# HOMOLOGICAL INVARIANTS OF SQUAREFREE POWERS OF SIMPLICIAL TREES

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**HOMOLOGICAL INVARIANTS OF SQUAREFREE POWERS OF SIMPLICIAL TREES**

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# ABSTRACT

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Keywords: Squarefree power, simplicial forest, linear resolution, linearly related, Castelnuovo-Mumford regularity, path ideals

In this thesis, we study the squarefree powers of facet ideals associated with simplicial trees. Specifically, we examine the linearity of their minimal free resolution and their regularity. Additionally, we investigate when the first syzygy module of squarefree powers of facet ideal of a simplicial tree is generated by linear relations. Finally, we provide a combinatorial formula for the regularity of the squarefree powers of $t$-path ideals of path graphs.

# ÖZET

SİMPLEKS AĞAÇLARIN KARESİZ KUVVETLERİNİN HOMOLOJİK DEĞIŞMEZLERI

ELSHANI KAMBERI

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Anahtar Kelimeler: Karesiz kuvvet, simpleks orman, doğrusal çözümleme, doğrusal ilişkili, Castelnuovo-Mumford düzenliliği, yol idealleri

Bu tezde, simpleks ağaçlarla ilişkili faset ideallerinin karesiz kuvvetlerini inceliyoruz. Özellikle, bunların minimal serbest çözümlemesinin doğrusallığını ve düzenliliğini ele alıyoruz. Ayrıca, bir simpleks ağacın faset idealinin karesiz kuvvetlerinin ilk syzygy modülünün doğrusal ilişkiler tarafından ne zaman üretildiğini araştırıyoruz. Son olarak, yol çizgelerinin t-yol ideallerinin karesiz kuvvetlerinin düzenliliği için bir kombinatorik formül sunuyoruz.

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also a reflection of the care, time, and wisdom shared with me by many wonderful people. I dedicate this work to all those who supported and believed in me along the way.

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# 1. INTRODUCTION

In the last two decades, the study of the regularity of powers of squarefree monomial ideals has become a notable trend in combinatorial commutative algebra. This line of research took one of its initial steps with a beautiful theorem (see Cutkosky, Herzog & Trung (1999); Kodiyalam (1993)), which states that if $I$ is a homogeneous ideal in a polynomial ring, then the function $\operatorname{reg}(I^k)$ is asymptotically linear. This result gathered the interest of many algebraists to investigate the regularity and, more generally, the minimal graded free resolution of the powers of homogeneous ideals, leading to the publication of many interesting papers on this topic. However, obtaining a complete understanding of the asymptotic linearity of the regularity function for arbitrary homogeneous ideals seems almost impossible, and this problem remains open even for squarefree monomial ideals.

In this context, the study of the regularity of the squarefree powers of squarefree monomial ideals assumes an important role. Let $I$ be a squarefree monomial ideal. The $k$-th squarefree power $I^{[k]}$ of $I$ is the ideal generated by the squarefree generators of $I^k$. The squarefree powers of $I$ provide important information on the ordinary powers of $I$. It is known from (Herzog, Hibi & Zheng, 2004, Lemma 4.4) that the multigraded minimal free resolution $I^{[k]}$ is a subcomplex of the multigraded minimal free resolution of $I^k$. Consequently, $\operatorname{reg} I^{[k]} \leq \operatorname{reg} I^k$, and if $I^{[k]}$ does not have a linear resolution, then $I^k$ also does not have a linear resolution. Another interesting aspect of the study of squarefree powers comes from their deep link with the matching theory of simplicial complexes (or hypergraphs). Recall that $I$ can be viewed both as an edge ideal of a hypergraph and as a facet ideal of a simplicial complex. In this thesis, we will adopt the latter terminology. Let $\Delta$ be a simplicial complex and $I(\Delta)$ be the facet ideal of $\Delta$. A matching of $\Delta$ is a set of pairwise disjoint facets of $\Delta$. Indeed, the generators of $I(\Delta)^{[k]}$ correspond to the matching of $\Delta$ of size $k$. This means that $I(\Delta)^{[k]} \neq 0$ only when $1 \leq k \leq \nu(\Delta)$, where $\nu(\Delta)$ is the maximum size of a matching of $\Delta$. The study of squarefree powers began with (Bigdeli, Herzog & Zaare-Nahandi, 2017) for facet ideals of 1-dimensional simplicial complexes, or simply, the edge ideals of graphs. Since then, several papers, including

(Crupi, Ficarra & Lax, 2023; Erey & Ficarra, 2023; Erey, Herzog, Hibi & Madani, 2022,2; Erey & Hibi, 2021; Ficarra, Herzog & Hibi, 2023), have been published on this topic, focusing on the squarefree powers of edge ideals of different classes of graphs.

Inspired by this, in this thesis we study the homological properties of the squarefree powers of squarefree monomial ideals that are not necessarily quadratic. To this end, we begin our investigation with the squarefree monomial ideals attached to simplicial trees. In (Faridi, 2002), Faridi introduced simplicial trees as a natural generalization of trees in the context of graphs. In the language of hypergraphs, simplicial trees correspond to totally balanced hypergraphs, (Herzog, Hibi, Trung & Zheng, 2008, Theorem 3.2), that is the hypergraphs without any “special" cycles. We direct reader to (Berge, 1989, Chapter 5), for more details.

A breakdown of the contents of this thesis is as follows: in Section 2 we give some preliminaries. Then, in Section 3 we prove some preliminary results related to the matching of simplicial trees. Section 4 is devoted to studying the linearity of the resolutions of the squarefree powers of the facet ideals of simplicial trees. In (Zheng, 2004b, Theorem 3.17), Zheng showed that a facet ideal of a simplicial tree $\Delta$ has a linear resolution if and only if $\Delta$ satisfies the so-called intersection property. More recently, the authors of (Kumar & Kumar, 2023) proved that if the facet ideal $I(\Delta)$ of a simplicial tree has a linear resolution, then $I(\Delta)^k$ has a linear resolution for all $k \geq 1$. A natural question in this context is whether the squarefree powers of $I(\Delta)$ also admit the same property, that is, if $I(\Delta)$ has a linear resolution, then does $I(\Delta)^{[k]}$ also have a linear resolution? However, in Proposition 4.0.1 we prove that if $I(\Delta)$ has a linear resolution, or equivalently, if $\Delta$ has the intersection property, then $\nu(\Delta) \leq 2$. This implies that $I(\Delta)^{[k]} = 0$ for all $k > 2$, meaning that the only squarefree power to be considered is $I(\Delta)^{[2]}$. The linearity of the resolution of $I(\Delta)^{[2]}$ follows from (Soleyman Jahan & Zheng, 2010, Proposition 2.10) together with (Kumar & Kumar, 2023, Lemma 2.3), as discussed in Theorem 4.0.3. In addition to this, we prove Theorem 4.0.4, where we give a precise ordering of the generators of the second squarefree power, which gives linear quotients.

The next question that we tackle in Section 4 is motivated by (Bigdeli et al., 2017, Theorem 5.1), in which the authors show that the highest non-vanishing power of an edge ideal $I(G)$ of any graph $G$ admits a linear resolution. Theorem 4.0.3 also implies that the highest non-vanishing squarefree power of the facet ideal of a simplicial tree with the intersection property has a linear resolution. However, this is not true in general for simplicial trees, as observed in Example 4.1.1 for a suitable 3-path ideal of a rooted tree graph. The $t$-path ideals of graphs were introduced in (Conca & De Negri, 1999) as ideals generated by monomials that correspond to paths of length $t-1$ of a graph $G$. If $G$ is a directed graph, then one may consider the $t$-path ideal of $G$ as an ideal generated by monomials corresponding to the directed paths of length $t-1$. For $t=2$, the $t$-path ideal coincides with the edge ideal of $G$. In (He & Van Tuyl, 2010), the authors proved that the $t$-path ideal of a rooted tree (a special class of directed trees) is the facet ideal of a simplicial tree, providing a rich class of simplicial trees. The $t$-path ideal of a rooted tree have been studied by many authors, for example see (Alilooee & Faridi, 2015,1; Bouchat, Hà & O'Keefe, 2011; Kumar & Kumar, 2023). Given a rooted tree $\Gamma$, we denote the simplicial complex whose facets are $t$-paths of $\Gamma$ as $\Gamma_t$. As observed in Example 4.1.1, the highest non-vanishing power of $I(\Gamma_t)$ need not have a linear resolution. This motivated us to state Theorem 4.1.5, where we show that the highest non-vanishing squarefree power of $I(\Gamma_t)$ has a linear resolution if $\Gamma_t$ is a broom graph.

Section 5 is devoted to understanding when the first syzygy module of the squarefree powers of the facet ideals of simplicial forests is generated by linear forms. It turns out that the restricted matching of a simplicial complex $\Delta$ plays an important role in this context. The restricted matching of a graph (1-dimensional simplicial complex) was introduced in (Bigdeli et al., 2017), and we extend this definition for any $d$-dimensional simplicial complex; see Section 3 for the formal definition. We prove in Theorem 5.0.3 that if $\Delta$ is a simplicial forest, then the squarefree powers of $I(\Delta)$ are not linearly related up to the restricted matching number of $\Delta$. It is shown in (Erey et al., 2022, Theorem 3.1) that given a graph $G$, if $I(G)^{[k]}$ is linearly related, then $I(G)^{[k+1]}$ is linearly related for all $k \geq 1$. We prove an analogue of this result in the case when $\Delta$ is a pure simplicial tree. This also shows that if the highest squarefree power of $I(\Delta)$ is not linearly related, then none of the non-vanishing squarefree powers have a linear resolution. We further investigate the first syzygy modules for a special class of simplicial trees, laying the groundwork for discussing regularity in the next section. Utilizing a celebrated result by Gasharov, Peeva, and Welker (Gasharov, Peeva & Welker, 1999), we prove that for the $t$-path simplicial tree $\Gamma_{n,t}$ of a path graph $P_n$ and its associated ideal $I_{n,t} = I(\Gamma_{n,t})$, we have $\beta_{1,p}\left(I_{n,t}^{[k]}\right) = 0$ whenever $p \notin \{kt+1, (k+1)t\}$.

Recently, in (Kumar & Kumar, 2023, Section 4), the authors discussed the regularity of certain ordinary powers of $t$-path ideals of broom graphs and proposed a conjecture on the upper bound of the regularity of facet ideals of simplicial trees. It is noteworthy that path graphs constitute a special subclass of broom graphs.

Motivated by their findings, in Section 6, we concentrate on the $t$-path ideals of path graphs. In Theorem 6.0.9, we present a combinatorial formula for their regularity. To achieve this, we utilize well-known exact sequences for homogeneous ideals (see Theorem 6.0.4) and employ both combinatorial and topological techniques (see Theorem 6.0.7). The main result of this section is summarized in the following theorem.

**Theorem** (see Theorem 6.0.9) *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$ and $I_{n,t} = I(\Gamma_{n,t})$. Then for any $1 \leq k+1 \leq \nu(\Gamma_{n,t})$, we have*

$$\operatorname{reg}\left(\frac{R}{I_{n,t}^{[k+1]}}\right) = kt + (t-1)\nu_1(\Gamma_{n-kt,t}) = kt + \operatorname{reg}\left(\frac{R}{I_{n-kt,t}}\right).$$

where $\nu_1(\Gamma_{n-kt,t})$ denotes the induced matching number of $\Gamma_{n-kt,t}$.

We conclude this article with some open questions in Section 7 and a conjecture on the bounds of the regularity of squarefree powers of $I(\Delta)$, where $\Delta$ is a simplicial tree.

# 2. ALGEBRAIC AND COMBINATORIAL INGREDIENTS

In this section, we give some combinatorial and algebraic preliminaries. The notation and definitions given in this section will be used throughout in the later sections. Note that, all the rings considered in this thesis are commutative rings with identity.

## 2.1 Simplicial Trees

A *simplicial complex* $\Delta$ on vertex set $V(\Delta)$ is a non-empty collection of subsets of $V(\Delta)$ such that if $F' \in \Delta$ and $F \subseteq F'$, then $F \in \Delta$. Given a collection $\mathrm{F} = \{F_1, \ldots, F_m\}$ of subsets of $V(\Delta)$, we denote by $\langle F_1, \ldots, F_m \rangle$ or briefly $\langle \mathrm{F} \rangle$, the simplicial complex consisting of all subsets of $V(\Delta)$ which are contained in $F_i$ for some $i = 1, \ldots, m$. The elements of $\Delta$ are called *faces* of $\Delta$. For any $F \in \Delta$, the dimension of $F$, denoted by $\dim(F)$ is one less than the cardinality of $F$. An *edge* of $\Delta$ is a face of dimension 1, while a *vertex* of $\Delta$ is a face of dimension 0. The dimension of $\Delta$ is given by $\max\{\dim(F) : F \in \Delta\}$. The maximal faces of $\Delta$ with respect to the set inclusion are called *facets*. We denote the set of all facets of $\Delta$ by $\mathcal{F}(\Delta)$. A *subcomplex* $\Delta'$ of $\Delta$ is a simplicial complex such that $\mathcal{F}(\Delta') \subseteq \mathcal{F}(\Delta)$. A subcomplex $\Delta'$ of $\Delta$ is said to be *induced* if each facet $F \in \mathcal{F}(\Delta)$ with $F \subseteq V(\Delta')$ belongs to $\Delta'$. A simplicial complex $\Delta$ is called *pure* if all facets of $\Delta$ have the same dimension. For a pure simplicial complex $\Delta$, the dimension of $\Delta$ is given trivially by the dimension of a facet of $\Delta$.

**Example 2.1.1.** Consider the simplicial complex $\Delta$ with facets

$$\begin{aligned}
&F_1 = \{1,2,3\}, \quad F_2 = \{3,4,5\}, \quad F_3 = \{5,6,7\}, \\
&F_4 = \{6,7,8\}, \quad F_5 = \{8,9,10\}, \quad F_6 = \{9,10,11\}, \quad F_7 = \{3,11,12\}.
\end{aligned}$$

We write $\Delta = \langle F_1, F_2, \ldots, F_7 \rangle$.

The complex $\Delta$ is generated by 2-dimensional facets (triangles), each with three vertices. All other faces of $\Delta$ are subsets of these facets. For example, $\{5,6\}$ is a 1-dimensional face, and $\{7\}$ is a 0-dimensional face. Since every facet has dimension 2, the complex is pure of dimension 2.

Note that $\Delta' = \langle \{5,6,7\},\ \{8,9,10\} \rangle$ is a subcomplex of $\Delta$, but it is not an induced subcomplex because $F_4 = \{6,7,8\} \subseteq V(\Delta')$ and $F_4 \notin \Delta'$.

This simplicial complex can be visualized by plotting each triangle in the plane and connecting those that share common faces, as shown in Figure 2.1.

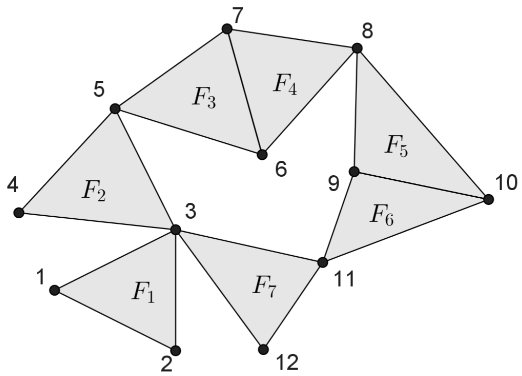



Figure 2.1 A simplicial complex.

For instance, $F_1$ and $F_2$ share vertex 3; $F_3$ and $F_4$ share the edge $\{7,8\}$.

In (Faridi, 2002), Faridi introduced the class of the simplicial forests, which will be deeply discussed along this work. A facet $F$ of $\Delta$ is called a *leaf* if either $F$ is the only facet of $\Delta$ or there exists a facet $G \in \Delta$ such that $H \cap F \subset G \cap F$ for all facets $H \neq F$ of $\Delta$. Such a facet $G$ is called *branch* of $F$ in $\Delta$. Observe that every leaf $F$ of $\Delta$ contains a *free vertex*, that is a vertex $v$ of $\Delta$ such that $v \notin F'$ for every facet $F' \neq F$ of $\Delta$. A connected simplicial complex $\Delta$ is a *tree* if every nonempty subcomplex of $\Delta$ has a leaf. A simplicial forest is a simplicial complex whose every connected component is a tree. It directly follows from the definition of a simplicial tree that if $\Delta$ is a simplicial tree, then any subcomplex of $\Delta$ is also a tree. For instance, the simplicial complex as in Figure 2.2 is a tree.

On the other hand, if $\Delta$ is the simplicial complex as in Figure 2.1, then $\langle \mathcal{F}(\Delta) \setminus \{F_1\} \rangle$ does not have a leaf, which implies that $\Delta$ is not a simplicial tree.

In (Zheng, 2004a), Zheng introduced the notion of a good leaf of a simplicial complex. Let $\Delta$ be a simplicial complex. A leaf $F$ of $\Delta$ is said to be a *good leaf* if $F$ is a leaf of every subcomplex of $\Delta$ containing $F$. In (Herzog et al., 2008, Corollary 3.4) it is shown that every simplicial forest has a good leaf. A *good leaf order* on the facets of $\Delta$ is an ordering of facets $F_1, \ldots, F_r$ such that $F_i$ is a good leaf of

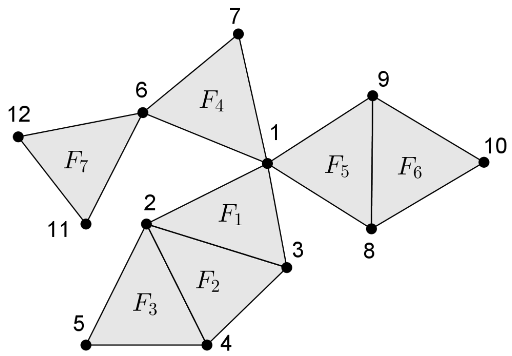



Figure 2.2 A simplicial tree.

the subcomplex $\langle F_1, \ldots, F_i \rangle$ for every $2 \leq i \leq r$. In Figure 2.2, for instance, $F_7$ is a good leaf and the ordering of facets $F_3, F_2, F_1, F_5, F_6, F_4, F_7$ is a good leaf order. By virtue of (Herzog et al., 2008, Corollary 3.4), it immediately follows that every simplicial forest $\Delta$ admits a good leaf order.

A simplicial complex $\Delta$ is said to be *connected in codimension 1*, if for any two facets $F$ and $G$ of $\Delta$ with $\dim(F) \geq \dim(G)$, there exists a chain $\mathcal{C} : F = F_0, \ldots, F_n = G$ between $F$ and $G$ such that $\dim(F_i \cap F_{i+1}) = \dim(F_{i+1}) - 1$ for all $i = 0, \ldots, n-1$. Such a chain $\mathcal{C}$ is called a *proper* chain. A proper chain $\mathcal{C}$ between $F$ and $G$ is called *irredundant* if no subsequence of this chain except $\mathcal{C}$ itself is a proper chain between $F$ and $G$.

Let $\Delta$ be a pure $d$-dimensional simplicial tree connected in codimension 1. It is known from (Zheng, 2004b, Proposition 1.17) that for any two facets $F$ and $G$, there exists a unique irredundant proper chain between $F$ and $G$. The length of the unique irredundant proper chain between $F$ and $G$ is called the *distance* between $F$ and $G$, and is denoted by $\mathrm{dist}(F, G)$. If for any two facets $F$ and $G$ with $\dim(F \cap G) = d - k$ for some $k = 1, \ldots, d+1$, we have $\mathrm{dist}(F, G) = k$, then $\Delta$ is said to have the *intersection property*.

**Example 2.1.2.** Let $\Delta$ be a simplicial complex on the vertex set $[n] = [7]$, given in Figure 2.3.

We will show that this simplicial tree has the intersection property. We compute the distance between selected pairs of facets and verify that the intersection property holds:

In each case, the distance satisfies the relation

$$\dim(F_i \cap F_j) = d - k \quad \text{if and only if} \quad \mathrm{dist}(F_i, F_j) = k,$$

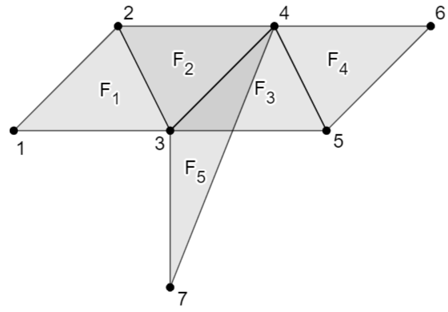



Figure 2.3 A pure 2-dimensional simplicial tree connected in codimension 1.

Table 2.1 Intersection property calculations.

| Facets | $\dim(F_i \cap F_j)$ | $\mathrm{dist}(F_i, F_j) = k$ |
|---|---|---|
| $F_1, F_2$ | 1 | 1 |
| $F_1, F_3$ | 0 | 2 |
| $F_1, F_4$ | $-1$ | 3 |
| $F_1, F_5$ | 0 | 2 |
| $F_2, F_3$ | 1 | 1 |
| $F_2, F_4$ | 0 | 2 |
| $F_2, F_5$ | 1 | 1 |
| $F_3, F_4$ | 1 | 1 |
| $F_3, F_5$ | 1 | 1 |
| $F_4, F_5$ | 0 | 2 |

where $d = 2$. Therefore, the complex $\Delta$ satisfies the *intersection property.*

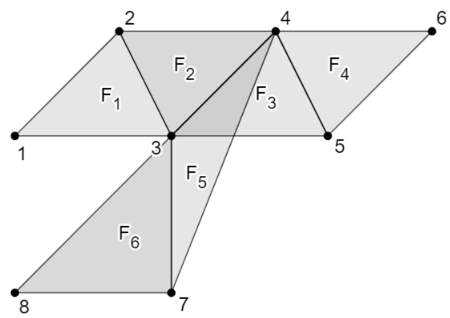



Figure 2.4 The simplicial complex $\Delta'$.

