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Research Article

Edwards' Theorem and matrix-valued functions

Umutcan Erdur*, Nihat Gökhan Göğüş

Faculty of Engineering and Natural Sciences, Sabanci University, Istanbul, Turkey

Abstract

We extend several notions such as semi-continuity and Jensen measures for matrix-valued functions. For that purpose, we introduce Γ-order on noncommutative matrix spaces. Afterward, we generalize the Edwards' Theorem for a noncommutative matrix space by exploiting properties of Γ-order given on the matrix space which we consider.

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1. Introduction

Edwards' theorem provides a duality between a positive cone of upper semi-continuous functions on a compact space and the set of Jensen measures for this cone. This theorem found many applications in functional analysis, uniform algebras, pluripotential theory and optimization. Additionally, Perron solution of Dirichlet problem heavily relies on the investigation of upper envelopes. The purpose of this paper is to prove an analogue of Edwards' Theorem for matrix-valued functions. In order to achieve such result, we need a class $\mathcal{F}$ of matrix-valued functions that is a cone of functions and also if $\{u_\alpha\}$ is a family of functions in $\mathcal{F}$, we want the supremum $\sup_\alpha u_\alpha$ to be a well-defined function that belongs to the cone $\mathcal{F}$.

The first task is to choose a suitable order on a given family of matrices. With the well-known usual order on the class $H(n)$ of $n \times n$ Hermitian matrices, defined by positive definiteness, $H(n)$ becomes a partially ordered vector space but not a vector lattice. Sherman [10] proved that a subalgebra $\mathcal{A}$ of a C^*-algebra of self-adjoint operators on a complex Hilbert space forms a lattice if and only if $\mathcal{A}$ is commutative, and as a special case, a subalgebra $\mathcal{A}$ of $H(n)$ is a lattice if and only if $\mathcal{A}$ is commutative. For that reason, we are in the need of commutativity condition on a subalgebra of $H(n)$ in order to have a lattice structure. Another order, so called the spectral order on $H(n)$ (indeed, on the class of self-adjoint operators on a Hilbert space) was defined by Olson in [8]. It was proved in [8] that $H(n)$ becomes a conditionally complete lattice under the spectral order but it is not a vector lattice. It was proved again in [8] that if $\mathcal{Y}$ is a commutative subalgebra of $H(n)$, the usual order and the spectral order on $\mathcal{Y}$ are equivalent to each other, and furthermore $\mathcal{Y}$ with the usual order becomes a conditionally complete vector lattice. Therefore, neither of these partial orders will be very useful in noncommutative settings.

*Corresponding Author.
Email addresses: umutcanerdur@sabanciuniv.edu, nggogus@sabanciuniv.edu
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In this paper, we introduce the notion of Γ-order that is given via a specific map Γ and our notion of Γ-order overcomes this difficulty. For further details on vector lattices, we refer to Aliprantis and Burkinshaw [1].

Another novelty is the introduction of the notion of Jensen measures, which are by definition operator valued measures in our settings, and the proof of an analog of Edwards' theorem for matrix valued functions. In scalar case, Jensen measures attracted the attention of quite a number of mathematicians to explain different phenomena of pluripotential theory.

Coifman and Semmes [2] considered subharmonic norm-valued functions N_z on a finite dimensional complex Banach space V. In particular, they paid attention to the following the matrix-valued Dirichlet problem

$$\sum_{i=1}^{d} \bar{\partial}_i (P^{-1}\partial_i P) = 0, P = \omega \text{ on } \partial\Omega, \tag{1.1}$$

where ω is a positive definite $n \times n$ matrix-valued function from the class $C(\partial\Omega)$.

Lempert [7] studied the form $R^P = \bar{\partial}(P^{-1}\partial P)$ on some open subset Ω of $\mathbb{C}$ where $P : \Omega \to EndV$ is a C^∞ map attaining values in positive invertible operators. Herein, V is a finite or an infinite dimensional separable Hilbert space, $EndV$ is the space of continuous linear maps on V to itself. Lempert [7] provides further information about the solution of the mentioned Dirichlet problem.

In this paper, $M(n,m)$ denotes the class of $n \times m$ matrices with complex entries. We will use the notation $M(n)$ instead of $M(n,n)$. The real subspace of Hermitian matrices in $M(n)$ is denoted by $H(n)$. We denote by $\|\cdot\|$ the operator norm. For $A \in H(n)$, $A \geq 0$ means A is positive definite that is $\langle Ax, x\rangle \geq 0$ for every $x \in \mathbb{C}^n$. If A, $B \in H(n)$, the usual order $A \leq B$ means $B - A \geq 0$. We denote by $0_{n\times m}$ and I_n the $n \times m$ zero matrix and the $n \times n$ identity matrix, respectively.

2. Main results

The main result of this paper is the following which is a generalization of Edwards' Theorem to noncommutative matrix spaces:

Theorem 2.1. *Let (Ω, ρ) be a compact metric space and $\mathcal{F}$ be a cone of $\mathcal{X}$-valued, Γ-order bounded, Γ-upper semi-continuous functions on Ω. Suppose that each nontrivial cone $\{\lambda_j : U^*(\Gamma u)U = diag(\lambda_1, \lambda_2, \ldots, \lambda_N), u \in \mathcal{F}\}, j = 1, 2, \ldots, N,$ contains all real (scalar) constant functions. Let $\varphi : \Omega \to \mathcal{X}$ be a Γ-order bounded, Γ-lower semi-continuous function on Ω. Then the $\mathcal{X}$-valued upper envelope*

$$S^{\mathcal{F}}\varphi(z) = \sup\{u(z) : u \in \mathcal{F},\ u \leq_{\mathcal{F}} \varphi \text{ on } \Omega\}$$

coincides on Ω with the $\mathcal{X}$-valued lower envelope

$$I^{\mathcal{F}}\varphi(z) = \inf\left\{\int_{\Omega} \varphi d\mu : \mu \in \mathcal{J}_z^{\mathcal{F}}\right\}.$$

We will give the proof of Theorem 2.1 in section 4.

A real vector space $\mathcal{V}$ is said to be an ordered vector space if it is equipped with a partial order relation $'' \leq''$ possessing the following properties:

(i) If $x \leq y$, then $x + z \leq y + z$, for every $x, y, z \in \mathcal{V}$,

(ii) If $x \leq y$, then $\alpha x \leq \alpha y$, for every $x, y \in \mathcal{V}$ and every $\alpha \in \mathbb{R}^+ = [0, +\infty)$.

A vector lattice $\mathcal{V}$ is an ordered vector space such that for any given vectors $x, y \in \mathcal{V}$, both the supremum and the infimum of the set $\{x, y\}$ exist in $\mathcal{V}$. A vector lattice is said to be Dedekind complete if both of the conditions that "every subset with an upper bound in $\mathcal{V}$ has a supremum in the vector lattice", and "every subset with a lower bound in $\mathcal{V}$ has an infimum in the vector lattice" hold.

A nonempty subset $\mathcal{W}$ of a vector space is called a cone if the following conditions hold:

(i) $x+y$ is in $\mathcal{W}$ whenever $x, y \in \mathcal{W}$,

(ii) αx is in $\mathcal{W}$ whenever $x \in \mathcal{W}$ and $\alpha \in \mathbb{R}^+$.

2.1. Gamma order

A collection $\mathcal{Y}$ in $M(N)$ is said to be simultaneously diagonalizable if there exists a unique nonsingular matrix $S \in M(N)$ so that $S^{-1}AS$ is diagonal for every $A \in \mathcal{Y}$ and we say that S simultaneously diagonalizes $\mathcal{Y}$. A family $\mathcal{Y}$ of diagonalizable matrices is simultaneously diagonalizable if and only if $\mathcal{Y}$ is commutative. In addition, if $\mathcal{Y}$ is a commutative family in $H(N)$, there exists a unique (up to a unimodular constant) $N \times N$ unitary matrix U so that U simultaneously diagonalizes $\mathcal{Y}$. Note that a commutative subspace of $H(N)$ is at most N real dimensional.

Let $\mathcal{Y}$ be a commutative subspace of $H(N)$ of real dimension N. In other words, the vector space $\tilde{\mathcal{Y}} = \{U^*AU : A \in \mathcal{Y}\}$ has a vector space basis $\{E_j : j = 1, 2, \ldots, N\}$ where E_j is the $N \times N$ canonical diagonal matrix

$$E_j = diag(0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with 1 as its j-th diagonal entry and U is the unitary matrix that simultaneously diagonalizes $\mathcal{Y}$. The reason for taking such a $\mathcal{Y}$ is clear from the simple observation below:

Lemma 2.1. Let $\mathcal{Y}$ be a commuting real subspace in $H(N)$ with N real dimension so that $\tilde{\mathcal{Y}} = span\{E_j : j = 1, 2, \ldots, N\}$.

