ALGEBRAIC VIEW ON NEIGHBORHOOD HYPERGRAPHS THEIR TRANSVERSALS, AND d-PARTITE HYPERGRAPHS

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ABSTRACT

ALGEBRAIC VIEW ON NEIGHBORHOOD HYPERGRAPHS, THEIR TRANSVERSALS, AND d-PARTITE HYPERGRAPHS

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Keywords: dominating ideals, closed neighborhood ideals, componentwise linearity, d-partite hypergraphs, t-spread ideals, normally torsion-free, linear quotients

In this thesis, we explore the algebraic and homological properties of square-free monomial ideals that originate from graphs and hypergraphs. Our study has two main parts. In the first part, we study the closed neighborhood ideals and the dominating ideals of graphs. We prove that the closed neighborhood ideals and the dominating ideals of some classes of trees are normally torsion-free. However, the closed neighborhood ideals and the dominating ideals of cycles fail to be normally torsion-free. We prove that the closed neighborhood ideals of cycles admit the (strong) persistence property and the dominating ideals of cycles are nearly normally torsion-free. Expanding our study to path graphs, we show the componentwise linearity of dominating ideals of path graphs by describing a linear quotient order of their minimal generators. We also give formulas for their Betti numbers, regularity, and projective dimension.

In the second part, we shift our focus to d-partite hypergraphs. Inspired by the definition of $\mathbf{t}$-spread monomial ideals; we introduce the $\mathbf{t}$-spread d-partite hypergraphs. The edge ideals of these hypergraphs, denoted by $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$, admit some nice properties. Namely, $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ has linear quotients and satisfies the ℓ-exchange property and the strong persistence property. Moreover, all powers of $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ have linear resolutions and the Rees algebra of $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ is a normal Cohen-Macaulay domain. We also prove that $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ is normally torsion-free and give a complete characterization of Cohen-Macaulay $S/I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$.

ÖZET

KOMŞULUK HIPERGRAFLARI, TRANSVERSLERI VE d-PARÇALI HIPERGRAFLARA CEBIRSEL BIR BAKIŞ

ASLI TUĞCUOĞLU MUSAPAŞAOĞLU

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Tez Danışmanı: Asst. Prof. Dr. AYESHA ASLOOB QURESHI

Anahtar Kelimeler: baskın idealler, kapalı komşuluk idealleri, bileşen bazında doğrusallık, d-parçalı hipergraflar, t-yayılımlı idealler, normalde torsiyonsuz, doğrusal bölümler

Bu tezde, graf ve hipergraf kökenli karesiz tekterimli ideallerin cebirsel ve homolojik özelliklerini araştırıyoruz. Çalışmamız iki ana bölüme ayrılmıştır. İlk bölümde, grafların kapalı komşuluk idealleri ve baskın ideallerini inceliyoruz. Bazı ağaç sınıflarının kapalı komşuluk ideallerinin ve baskın ideallerinin normalde torsiyonsuz olduğunu kanıtlıyoruz. Ancak döngülerin kapalı komşuluk idealleri ve dominasyon idealleri normalde torsiyonsuz değildir. Döngülerin kapalı komşuluk ideallerinin (güçlü) kalıcılık özelliğine sahip olduğunu ve döngülerin baskın ideallerinin neredeyse normalde torsiyonsuz olduğunu kanıtlıyoruz. Çalışmamızı yol graflarına genişleterek yol graflarının baskın ideallerinin bileşen bazında doğrusallığını, minimal üreteçlerinin doğrusal bölüm sırasını açıklayarak gösteriyoruz. Ayrıca Betti sayıları, regülarite ve projektif boyutları için formüller sunuyoruz.

İkinci bölümde, odak noktamız d-parçalı hipergraflara kaymaktadır. $\mathbf{t}$-yayılımlı tekterimli ideal tanımından esinlenerek, $\mathbf{t}$-yayılımlı d-parçalı hipergrafları tanıtıyoruz. Bu hipergrafların kenar idealleri, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ olarak gösterilir ve bazı güzel özelliklere sahiptir. Özellikle, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$, doğrusal bölümlere sahiptir ve ℓ-değişim özelliğini ve güçlü kalıcılık özelliğini sağlar. Dahası, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ tüm kuvvetleri için doğrusal bölümlere sahiptir ve $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$'nin Rees cebiri normal bir Cohen-Macaulay alanıdır. Ayrıca, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ normalde torsiyonsuz olduğunu ispatlayıp, Cohen-Macaulay $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$'nin tam bir karakterizasyonunu veriyoruz.

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To all women that have to sacrifice for science

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Introduction

Combinatorial commutative algebra is a fascinating and rapidly developing field that lies at the intersection of two well-established areas: commutative algebra and combinatorics. Commutative algebra focuses on commutative rings, polynomial rings, ideals, and modules. It has applications in algebraic geometry, number theory, and coding theory. On the other hand, combinatorics focuses on counting, arranging, and structuring discrete objects. It includes graph theory, combinatorial optimization, and polyhedral geometry. The cornerstone of both commutative algebra and combinatorics lies in the study of monomial ideals. These ideals bridge algebraic structures and combinatorial objects, providing essential insights into both fields. By investigating monomial ideals, researchers unlock connections to algebraic geometry, simplicial complexes, integral points in polytopes, and graph theory. Their computational aspects allow us to solve various problems, making them indispensable tools in mathematical exploration. This thesis investigates square-free monomial ideals derived from graphs and hypergraphs. The well-known examples of these ideals are the edge and cover ideals, which have been extensively studied. Every square-free monomial ideal generated in degree two can be viewed as an edge ideal of a simple graph. Villarreal introduced edge ideals in [54], and since their first appearance, they have been a central topic of many articles. One of the exciting properties of the edge ideals is that their minimal primes correspond to the minimal vertex covers of their underlying graphs. In other words, the Alexander dual of the edge ideal of a graph G is the cover ideal of G, a square-free monomial ideal whose minimal generators correspond to the minimal vertex covers of the underlying graph. Inspired by this relation, the closed neighborhood ideals and the dominating ideals of graphs were recently introduced in [48]. Let G be a simple graph. The closed neighborhood ideal $NI(G)$ of G is generated by square-free monomials corresponding to the closed neighborhoods of the vertices of G. In contrast, the dominating ideal $DI(G)$ of

G is generated by the monomials that correspond to the dominating sets of G(see Chapter 2 for the formal definitions). As shown in [48], $NI(G)$ and $DI(G)$ are the Alexander dual of each other, a similar relation that exists between edge ideals and cover ideals of G. In this thesis, to advance our understanding of square-free monomial ideals arising from graphs and hypergraphs, we further extend the study of closed neighborhood ideals and dominating ideals, and the details can be found in Chapter 2 and Chapter 3. Moreover, we study the edge ideals of $\mathbf{t}$-spread d-partite hypergraphs to expand our knowledge in this context in Chapter 4.

The basic outline of this thesis is as follows. In Chapter 1, we recall some preliminary results and definitions to be used in this work. Then, the thesis is divided into three different parts. A breakdown of Chapter 2 is as follows: Section 2.1 focuses on the closed neighborhood ideals and the dominating ideals of star graphs. Any square-free monomial ideal can be visualized as an edge ideal of a hypergraph. A hypergraph $\mathcal{H}$ is called *Mengerian* if it satisfies a certain min-max equation, which is known as the Mengerian property in hypergraph theory or as the max-flow min-cut property in integer programming. Algebraically, it is equivalent to $I(\mathcal{H})$ being normally torsion-free, see [22, Corollary 10.3.15], [55, Theorem 14.3.6]. This fact enhances the importance of normally torsion-free ideals. In Section 2.1, our main goal is to establish the normally torsion-freeness of the closed neighborhood ideals and the dominating ideals of star graphs, which is achieved in Corollaries 2.1.6 and 2.1.19. To do this, we first prove some results of a general nature, which provide certain inductive and recursive techniques to create new normally torsion-free ideals based on the existing ones, see Theorem 2.1.3 and Lemma 2.1.5. We apply these techniques to study the normally torsion-freeness of the closed neighborhood ideals and the dominating ideals of cone graph of a given graph, see Lemma 2.1.10. In addition, in Corollary 2.1.4, we prove that 3-path ideals of path graphs are normally torsion-free.

In Section 2.2, we focus on the closed neighborhood ideals and the dominating ideals of cycles. The edge ideals and the cover ideals of cycles are well-studied in the context of normally torsion-freeness. It is a well-known fact that the edge ideals and cover ideals of even cycles are normally torsion-free, and odd cycles fail to have this property in general. Given a cycle C_n of length n, it is natural to expect somewhat similar behavior for $NI(C_n)$ and $DI(C_n)$, but we observe in Section 2.2 that this is not the case. Normally torsion-freeness is not maintained by $NI(C_n)$, but we prove in Theorem 2.2.2 that they admit strong persistence property and, therefore, the persistence property. This facilitates the study of the behavior of depth of powers of $NI(C_n)$ in Corollary 2.2.7. As a final result, we prove in Theorem 2.2.9 that $DI(C_n)$ are nearly normally torsion-free.

In Chapter 3, we turn our attention to the dominating ideals of path graphs. In [13], Farber introduced strongly chordal graphs and proved that a graph G is strongly chordal if and only if the neighborhood hypergraph of G is totally balanced. Rephrasing this in the algebraic language, it is established in [40] that the dominating ideals of strongly chordal graphs are componentwise linear. Since path graphs are strongly chordal, one can conclude that the dominating ideals of path graphs are componentwise linear. In our work, we show that the dominating ideals of path graphs have linear quotients by precisely giving a linear quotient order of their minimal generating set. Invoking [22, Theorem 8.3.15], we obtain another proof for the componentwise linearity of dominating ideals of path graphs. Utilizing a well-known result of Sharifan and Varbaro [49, Corollary 2.7], we also compute the total and graded Betti numbers of the dominating ideals of path graphs. A breakdown of Chapter 3 is as follows: We start by describing a recursive order on the generating set of dominating ideals of path graphs which gives as linear quotients shown in Theorem 3.1.5. This order is used to describe the total and graded Betti numbers of dominating ideals of path graphs (See Theorem 3.2.3 and Theorem 3.2.4). In Theorem 3.2.2, we also compute projective dimension and regularity of dominating ideals of path graphs and recover the formulas given in [48, Theorem 2.6].

