C(\delta) > 0$ such that

$$\mathbb{B}_{H_{\alpha_p}} \subset \left(\frac{1}{\lambda} \right)^\delta \mathbb{B}_{X_1} + C\lambda^{1-\delta} \mathbb{B}_{X_0}, \quad \lambda > 0. \tag{30}$$

On the other hand, consider a sequence $\beta_q \downarrow 0$. Then, due to (28), the sequence of non-bounded norms $\left\{ \|x\|_{H_{\alpha_q}}, \ q \in \mathbb{N} \right\}$ defines the original inductive limit topology on the space $A(K)$. Since X_0 is adherent to $A(K)$, applying (26) and taking into account reflexivity of the space $A(K)$, we obtain that there is $q = q(\delta)$ and $C_1 = C_1(\delta) > 0$ such that

$$\lambda \mathbb{B}_{X_0} \cap \mathbb{B}_{X_1} \subset C_1 \lambda^{1-\delta} \mathbb{B}_{H_{\alpha_q}}, \quad \lambda > 0. \tag{31}$$

Now take an arbitrary $\lambda > 0$ such that

$$d_i(X_1, X_0) < \lambda < 2d_i(X_1, X_0). \tag{32}$$

By the definition, there is $L \in \mathcal{L}_i$ such that

$$\mathbb{B}_{X_1} \subset \lambda \mathbb{B}_{X_0} + L. \tag{33}$$

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Combining (30), (31), (32), (33), we obtain that

$$
\begin{aligned}
\mathbb{B}_{H_{\alpha_p}} &= \mathbb{B}_{H_{\alpha_p}} \cap \mathbb{B}_{X_1} \subset (1+C)\,\lambda^{1-\delta}\,\mathbb{B}_{X_0} \cap \mathbb{B}_{X_1} + L \\
&\subset C_1\,(1+C)\,\lambda^{(1-\delta)^2}\,\mathbb{B}_{H_{\beta_q}} + L.
\end{aligned}
$$

By the definition of diameters (16), we have

$$
d_i\,(H_1, H_0) \le d_i\left(H_{\alpha_p}, H_{\beta_q}\right) \prec (d_i\,(X_1, X_0))^{(1-\delta)^2}.
$$

Together with the evident relation $d_i\,(X_1, X_0) \prec d_i\,(H_1, H_0)$ and due to arbitrariness of $\delta > 0$, this yields

$$
\ln d_i\,(X_1, X_0) \sim \ln d_i\,(H_1, H_0), \quad i \to \infty,
$$

hence the couple of Banach spaces (X_0, X_1) is admissible for (K, Ω). ∎

Remark 21 *The admissibility is, in general, essentially weaker than the adherence. Indeed, consider the simplest pair of two concentric disks $K = \overline{\mathbb{D}}$, $D = \mathbb{D}_R$, $R > 1$. Define the couple of Hilbert spaces (H_0, H_1) by the norms*

$$
\|x\|_\nu := \left(\sum_{k=0}^{\infty} |\xi_k|^2\, a_k^{(\nu)}\right)^{1/2}, \quad x\,(z) = \sum_{k=0}^{\infty} \xi_k\, z^k, \quad \nu = 0, 1 \tag{34}
$$

with $a_k^{(\nu)} := R^{-\nu k}$ if $k \ne 2^j$ and $a_k^{(\nu)} := (2R)^{-\nu k}$ if $k = 2^j$, $j \in \mathbb{N}$. An easy calculation shows that $\ln d_i\,(H_1, H_0) \sim -\frac{i}{\ln R} = -\frac{i}{\tau(K,D)}$ as $i \to \infty$, hence the couple (H_0, H_1) is admissible for (K, D), but it is not adherent to $(A\,(K), A\,(D))$.

Problem 22 *Let $(AC\,(K), H^\infty\,(D))$ be an admissible couple for a pluriregular pair (K, D). Is then $H^\infty\,(D)$ adherent to $A\,(D)$?*

5 Maximal plurisubharmonic functions with isolated singularities

Let Ω be a pluriregular Stein manifold. Given a finite set $\Lambda = \left\{\zeta_\mu : \mu = 1, \dots, m\right\} \subset \Omega$, denote by $\mathfrak{G}_\Lambda\,(\Omega)$ the class of all functions $u \in Psh\,(\Omega) \cap MP\,(\Omega \smallsetminus \Lambda)$ taking the value $-\infty$ on Λ and satisfying the conditions $\lim_{j \to \infty} u\,(z_j) = 0$ for any sequence $\{z_j\}$ discrete in Ω (or, shortly, $\lim_{z \to \partial\Omega} u\,(z) = 0$). In particular, if Ω is a regular one-dimensional Stein manifold, then every function $u \in \mathfrak{G}\,(\Omega)$ is represented in the form

$$
u\,(z) = -\sum_{\mu=1}^{m} \alpha_\mu\, G_\Omega\left(\zeta_\mu, z\right), \tag{35}
$$

where $\alpha_\mu > 0$ and $G_\Omega\,(\zeta, z)$ is the Green function for Ω with the unit logarithmic singularity at the point ζ. It is well-known how important are those

functions in one-dimensional complex analysis, especially in the approximation and interpolation theory.

By contrast, the multidimensional case is much more complicated: on the one hand, usually a function $u \in \mathfrak{G}(\Omega)$ with more than one pole is not representable as a sum of functions with single poles and , on the other hand, isolated singularities of such functions are extremely varied (as seen from the following example).

Example 23 *Let Δ be a complete logarithmically convex n-circular domain, relatively compact in a unit polydisc $\mathbb{U}^n$, and*

$$h(\theta) = h_\Delta(\theta) := \sup \left\{ \sum_{\nu=1}^{n} \theta_\nu \ \ln|z_\nu| : z = (z_\nu) \in \Delta \right\}$$

its characteristic function. Then every function

$$k_\Delta(z) := \sup \left\{ \frac{\sum_{\nu=1}^{n} \theta_\nu \ \ln|z_\nu|}{-h(\theta)} : \theta = (\theta_\nu) \in \Sigma \right\}, \ z \in \mathbb{U}^n \tag{36}$$

is of the class $\mathfrak{G}_{\{0\}}(\mathbb{U}^n)$. All this singularities at the origin are different, in the sense of the following.