(*i*) Let $\{A_\alpha = [a_{ij,\alpha}] : \alpha \in \Lambda\}$ be a collection in $\mathcal{Y}$ such that $A_\alpha \leq A$ for some $A \in \mathcal{Y}$. Then, the matrix $\tilde{A} = \sup_\alpha A_\alpha$ belongs to the family $\mathcal{Y}$.

(*ii*) Let $\{A_\alpha = [a_{ij,\alpha}] : \alpha \in \Lambda\}$ be a collection in $\mathcal{Y}$ such that $A_\alpha \geq A$ for some $A \in \mathcal{Y}$. Then, the matrix $\hat{A} = inf_\alpha A_\alpha$ belongs to the family $\mathcal{Y}$.

Proof. We will prove only (*i*) as (*ii*) is proved analogously. Let us denote $D_\alpha = U^*A_\alpha U = diag(\lambda_{j,\alpha})_{1\leq j\leq N}$, where $\lambda_{j,\alpha}$'s are eigenvalues of A_α. Note that $\sup_\alpha \lambda_{j,\alpha}$, $j = 1, 2, \ldots, N$, are finite real numbers. Then, $D = diag(\sup_\alpha \lambda_{j,\alpha})_{1\leq j\leq N} = \sup_\alpha U^*A_\alpha U \leq U^*AU$ and so $UDU^* \leq A$. Hence, $\tilde{A}$ exists and $\tilde{A} = UDU^* \in \mathcal{Y}$. □

This lemma shows that $\mathcal{Y}$ becomes a Dedekind complete vector lattice under the usual order $\leq$.

Let $\mathcal{X}$ be a nontrivial real subspace of $M(n, m)$ with real dimension N and $\Gamma : \mathcal{X} \to \mathcal{Y}$ a bijective, real linear map. The continuity of Γ then follows since $\mathcal{X}$ is finite dimensional. Let us consider the partial order relation $\leq_\Gamma$ on $\mathcal{X}$ given by

$$A \leq_\Gamma B \Longleftrightarrow \Gamma(A) \leq \Gamma(B),\ \ A, B \in \mathcal{X}.$$

We say that a family $\{A_\alpha\}$ in $\mathcal{X}$ is Γ-order bounded above if there exists some $A \in \mathcal{X}$ so that $A_\alpha \leq_\Gamma A$ for all α. The supremum of any Γ-order bounded above family $\{A_\alpha\}$ in $\mathcal{X}$ belongs to $\mathcal{X}$ itself. Moreover, if the infimum of the family $\{\Gamma A_\alpha\}$ for a given family $\{A_\alpha\}$ in $\mathcal{X}$ exists, we have the relation

$$\inf_\alpha \Gamma A_\alpha = -\sup_\alpha \Gamma(-A_\alpha).$$

and hence, we obtain the following theorem:

Theorem 2.2. $(\mathcal{X}, \leq_\Gamma)$ *is a Dedekind complete vector lattice.*

We will provide several examples for Γ-order in section 5.

3. Matrix-valued functions

In this section, we will introduce upper semi-continuous matrix valued functions and integral of such functions with respect to a given operator valued measure.

3.1. Upper semi-continuity

Let (Ω, ρ) be a compact metric space and let $\mathcal{V}$ be a real vector subspace of $M(n,m)$. We denote by $C(\Omega; \mathcal{V})$ the space of norm continuous functions on Ω with values in $\mathcal{V}$. In particular, $C(\Omega; \mathbb{R})$ is the space of all real valued continuous functions on Ω.

A function $u : \Omega \to \mathcal{Y}$ is called order bounded below if there exists some constant $c \in \mathbb{R}$ such that $u \geq cI_N$ on Ω, order bounded above if there exists some constant $d \in \mathbb{R}$ such that $u \leq dI_N$ on Ω, and order bounded if it is order bounded below and above. We will say that a function $u : \Omega \to \mathcal{X}$ is Γ-order bounded below or above if the function $\Gamma u : \Omega \to \mathcal{Y}$ is order bounded below or above, respectively.

We denote by $B(z_0, r) = \{z \in \Omega : \rho(z, z_0) < r\}$ the open ball centered at z_0 with radius $r > 0$. For a given function $u : \Omega \to \mathcal{Y}$ which is order bounded above, we define the following $\mathcal{Y}$-valued function on Ω for u:

$$M_r(u, z_0) = \sup_{z \in B(z_0, r)} u(z), \quad r > 0.$$

It is clear that $u(z_0) \leq M_r(u, z_0)$ for $r > 0$. We define the upper semi-continuous regularization of u at $z_0 \in \Omega$ as

$$\hat{u}(z_0) = \limsup_{z \to z_0} u(z) = \inf_{r>0} \sup_{z \in B(z_0, r)} u(z).$$

Herein, the infimum and supremum are taken with respect to the usual order on the space $\mathcal{Y}$. By definition, we have the relation $u \leq \hat{u}$ on Ω.

Whenever we say a function/sequence is increasing/decreasing, we mean that it increases/decreases with respect to the given order. By $\nearrow$, we denote the convergence of a function/sequence while increasing with respect to the given order, and by $\searrow$, we denote the convergence of a function/sequence while decreasing with respect to the given order. It quickly follows from their definitions that $M_r(u, z_0) \searrow \hat{u}(z_0)$ as $r \searrow 0^+$.

A function $u : \Omega \to \mathcal{Y}$ is said to be upper semi-continuous at $z_0 \in \Omega$ if $\hat{u}(z_0) = u(z_0)$. We say a function $u : \Omega \to \mathcal{Y}$ is lower semi-continuous at $z_0 \in \Omega$ if $-u$ is upper semi-continuous at z_0 . Also, u is said to be upper semi-continuous (or lower semi-continuous) on Ω if it is upper semi-continuous (lower semi-continuous) at every point $z \in \Omega$. We will denote the class of upper semi-continuous functions from Ω to $\mathcal{Y}$ by $USC(\Omega; \mathcal{Y})$. In addition, if $\mathcal{X}$, $\mathcal{Y}$ and the map Γ are as in section 2, we say that a function $u : \Omega \to \mathcal{X}$ is Γ-upper semi-continuous at a point $z_0 \in \Omega$ if the function $\Gamma u : \Omega \to \mathcal{Y}$ is upper semicontinuous at z_0.

We have a useful observation on upper semi-continuous matrix-valued functions in connection to the classical notion of upper semi-continuity.

Let $u : \Omega \to \mathcal{Y}$ be a function. Consider the diagonal matrix-valued function on Ω

$$U^* u(z) U = diag(\lambda_1(z), \lambda_2(z), \ldots, \lambda_N(z)) \tag{3.1}$$

where $\lambda_1(z), \lambda_2(z), \ldots, \lambda_N(z)$ are eigenvalues of $u(z)$ at $z \in \Omega$. Let $\hat{\lambda}_j$ denote the upper semi-continuous regularization of $\lambda_j, j = 1, 2, \ldots, N$. If u is upper semi-continuous at a point $z \in \Omega$, then

$$\begin{aligned} diag(\hat{\lambda}_1(z), \hat{\lambda}_2(z), \ldots, \hat{\lambda}_N(z)) &= U^* \hat{u}(z) U = U^* u(z) U \\ &= diag(\lambda_1(z), \lambda_2(z), \ldots, \lambda_N(z)). \end{aligned}$$

Hence, each λ_j is upper semi-continuous at $z \in \Omega$. Conversely, if every λ_j in (3.1) is upper semi-continuous at $z \in \Omega$, then u is upper semi-continuous at z.

It is a well known fact that a scalar-valued function is upper semi-continuous if and only if it is the pointwise limit of a decreasing sequence of continuous functions. For a function $u : \Omega \to \mathcal{Y}$, we have

$$U^* u(z) U = diag(\lambda_1(z), \lambda_2(z), \ldots, \lambda_N(z)).$$

If u is upper semi-continuous, i.e. each $\lambda_j : \Omega \to \mathbb{R}$ is upper semi-continuous on Ω, then we can find decreasing sequences $\{\lambda_j^{(k)}\}$ of continuous functions on Ω so that λ_j is pointwise limit of the sequence $\{\lambda_j^{(k)}\}$. It is clear that $\{u_k\}$,

$$u_k = U diag\left(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_N^{(k)}\right) U^*,$$

is a decreasing sequence in $C(\Omega; \mathcal{Y})$. Moreover, $\|u_k(z) - u(z)\| \to 0$ for every $z \in \Omega$ as $k \to \infty$.