In Chapter 4 we introduce $\mathbf{t}$-spread d-partite hypergraphs and study their edge ideals. Recall that a hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ is a finite family of non-empty subsets of $V(\mathcal{H})$. Let $V = \{V_1, \ldots, V_d\}$ be a partitioning of a finite set $U \subset \mathbb{N}$ such that $p < q$ if $p \in V_i, q \in V_j$ with $i < j$. We call $\{i_1, \ldots, i_d\} \subset U$ a $\mathbf{t}$-spread set if $i_j \in V_j$ for all $j = 1, \ldots, d$ and $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \ldots, d$. We call the hypergraph $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ on vertex set $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) = U$, a complete $\mathbf{t}$-spread d-partite hypergraph if all $\mathbf{t}$-spread sets of U are the edges of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. For $\mathbf{t} = (1, \ldots, 1)$, the hypergraph $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ is a complete d-partite hypergraph, see [6, Example 3]. Using this definition, we obtain the edge ideal of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ denoted by $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is a t-spread monomial ideal. The ideal generated by t-spread monomials is first defined by Qureshi, Herzog and Ene in [12]. After their first appearance, different classes of t-spread monomial ideals have been studied by many authors (see [5, 42, 3, 9]).

In 2023, Ficarra gave a more generalized notion of t-spread monomials by replacing the integer t with $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{N}^{d-1}$, (see [14] and the reference therein).

It turns out that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ admits many nice algebraic and homological properties. It is shown in Theorem 4.1.3 that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has linear quotients. The ideals with linear quotients were first defined by Herzog and Takayama in [26] and their free resolutions were computed as iterated mapping cones. Using the description of Betti numbers of ideals with linear quotients given in [26], in Proposition 4.1.4, we provide an intrinsic way to compute Betti numbers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

In Section 4.2, we study the powers and fiber cone of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. One of the main results of Section 4.2 is given in Corollary 4.2.7 that shows the ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the strong persistence property and all powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution. To prove Corollary 4.2.7, we first show that the minimal generating set of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is sortable and $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the ℓ-exchange property with respect to sorting order, see Proposition 4.2.1 and Theorem 4.2.4. Then it follows from classical results of Fröberg [16], Sturmfels [52] and Hochster [29] that the Rees algebra $\mathcal{R}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ is a normal Cohen-Macaulay domain, see Corollary 4.2.6. Then Corollary 4.2.7 is obtained as an application of [27, Corollary 1.6] and [22, Corollary 10.1.8]. We also compute the Krull dimension of fibercone $\mathcal{R}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))/\mathfrak{m}\mathcal{R}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ which provides the limit depth of $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ in Theorem 4.2.11.

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$. A set $T \subset V(\mathcal{H})$ is called a *transversal* of $\mathcal{H}$, if it meets all the edges of $\mathcal{H}$ and the family of all minimal transversals of $\mathcal{H}$, is called the *transversal hypergraph* of $\mathcal{H}$, see [6, Chapter 2]. The minimal transversals of a hypergraph $\mathcal{H}$ correspond to the minimal prime ideals of the edge ideal of $\mathcal{H}$. In Section 4.3, we consider $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ with $V = \{V_1, \ldots, V_d\}$ such that each V_i is an interval of integers. The description of the minimal primes of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is obtained by computing the minimal generating set of Alexander dual of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ in Theorem 4.3.1. In Theorem 4.3.6, we prove that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free which is equivalent to say that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ is a Mengerian hypergraph. A complete characterization of unmixed $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is given in Theorem 4.3.9. With the help of Theorem 4.3.9, a complete characterization of Cohen-Macaulay $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is obtained in Theorem 4.3.11.

Chapter 1

Algebraic and Combinatorial Ingredients

This chapter introduces fundamental algebraic and combinatorial concepts of commutative algebra necessary for the subsequent chapters.

All rings considered in this thesis are Noetherian and commutative with an identity. Monomial ideals serve as a bridge between commutative algebra and combinatorics. This thesis focuses on monomial ideals, with particular emphasis on square-free monomial ideals.

1.1 Monomial ideals

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K with n variables. An element $u \in S$ is called a *monomial* if $u = \prod_{i=1}^{n} x_i^{a_i}$ where $a_i \geq 0$. For ease of notation, we denote a monomial $u = x^a$, and if $a_i \in \{0, 1\}$, then we call u a *square-free monomial*. In a polynomial ring S over a field K, every element $f \in S$ can be written as a K-linear combination of monomials. The set of all monomials in S denoted by $\mathrm{Mon}(S)$ provides a natural K-basis for S. An ideal $I \subset S$ is called a *(square-free) monomial ideal* if it is generated by (square-free) monomials. A monomial ideal $I \subset S$ also has a K-basis of monomials. Monomial ideals have a unique generating set that is minimal with respect to divisibility. This unique minimal generating set of a monomial ideal I is denoted by $\mathrm{G}(I)$. The *support* of a monomial u, denoted by $\mathrm{supp}(u)$, is the set of variables that divide u. Moreover, we set $\mathrm{supp}(I) = \bigcup_{u \in \mathrm{G}(I)} \mathrm{supp}(u)$. A monomial ideal can be characterized by the following result:

Corollary 1.1.1. [22, Corollary 1.1.3] Let $I \subset S$ be an ideal. Then the following are equivalent:

(i) I is a monomial ideal;

(ii) for all $f \in S$ one has: $f \in I$ if and only if $\mathrm{supp}(f) \subset I$.

This result is one reason why it is easier to perform algebraic operations on monomial ideals compared to general ideals. Ideal operations can be simplified for monomial ideals. We recall some of these operations in the following:

Proposition 1.1.2. [22, Proposition 1.2.1] Let I and J be monomial ideals. Then $I \cap J$ is a monomial ideal, and $\{\mathrm{lcm}(u,v) : u \in G(I), v \in G(J)\}$ is a set of generators of $I \cap J$.

Let $I, J \subset R$ be two ideals. The set $I : J = \{f \in R : fg \in I \text{ for all } g \in J\}$ is an ideal, called the *colon ideal* of I with respect to J.

Proposition 1.1.3. [22, Proposition 1.2.2] Let I and J be monomial ideals. Then $I : J$ is a monomial ideal, and

$$I : J = \bigcap_{v \in G(J)} I : (v).$$

Moreover, $\{u/\gcd(u,v) : u \in G(I)\}$ is a set of generators of $I : (v)$.

Now we recall the notion of associated primes in a ring R and their properties before we discuss the associated primes of (square-free) monomial ideals.

Definition 1.1.4. Let $I \subset R$ be an ideal. A prime ideal P is called an associated prime of I if there exists $u \in R$ such that $P = (I :_R u)$ where
$(I :_R u) = \{r \in R : \ ru \in I\}$.

The set of associated primes of I is denoted by $\mathrm{Ass}_R(R/I)$, and if there is no confusion about the underlying ring, we use the notation $\mathrm{Ass}(R/I)$. This set includes all the associated prime ideals associated with I. The minimal elements of $\mathrm{Ass}(R/I)$ are called minimal primes and denoted by $\mathrm{Min}(I)$. The embedded primes are the ones that are not minimal elements in $\mathrm{Ass}(R/I)$.

In a Noetherian ring R, Broadmann showed that the set of associated primes for the powers of an ideal I in R is stationary, that is

there exists a positive integer k_0 such that $\mathrm{Ass}(R/I^k) = \mathrm{Ass}(R/I^{k_0})$ for all $k > k_0$.

The minimal such k_0 is called the index of stability of I, and the set $\mathrm{Ass}(R/I^{k_0})$ is called the stable set of associated primes of I denoted by $\mathrm{Ass}^{\infty}(I)$.

Definition 1.1.5. Let $I \subset R$ be an ideal. I has the persistence property if

$$\mathrm{Ass}(R/I) \subset \mathrm{Ass}(R/I^2) \subset \ldots \subset \mathrm{Ass}(R/I^k)$$

for all $k \in \mathbb{Z}^+$.

Later, Qureshi and Herzog gave a stronger version of the persistence property.

Definition 1.1.6. [27, Theorem 1.3] An ideal I satisfies the *strong persistence property* if $(I^{k+1} : I) = I^k$ for all positive integers k.

The strong persistence property implies the persistence property.

In monomial ideals, in particular square-free monomial ideals, the associated primes have a simpler form. A monomial ideal is called *irreducible* if it is generated by pure powers of variables. Let $I \subset R$ be a square-free monomial ideal. Then, the irreducible monomial ideals that appear in the decomposition of I are all of the form $(x_{i_1}, \ldots, x_{i_k})$. The ideals $(x_{i_1}, \ldots, x_{i_k})$ are called the monomial prime ideals. A prime ideal P is called a minimal prime ideal of I if $I \subset P$, and there is no prime ideal containing I which is properly contained in P. We denote the set minimal prime ideals of I by $\mathrm{Min}(I)$.

Corollary 1.1.7. [22, Corollary 1.3.6] Let $I \subset S$ be a square-free monomial ideal. Then

$$I = \bigcap_{P \in \mathrm{Min}(I)} P$$

and each $P \in \mathrm{Min}(I)$ is a monomial prime ideal.

Definition 1.1.8. A monomial ideal $I \subset S$ is called *normally torsion-free* if $\mathrm{Ass}(S/I^k) \subseteq \mathrm{Ass}(S/I)$ for all $k \in \mathbb{Z}^+$.

For a square-free monomial ideal I, if I is normally torsion-free then by Corollary 1.1.7, one has $\mathrm{Min}(I) = \mathrm{Ass}(R/I^k)$ for all $k \geq 1$.

By Gitler et al. [17], it is well-known that the cover ideals of bipartite graphs are normally torsion-free. Furthermore, normally torsion-free square-free monomial ideals have been studied in Ha and Morey [20], and Sullivant [52]. On the other hand, little is known about the normally torsion-free monomial ideals that are not square-free. One of our motivations in this thesis is to give some classes of square-free monomial ideals that satisfy normally torsion-freeness. Additionally, we seek for ideals that are normal and satisfies the (symbolic) (strong) persistence property. If I is a square-free monomial ideal that is normally torsion-free , then one obtains the other results immediately. In the following, we recall these results.