Definition 24 *Given a point $\zeta \in \Omega$ we consider the set of all functions $\varphi \in Psh(U) \cap MP(U \smallsetminus \{\zeta\})$ with $\varphi(\zeta) = -\infty$, where $U = U(\varphi)$ is an open neighborhood of ζ and define the equivalence relation by*

$$\varphi \sim \psi \overset{def}{=} \lim_{z \to \zeta} \frac{\varphi(z)}{\psi(z)} = 1. \tag{37}$$

Denote by $\mathfrak{S}_\zeta$ the set of all equivalence classes under the relation (37) (we call them also "standard singularities" at the point ζ) and write $\sigma = [\varphi]$ if $\varphi \in \sigma \in \mathfrak{S}_\zeta$. A singularity $\sigma \in \mathfrak{S}_\zeta$ is called continuous if there is a representative $\varphi \in \sigma$, continuous in some punctured neighborhood of ζ.

Theorem 25 *([46, 47]) Given a pluriregular Stein manifold Ω, a finite set $F = \{\zeta_\mu : \mu = 1, \ldots, m\}$ on it and continuous standard singularities $\sigma_\mu = [\varphi_\mu] \in \mathfrak{S}_{\zeta_\mu}$, $\mu = 1, \ldots, m$, there exists the unique function $g \in \mathfrak{G}_F(\Omega)$ having the singularities σ_μ at the points ζ_μ. This function is continuous in $\Omega \smallsetminus F$ and is defined by the formula:*

$$g(z) = g_\Omega(F, (\sigma_\mu); z) := \sup \{u(z) : u \in P(\Omega, F, (\sigma_\mu))\}, \tag{38}$$

where $P(\Omega, F, (\sigma_\mu))$ is the class of all functions $u \in Psh(\Omega)$ such that $u(z) < 0$ in Ω and there is a constant $c = c(u)$ provided the estimate $u(z) \leq \varphi_\mu(z) + c$ in some neighborhood of ζ_μ, $\mu = 1, \ldots, m$.

Proof. We suppose, without loss of generality, that the neighborhoods $U_\mu = U(\varphi_\mu)$ are disjoint, set $U := \cup U_\mu$ and define a function φ on U so that $\varphi(z) := \varphi_\mu(z)$ for $z \in U_\mu$. Consider the sets

$$\Phi_\lambda = \{z \in U : \varphi(z) < -\lambda\}; \ F_\lambda = \{z \in U : \varphi(z) \leq -\lambda\}, \ 0 < \lambda < \infty. \tag{39}$$

Choose $\delta > 0$ such that F_δ is compact in Ω. For any $u \in P(\Omega, F, (\sigma_\mu))$ find constants c and $\tau \geq \delta$ such that $u(z) \leq \varphi(z) + c$ if $z \in F_\tau$. Take $\lambda_0 = \max\{c, \tau\}$. The function $\psi(z) = \frac{\lambda - c}{\lambda - \delta}(\varphi(z) + \delta)$ is maximal plurisubharmonic function in U for $\lambda > \lambda_0$ and, by the construction, $u(z) \leq \psi(z)$ if $z \in \partial(\Phi_\delta \smallsetminus F_\lambda)$. Hence,

$$u(z) \leq \frac{\lambda - c}{\lambda - \delta}(\varphi(z) + \delta), \ z \in \Phi_\delta \smallsetminus F_\lambda, \ \lambda > \lambda_0 \ .$$

Tending λ to infinity in this inequality, we obtain the estimate $u(z) \leq \varphi(z) + \delta$ in Φ_δ for any $u \in P(\Omega, \Lambda, (\sigma_\mu))$ and derive from here that the function $\widetilde{g} := \limsup\limits_{\varsigma \to z} g(\varsigma)$ satisfies the estimate

$$\widetilde{g}(z) \leq \varphi(z) + \delta, \ z \in \Phi_\delta, \tag{40}$$

hence belongs to the class $P(\Omega, F, (\sigma_\mu))$, therefore $g(z) \equiv \widetilde{g}(z)$. Then the function

$$v(z) := \begin{cases} g(z), & z \in \Omega \smallsetminus \Phi_\delta \\ \max\{g(z), \varphi(z) + \delta - \gamma\}, & z \in \Phi_\delta \end{cases} ,$$

where $\gamma := -\inf\{g(z) : z \in \partial\Phi_\delta\}$, belongs to the class $P(\Omega, F, (\sigma_\mu))$ itself, hence $v(z) \leq g(z)$. Therefore

$$\varphi(z) \leq g(z) - \delta + \gamma, \ z \in \Phi_\delta. \tag{41}$$

From $g(z) = \widetilde{g}(z) \leq 0$ and (39), (40) we derive that

$$g(z) \leq (\lambda + \delta)(\omega(\Omega, F_\lambda; z) - 1), \ z \in \Omega \smallsetminus F_\lambda, \ \lambda > \lambda_0 \ . \tag{42}$$

On the other hand, due to (41), if $\lambda > \lambda_1 := \max\{\gamma, \lambda_0\}$, the function

$$w(z) := \begin{cases} \max\{g(z), (\lambda + \delta - \gamma)((\omega(\Omega, F_\lambda; z) - 1))\}, & z \in \Omega \smallsetminus F_\lambda \\ g(z), & z \in F_\lambda \end{cases}$$

belongs to the class $P(\Omega, F, (\sigma_\mu))$, which provides, together with (42), the estimates

$$(\lambda + \delta - \gamma)((\omega(\Omega, F_\lambda; z) - 1)) \leq g(z) \leq (\lambda + \delta)(\omega(\Omega, F_\lambda; z) - 1) \tag{43}$$

for $z \in \Omega \smallsetminus F_\lambda$, $\lambda > \lambda_1$. We conclude from here that

$$g(z) = \lim\limits_{\lambda \to \infty} \lambda(\omega(\Omega, F_\lambda; z) - 1),$$

with uniform convergence on any compact subset of $\Omega \smallsetminus F$. Since the pair (Ω, F_λ) is pluriregular for each λ, the function $\lambda(\omega(\Omega, F_\lambda; z) - 1)$ is continuous in Ω for each λ, hence the function g is continuous in $\Omega \smallsetminus F$. From (43) we derive also that $\lim_{z \to \partial D} g(z) = 0$, so $g \in \mathfrak{G}_F(\Omega)$. $\blacksquare$

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Corollary 26 *Given a finite set $F = \{\zeta_\mu : \mu = 1, \ldots, m\}$ on a pluriregular Stein manifold and a vector $\alpha = (\alpha_\mu) \in \mathbb{R}_+^m$ there is the unique function $g(z) = g_\Omega(F, \alpha; z) \in \mathfrak{G}_F(\Omega)$ having at the point ζ_μ the standard singularity $\sigma_\mu = [\varphi_\mu]$ defined by the function $\varphi_\mu(z) := \alpha_\mu \ln |t_\mu(\zeta_\mu) - t_\mu(z)|$, where $t_\mu : U_\mu \to \mathbb{C}^n$ are local coordinates in a neighborhood U_μ of the point ζ_μ, $\mu = 1, \ldots, m$ (we call this function multipolar Green function).*

Definition 27 *([46, 47]; cf.,[6], section 9) Let $u \in Psh(\Omega)$, $E \Subset \Omega$. The MP-balayage (sweeping out) of of the function u with respect to to the set $\Omega \setminus E$ is the function*

$$s(z) = s(E, u; z) := \limsup_{\zeta \to z} \sup\{v(\zeta) : v \in P(\Omega, E; u)\}, \qquad (44)$$

where $P(\Omega, E; u)$ is the class of all functions $v \in Psh(\Omega)$, satisfying $v(z) \leq u(z)$, $z \in \Omega \setminus E$.