In addition, the last relation also implies that if $u : \Omega \to \mathcal{Y}$ is the pointwise limit of a decreasing sequence in $C(\Omega; \mathcal{Y})$, then u is upper semi-continuous on Ω. Thus, we have the following proposition.

Proposition 3.1. *Let (Ω, ρ) be a compact metric space and $u : \Omega \to \mathcal{Y}$ a function that is order bounded above. Then, u is upper semi-continuous on Ω if and only if it is the pointwise limit of a decreasing sequence in $C(\Omega; \mathcal{Y})$.*

Let $\mathcal{F}$ be a cone of upper semi-continuous functions with values in $\mathcal{Y}$. If $u \in \mathcal{F}$, then we can write $U^* u(\cdot) U = diag(\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_N(\cdot))$ and each $\lambda_j(z)$ is the j^{th} eigenvalue of $u(z), z \in \Omega$. One may notice that for some j, the functions λ_j may be identically zero for every $u \in \mathcal{F}$. As a result, for those j, the cones $\mathcal{F}_j = \{\lambda_j : U^* u U = diag(\lambda_1, \lambda_2, \ldots, \lambda_N), u \in \mathcal{F}\}, j = 1, 2, \ldots, N$, consist of only the zero function on Ω. Let us denote the set of those j by $J_{\mathcal{F}}$. Then, whenever $j \notin J_{\mathcal{F}}$, the cone $\mathcal{F}_j$ contains a function other than the zero function on Ω. Throughout the paper, to distinguish a cone $\mathcal{F}_j$ which contains a function other than the zero function on Ω, we alternatively say that $\mathcal{F}_j$ is a nontrivial cone of functions.

3.2. Integration of matrix-valued functions

A countably additive set function μ on the collection Σ of all Borel subsets of Ω with values in a Banach space $\mathcal{V}$ is called a (vector) measure. For given subset A of Ω, we define

$$|\mu|(A) = \sup_{\mathcal{I}} \sum_{i \in \mathcal{I}} \|\mu(A_i)\|$$

where the supremum is taken over all finite families $\{A_i\}_{i \in \mathcal{I}}$ of pairwise disjoint sets from the collection Σ so that $\cup_{i \in \mathcal{I}} A_i = A$. The set function $|\mu|$ is called the variation of μ on A and it is a positive Borel measure on Ω. We say that μ has finite variation if $|\mu|(\Omega)$ is finite. For the rest of the paper, we assume that all measures are of finite variation.

Let $\mathcal{E}$, $\tilde{\mathcal{E}}$ be Banach spaces and let $\mathcal{V}$ be as in the previous paragraph. Let us consider a bilinear mapping $(\nu, x) \mapsto \nu x$ of $\mathcal{V} \times \mathcal{E}$ into $\tilde{\mathcal{E}}$ such that $\|\nu x\| \leq \|\nu\| \|x\|$ holds for every $(\nu, x) \in \mathcal{V} \times \mathcal{E}$. Assume that $\mathcal{E}$ is separable as well. Let $\mu : \Sigma \to \mathcal{V}$ be a measure with finite variation. We immediately see that functions of the class $C(\Omega; \mathcal{E})$ and, more specifically semi-continuous functions are Borel measurable when $\mathcal{E} = \mathcal{X}$ or $\mathcal{Y}$.

A function of the form

$$s(z) = \sum_{i \in \mathcal{I}} \chi_{A_i}(z) x_i$$

is called a simple function where $\mathcal{I}$ is a finite set, $A_i \in \Sigma$ are pairwise disjoint subsets of Ω, χ_{A_i} is the characteristic function of A_i and $x_i \in \mathcal{E}$ for every $i \in \mathcal{I}$. We will call the integral of s with respect to μ over $A \in \Sigma$ the element of $\tilde{\mathcal{E}}$, denoted by $\int_A s d\mu$, given by

$$\int_A s d\mu = \sum_{i \in \mathcal{I}} \mu(A_i \cap A) x_i.$$

We say that a function $f : \Omega \to \mathcal{E}$ is μ-integrable or integrable with respect to μ if there exists a sequence $\{s_n\}$ of simple functions so that

$$\lim_{n\to\infty} \int_\Omega \|s_n - f\| d|\mu| = 0.$$

The integral of f with respect to μ over $A \in \Sigma$, $\int_A f d\mu \in \tilde{\mathcal{E}}$ is then given by the relation

$$\int_A f d\mu = \lim_{n\to\infty} \int_A s_n d\mu.$$

As we deal with measures of finite variation, one can show that a Borel measurable function f is μ-integrable over a set $A \in \Sigma$ if and only if $\|f(\cdot)\|_{\mathcal{E}}$ is $|\mu|$-integrable over A. In such case, we have that

$$\left\|\int_A f d\mu\right\|_{\tilde{\mathcal{E}}} \le \int_A \|f\|_{\mathcal{E}} d|\mu|.$$

Let (Ω, ρ) be a compact metric space. Then, functions of the class $C(\Omega; \mathcal{E})$ are integrable with respect to a measure μ of finite variation with values in $\mathcal{V}$.

We denote by $\mathcal{M}(\Omega; \mathcal{V})$ the real vector space of all $\mathcal{V}$-valued Borel measures given on Σ with finite variation. One can show that the map $\mu \mapsto |\mu|(\Omega)$ is a norm function on $\mathcal{M}(\Omega; \mathcal{V})$. We say that a subset $\mathcal{A}$ of $\mathcal{M}(\Omega; \mathcal{V})$ is bounded if $\sup_{\mu\in\mathcal{A}} |\mu|(\Omega) < +\infty$. Let $B(\mathcal{E}) = \{T : \mathcal{E} \to \mathcal{E} : T \text{ is continuous, linear operator}\}$ and consider the bilinear mapping $(T, x) \to Tx$ of $B(\mathcal{E}) \times \mathcal{E}$ into $\mathcal{E}$. We will pay great attention to the space $\mathcal{M}(\Omega; B(\mathcal{E}))$, the real vector space of all $B(\mathcal{E})$-valued Borel measures given on Σ with finite variation.

Let us consider the space $\mathcal{E} = \mathcal{X}$ or $\mathcal{Y}$ equipped with the partial order $\le_\Gamma$ and $\le$, respectively. We denote by $\mathcal{E}^+$ the cone of positive elements of $\mathcal{E}$. We say that a measure μ with values in $B(\mathcal{E})$ is positive if for any μ-integrable φ with values in $\mathcal{E}^+$, $\int_A \varphi d\mu$ belongs to $\mathcal{E}^+$ for every $A \in \Sigma$. We now present a generalization of Monotone Convergence Theorem to vector lattice settings.

Proposition 3.2 (Monotone Convergence Theorem). *Let $\mu \in \mathcal{M}(\Omega; B(\mathcal{E}))$ be a positive measure on a compact metric space (Ω, ρ) and $\{u_k\}$, $u_k : \Omega \to \mathcal{E}$, an order bounded, increasing sequence of μ-integrable functions on Ω. Then the pointwise limit $u = \lim_{k\to\infty} u_k$ exists and it is a $\mathcal{E}$- valued, μ-integrable function on Ω satisfying $\int_\Omega u_k d\mu \nearrow \int_\Omega u d\mu$ and*

$$\int_\Omega u d\mu = \lim_{k\to\infty} \int_\Omega u_k d\mu.$$

Proof. Since $\mathcal{E}$ with the given order relation is a Dedekind complete vector lattice, the pointwise limit

$$u = \lim_{k\to\infty} u_k$$

exists as a Borel measurable, order bounded function with values in $\mathcal{E}$. Hence, $\|u(\cdot)\|_{\mathcal{E}}$ is a bounded, Borel measurable function on Ω. Then, $\|u(\cdot)\|_{\mathcal{E}}$ is $|\mu|$-integrable on Ω, and equivalently u is a μ-integrable function on Ω. We observe that

$$0 \le \left\|\int_\Omega (u - u_k) d\mu\right\| \le \int_\Omega \|u - u_k\| d|\mu| \to 0, \text{ as } k \to \infty.$$

Since $u_k \nearrow u$, the last part of the proposition follows.

