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Solving variational inequalities defined on a domain with infinitely many linear constraints

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Abstract We study a variational inequality problem whose domain is defined by infinitely many linear inequalities. A discretization method and an analytic center based inexact cutting plane method are proposed. Under proper assumptions, the convergence results for both methods are given. We also provide numerical examples to illustrate the proposed methods.

Keywords Variational inequality problem · Analytic center based cutting plane method · Discretization method · Semi-infinite programming

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1 Introduction

Let X be a nonempty subset of R^n and F be a function from R^n to itself. The finite-dimensional variational inequality problem, denoted by VI(X, F), is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in X. \tag{1.1}$$

The development of the theory, algorithms and applications of finite-dimensional variational inequalities can be found in [6, 15]. The theory is very rich and a large collection of algorithms exist for solving finite-dimensional variational inequalities. However, most algorithms work practically only when X exhibits a certain geometric structure (such as the positive orthant of R^n or a polyhedral set) or when X is defined by finitely many convex constraints (such as the one studied in [29]). Motivated by the recent development in semi-infinite programming [9, 24, 25], the authors of [8] propose to study variational inequalities with X being defined by infinitely many convex constraints. They call this class of problems "semi-infinite variational inequality problems" and show that such problems can be reduced to a convex feasibility problem. In this paper, we focus on studying the variational inequality problem whose domain is explicitly defined by infinitely many linear inequalities, using the concepts of semi-infinite linear programming [1].

Like in [8], in this paper we consider a setting for which X is a nonempty, bounded set defined by

$$X = \left\{x \in R^n \mid \langle u(t), x\rangle - \lambda(t) \leq 0 \text{ for all } t \in T\right\} \tag{1.2}$$

where T is a nonempty compact subset of R^1, $u(t) : T \to R^n$ and $\lambda(t) : T \to R^1$ are continuous on T. Since there may be infinitely many linear inequalities involved in defining X, we call this setting a linear semi-infinite variational inequality problem, or LSIVI(X, F) in short.

Notice that $x \in X$ if and only if $h(x) = \max_{t\in T}\{\langle u(t), x\rangle - \lambda(t)\} \leq 0$. The existence of a solution to X can be traced back to [30] and characterized by the concepts of the "1st moment cone," "2nd moment cone," "characteristic cone," and "cone of ascent rays" as described in [9]. The geometry of X can also be found in [9]. Finding a feasible solution of X is in general as hard as solving a semi-infinite linear programming problem with an initial solution [8, 24]. Also notice that when F is a continuous pseudomonotone mapping (to be defined in later sections) from X to R^n, it is not difficult to prove that $x^* \in X$ solves LSIVI(X, F) if and only if it solves the following problem:

$$F(x)^T(x - x^*) \geq 0 \quad \text{for all } x \in X, \tag{1.3}$$

which we denote as DLSIVI(X, F). Actually, this problem is equivalent to a convex feasibility problem [10–13, 18, 19], i.e., finding a point x^* in a convex set defined by an infinite number of linear inequalities

$$S = \left\{x^* \in X \mid F(x)^T x^* \leq F(x)^T x \text{ for all } x \in X\right\}. \tag{1.4}$$

The cutting plane approach has been used to solve finite convex optimization problems since the very beginning of the development of nonlinear programming. The methods of Kelley–Cheney–Goldstein [3, 17], Veinott [27], and Elzinga–Moore [5] were widely applied and modified in various manners [4, 16, 25, 26]. A recent development is using analytic center based cutting plane methods to solve variational inequalities [8, 12, 20–23]. This approach combines the feature of interior point methods with the classical cutting plane scheme. Recently, the authors in [8, 24] presented an analytic center based cutting plane method for solving a general semi-infinite variational inequality problem.

In this paper, we focus on a semi-infinite variational inequality problem whose domain is defined by infinitely many linear constraints. We first study a discretization approach for solving LSIVI(X, F) and show a convergence result under proper assumptions. The quality of solutions obtained by the discretization approach depends on the expansive sequence used. It is hard to provide any quantitative statement. Then, we propose an analytic center based *inexact* cutting plane method and give its convergence proof. Unlike other cutting plane methods, such as the one used in [8], the proposed method requires only an *inexact* solution to a variational inequality problem at each iteration. Also, the quality of solutions obtained by the proposed inexact method can be carefully analyzed.

This paper is organized as follows. Some preliminaries are given in Sect. 2. We discuss the discretization method for LSIVI(X, F) and show a convergence result in Sect. 3. An analytic center based inexact cutting plane method is proposed with a convergence proof in Sect. 4. The computational results over a set of problems are reported in Sect. 5. We conclude the paper in Sect. 6.