First, we give the symbolic power for a square-free monomial ideal.

Proposition 1.1.9. [22, Proposition 1.4.4] Let $I \subset S$ be a square-free monomial ideal. Then

$$I^{(k)} = \bigcap_{P \in \mathrm{Min}(I)} P^k.$$

An ideal I has the *symbolic strong persistence property* if $(I^{(k+1)} : I^{(1)}) = I^{(k)}$ for all k.

In the following we recall the definition of normal ideals. Then we give conditions for a square-free monomial ideal being normal.

Let R be a unitary commutative ring and I an ideal in R. An element $f \in R$ is *integral* over I, if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with} \quad c_i \in I^i.$$

The set of elements $\overline{I}$ in R which are integral over I is the *integral closure* of I. The ideal I is *integrally closed* if $I = \overline{I}$, and I is *normal* if all powers of I are integrally closed.
The following result provides a characterization of when a square-free monomial ideal is normally torsion-free. Additionally, it gives when the regular powers and symbolic powers of a square-free monomial ideal are the same.

Theorem 1.1.10. [22, Theorem 1.4.6] Let $I \subset S$ be a square-free monomial ideal. Then the following conditions are equivalent:

(a) I is normally torsion-free;

(b) $I^{(k)} = I^k$ for all k.

If the equivalent conditions hold, then I is a normal ideal.

Kaiser, Stehlík, and Škrekovski [32] have shown that not all square-free monomial ideals have the persistence property. However, combinatorial methods have shown that many large families of square-free monomial ideals satisfy the persistence property and the strong persistence property.

Theorem 1.1.11. [44, Theorem 6.2] Every normal monomial ideal has the strong persistence property.

Since the strong persistence property implies the persistence property, a normally torsion-free square-free monomial ideal satisfies the persistence property.

The notion of normally torsion-free ideals is generalized in [4] as follows:

Definition 1.1.12. A monomial ideal I in a polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K is called *nearly normally torsion-free* if there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\mathrm{Ass}(S/I^m) = \mathrm{Min}(I)$ for all $1 \leq m \leq k$, and $\mathrm{Ass}(S/I^m) \subseteq \mathrm{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k+1$.

If I is a normally torsion-free square-free monomial ideal, then it is nearly normally torsion-free.

1.2 Krull dimension and depth

Let M be a module over a ring R. An element $x \in R$ is called an M-regular element if $xm = 0$ for $m \in M$ implies $m = 0$. In other words, x is a nonzero divisor on M.

Definition 1.2.1. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an M-regular sequence or, in short, an M-sequence if it satisfies the following conditions:

(i) x_i is regular on $M/(x_1, \ldots, x_{i-1})M$ for all $1 \leq i \leq n$,

(ii) $M/\mathbf{x}M \neq 0$.

An R-regular sequence is simply called a regular sequence. The typical example of a regular sequence is the sequence of indeterminates in the polynomial ring.

Let R be a Noetherian ring and M an S-module. If $\mathbf{x} = x_1, \ldots, x_n$ is an M-sequence then

$$(x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, \ldots, x_n)$$

is a strictly ascendant sequence of ideals. Thus, an M-sequence can be extended to a maximal one. A maximal M-sequence is an M-sequence $\mathbf{x} = x_1, \ldots, x_n$ such that for any $x_{n+1} \in R$, the sequence $\mathbf{x}, x_{n+1}$ is no longer an M-sequence.

Definition 1.2.2. Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and M a finitely generated R-module. The common length of all maximal M-sequences in $\mathfrak{m}$ is called the *depth* of M and will be denoted by $\operatorname{depth}(M)$.

Now, we give the definition of the Krull dimension of a ring.

Definition 1.2.3. Let R be a ring. Then the (Krull) dimension of R is given as:

$$\dim(R) = \sup\{n : P_0 \subset \ldots \subset P_n \text{ a chain of prime ideals in } R\}.$$

An Artinian ring has Krull dimension 0. In particular, fields are of Krull dimension 0.

Let P be a prime ideal of R. Then the height of P is defined as:

$$\operatorname{ht}(P) = \max\{n : P_0 \subset \ldots \subset P_n = P \text{ a chain of prime ideals}\}$$

In other words, the height of P is the Krull dimension of the localization of R at P. For an arbitrary ideal $I \subset R$ we have $\operatorname{ht}(I) = \min\{\operatorname{ht}(P) : P \supset I; P \text{ is a prime ideal}\}$.

Example 1.2.4. 1) Let $S = K[x_1, \ldots, x_n]$. Then $\operatorname{ht}(x_1, \ldots, x_i) = i$ for all $1 \leq i \leq n$.

2) Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Let $a_1, \ldots, a_r \in \mathfrak{m}$ be a regular sequence in R. Then $\dim(R/(a_1, \ldots, a_r)) = \dim R - r$.

3) Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. Then $\dim(S) = n$.

Proposition 1.2.5. [8, Proposition 1.2.12] Let R be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then $\operatorname{depth}(M) \leq \dim(M)$.

Definition 1.2.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. M is called Cohen-Macaulay if $\operatorname{depth}(M) = \dim(M)$. If R itself is a Cohen Macaulay R-module, then R is called a *Cohen-Macaulay ring.*

An ideal I is said to be *unmixed* if all the associated primes of I have the same height. In a Cohen-Macaulay ring R/I, the ideal I is called a *Cohen-Macaulay ideal.* If R/I is Cohen-Macaulay, then I is unmixed.

For an ideal $I \subset R$, the function $f(k) = \operatorname{depth} R/I^k$ is called the depth function of I.

Definition 1.2.7. Let $I \subset R$ be a graded ideal. Then, the *analytic spread* of I denoted by $\ell(I)$ is the Krull dimension of the $\mathcal{R}(\mathcal{I})/\mathfrak{m}\mathcal{R}(\mathcal{I})$ where $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k$.

Brodmann showed in [7] that the depth function stabilizes for a large k. This constant value is the *limit depth* of I. Let $I \subset R$ be an ideal and $\dim R = d$. Then

$$\lim \operatorname{depth}_{k \to \infty} R/I^k \leq d - \ell(I).$$

In a ring R, the *associated graded ring* of an ideal $I \subset R$, denoted by $\operatorname{gr}_I(R)$ is defined as

$$\operatorname{gr}_I(R) = R/I \bigoplus I/I^2 \bigoplus \cdots I^n/I^{n+1} \bigoplus \cdots$$

Eisenbud and Huneke showed in [10] the following:

Proposition 1.2.8. [10, Proposition 3.3] Let $(R, \mathfrak{m})$ be a local ring and let I be an ideal of R of height at least one. Suppose $\operatorname{gr}_I(R)$ is Cohen Macaulay. Then

$$\lim \operatorname{depth}_{k \to \infty} R/I^k = d - \ell(I)$$

where $\dim R = d$.

Definition 1.2.9. [28] The smallest integer k for which $\operatorname{depth} R/I^k = \operatorname{depth} R/I^{k_0}$ for all $k \geq k_0$ is called the *index of depth stability* of I and denoted by $\operatorname{dstab}(I)$.

1.3 Ideals with linear quotients

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K and I be a homogeneous ideal of S. Let

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p,j}(I)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}(I)} \xrightarrow{\phi_0} I \to 0,$$

be the minimal graded free resolution of I. For each i and j, $\beta_{i,j}(I)$ is the (i,j)*-th graded Betti number of* I, and the i*-th Betti number of* I is $\beta_i(I) = \sum_{j\in\mathbb{Z}} \beta_{i,j}$. The Castelnuovo–Mumford regularity (or simply regularity) of I, denoted by $\operatorname{reg}(I)$, is

$$\operatorname{reg}(I) = \max\{j \mid \beta_{i,i+j}(I) \neq 0\},$$

and the projective dimension of I is the length of its minimal graded free resolution, given by

$$\operatorname{proj}\dim(I) = \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

The ideal I is said to have d*-linear* resolution if $\beta_{i,j}(I) = 0$ for all i and all $j - i \neq d$.

Let I_d be the ideal generated by all homogeneous polynomials of degree d in I. The ideal I is called a *componentwise linear ideal* if I_d has a linear resolution for each d. Componentwise linear ideals were first introduced by Herzog and Hibi in [23]. After their first appearance, Jahan and Zheng in [31] provided a large class of componentwise linear ideals:

Definition 1.3.1. Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field K. Let $I = (u_1,\dots,u_r)$ be a monomial ideal. I is said to have *linear quotients* if $(u_1,\dots,u_{i-1}) : (u_i)$ is generated by subsets of variables for $2 \leq i \leq r$.

Theorem 1.3.2. [31, Theorem 2.7] Let $I \subset S$ be a monomial ideal. If I has linear quotients, then I have componentwise linear quotients.

Corollary 1.3.3. [31, Corollary 2.8] If $I \subset S$ is a monomial ideal with linear quotients, then I is componentwise linear.

If the ideal I is componentwise linear, then I may not have linear quotients:

Remark 1.3.4. [26, Remark 2.15] Despite having linear quotients, componentwise linearity (having a linear resolution) depends upon the characteristic of the field. Due to Reisner [45], a well-known example is the square-free monomial ideal

$$I = (x_1x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_5, x_2x_5x_6, x_3x_4x_6, x_3x_5x_6)$$

in $S = K[x_1, \ldots, x_6]$. The ideal I has a linear resolution if and only if $\mathrm{char}(K) \neq 2$. So I is componentwise linear if and only if $\mathrm{char}(K) \neq 2$. This example also shows the converse of Corollary 1.3.3 is not true since having linear quotients is a property that does not "see" the characteristic of the field.