□

Thus, we have an analogue result for decreasing, order bounded sequences of μ-integrable functions as follows:

Corollary 3.1. Let $\mu \in \mathcal{M}(\Omega; B(\mathcal{E}))$ be a positive measure on a compact metric space (Ω, ρ) and $\{u_k\}$, $u_k : \Omega \to \mathcal{E}$, an order bounded, decreasing sequence of μ-integrable functions on Ω. Then, the pointwise limit $u = \lim_{k\to\infty} u_k$ exists and it is a $\mathcal{E}$- valued, μ-integrable function on Ω satisfying $\int_\Omega u_k d\mu \searrow \int_\Omega u d\mu$

$$\int_\Omega u d\mu = \lim_{k\to\infty} \int_\Omega u_k d\mu.$$

We lastly emphasize that order bounded semi-continuous functions $f : \Omega \to \mathcal{E}$ are integrable with respect to a measure μ. We refer to Dinculeanu [4] for further details about vector measures and integration theory.

We now present a version of Riesz Representation Theorem with operator valued measures which is a consequence of results on dominated operators. In that regard, we refer to Dinculeanu [4].

Proposition 3.3. *Let (Ω, ρ) be a compact metric space and $S : C(\Omega; \mathcal{E}) \to \mathcal{E}$ be a continuous linear operator. Then there exists a unique measure $\mu \in \mathcal{M}(\Omega; B(\mathcal{E}))$ such that*

$$S(\Phi) = \int_\Omega \Phi d\mu$$

holds for every $\Phi \in C(\Omega; \mathcal{E})$ and the operator norm $\|S\|$ of S is equal to $|\mu|(\Omega)$.

Notice that $\mathcal{Y}$ can be interpreted as a subset of $B(\mathcal{Y})$. Hence, there is a whole subclass of $\mathcal{M}(\Omega; B(\mathcal{Y}))$ that consists of measures with values in $\mathcal{Y}$.

3.3. A connection between measures with values in $B(\mathcal{X})$ and $B(\mathcal{Y})$

We can alternatively define the integral of a Γ-order bounded, Γ-upper semi-continuous function $u : \Omega \to \mathcal{X}$ with respect to a $B(\mathcal{X})$-valued measure μ with finite variation as follows:

We start with the case where $u \in C(\Omega; \mathcal{X})$. Let L be a continuous linear map on $C(\Omega; \mathcal{Y}) = \Gamma(C(\Omega; \mathcal{X})) = \{\Gamma(u) : u \in C(\Omega; \mathcal{X})\}$ given by

$$L(\Gamma(u)) = \Gamma\left(\int_\Omega u d\mu\right) \in \mathcal{Y}, u \in C(\Omega; \mathcal{X}). \tag{3.2}$$

As a consequence of Proposition 3.3, there exists a unique measure ν on Ω with values in $B(\mathcal{Y})$ so that

$$L(\Gamma(u)) = \int_\Omega \Gamma(u) d\nu.$$

Now, we consider the case where $u : \Omega \to \mathcal{X}$ is a Γ-order bounded, Γ-upper semi-continuous function. We can find a sequence $\{u_k\}$ in $C(\Omega; \mathcal{X})$ so that $u_{k+1} \leq_\Gamma u_k$ on Ω, for any $k \in \mathbb{N}$ and u is the pointwise limit of the sequence $\{u_k\}$. Then, we define the integral of u with respect to μ by the limit relation

$$\Gamma\left(\int_\Omega u d\mu\right) = \int_\Omega \Gamma(u) d\nu = \lim_{k\to\infty} \int_\Omega \Gamma(u_k) d\nu.$$

Conversely, let ν be a measure on Ω with values in $B(\mathcal{Y})$ and with finite variation. We define the continuous linear operator $\tilde{L} : C(\Omega; \mathcal{X}) \to \mathcal{X}$ as follows:

$$\tilde{L}(u) = \Gamma^{-1}\left(\int_\Omega \Gamma(u) d\nu\right), u \in C(\Omega; \mathcal{X}). \tag{3.3}$$

By Proposition 3.3, we can find a unique measure μ on Ω with values in $B(\mathcal{X})$ so that

$$\tilde{L}(u) = \int_\Omega u d\mu, u \in C(\Omega; \mathcal{X}).$$

Moreover,

$$\int_\Omega \Gamma(u) d\nu = \Gamma\left(\int_\Omega u d\mu\right) \tag{3.4}$$

for any Γ-order bounded, Γ- upper semi-continuous function $u : \Omega \to \mathcal{X}$.

3.4. Weak-$*$ convergence on $\mathcal{M}(\Omega;\mathcal{V})$

The Banach-Alaoglu theorem states that for a normed space $\mathcal{E}$, the unit closed ball $ball\mathcal{E}^*$ of $\mathcal{E}^*$, the continuous dual space of $\mathcal{E}$, is weak-$*$ compact. A particular case of the Banach-Alaoglu Theorem is that if (Ω,ρ) is a compact metric space, then any bounded subset of the dual space $C^*(\Omega;\mathbb{R})$ (when $C(\Omega;\mathbb{R})$ is equipped with the supremum norm) is weak-$*$ compact. We refer to Conway [3] in connection with the classical definition of weak-$*$ convergence and topology. We, in addition, recall that $C(\Omega;\mathbb{R})$ equipped with the supremum norm is a separable Banach space. Then, we can easily show that if $\mathcal{E}$ is a finite dimensional Banach space, then the space $C(\Omega;\mathcal{E})$ is a separable Banach space with the norm $\|u\|_\infty = \sup_{z\in\Omega}\|u(z)\|$.

Let $\mathcal{E}$, $\tilde{\mathcal{E}}$ and $\mathcal{V}$ be Banach spaces as in the beginning of subsection 3.2. We say that a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega;\mathcal{V})$ converges to $\mu\in\mathcal{M}(\Omega;\mathcal{V})$ if

$$\lim_{k\to\infty}\int_\Omega \Phi d\mu_k = \int_\Omega \Phi d\mu.$$

is satisfied for every $\Phi\in C(\Omega;\mathcal{E})$. The weak-$*$ convergence on $\mathcal{M}(\Omega;\mathcal{V})$ induces a topology on $\mathcal{M}(\Omega;\mathcal{V})$ and we call the induced topology as the weak-$*$ topology. We denote $ball\mathcal{E} = \{x\in\mathcal{E} : \|x\|\le 1\}$ and $ball\mathcal{M}(\Omega;\mathcal{V}) = \{\mu\in\mathcal{M}(\Omega;\mathcal{V}) : |\mu|(\Omega)\le 1\}$.

It is known that on metric spaces, compactness and sequential compactness coincide.

Proposition 3.4. *Assume that $\mathcal{E}$ and $\mathcal{V}$ are finite dimensional Banach spaces. Let (Ω,ρ) be a compact metric space and $\{\mu_k\}$ a bounded sequence in $\mathcal{M}(\Omega;\mathcal{V})$. Then $\{\mu_k\}$ has a subsequence $\{\mu_{k_l}\}$ which weak-$*$ converges to some $\nu\in\mathcal{M}(\Omega;\mathcal{V})$.*

Proof. Let $\{e_1,e_2,\dots,e_n\}$ and $\{v_1,v_2,\dots,v_m\}$ be vector space bases for $\mathcal{E}$ and $\mathcal{V}$, respectively. Let $\Phi\in C(\Omega;\mathcal{E})$ be given. Then, $\Phi=\sum_{i=1}^n \Phi_i e_i$ and $\mu_k=\sum_{j=1}^m \mu_{j,k}v_j$ where each Φ_i belongs to the class $C(\Omega;\mathbb{R})$, and $\mu_{j,k}$, $j=1,\dots,m$, are real valued regular Borel measures on Ω.