2 Preliminaries

Since an analytic center is usually defined at an interior point of a given region, we make the following *interior point* assumption throughout this paper:

Assumption 1 There exists an $\hat{x} \in R^n$ such that

$$\langle u(t), \hat{x} \rangle - \lambda(t) < 0 \quad \text{for all } t \in T. \tag{2.1}$$

The interior point assumption assures that X has a nonempty interior. It is easy to see that X is a convex set. Moreover, the continuity of $\langle u(t), x \rangle - \lambda(t)$ on $R^n \times T$ implies that X is a closed set. Remember that in our setting, X is assumed to be bounded. Consequently, X is a nonempty, convex, and compact subset of R^n and the next result follows:

Proposition 1 *In our setting, if F is a continuous mapping from X to R^n, then there exists a solution to* LSIVI(X, F).

Let us recall some definitions of the mappings commonly used for a variational inequality problem VI(X, F).

Definition 1 [12, 22] A mapping F is said to be:

- *Monotone* on X if for every pair of points $x, y \in X$,

$$\big(F(x) - F(y)\big)^T (x - y) \geq 0. \tag{2.2}$$

- *Strongly monotone* on X if there exists $\beta > 0$ such that for every pair of points $x, y \in X$,

$$\big(F(x) - F(y)\big)^T (x - y) \geq \beta \|x - y\|^2. \tag{2.3}$$

- *Pseudomonotone* on X if for every pair of points $x, y \in X$,

$$F(x)^T (y - x) \geq 0 \quad \text{implies } F(y)^T (y - x) \geq 0. \tag{2.4}$$

- *Pseudomonotone-plus* on X if it is pseudomonotone on X and for every pair of points

$$F(x)^T (y - x) = 0 \quad \text{and} \quad F(y)^T (y - x) = 0 \quad \text{imply } F(x) = F(y). \tag{2.5}$$

- *Pseudo-co-coercive* with modulus $\alpha > 0$ on X if for every pair of points $x, y \in X$,

$$F(x)^T (y - x) \geq 0 \quad \text{implies } F(y)^T (y - x) \geq \alpha \big\| F(x) - F(y) \big\|^2. \tag{2.6}$$

It is not difficult to see that a monotone mapping is pseudomonotone and a strongly monotone mapping is pseudo-co-coercive. Moreover, the following result follows.

Proposition 2 *Let F be a continuous pseudomonotone mapping over X. Then $x^* \in X$ is a solution to* LSIVI(X, F) *if and only if x^* solves* DLSIVI(X, F).

The following concept of gap function $g(x)$ associated with a general VI(X, F) will be utilized in this paper:

Definition 2 Given a problem VI(X, F), the gap function is defined to be

$$g(x) = \max_{y \in X} \big\{F(x)^T (x - y)\big\} \quad \text{for } x \in X. \tag{2.7}$$

Note that $g(x) \geq 0$ for $x \in X$ and $g(x^*) = 0$ if and only if x^* is a solution to VI(X, F). In general, $g(x)$ may be nonconvex and nonsmooth. However, in our setting the value of $g(x)$ can be computed by some semi-infinite programming algorithms [7, 24, 28].

Also note the following definition for any $\varepsilon > 0$:

Definition 3 A point $\bar{x} \in X$ is called an ε-solution of the problem VI(X, F), if the gap $g(\bar{x}) \leq \varepsilon$.

In this case, it is not difficult to see that $F(\bar{x})^T (x - \bar{x}) \geq -\varepsilon$ for all $x \in X$.

3 Discretization approach for LSIVI(X, F)

We first introduce a discretization approach for solving LSIVI(X, F). Since in our setting T is a compact subset of R^1, there exist a positive-valued, strictly monotone decreasing function Δ from the natural numbers I_+ to the positive orthant R_+ such that $\Delta(n) \to 0$ as $n \to \infty$ and an expansive sequence $\{T_i\}$ of finite subsets of T with the property that for each $t \in T$, there exists an $n_0 \in I_+$ such that for $n \geq n_0$, there exists $t' \in T_n$ with $\|t - t'\| \leq \Delta(n)$. Using T_i, we define

$$\bar{X}_i = \left\{x \in R^n \mid \left\langle u(t), x\right\rangle - \lambda(t) \leq 0 \text{ for all } t \in T_i\right\}. \tag{3.1}$$

Note that T_i is a finite subset of T and $T_i \subset T_{i+1}$ for each i. Consequently, $X \subset \bar{X}_{i+1} \subset \bar{X}_i$.

Now consider the following variational inequality problem:

VI($\bar{X}_i$, F): Find $x^i \in \bar{X}_i$ such that

$$F(x^i)^T(x - x^i) \geq 0 \quad \text{for all } x \in \bar{X}_i. \tag{3.2}$$

Note that $\bar{X}_i$ is closed and convex and we have the following result.

Lemma 1 *If $\bar{X}_i$ is bounded and F is continuous on $\bar{X}_i$, then there exists a solution to* VI($\bar{X}_i$, F).

In this case, we let x^i be a solution of VI($\bar{X}_i$, F). When $\bar{X}_i$ is not bounded, the existence of a solution to VI ($\bar{X}_i$, F) may become an issue. The solvability of a variational inequality problem with a continuous function over a general unbounded closed convex set can be found in [14, 29]. Here we assume the existence of x^i and show that there exists a subsequence of the sequence of solutions $\{x^i\}$ converging to a solution of LSIVI(X, F).