A monomial ideal with linear quotients provides methods to compute Betti numbers without finding the minimal free resolution. Let I be a monomial ideal with linear quotients with respect to the ordering $u_1, \ldots, u_r$ of $\mathrm{G}(I)$. If I is generated in a single degree d, then I has linear resolution as shown in [26]. Following [26], we define

$$\mathrm{set}(u_k) = \{i : x_i \in (u_1, \ldots, u_{k-1}) : (u_k)\} \text{ for } k = 2, \ldots, r.$$

Using [26, Lemma 1.5], we can conclude that

$$\beta_{i,i+d}(I) = \sum_{1 \leq k \leq r} \binom{n_k}{i} \; \textit{where} \; |\mathrm{set}(u_k)| = n_k.$$

Sharifan and Varbaro generalized this result by giving a recursive formula for total and graded Betti numbers for an ideal not necessarily generated in the same degree as given in [49, Corollary 2.7].

1.4 Simplicial complexes

This section aims to discuss the combinatorics on square-free monomial ideals.

Let $[n] = \{1, \ldots, n\}$ be the ground set on n vertices. A simplicial complex Δ is a nonempty collection of subsets of $[n]$ such that if $A \in \Delta$ and $A' \subset A$, then $A' \in \Delta$. Elements of Δ are called faces of Δ. For any $A \in \Delta$, the dimension of A is $\dim A = |A| - 1$. The dimension of Δ is defined by $\dim \Delta = \max\{|A| : A \in \Delta\} - 1$. A maximal face of Δ is called a facet and we denote the set of facets by $\mathcal{F}(\Delta)$. Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_r\}$. Then Δ is generated by F_i, $1 \leq i \leq r$.

A pure simplicial complex has facets that have the same cardinality. A subset $G \subset [n]$ is called a non-face of Δ if $G \notin \Delta$. We denote the minimal non-faces of Δ by $\mathcal{N}(\Delta)$.

A square-free monomial ideal can be attached in two ways for a given simplicial complex Δ. Below, we provide the definition of these ideals.

Given a subset $A \subset [n]$, we define a monomial $x_A := \prod_{i \in A} x_i$.

Definition 1.4.1. Let $\Delta = \langle F_1, \ldots, F_r \rangle$ be a simplicial complex. Then the *facet ideal* of Δ is $I(\Delta) = (x_F : F \in F(\Delta))$, and the *Stanley-Reisner ideal* of Δ is $I_\Delta = (x_F : F \in \mathcal{N}(\Delta))$.

Example 1.4.2.

$$\mathcal{F}(\Delta) = \{\{1,2,3,4\},\{4,5\},\{3,5\}\}$$

$$\mathcal{N}(\Delta) = \{\{3,4,5\},\{1,5\},\{2,5\}\}.$$

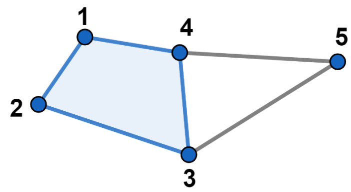

Figure 1.1 A geometric realization of Δ

In Figure1.1, $I(\Delta) = (x_1x_2x_3x_4, x_4x_5, x_3x_5)$, $I_\Delta = (x_3x_4x_5, x_1x_5, x_2x_5)$.

This process can be reversed as follows: For a given ideal I, if we set $I = I(\Delta)$ or $I = I_\Delta$, then there is a simplicial complex associated with I denoted by Δ_I. Thus, this one-to-one correspondence provides a tool to study ideals through simplicial complexes and vice versa.

Let Δ be a simplicial complex on $[n]$. Then the *Alexander dual* of Δ denoted by $\Delta^\vee$ is given by

$$\Delta^\vee = \langle [n] \setminus F | \ F \in \mathcal{N}(\Delta) \rangle.$$

For each subset $F \subset [n]$, we set $P_F = (x_i : i \in F)$.

Lemma 1.4.3. [22, Lemma 1.5.4] Let I_Δ be a Stanley-Reisner ideal. Then the standard primary decomposition of I_Δ

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}}$$

where $P_{\bar{F}}$ is the monomial prime ideal generated by the variables x_i with $i \in \overline{F} = [n] \setminus F$ and $F \in \mathcal{F}(\Delta)$.

Corollary 1.4.4. [22, Corollary 1.5.5] Let $I_\Delta = P_{F_1} \cap \ldots \cap P_{F_r}$ with each $F_i \subset [n]$ be the standard primary decomposition of I_Δ. Then

$$G(I_{\Delta^\vee}) = \{x_{F_1}, \ldots, x_{F_r}\}.$$

Example 1.4.5. Let Δ be the simplicial complex of Figure 1.1. Then

$$I_\Delta = (x_3, x_4) \cap (x_1, x_5) \cap (x_2, x_5) \cap (x_3, x_5) \cap (x_4, x_5),$$

and the ideal $I_{\Delta^\vee} = (x_3x_4, x_1x_5, x_2x_5, x_3x_5, x_4x_5)$.

Let $I \subset S$ be an arbitrary square-free monomial ideal. Then there is a unique simplicial complex Δ such that $I = I_\Delta$. For simplicity, we write $I^\vee$ for the ideal $I_{\Delta^\vee}$.

1.5 Graphs and hypergraphs

Let G be a graph. We denote the set of all vertices of G by $V(G)$ and the set of all edges of G by $E(G)$. If G has no loops or multiple edges, then G is a simple graph. In this thesis, all the graphs are simple, undirected, and finite.

A set $T \subseteq V(G)$ is called a *vertex cover* of G if it intersects every edge of G non-trivially. A vertex cover is called minimal if it does not properly contain any other vertex cover of G.

As in simplicial complexes, there are many ways to associate a square-free monomial ideal with a graph. We recall some commonly known definitions in this context. The edge ideal introduced by Villareal in [54] is one such example:

Definition 1.5.1. Let G be a simple graph with $V(G) = \{1, 2, \ldots, n\}$. The *edge* *ideal* of G, denoted by $I(G)$, is

$$I(G) = (x_i x_j : \{i,j\} \in E(G)).$$

Every square-free monomial ideal generated in degree two can be considered as an edge ideal of a graph. One of the interesting properties of the edge ideals is that their minimal primes correspond to the minimal vertex covers of their underlying graphs. In other words, the Alexander dual of the edge ideal of a graph G is the cover ideal of G, denoted by $J(G)$ is

$$J(G) = (\prod_{i\in T} x_i : T \text{ is a minimal vertex cover of } G).$$

Let t be a fixed positive integer. The t-path ideal of G, denoted by $I_t(G)$, is defined as

$$I_t(G) = (x_{i_1} x_{i_2} \cdots x_{i_t} : \{i_1, \ldots, i_t\} \text{ is a path of length } t-1 \text{ in } G).$$

The notion of path ideals is a generalization of edge ideals. Indeed, we have $I(G) = I_2(G)$.

For each vertex $v \in V(G)$, the *closed neighborhood* of v in G is defined as follows:

$$N_G[v] = \{u \in V(G) : \{u,v\} \in E(G)\} \cup \{v\}.$$

When there is no confusion about the underlying graph, we will denote $N_G[v]$ simply by $N[v]$. A subset $S \subseteq V(G)$ is called *dominating set* of G if $S \cap N[v] \neq \emptyset$, for all $v \in V(G)$. A dominating set is called minimal if it does not properly contain any other dominating set of G. A minimum dominating set of G is a minimal dominating set with the smallest size. The *dominating number* of G, denoted by $\gamma(G)$, is the size of its minimum dominating set, that is,

$$\gamma(G) = \min\{|S| : S \text{ is a minimal dominating set of } G\}$$

The dominating sets and domination numbers of graphs are well-studied topics in graph theory. We refer to [19] for further information.

Sharifan and Moradi recently introduced the closed neighborhood ideals and the dominating ideals of graphs. These ideals have very similar behavior in the case of edge ideals and cover ideals:

Definition 1.5.2. [48, Definition 2.1] Let G be a graph. Then, the closed neighborhood ideal denoted by $NI(G)$ is defined as follows:

$$NI(G) = (\prod_{j \in N[i]} x_j : i \in V(G)).$$

Moreover, the dominating ideal of G is defined as

$$DI(G) = (\prod_{i \in S} x_i : S \text{ is a minimal dominating set of } G).$$

Example 1.5.3. Consider C_6 with $V(C_6) = \{x_1, \ldots, x_6\}$. Then

$$NI(C_6) = (x_1x_2x_6, x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_1x_5x_6)$$

and

$$DI(C_6) = (x_1x_4, x_2x_5, x_1x_3x_5, x_3x_6, x_2x_4x_6).$$

It is shown in [48, Lemma 2.2] that $DI(G)$ is the Alexander dual of $NI(G)$. As indicated in [30, Example 2.2.], different graphs can admit the same $NI(G)$ and $DI(G)$.

Next, we recall some results about hypergraphs

Definition 1.5.4. A finite *hypergraph* $\mathcal{H}$ on a vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of edges $\{E_1, \ldots, E_m\}$ with $E_i \subseteq [n]$, for all $i = 1, \ldots, m$. The vertex set $[n]$ of $\mathcal{H}$ is denoted by $V(\mathcal{H})$, and the edge set of $\mathcal{H}$ is denoted by $E(\mathcal{H})$.

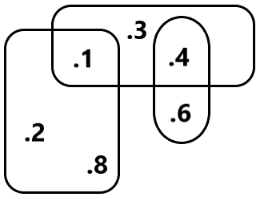

Figure 1.2 $\mathcal{H}_1$

Example 1.5.5. In Figure 1.2, $V(\mathcal{H}_1) = \{1,2,3,4,6,8\}$ and $E(\mathcal{H}_1) = \{\{1,2,8\},\{4,6\},\{1,3,4\}\}$.

A hypergraph $\mathcal{H}$ is called *simple*, if $E_i \subseteq E_j$ implies $i = j$. Simple hypergraphs are also known as *clutters*. Moreover, if $|E_i| = d$, for all $i = 1,\ldots,m$, then $\mathcal{H}$ is called a *d-uniform* hypergraph. A 2-uniform hypergraph $\mathcal{H}$ is just a finite simple graph. A vertex of a hypergraph $\mathcal{H}$ is said to be an *isolated vertex* if it is not contained in any edge of $\mathcal{H}$. In this thesis, all hypergraphs are simple, uniform, and without isolated vertices.