Since any bounded subset of $C^*(\Omega;\mathbb{R})$ is weak-$*$ compact, $\{\mu_{1,k}\}$ has a subsequence $\{\mu_{1,k}\}$, $k\in S_1\subset\mathbb{N}$, that is weak-$*$ convergent to some $\nu_1\in C^*(\Omega;\mathbb{R})$. By the same argument, $\{\mu_{2,k}\}$ has a subsequence $\{\mu_{2,k}\}, k\in S_2\subset S_1$, that is weak-$*$ convergent to some $\nu_2\in C^*(\Omega;\mathbb{R})$. Continuing the procedure, on the m^{th} step, we can find a subsequence of $\{\mu_{m,k}\}$, say $\{\mu_{m,k}\}, k\in S_m\subset S_{m-1}\subset\ldots\subset S_2\subset S_1$, which is weak-$*$ convergent to some $\nu_m\in C^*(\Omega;\mathbb{R})$. Then,

$$\lim_{k\to\infty}\int_\Omega \Phi_i d\mu_{j,k} = \int_\Omega \Phi_i d\nu_j,\;\; k\in S_m, i\in\{1,\dots,n\}, j\in\{1,\dots,m\}.$$

Let us consider the subsequence of $\{\mu_k\}$ with terms

$$\mu_k = \sum_{j=1}^m \mu_{j,k}v_j,\;\; k\in S_m$$

and $\nu=\sum_{j=1}^m \nu_j v_j$. Notice that ν is a measure of finite variation with values in $\mathcal{V}$. For given $\Phi\in C(\Omega;\mathcal{E})$,

$$\lim_{k\to\infty}\int_\Omega \Phi d\mu_k = \lim_{k\to\infty}\sum_{i,j}\left(\int_\Omega \Phi_i d\mu_{j,k}\right)v_j e_i = \sum_{i,j}\left(\int_\Omega \Phi_i d\nu_j\right)v_j e_i = \int_\Omega \Phi d\nu.$$

□

In the case where all three of $\mathcal{E}$, $\tilde{\mathcal{E}}$ and $\mathcal{V}$ are $\mathcal{Y}$, we have the following corollary:

Corollary 3.2. If $\{\mu_k\}$ is a sequence of measures with values in $\mathcal{Y}$ only, then $\{\mu_k\}$ has a subsequence $\{\mu_{k_l}\}$ which weak-$*$ converges to a measure ν with values in $\mathcal{Y}$ only.

4. Jensen measures

Let $\mathcal{Y}$ be the commutative subspace of $H(N)$ as in section 2. If $\mathcal{F}$ is a cone of order bounded above, upper semi-continuous $\mathcal{Y}$- valued functions on Ω so that each nontrivial cone $\mathcal{F}_j = \{\lambda_j : U^*uU = diag(\lambda_1, \lambda_2, \ldots, \lambda_N), u \in \mathcal{F}\}$ contains all real constant functions, we say that $\mathcal{F}$ has the constant function property.

Let $\mathcal{F}$ be a cone of order bounded above, upper semi-continuous $\mathcal{Y}$-valued functions on Ω. A positive $\mu \in \mathcal{M}(\Omega; B(\mathcal{Y}))$ is said to be a Jensen measure for $\mathcal{F}$ with barycenter $z \in \Omega$ if the following condition is satisfied for any $u \in \mathcal{F}$:

$$u(z) \leq \int_\Omega u d\mu.$$

We denote by $\mathcal{J}_z^{\mathcal{F}}$ the class of Jensen measure for $\mathcal{F}$ with barycenter z. For instance, the measure $\delta_z I_N$ belongs to $\mathcal{J}_z^{\mathcal{F}}$.

For given $x \in \mathcal{Y}$ and $\mu \in \mathcal{J}_z^{\mathcal{F}}$, $x \leq \int_\Omega x d\mu$ and $-x \leq \int_\Omega -x d\mu$ and so $\int_\Omega x d\mu = x$. In other words, $\mu(\Omega) = I_N$ and then $|\mu|(\Omega) \geq 1$. Let $\{A_i : i \in \mathcal{I}\}$ be a finite collection of pairwise disjoint Borel subsets of Ω so that $\cup_{i \in \mathcal{I}} A_i = \Omega$. Since A_i are pairwise disjoint, there exists only one $i_0 \in \mathcal{I}$ so that $z \in A_{i_0}$. Then, $\int_\Omega x d\mu = \sum_i \int_\Omega \chi_{A_i} x d\mu = \mu(A_{i_0})x$ and this implies that $\mu(\Omega) = \mu(A_{i_0}) = I_N$. If we consider the definition of variation of μ over Ω, we conclude that $|\mu|(\Omega) = 1$.

Proposition 4.1. *Let $\{\mu_k\}$ be a sequence in $\mathcal{J}_z^{\mathcal{F}}$ so that μ_k weak-$*$ converges to μ. Then, μ is an element of $\mathcal{J}_z^{\mathcal{F}}$.*

Proof. Let $\Phi : \Omega \to \mathcal{Y}$ be a function from the cone $\mathcal{F}$. By Proposition 3.1, we find a decreasing sequence $\{\Phi_j\}$ in $C(\Omega; \mathcal{Y})$ so that Φ is the pointwise limit of $\{\Phi_j\}$ on Ω. Then, $\Phi \leq \Phi_j$ on Ω for every $j \in \mathbb{N}$, and

$$\Phi(z) \leq \int_\Omega \Phi d\mu_k \leq \int_\Omega \Phi_j d\mu_k.$$

By our assumption, we have that $\lim_{k\to\infty} \int_\Omega \Phi_j d\mu_k = \int_\Omega \Phi_j d\mu$ for $j \in \mathbb{N}$, and so,

$$\Phi(z) \leq \int_\Omega \Phi_j d\mu, j \in \mathbb{N}.$$

It follows by Proposition 3.2 that

$$\Phi(z) \leq \lim_{j\to\infty} \int_\Omega \Phi_j d\mu = \int_\Omega \Phi d\mu.$$

Since μ_j are positive measures, the measure μ is also positive. Therefore, μ belongs to $\mathcal{J}_z^{\mathcal{F}}$. □

Let $\varphi : \Omega \to \mathcal{Y}$ be an order bounded Borel measurable function. We define the upper and lower envelopes of φ at $z \in \Omega$ respectively by

$$S^{\mathcal{F}}\varphi(z) = \sup\{u(z) : \ u \in \mathcal{F}, \ u \leq \varphi \text{ on } \Omega\},$$

$$I^{\mathcal{F}}\varphi(z) = \inf\left\{ \int_\Omega \varphi d\mu : \ \mu \in \mathcal{J}_z^{\mathcal{F}} \right\}.$$

Note that $S^{\mathcal{F}}\varphi(z)$ exists in $\mathcal{Y}$ at any given $z \in \Omega$ and $S^{\mathcal{F}}\varphi(z) \leq \varphi(z)$. Notice also that $u(z) \leq \int_\Omega u d\mu \leq \int_\Omega \varphi d\mu$ holds for any for any $\mu \in \mathcal{J}_z^{\mathcal{F}}$ and any $u \in \mathcal{F}$ satisfying $u \leq \varphi$ on Ω. Thus, $I^{\mathcal{F}}\varphi(z)$ always exists in $\mathcal{Y}$ at any given point $z \in \Omega$. As $\delta_z I_N \in \mathcal{J}_z^{\mathcal{F}}$, we have the relation $S^{\mathcal{F}}\varphi(z) \leq I^{\mathcal{F}}\varphi(z) \leq \varphi(z)$ at our disposal.

Proposition 4.2. *Let (Ω,ρ) be a compact metric space. Let $\varphi\in C(\Omega;\mathcal{Y})$ and $z\in\Omega$. Then, there exists a $\mathcal{Y}$-valued measure $\mu\in\mathcal{J}_z^{\mathcal{F}}$ such that*

$$I^{\mathcal{F}}\varphi(z)=\int_\Omega \varphi d\mu.$$

Proof. Let us consider first the case where $\varphi\in C(\Omega;\mathbb{R})$. Let $\{\varepsilon_j\}$ be a nonnegative sequence in $\mathbb{R}$ so that $\varepsilon_j\searrow 0$. For each j, there exists some $\mu_j\in\mathcal{J}_z^{\mathcal{F}}$ so that

$$I^{\mathcal{F}}\varphi(z)+\varepsilon_j\geq\int_\Omega\varphi d\mu_j\geq I^{\mathcal{F}}\varphi(z).$$

As any bounded subset of $C^*(\Omega;\mathbb{R})$ is weak-$*$ compact and by Proposition 4.1 for the case $N=1$, $\{\mu_j\}$ has a subsequence $\{\mu_{j_k}\}$ that is weak-$*$ convergent to some $\mu\in\mathcal{J}_z^{\mathcal{F}}$. Then, the following holds for every k;

$$I^{\mathcal{F}}\varphi(z)+\varepsilon_{j_k}\geq\int_\Omega\varphi d\mu\geq I^{\mathcal{F}}\varphi(z).$$