However, not all simplicial trees satisfy the intersection property. Consider the simplicial complex $\Delta'$ in Figure 2.4.

Then, $\dim(F_1 \cap F_6) = 0$ and $k = 3$, but $d - k = 2 - 3 = -1 \neq \dim(F_1 \cap F_6)$. Hence, $\Delta'$ does not satisfy the intersection property.

## 2.2 Monomial Ideals and Their Free Resolutions

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring on variables $x_1,\dots,x_n$ over a field $K$. A monomial is any product $u = a{x_1}^{a_1}\cdots {x_n}^{a_n}$ for some $a \in K$ and non-negative integers $a_1,\dots,a_n$. The set of variables $\{x_{i_j} \in \{x_1,\dots,x_n\} : a_{i_j} \neq 0\}$ is called the *support* of $u$, denoted by $\operatorname{supp}(u)$. A monomial $u = a{x_1}^{a_1}\cdots {x_n}^{a_n}$ is called squarefree if $a_i \leq 1$ for all $i = 1,\dots,n$. Then, an ideal $I$ of $S$ is called a monomial ideal if it is generated by monomials in $S$.

A monomial ideal $I$ in $S$ is said to be generated by a set of monomials $\{u_1,\dots,u_r\}$ if $I = (u_1,\dots,u_r)$. This generating set is called *minimal* if none of the monomials $u_i$ divides any other $u_j$ for $i \neq j$, and it is denoted by $G(I)$. Every monomial ideal has a unique minimal set of monomial generators, up to ordering.

For instance, consider the ideal $I = (x_1x_2, x_2x_3, x_1x_2x_4)$ in $K[x_1,x_2,x_3,x_4]$. The monomial $x_1x_2x_4$ is divisible by $x_1x_2$, so it is redundant. Thus, the minimal generating set of $I$ consists of the monomials $x_1x_2$ and $x_2x_3$, and we write $I = (x_1x_2,\ x_2x_3)$ minimally.

Let $I = (u_1,\dots,u_r)$ and $J = (v_1,\dots,v_s)$ be monomial ideals in $S$ with $G(I) = \{u_1,\dots,u_r\}$ and $G(J) = \{v_1,\dots,v_r\}$. The sum of $I$ and $J$ is:

$$I + J = (u_1,\dots,u_r,v_1,\dots,v_s),$$

which is also a monomial ideal generated by the union of the minimal generators of $I$ and $J$, but it may not be minimal itself. Equivalently, a monomial $m \in I + J$ if and only if $m$ is divisible by some $u_i$ or $v_j$.

The intersection of two monomial ideals $I$ and $J$ is generated by the least common multiples of their generators:

$$I \cap J = (\operatorname{lcm}(u,v) \mid u \in G(I),\ v \in G(J)).$$

That is, a monomial belongs to $I \cap J$ if and only if it is divisible by some monomial in $I$ and some monomial in $J$, and it can be expressed as the least common multiple of such a pair. By (Herzog & Hibi, 2011, Proposition 1.2.1) the intersection is also a monomial ideal.

The colon ideal $(I\colon J)$ or simply $I\colon J$ is said to be the set $\{r \in S | rJ \subseteq I\}$. By (Herzog & Hibi, 2011, Proposition 1.2.2) we have $I : J = \bigcap_{v\in J} I\colon (v) = \bigcap_{v\in J} I\colon v$.

Combining (Herzog & Hibi, 2011, Proposition 1.2.1) and (Herzog & Hibi, 2011, Proposition 1.2.2), we see that the colon ideal of two monomial ideals is also a monomial ideal.

**Example 2.2.1.** Let $S = K[x_1, x_2, x_3, x_4]$ be a polynomial ring over a field $K$. Consider the monomial ideals $I = (x_1x_2,\ x_2x_3,\ x_3x_4)$ and $J = (x_1x_3,\ x_2x_4)$.

The sum of $I$ and $J$ is simply the ideal generated by all the generators of both:

$$I + J = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_1x_3,\ x_2x_4).$$

The intersection of $I$ and $J$ is generated by the least common multiples of all pairs of generators from $G(I)$ and $G(J)$:

$$\begin{aligned} I \cap J = (&\mathrm{lcm}(x_1x_2, x_1x_3),\ \mathrm{lcm}(x_1x_2, x_2x_4),\ \mathrm{lcm}(x_2x_3, x_1x_3), \\ &\mathrm{lcm}(x_2x_3, x_2x_4),\ \mathrm{lcm}(x_3x_4, x_1x_3),\ \mathrm{lcm}(x_3x_4, x_2x_4)) \\ = (&x_1x_2x_3,\ x_1x_2x_4,\ x_1x_2x_3,\ x_2x_3x_4,\ x_1x_3x_4,\ x_2x_3x_4). \end{aligned}$$

After simplification, we get $I \cap J = (x_1x_2x_3,\ x_1x_2x_4,\ x_2x_3x_4,\ x_1x_3x_4)$.

For the colon ideal we have $I : J = (I : x_1x_3) \cap (I : x_2x_4)$. So, we have to find all monomials $r \in S$ such that $rx_1x_3 \in I \quad \text{and} \quad rx_2x_4 \in I$.

- We get $rx_1x_3 \in I$, if $r = x_2$ or $r = x_4$. So, $I : x_1x_3 = (x_2, x_4)$.
- We get $rx_2x_4 \in I$, if $r = x_1$ or $r = x_3$. So, $I : x_2x_4 = (x_1, x_3)$

Hence, $I : J = (x_2, x_4) \cap (x_1, x_3) = (x_1x_2, x_2x_3, x_1x_4, x_3x_4)$.

**Definition 2.2.2.** *A polynomial $f \in S$ is called homogeneous in degree $d$ if every term of $f$ is a scalar multiple of a monomial of degree $d$. The set of all homogeneous polynomials of degree $d$ in $S$ forms a $K$-vector space, denoted by $S_d$.*

An ideal of $S$ is called a *homogeneous ideal* if it can be generated by homogeneous polynomials. In particular, every monomial ideal is a homogeneous ideal, since each monomial is homogeneous.

The ring $S = K[x_1, \ldots, x_n]$ has a $\mathbb{Z}$-grading, where each variable has degree one:

$$\deg(x_i) = 1 \quad \text{for all } i,$$

which gives us a decomposition $S = \bigoplus_{d \geq 0} S_d$, where $S_d$ is the $K$-vector space of homogeneous polynomials of degree $d$. Note that the twist $S(-d)$ is the same underlying module as $S$, but with the grading shifted down by $d$, more precisely $S(-d)_i = S_{i-d}$. So, the degree $i$ part of $S(-d)$ is the degree $i-d$ part of $S$, for $d \geq 0$.

Let $I$ be a homogeneous ideal of $S$. Using the Hilbert syzygy theorem (Hilbert, 1890), it is well known that a minimal graded free resolution $\mathbb{F}(I)$ of $I$ exists, it is unique up to isomorphisms, and it is finite with length at most $n$. In such a case, $\mathbb{F}(I)$ can be written as

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{\ell,j}} \xrightarrow{d_\ell} \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}} \xrightarrow{d_i} \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}} \xrightarrow{d_0} I \to 0,$$

where $\ell \leq n$ and all $d_i$'s are graded module homomorphisms. The position index $i$ is called the homological degree and the shifting $j$ is called the internal degree. The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of $I$. When considering the minimal free resolution of the ideal $I$, we denote them by $\beta_{i,j}(I)$.

A *syzygy* is a relation among generators. For an ideal $I$, the first syzygy module correspond to the set of all relations among the generators of $I$ that sum to zero, that is, the kernel of $d_0$ in the free resolution. The second syzygy module consists of the relations among the first syzygies, i.e., the kernel of $d_1$ in the resolution, and so on. These syzygy modules explain how the structure of the ideal is built: starting from generators, then recording how those generators fit together (first syzygies), then how those relations relate (second syzygies), continuing until the syzygy module at some step becomes zero.

The graded free resolution describes the structure of the ideal step by step, showing how it is built from its generators and the relations between them. The Betti numbers tell us exactly how many generators or relations (syzygies) appear at each step.

**Definition 2.2.3.** *The Castelnuovo-Mumford regularity (or simply regularity) of $I$ is:*

$$\operatorname{reg}(I) = \max\{k : \beta_{i,i+k}(I) \neq 0, \text{ for some } i\}.$$

*Moreover,* $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$.

In the following example, we illustrate the concepts of free resolution, graded Betti numbers, and the Castelnuovo-Mumford regularity of a monomial ideal $I$.

**Example 2.2.4.** Let $S = K[x_1, x_2, x_3, x_4, x_5]$ be a polynomial ring over a field $K$, and consider the squarefree monomial ideal

$$I = (x_1x_2x_3,\ x_2x_3x_4,\ x_1x_4,\ x_3x_5).$$

This ideal has two generators of degree 3 and two of degree 2. We analyze its minimal graded free resolution, Betti numbers, and their homological interpretations.

The minimal graded free resolution and the Betti diagram of $S/I$ can be computed using the computer algebra system `Macaulay2` (Grayson & Stillman, Grayson & Stillman). The following code initializes the ring and ideal, computes the free resolution and the graded betti numbers of $S/I$:

```
S = QQ[x_1, x_2, x_3, x_4, x_5]
I = ideal(x_1*x_2*x_3, x_2*x_3*x_4, x_1*x_4, x_3*x_5)
res I
betti res I.
```

Minimal graded free resolution of $S/I$ and $I$ is as follows:

$$0 \longrightarrow S(-5)^2 \longrightarrow S(-4)^5 \longrightarrow S(-3)^2 \oplus S(-2)^2 \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

$$0 \longrightarrow S(-5)^2 \longrightarrow S(-4)^5 \longrightarrow S(-3)^2 \oplus S(-2)^2 \longrightarrow I \longrightarrow 0$$

Note that $I$ is the first syzygy module of $S/I$, and thus its resolution is obtained from that of $S/I$ by truncating the zero-th step. So, the relation between the Betti numbers of $I$ and $S/I$ is; $\beta_{i,j}(I) = \beta_{i+1,j}(S/I)$.

The Betti numbers $\beta_{i,j}(I)$ count the number of generators in homological degree $i$ and inner degree $j$ of the ideal $I$. Specifically:

- $\beta_{0,2}(I) = 2$, $\beta_{0,3}(I) = 2$: These reflect the two generators of degree 2 and two of degree 3 in $I$.
- $\beta_{1,4}(I) = 5$: First syzygies occur in degree 4.
- $\beta_{2,5}(I) = 2$: Second syzygies appear in degree 5.

The Betti numbers mentioned above are from the Betti diagram of $S/I$ given in Table 2.2.

In this Betti diagram, the columns correspond to the homological degree $i$, and the rows represent the difference $j - i$, where $j$ is the internal degree. Therefore, the

Table 2.2 Betti diagram for $S/I$.

| | 1 | 2 | 3 |
|---|---|---|---|
| 1: | 2 | – | – |
| 2: | 2 | 5 | 2 |
| Total: | 4 | 5 | 2 |

entry in column $i$ and row $j-i$ represents the Betti number $\beta_{i,j}$.

So, the regularity of $I$ is $\operatorname{reg}(I) = \max\{k \mid \beta_{i,i+k}(I) \neq 0, \text{ for some } i\} = \max\{2,3,3,3\} = 3$.

Note that the regularity of $S/I$ can be directly observed from its Betti diagram. In the layout of the Betti table, this is simply the *last nonzero row.* Since the regularity of the ideal $I$ satisfies the relation $\operatorname{reg}(I) = \operatorname{reg}(S/I)+1$, it follows that $\operatorname{reg}(I)$ is one more than the index of the last nonzero row in the Betti diagram.

If $\beta_{i,j}(I) = 0$ for all $i \geq 0$ and for $j \neq i+t$, then $I$ is said to admit a *linear resolution.* It is well known that a homogeneous ideal generated in degree $d$ admits a linear resolution if and only if $\operatorname{reg}(I) = d$. As en example, consider the ideal

$$I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5),$$

in $S = K[x_1, x_2, x_3, x_4, x_5]$. This ideal is generated by ten homogeneous monomials of degree 2. It admits a linear resolution:

$$0 \longrightarrow S(-5)^4 \longrightarrow S(-4)^{15} \longrightarrow S(-3)^{20} \longrightarrow S(-2)^{10} \longrightarrow I \longrightarrow 0.$$

The regularity of $I$ is 2, which equals the degree of its generators.

From (Herzog & Hibi, 2011, Proposition 8.2.1) we know that $I$ has a linear resolution if $I$ has linear quotients, that is, if there exists a system of homogeneous generators $f_1, f_2, \ldots, f_m$ of $I$ such that $(f_1, \ldots, f_{i-1}) : f_i$ is generated by linear forms for all $i = 1, \ldots, m$. However, the converse need not to hold.

### 2.3 Facet Ideals and Their Squarefree Powers

Let $I$ be a monomial ideal of $S$. Given any $\{i_1,\ldots,i_r\} \subset [n] = \{1,\ldots,n\}$, we associate a squarefree monomial $u = x_{i_1}\ldots x_{i_r} \in S$. Let $\Delta$ be a simplicial complex on vertex set $[n]$. The monomial ideal generated by all squarefree monomials $x_{i_1}\ldots x_{i_r}$ such that $\{i_1,\ldots,i_r\} \in \mathcal{F}(\Delta)$ is called the *facet ideal* of $\Delta$ and it is denoted by $I(\Delta)$.

**Example 2.3.1.** Let $\Delta$ be the simplicial complex generated by facets $\{1,2,3\},\{2,3,4\}$ and $\{3,4,5\}$. Then the facet ideal of $\Delta$ is

$$I(\Delta) = (x_1x_2x_3,\ x_2x_3x_4,\ x_3x_4x_5) \subseteq K[x_1,x_2,x_3,x_4,x_5].$$

Each generator of $I(\Delta)$ corresponds to a facet of the simplicial complex. Since each facet has exactly three vertices, the ideal is homogeneous of degree 3. Moreover, all generators are squarefree monomials.

**Definition 2.3.2.** *Given a squarefree monomial ideal $I$ in $K[x_1,\ldots,x_n]$, the $k$-th squarefree power of $I$ is defined to be the ideal generated by squarefree elements of $G(I^k)$ and it is denoted by $I^{[k]}$.*

**Example 2.3.3.** Consider the simplicial complex $\Delta = \langle\{1,2\},\{3,4\},\{1,3\},\{5,6\}\rangle$. Then the facet ideal of $\Delta$ is $I(\Delta) = (x_1x_2,\ x_3x_4,\ x_1x_3,\ x_5x_6)$. We now compute $I^{[k]}$ for all $k = 1,2,3,4$.

- For $k = 1$, we have $I^{[1]} = I$.
- For $k = 2$, we consider all squarefree products of pairs of distinct generators. Thus, $I(\Delta)^{[2]} = (x_1x_2x_3x_4,\ x_1x_2x_5x_6,\ x_3x_4x_5x_6,\ x_1x_3x_5x_6)$.
- For $k = 3$, we consider all squarefree products of 3 distinct generators. So, we have $I(\Delta)^{[3]} = (x_1x_2x_3x_4x_5x_6)$.
- For $k = 4$, we observe that $(x_1x_2)(x_3x_4)(x_1x_3)(x_5x_6) = x_1^2x_2x_3^2x_4x_5x_6$, which is not squarefree, hence $I^{[4]} = (0)$.

So the largest integer $k$ for which $I^{[k]} \neq 0$ is 3, and $I^{[k]} = 0$ for all $k \geq 4$. This corresponds to the fact that the maximum number of disjoint facets (i.e., the matching number of $\Delta$) is 3.

The study of squarefree powers is motivated by their deep connection to the ordinary powers of $I$, particularly regarding their homological properties. It is known from (Herzog et al., 2004, Lemma 4.4) that the multigraded minimal free resolution of

$I^{[k]}$ is a subcomplex of the multigraded resolution of $I^k$. This yields the inequality:

$$\operatorname{reg}(I^{[k]}) \leq \operatorname{reg}(I^k).$$

This inclusion implies that if $I^{[k]}$ fails to have a linear resolution, then $I^k$ also does not have one.

When $I(\Delta)$ is regarded as the facet ideal of a simplicial complex $\Delta$, the monomial generators of $I(\Delta)^{[k]}$ correspond to the $k$-matching of $\Delta$, that is,

$$I(\Delta)^{[k]} = (x_{i_1} \cdots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is a } k\text{-matching of } \Delta).$$

It is straightforward to check that $I^{[k]} \neq 0$ if and only if $1 \leq k \leq \nu(\Delta)$, where $\nu(\Delta)$ is the matching number of $\Delta$. The concept of $k$-matching for a simplicial complex will be introduced in the next section.

# 3. MATCHING IN SIMPLICIAL FORESTS

In this section we give some preliminary results related to the matching of simplicial forests that will be used in the following sections. But, firstly we highlight the following definitions and notation describing different matching in $\Delta$.

**Definition 3.0.1.** *A* matching *of $\Delta$ is a set of pairwise disjoint facets of $\Delta$. A matching consisting of $k$ facets is referred to as a $k$-matching. A $k$-matching is called* maximal*, if $\Delta$ does not admit any $(k+1)$-matching. The* matching number *of $\Delta$ is the size of a maximal matching of $\Delta$ and is denoted by $\nu(\Delta)$.*

**Definition 3.0.2.** *A matching $M$ of $\Delta$ is called* induced matching *if the set of facets of the induced subcomplex on $\cup_{E\in M} E$ is $\langle M\rangle$. The* induced matching number *of $\Delta$ is the maximum size of an induced matching of $\Delta$ and denoted by $\nu_1(\Delta)$.*

In (Bigdeli et al., 2017), authors introduced the definition of a restricted matching for 1-dimensional simplicial complexes (or simply a graph). We extend this definition to simplicial complexes of any given dimension in the following way.

**Definition 3.0.3.** *Let $F,G \in \mathcal{F}(\Delta)$. Then $F$ and $G$ form a* gap *in $\Delta$ if $F\cap G=\emptyset$ and the induced subcomplex on vertex set $F\cup G$ is $\langle F,G\rangle$. A matching $M$ of $\Delta$ is called a* restricted matching *if there exists a facet in $M$ forming a gap with every other facet in $M$. The maximal size of a restricted matching of $\Delta$ is denoted by $\nu_0(\Delta)$.*

We illustrate above definitions with following example.

**Example 3.0.4.** (a) Consider the simplicial complex $\Delta$, whose $\mathcal{F}(\Delta)$ consists of facets $F_i=\{i,i+1,i+2\}$ for all $1\le i\le 10$. In Figure 3.1, we illustrate the geometric realization of $\Delta$. Then, $M=\{F_1,F_4,F_7,F_{10}\}$ is the maximal matching of $\Delta$, and hence $\nu(\Delta)=4$. However, $M$ is not a restricted matching of $\Delta$. To see this, note that $F_1$ and $F_4$ do not form a gap because $F_2\in\langle F_1,F_4\rangle$. Similarly, $F_7$ and $F_{10}$ do not form a gap because $F_8\in\langle F_7,F_{10}\rangle$. Therefore, there does not exist a facet in $M$ that forms a gap with the other facets of $M$.

However, $M'=\{F_1,F_5,F_8\}$ is a restricted matching of $\Delta$ because $F_1$ forms a gap with $F_5$ and $F_8$. This gives $\nu_0(\Delta)=3$. Furthermore, we observe that $M'$

is not an induced matching of $\Delta$ because $F_5$ and $F_8$ do not form a gap. Let $M'' = \{F_1, F_5, F_9\}$; this is an induced matching of $\Delta$, and therefore $\nu_1(\Delta) = 3$.

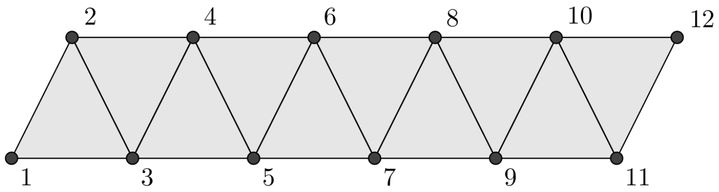



Figure 3.1 Geometric realization of $\Delta$.

(b) Consider the simplicial complex $\Delta$ whose facets are all subsets of $[n]$ of size $k$ for some $1 \leq k \leq n$, that is, $\mathcal{F}(\Delta) = \{\{i_1, i_2, \ldots, i_k\} : 1 \leq i_1 < \ldots < i_k \leq n\}$. Then, $\nu_0(\Delta) = \nu_1(\Delta) = 1$ and $\nu(\Delta) = \lfloor \frac{n}{k} \rfloor$. This example shows that, given a simplicial complex, the difference between its matching number and restricted matching number can be arbitrarily large.

Now, we recall briefly some definitions in graph theory. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Two vertices $u, v \in V(G)$ are *adjacent* if $\{u, v\} \in E(G)$. An isolated vertex in a graph is a vertex that is not adjacent to any other vertex in the graph. The *degree* of a vertex $v$ of $G$, denoted by $\deg(v)$, is the number of the vertices adjacent to $v$ in $G$. A subgraph $H \subset G$ is a graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Moreover, $H$ is said to be an *induced subgraph* on $V(H)$ if for any $u, v \in V(H)$, we have $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(G)$.