Theorem 1 *If there exists an $M > 0$ such that $||x^i|| \leq M$ for each i, then there exists a subsequence $\{x^{k_i}\}$ of $\{x^i\}$ converging to the solution of* LSIVI(X, F).

Proof Since $||x^i|| \leq M$ for each i, there exists a subsequence $\{x^{k_i}\}$ converging to x^*. We claim that $\langle u(t), x^*\rangle - \lambda(t) \leq 0, \forall t \in T$. If not, there exists at least one $\bar{t} \in T$ such that $\langle u(\bar{t}), x^*\rangle - \lambda(\bar{t}) > 0$. Hence there exists an $n_1 \in \{k_i\}$ such that

$$\left\langle u(\bar{t}), x^n\right\rangle - \lambda(\bar{t}) > 0 \quad \text{for each } n \in \{k_i\} \text{ and } n \geq n_1. \tag{3.3}$$

From the definition of T_i, there exists $t_i \in T_i$ such that $t_i \to \bar{t}$ as $i \to \infty$. Consequently, there exists n_2 such that, for $i \geq n_2$,

$$\left\langle u(t_i), x^n\right\rangle - \lambda(t_i) > 0 \quad \text{for each } n \in \{k_i\} \text{ and } n \geq n_1. \tag{3.4}$$

Choose $n_3 \geq \max\{n_1, n_2\}$, then

$$\left\langle u(t_{n_2}), x^{n_3}\right\rangle - \lambda(t_{n_2}) > 0. \tag{3.5}$$

Since x^{n_3} is a solution of $\mathrm{VI}(\bar{X}_{n_3}, F)$,

$$\langle u(t), x^{n_3}\rangle - \lambda(t) \le 0 \quad \text{for each } t \in T_{n_3}. \tag{3.6}$$

For $t_{n_2} \in T_{n_2} \subseteq T_{n_3}$, we know that $\langle u(t_{n_2}), x^{n_3}\rangle - \lambda(t_{n_2}) \le 0$, which contradicts (3.5). Hence $\langle u(t), x^*\rangle - \lambda(t) \le 0$ for any $t \in T$.

Now we show that $F(x^*)^T(x - x^*) \ge 0$ for each $x \in X$. If not, we assume there exists at least one $\bar{x} \in X$ such that $F(x^*)^T(\bar{x} - x^*) < 0$. Since F is continuous, there exists $\bar{n} \in \{k_i\}$ such that

$$F\left(x^{\bar{n}}\right)^T \left(\bar{x} - x^{\bar{n}}\right) < 0. \tag{3.7}$$

On the other hand, since $\bar{x} \in X \subset \bar{X}_{\bar{n}}$, we have

$$F\left(x^{\bar{n}}\right)^T \left(\bar{x} - x^{\bar{n}}\right) \ge 0, \tag{3.8}$$

which contradicts (3.7). Thus, for all $x \in X$, $F(x^*)^T(x - x^*) \ge 0$. □

Although Theorem 1 assures the convergence of the discretization approach, the quality of solutions obtained by this approach depends on the choice of the expansive sequence $\{T_i\}$. Usually finer discretization results in better approximation, but it is hard to provide any quantitative statement.

4 Inexact cutting plane approach for LSIVI(X, F)

In this section, we present an analytic center based inexact cutting plane method for solving LSIVI(X, F).

Let $t_1^*, \ldots, t_m^*$ be m given distinct points in T, define

$$T^* = \{t_1^*, \ldots, t_m^*\} \subset T \tag{4.1}$$

and

$$X^* = \left\{x \in R^n \mid \langle u(t), x\rangle - \lambda(t) \le 0 \text{ for all } t \in T^*\right\}. \tag{4.2}$$

Since $u : T^* \subset T \to R^n$, we can define an $m \times n$ matrix A with $A_i = u(t_i)$ being its ith row for $i = 1, \ldots, m$. Similarly, since $\lambda : T^* \to R$, we define b to be an m vector with $b_i = \lambda(t_i)$ being the ith component of b for $i = 1, \ldots, m$. Then X^* can be rewritten as a polyhedral set $\{x \in R^n \mid Ax \le b\}$.

For a variational inequality problem like $\mathrm{VI}(X^*, F)$, Goffin et al. [12] presented an analytic center based cutting plane method to solve it. They showed that under some technical conditions (such as F is pseudomonotone-plus and Lipschitz continuous on X^* and the inequalities $0 \le x \le e$ (where e is the vector of all 1's) are included in the system $Ax \le b$), their algorithm either terminates with an exact solution of $\mathrm{VI}(X^*, F)$ in a finite number of iterations, or generates an infinite sequence $\{x^k\}$ that has a subsequence converging to a solution of $\mathrm{VI}(X^*, F)$. In the latter case, when k is sufficiently large, x^k becomes an ε-solution to $\mathrm{VI}(X^*, F)$, for any given $\varepsilon > 0$.