The following lemma can be easily deduced from the Minimum Principle for plurisubharmonic functions (see, e.g., [5, 26]).

Lemma 28 *If u_1, $u_1 \in C(\Omega) \cap Psh(\Omega)$ and $G \Subset \Omega$ is a strongly pluriregular open set, then*

$$\sup\{s(G, u_1; z) - s(G, u_2; z) : z \in \Omega\} \leq \sup\{u_1(z) - u_2(z) : z \in \Omega\}.$$

Proposition 29 *Let $u \in C(\Omega) \cap Psh(\Omega)$, $G \Subset D \Subset \Omega$. Then*

$$\int_D (dd^c u(z))^n = \int_D (dd^c s(G, u; z))^n. \qquad (45)$$

Proof. Let first $u \in C^2(\Omega')$, $D \Subset \Omega' \Subset \Omega$, $\partial D \in C^1$. Then, by Stokes' formula,

$$\int_D (dd^c u)^n = \int_{\partial D} d^c u \wedge (dd^c u)^{n-1}. \qquad (46)$$

It is easily seen that Stokes' formula can be applied to functions which are C^2 only in a neighborhood of the boundary ∂D and so, to the function $s(z) := s(G, u; z)$. Since $u(z) \equiv s(z)$ in a neighborhood of ∂D, we obtain (45) and, furthermore, the smoothness assumption on ∂D can be dropped.

In the general case, take an approximating sequence $u_\nu \in C^2(\overline{\Omega'}) \cap Psh(\Omega')$, $D \Subset \Omega' \Subset \Omega$, such that $|u - u_\nu|_{\overline{\Omega'}} \to 0$. Then, by Lemma 28, $|s - s_\nu|_{\overline{\Omega'}} \to 0$, where $s_\nu(z) = s(G, u_\nu; z)$. Taking into account continuity of the Monge-Ampére operator (see, e.g., [5, 26]), the limit transition in the formula (45), with u_ν, s_ν instead of u, s, gives this formula in the general case. ∎

Definition 30 *Given a point ζ on a Stein manifold Ω, $\dim \Omega = n$, and $\sigma = [\varphi] \in \mathfrak{S}_\zeta$ the charge of the standard singularity σ is the value*

$$\nu_\zeta(\sigma) = \nu_\zeta\{\varphi\} := \left(\frac{1}{2\pi}\right)^n \int_D (dd^c s(\Delta_\lambda, \varphi; z))^n, \ \lambda > \delta, \qquad (47)$$

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where $\Delta_\lambda := \{z \in \Delta : \varphi(z) < \lambda\}$, $\Delta = \Delta(\varphi) \Subset \Omega$ is an open neighborhood of ζ provided $\varphi \in Psh(\Delta) \cap MP(\Delta \smallsetminus \{\zeta\})$ and $\delta = \delta(\varphi)$ is such that $\Delta_\delta \Subset \Omega$.

It is clear that

$$\omega\left(\Delta_{\lambda_1}, \overline{\Delta_{\lambda_0}}; z\right) = \frac{s(\Delta_{\lambda_0}, \varphi; z) + \lambda_0}{\lambda_0 - \lambda_1} \tag{48}$$

and

$$\nu_\zeta\{\varphi\} = (\lambda_0 - \lambda_1)^n \ C\left(\overline{\Delta_{\lambda_0}}, \Delta_{\lambda_1}\right), \tag{49}$$

where $\delta < \lambda_1 < \lambda_0$.

Proposition 31 *The charge of a standard singularity is well-defined, i.e. the value (47) does not depend on a choice of λ or φ.*

Proof. The value (47) does not depend on $\lambda > \delta$, due to Proposition 29. To show that it is independent also of a choice of a representative φ in the class σ, we take another representative $\varphi' \in \sigma = [\varphi]$ Then for each $\varepsilon > 0$ there is $\gamma = \gamma(\varepsilon)$ such that

$$\Delta_{(1+\varepsilon)\lambda} \subset \Delta'_\lambda \subset \Delta_{(1-\varepsilon)\lambda}, \ \lambda \geq \gamma,$$

where Δ'_λ are sublevel domains for the function φ'. Using monotonicity of the capacity ([26, 6]) and the relations (46), (48), (49) with $\lambda_1 = \lambda > \gamma$ and $\lambda_0 = 2\lambda$, we get

$$\frac{\nu_\zeta\{\varphi\}}{(1 + 3\varepsilon)\lambda} \leq \frac{\nu_\zeta\{\varphi'\}}{\lambda} \leq \frac{\nu_\zeta\{\varphi\}}{(1 - 3\varepsilon)\lambda}, \ \lambda \geq \gamma,$$

which implies the equality $\nu_\zeta\{\varphi\} = \nu_\zeta\{\varphi'\}$, since $\varepsilon > 0$ is arbitrary. $\blacksquare$

Definition 32 *Given a function $g \in \mathfrak{G}(\Omega)$ with the set of singularities $F = F(g) = \{\zeta_\mu : \mu = 1, \ldots, m\}$ consider its sublevel domains*

$$\Omega_\lambda := \{z \in \Omega : g(z) < \lambda\}, \ 0 < \lambda < \infty. \tag{50}$$

The charge of g (supported by the set F) is defined as the value

$$\nu\{g\} := \left(\frac{1}{2\pi}\right)^n \int\limits_\Delta (dd^c s(\Omega_\lambda, g; z))^n = \sum_{\mu=1}^m \nu_{\zeta_\mu}[g], \ \lambda > 0. \tag{51}$$

Proposition 33 *The charge of the multipole Green plurifunction $g(z) = g_\Omega(F, \alpha; z)$ is the value*

$$\nu(g) = \frac{(2\pi)^n C(K, \Omega_\lambda)}{\lambda} = \sum_{\mu=1}^m (\alpha_\mu)^n \tag{52}$$

where Ω_λ are the sublevel domains (50), $0 < \lambda < \infty$.