A hypergraph $\mathcal{H}$ is a d-partite hypergraph if its vertex set $V(\mathcal{H})$ is a disjoint union of sets $V_1,\ldots,V_d$ such that if E is an edge of $\mathcal{H}$, then $|E \cap V_i| \leq 1$. In particular, if $\mathcal{H}$ is a d-uniform d-partite hypergraph with a vertex partition $V_1,\ldots,V_d$, then $|E| = d$ and $|E \cap V_i| = 1$ for each $E \in E(\mathcal{H})$.

As in the graph case, the *edge ideal* of $\mathcal{H}$ is given by

$$I(\mathcal{H}) = (\prod_{j\in E_i} x_j : E_i \in E(\mathcal{H})).$$

A subset $W \subseteq V_{\mathcal{H}}$ is a *vertex cover* of $\mathcal{H}$ if $W \cap E_i \neq \emptyset$ for all $i = 1,\ldots,m$. A vertex cover W is *minimal* if no proper subset of W is a vertex cover of $\mathcal{H}$. The cover ideal of the hypergraph $\mathcal{H}$, denoted by $J(\mathcal{H})$, is given by

$$J(\mathcal{H}) = (\prod_{i\in W} x_i : W \text{ is a minimal vertex cover of } \mathcal{H}).$$

Similar to the case of edge ideal of graphs, the cover ideal $J(\mathcal{H})$ is the Alexander dual of $I(\mathcal{H})$, that is, $J(\mathcal{H}) = I(\mathcal{H})^{\vee}$, for example, see [55, Theorem 6.3.39].

Chapter 2

Dominating Ideals and Closed Neighborhood Ideals of Graphs

In the 1960s, Berge and Ore developed a mathematical formulation for graph domination, which has since gathered significant attention from researchers. Its applications span diverse fields such as computer science, operations research, linear algebra, and optimization. For additional related concepts regarding graph domination, readers are directed to [19]. In relation to domination in graphs, the definitions of closed neighborhood ideals and dominating ideals were introduced by Sharifan and Moradi recently in [48]. The closed neighborhood ideal of a simple graph G, denoted by $NI(G)$, is the squarefree monomial ideal generated by monomials corresponding to the closed neighborhoods of vertices of G. The dominating ideal of G, denoted by $DI(G)$, is the squarefree monomial ideal generated by monomials corresponding to the dominating sets of G. It is observed in [48, Lemma 2.2] that for any graphs G, the ideals $DI(G)$ and $NI(G)$ are Alexander dual of each other. In comparison to the edge ideals and cover ideals associated with graphs which are well-known and extensively studied, relatively little is known in the case of closed neighborhood ideals and dominating ideals of graphs. In this chapter, we provide some results about persistence properties and normally torsion-freeness of these ideals for specific graphs.

2.1 On the closed neighborhood ideals and dominating ideals of some classes of trees

In this section, our main goal is to prove that the closed neighborhood ideals and the dominating ideals of star graphs are normally torsion-free. To obtain this goal, we first give several results about normally torsion-free ideals and demonstrate a new proof of a well-known result related to strong persistence property.

Proposition 2.1.1. Let I be an ideal in a commutative Noetherian ring S that satisfies the strong persistence property. Then I has the persistence property.

Proof. Fix $k \geq 1$, and choose an arbitrary element $\mathfrak{p} \in \mathrm{Ass}_S(S/I^k)$. This implies that $\mathfrak{p} = (I^k :_S h)$ for some $h \in S$. Since I satisfies the strong persistence property, we have $(I^{k+1} :_S I) = I^k$, and so $\mathfrak{p} = ((I^{k+1} :_S I) :_S h)$. Let $\mathrm{G}(I) = \{u_1, \ldots, u_m\}$. Hence, one obtains $\mathfrak{p} = (I^{k+1} :_S h\sum_{i=1}^{m} u_i S) = \cap_{i=1}^{m}(I^{k+1} :_S hu_i)$. Accordingly, we get $\mathfrak{p} = (I^{k+1} :_S hu_i)$ for some $1 \leq i \leq m$. Therefore, $\mathfrak{p} \in \mathrm{Ass}_S(S/I^{k+1})$. This means that I has the persistence property, as claimed. $\square$

To prove Theorem 2.1.3, we need the following result. We state it here for ease of reference. For a given square-free monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, we denote by $I \setminus x_i$ the ideal generated by those elements in $\mathrm{G}(I)$ that does not contain x_i in their support.

Theorem 2.1.2. [47, Theorem 3.7] Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$. If there exists a square-free monomial $v \in I$ such that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \mathrm{Min}(I)$, and $\mathfrak{m} \setminus x_i \notin \mathrm{Ass}(R/(I \setminus x_i)^s)$ for all s and $x_i \in \mathrm{supp}(v)$, then I is normally torsion-free.

The next theorem will be used frequently to formulate proofs of some main results of this section. It provides a way to create new normally torsion-free ideals based on the existing ones.

Theorem 2.1.3. Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ and h be a square-free monomial in R. Let there exist two variables x_r and x_s with $1 \leq r \neq s \leq n$ such that $\gcd(h, u) = 1$ or

$\gcd(h,u) = x_r$ or $\gcd(h,u) = x_r x_s$ for all $u \in \mathrm{G}(I)$. Then $I + hR$ is normally torsion-free.

Proof. For convenience of notation, put $L := I + hR$. If $L \setminus x_k = \mathfrak{m} \setminus x_k$ for some $1 \le k \le n$, then one can write $L = x_k J + \mathfrak{m} \setminus x_k$. If $J = R$, then $L = \mathfrak{m}$, and there is nothing to prove. Let $J \neq R$, and take an arbitrary element $v \in \mathrm{G}(J)$. If $x_\ell \mid v$ for some $\ell \in \{1,\dots,n\} \setminus \{k\}$, then $v \in \mathfrak{m} \setminus x_k$, and so $J \subseteq \mathfrak{m} \setminus x_k$. This implies that $L = \mathfrak{m} \setminus x_k$, and hence the assertion holds. We thus assume that $L \setminus x_k \neq \mathfrak{m} \setminus x_k$ for all $k = 1,\dots,n$. We claim that $h \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \mathrm{Min}(L)$. Take an arbitrary element $\mathfrak{p} \in \mathrm{Min}(L)$. Since $h \in L$ and $L \subseteq \mathfrak{p}$, one has $h \in \mathfrak{p}$. Suppose, on the contrary, that $h \in \mathfrak{p}^2$. Due to h is square-free, this gives that $|\mathrm{supp}(h) \cap \mathrm{supp}(\mathfrak{p})| \ge 2$. We observe the following:
(i) If $x_s \in \mathrm{supp}(u)$ for some $u \in \mathrm{G}(I)$, then $x_r \in \mathrm{supp}(u)$ as well. It is due to the assumption on $\gcd(h,u)$ with $u \in \mathrm{G}(I)$.
(ii) At most one of x_r and x_s can be in $\mathrm{supp}(\mathfrak{p})$. Indeed, if both $x_r, x_s \in \mathrm{supp}(\mathfrak{p})$, then $x_r, x_s \in \mathrm{supp}(h) \cap \mathrm{supp}(\mathfrak{p})$. From (i), we see that $u \in \mathfrak{p} \setminus \{x_s\}$ for all $u \in \mathrm{G}(I)$. Also, $h \in \mathfrak{p} \setminus \{x_s\}$. Hence, $L \subset \mathfrak{p} \setminus \{x_s\}$, a contradiction to the minimality of $\mathfrak{p}$.

In order to establish our claim, we have the following cases to discuss:

Case 1. $x_r \in \mathfrak{p}$. Take any $x_i \in \mathrm{supp}(h) \cap \mathrm{supp}(\mathfrak{p})$ such that $x_r \neq x_i$. Then $x_s \neq x_i$ due to (ii). From the assumption on $\gcd(h,u)$ with $u \in \mathrm{G}(I)$ it follows that $x_i \notin \mathrm{supp}(I)$. Therefore, $I \subset \mathfrak{p} \setminus \{x_i\}$. Since $h \in \mathfrak{p} \setminus \{x_i\}$, we conclude that $L \subset \mathfrak{p} \setminus \{x_i\}$, a contradiction to the minimality of $\mathfrak{p}$.

Case 2. $x_s \in \mathfrak{p}$. By mimicking the same argument as in Case 1, we again obtain a contradiction to the minimality of $\mathfrak{p}$.

Case 3. $x_r \notin \mathfrak{p}$ and $x_s \notin \mathfrak{p}$. Take any $x_i, x_j \in \mathrm{supp}(h) \cap \mathrm{supp}(\mathfrak{p})$. Then $x_i, x_j \notin \mathrm{supp}(I)$, due to the assumption on $\gcd(h,u)$ with $u \in \mathrm{G}(I)$. It yields that $I \subset \mathfrak{p} \setminus \{x_i\}$. Since $h \in \mathfrak{p} \setminus \{x_i\}$, we conclude that $L \subset \mathfrak{p} \setminus \{x_i\}$, again a contradiction to the minimality of $\mathfrak{p}$.

This shows that our claim holds true. To complete the proof, note that for all $x_k \in \mathrm{supp}(h)$, one has $L \setminus x_k = I \setminus x_k$. Based on [46, Theorem 3.21], we gain $I \setminus x_k$ is normally torsion-free as well. This leads to $L \setminus x_k$ is normally torsion-free. Fix $s \ge 1$. Suppose, on the contrary, that $\mathfrak{m} \setminus x_k \in \mathrm{Ass}(R/(L \setminus x_k)^s)$ for some k. Because

$\operatorname{Ass}(R/(L\setminus x_k)^s)=\operatorname{Min}(L\setminus x_k)$, we get $\mathfrak{m}\setminus x_k\in\operatorname{Min}(L\setminus x_k)$, and so $L\setminus x_k=\mathfrak{m}\setminus x_k$, which is a contradiction. Therefore, $\mathfrak{m}\setminus x_i\notin\operatorname{Ass}(R/(I\setminus x_i)^s)$ for all s and $x_i\in\operatorname{supp}(h)$. Consequently, the assertion can be concluded readily from Theorem 2.1.2.

□

As an immediate consequence of Theorem 2.1.3, we give the following corollary.

Corollary 2.1.4. The path ideals corresponding to path graphs of length two are normally torsion-free.