With a given $T^* \subset T$ and a prescribed small number $\delta > 0$, we propose a general scheme as follows.

Algorithm 1

Step 0. Given $\Delta > 0$, $\varepsilon \in (0, 1)$, $T_1 = T^*$, and $X_1 = \{x \in R^n | \langle u(t), x\rangle - \lambda(t) \le 0$ for all $t \in T_1\}$. Set $k = 1$ and $\Delta_1 = \Delta$.
Step 1. Solve problem $\mathrm{VI}(X_k, F)$ with a Δ_k-solution x^k. Define $\omega^k(t) = \langle u(t), x^k\rangle - \lambda(t)$ for $t \in T$.
Step 2. Find any $t_k \in T$ such that $\omega^k(t_k) > \delta$.
  (i) If such t_k does not exist and $\Delta_k \le \delta$, then stop and output x^k as a solution.
  (ii) If such t_k does not exist and $\Delta_k > \delta$, then set $\Delta_k = (1 - \varepsilon)\Delta_k$ and go to step 1.
  (iii) If such t_k exists, then set $T_{k+1} = T_k \cup \{t_k\}$, $X_{k+1} = \{x \in R^n \mid \langle u(t), x\rangle - \lambda(t) \le 0$ for all $t \in T_{k+1}\}$, $\Delta_{k+1} = (1 - \varepsilon)\Delta_k$, and go to step 3.
Step 3. Update $k \leftarrow k + 1$ and go to step 1.

Note that in step 1, only an inexact solution to a subprogram $\mathrm{VI}(X_k, F)$ is needed at each iteration. This task can be carried out by using the analytic center cutting plane method proposed in [12], assuming that X_k is bounded in our setting. In step 2, to find $t_k \in T$ such that $\omega^k(t_k) > \delta$ is not always easy. Strictly speaking, this involves global optimization theory and techniques. In our case, when $u(t)$ and $\lambda(t)$ are continuous in t over a compact set T in R^1, there exist many practical methods [25] and heuristics [2] for the computation. But no algorithm with exact complexity analysis is known. Also note that in step 2(iii), only one cutting plane is added at each time.

Theorem 2 *In our setting, Algorithm 1 terminates in a finite number of iterations.*

Proof Suppose that the algorithm does not terminate in a finite number of iterations, instead it generates a sequence $\{x_k\}$. Since X_k is assumed to be bounded, we let $\{x^{n_k}\}$ be a subsequence of $\{x_k\}$ such that $x^{n_k} \to x^*$ as $t_{n_k} \to t^*$.

Define

$$\omega^*(t) = \left\langle u(t), x^*\right\rangle - \lambda(t).$$

We claim that

$$\omega^*(t_{n_k}) \le 0 \quad \text{for } k = 1, 2, \ldots.$$

If not, then there exists a positive integer N such that

$$\omega^*(t_{n_N}) > 0.$$

Consequently, there exists a sufficiently large positive integer $\bar{N} > N$ with

$$\omega^{n_{\bar{N}}}(t_{n_N}) = \left\langle u(t_{n_N}), x^{n_{\bar{N}}}\right\rangle - \lambda(t_{n_N}) > 0.$$

Since $\bar{N} > N$ and $t_{n_N} \in T_{n_{\bar{N}}}$, we have

$$\omega^{n_{\bar{N}}}(t_{n_N}) \le 0,$$

which yields a contradiction. This implies that

$$\omega^*(t_{n_k}) = \left\langle u(t_{n_k}), x^*\right\rangle - \lambda(t_{n_k}) \le 0 \quad \text{for } k = 1, 2, \ldots. \tag{4.3}$$

Taking limit of (4.3) yields the result of $\omega^*(t_*) \leq 0$. From step 2 of the algorithm, we know

$$\omega^{n_k}(t_{n_k}) > \delta \quad \text{for } k = 1, 2, \ldots. \tag{4.4}$$

As $k \to \infty$, (4.4) implies

$$\omega^*(t_*) \geq \delta.$$

Hence we have $\delta \leq \omega^*(t_*) \leq 0$, which is again a contradiction. Therefore, Algorithm 1 indeed terminates in a finite number of iterations. $\square$

The above theorem assures that, for each $\delta > 0$, Algorithm 1 terminates in finitely many iterations with an inexact solution $x(\delta)$. Here we aim to show that $x(\delta)$ converges to a solution of LSIVI(X, F) as $\delta \to 0$. To achieve this, we introduce the following modified algorithm.

Given that $T^* \subset T$ and $\{\delta_i\}$ be a sequence of positive numbers such that $\delta_i \to 0$ as $i \to \infty$.

Algorithm 2

Step 0. Let $\Delta > 0$, $\varepsilon \in (0, 1)$, $T_1 = T^*$ and $X_1 = \{x \in R^n | \langle u(t), x\rangle - \lambda(t) \leq 0$ for all $t \in T_1\}$. Set $k = 1$, $l = 0$ and $\Delta_1 = \Delta$.