Proof. Proposition 29 gives that the charge $\nu(g)$ does not depend on the choice of local coordinates in the definition of the function $g_\Omega(F, \alpha; z)$. Therefore, applying (47), (51) and the well-known Jensen equality (see, e.g.,[12], Example 6.5.6):

$$\left(\frac{1}{2\pi}\right)^n \int\limits_{\partial \mathbb{B}(o,r)} d^c\left(\ln|z|\right) \wedge \left(dd^c\left(\ln|z|\right)\right)^{n-1} = 1,$$

we obtain (52). ∎

6 Generalized Schwartz and Bernstein Lemmas

Let Ω be a pluriregular Stein manifold, $\dim \Omega = n$, $F = \left\{\zeta_\mu : \mu = 1, \ldots, m\right\} \subset \Omega$; $\sigma = (s_\mu) \in \mathbb{Z}_+^n$. Denote by $A_0\left((F, \sigma), \Omega\right)$ the ideal consisted of functions $f \in A(\Omega)$ vanishing on F and having zero of order s_μ at the point ζ_μ, $\mu = 1, \ldots, m$, and set

$$A_0^\perp(F, \sigma) := \left\{f' \in A(F)' : f'(f) = 0 \text{ for all } f \in A_0\left((F, \sigma), \Omega\right)\right\} \quad (53)$$

An analytic functional $f' \in A_0^\perp(F, \sigma) \subset A(\Omega)'$ is called *discrete rational functional having the poles of order at least s_μ at the point ζ_μ, $\mu = 1, \ldots, m$.*

The following statement may be considered as a generalization of the classical Schwartz Lemma.

Theorem 34 *Let Ω be a pluriregular Stein manifold, $\dim \Omega = n$, and*

$$F = \left\{\zeta_\mu : \mu = 1, \ldots, m\right\} \subset \Omega; \sigma = (s_\mu) \in \mathbb{Z}_+^n.$$

Let $f \in A_0\left((F, \sigma), \Omega\right)$. Then the estimates

$$|f(z)| \leq |f|_\Omega \ \exp s(\alpha) g_\Omega(F, \alpha; z), \ z \in \Omega, \quad (54)$$

hold with $s(\alpha) := \inf\left\{\dfrac{s_\mu}{\alpha_\mu} : \mu = 1, \ldots, m\right\}$.

Proof. It makes sense to consider (54) only for bounded functions f. The function $u(z) := \dfrac{\ln|f(z)| - \ln|f|_\Omega}{s(\alpha)}$ belongs to the class $P\left(\Omega, F, (\sigma_\mu)\right)$ with the singularities σ_μ defined by the function $\varphi_\mu(z) := \ \alpha_\mu \ln\left|t(\zeta_\mu) - t(z)\right|$, where $t : U_\mu \to \mathbb{C}^n$ are a local coordinates in a neighborhood U_μ of the point ζ_μ, $\mu = 1, \ldots, m$. Therefore $u(z) \leq g_\Omega(F, \alpha; z)$ in Ω which is equivalent to (54). ∎

Now we consider the statement which is a generalization of the classical Bernstein-Walsh-Siciak Lemma (see, e.g., [28, 44]) in order to fit estimates of discrete rational functionals.

Theorem 35 *Let D be a strictly pluriregular open set on a Stein manifold Ω, $F = \left\{\zeta_\mu : \mu = 1, \ldots, m\right\} \subset D$; $\sigma = (s_\mu) \in \mathbb{Z}_+^n$ and*

$$\Phi_\lambda = \Phi_\lambda(F, \sigma) := \left\{z \in D : g_D(F, \alpha; z) < -\lambda\right\}, \ 0 < \lambda < \infty. \quad (55)$$

Suppose $f' \in A_0^{\perp}(F, \sigma)$. Then for each $\delta > 0$ the estimates

$$|f'|_{\Phi_\lambda}^* \leq C \ |f'|_D^* \ \exp(\lambda + \delta) s, \ 0 < \lambda < \infty, \tag{56}$$

hold with some constant $C = C(\lambda, \delta)$ and $s = s(\alpha) := \sup\left\{\dfrac{s_\mu}{\alpha_\mu} : \mu = 1, \ldots, m\right\}$.

Proof. For each point $\zeta_\mu \in F$ we choose some local coordinates

$$t^{(\mu)} = \left(t_j^{(\mu)}\right) : \Delta^{(\mu)} \to \mathbb{U}^n, \tag{57}$$

with mutually disjoint neighborhoods $\Delta^{(\mu)}$ and set $\Delta = \cup_{\mu=1}^m \Delta^{(\mu)}$. Then

$$g_\Delta(F, \alpha; z) = \left\{\sup\left\{\ln\left|t_j^{(\mu)}(z)\right| : j = 1, \ldots, n\right\}, \ z \in \Delta^{(\mu)}, \ \mu = 1, \ldots, m. \tag{58}$$

and

$$\Delta_\tau := \{z \in \Delta : g_\Delta(F, \alpha; z) < \tau\} = \cup_{\mu=1}^m \Delta_\tau^{(\mu)} \tag{59}$$

where $\Delta_\tau^{(\mu)} = \left\{z \in \Delta^{(\mu)} : \left|t_j^{(\mu)}(z)\right| < \exp\left(-\dfrac{\tau}{\alpha_\mu}\right)\right\}, \ 0 < \tau < \infty$. The system

$$f_{k,\mu}(z) := \begin{cases} t^{(\mu)}(z)^k := t_1^{(\mu)}(z)^{k_1} \cdots t_n^{(\mu)}(z)^{k_n}, & z \in \Delta^{(\mu)} \\ 0, & z \in \Delta \setminus \Delta^{(\mu)} \end{cases},$$

with $k = (k_1, \ldots, k_j, \ldots, k_n) \in \mathbb{Z}_+^n$ and $\mu = 1, \ldots, m$, forms a common basis for all the spaces $A(\Delta_\tau)$ and the space $A(F)$. Its biorthogonal system $\left\{f'_{k,\mu}\right\} \subset A(F)' \subset A(\Delta_\tau)'$ may be expressed by the formula

$$f'_{k,\mu}(f) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{S\left(\mathbb{U}_{r(\tau-\delta)}^n\right)} \frac{f(v_\mu(t)) \ dt}{t^{k+I}}, \ f \in A(\Delta_\tau), \ 0 < \tau < \infty, \tag{60}$$

where $v_\mu : \mathbb{U}^n \to \Delta_\mu$ is the mapping inverse to the coordinate mapping (57), $\mu = 1, \ldots, m, \ k = (k_1, \ldots, k_j, \ldots, k_n) \in \mathbb{Z}_+^n, \ I = (1, \ldots, 1); \ r(\lambda) := \exp(-\lambda), \ 0 < \lambda < \infty$. It is easily seen that

$$|f_{k,\mu}|_{\Delta_\tau} = \exp\left(-\frac{\tau|k|}{\alpha_\mu}\right); \ |f'_{k,\mu}|_{\Delta_\tau}^* = \exp\left(\frac{\tau|k|}{\alpha_\mu}\right), \ 0 < \tau < \infty. \tag{61}$$