Proof. Let $P=(V(P),E(P))$ denote a path graph with the vertex set $V(P)=\{x_1,\ldots,x_n\}$ and the edge set $E(P)=\{x_i,x_{i+1}\}\ :\ i=1,\ldots,n-1\}$. Hence, the path ideal corresponding to the path graph P of length two is given by

$$L:=(x_ix_{i+1}x_{i+2}\ :\ i=1,\ldots,n-2).$$

We proceed by induction on n. If $n=3$, then $L=(x_1x_2x_3)$, and there is nothing to prove. Let $n>3$ and the claim has been proven for $n-1$. Set $h:=x_{n-2}x_{n-1}x_n$ and $I:=(x_ix_{i+1}x_{i+2}\ :\ i=1,\ldots,n-3)$. One can easily check that, for each $u\in\mathrm{G}(I)$, we have $\gcd(h,u)=1$ or $\gcd(h,u)=x_{n-2}$ or $\gcd(h,u)=x_{n-2}x_{n-1}$. It follows from the induction hypothesis that I is normally torsion-free. Since $L=I+hR$, where $R=K[x_1,\ldots,x_n]$, we can derive the assertion from Theorem 2.1.3. □

As an application of Theorem 2.1.3, we give the following lemma.

Lemma 2.1.5. Let $G=(V(G),E(G))$ and $H=(V(H),E(H))$ be two finite simple graphs such that $V(H)=V(G)\cup\{w\}$ with $w\notin V(G)$, and $E(H)=E(G)\cup\{\{v,w\}\}$ for some vertex $v\in V(G)$. If $NI(G)$ is normally torsion-free, and $\prod_{j\in N_G[v]}x_j\notin\mathrm{G}(NI(G)$, then $NI(H)$ is normally torsion-free.

Proof. Let $NI(G)$ be normally torsion-free. It is routine to check that $NI(H)=NI(G)+(x_vx_w)R$, where $R=K[x_\alpha\ :\ \alpha\in V(H)]$. In addition, one can easily see that either $\gcd(x_vx_w,u)=1$ or $\gcd(x_vx_w,u)=x_v$ for all $u\in\mathrm{G}(NI(G))$. Therefore, the claim follows immediately from Theorem 2.1.3. □

We are ready to state the first main result of this section as an immediate corollary of Theorem 2.1.3 and Lemma 2.1.5.

Corollary 2.1.6. The closed neighborhood ideals of star graphs are normally torsion-free.

Proof. Proceed by induction on the number of vertices and use Lemma 2.1.5. □

In what follows, we investigate the closed neighborhood ideals related to the whisker graph and cone of a graph.

Definition 2.1.7. [55, Definition 7.3.10] Let G_0 be a graph on the vertex set $Y = \{y_1, \ldots, y_n\}$ and take a new set of variables $X = \{x_1, \ldots, x_n\}$. The *whisker graph* or *suspension* of G_0, denoted by $G_0 \cup W(Y)$, is the graph obtained from G_0 by attaching to each vertex y_i a new vertex x_i and the edge $\{x_i, y_i\}$. The edge $\{x_i, y_i\}$ is called a *whisker*.

Question 2.1.8.
(i) Can we conclude that the closed neighborhood ideals of trees are normally torsion-free?

(ii) Let G_0 be a graph and let $H := G_0 \cup W(Y)$ be its whisker graph. If $NI(G_0)$ is normally torsion-free, then can we deduce that $NI(H)$ is normally torsion-free?

Definition 2.1.9. [55, Definition 10.5.4] The *cone* $C(G)$, over the graph G, is obtained by adding a new vertex t to G and joining every vertex of G to t.

Lemma 2.1.10. Let G be a graph and let $H := C(G)$ be its cone. Then $NI(G)$ is normally torsion-free if and only if $NI(H)$ is normally torsion-free.

Proof. Assume that the cone $H = C(G)$ is obtained by adding the new vertex w to G and joining every vertex of G to w. Then one can easily see that

$$NI(H) = x_w NI(G) + (x_w \prod_{i \in V(G)} x_i).$$

Since $\prod_{i \in V(G)} x_i \in NI(G)$, this implies that $NI(H) = x_w NI(G)$.

The result can be deduced from [46, Lemma 3.12]. □

We recall the following definition which will be used in the proof of Lemma 2.1.12.

Definition 2.1.11. [37, Definition 2.1] Let $I \subset R = K[x_1,\dots,x_n]$ be a monomial ideal with $\mathrm{G}(I) = \{u_1,\dots,u_m\}$. Then I is said to be unisplit, if there exists $u_i \in \mathrm{G}(I)$ such that $\gcd(u_i,u_j) = 1$ for all $u_j \in \mathrm{G}(I)$ with $i \neq j$.

Lemma 2.1.12. Let G be a graph and let $H := C(G)$ be its cone. Then $DI(G)$ is normally torsion-free if and only if $DI(H)$ is normally torsion-free.

Proof. Suppose that the cone $H = C(G)$ is obtained by adding the new vertex w to G and joining every vertex of G to w. Using [48, Lemma 2.2] yields that

$$DI(H) = DI(G) + (x_w).$$

It follows now from [20, Lemma 3.4] that, for all s,

$$\mathrm{Ass}(DI(H)^s) = \{(\mathfrak{p}, x_w) \ : \ \mathfrak{p} \in \mathrm{Ass}(DI(G)^s)\}. \tag{2.1}$$

Let $DI(G)$ be normally torsion-free. Then the claim can be deduced from [46, Theorem 2.5]. Conversely, let $DI(H)$ be normally torsion-free. By using [46, Theorem 3.21], we obtain that $DI(G)$ is normally torsion-free. □

Our next goal is to show that the dominating ideals of star graphs are normally torsion-free. To do this, we first prove some results of a general nature. We recall some definitions from [15] which are necessary to establish Theorem 2.1.16. Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph with $V(\mathcal{H}) = \{x_1,\dots,x_n\}$.

Definition 2.1.13. (see [15, Definition 2.7]) A *d-coloring* of $\mathcal{H}$ is any partition of $V(\mathcal{H}) = C_1 \cup \cdots \cup C_d$ into d disjoint sets such that for every $E \in E(\mathcal{H})$, we have $E \nsubseteq C_i$ for all $i = 1,\dots,d$. (In the case of a graph G, this simply means that any two vertices connected by an edge receive different colors.) The C_i's are called the color classes of $\mathcal{H}$. Each color class C_i is an *independent set*, meaning that C_i does not contain any edge of the hypergraph. The chromatic number of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the minimal d such that $\mathcal{H}$ has a d-coloring.

Definition 2.1.14. (see [15, Definition 2.8]) The hypergraph $\mathcal{H}$ is called *critically d-chromatic* if $\chi(\mathcal{H}) = d$, but for every vertex $x \in V(\mathcal{H})$, $\chi(\mathcal{H} \setminus \{x\}) < d$, where $\mathcal{H} \setminus \{x\}$ denotes the hypergraph $\mathcal{H}$ with x and all edges containing x removed.

Definition 2.1.15. (see [15, Definition 4.2]) For each s, the *s-th expansion* of $\mathcal{H}$ is defined to be the hypergraph obtained by replacing each vertex $x_i \in V(\mathcal{H})$ by a collection $\{x_{ij} \mid j = 1,\dots,s\}$, and replacing $E(\mathcal{H})$ by the edge set that consists of edges $\{x_{i_1 l_1},\dots,x_{i_r l_r}\}$ whenever $\{x_{i_1},\dots,x_{i_r}\} \in E(\mathcal{H})$ and edges $\{x_{il}, x_{ik}\}$ for $l \neq k$. We denote this hypergraph by $\mathcal{H}^s$. The new variables x_{ij} are called the shadows of x_i. The process of setting x_{il} to equal to x_i for all i and l is called the *depolarization*.

The following result is a slightly generalized form of [42, Theorem 4.9].

Theorem 2.1.16. Assume that $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ and $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ are two finite simple hypergraphs such that $V(\mathcal{H}) = V(\mathcal{G}) \cup \{w_1,\dots,w_t\}$ with $w_i \notin V(\mathcal{G})$ for each $i = 1,\dots,t$, and $E(\mathcal{H}) = E(\mathcal{G}) \cup \{\{v, w_1,\dots,w_t\}\}$ for some vertex $v \in V(\mathcal{G})$. Then

$$\mathrm{Ass}_{R'}(R'/J(\mathcal{H})^s) = \mathrm{Ass}_R(R/J(\mathcal{G})^s) \cup \{(x_v, x_{w_1},\dots,x_{w_t})\},$$

for all s, where $R = K[x_\alpha : \alpha \in V(\mathcal{G})]$ and $R' = K[x_\alpha : \alpha \in V(\mathcal{H})]$.