A *path* of length $r-1$ in $G$ is a sequence of $r$ distinct vertices $v_1, \ldots, v_r$ such that $\{v_i, v_{i+1}\}$ is an edge of $G$, for all $i = 1, \ldots, r-1$. A *cycle* of length $r$ is a sequence $v_1, \ldots, v_r, v_1$ such that $v_1, \ldots, v_r$ are distinct vertices of $G$, and the only edges of $G$ are $\{v_1, v_r\}$ and $\{v_i, v_{i+1}\}$ , for all $i = 1, \ldots, r-1$. A graph on $n$ vertices, forming a path of length $n-1$ is called a *path graph*, and is denoted by $P_n$. A graph $G$ is connected if for every pair of vertices $u, v \in V(G)$, there exists a path in $G$ that starts at $u$ and ends at $v$. A graph $G$ is called a tree if it is connected and does not have a cycle. A forest is a disjoint union of trees.

After providing the necessary definitions, we now continue with the first result.

**Proposition 3.0.5.** *Let $\Delta$ be a simplicial forest and $M = \{F_1, \ldots, F_r\}$ and $N = \{G_1, \ldots, G_r\}$ be two $r$-matching of $\Delta$ with $\cup_{i=1}^r F_i = \cup_{i=1}^r G_i$. Then $M = N$.*

*Proof.* Assume that $M \neq N$. Without loss of generality, we may assume that $M \cap N = \emptyset$. Consider the bipartite graph $G$ on the vertex set $\{F_1, \ldots, F_r\} \cup \{G_1, \ldots, G_r\}$ such that $\{F_i, G_j\} \in E(G)$ if and only if $F_i \cap G_j \neq \emptyset$. The condition $\cup_{i=1}^r F_i =$

$\cup_{i=1}^{r} G_i$ enforces that the degree of each vertex in $G$ is at least two. For instance, let $\{F_i, G_j\} \in E(G)$. Since $F_i \neq G_j$ and $\cup_{i=1}^{t} F_i = \cup_{i=1}^{t} G_i$, there exists some $F_k \in M$ with $i \neq k$ such that $(G_j \setminus F_i) \cap F_k \neq \emptyset$. Hence $\deg(G_j) \geq 2$.

The fact that every vertex of $G$ has degree at least two shows that $G$ is not a tree and therefore contains a cycle $C$ of length $t \geq 4$. After rearranging the indices, let $C$ be given by: $F_1, G_1, F_2, \ldots, F_t, G_t, F_1$. Consider the subsubcomplex $\Delta' = \langle F_1, \ldots, F_t, G_1, \ldots, G_t \rangle$ of $\Delta$. Then $\Delta'$ is also a simplicial forest with a leaf, say $G_1$. Since $M$ is a matching, we observe that $G_1 \cap F_i$ is disjoint with $G_1 \cap F_j$ for all $i \neq j$. Moreover, the sets $G_1 \cap F_1$ and $G_1 \cap F_2$ are non-empty by the definition of $G$. This shows that $G_1$ is not a leaf of $\Delta'$, and $\Delta'$ is a not a simplicial tree, which is a contradiction to $\Delta$ being a simplicial tree. □

We obtain following description of elements of $G(I(\Delta)^{[k]})$ as an immediate corollary of above proposition.

**Corollary 3.0.6.** *Let $\Delta$ be a simplicial forest with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$ and for all $i = 1, \ldots, r$, set $f_i = \prod_{j \in F_i} x_j$. For any $1 \leq k \leq \nu(\Delta)$, each $u \in G(I(\Delta)^{[k]})$ can be uniquely expressed as $u = f_{i_1} \cdots f_{i_k}$ where $M = \{F_{i_1}, \ldots, F_{i_k}\}$ is a $k$-matching of $\Delta$.*

It is argued in (Bigdeli et al., 2017, page 24) that if a simple graph $G$ is a tree then $\nu(G) - \nu_0(G) \leq 1$. Below, we extend this result to any simplicial forest.

**Proposition 3.0.7.** *Let $\Delta$ be a simplicial forest. Then $\nu(\Delta) - \nu_0(\Delta) \leq 1$.*

*Proof.* Let $s = \nu(\Delta)$ and $M = \{E_1, \ldots, E_s\}$ be a $s$-matching of $\Delta$. Consider the graph $G_M$ associated with $M$ such that $V(G_M) = M$ and

$$E(G_M) = \{\{E_i, E_j\} : \, E_i \text{ and } E_j \text{ do not form a gap in } \Delta\}.$$

We first prove that $G_M$ is a forest. To do this, suppose that $G_M$ is not a forest; that is, $G_M$ contains a cycle of length at least three. We may assume that the sequence of vertices $E_1, \ldots, E_r$ gives a minimal cycles in $G_M$ for some $r \leq s$. By the definition of $G_M$, for each $i = 1, \ldots, r$, there exists $F_i \in \mathcal{F}(\Delta)$ such that $F_i \subset E_i \cup E_{i+1}$ and $F_i \not\subset \langle E_i, E_{i+1} \rangle$ .

Let $\Delta'$ be the subcomplex of $\Delta$ such that $F_1, \ldots, F_r \in \Delta'$ and $E_i \in \Delta'$ if and only if $F_{i-1} \cap F_i = \emptyset$. Since $\Delta$ is a simplicial tree, the subcomplex $\Delta'$ is also a simplicial tree and contains a leaf. For all $i = 1, \ldots, r$, using the fact that $M$ is a matching of $\Delta$ provides that if $F_i \cap F_j \neq \emptyset$, then $j \in \{i-1, i+1\}$, and if $F_i \cap E_j \neq \emptyset$, then $j \in \{i, i+1\}$. Therefore, if $F_i$ is a leaf of $\Delta'$, then the only possible candidates to be a branch of $F_i$ in $\Delta'$ are $F_{i-1}, F_{i+1}$ and $E_i, E_{i+1}$ provided that $E_i, E_{i+1} \in \Delta'$. We have the following possible cases:

(1) Let $E_i, E_{i+1} \in \Delta'$. Then $F_i \cap F_{i+1} = \emptyset$ and $F_{i-1} \cap F_i = \emptyset$. Moreover, $F_i \cap E_i$ and $F_i \cap E_{i+1}$ are nonempty and disjoint. This shows that $F_i$ is not a leaf of $\Delta'$.

(2) Let $E_i \in \Delta'$ and $E_{i+1} \notin \Delta'$. Then $F_i \cap F_{i+1} \neq \emptyset$ and $F_{i-1} \cap F_i = \emptyset$ . Since $F_i \cap F_{i+1} \subset E_{i+1}$, we see that $F_i \cap F_{i+1}$ and $E_i \cap F_i$ are disjoint. Hence $F_i$ is not a leaf of $\Delta'$. The case when $E_i \notin \Delta'$ and $E_{i+1} \in \Delta'$ can be argued in a similar way to conclude that $F_i$ is not a leaf of $\Delta'$.

(3) Let $E_i, E_{i+1} \notin \Delta'$. Then $F_i \cap F_{i-1} \neq \emptyset$ and $F_i \cap F_{i-1} \neq \emptyset$. It follows from the fact that $E_i \cap E_{i+1} = \emptyset$ and $F_i \cap F_{i-1} \subset E_i$ and $F_i \cap F_{i+1} \subset E_{i+1}$ that $F_i$ is not a leaf of $\Delta'$.

The above discussion shows that $F_i$ is not a leaf of $\Delta'$ for any $i = 1, \ldots, r$. If $E_i \notin \Delta'$ for all $i = 1, \ldots, r$, then $\Delta'$ does not contain a leaf, which is a contradiction to $\Delta$ being a simplicial tree. Let $E_i \in \Delta'$ for some $i = 1, \ldots, r$ such that $E_i$ is a leaf of $\Delta'$. Then $F_{i-1} \cap F_i = \emptyset$. The only possible branches of $E_i$ in $\Delta'$ are $F_{i-1}$ and $F_i$ because $E_i$ does not intersect any other facet non-trivially. On the other hand,we observe that $E_i \cap F_{i-1}$ and $E_i \cap F_i$ do not contain each other because $F_{i-1} \cap F_i = \emptyset$. From this we conclude that $E_i$ is not a leaf of $\Delta'$, a contradiction to the assumption that $\Delta$ is a simplicial tree. Therefore, $G_M$ does not contain any cycle and the claim holds.

If $G_M$ contains an isolated vertex, say $E_i$, then $E_i$ forms a gap with all other elements of $M$. This gives $\nu_0(\Delta) = \nu(\Delta)$. If $G_M$ does not contain any isolated vertex, then pick $E_i$ such that $E_i$ is a leaf of $G_M$ and let $E_j$ be the unique neighbor of $E_i$ in $G_M$. Then $M \setminus \{E_j\}$ forms a restricted matching of $\Delta$ and $\nu_0(\Delta) = \nu(\Delta) - 1$. $\square$

# 4. SIMPLICIAL FOREST WITH LINEAR SQUAREFREE POWERS

In the first part of this section, our aim is to show that if facet ideal of a simplicial forest has linear resolution then its non-trivial squarefree powers also have linear resolution. Zheng in (Zheng, 2004b) characterized all simplicial forests whose facets ideals have linear resolution. Indeed, it is shown in (Zheng, 2004b, Theorem 3.17) that the facet ideal of a simplicial forest $\Delta$ has linear resolution if and only if $\Delta$ satisfies intersection property.

We first observe in the following proposition that a simplicial forest with intersection property has matching number at most two.

**Proposition 4.0.1.** *Let $\Delta$ be a simplicial forest with the intersection property and $\dim(\Delta) \geq 1$. Then $\nu(\Delta) \leq 2$.*

*Proof.* From the definition of the intersection property, we know that $\Delta$ is pure. Let $\dim \Delta = n-1$.

On contrary, assume that $\nu(\Delta) \geq 3$, and let $E_1, E_2$ and $E_3$ be three pairwise disjoint facets of $\Delta$. Let $E_1 = \{a_1, \dots, a_n\}$, $E_2 = \{b_1, \dots, b_n\}$ and $E_3 = \{c_1, \dots, c_n\}$. Since $\Delta$ has intersection property, using $\dim(E_i \cap E_j) = -1$, we obtain $\operatorname{dist}(E_i, E_j) = n$, for all $i \neq j$. Let $\mathcal{C}_1 : \quad E_1 = F_0, F_1, \dots, F_n = E_2$ be the unique irredundant chain between $E_1$ and $E_2$. Since $\dim(F_i \cap F_{i+1}) = \dim(F_{i+1}) - 1$, after rearranging indices, we set $F_i = \{b_1, \dots, b_i, a_{i+1}, \dots, a_n\}$, for all $i = 1, \dots, n-1$. Similarly, let $\mathcal{C}_2 : \quad E_2 = F'_0, F'_1, \dots, F'_n = E_3$ be the unique irredundant chain between $E_2$ and $E_3$ with $F'_i = \{c_1, \dots, c_i, b_{i+1}, \dots, b_n\}$ for all $i = 1, \dots, n-1$. Lastly, let $\mathcal{C}_3 : \quad E_3 = F''_0, F''_1, \dots, F''_n = E_1$ be the unique irredundant chain between $E_1$ and $E_3$ with $F''_i = \{a_1, \dots, a_i, c_{i+1}, \dots, c_n\}$ for all $i = 1, \dots, n-1$.

Consider the subcomplex $\Delta' = \langle E_1, E_2, E_3, F_{n-1}, F'_{n-1}, F''_{n-1} \rangle \subseteq \Delta$. Observe that $E_1 \cap F_{n-1} = \{a_n\}$, $E_1 \cap F''_{n-1} = \{a_1, \dots, a_{n-1}\}$ and the intersection of $E_1$ with $E_2, E_3$ and $F'_{n-1}$ is trivial. This shows that $E_1$ is not a leaf of $\Delta'$. Similarly, one can see that none of the facet of $\Delta'$ is a leaf of $\Delta'$, and hence $\Delta'$ is not a simplicial tree, which is a contradiction to $\Delta$ being a simplicial tree. Therefore, we conclude $\nu(\Delta) \leq 2$. $\square$

Below, we recall a nice result proved in (Kumar & Kumar, 2023), which is crucial in the proof of Theorem 4.0.4.

**Lemma 4.0.2.** *(Kumar & Kumar, 2023, Lemmas 2.1 and 2.2) Let $\Delta$ be a simplicial tree with the intersection property. Then, there exists an ordering on the facets of $\Delta$, say $F_1,\dots,F_r$, such that the following conditions hold:*

*(i)* *$F_1,\dots,F_r$ is a good leaf ordering on the facets of $\Delta$, and $\operatorname{dist}(F_i,F_{i+1}) = 1$ for all $i \in \{1,\dots,r-1\}$.*

*(ii)* *If there exists $x \in F_j \setminus F_i$ for some $j < i$, then $x \notin F_k$ for all $k \geq i$.*

*(iii)* *For any $j < i$, there exists $k \in \{j,\dots,i-1\}$ such that $|F_k \cap F_i| = |F_i| - 1$ and $F_j \cap F_k \not\subseteq F_i$.*

The next result can be seen as a consequence of (Kumar & Kumar, 2023, Lemma 2.3) and (Soleyman Jahan & Zheng, 2010, Proposition 2.10).

**Theorem 4.0.3.** *Let $\Delta$ be a simplicial tree with intersection property. Then, $I(\Delta)^{[2]}$ has linear quotients.*

*Proof.* Let $\Delta$ be a simplicial tree with the intersection property. Zheng's result (Zheng, 2004b, Proposition 3.9 and Theorem 3.17) shows that $I(\Delta)$ has linear quotients, and this result is generalized in (Kumar & Kumar, 2023, Lemma 2.3) which proves that $I(\Delta)^k$ has linear quotients for all $k$. Moreover, (Soleyman Jahan & Zheng, 2010, Proposition 2.10) asserts that if a monomial ideal $J$ has linear quotients, then the ideal generated by the squarefree monomials in $J$ also has linear quotients. Therefore, the desired conclusion follows from (Soleyman Jahan & Zheng, 2010, Proposition 2.10) together with (Kumar & Kumar, 2023, Lemma 2.3). $\square$

We know that the second squarefree power of these ideals has linear quotients, but in the following theorem, we give a precise ordering of the generators of the second squarefree power, which gives us linear quotients. Note also that this result recovers the result of Erey and Hibi (Erey & Hibi, 2021, Theorem 41).

**Theorem 4.0.4.** *Let $\Delta$ be a simplicial tree with intersection property. Then, $I(\Delta)^{[k]}$ has linear quotients, for all $1 \leq k \leq \nu(\Delta)$.*

*Proof.* The assertion hold when $k = 1$ due to (Zheng, 2004b, Theorem 3.17). By virtue of Lemma 4.0.1, it is enough to consider the case when $k = \nu(\Delta) = 2$. Set $I = I(\Delta)$. Let $F_1,\dots,F_r$ be a good leaf order of $\Delta$ as in Lemma 4.0.2 and $f_i = \prod_{j\in F_i} x_j$ for each $i = 1,\dots,r$. It follows from Corollary 3.0.6 that any monomial generator of $I^{[2]}$ can be uniquely written as $f_i f_j$ where $\{F_i,F_j\}$ is a 2-matching.

Set $m_{i,j} = f_i f_j$ for all $f_i f_j \in G(I^{[2]})$ with $i < j$. For any $m_{i,j}, m_{k,\ell} \in G(I^{[2]})$, we set $m_{k,\ell} < m_{i,j}$ if and only if $i < k$ or $i = k$ and $j < \ell$. We order the minimal generators of $I^{[2]}$ in a descending chain, $m_{i_1,j_1} > \cdots > m_{i_k,j_k}$ and we claim that $I^{[2]}$ has linear quotient with respect to this order. Using (Herzog & Hibi, 2011, Lemma 8.2.3), it is enough to show that for any $m_{k,\ell}, m_{i,j} \in I^{[2]}$ with $m_{k,\ell} < m_{i,j}$, there exists $m_{p,q} \in I^{[2]}$ with $m_{k,\ell} < m_{p,q}$ such that $(m_{p,q})\colon (m_{k,\ell})$ is generated by a variable and $(m_{i,j})\colon (m_{k,\ell}) \subseteq (m_{p,q})\colon (m_{k,\ell})$.

First we consider the case $i = k$ and write $m_{k,\ell} = m_{i,\ell}$. Since $m_{i,\ell} < m_{i,j}$, we obtain $j < l$. Due to Lemma 4.0.2(iii) there exists some $q \in \{j, \ldots, \ell - 1\}$ such that $|F_q \cap F_\ell| = |F_\ell| - 1$ and $F_j \cap F_q \not\subseteq F_\ell$. More precisely, there exists $a \in F_j \setminus F_\ell$ and $b \in F_\ell$ such that $F_q = (F_\ell \setminus \{b\}) \cup \{a\}$. Then it follows immediately from $F_i \cap F_j = F_i \cap F_\ell = \emptyset$ that $F_i \cap F_q = \emptyset$. This shows $m_{i,q} = f_i f_q \in G(I^{[2]})$. Since $i < j \leq q < \ell$, we have either $f_i f_q = f_i f_j$ or $f_i f_q > f_i f_\ell$. The construction of $F_q$ yields

$$(f_i f_j)\colon f_i f_\ell \subseteq (f_i f_q)\colon f_i f_\ell = (x_a),$$

as required.

The case when $j = \ell$ can be argued in a similar way.

Now we consider the case when $i \neq k$ and $j \neq \ell$. Since $m_{i,j} > m_{k,\ell}$, we obtain $i < k$. Again by Lemma 4.0.2(iii) there exists some $p \in \{i, \ldots, k-1\}$ such that $|F_p \cap F_k| = |F_k| - 1$ and $F_i \cap F_p \not\subseteq F_k$. Then, for some $a \in F_i \setminus F_k$ and $b \in F_k$, we have $F_p = (F_k \setminus \{b\}) \cup \{a\}$. It follows from Lemma 4.0.2(ii) that $a \notin F_\ell$ because $i < k < \ell$ and $a \in F_i \setminus F_k$. Combining $a \notin F_\ell$ with $F_k \cap F_\ell = \emptyset$ gives $F_p \cap F_\ell = \emptyset$. Therefore, $m_{p,\ell} = f_p f_\ell \in G(I^{[2]})$ and $f_p f_\ell > f_k f_\ell$ because $p < k$. From the construction of $F_p$ we obtain $(f_i f_j)\colon f_k f_\ell \subseteq (f_p f_\ell)\colon f_k f_\ell = (x_a)$. This completes the proof. $\square$

### 4.1 The $\nu$-th Squarefree Power of Simplicial Forests

We begin by introducing some definitions that will be used throughout this section and the upcoming ones.

A *rooted tree* $\Gamma$ is a directed tree with a fixed vertex, called the *root*, which is directed implicitly away from the root. A directed path of length $t-1$ is a sequence of $t$ distinct vertices $i_1, \ldots, i_t$, such that each pair $(i_j, i_{j+1})$ forms a directed edge from $i_j$ to $i_{j+1}$ for all $j = 1,\ldots,t-1$. The graph in Example 4.1.1 is a rooted tree with directed edges.

A well known example of a simplicial forest is due to (He & Van Tuyl, 2010, Corollary 2.9), which we describe below.

Let $\Gamma$ be a rooted tree. We denote by $\Gamma_t$ the simplicial complex whose facets correspond to the directed $t$-paths of $\Gamma$. In (He & Van Tuyl, 2010, Theorem 2.7) it is shown that $\Gamma_t$ is a simplicial tree. Due to this, we refer to $\Gamma_t$ as *$t$-path simplicial tree of* $\Gamma$. The facet ideal of $\Gamma_t$ is called the *$t$-path ideal* of $\Gamma$, and is given by

$$I(\Gamma_t) = (x_{i_1}\cdots x_{i_t} : i_1,\ldots,i_t \text{ is a directed path on } t \text{ vertices in } \Gamma).$$

It is known that the $\nu$-th squarefree power of the edge ideal of a simple graph has linear quotients, as shown in (Bigdeli et al., 2017, Theorem 4.1). Such a statement does not hold for the facet ideals of simplicial complexes (equivalently edge ideals of simple hypergraphs), or even for facet ideals of simplicial forests, as shown in the following example.

**Example 4.1.1.** Let $\Gamma$ be the rooted tree in Figure 4.1 and $I(\Gamma_3)$ be the 3-path ideal of $\Gamma$. Then

$$I(\Gamma_3) = (x_1x_2x_4, x_1x_2x_5, x_1x_3x_6, x_1x_3x_7, x_2x_4x_8, x_2x_5x_9, x_3x_6x_{10}, x_3x_7x_{11}).$$

Observe that $\nu(\Gamma_3) = 2$, and $I_3(\Gamma)^{[k]} = 0$ for all $k > 3$. Easy computations with

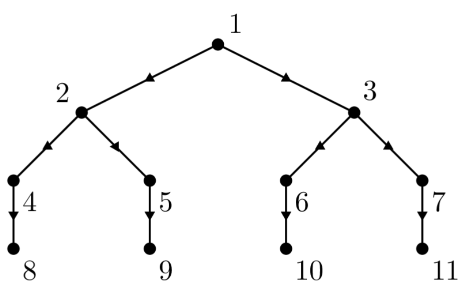



Figure 4.1 A rooted tree.

`Macaulay2` (Grayson & Stillman, Grayson & Stillman) show that $I_3(\Gamma)^{[2]}$ does not have a linear resolution. This shows that (Bigdeli et al., 2017, Theorem 4.1) can not be extended even to the case of $t$-path ideals of rooted trees.