Since any functional $f' \in A_0^{\perp}(F, \sigma)$ is represented in the form

$$f' = \sum_{\mu=1}^m \sum_{|k| \leq s_\mu} f'(f_{k,\mu}) \ f'_{k,\mu},$$

using (61), we obtain the estimate:

$$|f'|_{\Delta_\tau}^* \leq \sum_{\mu=1}^m (\alpha_\mu)^n \left(\frac{s_\mu}{\alpha_\mu}\right)^n \exp\frac{\tau s_\mu}{\alpha_\mu} \leq \sum_{\mu=1}^m (\alpha_\mu)^n \ s(\alpha)^n \ \exp \tau s(\alpha)$$

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with $0 < \tau < \infty$. So, for each $\varepsilon < 0$ the estimates

$$|f'|_{\Delta_\tau}^* \leq L(\varepsilon) \ |f'|_\Delta^* \ \exp(\tau + \varepsilon) s, \ 0 < \tau < \infty, \qquad (62)$$

hold with $L(\varepsilon) = \sum\limits_{\mu=1}^m (\alpha_\mu)^n \ \sup\{s^n \ \exp(-\varepsilon s) : s \in \mathbb{N}\}$, where $s = s(\alpha) :=$ $\sup\left\{\dfrac{s_\mu}{\alpha_\mu} : \mu = 1, \ldots, m\right\}$.

Choose $\gamma > 0$ so that $\Phi_\gamma \Subset \Delta$. Since $\Delta_\tau \subset \Phi_\tau$, $0 < \tau < \infty$, we first derive from (62) a quite rough estimate for the left-hand side of (56):

$$|f'|_{\Phi_\tau}^* \leq L(\varepsilon) \ |f'|_{\Phi_\gamma}^* \ \exp(\tau + \varepsilon) s, \ 0 < \tau < \infty. \qquad (63)$$

Using the notation $F_\tau := \{z \in D : g_D(F, \alpha; z) \leq -\tau\}$, we have the relation:

$$\omega(D, F_\tau; z) = \frac{1}{\tau} g_D(F, \alpha; z) + 1, \ z \in D \smallsetminus F_\tau, \ 0 < \tau < \infty,$$

which implies that

$$D_\alpha^{(\tau)} := \{z \in D : \omega(D, F_\tau; z) < \alpha\} = \Phi_{(1-\alpha)\tau}, \ 0 < \alpha < 1, \ \ 0 < \tau < \infty. \qquad (64)$$

Therefore, applying (21) with $K = F_\tau$, $\alpha = 1 - \lambda/\tau$, $0 < \lambda < \tau$, $\varepsilon < 0$, we obtain the estimate

$$|f'|_{\Phi_\lambda}^* \leq M \ \left(|f'|_{F_\tau}^*\right)^{\lambda/\tau + \varepsilon} \left(|f'|_D^*\right)^{1 - \lambda/\tau - \varepsilon} \qquad (65)$$

with some constant $M = M(\tau, \lambda, \varepsilon)$.

Since the relation (56) is homogeneous, it is sufficient to prove it in the case when $|f'|_D^* = 1$. Under this assumption, taking into account (63) and (65), we obtain the estimate:

$$|f'|_{\Phi_\lambda}^* \leq M \ \left(L(\varepsilon) \ |f'|_{\Phi_\gamma}^* \ \exp(\tau + \varepsilon) s\right)^{\frac{\lambda}{\tau} + \varepsilon} \leq M' \left(|f'|_{\Phi_\gamma}^*\right)^{\lambda/\tau + \varepsilon} \exp(\lambda + \varepsilon') s \qquad (66)$$

with some constant $M' = M'(\tau, \lambda, \varepsilon)$ and $\varepsilon' = \tau\varepsilon + \varepsilon + \varepsilon^2$, $0 < \lambda < \tau$. Applying this estimate for $\lambda = \gamma$ with $\tau = 4\gamma$ and $\varepsilon = 1/4$ we obtain that

$$|f'|_{\Phi_\gamma}^* \leq (M')^2 \ \exp 2(2\gamma + 1) s. \qquad (67)$$

Given $\delta > 0$ and $\lambda > 0$ we choose $\tau = \tau(\lambda, \delta)$ and $\varepsilon = \varepsilon(\lambda, \delta)$ so that

$$2(2\gamma + 1)(\lambda/\tau + \varepsilon) < \delta/2; \ \tau\varepsilon + \varepsilon + \varepsilon^2 < \delta/2. \qquad (68)$$

Then putting (67) into (66) considered with parameters satisfying the conditions (68) we obtain the estimate (56) in the case $|f'|_D^* = 1$ with some constant $C = (\lambda, \delta)$. This completes the proof. ∎

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Corollary 36 *Let Ω be a pluriregular Stein manifold of dimension n and $F = \left\{ \zeta_\mu : \mu = 1, \ldots, m \right\} \subset \Omega$; $\sigma = (s_\mu) \in \mathbb{Z}_+^n$. Let $H \hookrightarrow A(\Omega)$ be a Hilbert space adherent to $A(\Omega)$ and the dual space $A(\Omega)^*$ is considered as imbedded naturally into H^*. Then for each $f' \in A_0^\perp(F, \sigma)$ and for any $\delta > 0$ the estimates*

$$\left| f' \right|_{D_\lambda}^* \leq C \ \| f \|_{H^*}' \exp\left(\lambda + \delta\right) s, \ 0 < \lambda < \infty, \tag{69}$$

hold with some constant $C = C(\lambda, \delta)$ and $s = s(\alpha) := \sup \left\{ \dfrac{s_\mu}{\alpha_\mu} : \mu = 1, \ldots, m \right\}$.