Proof. For convenience of notation, set $I := J(\mathcal{G})$ and $J := J(\mathcal{H})$. We first prove that $\mathrm{Ass}_R(R/I^s) \cup \{(x_v, x_{w_1},\dots,x_{w_t})\} \subseteq \mathrm{Ass}_{R'}(R'/J^s)$ for all s. Fix $s \geq 1$, and assume that $\mathfrak{p} = (x_{i_1},\dots,x_{i_r})$ is an arbitrary element of $\mathrm{Ass}_R(R/I^s)$. According to [15, Lemma 2.11], we get $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(\mathcal{G}_\mathfrak{p})^s)$, where $K[\mathfrak{p}] = K[x_{i_1},\dots,x_{i_r}]$ and $\mathcal{G}_\mathfrak{p}$ is the induced subhypergraph of $\mathcal{G}$ on the vertex set $\{i_1,\dots,i_r\} \subseteq V(\mathcal{G})$. Since $\mathcal{G}_\mathfrak{p} = \mathcal{H}_\mathfrak{p}$, we have $\mathfrak{p} \in \mathrm{Ass}(K[\mathfrak{p}]/J(H_\mathfrak{p})^s)$. This yields that $\mathfrak{p} \in \mathrm{Ass}_{R'}(R'/J^s)$. On account of $(x_v, x_{w_1},\dots,x_{w_t}) \in \mathrm{Ass}_{R'}(R'/J^s)$, one derives

$$\mathrm{Ass}_R(R/I^s) \cup \{(x_v, x_{w_1},\dots,x_{w_t})\} \subseteq \mathrm{Ass}_{R'}(R'/J^s).$$

To complete the proof, it is enough for us to show the reverse inclusion. Assume that $\mathfrak{p} = (x_{i_1},\dots,x_{i_r})$ is an arbitrary element of $\mathrm{Ass}_{R'}(R'/J^s)$ with $\{i_1,\dots,i_r\} \subseteq V(\mathcal{H})$. If $\{i_1,\dots,i_r\} \subseteq V(\mathcal{G})$, then [15, Lemma 2.11] implies that $\mathfrak{p} \in \mathrm{Ass}_R(R/I^s)$, and the proof is done. Thus, let $\{w_1,\dots,w_t\} \cap \{i_1,\dots,i_r\} \neq \emptyset$. It follows from [15, Corollary 4.5] that the associated primes of $J(\mathcal{H})^s$ will correspond to critical chromatic subhypergraphs of size $s+1$ in the s-th expansion of $\mathcal{H}$. This means that one can take the induced subhypergraph on the vertex set $\{i_1,\dots,i_r\}$, and then form the s-th expansion on this induced subhypergraph, and within this new hypergraph find a critical $(s+1)$-chromatic hypergraph. Notice that since this expansion cannot have any critical chromatic subgraphs, this implies that $\mathcal{H}_\mathfrak{p}$ must be connected.

Without loss of generality, one may assume that $i_1 = v$ and $i_2 = w_1, i_3 = w_2, \ldots, i_{t+1} = w_t$. Thanks to $w_1, \ldots, w_t$ are connected to v in the hypergraph $\mathcal{H}$, and because this induced subhypergraph is critical, if we remove any vertex w_k for some $1 \leq k \leq t$, one can color the resulting hypergraph with at least s colors. This leads to that w_k has to be adjacent to at least s vertices. But the only things w_k is adjacent to are the shadows of w_i for each $i = 1, \ldots, t$, and the shadows of v, and so one has a clique among these vertices. Accordingly, w_k and its neighbors will form a clique of size $s+1$. Since a clique is a critical graph, it follows that we do not need any element of $\{i_{t+2}, \ldots, i_r\}$ or their shadows when making the critical $(s+1)$-chromatic hypergraph. Hence, we obtain $\mathfrak{p} = (x_v, x_{w_1}, \ldots, x_{w_t})$. This finishes the proof. □

Lemma 2.1.17. Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ with $\mathrm{G}(I) \subset R$. Then the ideal

$$L := IS \cap (x_n, x_{n+1}, x_{n+2}, \ldots, x_m) \subset S = R[x_{n+1}, x_{n+2}, \ldots, x_m],$$

is normally torsion-free.

Proof. It is well-known that one can view the square-free monomial ideal I as the cover ideal of a simple hypergraph $\mathcal{H}$ such that the hypergraph $\mathcal{H}$ corresponds to $I^\vee$, where $I^\vee$ denotes the Alexander dual of I. Then we have $I = J(\mathcal{H})$, where $J(\mathcal{H})$ denotes the cover ideal of the hypergraph $\mathcal{H}$. Fix $k \geq 1$. On account of Theorem 2.1.16, we get the following equality

$$\mathrm{Ass}_S(S/L^k) = \mathrm{Ass}_R(R/J(\mathcal{H})^k) \cup \{(x_n, x_{n+1}, x_{n+2}, \ldots, x_m)\}.$$

Because I is normally torsion-free, one derives that $\mathrm{Ass}_R(R/J(\mathcal{H})^k) = \mathrm{Min}(J(\mathcal{H}))$, and hence $\mathrm{Ass}_S(S/L^k) = \mathrm{Min}(J(\mathcal{H})) \cup \{(x_n, x_{n+1}, x_{n+2}, \ldots, x_m)\}$. This gives rise to $\mathrm{Ass}_S(S/L^k) = \mathrm{Min}(L)$. Therefore, L is normally torsion-free, as claimed. □

Lemma 2.1.18. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{\{v, w\}\}$ for some vertex $v \in V(G)$, and $\prod_{j \in N_G[v]} x_j \notin \mathrm{G}(NI(G)$. If $DI(G)$ is normally torsion-free, then $DI(H)$ is normally torsion-free.

Proof. Let $DI(G)$ be normally torsion-free. It follows from [48, Lemma 2.2] that

$DI(H) = DI(G) \cap (x_v, x_w)R$, where $R = K[x_\alpha \ : \ \alpha \in V(H)]$. Now, we can conclude the assertion from Lemma 2.1.17. $\square$

We are in a position to give the second main result of this section in the following corollary, which is related to the dominating ideals of star graphs.

Corollary 2.1.19. The dominating ideals of star graphs are normally torsion-free.

Proof. We use the induction on the number of vertices together with Lemma 2.1.18. $\square$

Question 2.1.20. (i) Can we conclude that the dominating ideals of trees are normally torsion-free?

(ii) Let G_0 be a graph and let $H := G_0 \cup W(Y)$ be its whisker graph. If $DI(G_0)$ is normally torsion-free, can we deduce that $DI(H)$ is normally torsion-free?

2.2 On the closed neighborhood ideals and dominating ideals of cycles

The edge ideals and the cover ideals of bipartite graphs are known to be normally torsion-free, see [17, 50]. In particular, the edge ideals and the cover ideals of even cycles are normally torsion-free. However, this behaviour changes when we consider the odd cycles. The cover ideals of odd cycles happen to be nearly normally torsion-free, see [38], but edge ideals of odd cycles do not admit such tamed behaviour for the set of their associated primes. Given these facts, it is natural to expect some irregularities for the closed neighborhood ideals and dominating ideals of even and odd cycles. It can be verified by using Macaulay2 [18] that in general, the closed neighborhood ideals of cycles, regardless of the parity of their lengths, are neither normally torsion-free nor nearly normally torsion-free. However, in this section, we will show that the closed neighborhood ideals of cycles admit strong persistence property. On the other side, as another main result of this section, we will show that the dominating ideals of cycles are nearly normally torsion-free.

To establish the above-mentioned results, we begin by proving the following theorem, which gives an inductive way to study the normality of an ideal.

Theorem 2.2.1. Let I and H be two normal square-free monomial ideals in a polynomial ring $R = K[x_1,\dots,x_n]$ such that $I+H$ is normal. Let $x_c \in \{x_1,\dots,x_n\}$ be a variable with $\gcd(v,x_c) = 1$ for all $v \in \mathrm{G}(I)\cup\mathrm{G}(H)$. Then $L := I + x_c H$ is normal.

Proof. Let $G(I) = \{u_1,\dots,u_s\}$ and $\mathrm{G}(H) = \{h_1,\dots,h_r\}$. Since $\gcd(v,x_c) = 1$ for all $v \in \mathrm{G}(I)\cup\mathrm{G}(H)$, without loss of generality, one may assume that $x_c = x_1 \in K[x_1]$ and

$$\mathrm{G}(I)\cup\mathrm{G}(H) = \{u_1,\dots,u_s,h_1,\dots,h_r\} \subseteq K[x_2,\dots,x_n].$$

We must show that $\overline{L^t} = L^t$ for all integers $t \geq 1$. For this purpose, it is enough to prove that $\overline{L^t} \subseteq L^t$. Let α be a monomial in $\overline{L^t}$ and write $\alpha = x_1^b\delta$ with $x_1 \nmid \delta$ and $\delta \in R$. On account of [22, Theorem 1.4.2], $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Write

$$\alpha^k = x_1^{bk}\delta^k = \prod_{i=1}^{s} u_i^{p_i} x_1^{q+\varepsilon} \prod_{j=1}^{r} h_j^{q_j}\beta, \tag{2.2}$$

with $\sum_{i=1}^{s} p_i = p$, $\sum_{j=1}^{r} q_j = q$, $p+q = tk$, $\varepsilon \geq 0$, and β is some monomial in R such that $x_1 \nmid \beta$. Because $x_1 \nmid \beta$, $x_1 \nmid \delta$, and $\gcd(v,x_1) = 1$ for all $v \in \mathrm{G}(I)\cup\mathrm{G}(H)$, one can conclude that $bk = q+\varepsilon$. Accordingly, by virtue of (2.2), we obtain

$$\delta^k = \prod_{i=1}^{s} u_i^{p_i} \prod_{j=1}^{r} h_j^{q_j}\beta \in (I+H)^{tk}.$$

This leads to $\delta \in \overline{(I+H)^t}$. Thanks to $I+H$ is normal, we deduce that $\overline{(I+H)^t} = (I+H)^t$, and so $\delta \in (I+H)^t$. Therefore, one can write

$$\delta = \prod_{i=1}^{s} u_i^{l_i} \prod_{j=1}^{r} h_j^{z_j}\gamma, \tag{2.3}$$

with $\sum_{i=1}^{s} l_i = l$, $\sum_{j=1}^{r} z_j = z$, $l+z = t$, and γ is some monomial in R. Note that $x_1 \nmid \gamma$ as $x_1 \nmid \delta$. Due to $x_1^{bk}\delta^k \in L^{tk}$, it follows immediately from (2.3) that

$$\prod_{i=1}^{s} u_i^{l_i k} x_1^{bk} \prod_{j=1}^{r} h_j^{z_j k}\gamma^k \in L^{tk} = (I + x_1 H)^{tk}.$$

Consequently, we conclude that $bk \geq zk$, that is, $b \geq z$. This gives rise to

$$x_1^b \delta = \prod_{i=1}^{s} u_i^{l_i} x_1^b \prod_{j=1}^{r} h_j^{z_j} \gamma \in (I + x_1 H)^t ,$$

and the proof is over. $\square$

We state the third main result of this section in the next theorem, which is related to the closed neighborhood ideals of cycles.

Theorem 2.2.2. Let C_n be a cycle graph of order n. Then $NI(C_n)$ is normal.