Step 1. Solve problem VI(X_k, F) with a Δ_k-solution x^k. Define $\omega^k(t) = \langle u(t), x^k\rangle - \lambda(t)$ for $t \in T$.

Step 2. Find a $t_k \in T$ such that $\omega^k(t_k) > \delta_{l+1}$.

- (i) If such t_k exists, then set $T_{k+1} = T_k \cup \{t_k\}$, $X_{k+1} = \{x \in R^n | \langle u(t), x\rangle - \lambda(t) \leq 0$ for all $t \in T_{k+1}\}$, $\Delta_{k+1} = (1-\varepsilon)\Delta_k$. Update $k \leftarrow k+1$ and go to step 1.
- (ii) If such t_k does not exist and $\Delta_k > \delta_{l+1}$, then set $\Delta_k = (1-\varepsilon)\Delta_k$ and go to step 1.
- (iii) If such t_k does not exist and $\Delta_k \leq \delta_{l+1}$, then set $l \leftarrow l+1$, $k_l^* \leftarrow k$, and go to step 3.

Step 3. If $\omega^k(t) \leq 0$ for all $t \in T$ and x^k is an exact solution for problem VI(X_k, F), then stop and output x^k as solution. Otherwise, go to step 2.

Note that if Algorithm 2 does not stop in step 3, then we obtain $x^{k_l^*}$ and update l by $l+1$ each time going through (iii) of step 2. Therefore, for $t \in T$ and $l = 1, 2, \ldots$

$$\omega^{k_l^*}(t) \leq \delta_{l+1}. \tag{4.5}$$

In the next theorem, we show that there exists a subsequence of $\{x^{k_l^*}\}$ that converges to a solution of LSIVI(X, F) as $\delta_l \to 0$.

Theorem 3 *In our setting, if $\delta_l \to 0$ as $l \to \infty$, then there exists a subsequence of $\{x^{k_l^*}\}$ converging to a solution of* LSIVI(X, F).

Proof Since X_k is bounded in our setting, we know that there exists a subsequence $x^{k_{n_l}^*}$ of $x^{k_l^*}$ converging to some x^* and $\delta_{n_l+1} \to 0$ as $l \to \infty$. From (4.5) we have

$$\omega^{k_{n_l}^*}(t) \leq \delta_{n_l+1} \quad \text{for } t \in T.$$

Let $l \to \infty$, we have $\omega^*(t) = \langle u(t), x^* \rangle - \lambda(t) \le 0$ for $t \in T$.

Now we show that

$$F(x^*)^T (x - x^*) \ge 0 \quad \text{for all } x \in X.$$

For each l, since $x^{k^*_{n_l}}$ is a $(1-\varepsilon)^{k^*_{n_l}-1}\Delta$-solution of $\mathrm{VI}(X_{k^*_{n_l}}, F)$,

$$F\big(x^{k^*_{n_l}}\big)^T \big(x - x^{k^*_{n_l}}\big) \ge -(1-\varepsilon)^{k^*_{n_l}-1}\Delta \quad \text{for } x \in X \subset X_{k^*_{n_l}}.$$

As $l \to \infty$, we have

$$F(x^*)^T (x - x^*) \ge 0 \quad \text{for all } x \in X.$$

Therefore, x^* is a solution of LSIVI(X, F). □

Suppose that Algorithm 2 does not terminate in a finite number of iterations, but after sth iteration we have a Δ_s-solution x^s for $\mathrm{VI}(X_s, F)$. Assume that there exists a small $\delta' > 0$ such that

$$\omega^s(t) = \big\langle u(t), x^s \big\rangle - \lambda(t) \le \delta' \quad \text{for all } t \in T. \tag{4.6}$$

From Theorem 2, x^s can be viewed as an approximate solution of LSIVI(X, F), if $\delta' > 0$ is sufficiently small. An interesting, yet important, question is "how good such an approximate solution can be"? In this case, we let $\delta^* = \max\{\delta', (1-\varepsilon)^{s-1}\Delta\}$, $c > \delta^*$ and $a > 0$ be given positive numbers, and $S = \{t : -c \le \omega^s(t) \le \delta'\}$. The following theorem addresses the approximation issue under some technical conditions.

Theorem 4 *Let F be a continuous and pseudo-monotone mapping over X. In our setting, if there exists an $\bar{x} \in R^n$ such that* (i) *$\langle u(t), \bar{x} \rangle \ge a$ for $t \in S$,* (ii) *$-a \le \langle u(t), \bar{x} \rangle$ for $t \in T - S$; and* (iii) *$F(x)^T \bar{x} \ge a$ for all $x \in X$, then $x^s - (\delta^*/a)\bar{x}$ is a solution of* LSIVI(X, F).