7 Extendible bases

Theorem 37 *([46, 47]) Let $F = \left\{ \zeta_\mu : \mu = 1, \ldots, m \right\}$ be a finite set on a pluriregular Stein manifold, $\dim \Omega = n$, having no connected component disjoint with the set F; $\alpha = (\alpha_\mu)_{\mu=1}^m$, $\alpha_\mu > 0$, $\mu = 1, \ldots, m$. Let $g_\Omega(F, \alpha; z)$ be the corresponding Green multipole function,*

$$\Omega_\lambda := \left\{ z \in \Omega : g_\Omega(F, \alpha; z) < -\lambda \right\}, \ F_\lambda := \left\{ z \in \Omega : g_\Omega(\Lambda, \alpha; z) \leq -\lambda \right\}, \tag{70}$$

with $0 < \lambda < \infty$, and

$$\sigma_n = \left(\frac{n!}{\sum_{\mu=1}^m (\alpha_\mu)^n} \right)^{1/n}. \tag{71}$$

Then there exists a common basis $\left\{ f_i(z) \right\}_{i \in \mathbb{N}}$ for all the spaces

$$A(\Omega), \ A(F), \ A(\Omega_\lambda), \ A(F_\lambda), \ 0 < \lambda < \infty, \tag{72}$$

such that for each $\varepsilon > 0$ and $0 < \lambda < \infty$ the estimates

$$\frac{1}{C} \ \exp \sigma_n \left(-\lambda - \varepsilon\right) i^{1/n} \leq \left| f_i(z) \right|_{F_\lambda} \leq C \ \exp \sigma_n \left(-\lambda + \varepsilon\right) i^{1/n}, \ i \in \mathbb{N}, \tag{73}$$

hold with some constant $C = C(\lambda, \varepsilon)$. The Green multipole function can be expressed via the basis by the formula:

$$\sigma_n \ g_\Omega(F, \alpha; z) = \limsup_{\zeta \to z} \limsup_{i \to \infty} \frac{\ln \left| f_i(z) \right|}{i^{1/n}}, \ z \in \Omega \smallsetminus \Lambda. \tag{74}$$

Proof. We enumerate the system of analytic functionals

$$\left\{ f_{k,\mu}', \ k \in \mathbb{Z}_+^n, \ \mu = 1, \ldots, m \right\} \subset A(F)' \subset A(\Omega)',$$

biorthogonal to the basis in $A(F)$ consisted of monomials in local coordinates (see the beginning part of the proof of Theorem 35), into the sequence

$$e_i' = f_{k(i), \mu(i)}', \ i \in \mathbb{N}, \tag{75}$$

so that the sequence $s_\alpha(i) := \dfrac{\left| k(i) \right|}{\alpha_{\mu(i)}}$ will be non-decreasing.

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Taking any Hilbert space H adherent to $A(\Omega)$, we consider the dual space $A(\Omega)^*$ as imbedded naturally into H^* (see, Preliminaries, Spaces). The system (75) is linearly independent and complete in H^*. Orthonormalizing it in the space H^*, we obtain the system of analytic functionals ("polynomials" with respect to the system (75)):

$$\varphi'_i = \sum_{j \leq i} b_{ij} \; e'_j, \; i \in \mathbb{N}, \tag{76}$$

belonging to $A(F)' \subset A(\Omega)^* \subset H^*$.

We are going to show that the system $\{\varphi_i\}_{i \in \mathbb{N}} \subset H \subset A(\Omega) \subset A(F)$, biorthognal to this system, is a required basis. This system is orthonormal in H. For any $f \in A(F)$ the relation $\varphi'_i(f) = 0, \; i \in \mathbb{N}$, implies that $e'_i(f) = 0, \; i \in \mathbb{N}$, so the function f vanishes on the set F with all its derivatives, hence $f \equiv 0$ in a neighborhood of F. Therefore the system is total and hence complete in the spaces (72), due to reflexivity all of them. Since $\varphi'_i(f) = (f, \varphi_i)_H$, the orthogonal system $\{\varphi_i\}$ is maximal in H and hence complete also in H.

From (76) we have that $\varphi'_i \in A_0^{\perp}(F, \sigma)$ with $\sigma = ([\alpha_\mu \; s_\alpha(i)] + 1)$. Therefore remembering that $\|\varphi'_i\| = 1$ we obtain, by Corollary 36, the estimates

$$|\varphi'_i|_{\Omega_\lambda}^* \leq M \; \exp(\lambda + \delta) s_\alpha(i), \;\; i \in \mathbb{N}, \; 0 < \lambda < \infty \tag{77}$$

with $M = C(\lambda, \delta) \exp(\lambda + \delta), \; \delta > 0$.

On the other hand, due to $H \hookrightarrow A(\Omega)$, for each $\delta > 0$ there is a constant $L = L(\delta)$ such that $|\varphi_i|_{\Omega_\delta} \leq L \; \|\varphi_i\|_H = L$ and $\varphi_i \in A_0((F, \sigma), \Omega_\delta)$ with $\sigma = ([\alpha_\mu \; s_\alpha(i)])$. Therefore, taking into account that $g_{\Omega_\delta}(F, \alpha; z) = g_\Omega(F, \alpha; z) + \delta$, we obtain that the estimates

$$|\varphi_i|_{\Omega_\lambda} \leq N \; \exp(-\lambda + \delta) s_\alpha(i), \;\; i \in \mathbb{N}, \; 0 < \lambda < \infty, \tag{78}$$

hold with some constant $N = N(\lambda, \delta)$. The estimates (77) and (78) provide that, for each function f belonging to any space from the list (72), its basis expansion $f(z) = \sum_{i \in \mathbb{N}} \varphi'_i(f) \; \varphi_i(z)$ converges in the topology of that space.

The estimates (73) follows immediately from the strict asymptotics

$$s_\alpha(i) \sim \sigma_n \; i^{i/n}, \; i \to \infty,$$

which is derived from the evident strict asymptotics of the counting function:

$$c(t) := |\{i : s_\alpha(i) \leq t\}| \sim \sum_{\mu=1}^{m} \frac{(\alpha_\mu \; t)^n}{n!}, \; t \to \infty.$$

∎

8   Strict asymptotics for Kolmogorov diameters

Now we are ready to prove the main results (Theorem 5 and Corollary 7). As it was emphasized in the introduction, an important part of the proof is covered by the following recent result of Poletsky-Nivoche (solving positively our conjecture from [46, 47]; see Problem 4 above).

Proposition 38 *Let (K, D) be a pluriregular pair on a Stein manifold Ω. Then there exist a sequence of finite sets $F_j = \left\{ \zeta_\mu^{(j)} : \mu = 1, \ldots, m_j \right\} \subset D$ and a sequence $\alpha^{(j)} = \left(\alpha_\mu^{(j)} \right) \in \mathbb{R}^{m_j}$, $\alpha_\mu^{(j)} > 0$, $\mu = 1, \ldots, m_j$ such that the sequence $g_D \left(F_j, \alpha^{(j)}; z \right)$ converges to the function $\omega (D, K; z) - 1$ uniformly on any compact subset of $D \smallsetminus K$.*

Applying this result and our considerations developed in the previous sections, we prove now Theorem 5.