Our next main result is motivated by the question, for which classes of simplicial trees we can generalize (Bigdeli et al., 2017, Theorem 4.1). More precisely, we want to answer the following:

**Question 4.1.2.** *Can we find some special classes of simplicial trees whose facet ideal has the property that their $\nu$-th squarefree power has linear resolution.*

In an effort to answer above question, we prove that an analogue of (Bigdeli et al., 2017, Theorem 4.1) holds true for $I(\Gamma_t)$, where $\Gamma_t$ is a simplicial tree whose facets are the directed $t$-paths of a broom graph, see (Bouchat et al., 2011).

A *broom graph* $\Gamma$ of height $h$ is a rooted tree with root $x$, consisting of a *handle* which forms a directed path of length $h$ rooted at $x$, and all the other vertices that are not on the handle are leaves of $\Gamma$. See Figure 4.2 for an example of a broom graph. A graph consisting of only a directed path is a broom graph consisting of only handle.

Before stating our next theorem related to broom graphs, we setup the following notation.

**Notation 4.1.3.** Let $\Gamma$ be a broom graph rooted at the vertex $x_{0,0}$ with $\mathrm{ht}(\Gamma) = h$ and let $x_{0,0}, x_{1,0} \dots, x_{h,0}$ be the vertices of the handle of $\Gamma$. Furthermore, for each $1 \leq i \leq h$, let $l_i$ be the number of vertices of $\Gamma$ which do not lie on the handle and their unique neighbor on the handle is $x_{i-1,0}$. We set the following notation for the facets of $\Gamma_t$ where $t \geq 2$. For $0 \leq i \leq h-t+1$ and $0 \leq j \leq \ell_{i+t-1}$, set

$$F_{i,j} = \{x_{i,0}, x_{i+1,0}, \dots, x_{i+t-2,0}, x_{i+t-1,j}\}. \tag{4.1}$$

We define a total order on $\mathcal{F}(\Gamma_t)$ as follows: for all $F_{i,j}, F_{k,m} \in \mathcal{F}(\Gamma_t)$ with $i,k \in \{0,\dots,h-t+1\}$, $0 \leq j \leq \ell_{i+t-1}$ and $0 \leq m \leq \ell_{k+t-1}$, we set $F_{i,j} < F_{k,m}$ if either $i < k$ or $i = k$ and $m < j$.

We identify the vertices of $\Gamma$ as variables and set $S = K[x_{i,j} : x_{i,j} \in V(\Gamma)]$. For each $F_{i,j} \in \mathcal{F}(\Gamma_t)$, let $m_{i,j}$ be the monomial corresponding to $F_{i,j}$, that is $m_{i,j} = x_{i,0}x_{i+1,0} \cdots x_{i+t-2,0}x_{i+t-1,j}$. Let $1 \leq k \leq \nu(\Gamma_t)$. Due to Corollary 3.0.6, any monomial generator of $I^{[k]}$ can be uniquely expressed as $m_{i_1,j_1} \cdots m_{i_k,j_k}$ such that $M = \{F_{i_1,j_1}, \dots, F_{i_k,j_k}\}$ is a $k$-matching. Let $u_a, u_b \in G(I^{[k]})$ with $u_a = m_{i_1,j_1} \cdots m_{i_k,j_k}$ and $u_b = m_{i'_1,j'_1} \cdots m_{i'_k,j'_k}$, such that $F_{i_1,j_1} < \ldots < F_{i_k,j_k}$ and $F_{i'_1,j'_1} < \ldots < F_{i'_k,j'_k}$ and $s = \max\{\ell : F_{i_\ell,j_\ell} \neq F_{i'_\ell,j'_\ell}\}$. We set $u_b < u_a$ if $F_{i'_s,j'_s} < F_{i_s,j_s}$.

**Example 4.1.4.** Let $\Gamma$ be the broom graph as in Figure 4.2 and $\Gamma_3$ be its 3-path simplicial tree. Then $\mathrm{ht}(\Gamma) = 5$ and $l_1 = l_3 = l_4 = 0$, $l_2 = 1$ and $l_5 = 2$. Following Notation 4.1.3, we label the facets of $\Gamma_3$ as $F_{0,0} = \{x_{0,0}, x_{1,0}, x_{2,0}\}$, $F_{0,1} = \{x_{0,0}, x_{1,0}, x_{2,1}\}$,$F_{1,0} = \{x_{1,0}, x_{2,0}, x_{3,0}\}$, $F_{2,0} = \{x_{2,0}, x_{3,0}, x_{4,0}\}$, $F_{3,0} = \{x_{3,0}, x_{4,0}, x_{5,0}\}$, $F_{3,1} = \{x_{3,0}, x_{4,0}, x_{5,1}\}$, and $F_{3,2} = \{x_{3,0}, x_{4,0}, x_{5,2}\}$. With the total order defined on the facets of $\Gamma_t$ in Notation 4.1.3 we obtain $F_{3,0} > F_{3,1} > F_{3,2} >$

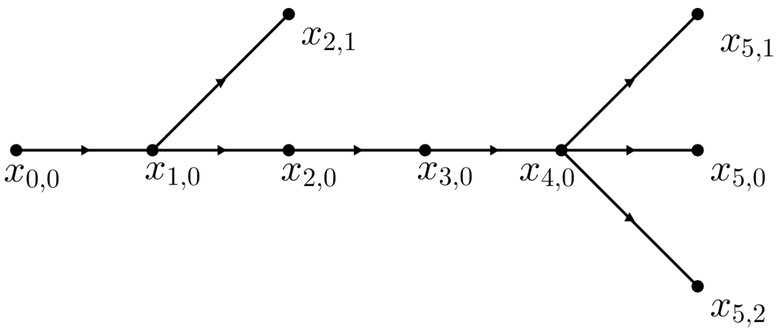



Figure 4.2 A broom graph.

$F_{2,0} > F_{1,0} > F_{0,0} > F_{0,1}$. Moreover, the elements of $G(I^{[2]})$ are ordered as

$$m_{0,0}m_{3,0} > m_{0,1}m_{3,0} > m_{0,0}m_{3,1} > m_{0,1}m_{3,1} > m_{0,0}m_{3,2} > m_{0,1}m_{3,2} > m_{0,1}m_{2,0}.$$

Now, we give the main result of this section.

**Theorem 4.1.5.** *Let $\Gamma$ be a broom graph and $\Gamma_t$ be its $t$-path simplicial tree.*

*(i) $I(\Gamma_t)^{[\nu(\Gamma_t)]}$ has linear quotient.*

*(ii) In particular, if $\Gamma$ is a path graph, then $I(\Gamma_t)^{[\nu_0(\Gamma_t)]}$ has linear quotient.*

*Proof.* Set $I = I(\Gamma_t)$. We use Notation 4.1.3 to order the elements of $G(I^{[q]})$ as $u_1 > \ldots > u_p$, and use this ordering of $G(I^{[q]})$ to prove $(i)$ and $(ii)$. Let $u_a, u_b \in G(I^{[q]})$ with $a < b$ and $u_a = m_{i_1,j_1} \cdots m_{i_q,j_q}$, $u_b = m_{i'_1,j'_1} \cdots m_{i'_q,j'_q}$ such that $F_{i_1,j_1} < \ldots < F_{i_q,j_q}$ and $F_{i'_1,j'_1} < \ldots < F_{i'_q,j'_q}$. Moreover, let $s = \max\{\ell : F_{i_\ell,j_\ell} \neq F_{i'_\ell,j'_\ell}\}$. Since, $u_b < u_a$, we obtain $F_{i'_s,j'_s} < F_{i_s,j_s}$. Let $M = \{F_{i_1,j_1}, \ldots, F_{i_q,j_q}\}$ and $N = \{F_{i'_1,j'_1}, \ldots, F_{i'_q,j'_q}\}$. To prove $(i)$ and $(ii)$, we invoke (Herzog & Hibi, 2011, Lemma 8.2.3) and construct a monomial $u_c \in G(I^{[q]})$ with $u_b < u_c$ such that $(u_c) \colon (u_b)$ is generated by a variable and $(u_a) \colon (u_b) \subseteq (u_c) \colon (u_b)$.

$(i)$ Let $q = \nu(\Gamma)$. Since $F_{i'_s,j'_s} < F_{i_s,j_s}$, it follows that either $i'_s < i_s$ or $i_s = i'_s$ and $j_s < j'_s$. We distinguish three cases: (1) $i_s = i'_s$; (2) $i'_s < i_s$ and $0 < j'_s$; (3) $i'_s < i_s$ and $j'_s = 0$.

Case (1): Let $i_s = i'_s$. We know that $F_{i'_s,j_s}$ does not intersect any facet in $\{F_{i'_{s+1},j'_{s+1}}, \ldots, F_{i'_q,j'_q}\} = \{F_{i_{s+1},j_{s+1}}, \ldots, F_{i_q,j_q}\} \subset M$ because $M$ is a matching of $\Gamma_t$. The construction of facets of $\Gamma_t$ in (4.1) together with the fact that $F_{i'_s,j'_s}$ does not intersect any facet in $\{F_{i'_1,j'_1}, \ldots, F_{i'_{s-1},j'_{s-1}}\}$ shows that $F_{i'_s,j_s}$ also does not intersect any facet in $\{F_{i'_1,j'_1}, \ldots, F_{i'_{s-1},j'_{s-1}}\}$. From this we conclude that $F_{i'_s,j_s}$ does not intersect any facet in $N \setminus \{F_{i'_s,j'_s}\}$. Hence $u_c = (u_b/m_{i'_s,j'_s})m_{i'_s,j_s} = (u_b/x_{i'_s+t-1,j'_s})x_{i'_s+t-1,j_s} \in$

$G(I^{[q]})$, and $(u_c)\colon(u_b)=(x_{i'_s+t-1,j_s})$. Since $F_{i'_s,j'_s}<F_{i'_s,j_s}$, it follows that $u_b<u_c$. Since $x_{i'_s+t-1,j_s}\in F_{i'_s,j_s}\subseteq \operatorname{supp}(u_a)$, we have $(u_a)\colon(u_b)\subseteq(x_{i'_s+t-1,j_s})$, as required.

Before proceeding further to Case(2) and Case(3), we need to acknowledge that $F_{i_s,j_s}\cap F_{i'_s,j'_s}\neq\emptyset$. Indeed, since $F_{i'_s,j'_s}$ is disjoint with all the facets in $\{F_{i'_1,j'_1},\cdots,F_{i'_{s-1},j'_{s-1}}\}$, and $F_{i'_1,j'_1}<\ldots<F_{i'_s,j'_s}<F_{i_s,j_s}$, it follows immediately that $F_{i_s,j_s}$ is also disjoint with each of the facet in $\{F_{i'_1,j'_1},\cdots,F_{i'_{s-1},j'_{s-1}}\}$. Moreover, due to $\{F_{i'_{s+1},j'_{s+1}},\ldots,F_{i'_q,j'_q}\}=\{F_{i_{s+1},j_{s+1}},\ldots,F_{i_q,j_q}\}\subset M$, the facet $F_{i_s,j_s}$ is disjoint with each of the facet in $\{F_{i'_{s+1},j'_{s+1}},\ldots,F_{i'_q,j'_q}\}$. Therefore, if $F_{i_s,j_s}\cap F_{i'_s,j'_s}=\emptyset$, then $F_{i_s,j_s}\cup N$ forms a $(q+1)$-matching, a contradiction to the assumption $q=\nu(\Gamma_t)$. Therefore, $F_{i_s,j_s}\cap F_{i'_s,j'_s}\neq\emptyset$. To discuss Case(2) and Case(3) we make use of $F_{i_s,j_s}\cap F_{i'_s,j'_s}\neq\emptyset$. Note that the vertices in $F_{i_s,j_s}\cap F_{i'_s,j'_s}$ lie on the handle of $\Gamma$.

Case (2): Let $i'_s<i_s$ and $0<j'_s$. Then $F_{i_s,j_s}\cap F_{i'_s,j'_s}\neq\emptyset$ if and only if $i'_s+1\le i_s\le i'_s+t-2$. This shows that $x_{i'_s+t-1,0}\in F_{i_s,j_s}\subset\operatorname{supp}(u_a)$. Also $x_{i'_s+t-1,0}\notin F_{i'_s,j'_s}$ due to $j'_s>0$ and therefore $x_{i'_s+t-1,0}\notin\operatorname{supp}(u_b)$. This gives $(u_a)\colon(u_b)\subseteq(x_{i'_s+t-1,0})$. Let $u_c=(u_b/m_{i'_s,j'_s})m_{i'_s+1,0}$, Then $(u_c)\colon(u_b)=(x_{i'_s+t-1,0})$. Since $F_{i'_s,j'_s}<F_{i'_s+1,0}$, it yields $u_b<u_c$. Again from $F_{i'_s,j'_s}<F_{i'_s+1,0}$, it is follows immediately that $F_{i'_s+1,0}$ does not intersect any facet in $\{F_{i'_1,j'_1},\ldots,F_{i'_{s-1},j'_{s-1}}\}$. Because of $i'_s+1\le i_s\le i'_s+t-2$, we have either $F_{i'_s+1,0}=F_{i_s,j_s}$ or $F_{i'_s+1,0}<F_{i_s,j_s}$. In both case, it is easy to see that $F_{i'_s+1,0}$ does not intersect any element in $\{F_{i'_{s+1},j'_{s+1}},\ldots,F_{i'_q,j'_q}\}=\{F_{i_{s+1},j_{s+1}},\ldots,F_{i_q,j_q}\}\subset M$. Hence $u_c\in G(I^{[q]})$. Then $(u_a)\colon(u_b)\subseteq(x_{i'_s+t,0})=(u_c)\colon(u_b)$, as required.

Case (3): Let $i'_s<i_s$ and $j'_s=0$. Then $F_{i_s,j_s}\cap F_{i'_s,j'_s}\neq\emptyset$ if and only if $i'_s+1\le i_s\le i'_s+t-1$. If $i'_s+1=i_s$, then set $u_c=(u_b/m_{i'_s,j'_s})m_{i'_s+1,j_s}$, and if $i'_s+1<i_s\le i'_s+t-1$ then set $u_c=(u_b/m_{i'_s,j'_s})m_{i'_s+1,0}$. In both cases, $u_b<u_c$ and $(u_a):(u_b)\subset(u_c):(u_b)$, and by arguing similarly as in the cases above, it follows that $u_c\in G(I^{[q]})$. Also, if $i'_s+1=i_s$ then $(u_c)\colon(u_b)=(x_{i'_s+t,j_s})$, and if $i'_s+1<i_s\le i'_s+t-1$ then $(u_c)\colon(u_b)=(x_{i'_s+t,0})$. This completes the proof.

$(ii)$ Now, let $\Gamma$ be a path graph. It is known from Theorem 3.0.7 that $\nu(\Gamma_t)-\nu_0(\Gamma_t)\le 1$. If $\nu(\Gamma_t)-\nu_0(\Gamma_t)=0$, then the assertion follows by virtue of $(i)$. Therefore, it is enough to consider the case when $\nu_0(\Gamma_t)=\nu(\Gamma_t)-1$. Set $q=\nu_0(\Gamma_t)$.

Since $\Gamma$ is a path graph, we can view $\Gamma$ as a broom graph consisting of only the handle $x_{0,0},x_{1,0},\ldots,x_{h,0}$. In this case, $j=0$ in (4.1). To simplify the notation, we set $F_i:=F_{i,0}$ and $m_i:=m_{i,0}$ for each $0\le i\le h-t+1$.

To construct the monomial $u_c$, we proceed in the following way. Since $F_{i'_s}<F_{i_s}$, we obtain $i'_s<i_s$. If $F_{i_s}\cap F_{i'_s}\neq\emptyset$, then the desired conclusion follows from the same argument as in Case (3) above. Now suppose that $F_{i_s}\cap F_{i'_s}=\emptyset$, that is ,

$i'_s+t-1<i_s$. Then due to $F_{i'_1}<\ldots<F_{i'_s}<F_{i_s}$, we conclude that $F_{i_s}$ does not intersect any facet in $\{F_{i'_1},\ldots,F_{i'_s}\}$. Also, due to $\{F_{i'_{s+1}},\ldots,F_{i'_q}\}=\{F_{i_{s+1}},\ldots,F_{i_q}\}\subset M$, the facet $F_{i_s}$ is disjoint with each of the facet in $\{F_{i'_{s+1}},\ldots,F_{i'_q}\}$. Therefore, $A=\{F_{i'_1},\ldots,F_{i'_s},F_{i_s},F_{i'_{s+1}}\ldots,F_{i'_q}\}$ forms a maximal matching of $\Gamma_t$.

First we claim that $F_{i'_s}$ and $F_{i_s}$ do not form a gap in $\Gamma_t$. To prove the claim, assume that $F_{i'_s}$ and $F_{i_s}$ form a gap in $\Gamma_t$. Then $F_{i_s}$ forms a gap with all elements in $\{F_{i'_1},\ldots,F_{i'_s}\}$. It yields $s\neq q$, otherwise $A$ is a restricted matching of size $q+1$, a contradiction. Also $i'_s+t\neq i_s$, otherwise $F_{i'_s+t-1}$ belongs to the induced subcomplex generated by $F_{i'_s}$ and $F_{i_s}$, a contradiction to our assumption that $F_{i'_s}$ and $F_{i_s}$ form a gap. Since $i'_s+t-1<i_s$ and $i'_s+t\neq i_s$, we conclude $i'_s+t<i_s$. Note that $F_{i'_s}\cap F_{i_s-1}=\emptyset$. Consider the following set

$$B=\{F_{i'_1},\ldots,F_{i'_s},F_{i_s-1},F_{i'_{s+1}-1}\ldots,F_{i'_q-1},F_{i'_q}\}.$$

Since $F_{i'_q}\cap F_{i'_q}=\emptyset$, it follows immediately that $F_{i'_q-1}\cap F_{i'_q}=\emptyset$ and $F_{i'_q}$ forms a gap with the rest of the elements in $B$. This implies $\nu_0(\Gamma_t)=q+1$, a contradiction. Hence we conclude that $F_{i'_s}$ and $F_{i_s}$ do not form a gap in $\Gamma_t$, and $i'_s+t=i_s$. Let $u_c=(u_b/m_{i'_s})m_{i'_s+1}$. Then $u_b<u_c$ and $(u_c):(u_b)=(x_{i'_s+t,0})$. Since $x_{i'_s+t,0}\in F_{i_s}$, it follows that $(u_a):(u_b)\subset(u_c):(u_b)=(x_{i'_s+t,0})$. It only remains to show that $u_c\in G(I^{[q]})$. Due to $F_{i'_1}<\ldots<F_{i'_s}<F_{i'_s+1}$, we conclude that $F_{i'_s+1}$ does not intersect any facet in $\{F_{i'_1},\ldots,F_{i'_{s-1}}\}$. Since $i'_s+t=i_s<\ldots<i_q$, the facet $F_{i'_s+1}$ is disjoint with each of the facet in $\{F_{i'_{s+1}},\ldots,F_{i'_q}\}=\{F_{i_{s+1}},\ldots,F_{i_q}\}$. This shows that $u_c\in G(I^{[q]})$, as required. This completes the proof. □

We point out that the linearity of the resolution concerning the $\nu_0$-th squarefree power of a $t$-path ideal needs to be specialized for path graphs, because it does not hold in general for broom graphs. We illustrate this in the following example which is given in (Erey et al., 2022, Page 12).

**Example 4.1.6.** Consider the broom graph given in Figure 4.3. Let $\Gamma_2$ be its 2-path simplicial tree. Then

$$I(\Gamma_2)=(x_1x_2,x_2x_3,x_3x_4,x_4x_5,x_5x_6,x_6x_7,x_3x_8,x_5x_9)\subset K[x_1,\ldots,x_9].$$

We compute the restricted matching number $\nu_0(\Gamma_2)$. Consider the matching $M=\{\{1,2\},\ \{6,7\},\ \{5,9\}\}$, which consists of pairwise disjoint facets. Let $F=\{1,2\}$. Then $F$ forms a *gap* with both other facets in the matching. Hence, $M$ is a restricted matching of size 3 and we do not have a restricted matching of $M$ with size bigger than 3. Therefore, $\nu_0(\Gamma_2)=3$. Then,

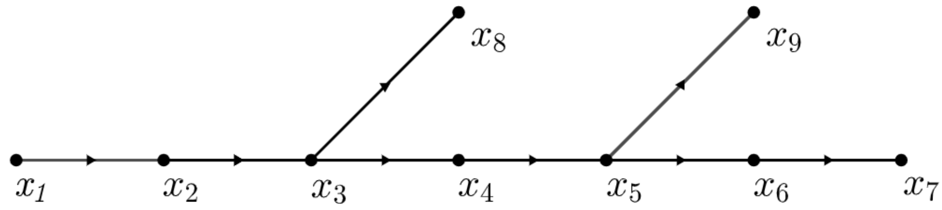



Figure 4.3 A broom graph.

$$\begin{aligned} I(\Gamma_2)^{[3]} = (&x_1x_2x_3x_4x_5x_6, x_1x_2x_3x_4x_5x_8, x_1x_2x_3x_4x_5x_9, x_1x_2x_3x_4x_6x_7, x_1x_2x_3x_5x_6x_8,\\ &x_1x_2x_3x_5x_8x_9, x_1x_2x_3x_6x_7x_8, x_1x_2x_4x_5x_6x_7, x_1x_2x_5x_6x_7x_9, x_2x_3x_4x_5x_6x_7,\\ &x_2x_3x_5x_6x_7x_9, x_3x_4x_5x_6x_7x_8, x_3x_4x_5x_6x_7x_9, x_3x_5x_6x_7x_8x_9). \end{aligned}$$

The free resolution for $I(\Gamma_2)^{[3]}$ is given as follow:

$$0 \longrightarrow S(-8)^6 \oplus S(-9) \longrightarrow S(-7)^{19} \oplus S(-8) \longrightarrow S(-6)^{14} \longrightarrow I(\Gamma_2)^{[3]} \longrightarrow 0.$$

This shows that $I(\Gamma_2)^{[3]}$ does not have a linear resolution and consequently does not have linear quotients.