Proof Since for $t \in S$

$$\omega^s(t) = \big\langle u(t), x^s \big\rangle - \lambda(t) \le \delta' \tag{4.7}$$

and

$$\big\langle u(t), (\delta^*/a)\bar{x} \big\rangle \ge \delta^*, \tag{4.8}$$

we know that

$$\begin{aligned}\big\langle u(t), x^s - (\delta^*/a)\bar{x} \big\rangle - \lambda(t) &= \big\langle u(t), x^s \big\rangle - \lambda(t) - \big\langle u(t), (\delta^*/a)\bar{x} \big\rangle \\ &\le \delta' - \delta^* \le 0.\end{aligned} \tag{4.9}$$

Moreover, for $t \in T - S$, we have $\omega^s(t) < -c$ and $\langle u(t), (\delta^*/a)\bar{x} \rangle \ge -\delta^*$. Therefore, for $t \in T - S$, we have

$$\begin{aligned}\big\langle u(t), x^s - (\delta^*/a)\bar{x} \big\rangle - \lambda(t) &= \big\langle u(t), x^s \big\rangle - \lambda(t) - \big\langle u(t), (\delta^*/a)\bar{x} \big\rangle \\ &< -c + \delta^* \le -\delta^* + \delta^* = 0.\end{aligned} \tag{4.10}$$

Consequently, $x^s - (\delta^*/a)\bar{x} \in X$.

Now, since for each $x \in X$

$$F(x)^T\left(x - x^s\right) \geq -(1-\varepsilon)^{s-1}\Delta \tag{4.11}$$

and

$$F(x)^T\left((\delta^*/a)\bar{x}\right) \geq \delta^*, \tag{4.12}$$

we know

$$\begin{aligned} F(x)^T\left(x - \left(x^s - (\delta^*/a)\bar{x}\right)\right) &= F(x)^T\left(x - x^s\right) + F(x)^T\left((\delta^*/a)\bar{x}\right) \\ &\geq -(1-\varepsilon)^{s-1}\Delta + \delta^* \geq 0. \end{aligned} \tag{4.13}$$

Therefore, $x^s - (\delta^*/a)\bar{x}$ is a solution of DLSIVI(X, F). From Proposition 2, we know $x^s - (\delta^*/a)\bar{x}$ must be a solution of LSIVI(X, F). □

Notice that the conditions assumed in Theorem 4 are technical conditions that may be difficult to check in general. But when they are satisfied, we know how close x^s can be a solution to LSIVI(X, F). In some cases, the technical conditions can be verified easily. For example, if $u(t) = (u_1(t), \ldots, u_n(t)) > 0$ for $t \in T$ and $F(x) = (F_1(x), \ldots, F_n x) > 0$ for $x \in X$, then the conditions are clearly satisfied. Also notice that when the analytic center cutting plane algorithm of [12] is used, under the assumption that F is pseudo co-coercive and Lipschitz continuous on X_i, an approximation solution x^i to VI(X_i, F) can be found in polynomial time. Therefore, in this case, an ε-solution of LSIVI(X, F) can be achieved in polynomial time.

5 Numerical examples

In this section we provide some examples to illustrate the potentials of the discretization approach and the inexact cutting plane approach. We have implemented both approaches using MATLAB on a 1000 MHz Pentium III personal computer running Linux.

Recall that for both approaches, a finite-dimensional variational inequality subproblem has to be solved. For this purpose, we have implemented the method proposed by Goffin et al. [12] that has been cited in the previous sections. To compute the approximate analytic center in Goffin's method, we have used Newton's linear approximation along with a dual scaling procedure [11].

The following examples are studied in the sequel:

Example 1 $n = 7$, $T = [0, 1]$, and

$$X = \left\{ x \in R^7 \,\middle|\, \sum_{j=1}^{7} t^{j-1} x_j \leq \sum_{l=1}^{4} t^{2l} + 1, \quad t \in T \text{ and } 0 \leq x_j \leq 1,\, j = 1, \ldots, 7 \right\},$$

$$F = (F_1, \ldots, F_7) \quad \text{with } F_j = x_j - \frac{1}{\sqrt{x_j}},\ j = 1, \ldots, 7.$$

Example 2 $n = 7$, $T = [0, 1]$, and

$$X = \left\{ x \in R^7 \,\middle|\, \sum_{j=1}^{7} t^{j-1} x_j \le 4t^5 + 1, \quad t \in T \text{ and } 0 \le x_j \le 1,\, j = 1, \ldots, 7 \right\},$$

$$F = (F_1, \ldots, F_7) \quad \text{with } F_j = 3x_j - \frac{1}{x_j^2}, \; j = 1, \ldots, 7.$$

Example 3 $n = 7$, $T = [0, 1]$, and

$$X = \left\{ x \in R^7 \,\middle|\, \sum_{j=1}^{7} t^{j-1} x_j \le 3t^5 + 2t^2 + \frac{1}{3}, \quad t \in T \text{ and } 0 \le x_j \le 1,\, j = 1, \ldots, 7 \right\},$$

$$F = (F_1, \ldots, F_7) \quad \text{with } F_j = \sqrt{x_j} - \frac{1}{x_j^2}, \; j = 1, \ldots, 7.$$

Notice that, for the above examples the *interior point* assumption (2.1) is satisfied when we set $\hat{x}$ to $0.1e$ (e is the vector of all 1's). To apply Goffin's method, the inequalities $0 \le x \le e$ are included in the system of linear inequalities. Consequently, for each example the set X becomes nonempty, convex and compact.