Proof. Let $\omega(z) := \omega(\Omega, K; z)$. Take any pair of Hilbert spaces (H_0, H_1) adherent to the pair $(A(K), A(\Omega))$, hence, by Theorem 20, admissible for the couple (K, Ω). Since the strict asymptotics is the same for all admissible pairs of Banach spaces it is sufficient to prove that

$$\lim_{i \to \infty} \frac{\ln d_i (H_1, H_0)}{i^{1/n}} = -\frac{C(K, \Omega)}{(2\pi)^n}. \tag{79}$$

First, by Theorem 19, the continuous linear imbeddings

$$A(K_\tau) \hookrightarrow H_0^{1-\tau} H_1^\tau \hookrightarrow A(\Omega_\tau), \ 0 < \tau < 1, \tag{80}$$

hold for the Hilbert scale $H_\tau := H_0^{1-\tau} H_1^\tau$ with

$$K_\tau := \{ z \in \Omega : \omega(z) \leq \tau \}, \ \Omega_\tau := \{ z \in \Omega : \omega(z) < \tau \}. \tag{81}$$

Take some sequences $\varepsilon_j \downarrow 0$ and $\delta_j \downarrow 0$ so that

$$\varepsilon_{j+1} < \varepsilon_j - 2\delta_j, \ j \in \mathbb{N}. \tag{82}$$

By the Nivoche-Poletsky result (Proposition 38), for each $j \in \mathbb{N}$ there exists a finite set

$$F^{(j)} = \{ \zeta_{j,\mu} : \mu = 1, \ldots, m_j \} \subset \Omega$$

and a vector $\alpha^{(j)} = (\alpha_{j,\mu}) \in \mathbb{R}_+^{m_j}$ such that

$$|g_j(z) - \omega(z)| < \delta_j, \ z \in K_{1-\varepsilon_j} \smallsetminus \Omega_{\varepsilon_j}, \tag{83}$$

where $g_j(z) := g_\Omega \left(F^{(j)}, \alpha^{(j)}; z \right), \ z \in \Omega$. Denote for $j \in \mathbb{N}$

$$\Phi_j := \{ z \in \Omega : g_j(z) \leq -1 + \varepsilon_j \}, \ D_j := \{ z \in \Omega : g_j(z) < -1 + \varepsilon_j \} \tag{84}$$

and consider the function

$$\omega_j(z) := s\left(\Phi_j, \frac{g_j + 1 - \varepsilon_j}{1 - \varepsilon_j}; z\right) = \begin{cases} \dfrac{g_j(z) + 1 - \varepsilon_j}{1 - \varepsilon_j}, & z \in \Omega \setminus \Phi_j \\ 0, & z \in \Phi_j \end{cases} \tag{85}$$

It is clear that $\omega_j(z) = \omega(\Omega, \Phi_j; z)$, $z \in \Omega$. We show that

$$\omega_j(z) \uparrow \omega(z), \ z \in \Omega. \tag{86}$$

Indeed, taking into account $(81),(82),(83),(84),(85)$, we get the inclusions:

$$K_{\varepsilon_{j+1} - \delta_{j+1}} \subset \Phi_{j+1} \subset \Omega_{\varepsilon_{j+1} + \delta_{j+1}} \subset K_{\varepsilon_j - \delta_j} \subset D_j \subset \Phi_j \subset \Omega_{\varepsilon_j + \delta_j} \tag{87}$$

Therefore the sequence $\omega_j(z)$ is non-decreasing and

$$\omega\left(\Omega, K_{\varepsilon_j - \delta_j}; z\right) \leq \omega_{j+1}(z) \leq \omega\left(\Omega, K_{\varepsilon_{j+1} - \delta_{j+1}}; z\right), \ j \in \mathbb{N}. \tag{88}$$

So (86) is proved.

Due to $[5, 6]$, we have

$$C(\Phi_j, \Omega) := \int (dd^c \omega_j(z))^n \downarrow \int (dd^c \omega(z))^n =: C(K, \Omega). \tag{89}$$

On the other hand, $(51),(52)$,

$$C(\Phi_j, \Omega) = (2\pi)^n \sum_{\mu=1}^{m} (\alpha_{j,\mu})^n. \tag{90}$$

Given $\varepsilon > 0$, due to (89), we can choose $j \in \mathbb{N}$ so that

$$C(K, \Omega) \leq C(\Phi_j, \Omega) \leq (1 + \varepsilon) C(K, \Omega) \text{ and } \varepsilon_j < \varepsilon. \tag{91}$$

Now we consider the basis $\{\varphi_i\}$ from Theorem 37 with the Hilbert space $H = H_1$ chosen in the beginning of the proof and with $F = F^{(j)}$, $\alpha = \alpha^{(j)}$. Let G be the Hilbert space of all $x = \sum\limits_{i \in \mathbb{N}} \xi_i \varphi_i$ such that

$$\|x\|_G := \left(\sum_{i \in \mathbb{N}} |\xi_i|^2 \exp 2\sigma_n (-1 + \varepsilon_j) i^{1/n}\right)^{1/2} < \infty, \tag{92}$$

where

$$\sigma_n = \sum_{\mu=1}^{m_j} (\alpha_{j,\mu})^n = \frac{C(\Phi_j, \Omega)}{(2\pi)^n}. \tag{93}$$

By (17), (92) we have

$$d_i(H_1, G) = \exp \sigma_n (-1 + \varepsilon_j) i^{1/n} \tag{94}$$

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which together with (91), (93) implies that

$$-\frac{(1+\varepsilon)\,C\,(K,\Omega)}{(2\pi)^n} \leq \frac{\ln d_i\,(H_1\,,G)}{i^{1/n}} \leq -\frac{(1-\varepsilon)\,C\,(K,\Omega)}{(2\pi)^n} \qquad (95)$$

Due to the estimates (73), (80), we get the imbeddings:

$$H_{\varepsilon_j+\delta_j} \hookrightarrow A\left(\Omega_{\varepsilon_j+\delta_j}\right) \hookrightarrow A\left(\Phi_j\right) \hookrightarrow G \hookrightarrow A\left(D_j\right) \hookrightarrow A\left(K_{\varepsilon_j-\delta_j}\right) \hookrightarrow H_{\varepsilon_j-\delta_j}.$$

Hence, by Proposition 8, there is a constant $M > 0$ such that

$$\frac{1}{M}\,d_i\left(H_1, H_{\varepsilon_j-\delta_j}\right) \leq d_i\left(H_1, G\right) \leq M\,d_i\left(H_1, H_{\varepsilon_j+\delta_j}\right). \qquad (96)$$

By (18), we have

$$d_i\left(H_1, H_\tau\right) = \left(d_i\left(H_1, H_0\right)\right)^{1-\tau},\ 0 < \tau < 1. \qquad (97)$$

Therefore, combining (96),(97),(95), we get that

$$\limsup_{i\to\infty}\frac{\ln d_i\left(H_1, H_0\right)}{i^{1/n}} \leq -\frac{(1-\varepsilon)C\left(K,\Omega\right)}{(2\pi)^n}, \qquad (98)$$

$$\liminf_{i\to\infty}\frac{\ln d_i\left(H_1, H_0\right)}{i^{1/n}} \geq -\frac{(1+\varepsilon)C\left(K,\Omega\right)}{(1-2\varepsilon)\,(2\pi)^n}. \qquad (99)$$

Since $\varepsilon > 0$ is arbitrary, we conclude from (98),(99) that (79) is true. This completes the proof. ∎

9 Final remarks and some problems

The following sufficient condition for the asymptotics (7) flows out from Proposition 14 and Theorem 20.