Taking limit in the inequality as $k\to\infty$ shows that

$$I^{\mathcal{F}}\varphi(z)=\int_\Omega\varphi d\mu.$$

Now we prove the claim for the case where $\varphi\in C(\Omega;\mathcal{Y})$. Consider the cones $\mathcal{F}_j=\{\lambda_j: U^*uU=diag(\lambda_1,\lambda_2,\dots,\lambda_N),u\in\mathcal{F}\}$, $j=1,2,\dots,N$. We have the identity $U^*I^{\mathcal{F}}\varphi(z)U=diag(I^{\mathcal{F}_1}\varphi_1(z),\dots,I^{\mathcal{F}_N}\varphi_N(z))$ where $U^*\varphi U=diag(\varphi_1,\dots,\varphi_N)$. Note also that if $\mathcal{F}_j$ is trivial for some j, then $I^{\mathcal{F}_j}\varphi_j(z)=0$. Hence,

$$\begin{aligned}U^*I^{\mathcal{F}}\varphi(z)U&=diag\Big(I^{\mathcal{F}_1}\varphi_1(z),\dots,I^{\mathcal{F}_N}\varphi_N(z)\Big)\\&=diag\Big(\int_\Omega\varphi_1d\mu_1,\dots,\int_\Omega\varphi_Nd\mu_N\Big)\\&=U^*\left(\int_\Omega\varphi d\mu\right)U\end{aligned}$$

where $\mu_j\in\mathcal{J}_z^{\mathcal{F}_j}$, $j=1,2,\dots,N$, and $\mu=Udiag(\mu_1,\dots,\mu_N)U^*$. Since $\mu_j\in\mathcal{J}_z^{\mathcal{F}_j}, j=1,2,\dots,N$, one can prove that μ is in $\mathcal{J}_z^{\mathcal{F}}$. □

In noncommutative settings of the space $\mathcal{X}$, a positive measure μ on Ω with values in $B(\mathcal{X})$ is called a Jensen measure for a cone $\mathcal{F}$ of $\mathcal{X}$-valued, Γ-order bounded above, Γ-upper semi-continuous functions on Ω with barycenter $z\in\Omega$ if

$$u(z)\leq_\Gamma\int_\Omega ud\mu$$

holds for all $u\in\mathcal{F}$ and we denote the class of such measures by $\mathcal{J}_z^{\mathcal{F}}$. The measure δ_zI_n is in $\mathcal{J}_z^{\mathcal{F}}$ for given $z\in\Omega$. We can prove that $|\mu|(\Omega)=1$ for any $\mu\in\mathcal{J}_z^{\mathcal{F}}$ as we proved before.

Let $\varphi:\Omega\to\mathcal{X}$ be a Γ-order bounded, Borel measurable function. We define the upper and lower envelopes of φ at $z\in\Omega$ respectively by

$$S^{\mathcal{F}}\varphi(z)=\sup\{u(z):\ u\in\mathcal{F},\ u\leq_\Gamma\varphi\text{ on }\Omega\},$$

$$I^{\mathcal{F}}\varphi(z)=\inf\left\{\int_\Omega\varphi d\mu:\ \mu\in\mathcal{J}_z^{\mathcal{F}}\right\}.$$

Herein, the infimum and supremum in the definitions of envelopes are taken with respect to the Γ-order given on $\mathcal{X}$.

If $\mu \in \mathcal{J}_z^{\mathcal{F}}$, by our argument in subsection 3.3, there exists some $\nu \in \mathcal{M}(\Omega; B(\mathcal{Y}))$ so that for given $u \in \mathcal{F}$, the following holds:

$$\Gamma u(z) \leq \Gamma\left(\int_\Omega u d\mu\right) = \int_\Omega \Gamma u d\nu, \tag{4.1}$$

hence, $\nu \in \mathcal{J}_z^{\Gamma(\mathcal{F})}$.

On the other had, if a measure $\nu \in \mathcal{J}_z^{\Gamma(\mathcal{F})}$ is given, via the operator $\tilde{L}$ given in (3.3) and the relation (3.4), we can find a measure μ which belongs to $\mathcal{J}_z^{\mathcal{F}}$ satisfying (4.1).

4.1. Edwards' Theorem

The classical Edwards' Theorem states that if $\mathcal{F}$ is cone of real valued, upper bounded, upper semi-continuous functions on Ω which contains real constant functions, and $\varphi : \Omega \to \mathbb{R}$ is a bounded lower semi-continuous function, then $S^{\mathcal{F}}\varphi(z) = I^{\mathcal{F}}\varphi(z)$ for any point $z \in \Omega$. In regards to the classical Edwards' Theorem, we refer to Edwards [5], Gamelin [6] and Wikström [11].

Let X be a real vector space. A function $p : X \to \mathbb{R}$ is said to be sublinear if

(i) $p(x_1 + x_2) \leq p(x_1) + p(x_2)$ for all $x_1, x_2 \in X$,
(ii) $p(\alpha x) = \alpha p(x)$ for any $x \in X$ and any $\alpha \in \mathbb{R}^+$.

A function $p : X \to Y$ is called superlinear if $-p$ is sublinear. We refer to Rudin [9] for further details on the following version of the Hahn-Banach Theorem.

Proposition 4.3 (Hahn-Banach Theorem). *Let X be a real vector space and $p : X \to \mathbb{R}$ a sublinear function. If Z is a vector subspace of X and $S : Z \to \mathbb{R}$ is a linear map such that $S(x) \leq p(x)$ for all $x \in Z$, then there exists a linear map $\tilde{S} : X \to \mathbb{R}$ such that:*

(i) $\tilde{S} = S$ on Z.
(ii) $\tilde{S}(x) \leq p(x)$ for every $x \in X$.

By the Hahn-Banach Theorem and Proposition 3.3, we provide a functional analytical proof of Edwards' Theorem in matrix space settings.

Theorem 4.4 (Edwards' Theorem- Matrix Case). *Let (Ω, ρ) be a compact metric space. Let $\mathcal{F} \subset USC(\Omega; \mathcal{Y})$ be a cone of order bounded above functions which satisfies the constant function property. For given $\varphi \in C(\Omega; \mathcal{Y})$, $S^{\mathcal{F}}\varphi = I^{\mathcal{F}}\varphi$ on Ω.*

Proof. Since we already showed that $S^{\mathcal{F}}\varphi(z) \leq I^{\mathcal{F}}\varphi(z)$, we will only prove the reverse relation. As φ is order bounded, we without loss of generality assume that $\varphi \leq -\varepsilon I_N$ for some $\varepsilon > 0$.

We define the map

$$S^{\mathcal{F}} : C(\Omega;\mathcal{Y}) \to \mathcal{Y}$$
$$\Phi \mapsto S^{\mathcal{F}}\Phi$$

Let $z \in \Omega$ be given. The map $S^{\mathcal{F}}$ satisfies the following three properties:

(i) $S^{\mathcal{F}}(\alpha\Phi) = \alpha S^{\mathcal{F}}\Phi$ for any $\alpha \in \mathbb{R}^+$:

$$\begin{aligned} S^{\mathcal{F}}(\alpha\Phi)(z) &= \sup\{u(z) : u \in \mathcal{F}, u \leq \alpha\Phi\} = \sup\{\alpha u(z) : u \in \mathcal{F}, u \leq \Phi\} \\ &= \alpha \sup\{u(z) : u \in \mathcal{F}, u \leq \Phi\} = \alpha S^{\mathcal{F}}\Phi(z). \end{aligned}$$

(ii) $S^{\mathcal{F}}\Phi_1 + S^{\mathcal{F}}\Phi_2 \leq S^{\mathcal{F}}(\Phi_1 + \Phi_2)$:

It is obvious for any given $u, v \in \mathcal{F}$ with $u \leq \Phi_1$, $v \leq \Phi_2$ that $u + v \leq \Phi_1 + \Phi_2$, $u+v \in \mathcal{F}$, and hence $u(z)+v(z) \leq S^{\mathcal{F}}(\Phi_1+\Phi_2)(z)$. For a fixed $v \in \mathcal{F}$ with $v \leq \Phi_2$, the relation $u(z) + v(z) \leq S^{\mathcal{F}}(\Phi_1 + \Phi_2)(z)$ holds for any $u \in \mathcal{F}$ with $u \leq \Phi_1$. Then,

$$\mathcal{A} = \sup\{u(z) + v(z) : u \in \mathcal{F}, u \leq \Phi_1\} \leq S^{\mathcal{F}}\Phi_1(z) + v(z).$$

On the other hand, $\mathcal{A} - v(z) \geq u(z)$ for arbitrary $u \in \mathcal{F}$ with $u \leq \Phi_1$, and hence $\mathcal{A} - v(z) \geq S^{\mathcal{F}}\Phi_1(z)$. Thus, $\sup\{u(z) + v(z) : u \in \mathcal{F}, u \leq \Phi_1\} = S^{\mathcal{F}}\Phi_1(z) + v(z)$. Similarly, we can show that

$$\sup\{S^{\mathcal{F}}\Phi_1(z) + v(z) : v \in \mathcal{F}, v \leq \Phi_2\} = S^{\mathcal{F}}\Phi_1(z) + S^{\mathcal{F}}\Phi_2(z).$$

It follows that $S^{\mathcal{F}}\Phi_1(z) + S^{\mathcal{F}}\Phi_2(z) \leq S^{\mathcal{F}}(\Phi_1 + \Phi_2)(z)$.