# 5. LINEARLY RELATED SQUAREFREE POWERS OF SIMPLICIAL TREES

In this section we discuss the linearity of the first syzygy module of squarefree powers of the facet ideals attached to simplicial forests. We say that a graded ideal $I$, generated by homogeneous elements of degree $t$, is *linearly related* if $\beta_{1,j}(I) = 0$ for all $j \neq 1+t$. A useful tool to investigate the linearly related property for a monomial ideal is provided by (Bigdeli et al., 2017, Corollary 1.2), which we recall in Theorem 5.0.1.

Let $I$ be a monomial ideal generated in degree $d$. In (Bigdeli et al., 2017), authors associated a graph $G_I$ to $I$ as follows: $V(G_I) = G(I)$, and $\{u,v\} \in E(G_I)$ if and only if $\deg(\mathrm{lcm}(u,v)) = d+1$. Moreover, for all $u,v \in G(I)$, the induced subgraph of $G_I$ on the vertex set $\{w \in V(G_I) \mid w \text{ divides } \mathrm{lcm}(u,v)\}$ is denoted by $G_I^{(u,v)}$.

**Theorem 5.0.1.** *(Bigdeli et al., 2017, Corollary 1.2) Let $I$ be a monomial ideal generated in degree $d$. Then $I$ is linearly related if and only if for all $u,v \in G(I)$ there is a path in $G_I^{(u,v)}$ connecting $u$ and $v$.*

In (Bigdeli et al., 2017, Lemma 4.2), authors showed that if $I$ is the edge ideal of a simple graph, then $I^{[k]}$ is not linearly related for any $1 \leq k < \nu_0(G)$. This result cannot be extended even for the facet ideal of an arbitrary 2-dimensional simplicial complex as observed in the following example.

**Example 5.0.2.** Let $\Delta$ be the simplicial complex whose facet ideal is

$$I(\Delta) = (x_1x_2x_3, x_4x_5x_6, x_7x_8x_9, x_4x_5x_7, x_2x_4x_8, x_3x_5x_7, x_4x_8x_9, x_5x_6x_7,$$

$$x_1x_4x_7, x_2x_5x_8, x_3x_6x_9, x_4x_7x_9, x_6x_7x_9, x_6x_8x_9, x_4x_6x_9)$$

The set $M = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$ is a restricted matching of $\Delta$ because $\{1,2,3\}$ makes a gap with the rest of the facets in $M$. One can verify that there does not exists any restricted matching of $\Delta$ of size bigger than three. It gives $\nu_0(\Delta) = 3$. With `Macaulay2` (Grayson & Stillman, Grayson & Stillman), we see that $I(\Delta)^{[2]}$ is linearly related.

Now, we prove an analogue of (Bigdeli et al., 2017, Lemma 4.2) for pure simplicial forests. Together with Proposition 3.0.7, it basically gives a necessary condition for the squarefree powers of the facet ideals of a pure simplicial forest to have a linear resolution.

**Theorem 5.0.3.** *Let $\Delta$ be a pure simplicial forest with* $\dim(\Delta) > 0$*. Then $I(\Delta)^{[k]}$ is not linearly related for all $1 \leq k < \nu_0(\Delta)$.*

*Proof.* Let $\dim(\Delta) = n-1 > 0$ and $M = \{F_1, F_2, \ldots, F_{\nu_0(\Delta)}\}$ be a restricted matching of $\Delta$. Set $I = I(\Delta)$. Let $f_i = \prod_{j\in F_i} x_j$ and set $u = f_1 f_2 \cdots f_k$ and $v = f_2 f_3 \ldots f_{k+1}$ for $1 \leq k < \nu_0(\Delta)$ such that $F_{k+1}$ forms a gap with rest of the elements in $M$. By virtue of Theorem 5.0.1 it is enough to show that $u$ and $v$ are disconnected in $G^{(u,v)}_{I^{[k]}}$.

Note that $\{u,v\}$ is not an edge in $G^{(u,v)}_{I^{[k]}}$ because $\deg(\operatorname{lcm}(u,v)) = kn+n > nk+1$. Suppose that $u$ and $v$ are connected in $G^{(u,v)}_{I^{[k]}}$. Then there exists some $w \in G(I^{[k]})$ such that $\{w,u\}$ is an edge in $G^{(u,v)}_{I^{[k]}}$ and $\deg(\operatorname{lcm}(u,w)) = nk+1$. Since $w$ divides $\operatorname{lcm}(u,v)$ and $\deg(w) = nk$, there exist some $x \in F_{k+1} \subset \operatorname{supp}(v)$ and $y \in \operatorname{supp}(u)$ such that $\operatorname{supp}(w) = \{x\} \cup (\operatorname{supp}(u) \setminus \{y\})$. Let $w = w_1 \cdots w_k$ with $G_i = \operatorname{supp}(w_i) \in \mathcal{F}(\Delta)$ for $i = 1, \ldots, k$. After a relabelling of vertices, we may assume that $x \in G_1$ and $y \in F_1$. Let $G_1' = G_1 \setminus \{x\}$ and $F_1' = F_1 \setminus \{y\}$, and set $A = \{G_1', G_2, \ldots, G_k\}$ and $B = \{F_1', F_2, \ldots, F_k\}$. Observe that $G_1' \not\subseteq F_\ell$, for all $\ell = 2, \ldots, k$. Indeed, if $G_1' \subseteq F_\ell$ for some $\ell$, then $G_1$ belongs to the induced subcomplex on $F_\ell \cup F_{k+1}$, which is a contradiction to the assumption that $F_{k+1}$ forms a gap with $F_\ell$.

We claim that $A = B$ and consequently $G_1' = F_1'$. To see this, we apply the similar argument as in the proof of Proposition 3.0.5. Note that the elements of $A$ are pairwise disjoint and the elements of $B$ are pairwise disjoint as well. Moreover, the union of elements of $A$ coincides with the union of elements in $B$. Assume that $A \neq B$, and without loss of generality, we may assume that $A \cap B = \emptyset$. Consider the bipartite graph $H$ on the vertex set $A \cup B$ such that two vertices of $H$ are adjacent if and only if their intersection as facets of $\Delta$ is non-empty. Since $G_1' \not\subseteq F_\ell$ for any $\ell$, we see that degree of $G_1'$ in $H$ is at least two. Moreover, due to $A \cap B = \emptyset$ we obtain that degrees of $G_2, \ldots G_k, F_2, \ldots, F_k$ are also at least two. Therefore, $H$ may have at most one vertex of degree one, namely, $F_1'$. This show that $H$ is not a forest and it contains a cycle. Let $C$ be the vertex set of a cycle in $H$. If $G_1'$ or $F_1'$ appears in $C$, then we replace them by $G_1$ and $F_1$, respectively. As argued in the proof of Proposition 3.0.5, we see that the subcomplex of $\Delta$ with facets in $C$ does not have a leaf, which is a contradiction to $\Delta$ being a simplicial tree. Therefore, $A = B$, and consequently $G_1 \setminus \{x\} = F_1 \setminus \{y\}$. This shows that $G_1$ belongs to the induced subcomplex on $F_1 \cup F_{k+1}$, a contradiction to the assumption that $F_{k+1}$ forms a gap with $F_1$.

From above argument we see that there does not exist any $w \in G_{I^{[k]}}^{(u,v)}$ adjacent to $u$. Therefore, $G_{I^{[k]}}^{(u,v)}$ is disconnected, as claimed. □

**Corollary 5.0.4.** *Let $\Delta$ be a pure simplicial forest. If $I(\Delta)^{[k]}$ has a linear resolution then $k = \nu(\Delta) - 1$ or $k = \nu(\Delta)$.*

*Proof.* It follows from Proposition 3.0.7 and Theorem 5.0.3. □

Next, we show that for any pure simplicial tree $\Delta$, if $I(\Delta)^{[k]}$ is linearly related then $I(\Delta)^{[k+1]}$ is also linearly related. Hence, if the highest squarefree power $I(\Delta)^{[\nu(\Delta)]}$ is not linearly related then $I(\Delta)^{[k]}$ cannot be linearly related for all $1 \leq k \leq \nu(\Delta)$, in particular $I(\Delta)^{[k]}$ cannot have a linear resolution for all $1 \leq k \leq \nu(\Delta)$. To show this, we first prove the following lemma.

**Lemma 5.0.5.** *Let $\Delta$ be a simplicial tree. Further, let $M = \{F_1, \ldots, F_s\}$ and $N = \{G_1, \ldots, G_s\}$ be two $s$-matching of $\Delta$. Then there exist $i, j \in \{1, \ldots, s\}$ such that $F_i \cap G_k = \emptyset$ for all $k \neq j$.*

*Proof.* If $F_i = G_j$ for any $i, j \in \{1, \ldots, s\}$, then the assertion holds trivially. Assume that $F_i \neq G_j$, for all $i, j \in \{1, \ldots, s\}$, that is, $M \cap N = \emptyset$. Let $H$ be the bipartite graph with $V(H) = M \cup N$ and $E(H) = \{\{F_i, G_j\} : F_i \cap G_j \neq \emptyset, i, j \in \{1, \ldots, s\}\}$. On contrary, assume that for each $i \in \{1, \ldots, s\}$ there exist at least two $p, q \in \{1, \ldots, s\}$ such that $F_i \cap G_p \neq \emptyset$ and $F_i \cap G_q \neq \emptyset$. Then each vertex in $H$ has degree at least two. Therefore, $H$ contains an even cycle. After rearranging the indices, we may assume that $F_1, G_1, F_2, G_2, \ldots, G_t, F_{t+1} = F_1$ is a cycle of length $t$ in $H$.

Consider the subcomplex $\Delta' = \langle F_1, \ldots, F_t, G_1, \ldots, G_t \rangle$. Since $M$ and $N$ are matching of $\Delta$, it follows that the sets $F_i \cap G_i$ and $G_i \cap F_{i+1}$ are distinct for all $i = 1, \ldots, t$. This shows that the subcomplex $\Delta' = \langle F_1, \ldots, F_t, G_1, \ldots, G_t \rangle \subset \Delta$ has no leaf, and $\Delta$ is not a simplicial tree, a contradiction. □

**Theorem 5.0.6.** *Let $\Delta$ be a pure simplicial tree. If $I(\Delta)^{[k]}$ is linearly related then $I(\Delta)^{[k+1]}$ is also linearly related.*

*Proof.* Let $I = I(\Delta)$ and $u, v \in G(I^{[k+1]})$ with $u = f_1 \cdots f_{k+1}$ and $v = g_1 \cdots g_{k+1}$ and $f_1, \ldots, f_{k+1}, g_1, \ldots, g_{k+1} \in G(I)$, for any $1 \leq k \leq \nu(\Delta) - 1$. Following (Bigdeli et al., 2017, Corollary 1.2), it is enough to show that $u$ and $v$ are connected by a path in $G_{I^{[k+1]}}^{(u,v)}$. Let $F_i = \mathrm{supp}(f_i)$ and $G_i = \mathrm{supp}(g_i)$, for all $i = 1, \ldots, k+1$. Then $M = \{F_1, \ldots, F_{k+1}\}$ and $N = \{G_1, \ldots, G_{k+1}\}$ are $k$-matching of $\Delta$.

It follows from Lemma 5.0.5 that there exist $i,j \in \{1,\ldots,k+1\}$ such that $F_i \cap G_t = \emptyset$ for all $t \neq j$. After rearranging the indices, we may assume that $i = j = k+1$. Then $F_{k+1} \cap (G_1 \cup \ldots \cup G_k) = \emptyset$. Since $I^{[k]}$ is linearly related, the monomials $u' = f_1 \cdots f_k$ and $v' = g_1 \cdots g_k$ are connected by a path, say $P_1 : u' = w_0, w_1, \ldots, w_s = v'$, in $G^{(u',v')}_{I^{[k]}}$. Since $w_i$ divides $\operatorname{lcm}(u',v')$, we obtain $\operatorname{supp}(w_i) \cap F_{k+1} = \emptyset$ and $w_i f_{k+1} \in G(I^{[k+1]})$, for all $i = 0,\ldots,s$. Moreover, $w_i f_{k+1}$ divides $\operatorname{lcm}(u,v)$ and hence $w_i f_{k+1} \in G^{(u,v)}_{I^{[k+1]}}$. This gives a path

$$Q_1 : u = w_0 f_{k+1}, w_1 f_{k+1}, \ldots, w_s f_{k+1} = v' f_{k+1}$$

in $G^{(u,v)}_{I^{[k+1]}}$. Proceeding in a similar way, we construct a path from $u'' = g_2 \cdots g_k f_{k+1}$ to $v'' = g_2 \cdots g_k g_{k+1}$ in $G^{(u'',v'')}_{I^{[k]}}$ which provides a path $Q_2$ from $g_1 u'' = v' f_{k+1}$ to $g_1 v'' = v$ in $G^{(u,v)}_{I^{[k+1]}}$. Joining $Q_1$ and $Q_2$ gives us a path connecting $u$ and $v$ in $G^{(u,v)}_{I^{[k+1]}}$, as required. $\square$

We conclude this section with a description of the degrees of the vanishing graded Betti numbers with homological degree one. An application of this result will be provided in Proposition 6.0.5 in order to give a lower bound for the regularity of the $(\nu-1)$-squarefree power of the facet ideal of the $t$-path simplicial tree of a path graph. To this end, we recall some definitions below.

Let $P$ be a poset. The comparability graph of $P$, denoted by $G_P$, is a graph whose vertex set consists of the elements of $P$ and $\{a,b\} \in E(G_P)$ if and only if $a$ and $b$ are comparable in $P$.

Let $I$ be a monomial ideal. The lcm-lattice of $I$, denoted by $L(I)$, is the poset whose elements are the least common multiples of subsets of monomials in $G(I)$ which are ordered by divisibility. By the definition, $L(I)$ has 1 as the unique minimal element. For any $u \in L(I)$, the induced subposet of $L(I)$ with elements $v \in L(I)$ such that $1 < v < u$, is denoted by the open interval $(1,u)$. The simplicial complex $\Delta((1,u))$ is the order complex of the poset $(1,u)$.

In the following theorem and in next section, we adopt the following notation to refer to $t$-path simplicial trees of path graphs. Let $P_n$ be the path graph on vertex $\{1,\ldots,n\}$ and edges $\{i,i+1\}$ for all $i = 1,\ldots,n-1$. For any $t \leq n$, we denote the $t$-path simplicial tree of $P_n$ by $\Gamma_{n,t}$. Then

$$\mathcal{F}(\Gamma_{n,t}) = \{F_i = \{i,i+1,\ldots,i+t-1\} : i = 1,\ldots,n-t+1\}.$$

The ideal $I_{n,t} = I(\Gamma_{n,t})$ is called the *$t$-path ideal of $P_n$*. We label the generators of

$I_{n,t}$ as $f_1,\ldots,f_{n-t+1}$ such that $f_i = \prod_{j\in F_i} x_j$ for each $i$. Moreover, we write $f_i < f_j$, if $i<j$. Let $u,v \in G(I_{n,t}^{[k]})$ with $u = f_{i_1}\ldots f_{i_k}$ and $v = f_{j_1}\ldots, f_{j_k}$. Let $A_{(u,v)}$ be the set of indices such that $i_a \in A_{(u,v)}$ if and only if $f_{i_a} \neq f_{j_a}$. Now we are ready to prove the following.

**Theorem 5.0.7.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$ and $I_{n,t} = I(\Gamma_{n,t})$. Then $\beta_{1,p}(I_{n,t}^{[k]}) = 0$ if $p \notin \{kt+1,(k+1)t\}$.*

*Proof.* Let $I = I_{n,t}$. Following Taylor's complex attached to $I^{[k]}$, we have $\beta_{1,m}(I^{[k]}) = 0$ if $m \neq \operatorname{lcm}(u_1,u_2)$ for any $u_1,u_2 \in G(I^{[k]})$. Moreover, using the result of Gasharov, Peeva and Welker (Gasharov et al., 1999), we have for all $i \geq 0$ and for all $m \in L(I^{[k]})$

$$\beta_{1,m}(I^{[k]}) = \dim_K \tilde{H}_0(\Delta((1,m));K).$$

Recall that $\dim_K \tilde{H}_0(\Delta((1,m));K)$ is $c-1$, where $c$ is the number of connected components of $\Delta((1,m))$ (see (Munkres, 1984, Chapter 1-Section 7)). Note that the maximum degree that $m$ can have is $2kt$. This gives $\beta_{1,p}(I^{[k]}) = 0$ for all $p > 2kt$. Therefore, to prove the assertion, it is enough to show the following: if $m$ is the least common multiple of two elements in $G(I^{[k]})$, with $kt+1 < \deg(m) \leq 2kt$ and $\deg(m) \neq (k+1)t$, then the open interval $(1,m)$ in the lcm-lattice $L(I^{[k]})$ is connected. Moreover, to show that $P = (1,m)$ is connected, it is enough to show that the comparability graph $G_P$ of $P$ is connected.

By the definition of $G_P$, the vertices of $G_P$ are those monomials in $L(I^{[k]})$ which are different from 1 and strictly divide $m$. Therefore, any monomial in $V(G_P)$ with degree strictly greater than $kt$ is adjacent with some elements of degree $kt$, which are precisely the generators of $I^{[k]}$. Then, it is enough to show that for any $v,w \in V(G_P)$ with $\deg(v) = \deg(w) = kt$ and $v \neq w$, there is a path in $G_P$ that connects $v$ and $w$. If $\operatorname{lcm}(v,w) \in V(G_P)$, then $\operatorname{lcm}(v,w)$ is a common neighbor of $v$ and $w$ and we are done. Assume that $\operatorname{lcm}(v,w) \notin V(G_P)$. Since $v$ and $w$ strictly divide $m$, we note that $\operatorname{lcm}(v,w) \notin V(G_P)$ if and only if $\operatorname{lcm}(v,w) = m$. Let $v = f_{i_1}\ldots f_{i_k}$ and $w = f_{j_1}\ldots f_{j_k}$ with $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$. First we assume that $kt+1 < \deg(m) < (k+1)t$. Then $\deg(m) = kt+r$ for some $2 \leq r \leq t-1$. Since $v \neq w$, there exists some index $a$ for which $f_{i_a} \neq f_{j_a}$.

Case (1): Let $|A_{(v,w)}| = 1$. This means there exists exactly one index $a$ for which $f_{i_a} \neq f_{j_a}$, and $f_{i_p} = f_{j_p}$ for all $p \neq a$. Without loss of generality, we may assume that $f_{i_a} < f_{j_a}$. Then $j_a = i_a + r$. Set $z_q := f_{i_a+q}$ for $q = 0,\ldots,r$. Consider the elements of $G(I^k)$ for each $q = 0,\ldots,r$ given by $v_q = (v/f_{i_a})z_q$. Then $v_0 = v$, $v_r = w$ and $\deg(m_q) = kt+1 < \deg(m)$ where $m_q = \operatorname{lcm}(v_q,v_{q+1})$. This shows that $m_q$

strictly divide $m$ and $m_q \in (1,m)$, that is, $m_q \in V(G_P)$. This gives us a path $v = v_0, m_0, v_1, \ldots, m_{r-1}, v_r = w$ in $G_P$ connecting $v$ and $w$, as required.

Case (2): Let $|A_{(v,w)}| > 1$, and $a$ be the smallest index for which $f_{i_a} \neq f_{j_a}$. Without loss of generality, we may assume that $f_{j_a} < f_{i_a}$. Then $\mathrm{supp}(f_{j_a}) \cap \mathrm{supp}(f_{i_p}) = F_{j_a} \cap F_{i_p} = \emptyset$ for all $p \neq a$. Set $v_1 = (v/f_{i_a})f_{j_a}$. It yields $v_1$ corresponds to a $k$-matching of $\Gamma_{n,t}$ and $v_1 \in G(I^{[k]})$. Moreover, $\deg(\mathrm{lcm}(v,v_1)) = kt + \ell < \deg(m)$ with $\ell < r$ because $|A_{(v,w)}| > 1$. Hence, $\mathrm{lcm}(v,v_1)$ strictly divides $m$ and $\mathrm{lcm}(v,v_1) \in V(G_P)$. So far, we have a path $v, \mathrm{lcm}(v,v_1), v_1$ in $G_P$. Note that $|A_{(v_1,w)}| < |A_{(v,w)}|$.

We repeat our argument by replacing $v$ with $v_1$ to obtain $v_2$ such that $|A_{(v_2,w)}| < |A_{(v_1,w)}| < |A_{(v,w)}|$. After $d = |A_{(v,w)}| - 1$ number of steps, we obtain $v_d$ for which $|A_{(v_d,w)}| = 1$. Then by repeating the arguments as in Case (1), we further obtain a path from $v_d$ to $w$ in $G_P$ which can be augmented with the path $v, \mathrm{lcm}(v,v_1), v_1, \mathrm{lcm}(v_1,v_2), v_2, \ldots, \mathrm{lcm}(v_{d-1},v_d), v_d$. In this way, we obtain a path from $v$ to $w$ in $G_P$.

Now assume that $(k+1)t < \deg(m) < 2kt$. In this case $|A_{(v.w)}| > 2$. We proceed as follows: let $v_1 = v$, $w_1 = w$, $s = r = 1$ and perform the following step.