Corollary 39 *Let D be strongly pluriregular. Then the strict asymptotics (7) holds for any compact set K making up a pluriregular pair with D.*

Some more general sufficient condition (covering the one-dimensional Proposition 1) is represented by

Proposition 40 *Let (K, D) be a pluriregular pair on a Stein manifold, such that D is an intersection of a countable decreasing sequence of open sets D_s such that $H^\infty\left(D_s\right)$ is adherent to $A\left(D_s\right)$ for each s. Then $H^\infty\left(D\right)$ is adherent to $A\left(D\right)$ and the asymptotics (7) holds.*

Whereas the pluripotential $\omega\left(D, K; z\right)$ is an appropriate tool for the modified Kolmogorov problem (see, Theorem 5), for the solution of the original problem of Kolmogorov (3) in general case one needs to deal also with another maximal function:

$$\gamma\left(D, K; z\right) := \limsup_{\zeta\to z}\sup\left\{u\left(\zeta\right) : u \in \mathcal{A}(D, K)\right\}, \qquad (100)$$

here $\mathcal{A}(D, K)$ consists of all functions $u(z) = \alpha \ln |f(z)|$ with $\alpha > 0$, $f \in H^\infty(D)$ and $u|_K \leq 0$, $u(z) < 1$ in D. This is supported by the observation that in the conditions of Corollary 39 or Proposition 40 we have ([43, 46, 47]):

$$\gamma(D, K; z) = \omega(D, K; z), \quad z \in D, \tag{101}$$

and by the following result.

Theorem 41 *Let the space $H^\infty(D)$ be adherent to $A(D)$. Then (101) holds for any compact set $K \subset D$ making up a pluriregular pair (K, D).*

Proof. The space $X_0 = AC(K)$ is adherent to $A(K)$ ([43]) and $X_1 = H^\infty(D)$ is adherent to $A(D)$ by the assumption. Therefore, by Theorem, for any regular normal scale X_α, connecting X_0 and X_1, the imbedding

$$A(K_\alpha) \hookrightarrow X_\alpha, \quad 0 < \alpha < 1, \tag{102}$$

holds. Suppose now that (101) is not valid. Then, since the function does not exceed $\omega(D, K; z)$, there is a point $z_0 \in D \smallsetminus K$ such that $\beta := \gamma(D, K; z_0) < \omega(D, K; z_0) =: \alpha$. Take $\sigma : \beta < \sigma < \alpha$ and denote $\Delta_\sigma := \{z \in D : \gamma(D, K; z_0) < \sigma\}$. By the definition (100), the estimate

$$|x|_L \leq \left(\|x\|_{X_0}\right)^{1-\sigma} \left(\|x\|_{X_1}\right)^\sigma, \quad x \in X_1,$$

holds for any compact set $L \subset \Delta_\sigma$. Therefore the imbedding $\widehat{X}_\sigma \hookrightarrow A(\Delta_\sigma)$ is true, where $\widehat{X}_\theta$, $0 \leq \theta \leq 1$, is the maximal scale of means spanned by X_0, X_1 ([15], IV, Lemma 2.6). Since any regular scale is almost imbedded into any scale ([15], IV.11, Corollary 3), we obtain from here and (102) that $A(K_\alpha) \hookrightarrow X_\alpha \hookrightarrow \widehat{X}_\sigma \hookrightarrow A(\Delta_\sigma)$. Since $z_0 \in K_\alpha$ is an interior point of Δ_σ we obtain from here that each germ $x \in A(K_\alpha)$ has an analytic extension onto some fixed neighborhood of the point z_0, which is impossible, because K_α is holomorphically convex with respect to D. This contradiction completes the proof. ∎

Problem 42 *Let (K, D) be a pluriregular pair on a Stein manifold Ω. Is the condition (101) sufficient for the adherence of $H^\infty(D)$ to $A(D)$?*

Also, in connection with Theorem 20 and Problem 22, the following question arises.

Problem 43 *Let (K, D) be a pluriregular pair on a Stein manifold Ω. Does the condition (101) characterize the asymptotics (3) with the constant $\sigma = 2\pi \left(\frac{n!}{C(K,D)}\right)^{1/n}$?*

In conclusion we consider some examples.

Example 44 *Let $E \subset \mathbb{R} \subset \mathbb{C}$ be the usual Cantor set. Then for the domain $D := \overline{\mathbb{C}} \smallsetminus E$ the space $H^\infty(D)$ is trivial (consists only of constants and hence $\gamma(D, K; z) \equiv 0$), so for any compact set $K \subset D$ the asymptotics (3) has no sense.*

Example 45 *Let E be again the usual Cantor set, $G \supset E$ be any domain in $\mathbb{C}$ satisfying the conditions of Proposition 1. Then for any regular compact set $K \subset D := G \smallsetminus E$ the asymptotics (3) holds with the constant $\sigma = \frac{1}{\tau(K,G)} < \frac{1}{\tau(K,D)}$.*

So, in this case the constant σ is defined by $\omega(G, K; z)$, which is the harmonic extension of the function $\gamma(D, K; z)$ onto the domain G, obtained after removing of the non-polar portion $E \subset \partial D$ having zero analytic capacity.

Example 46 *Let E be a compact set of positive length on a rectifiable curve in $\mathbb{C}$, having no portion of zero length, and $D := \overline{\mathbb{C}} \smallsetminus E$. Though, due to the positive solution of Denjoy conjecture (see, e.g., [35]), the space $H^\infty(D)$ is non-trivial it seems that there is no answer to the following questions (even in the simplest case of Cantor sets of positive Lebesgue measure):*

1. *Is $H^\infty(D)$ adherent to $A(D)$?*

2. *Is the equality (101) true with some (hence with any) regular compact set $K \subset D$ such that $\widehat{K}_D = K$?*

3. *Is the couple $(AC(K), H^\infty(D))$ admissible for (K, D) if K is as above or, what is the same, does the asymptotics (6) hold in this case?*

Finally, if the answer is not always "yes" in the above questions what are additional properties of E providing the positive answer to them?

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