Table 1 shows the solutions found by the discretization approach. The set T has been divided into equally spaced partitions, and to analyze the effect of finer discretization, the number of partitions (NOP) has been varied from 10 to 100. The first column shows the example number. The second column gives the NOP required to achieve the solution x^* reported in the third column. The total number of iterations (Iter) spent for solving the variational inequality subproblems are reported in column

Table 1 Solutions using the discretization approach

Ex.	NOP	x^*	Iter.	Gap	RT
1	10	$x^* = (0.4768, 0.5603, 0.6340, 0.6992, 0.7540, 0.7999, 0.8384)^T$	62	0.001	15 s
	20	$x^* = (0.5053, 0.5718, 0.6303, 0.6845, 0.7312, 0.7698, 0.8054)^T$	74	0.0009	29 s
	40	$x^* = (0.4958, 0.5663, 0.6303, 0.6867, 0.7364, 0.7794, 0.8158)^T$	89	0.0004	1 min 13 s
	80	$x^* = (0.4964, 0.5654, 0.6289, 0.6881, 0.7381, 0.7795, 0.8141)^T$	116	0.0004	4 min 31 s
	100	$x^* = (0.4996, 0.5678, 0.6302, 0.6852, 0.7338, 0.7753, 0.8112)^T$	134	0.0002	8 min 1 s
2	10	$x^* = (0.4824, 0.5286, 0.5680, 0.6001, 0.6252, 0.6440, 0.6578)^T$	70	0.0086	17 s
	20	$x^* = (0.4719, 0.5274, 0.5736, 0.6093, 0.6357, 0.6546, 0.6671)^T$	74	0.0038	28 s
	40	$x^* = (0.4738, 0.5265, 0.5705, 0.6059, 0.6321, 0.6510, 0.6640)$	95	0.0046	1 min 18 s
	80	$x^* = (0.4753, 0.5262, 0.5693, 0.6049, 0.6298, 0.6492, 0.6633)^T$	134	0.0012	5 min 47 s
	100	$x^* = (0.4754, 0.5255, 0.5702, 0.6047, 0.6307, 0.6499, 0.6630)^T$	149	0.0031	10 min 3 s
3	10	$x^* = (0.2785, 0.4766, 0.7150, 0.8853, 0.9623, 0.9880, 0.9957)^T$	73	0.001	30 s
	20	$x^* = (0.2785, 0.4766, 0.7143, 0.8852, 0.9623, 0.9880, 0.9970)^T$	84	0.0026	52 s
	40	$x^* = (0.2769, 0.4793, 0.7215, 0.8912, 0.9647, 0.9895, 0.9969)^T$	119	0.0011	2 min 38 s
	80	$x^* = (0.2760, 0.4807, 0.7256, 0.8948, 0.9667, 0.9902, 0.9971)^T$	173	0.0003	14 min 14 s
	100	$x^* = (0.2762, 0.4801, 0.7238, 0.8933, 0.9662, 0.9899, 0.9969)^T$	204	0.0002	23 min 7 s

Table 2 Solutions using the inexact cutting plane approach

Ex.	k	x^* and T_k	Iter.	Gap	RT
1	5	$x^* = (0.4995, 0.5678, 0.6300, 0.6857, 0.7337, 0.7752, 0.8111)^T$ $T_5 = \{0.0, 0.8131, 0.8268, 0.8299, 0.8338, 1.0\}$	660	0.0002	1 min 47 s
2	4	$x^* = (0.4756, 0.5253, 0.5707, 0.6045, 0.6299, 0.6496, 0.6629)^T$ $T_4 = \{0.0, 0.6678, 0.6738, 0.7333, 1.0\}$	768	0.002	2 min 9 s
3	6	$x^* = (0.2764, 0.4799, 0.7234, 0.8933, 0.9659, 0.9896, 0.9970)^T$ $T_6 = \{0.0, 0.2745, 0.2876, 0.2930, 0.2993, 0.5891, 1.0\}$	629	0.0001	2 min 5 s

four. The fifth column with the title Gap gives the results of the gap function (2.7) evaluated at the corresponding solution x^*. The last column of the table gives the running time (RT) of the algorithm for the corresponding example.

Next we have solved the examples using the inexact cutting plane approach (Algorithm 1). For all the examples, the parameters δ, Δ and ε are set to 1.0e−5, 0.1 and 0.5, respectively. Also, the initial set T_1 is taken as $\{0.0, 1.0\}$. The results with the inexact cutting plane approach are reported in Table 2. The first column shows the example number whereas the second column gives the number of iterations (k) for finding the solution x^*. The third column shows the solution x^* and the final set T_k reported by the algorithm. Similar to Table 1, the last three columns show the total number of iterations spent for solving the variational inequality subproblems, the gap function values and the running times, respectively.