Step-$s$: We set $v_s = f_{s_1} \cdots f_{s_k}$ with $s_1 < \ldots < s_k$, and $w_r = f_{r_1} \cdots f_{r_k}$ with $r_1 < \ldots < r_k$. Let $a$ be the smallest integer for which $f_{s_a} \neq f_{r_a}$. If $f_{r_a} < f_{s_a}$, then we construct $v_{s+1}$ as follows such that $\mathrm{lcm}(v_s, v_{s+1}) \in V(G_P)$.

Set $v_{s+1} := (v_s/f_{s_a})f_{r_a}$. Since $f_{s_1} = f_{r_1}, \ldots, f_{r_{a-1}} = f_{s_{a-1}}$. and $f_{r_a} < f_{s_a} < f_{s_{a+1}} < \ldots, f_{s_k}$, it immediately follows that $v_{s+1} \in G(I^{[k]})$. Moreover,

(i) if $\mathrm{supp}(f_{r_a}) \cap \mathrm{supp}(f_{s_a}) = \emptyset$, then $\deg(\mathrm{lcm}(v_s, v_{s+1})) = (k+1)t < \deg(m)$.

(ii) if $\mathrm{supp}(f_{r_a}) \cap \mathrm{supp}(f_{s_a}) \neq \emptyset$, then $\deg(\mathrm{lcm}(v_s, v_{s+1})) = kt + r < \deg(m)$ where $r < t$.

In both cases (i) and (ii) above, the $\mathrm{lcm}(v_s, v_{s+1})$ strictly divides $m$. Therefore, $\mathrm{lcm}(v_s, v_{s+1}) \in V(G_P)$. At the end of this step, we obtain a path $v_s, \mathrm{lcm}(v_s, v_{s+1}), v_{s+1}$ in $G_P$. Also, $|A_{(v_s,w_r)}| > |A_{(v_{s+1},w_r)}|$. If $|A_{(v_{s+1},w_r)}| > 0$, we set $v_s := v_{s+1}$ and repeat Step-$s$. Otherwise, $v_{s+1} = w_r$ and we obtain the desired path

$$v = v_1, \mathrm{lcm}(v_1,v_2), v_2, \ldots, v_s, \mathrm{lcm}(v_s,w_r), w_r, \mathrm{lcm}(w_{r-1},w_r), w_{r-1}, \ldots, w_1 = w$$

connecting $v$ and $w$ in $G_P$.

If $f_{r_a} > f_{s_a}$ then we terminate Step-$s$ and go to Step-$r$ to construct $w_{r+1}$ such that $\operatorname{lcm}(w_r, w_{r+1}) \in V(G_P)$.

Step-$r$: Set $w_{r+1} := (w_r/f_{r_a})f_{s_a}$. From a similar discussion as in the construction of $v_{s+1}$ above, it immediately follows that $w_{r+1} \in G(I^{[k]})$. Moreover,

(i) if $\operatorname{supp}(f_{r_a}) \cap \operatorname{supp}(f_{s_a}) = \emptyset$, then $\deg(\operatorname{lcm}(w_r, w_{r+1})) = (k+1)t < \deg(m)$.

(ii) if $\operatorname{supp}(f_{r_a}) \cap \operatorname{supp}(f_{s_a}) \neq \emptyset$, $\deg(\operatorname{lcm}(w_r, w_{r+1})) = kt + r < \deg(m)$ where $r < t$.

In both cases (i) and (ii) above, $\operatorname{lcm}(w_r, w_{r+1})$ strictly divides $m$ and $\operatorname{lcm}(w_r, w_{r+1}) \in V(G_P)$. At the end of this step, we obtain a path $w_r, \operatorname{lcm}(w_r, w_{r+1}), w_{r+1}$ in $G_P$. Also, $|A_{(w_r,v_s)}| > |A_{(w_{r+1},v_s)}|$. If $|A_{(w_{r+1},v_s)}| > 1$, we set $w_r := w_{r+1}$ and repeat Step-$s$. Otherwise, $w_{r+1} = v_s$ and we obtain the desired path connecting $v_1$ and $w_1$ in $G_P$.

Note that, we perform Step-$s$ and Step-$r$ only a finite number of time. Indeed, $|A_{(v_s,w_r)}| > |A_{(v_{s+1},w_r)}|$ at the end of Step-$s$ and $|A_{(w_r,v_s)}| > |A_{(w_{r+1},v_s)}|$ at the end of Step-$r$. This completes the proof. $\square$

To illustrate this theorem we give the following example.

**Example 5.0.8.** Let $t = 2$, $n = 6$, so we consider the ideal $I_{6,2} = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6)$. The first squarefree power is just $I_{6,2}$ and the Betti diagram of $S/I_{6,2}$ is given below.

Table 5.1 Betti diagram for $S/I_{6,2}$.

| | 1 | 2 | 3 | 3 |
|---|---|---|---|---|
| 1: | 5 | 4 | – | – |
| 2: | – | 3 | 4 | 1 |
| Tot: | 5 | 7 | 4 | 1 |

We see that $\beta_{1,p}(I_{6,2}) \neq 0$ only for $p = 3$ and $p = 4$ , as expected since $kt+1 = 3$ and $(k+1)t = 4$. The second squarefree power is

$$I_{6,2}^{[2]} = (x_1x_2x_3x_4, x_1x_2x_4x_5, x_1x_2x_5x_6, x_2x_3x_4x_5, x_2x_3x_5x_6, x_3x_4x_5x_6)$$

and the Betti diagram of $S/I_{6,2}^{[2]}$ is given in Table 5.2.

Table 5.2 Betti diagram for $S/I_{6,2}^{[2]}$.

| | 1 | 2 | 3 |
|---|---|---|---|
| 1: | – | – | – |
| 2: | – | – | – |
| 3: | 6 | 6 | 1 |

So, $\beta_{1,p}(I_{6,2}^{[2]}) \neq 0$ only for $p = 6$, consistent with $(k+1)t = 6$. Note that $\beta_{1,p}(I_{6,2}^{[3]}) = 0$ for all $p = 1, 2, 3, \ldots$, since $I_{6,2}^{[3]} = (x_1 x_2 x_3 x_4 x_5 x_6)$.

## 6. REGULARITY OF T-PATH IDEALS OF PATH GRAPHS

In this section we study the regularity of the squarefree powers of the $t$-path ideal $I_{n,t}$ of path graph $P_n$. We give a combinatorial description for $\operatorname{reg}(R/I_{n,t}^{[k]})$ in terms of the induced matching number of $\Gamma_{n-kt,t}$. First, we set some notations in order to prove the next lemma. If $\Delta$ is a simplicial complex and $F \in \mathcal{F}(\Delta)$, then

$$\Delta \setminus F := \langle G : G \in \mathcal{F}(\Delta) \text{ with } F \cap G = \emptyset \rangle.$$

Moreover, given a simplicial forest $\Delta$ with a good leaf order $F_r, \dots, F_1$, we set $f_i = \prod_{x_j \in F_i} x_j$, $\Delta_i = \langle F_i, \dots, F_r \rangle$ for each $i = 1, \dots, r$, and $J_i = (f_1, \dots, f_{i-1})$, for $i = 2, \dots, r$. Furthermore, we set $J_1 = (0)$.

**Lemma 6.0.1.** *Let $\Delta$ be a simplicial forest with good leaf order $F_r, \dots, F_1$ and $f_i = \prod_{x_j \in F_i} x_j$ for all $i = 1, \dots, r$. Then for all $1 \le k \le \nu(\Delta)$, we have*

1. $I(\Delta)^{[k+1]} : (f_1) = I(\Delta \setminus F_1)^{[k]}$.
2. $(I(\Delta_i)^{[k+1]} + J_i) : (f_i) = I(\Delta_i \setminus F_i)^{[k]} + (J_i : (f_i))$, *for all* $2 \le i \le r$.
3. $(I(\Delta_i)^{[k+1]} + J_i) + (f_i) = I(\Delta_{i+1})^{[k+1]} + J_{i+1}$, *for all* $1 \le i \le r$.

*Proof.* (1) Let $J = I(\Delta)^{[k+1]} : (f_1)$ and $g \in J$. Then $gf_1 = hf_{i_1} \dots f_{i_{k+1}}$ such that $\{F_{i_1}, \dots, F_{i_{k+1}}\}$ is a $(k+1)$-matching of $\Delta$ and $h$ is a monomial. Since $F_1$ is a good leaf of $\Delta$, it yields that $F_1$ is a leaf of the subcomplex $\langle F_1, F_{i_1} \dots F_{i_{k+1}} \rangle$ and we may assume that $F_1 \cap F_{i_j} \subseteq F_1 \cap F_{i_1}$ for all $j = 2, \dots k+1$. Therefore, any element in $F_1 \setminus F_{i_1}$ is not contained in $F_{i_j}$ for all $j = 2, \dots, k+1$. It yields $f_1$ divides $hf_{i_1}$ and $f_{i_2} \dots f_{i_{k+1}}$ divide $g$. Since $F_{i_1}$ does not intersect any $F_{i_j}$ for all $j = 2, \dots, k+1$, it follows that $F_1$ also does not intersect any $F_{i_j}$ for all $j = 2, \dots, k+1$. From this we conclude that $F_{i_2}, \dots, F_{i_{k+1}} \in \Delta \setminus F_1$ and $g \in I(\Delta \setminus F_1)^{[k]}$. The inclusion $I(\Delta \setminus F_1)^{[k]} \subseteq J$ is obvious since any facet in $\Delta \setminus F_1$ is disjoint with $F_1$.

(2) We have $(I(\Delta_i)^{[k+1]} + J_i) : (f_i) = (I(\Delta_i)^{[k+1]} : f_i) + (J_i : (f_i))$, for all $2 \le i \le r$. Since $F_i$ is a good leaf of $\Delta_i$, applying a similar argument as in proof of statement in (1), we have $(I(\Delta_i)^{[k+1]} : f_i) = I(\Delta_i \setminus F_i)^{[k]}$, which gives us the equality in (2).

(3) We have $J_i + (f_i) = J_{i+1}$. Any element in $G(I(\Delta_i)^{[k+1]})$ is of the form $u = f_{i_1}\cdots f_{i_{k+1}}$ such that $\{F_{i_1},\ldots,F_{i_{k+1}}\}$ is a $(k+1)$-matching. Note that if $f_i$ divides $u$ then $u \in (f_i) \subset J_{i+1}$. On the other hand, if $f_i$ does not divide $u$ then $u \in I(\Delta_{i+1})^{[k+1]}$. This completes the proof. □

**Remark 6.0.2.** Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of the path graph $P_n$. Then

$$\nu(\Gamma_{n,t}) = \left\lfloor \frac{n}{t} \right\rfloor,\ \nu_0(\Gamma_{n,t}) = \left\lfloor \frac{n-1}{t} \right\rfloor,\ \nu_1(\Gamma_{n,t}) = \left\lceil \frac{n-t+1}{t+1} \right\rceil.$$

Below, we provide a brief reasoning to justify the above equalities. As before, we set

$$\mathcal{F}(\Gamma_{n,t}) = \{F_i = \{i, i+1, \ldots, i+t-1\} : i = 1, \ldots, n-t+1\}$$

(1) Let $n = qt + r$ where $q = \lfloor \frac{n}{t} \rfloor$ and $0 \le r \le t-1$. Then, it is easy to see that matching number of $\Gamma_{n,t} = q$. Indeed, $M = \{F_1, F_{1+t}, \ldots, F_{1+(q-1)t}\}$ is a maximal matching of $\Gamma_{n,t}$.

(2) Observe that two facets $F_i, F_j$ of $\Gamma_{n,t}$ with $i < j$ form a gap if and only if $i + t < j$. Indeed, $F_i \cap F_j = \emptyset$ if and only if $i + t \le j$, and if $j = i + t$, then $F_{i+1}$ belongs to the induced subcomplex on $F_i \cup F_j$. With this observation, we form a restricted matching $M$ of $\Gamma_{n,t}$ of maximal size as follows: take a maximal matching $N$ of $\Gamma_{n-(t+1),t}$ as described in (1), and set $M = N \cup \{F_{n-t+1}\}$. In $M$, the facet $F_{n-t+1}$ forms a gap with all the facets in $N$. This gives

$$\nu_0(\Gamma_{n,t}) = \nu(\Gamma_{n-(t+1),t}) + 1 = \left\lfloor \frac{n-(t+1)}{t} \right\rfloor + 1 = \left\lfloor \frac{n-1}{t} \right\rfloor.$$

(3) It follows from the definition of induced matching that any two facets in an induced matching form a gap. Let $n = q'(t+1) + r'$ with $0 \le r' \le t$. Then following the explanation given above in (2), it is easy to see that if $r' < t$, then the set $\{F_1, F_{1+(1+t)}, \ldots, F_{1+(q'-1)(t+1)}\}$ is an induced matching of $\Gamma_{n,t}$ of maximal size, and $\nu_1(\Gamma_{n,t}) = q'$. On the other hand, if $r' = t$, then $\{F_1, F_{2+t}, \ldots, F_{1+(q'-1)(t+1)}, F_{n-t+1}\}$ is an induced matching of $\Gamma_{n,t}$ of maximal size, and $\nu_1(\Gamma_{n,t}) = q' + 1$. This completes the argument.

**Lemma 6.0.3.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$. Let $I_{n,t} = I(\Gamma_{n,t})$. Then*

$$\operatorname{reg}\left(\frac{R}{I_{n,t}}\right) = (t-1)\nu_1(\Gamma_{n,t}).$$

*Proof.* It follows immediately from (Bouchat et al., 2011, Corollary 5.4) and Remark 6.0.2. □

Our strategy to provide an upper bound for $\operatorname{reg}(R/I_{n,t}^{[k]})$ relies on repeatedly utilizing the following short exact sequence

$$0 \to R/(I:f) \to R/I \to R/(I+(f)) \to 0,$$

where $I$ is an appropriate monomial ideal and $f$ is an element of $I$ of degree $d$. In fact it is known from (Dao, Huneke & Schweig, 2013, Lemma 2.10) that

$$\operatorname{reg}(R/I) \leq \max\{\operatorname{reg}(R/(I:f))+d, \operatorname{reg}(I+(f))\}.$$

**Theorem 6.0.4.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$. Let $I_{n,t} = I(\Gamma_{n,t})$. Then for any $1 \leq k+1 \leq \nu(\Gamma_{n,t})$, we have*

$$\operatorname{reg}\left(\frac{R}{I_{n,t}^{[k+1]}}\right) \leq kt + \operatorname{reg}\left(\frac{R}{I_{n-kt,t}}\right) = kt + (t-1)\nu_1(\Gamma_{n-kt,t}).$$

*Proof.* We prove the assertion by applying induction on $n$. It is easy to see that the assertion holds for $P_t$. Assume that the assertion holds for all paths on $m$ vertices such that $t \leq m < n$, that is, for each $1 \leq k+1 \leq \nu(\Gamma_{m,t})$, we have

$$\operatorname{reg}\left(\frac{R}{I_{m,t}^{[k+1]}}\right) \leq kt + (t-1)\nu_1(\Gamma_{m-kt,t}).$$

Given any two positive integers $i \leq j$, we set $[i,j] = \{i, i+1, \ldots, j\}$. To simplify the notation in this proof, we denote the path graph on vertices $[i,j]$ by $P_{[i,j]}$. In particular, for the path graph on the vertices $\{1,\ldots,n\}$, instead of writing $P_n$, we write $P_{[1,n]}$. Since $t$ is fixed throughout the proof, we denote the $t$-path ideal of $P_{[i,j]}$ simply by $I_{[i,j]}$, and $\Gamma_{n,t}$ by $\Gamma_n$.

Set $F_i = \{i, i+1, \ldots, i+t-1\}$ and $f_i = \prod_{j \in F_i} x_j$ for all $i = 1, \ldots, n-t+1$. The order of the facets of $\Gamma_n$ given by $F_{n-t+1}, \ldots, F_1$ is a good leaf order. Consider the following exact sequence

$$0 \to \frac{R}{I_{[1,n]}^{[k+1]} : f_1} \xrightarrow{f_1} \frac{R}{I_{[1,n]}^{[k+1]}} \to \frac{R}{I_{[1,n]}^{[k+1]} + (f_1)} \to 0.$$

Using Lemma 6.0.1, we obtain

$$0 \to \frac{R}{I_{[1+t,n]}^{[k]}} \xrightarrow{f_1} \frac{R}{I_{[1,n]}^{[k+1]}} \to \frac{R}{I_{[2,n]}^{[k+1]} + I_{[1,t]}} \to 0$$

which gives

$$\operatorname{reg}\left(\frac{R}{I_{[1,n]}^{[k]}}\right) \leq \max\left\{t+\operatorname{reg}\left(\frac{R}{I_{[1+t,n]}^{[k]}}\right), \operatorname{reg}\left(\frac{R}{I_{[2,n]}^{[k+1]}+I_{[1,t]}}\right)\right\}.$$

Now we investigate $\operatorname{reg}(R/(I_{[2,n]}^{[k+1]}+I_{[1,t]}))$. Let $a$ be the maximum integer for which $k+1$ is the matching number of the path graph $P_{[a+1,n]}$. Indeed, it can be verified that $a = n-t(k+1)$. Using Lemma 6.0.1, for any $2 \leq i \leq a$, we obtain

$$0 \to \frac{R}{I_{[i+t,n]}^{[k]}+(I_{[1,t+i-2]}:f_i)} \xrightarrow{f_i} \frac{R}{I_{[i,n]}^{[k+1]}+I_{[1,t+i-2]}} \to \frac{R}{I_{[i+1,n]}^{[k+1]}+I_{[1,t+i-1]}} \to 0.$$

Furthermore, since $\nu(P_{[a+1,n]}) = k+1$, we get:

$$0 \to \frac{R}{I_{[a+1+t,n]}^{[k]}+(I_{[1,t+a-1]}:f_{a+1})} \xrightarrow{f_{a+1}} \frac{R}{I_{[a+1,n]}^{[k+1]}+I_{[1,t+a-1]}} \to \frac{R}{I_{[1,t+a]}} \to 0.$$

Therefore,

$$\operatorname{reg}\left(\frac{R}{I_{n,t}^{[k+1]}}\right) \leq \max\left\{t+\operatorname{reg}\left(\frac{R}{I_{[1+t,n]}^{[k]}}\right), \operatorname{reg}\left(\frac{R}{I_{[1,t+a]}}\right), \alpha\right\}$$

where

$$\alpha = \max_{2\leq i\leq a+1}\left\{t+\operatorname{reg}\left(\frac{R}{I_{[i+t,n]}^{[k]}+(I_{[1,t+i-2]}:f_i)}\right)\right\}.$$

Since $a = n-t(k+1)$, using Lemma 6.0.3 we obtain

$$\operatorname{reg}\left(\frac{R}{I_{[1,t+a]}}\right) = (t-1)\nu_1(\Gamma_{t+a}) = (t-1)\nu_1(\Gamma_{n-tk}) \leq kt+(t-1)\nu_1(\Gamma_{n-kt}).$$

Note that $I_{[1+t,n]}^{[k]}$ can be identified as the $t$-path ideal of the path graph $P_{[1,n-t]}$, using the induction hypothesis, we obtain

$$t+\operatorname{reg}\left(\frac{R}{I_{[1+t,n]}^{[k]}}\right) \leq t+(k-1)t+(t-1)\nu_1(\Gamma_{n-t-(k-1)t}) = kt+(t-1)\nu_1(\Gamma_{n-kt}).$$

Now we analyze $\alpha$ and compare it with $kt+(t-1)\nu_1(\Gamma_{n-kt})$. For any $2 \leq i \leq a+1$, the ideals $I^{[k]}_{[i+t,n]}$ and $I_{[1,t+i-2]}:f_i$ lie in disjoint set of vertices. Therefore

$$\operatorname{reg}\left(\frac{R}{I^{[k]}_{[i+t,n]}+(I_{[1,t+i-2]}:f_i)}\right)=\operatorname{reg}\left(\frac{R}{I^{[k]}_{[i+t,n]}}\right)+\operatorname{reg}\left(\frac{R}{(I_{[1,t+i-2]}:f_i)}\right). \tag{6.1}$$

Since $I_{[1,t+i-2]}=(f_1,\dots,f_{i-1})$, for $2\le i\le t$, we have

$$\begin{aligned} I_{[1,t+i-2]}:f_i &= (f_1,\dots,f_{i-1}):f_i \\ &= (f_1,\dots,f_{i-t-1}):f_i+(f_{i-t},\dots,f_{i-1}):f_i \\ &= (f_1,\dots,f_{i-t-1})+(x_{i-1})=I_{[1,i-2]}+(x_{i-1}). \end{aligned} \tag{6.2}$$

If $i\le t+1$, then we set $(f_1,\dots,f_{i-t-1})=0$ in above equation. By identifying $I^{[k]}_{[i+t,n]}$ as $t$-path ideal of the path graph $P_{[1,n-t-i+1]}$, and using induction hypothesis yields

$$\operatorname{reg}\left(\frac{R}{I^{[k]}_{[i+t,n]}}\right)\le (k-1)t+(t-1)\nu_1(\Gamma_{n-i+1-kt}).$$

On the other hand using (6.2) and Lemma 6.0.3 we have

$$\operatorname{reg}\left(\frac{R}{(I_{[1,t+i-2]}:f_i)}\right)\le (t-1)\nu_1(\Gamma_{i-2}).$$

Since $\nu_1(\Gamma_{i-2})+\nu_1(\Gamma_{n-i+1-kt})\le\nu_1(\Gamma_{n-1-kt})$, we see that $\alpha$ is less than $kt+(t-1)\nu_1(\Gamma_{n-kt})$. This completes the proof. □

Next we show that the upper bound given in Theorem 6.0.4, is indeed equal to $\operatorname{reg}\left(\frac{R}{I^{[k+1]}_{n,t}}\right)$. To do this, we first compute the regularity of $(\nu(\Gamma_{n,t})-1)$-th power of $I_{n,t}$.