Hence, $S^{\mathcal{F}}$ is a superlinear map on the space $C(\Omega; \mathcal{Y})$. If $\Phi_1 \leq \Phi_2 \leq 0$ and $u \in \mathcal{F}$ with $u \leq \Phi_1$, we have $u \leq \Phi_2$. Then, $S^{\mathcal{F}}\Phi_1 \leq \Phi_2$, and so $S^{\mathcal{F}}\Phi_1 \leq S^{\mathcal{F}}\Phi_2$.

Define the linear map $S : span\{\varphi\} \to \mathcal{Y}$ by $\alpha\varphi \mapsto \alpha S^{\mathcal{F}}\varphi(z), \alpha \in \mathbb{R}$. By the definition of the map S, $S\Phi \geq S^{\mathcal{F}}\Phi(z)$ holds for all $\Phi \in span\{\varphi\}$. We note that the map $S : span\{\varphi\} \to \mathcal{Y}$ given by $\Phi \mapsto S^{\mathcal{F}}\Phi(z)$ is superlinear.

It is easily shown that $S^{\mathcal{F}}\varphi = Udiag(S^{\mathcal{F}_j}\varphi_j)U^*$ where $\varphi = Udiag(\varphi_j)U^*$ and $\mathcal{F}_j = \{\lambda_j : U^*uU = diag(\lambda_1, \lambda_2, \ldots, \lambda_N), u \in \mathcal{F}\}, j = 1, 2, \ldots, N$. Let us define the linear maps $S_j : span\{\varphi_j\} \to \mathbb{R}$ by $\alpha\varphi_j \mapsto \alpha S^{\mathcal{F}_j}\varphi_j(z), j = 1, 2, \ldots, N$ and each one of them satisfies $S_j\Phi_j \geq S^{\mathcal{F}_j}\Phi_j(z)$ where $\Phi_j = \alpha\varphi_j$. It follows from Hahn-Banach Theorem that, for each j, there exists a linear extension $\tilde{S}_j$ of S_j to $C(\Omega; \mathbb{R})$ so that $\tilde{S}_j(\Phi_j) \geq S^{\mathcal{F}_j}\Phi_j(z)$, $\Phi_j \in C(\Omega; \mathbb{R})$. Notice that each $\tilde{S}_j$ is a positive map. By the classical Riesz Representation Theorem, for each j, there exists a regular positive Borel measure μ_j on Ω so that

$$\tilde{S}_j\Phi_j = \int_\Omega \Phi_j d\mu_j,\ \Phi_j \in C(\Omega; \mathbb{R}).$$

Let $\tilde{S} : C(\Omega; \mathcal{Y}) \to \mathcal{Y}$ be the linear map given by

$$\tilde{S}(\Phi) = \int_\Omega \Phi d\mu,$$

where $\mu = Udiag(\mu_j)U^*$ is a positive measure of finite variation on Ω with values in $\mathcal{Y}$.

For given $u \in \mathcal{F}$, there exists a decreasing sequence $\{u_k\}$ in $C(\Omega; \mathcal{Y})$ so that $u_k \searrow u$ pointwise on Ω. Clearly, $\tilde{S}u_k \geq S^{\mathcal{F}}u_k(z) \geq S^{\mathcal{F}}u(z) = u(z), k \in \mathbb{N}$. Since u_k's are μ-integrable and $u_k \searrow u$, we obtain that

$$\int_\Omega ud\mu \geq S^{\mathcal{F}}u(z) = u(z).$$

Hence, $\mu \in \mathcal{J}_z^{\mathcal{F}}$. This further implies that

$$I^{\mathcal{F}}\varphi(z) \leq \int_\Omega \varphi d\mu = S^{\mathcal{F}}\varphi(z).$$

□

We remind that a lower semi-continuous function $u : \Omega \to \mathcal{Y}$ is the pointwise limit of an increasing sequence in $C(\Omega; \mathcal{Y})$. Thus, we can further extend Edwards' Theorem for order bounded lower semi-continuous functions via the following lemma.

Lemma 4.1. Let (Ω, ρ) be a compact metric space. Suppose that $\varphi : \Omega \to \mathcal{Y}$ is an order bounded lower semi-continuous function on Ω, and $\{\varphi_k\}$ is an increasing sequence in $C(\Omega; \mathcal{Y})$ so that $\varphi_k \nearrow \varphi$ pointwise on Ω. Let $\mathcal{F} \subset USC(\Omega; \mathcal{Y})$ be a cone of order bounded functions which has the constant function property. Then, $I^{\mathcal{F}}\varphi_k \nearrow I^{\mathcal{F}}\varphi$ pointwise on Ω.

Proof. It is clear that $S^{\mathcal{F}}\varphi_k = I^{\mathcal{F}}\varphi_k \leq I^{\mathcal{F}}\varphi$ on Ω for every k. Let $z \in \Omega$, and $\varepsilon > 0$ be arbitrary. Let us write $\varphi_k = Udiag(\varphi_k^{(1)}, \varphi_k^{(2)}, \ldots, \varphi_k^{(N)})U^*$ for fixed k. We already know for fixed k that $I^{\mathcal{F}}\varphi_k(z) = Udiag(I^{\mathcal{F}_j}\varphi_k^{(j)}(z))U^*$, and for each j, there exists a $\nu_k^{(j)} \in \mathcal{J}_z^{\mathcal{F}_j}$ so that $I^{\mathcal{F}_j}\varphi_k^{(j)}(z) = \int_\Omega \varphi_k^{(j)} d\nu_k^{(j)}$. Then $I^{\mathcal{F}}\varphi_k(z) = \int_\Omega \varphi_k d\nu_k$ where $\nu_k = Udiag(\nu_k^{(j)})U^* \in \mathcal{J}_z^{\mathcal{F}}$.

We can assume without loss of generality that $\{\nu_k\}$ weak-$*$ converges some $\nu \in \mathcal{J}_z^{\mathcal{F}}$ that attains values only in $\mathcal{Y}$. Let $j \in \mathbb{N}$ be fixed. It is trivial that for $j < k$,

$$S^{\mathcal{F}}\varphi(z) \geq S^{\mathcal{F}}\varphi_k(z) = I^{\mathcal{F}}\varphi_k(z) = \int_\Omega \varphi_k d\nu_k \geq \int_\Omega \varphi_j d\nu_k,$$

$$\begin{aligned} S^{\mathcal{F}}\varphi(z) \geq \lim_{k\to\infty} S^{\mathcal{F}}\varphi_k(z) = \lim_{k\to\infty} I^{\mathcal{F}}\varphi_k(z) &= \lim_{k\to\infty}\int_\Omega \varphi_k d\nu_k \\ &\geq \lim_{k\to\infty}\int_\Omega \varphi_j d\nu_k = \int_\Omega \varphi_j d\nu. \end{aligned}$$

By Proposition 3.2, we have

$$\int_\Omega \varphi_j d\nu \nearrow \int_\Omega \varphi d\nu, \text{ as } j \to \infty.$$

Obviously, $\int_\Omega \varphi d\nu \geq I^{\mathcal{F}}\varphi(z)$. The last inequality brings out that $\lim_{k\to\infty} I^{\mathcal{F}}\varphi_k(z) \geq I^{\mathcal{F}}\varphi(z)$. □

Theorem 4.5. *Let (Ω, ρ) be a compact metric space and $\varphi : \Omega \to \mathcal{Y}$ be an order bounded lower semi-continuous function. Suppose that $\mathcal{F} \subset USC(\Omega; \mathcal{Y})$ is a cone of order bounded functions possessing the constant function property. Then, $I^{\mathcal{F}}\varphi = S^{\mathcal{F}}\varphi$ on Ω.*

Proof. Since φ is order bounded and lower semi-continuous on Ω, there exists an increasing sequence $\{\varphi_k\}$ in $C(\Omega; \mathcal{Y})$ so that $\varphi_k \nearrow \varphi$ pointwise on Ω. By Lemma 4.1, $I^{\mathcal{F}}\varphi_k \nearrow I^{\mathcal{F}}\varphi$ on Ω. As each φ_k belongs to the class $C(\Omega; \mathcal{Y})$, by Theorem 4.4, we have $S^{\mathcal{F}}\varphi_k = I^{\mathcal{F}}\varphi_k$ on Ω for each $k \in \mathbb{N}$. Then, $S^{\mathcal{F}}\varphi \geq \lim_{k\to\infty} S^{\mathcal{F}}\varphi_k = \lim_{k\to\infty} I^{\mathcal{F}}\varphi_k = I^{\mathcal{F}}\varphi$ on Ω. □

4.2. Noncommutative settings

As we prepared the necessary background in the previous parts, we can now prove Theorem 2.1.