In step 2 of Algorithm 1 if x^k is not an exact solution, at the next iteration the algorithm moves back to step 1 without adding a new cutting plane. Therefore, in Table 2 the number of iterations in the second column may be higher than the cardinality of the final sets (T_k) in the third column.

Recall that we need to solve a semi-infinite programming problem to evaluate the gap function (2.7). In order to report the gap function results, we have used the semi-infinite programming procedure, called `fseminf` of MATLAB. This procedure uses interpolation to estimate the peak values of the constraints and then proceeds with a sequential quadratic programming method.

Analyzing Tables 1 and 2, we see that the inexact cutting plane approach converges to the solutions of the three examples after adding, respectively, 4, 3, and 5 points to the initial sets (T_1). Meanwhile, the third column in Table 1 shows that at the expense of high number of partitions, the discretization approach leads to the solutions that are closer to the solutions confirmed by the inexact cutting plane approach. As the fourth columns show, the number of iterations used in the variational inequality subproblems with the discretization approach is less than the number with the inexact cutting plane approach. However, the time required to solve a problem by the discretization approach is much longer than that of the cutting plane approach. This is a direct consequence of the fact that the solution methods for the inner variational inequality problems slow down as the number of inequalities increase. When we look at the gap function values, we see that the inexact cutting plane approach gives promising gap function values for an inexact algorithm. On the other hand, as expected, the gap function values with the discretization approach improve by finer partitioning.

We have also tried to test the estimated infeasibility of the solutions with both approaches. First, we have partitioned the set $T = [0, 1]$ into 100,000 partitions. Then among these 100,000 inequalities, we have checked the violated ones. Table 3 shows the estimated infeasibility figures with the discretization approach. The first column gives the example number and the second column shows the number of partitions (NOP). The number of violated inequalities are reported in column three. The fourth column gives the average (Avg.) infeasibility, i.e., the average of the differences between the right and left hand-sides of the violated inequalities. The last three columns give the standard deviation (Std. Dev.) of the infeasibility, the maximum (Max) infeasibility and the minimum (Min) infeasibility, respectively. Similarly, Table 4 shows the estimated infeasibility figures with the inexact cutting plane approach. The first column gives the example number and the remaining columns are same as the last five columns of Table 3.

Our test results indicate that the level of infeasibility using the cutting plane approach is not a problem. It is better than that produced by the discretization approach in our experiments. Of course, one may take longer running time with further partitioning to decrease the level of infeasibility for the discretization approach.

Table 3 Estimated infeasibility with the discretization approach

Ex.	NOP	Num.	Avg.	Std. Dev.	Max	Min
1	10	8507	7.4e−3	3.3e−3	1.1e−3	3.3e−6
	20	4998	2.4e−3	1.1e−3	3.5e−3	1.2e−6
	40	1191	1.3e−4	6.1e−5	2.6e−4	2.3e−7
	80	989	1.1e−4	5.9e−5	2.0e−4	2.4e−8
	100	259	6.4e−6	2.8e−6	9.6e−6	4.4e−8
2	10	6151	2.9e−3	1.3e−3	4.4e−3	1.9e−6
	20	2990	1.9e−3	8.6e−4	2.9e−3	1.2e−6
	40	656	3.3e−5	1.5e−5	5.1e−5	3.6e−8
	80	504	1.9e−5	8.2e−6	2.9e−5	1.9e−8
	100	454	1.6e−5	7.2e−6	2.4e−5	2.6e−8
3	10	4320	1.4e−4	6.1e−5	2.1e−4	1.7e−7
	20	3800	1.2e−4	5.5e−5	1.8e−4	3.1e−8
	40	1499	4.2e−5	1.9e−5	6.3e−5	6.9e−8
	80	995	9.9e−6	4.4e−6	1.5e−5	3.5e−8
	100	567	9.8e−7	7.7e−7	9.1e−6	2.1e−8

Table 4 Estimated infeasibility with the inexact cutting plane approach

Ex.	Num.	Avg.	Std. Dev.	Max	Min
1	244	5.6e−6	2.5e−6	8.5e−6	7.9e−6
2	250	4.8e−6	1.0e−6	3.4e−6	3.8e−7
3	233	1.8e−6	8.3e−7	2.8e−6	1.3e−8

6 Conclusion

In this paper, we have studied a special class of variational inequalities over a domain defined by infinitely many linear inequalities. A discretization approach for solving such problems is introduced with a convergence proof. We also propose an *inexact* cutting plane method based on analytic centers. A convergence proof and several numerical examples are included. Under proper conditions, we can examine the quality of solutions obtained. When F is pseudo co-coercive and Lipschitz continuous, an ε-optimal solution may be generated by the proposed algorithm in polynomial time.

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