**Proposition 6.0.5.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of path graph $P_n$, and $I_{n,t}=I(\Gamma_{n,t})$. Suppose that $\nu(\Gamma_{n,t})-1\neq\nu_0(\Gamma_{n,t})$ and $\nu(\Gamma_{n,t})>1$. Then*

$$\operatorname{reg}\left(\frac{R}{I^{[\nu(\Gamma_{n,t})-1]}_{n,t}}\right)=\nu(\Gamma_{n,t})t-2.$$

*Proof.* To simplify the notation, we denote $\nu(\Gamma_{n,t})$ and $\nu_0(\Gamma_{n,t})$ by $\nu$ and $\nu_0$, respectively. Since $\operatorname{reg}(R/I^{[\nu-1]}_{n,t})=\operatorname{reg}(I^{[\nu-1]}_{n,t})-1$, it is enough to show that $\operatorname{reg}(I^{[\nu-1]}_{n,t})=\nu t-1$.

The assumption $\nu-1\neq\nu_0$ together with Proposition 3.0.7 gives that $\nu=\nu_0$. Then from Theorem 5.0.3, we see that $I_{n,t}^{[\nu-1]}$ is not linearly related. Thanks to Theorem 5.0.7, we obtain $\beta_{1,\nu t}(I_{n,t}^{[\nu-1]}) \neq 0$. Therefore, $\operatorname{reg}(I_{n,t}^{[\nu-1]}) \geq \nu t - 1$. Now, we show that $\operatorname{reg}(I_{n,t}^{[\nu-1]}) \leq \nu t - 1$. From Theorem 6.0.4 we have

$$\operatorname{reg}(I_{n,t}^{[\nu-1]}) \leq (\nu-2)t + (t-1)\nu_1(\Gamma_{n-(\nu-2)t,t}) + 1.$$

It is sufficient to prove that $\nu_1(\Gamma_{n-(\nu-2)t,t}) = 2$. There exists some $j$ such that $n = \nu t + j$ with $1 \leq j \leq t-1$. Indeed, if $j \geq t$, then there exists a matching whose cardinality is strictly more that $\nu$, which is a contradiction with being $\nu$ the matching number. If $j = 0$ then $n = \nu t$, so $\nu_0 < \nu$, which is a contradiction with being $\nu = \nu_0$. Since $n - (\nu-2)t = 2t + j$, we obtain $\nu_1(\Gamma_{n-(\nu-2)t,t}) = 2$, due to Remark 6.0.2, as desired. □

**Corollary 6.0.6.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$ and $I_{n,t} = I(\Gamma_{n,t})$. Denote the matching number $\nu(\Gamma_{n,t})$ and the restricted matching number $\nu_0(\Gamma_{n,t})$ of $\Gamma_{n,t}$ by $\nu$ and $\nu_0$, respectively. Suppose that $\nu > 1$. Then*

$$\operatorname{reg}\left(\frac{R}{I_{n,t}^{[\nu-1]}}\right) = (\nu-2)t + (t-1)\nu_1(\Gamma_{n-(\nu-2)t,t}) = \begin{cases} t\nu - 1 & \text{if } \nu - 1 = \nu_0 \\ t\nu - 2 & \text{if } \nu - 1 \neq \nu_0 \end{cases}$$

*Proof.* The claim follows from Theorem 4.1.5 and Proposition 6.0.5. □

In order to prove a lower bound for $\operatorname{reg}(R/I_{n,t}^{[k]})$, for $1 \leq k+1 < \nu(\Gamma_{n,t})$, we need to introduce an equivalent form of Hochster's formula ((Alilooee & Faridi, 2015, Theorem 2.8)). Let us recall firstly the simplicial homological group of a simplicial complex. If $\Delta$ is simplicial complex then an orientation on $\Delta$ is a linear order $<$ on the vertex set of $\Delta$. In such a case, $(\Delta, <)$ is said to be an *oriented* simplicial complex. Let $\Delta$ be a $d$-dimensional oriented simplicial complex with vertex set $V(\Delta)$ and $i \in \{1, \ldots, d\}$. A face $F = \{v_1, \ldots, v_{i+1}\}$, with $v_1 < \cdots < v_{i+1}$, is said to be *oriented*; in such a case, we write $F = [v_1, \ldots, v_{i+1}]$. We denote by $C_i(\Delta)$ the free $\mathbb{Z}$-module generated by all the oriented $i$-dimensional faces of $\Delta$. The augmented oriented chain complex of $\Delta$ is the complex

$$0 \longrightarrow C_d(\Delta) \xrightarrow{\delta_d} C_{d-1}(\Delta) \xrightarrow{\delta_{d-1}} \ldots \longrightarrow C_1(\Delta) \xrightarrow{\delta_1} C_0(\Delta) \xrightarrow{\delta_0} K \longrightarrow 0,$$

where $\delta_0(v) = 1$ for all $v \in V(\Delta)$ and, for any $1 \leq i \leq d$, the map $\delta_i : C_i(\Delta) \to C_{i-1}(\Delta)$ acts on basis elements as follows:

$$\delta_i([v_1, \ldots, v_j, \ldots, v_{i+1}]) = \sum_{j=1}^{i+1} (-1)^{j+1} [v_1, \ldots, \hat{v}_j, \ldots, v_{i+1}],$$

where $\hat{v_j}$ means that $v_j$ is removed. Recall that the $i$-th reduced simplicial homology group of $\Delta$ over $K$ is defined as $\tilde{H}_i(\Delta;K) = \ker \delta_i / \operatorname{Im} \delta_{i+1}$. For convention, set $\tilde{H}_{-1}(\emptyset;K) = K$ and $\tilde{H}_i(\emptyset;K) = 0$, for all $i \geq 0$. Moreover, it is well-know that, if $\Delta \neq \emptyset$, then $\dim_K \tilde{H}_0(\Delta;K)$ is one less than the number of the connected components of $\Delta$ (see (Munkres, 1984, Chapter 1-Section 7)). Recall that, for any $Y \subseteq V$, an induced subcomplex of $\Delta$ on $Y$, denoted by $\Delta_Y$, is the simplicial complex whose vertex set is a subset of $Y$ and the facet set is $\{F \in \mathcal{F}(\Delta) : F \subseteq Y\}$. If $\Delta = \langle F_1, \ldots, F_s \rangle$, then we define the complement of a face $F$ of $\Delta$ in $Y$ to be $F_Y^c = Y \setminus F$ and the complement of $\Delta$ in $Y$ as $\Delta_Y^c = \langle (F_1)_Y^c, \ldots, (F_s)_Y^c \rangle$.

From (Alilooee & Faridi, 2015, Theorem 2.8) we know that, if $I$ is a squarefree monomial ideal generated in single degree in $K[x_1, \ldots, x_n]$, then

$$\beta_{i,d}(I) = \sum_{\substack{\Gamma \subseteq \Delta(I) \\ |V(\Gamma)| = d}} \dim_K \tilde{H}_{i-1}(\Gamma^c_{V(\Gamma)}), \tag{6.3}$$

where the sum is taken over the induced subcomplexes $\Gamma$ of the facet complex $\Delta(I)$ which have $d$ vertices.

**Theorem 6.0.7.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$. Let $I_{n,t} = I(\Gamma_{n,t})$. Then for any $1 \leq k+1 \leq \nu(\Gamma_{n,t})$, we have*

$$kt + (t-1)\nu_1(\Gamma_{n-kt,t}) \leq \operatorname{reg}\left(\frac{R}{I_{n,t}^{[k+1]}}\right)$$

*Proof.* By replacing $k+1$ with $k$, it is equivalent to show for any $1 \leq k \leq \nu(\Gamma_{n,t})$, the following inequality

$$(k-1)t + (t-1)\nu_1(\Gamma_{n-(k-1)t,t}) \leq \operatorname{reg}\left(\frac{R}{I_{n,t}^{[k]}}\right). \tag{6.4}$$

Due to Lemma 6.0.3, it is enough to consider the case when $k \geq 2$. Fix $k, t \in \mathbb{N}$ with $k, t \geq 2$. We divide the discussion into distinct cases depending on the value of $n$. First, observe that $kt \leq n$ due to Remark 6.0.2 and the assumption $k \leq \nu(\Gamma_{n,t})$.

First, we consider the case when $kt \leq n \leq kt+t$. If $kt \leq n < kt+t$, then $\nu(\Gamma_{n,t}) = k$ and thanks to Theorem 4.1.5 (i) we obtain $\operatorname{reg}(R/I_{n,t}^{[k]}) = kt-1$. If $n = kt+t$, then $\nu(\Gamma_{n,t}) = k+1$, and due to Corollary 6.0.6, we get $\operatorname{reg}(R/I_{n,t}^{[k]}) = kt-1$. On the other hand, if $kt \leq n \leq kt+t$, then due to Remark 6.0.2, we have $\nu_1(\Gamma_{n-(k-1)t,t}) = 1$ which gives $(k-1)t + (t-1)\nu_1(\Gamma_{n-(k-1)t,t}) = kt-1$. Hence the inequality in (6.4) holds when $kt \leq n \leq kt+t$.

Now, let $kt+j(t+1) \leq n \leq kt+j(t+1)+t$, for some $j \geq 1$. From here on, we separate the discussion into two cases, namely Case(1): $n = kt+j(t+1)$, and Case(2): $kt+j(t+1) < n \leq kt+j(t+1)+t$. To simplify the discussion, we will argue on $\operatorname{reg}(I_{n,t}^{[k]})$ and use $\operatorname{reg}(R/I_{n,t}^{[k]}) = \operatorname{reg}(I_{n,t}^{[k]})-1$ to make the final conclusion.

Case (1): Let $n = kt+j(t+1)$, for some $j$. Hereafter, we set $n_0 := n$. Let $\Delta(I_{n_0,t}^{[k]})$ be the simplicial complex whose facet ideal is $I_{n_0,t}^{[k]}$. We show that $\beta_{i_0,n_0}(I_{n_0,t}^{[k]}) \neq 0$ for $i_0 = n_0 - [(k-1)t + (t-1)\nu_1(\Gamma_{j(t+1)+t,t}) + 1]$. Since $\nu_1(\Gamma_{j(t+1)+t,t}) = j+1$, we obtain after simplifying that $i_0 = 2j$. Using (6.3) gives

$$\beta_{i_0,n_0}(I_{n_0,t}^{[k]}) = \dim_K \tilde{H}_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c).$$

We choose a subset $Q$ of $\{1,\ldots,n_0\}$ with $|Q| = i_0+1 = 2j+1$ as follows:

$$Q = \{kt+h(t+1), kt+h(t+1)+1 : h = 0,\ldots,j-1\} \cup \{n_0\}.$$

Consider

$$\gamma = \sum_{r=1}^{i_0+1} (-1)^{r+1} [v_1,\ldots,\hat{v_r},\ldots,v_{i_0+1}],$$

where $v_p \in Q$ for all $p = 1,\ldots,i_0+1$ and $v_p < v_{p+1}$ for all $p = 1,\ldots,i_0$. First, we prove that $\gamma \in \ker \delta_{i_0-1}$. In order to make sense that $\delta_{i_0-1}$ acts on $\gamma$, we need $\gamma \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$. We begin by proving

$$[v_1,\ldots,\hat{v_r},\ldots,v_{i_0+1}] \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$$

for all $r = 1,\ldots,i_0+1$. We consider the following three cases.

- If $r = 1$, then $[\hat{v_1}, v_2,\ldots,v_{i_0+1}]$ belongs to $C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$ because $v_1 = kt$ and $\{1,\ldots,kt\}$ is a facet of $\Delta(I_{n_0,t}^{[k]})$.
- If $r = i_0+1$, then $[v_1,\ldots,v_{i_0},\hat{v}_{i_0+1}] \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$ because $v_{i_0+1} = n_0$ and $\{1,\ldots,(k-1)t\} \cup \{n_0-t+1,\ldots,n_0\}$ is a facet of $\Delta(I_{n_0,t}^{[k]})$.
- Now, let $1 < r < i_0+1$. If $v_r = kt+h(t+1)$ for some $h \in \{1,\ldots,j-1\}$. Then

  $$v_{r-1} = kt+(h-1)(t+1)+1 = v_r - t,$$

  and $\{v_{r-1}+1,\ldots,v_r\}$ is a path on $t$ vertices. Therefore $[v_1,\ldots,\hat{v_r},\ldots,v_{i_0+1}] \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$ because $\{1,\ldots,(k-1)t\} \cup \{v_{r-1}+1,\ldots,v_r\}$ is a facet of $\Delta(I_{n_0,t}^{[k]})$.

  The case when $v_r = kt+h(t+1)+1$, for some $h \in \{0,\ldots,j-1\}$, can be argued in a similar way. In fact, in this case we have $v_{r+1} = kt + (h+1)(t+1) = v_r + t$. Then $[v_1, \ldots, \hat{v_r}, \ldots, v_{i_0+1}] \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$ since $\{1, \ldots, (k-1)t\} \cup \{v_r, \ldots, v_{r+1}-1\}$ is a facet of $\Delta(I_{n_0,t}^{[k]})$.

Therefore $\gamma \in C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$. Moreover,

$$\delta_{i_0-1}(\gamma) = \sum_{r=1}^{i_0+1} (-1)^{r+1} \delta_{i_0-1}([v_1, \ldots, \hat{v_r}, \ldots, v_{i_0+1}]) =$$

$$= \sum_{r=1,\ldots,i_0+1} (-1)^{r+1} \left( \sum_{\substack{s=1,\ldots,i_0 \\ s \neq r}} (-1)^{s+1} [v_1, \ldots, \hat{v_r}, \ldots, \hat{v_s} \ldots, v_{i_0+1}] \right).$$

It is easy to check that, for fixed $r$ and $s$ the coefficient of $[v_1, \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_{i_0+1}]$ is $(-1)^{r+s+2} - (-1)^{r+s+1} = 0$. Hence $\gamma \in \ker \delta_{i_0-1}$.

We claim that $\gamma$ does not belong to $\operatorname{Im} \delta_{i_0}$. Indeed, if this were the case, we would have $\tilde{H}_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c; K) \neq 0$. Consequently, this would imply that $\beta_{i_0,n_0}(I_{n_0,t}^{[k]}) \neq 0$, leading us to the inequality $(k-1)t + (t-1)\nu_1(\Gamma_{n_0-(k-1)t,t}) \leq \operatorname{reg}(I_{n_0,t}^{[k]})$, as required. We now proceed to prove that $\gamma$ is not in $\operatorname{Im} \delta_{i_0}$, that is, there does not exist any element in $C_{i_0}((\Delta(I_{n_0,t}^{[k]}))^c)$ whose image under the boundary map $\delta_{i_0}$ is $\gamma$. We begin by noting that $[v_1, \ldots, v_{i_0+1}] \notin C_{i_0}((\Delta(I_{n_0,t}^{[k]}))^c)$ since, by the definition of $Q$, there does not exist a $k$-matching $S = \{S_1, \ldots, S_k\}$ in the induced subgraph of $P_{n_0}$ on $\{1, \ldots, n_0\} \setminus Q$ such that $\cup_{i=1}^k S_i$ is a facet of $\Delta(I_{n_0,t}^{[k]})$. Next, we define $\mathcal{B}_{i_0-1}$ as the basis of $C_{i_0-1}((\Delta(I_{n_0,t}^{[k]}))^c)$, consisting of all oriented $(i_0-1)$-dimensional faces of $(\Delta(I_{n_0,t}^{[k]}))^c$. Similarly, $\mathcal{B}_{i_0}$ represents the basis of $C_{i_0}((\Delta(I_{n_0,t}^{[k]}))^c)$. Let $A$ be the transformation matrix of $\delta_{i_0}$. Recall that each column of $A$ represents the coordinate vector of an element in $\mathcal{B}_{i_0}$ expressed in terms of the basis $\mathcal{B}_{i_0-1}$, consequently the entries of $A$ are either $1, -1$ and $0$. Furthermore, let $\mathbf{x}$ represent the column vector of the components of an element in $(\Delta(I_{n_0,t}^{[k]}))^c$ with respect to $\mathcal{B}_{i_0}$. With this notation, we can write $\delta_{i_0}(\mathbf{x}) = A\mathbf{x}$. Let $\mathbf{b}_\gamma$ be the column vector of the components of $\gamma$ with respect to $\mathcal{B}_{i_0-1}$. We aim to show that the linear system $A\mathbf{x} = \mathbf{b}_\gamma$ has no solutions. In other words, by Rouché-Capelli Theorem, the ranks of the coefficient matrix $A$ and the augmented matrix $(A|\mathbf{b}_\gamma)$ are different. Let $R$ be the row of $(A|\mathbf{b}_\gamma)$ corresponding to $[v_2, \ldots, v_{i_0+1}]$. The row $R$ has exactly $(t-1)\frac{i_0}{2} + 1$ non-zero entries; specifically, in $R$ there is:

- $1$ in the column corresponding to $\mathbf{b}_\gamma$;
- $-1$ in the column, denoted by $C_{j,a}$, which corresponds to the element of $\mathcal{B}_{i_0}$ of the form $[v_2, \ldots, v_j, \ldots, a, \ldots, v_{j+1}, \ldots, v_{i_0+1}]$, for all even $j$ in $\{2, \ldots, i_0\}$ and

for all $a \in \mathbb{N}$ with $v_j < a < v_{j+1}$;

- 0 in the remaining columns.

For example, if we consider the second squarefree power of the 3-path ideal of the path graph $P_{14}$, we have $Q = \{6,7,10,11,14\}$ and $\gamma = [7,10,11,14] - [6,10,11,14] - [6,7,11,14] + [6,7,10,14] - [6,7,10,11]$. In the augmented matrix $(A|\mathbf{b}_\gamma)$ of $\delta_4$, the five non-zero entries of $R$ are given in Table 6.1, specifically: 1 appears in the column corresponding to $\mathbf{b}_\gamma$, and $-1$ appears in the columns corresponding to the elements $[7,8,10,11,14]$, $[7,9,10,11,14]$, $[7,10,11,12,14]$, and $[7,10,11,13,14]$, denoted by $C_{2,8}$, $C_{2,9}$, $C_{4,12}$, and $C_{4,13}$, respectively. Moreover, 0 appears in the remaining entries of $R$, which are not illustrated due to the large size of the matrix.

Table 6.1 The non-zero entries of $R$ in $(A|\mathbf{b}_\gamma)$.

| | 7 8 10 11 14 | 7 9 10 11 14 | 7 10 11 12 14 | 7 10 11 13 14 | $\mathbf{b}_\gamma$ |
|---|---|---|---|---|---|
| 7 10 11 14 | -1 | -1 | -1 | -1 | 1 |

To prove our claim, it is enough to show that we can perform elementary row operations within the augmented matrix $(A|\mathbf{b}_\gamma)$ to reduce $R$ to a row with zero entries everywhere except for a 1 in the column corresponding to $\mathbf{b}_\gamma$. This transformation can be achieved by utilizing only certain specific rows of $(A|\mathbf{b}_\gamma)$, which will be delineated in the following description. For all $s \in [i_0/2]$, let $R(j_1,\dots,j_s;a_1,\dots,a_s)$ be the row of $(A|\mathbf{b}_\gamma)$ corresponding to the element of $\mathcal{B}_{i_0}$ of the form

$$[v_2,\dots,v_{j_1},\dots,a_1,\dots,\widehat{v_{j_1+1}},\dots,v_{j_s-1},v_{j_s},\dots,a_s,\dots,\widehat{v_{j_s+1}},\dots,v_{i_0+1}],$$

for all even $j_1,\dots,j_s$ in $\{2,\dots,i_0\}$ with $j_1 < \cdots < j_s$ and $a_1,\dots,a_s \in \mathbb{N}$ with $v_{j_h} < a_h < v_{j_h+1}$ for each $h \in [s]$. Hence, the following hold within $R(j_1,\dots,j_s;a_1,\dots,a_s)$:

(1) there is 0 in the column corresponding to $\mathbf{b}_\gamma$;

(2) for each $\delta \in (\{v_2, v_2+1,\dots,v_{i_0+1}\} \setminus Q) \cup \{v_{j_1+1},\dots,v_{j_s+1}\}$, let $C(j_1,\dots,j_s;a_1,\dots,a_s;$
$\delta)$ be the column of $(A|\mathbf{b}_\gamma)$ corresponding to the element of $\mathcal{B}_{i_0}$ of the form

$$[v_2,\dots,\delta,\dots,v_{j_1},\dots,a_1,\dots,\widehat{v_{j_1+1}},v_{j_1+2},\dots,v_{j_s-1},v_{j_s},\dots,a_s,\dots,\widehat{v_{j_s+1}},\dots,v_{i_0+1}].$$

The following value appears in $C(j_1,\dots,j_s;a_1,\dots,a_s;\delta)$:

- $-1$, if $v_{j_h+2} < \delta < a_{h+1}$ for some $h = 0,\dots,s$ (where $j_0 := 0$ and $a_{s+1} := v_{i_0+1}$);
- 1, if $a_h < \delta \le v_{j_h+1}$ for some $h = 1,\dots,s$;

(3) All other entries are zero.

For example, in Table 6.2, we present the relevant rows and columns utilized in the reduction of $R$ when considering the second squarefree power of the 3-path ideal of $P_{14}$. Given that $i_0 = 4$, it follows that $s \in \{1,2\}$; specifically, the first four rows correspond to $s = 1$, while the last four rows to $s = 2$.