Proof of Theorem 2.1. By our observations in sections 3 and 4, one can show that the upper and lower envelope of a function $\varphi : \Omega \to \mathcal{X}$ satisfies

$$\begin{aligned} S^{\Gamma(\mathcal{F})}\Gamma(\varphi)(z) &= \Gamma(S^{\mathcal{F}}\varphi(z)), \\ I^{\Gamma(\mathcal{F})}\Gamma(\varphi)(z) &= \Gamma(I^{\mathcal{F}}\varphi(z)), \end{aligned}$$

for given $z \in \Omega$. Theorem 4.5 shows that $S^{\Gamma(\mathcal{F})}\Gamma\varphi(z) = I^{\Gamma(\mathcal{F})}\Gamma\varphi(z)$, $z \in \Omega$. Hence, $S^{\mathcal{F}}\varphi(z) = I^{\mathcal{F}}\varphi(z)$, $z \in \Omega$. □

5. Examples

5.1. Circulant matrices

A matrix $C \in M(n)$ is called circulant if it is of the form

$$C = circ(c) = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}$$

where $c = (c_0, c_1, \ldots, c_{n-1})$ is a given vector in $\mathbb{C}^n$. Notice that the rows of C are actually cyclic permutations of the vector c. We will denote the class of $n \times n$ circulant matrices by $Circ(n)$, and its subclass of Hermitian circulant matrices by $CircH(n)$. It is a known fact that $Circ(n)$ is a commutative algebra. The eigenvalues of a $C = circ(c_0, c_1, \ldots, c_{n-1}) \in Circ(n)$ are given by

$$\lambda_j = c_0 + c_1\omega^j + c_2\omega^{2j} + \cdots + c_{n-1}\omega^{(n-1)j}$$

where $\omega = \exp(2i\pi/n)$ is a primitive n^{th} root of unity, and the corresponding normalized eigenvectors are

$$x_j = \frac{1}{\sqrt{n}}(1, \omega^j, \omega^{2j} \ldots, \omega^{(n-1)j})^\top, j = 0, 1, \ldots, n-1.$$

The unitary matrix

$$U = (x_0|x_1|\ldots|x_{n-1})$$

is called the n-dimensional Fast Fourier Transform (FFT) matrix. It is a well known fact that U simultaneously diagonalizes $Circ(n)$. We point out that $CircH(n)$ is a nontrivial example for a commutative subspace $\mathcal{Y}$ of $H(n)$ that $\tilde{Y}$ is generated by all diagonal matrices D_k

$$D_k = diag\Big(0, 0, \ldots, 0, 1, 0 \ldots, 0, \ldots, 0\Big), k = 1, 2, \ldots, n.$$

where 0 as all entries except its k^{th} main diagonal entry and 1 as its k^{th} main diagonal entry. One can easily show that $CircH(n)$ is an algebra.

5.2. Examples in noncommutative settings

For a given matrix $A = [a_{ij}] \in M(n, m)$, we can construct a circulant Hermitian matrix $\Gamma_0 A$ as in the following:

$$\Gamma_0 A = \begin{bmatrix} 0 & a_{11} & a_{12}\ldots & a_{nm} & \overline{a_{nm}} & \overline{a_{n(m-1)}} & \ldots & \overline{a_{12}} & \overline{a_{11}} \\ \overline{a_{11}} & 0 & a_{11} & a_{12} & \ldots & a_{nm} & \overline{a_{nm}} & \ldots & \overline{a_{12}} \\ \overline{a_{12}} & \overline{a_{11}} & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \overline{a_{13}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{11} & a_{12}\ldots & a_{nm} & \overline{a_{nm}} & \overline{a_{n(m-1)}} & \ldots & \overline{a_{12}} & \overline{a_{11}} & 0 \end{bmatrix}$$

By employing this method of construction, we define an injective, continuous, real-linear operator $\Gamma_0 : M(n, m) \to CircH(N)$, $N = 2nm + 1$.

We now turn our attention to noncommutative subspaces of matrices and present examples of lattice structures on such spaces. We have two basic methods of constructing examples of lattices on matrix spaces, one via using simultaneous diagonalization and one via using the map Γ_0.

Example 5.1. Let $\mathcal{X}$ be a nontrivial real vector subspace of $M(n, m)$ of real dimension k and $\{\hat{E}_j : j = 1, 2, \ldots k\}$ a basis for $\mathcal{X}$. Let $\mathcal{Y} = span\{UE_jU^* : j = 1, 2, \ldots, k\}$ where $k \le N$, E_j are the $N \times N$ canonical diagonal matrices and U is the N-dimensional FFT matrix. Note that $\mathcal{Y}$ is a commutative subalgebra of $CircH(N)$ as well as a Dedekind complete vector lattice with the usual order. Let us define the map $\Gamma : \mathcal{X} \to \mathcal{Y}$ as $\Gamma\hat{E}_j = UE_jU^*, j = 1, 2, \ldots, k$. The map Γ is a bijective, real linear map, and hence, $(\mathcal{X}, \le_\Gamma)$ is a Dedekind complete vector lattice.

Example 5.2. Let $T(n) = \{A \in M(n) : tr(A) \in \mathbb{R}\}$ where $tr(A)$ is the trace of $A \in M(n)$. $T(n)$ is a real subspace of $M(n)$ with real dimension $2n^2-n$. Let $\{\hat{E}_j : j = 1, 2, \ldots, 2n^2-n\}$ be a basis of $T(n)$.

Let us define the operator $\Gamma : T(n) \to \mathcal{Y}$ as $\Gamma A = \Gamma_0 A + tr(A)I_N$, $A \in T(n)$ with the image $\mathcal{Y} = span\{\tilde{E}_j = \Gamma\hat{E}_j : j = 1, 2, \ldots, 2n^2-n\}$. It is straightforward to prove that Γ is a bijective real-linear operator and $(T(n), \le_\Gamma)$ is a Dedekind complete vector lattice. Notice that any eigenvalue $\lambda_j(\Gamma A)$ of ΓA is equal to $\lambda_j(\Gamma_0 A)+tr(A)$. Moreover, one can show that $\|\Gamma_0\| = 1$. For a given positive definite matrix $A \in H(n)$, $\|\Gamma_0 A\| \le \|A\| \le tr(A)$ holds.

Since $\Gamma_0 A$ is Hermitian, $\Gamma_0 A + \|\Gamma_0 A\| I_N$ is positive definite and $\Gamma_0 A + \|\Gamma_0 A\| I_N \leq \Gamma A$. Therefore, ΓA is positive definite whenever $A \in H(n)$ is.

Example 5.3. Let $s, t > 0$ be given scalars and $\{\hat{E}_j : j = 1, 2, \ldots, 2n^2\}$ a real vector basis of $M(n)$. Consider the map $\Gamma_{s,t} : M(n) \to \mathcal{Y}$ given by $\Gamma_{s,t}(A) = \Gamma_0 A + s(tr(ReA))I_N + t(tr(ImA))I_N$ where $ReA = (A + A^*)/2$, $ImA = (A - A^*)/2i$ and $\mathcal{Y} = span\{\Gamma_{s,t}\hat{E}_j : j = 1, 2, \ldots, 2n^2\} \subset CircH(N)$. One can easily show that $\Gamma_{s,t}$ is a bijective real linear operator. Similarly, we demonstrate that $(M(n), \leq_{\Gamma_{s,t}})$ is a Dedekind complete vector lattice. Let $A \in H(n)$ be a positive definite matrix. Then, $\Gamma_{s,t}(A) = \Gamma_0 A + str(A)I_N$. If $s \geq 1$, we can show that $\Gamma_{s,t}(A)$ is positive definite as we did in the previous example.

6. Final remarks

We propose to extend notions such as upper semi-continuity and Jensen measures for cones of functions with values in operator spaces on an infinite dimensional Hilbert spaces and also obtain Edwards' Theorem for functions with values in operator spaces in future works. The notion of Γ-order on a real matrix space provides a Dedekind complete vector lattice structure on the given matrix space. This result lays foundation for Edwards' Theorem in noncommutative matrix spaces. For that reason, we first suggest to find a generalization of Γ-order on operator spaces. Furthermore, we are of the opinion that the functional analytic proof of Edwards' Theorem can be generalized to the case of functions with values in the class of self-adjoint operators on an infinite dimensional complex Hilbert space.

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