Table 6.2 Rows used for the elementary operations on $R$.

| | 7 9 11 12 13 | 7 8 9 11 13 | 7 8 9 11 12 | 7 8 11 12 13 | 7 8 11 12 14 | 7 8 11 13 14 | 7 8 9 11 14 | 7 9 11 12 14 | 7 9 11 13 14 | 7 9 11 12 13 |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 9 11 12 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 7 8 11 13 | 0 | 1 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 8 11 12 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 7 9 11 13 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 |
| | | | | | | | | | | |
| 7 8 11 14 | 0 | 0 | 0 | 0 | -1 | -1 | 1 | 0 | 0 | 0 |
| 7 9 11 14 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | 0 |
| 7 10 11 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 10 11 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | 7 8 9 11 13 | 7 8 10 11 12 | 7 9 10 11 12 | 7 8 10 11 13 | 7 9 10 11 13 | 7 10 11 12 13 | 7 8 10 11 14 | 7 9 10 11 14 | 7 10 11 12 14 | 7 10 11 13 14 | $\mathbf{b}_\gamma$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 9 11 12 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 8 11 13 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 8 11 12 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 9 11 13 | -1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | | | | | | | |
| 7 8 11 14 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 9 11 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 10 11 12 | 0 | -1 | -1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 7 10 11 13 | 0 | 0 | 0 | -1 | -1 | -1 | 0 | 0 | 0 | 1 | 0 |

Our aim is to prove that prove that we can reduce $R$ to a row with zero entries everywhere except for a 1 in the column corresponding to $\mathbf{b}_\gamma$. This reduction is achieved by successively adding to $R$ the rows $R(j_1,\ldots,j_s;a_1,\ldots,a_s)$, for $s = 1,\ldots,i_0/2$, following a stepwise process.

For example, the reader can easily verify that the row $R$ in Table 6.1 can be reduced to the desired form by summing the rows from Table 6.2 to $R$. This process involves first adding the initial four rows, followed by the remaining ones, so that we cancel out 1 and $-1$ entries in each column to obtain zeros. This example illustrates the key steps of the reduction process. We now present the general procedure. Throughout, we adopt the notation $M(U,V)$ to denote the entry in row $U$ and column $V$ of a matrix $M$.

**Step 1:** For simplicity, set $M = (A|\mathbf{b}_\gamma)$. In $M$, we replace $R$ by $R + \sum R(j_1;a_1)$, where the sum ranges over all even $j_1 \in \{2,\ldots,i_0\}$ and $a_1 \in \mathbb{N}$ such that $v_{j_1} < a_1 < v_{j_1+1}$. This results in the following:

(1) The $-1$ at $M(R,C_{j_1,a_1})$ is reduced to zero by adding the 1 at $M(R(j_1;a_1),C(j_1;a_1;\delta))$, where $\delta = v_{j_1+1}$ (note that $C(j_1;a_1;\delta)$ and $C_{j_1,a_1}$ correspond to the same column in $M$). In our example, the $-1$ in $C_{4,12}$ reduces to 0 by adding 1 in $M(R(4;12),(4;12;14))$.

(2) For every $\delta \in (\{v_2, v_2+1,\ldots,v_{i_0+1}\} \setminus Q)$, the following hold:

(a) If $a_1 < \delta < v_{j_1+1}$, the entry at $M(R, C_{j_1,a_1})$ is zero, because the 1 at $M(R(j_1;a_1), C(j_1;a_1;\delta))$ cancels out the $-1$ at $M(R(j_1,\delta), C(j_1;a_1;\delta))$. For example, 1 in $M(R(4;12), C(4;12;13))$ cancels out $-1$ in $M(R(4;13), C(4;12;13))$.

(b) If $\delta < a_1$ or $v_{j_1+1} < \delta < v_{i_0+1}$, the entry at $M(R, C(j_1;a_1;\delta))$ is $-1$. For example, see $M(R(4;12), C(4;12;9))$ and $M(R(4;12), C(4;12;8))$.

We denote the resulting row and the resulting matrix, after applying these elementary operations, by $R_1$ and $M_1$ respectively.

**Step 2:** In the matrix $M_1$, we replace $R_1$ by $R_1 + \sum R(j_1, j_2; a_1, a_2)$, where the sum ranges over all even $j_1, j_2 \in \{2, \ldots, i_0\}$ with $j_1 < j_2$ and $a_1, a_2 \in \mathbb{N}$ such that $v_{j_1} < a_1 < v_{j_1+1}$ and $v_{j_2} < a_2 < v_{j_2+1}$. We obtain the following result:

(1) From Step 1 the entry at $M_1(R, C(j_1;a_1;\delta))$ is $-1$, if $\delta < a_1$ (similarly, if $v_{j_1+1} < \delta < v_{i_0+1}$). Thus, there exists an $h \in \{0, \ldots, s\}$ such that $v_{j_h} < \delta < v_{j_{h+1}}$, allowing us to cancel $-1$ by adding 1 from $M_1(R(j_h, j_1; \delta, a_1), C(j_h, j_1; \delta, a_1; v_{j_h+1}))$. For example, $-1$ at $M_1(R(4;12), C(4;12;9))$ is canceled by adding 1 from $M_1(R(2,4;9,12), C(2,4;9,12;10))$.

(2) For every $\delta \in \{v_2, v_2+1, \ldots, v_{i_0+1}\} \setminus Q$, the following hold:

(a) If $a_1 < \delta < v_{j_1+1}$ (or $a_2 < \delta < v_{j_2+1}$), the entry at $M_1(R, C(j_1, j_2; a_1, a_2; \delta))$ becomes zero, since the 1 at $M_1(R(j_1, j_2; a_1, a_2), C(j_1, j_2; a_1, a_2; \delta))$ cancels the $-1$ at $M_1(R(j_1, j_2; \delta, a_2), C(j_1, j_2; \delta, a_2; a_1))$. In our example, $-1$ at $M_1(R(2,4;9,12), C(2,4;9,12;8))$ can be reduced to zero by summing with 1 at $M_1(R(2,4;8,12), C(2,4;8,12;9))$.

(b) If $\delta < a_1$ (or $v_{j_1+2} < \delta < a_2$ or $v_{j_2+2} < \delta < v_{i_0+1}$), the entry at $M_1(R, C(j_1, j_2; a_1, a_2; \delta))$ remains $-1$.

After this step, we denote the updated row and matrix by $R_2$ and $M_2$ respectively.

**Step $s$ (with $2 < s < i_0/2$):** In the matrix $M_{s-1}$ obtained at the $(s-1)$-th step, we replace the row $R_{s-1}$ by $R_{s-1} + \sum R(j_1, \ldots, j_s; a_1, \ldots, a_s)$, where the sum ranges over all even $j_1, \ldots, j_s \in \{2, \ldots, i_0\}$ with $j_1 < \cdots < j_s$ and $a_1, \ldots, a_s \in \mathbb{N}$ such that $v_{j_h} < a_h < v_{j_h+1}$ for $h \in [s]$. As observed in Step 2, the only non-zero entries in $R_s$ are the $-1$ located at $M_s(R_s, C(j_1, \ldots, j_s; a_1, \ldots, a_s; \delta))$, where either $\delta < a_1$, or $v_{j_h+2} < \delta < a_h$ for some $h \in [s-1]$, or $v_{j_s+2} < \delta < v_{i_0+1}$.

Now, continue the procedure described above until the final step, which is given for $s = i_0/2$. In our example, the process concludes at the second step since $s = 2$, and no additional $-1$ values arise from this step, meaning that the desired result has been achieved.

**Step $i_0/2$:** For simplicity, let $i_0/2 = \ell$. In the matrix $M_{\ell-1}$, we replace $R_{\ell-1}$ by $R_{\ell-1} + \sum R(j_1,\dots,j_\ell;a_1,\dots,a_\ell)$, where the sum ranges over all even $j_1,\dots,j_\ell \in \{2,\dots,i_0\}$ with $j_1 < \cdots < j_\ell$ and $a_1,\dots,a_\ell \in \mathbb{N}$ such that $v_{j_h} < a_h < v_{j_h+1}$ for $h \in [\ell]$. The only significant case that remains to be analyzed is when $v_{j_{h-1}+2} < \delta < a_h$ for some $h \in [\ell]$ but, in this case, it suffices to proceed analogously to the argument presented in (1) of Step 2.

This guarantees that summing to $R$ the rows $R(j_1,\dots,j_s;a_1,\dots,a_s)$, for $s = 1,\dots,i_0/2$, we get a row with zero entries everywhere except for a single 1 in the column corresponding to $\mathbf{b}_\gamma$. As a consequence, the coefficient matrix $A$ and the augmented matrix $(A|\mathbf{b}_\gamma)$ have different ranks. This completes the proof of our claim.

Case (2): Now, suppose $kt + j(t+1) < n \le kt + j(t+1) + t$, for some $j$. Let $\Delta(I_{n,t}^{[k]})$ be the simplicial complex whose facet ideal is $I_{n,t}^{[k]}$. Observe that $\nu_1(\Gamma_{n-(k-1)t,t}) = \nu_1(\Gamma_{n_0-(k-1)t,t})$. It is sufficient to show that $\beta_{i_0,n_0}(I_{n,t}^{[k]}) \neq 0$, where $i_0$ and $n_0$ are defined in Case 1. From (Alilooee & Faridi, 2015, Theorem 2.8) we know that

$$\beta_{i_0,n_0}(I_{n,t}^{[k]}) = \sum_{\substack{\Gamma \subseteq \Delta(I_{n,t}^{[k]}) \\ |V(\Gamma)| = n_0}} \dim_K \tilde{H}_{i_0-1}(\Gamma_{V(\Gamma)}^c),$$

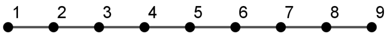
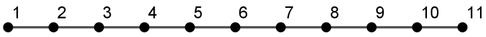
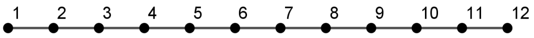
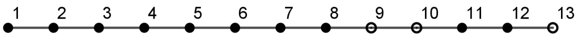
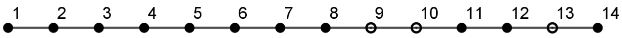
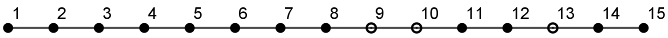
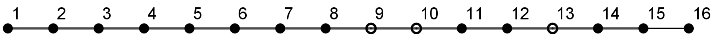
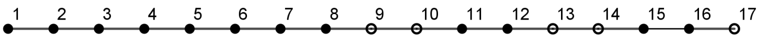
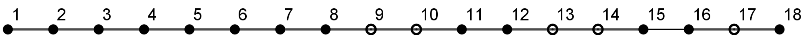
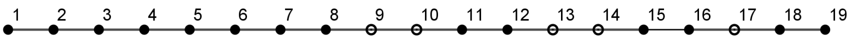
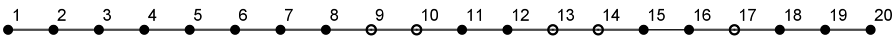
where the sum is taken over the induced subcollections $\Gamma$ of $\Delta(I_{n,t}^{[k]})$ which have $n_0$ vertices. Take $\Gamma = \Delta(I_{n_0,t}^{[k]})$ and observe that $\Delta(I_{n_0,t}^{[k]})$, defined as in Case (1), is an induced subcollection of $\Delta(I_{n,t}^{[k]})$ having $n_0$ vertices. From Case (1), we know that $\dim_K \tilde{H}_{i_0-1}(\Gamma_{V(\Gamma)}^c) \neq 0$, so $\beta_{i_0,n_0}(I_{n,t}^{[k]}) \neq 0$, which implies $(k-1)t + (t-1)\nu_1(\Gamma_{n-(k-1)t,t}) \le \operatorname{reg}(I_{n,t}^{[k]})$.

In conclusion, we get the desired lower bound. □

We illustrate the proof of above theorem in the following example.
**Example 6.0.8.** Referring to the notation of the proof of Theorem 6.0.7, in Table 6.3 we display the regularity of the third squarefree power of the 3-path ideal of $P_n$ for $9 \le n \le 20$. The elements of set $A$ are displayed by the hollow circles.

For instance, take $n = 17$. Since $17 = kt + 2(t+1)$ for $k = t = 3$, we are in Case (1) of the proof of above theorem. Here we have $i_0 = 4$, and $A = \{9, 10, 13, 14, 17\}$, and $\gamma = [10,13,14,17] - [9,13,14,17] + [9,10,14,17] - [9,10,13,17] + [9,10,13,14]$. Moreover,

Table 6.3 The regularity of the third squarefree power of the 3-path ideal of $P_n$ for $9 \leq n \leq 20$.

| $n$ | $\operatorname{reg}(I_{n,3}^{[3]})$ | |
|---|---|---|
| 9 | 9 |   |
| 10 | 9 |   |
| 11 | 9 |   |
| 12 | 9 |   |
| 13 | 11 |   |
| 14 | 11 |   |
| 15 | 11 |   |
| 16 | 11 |   |
| 17 | 13 |   |
| 18 | 13 |   |
| 19 | 13 |   |
| 20 | 13 |   |

- $[10,13,14,17] \in C_3((\Delta(I_{17,3}^{[3]}))^c)$ because $\{1,\ldots,9\}$ is a facet of $\Delta(I_{17,3}^{[3]})$;
- $[9,13,14,17] \in C_3((\Delta(I_{17,3}^{[3]}))^c)$ because $\{1,\ldots,6\} \cup \{10,11,12\}$ is a facet of $\Delta(I_{17,3}^{[3]})$;
- $[9,10,14,17] \in C_3((\Delta(I_{17,3}^{[3]}))^c)$ because $\{1,\ldots,6\} \cup \{11,12,13\}$ is a facet of $\Delta(I_{17,3}^{[3]})$;
- $[9,10,13,17] \in C_3((\Delta(I_{17,3}^{[3]}))^c)$ because $\{1,\ldots,6\} \cup \{14,15,16\}$ is a facet of $\Delta(I_{17,3}^{[3]})$;
- $[9,10,13,14] \in C_3((\Delta(I_{17,3}^{[3]}))^c)$ because $\{1,\ldots,6\} \cup \{15,16,17\}$ is a facet of $\Delta(I_{17,3}^{[3]})$.

It is easy to see that $\delta_3(\gamma) = 0$, so $\gamma \in \ker \delta_3$. Moreover, $[9,10,13,14,17]$ does not belong to $C_4((\Delta(I_{17,3}^{[3]}))^c)$ because there is no 3-matching in the induced subgraph of $P_{17}$ on $\{1,\ldots,8\} \cup \{11,12\} \cup \{15,16\}$. Finally, it can be verified by using `Macaulay2` ((Grayson & Stillman, Grayson & Stillman), (Hersey, Smith & Zotine, 2023)) that $\gamma$ is not in $\operatorname{Im} \delta_4$, since the rank of the representative matrix of $\delta_4$ differs from that of the corresponding augmented matrix. Therefore, $\tilde{H}_3((\Delta(I_{17,3}^{[3]}))^c; K) \neq 0$ and so $\beta_{4,17}(I_{17,3}^{[3]}) \neq 0$.

Now, let $n = 19$. We know that

$$\beta_{4,17}(I_{19,3}^{[3]}) = \sum_{\substack{\Gamma \subseteq \Delta(I_{19,3}^{[3]}) \\ |V(\Gamma)|=17}} \dim_K \tilde{H}_3(\Gamma^c_{V(\Gamma)}),$$

where the sum is taken over the induced subcomplexes $\Gamma$ of $\Delta(I_{19,3}^{[3]})$ which have 17 vertices. Observe that $\Delta(I_{17,3}^{[3]})$ is an induced subcomplex of $\Delta(I_{19,3}^{[3]})$ on $\{1,\ldots,17\}$ vertices and $\tilde{H}_3((\Delta(I_{17,3}^{[3]}))^c;K) \neq 0$ as discussed before. Therefore $\beta_{4,17}(I_{19,3}^{[3]}) \neq 0$.

Combining Theorem 6.0.4 and Theorem 6.0.7, we obtain the following nice combinatorial description of the regularity of squarefree powers of $t$-path ideals of path graphs.

**Theorem 6.0.9.** *Let $\Gamma_{n,t}$ be the $t$-path simplicial tree of a path graph $P_n$ and $I_{n,t} = I(\Gamma_{n,t})$. Then for any $1 \leq k+1 \leq \nu(\Gamma_{n,t})$, we have*

$$\operatorname{reg}\left(\frac{R}{I_{n,t}^{[k+1]}}\right) = kt + (t-1)\nu_1(\Gamma_{n-kt,t}) = kt + \operatorname{reg}\left(\frac{R}{I_{n-kt,t}}\right).$$

# 7. DIRECTIONS FOR FUTURE RESEARCHES

In this section we propose some open questions, which can inspire new work related to squarefree powers of monomial ideals.

1) We know that $I(\Delta)^{[\nu(\Delta)]}$ has a linear resolution if $\Delta$ is a simplicial tree with the intersection property (see Theorem 4.0.3) or $\Delta$ is a $t$-path simplicial tree of a broom graph (see Theorem 4.1.5). However, given an arbitrary simplicial tree $\Delta$, Example 4.1.1 shows that $I(\Delta)^{[\nu(\Delta)]}$ need not have a linear resolution. It is interesting to characterize those simplicial trees $\Delta$ for which $I(\Delta)^{[\nu(\Delta)]}$ has a linear resolution.

2) Let $\Delta$ be a pure simplicial tree. From Theorem 5.0.6, we know that if $I(\Delta)^{[k]}$ is linearly related then $I(\Delta)^{[k+1]}$ is also linearly related. Does this statement hold for the linearity of the resolution, that is, if $I(\Delta)^{[k]}$ has a linear resolution then does $I(\Delta)^{[k+1]}$ has also a linear resolution?

3) In Theorem 6.0.9, we provide a closed formula for $\operatorname{reg}(R/I_{n,t}^{[k]})$, where $I_{n,t}$ is the $t$-path ideal of the path graph $P_n$. In general, it seems very difficult to derive such a formula for the squarefree powers of an arbitrary simplicial tree. It could be interesting to establish at least an upper or lower bound for $\operatorname{reg}(R/I(\Delta))$, where $\Delta$ is a simplicial tree.

In a more general context, a lower bound for certain squarefree powers can be obtained as an easy generalization of (Erey et al., 2022, Theorem 2.1), whose proof is provided below for the sake of completeness.

**Proposition 7.0.1.** *Let $\Delta$ be a pure simplicial complex of dimension $t-1$. Then*

$$k-1+(t-1)\nu_1(\Delta) \le \operatorname{reg}\left(\frac{R}{I(\Delta)^{[k]}}\right)$$

*for all $1 \le k \le \nu_1(\Delta)$.*

*Proof.* Denote by $r$ the induced matching number of $\Delta$, for simplicity. Let $\Delta'$ be a subcomplex of $\Delta$ with $r$ facets $\{F_1, \dots, F_r\}$ which is an induced matching of $\Delta$. Since

$\mathrm{reg}(I(\Delta')^{[k]}) \leq \mathrm{reg}(I(\Delta)^{[k]})$ from (Herzog et al., 2004, Lemma 4.4), it is enough to prove that $k+(t-1)r \leq \mathrm{reg}(I(\Delta')^{[k]})$, in particular that $\beta_{r-k,rt}(I(\Delta')) \neq 0$. Consider the ideal $J = (z_1, \ldots, z_r)$ in a new polynomial ring $R = K[z_1, \ldots, z_r]$. Since $J^{[k]}$ is a squarefree strongly stable ideal in $R$, we have $\beta_{r-k,r}(J^{[k]}) \neq 0$ from (Herzog & Hibi, 2011, Theorem 7.4.1). Now, let $f_j = \prod_{i \in F_j} x_i$. Since $\Delta$ is pure of dimension $t-1$ and $\{F_1, \ldots, F_r\}$ is an induced matching of $\Delta$, then $f_1, \ldots, f_r$ is a regular sequence in $S = K[x_i : i \in \cup_{j=1}^{r} F_j]$ and $\deg(f_i) = t$ for all $i = 1, \ldots, r$. Set $I = I(\Delta') = (f_1, \ldots, f_r)$ and the map $\phi : R \to S$ with $\phi(z_j) = f_j$ for all $j = 1, \ldots, r$. As explained in (Erey et al., 2022, Theorem 2.1), the $i$-th free module in the minimal free resolution of $I^{[k]}$ is given by $S(-tk-ti)^{\beta_i(I^{[k]})}$ and $\beta_{i,j}(J^{[k]}) = \beta_{i,tj}(I^{[k]})$. Then $\beta_{r-k,rt}(I) \neq 0$ because $\beta_{r-k,r}(J^{[k]}) \neq 0$. $\square$

We recall that, for a simple graph $G$, it is expected that

$$k + \nu_1(G) \leq \mathrm{reg}(I(G)^{[k]}) \leq k + \nu(G)$$

for $1 \leq k \leq \nu(G)$, as discussed in (Erey et al., 2022, page 3). Based on the results of this thesis, we present the following conjecture.

**Conjecture 7.0.2.** *Let $\Delta$ be a simplicial tree of dimension $t-1$. Then*

$$k - 1 + (t-1)\nu_1(\Delta) \leq \mathrm{reg}\left(\frac{R}{I(\Delta)^{[k]}}\right) \leq k - 1 + (t-1)\nu(\Delta)$$

*for all $1 \leq k \leq \nu(\Delta)$.*

4) In (Alilooee & Faridi, 2018), a combinatorial description of all graded Betti numbers of $t$-path ideals of path graphs and cycle graphs is given. It would be of interest to give such a description for the squarefree powers of $t$-path ideals of path graphs and cycle graphs.

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