Proof. (i) Let $C_n = (V(C_n), E(C_n))$ be a cycle graph of order n with $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} \,:\, i = 1, \ldots, n-1\} \cup \{\{x_n, x_1\}\}$. Then the closed neighborhood ideal of C_n is given by

$$NI(C_n) = (x_i x_{i+1} x_{i+2} \,:\, i = 1, \ldots, n) \subset R = K[x_1, \ldots, x_n],$$

where x_{n+1} (respectively, x_{n+2}) represents x_1 (respectively, x_2). If $n = 3$, then $NI(C_3) = (x_1 x_2 x_3)$, and so there is nothing to prove. Thus, let $n \geq 4$. Put $H := (x_2 x_3, x_{n-1} x_n, x_2 x_n)$ and $I := (x_i x_{i+1} x_{i+2} \,:\, i = 2, \ldots, n-2)$. One can easily see that $NI(C_n) = I + x_1 H$. Our strategy is to use Theorem 2.2.1 to complete the proof. To do this, we first show that I, H, and $I + H$ are normal. Assume that G is a path graph with $V(G) = \{x_2, x_3, x_{n-1}, x_n\}$ and $E(G) = \{\{x_2, x_3\}, \{x_{n-1}, x_n\}, \{x_2, x_n\}\}$. It is routine to check that $I(G) = H$, where $I(G)$ denotes the edge ideal of G. Since by [17, Corollary 2.6], the edge ideal of any path graph is normally torsion-free, and by remembering this fact that every normally torsion-free square-free monomial ideal is normal, we deduce that H is a normal square-free monomial ideal. Now, assume that P is a path graph with $V(P) = \{x_2, x_3, \ldots, x_{n-1}, x_n\}$ and $E(P) = \{\{x_i, x_{i+1}\} \,:\, i = 2, \ldots, n-1\}$. It is not hard to check that $I = I_3(P)$, where $I_3(P)$ denotes the path ideal of length 2 of P. It follows readily from Corollary 2.1.4 that $I = I_3(P)$ is normally torsion-free, and so is normal. To complete the proof, we show that $I + H$ is normal. To accomplish this, we note that

$$I + H = (x_2 x_3, x_{n-1} x_n, x_2 x_n, x_i x_{i+1} x_{i+2} \,:\, i = 3, \ldots, n-3).$$

Set $A := (x_3, x_n)$ and $B := (x_{n-1} x_n, x_i x_{i+1} x_{i+2} \,:\, i = 3, \ldots, n-3)$. Notice that

$I+H=B+x_2A$. It is clear that A is a normal ideal. Furthermore, it follows from Corollary 2.1.4 and Theorem 2.1.3 that B is normally torsion-free, and so is normal. In addition, we have

$$B+A=(x_3,x_n,x_ix_{i+1}x_{i+2}\ :\ i=4,\ldots,n-3).$$

One can easily conclude from Corollary 2.1.4 and Theorem 2.1.3 that $B+A$ is normally torsion-free, and hence is normal. By virtue of Theorem 2.2.1, we deduce that $B+x_2A$ is normal, and so $I+H$ is normal as well. Finally, note that $\gcd(v,x_1)=1$ for all $v\in \mathrm{G}(I)\cup \mathrm{G}(H)$. This finishes the proof.

The neighborhood ideals of cycles are particularly nice because they are generated by monomials of the same degree. This fact together with Theorem 2.2.2 enables us to study the depth of powers of $NI(C_n)$. For this purpose, we first recall the following definition and result from [27].

Definition 2.2.3. Let $I\subset R$ be a monomial ideal with $\mathrm{G}(I)=\{u_1,\ldots,u_m\}$. The *linear relation graph* Γ_I of I is the graph with the edge set

$$E(\Gamma_I)=\{\{x_i,x_j\}:\text{there exist } u_k,u_l\in \mathrm{G}(I) \text{ such that } x_iu_k=x_ju_l\},$$

and the vertex set $V(\Gamma_I)=\bigcup_{\{x_i,x_j\}\in E(\Gamma)}\{i,j\}$.

Theorem 2.2.4. [27, Theorem 3.3] Let $I\subset R=K[x_1,\ldots,x_n]$ be a monomial ideal generated in a single degree whose linear relation graph has r vertices and s connected components. Then

$$\mathrm{depth}(R/I^t)\leq n-t-1 \text{ for } t=1,\ldots,r-s.$$

In order to apply the above theorem, we first analyze the linear relation graph of $NI(C_n)$. Let $V(C_n)=[n]$ and $E(C_n)=\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$. We set the following notations.

1.1 $u_i=\prod_{j\in N[i]}x_j$. In simple words, u_i is the monomial that corresponds to the closed neighborhood of the vertex i.

1.2 Note that $u_i=x_{i-1}x_ix_{i+1}$, for all $i=2,\ldots n-1$ and $u_1=x_nx_1x_2$, $u_n=$

$x_{n-1}x_nx_1$. To synchronize this notation for all i, if $i > n$ then we read i as $i(\bmod n)$. In this way, we can write $u_i = x_{i-1}x_ix_{i+1}$, for all $i = 1, \ldots n$.

Remark 2.2.5. Let $i \neq j$. Note that each variable x_i appears in exactly three monomials in $\mathrm{G}(NI(C_n))$, and these monomials are $u_{i-1} = x_{i-2}x_{i-1}x_i$, $u_i = x_{i-1}x_ix_{i+1}$ and $u_{i+1} = x_ix_{i+1}x_{i+2}$. From this observation, we conclude that $\{x_i, x_j\} \in E(\Gamma)$ if and only if there exists a path of length three from i to j in C_n. Here a path P of length n is defined on $n+1$ vertices and n edges.

Remark 2.2.6. Let $n \geq 4$, and set $I_n := NI(C_n)$. Remark 2.2.5 leads us to the following:

2.1 $|V(\Gamma_{I_n})| = n$. This can be easily verified because, for every i, we can find another vertex j such that there is a path of length three from i to j in C_n.

2.2 Γ_{I_n} has one connected component if $n \neq 3k$, for all $k \geq 2$. Indeed, if $n = 1(\bmod 3)$, that is, $n = 3k+1$ for some $k \geq 1$, then we have

$$\begin{gathered} E(\Gamma_{I_n}) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_{3k+1}\}, \\ \{x_{3k+1}, x_3\}, \{x_3, x_6\}, \ldots, \{x_{3k-3}, x_{3k}\}, \\ \{x_{3k}, x_2\}, \{x_2, x_5\}, \ldots, \{x_{3k-2}, x_1\}\}. \end{gathered}$$

If $n = 2(\bmod 3)$, that is, $n = 3k+2$ for some $k \geq 1$, then we have

$$\begin{gathered} E(\Gamma_{I_n}) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_{3k+1}\}, \\ \{x_{3k+1}, x_2\}, \{x_2, x_5\}, \ldots, \{x_{3k-1}, x_{3k+2}\}, \\ \{x_{3k+2}, x_3\}, \{x_3, x_6\}, \ldots, \{x_{3k}, x_1\}\}. \end{gathered}$$

2.3 Γ_{I_n} has three connected components if $n = 3k$, for some $k \geq 2$. Set $V(\Gamma_1) = \{x_1, x_4, \ldots, x_{3k-2}\}$, and

$$E(\Gamma_1) = \{\{x_1, x_4\}, \{x_4, x_7\}, \ldots, \{x_{3k-2}, x_1\}\}.$$

Set $V(\Gamma_2) = \{x_2, x_5, \ldots, x_{3k-1}\}$, and

$$E(\Gamma_1) = \{\{x_2, x_5\}, \{x_5, x_8\}, \ldots, \{x_{3k-1}, x_2\}\}.$$

Set $V(\Gamma_3) = \{x_3, x_6, \ldots, x_{3k}\}$, and

$$E(\Gamma_1) = \{\{x_3, x_6\}, \{x_6, x_9\}, \ldots, \{x_{3k}, x_3\}\}.$$

It can be easily verified that Γ_{I_n} is the disjoint union of Γ_1, Γ_2, and Γ_3.

Theorem 2.2.4 together with Remark 2.2.6 leads to the following corollary:

Corollary 2.2.7. Let $n \neq 0 (\text{mod } 3)$. Set $I_n = NI(C_n) \subset R = K[x_1, \ldots, x_n]$. Then $\text{depth}(R/I_n^{n-1}) = 0$. In particular, $\mathfrak{m} \in \text{Ass}(R/I_n^{n-1})$ and $\lim_{k\to\infty}\text{depth} R/I_n^k = 0$.

We provide the fourth main result of this section in the subsequent theorem, which is related to the dominating ideals of cycles. We will use the following result to establish our proof.

Corollary 2.2.8. [42, Corollary 3.3] Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Let $I(\mathfrak{m} \setminus \{x_i\})$ be normally torsion-free for all $i = 1, \ldots, n$. Then I is nearly normally torsion-free.

Now, we state the next main result.

Theorem 2.2.9. The dominating ideals of cycles are nearly normally torsion-free.

Proof. Let C_n denote a cycle graph of order n with $V(C_n) = \{x_1, \ldots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} \ : \ i = 1, \ldots, n-1\} \cup \{\{x_n, x_1\}\}$. In the light of [48, Lemma 2.2], the dominating ideal of C_n is given by

$$DI(C_n) = \bigcap_{i=1}^{n} (x_i, x_{i+1}, x_{i+2}) \subset R = [x_1, \ldots, x_n],$$

where x_{n+1} (respectively, x_{n+2}) represents x_1 (respectively, x_2). Set $I := DI(C_n)$. Our strategy is to use Corollary 2.2.8. To do this, we must show that $I(\mathfrak{m} \setminus \{x_i\})$ is normally torsion-free for all $i = 1, \ldots, n$, where $\mathfrak{m} = (x_1, \ldots, x_n)$. Without loss of generality, it is sufficient for us to prove that $I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. To simplify notation, set $F := \bigcap_{i=2}^{n-2} (x_i, x_{i+1}, x_{i+2})$. By virtue of Corollary

2.2.8, one has to show that the ideal $F = I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. To do this, let $T = (V(T), E(T))$ be the rooted tree with the root 2, the vertex set $V(T) = \{x_2, \ldots, x_n\}$, and the edge set $E(T) = \{(x_i, x_{i+1}) \ : \ i = 2, \ldots, n-1\}$, where (x_i, x_{i+1}) denotes the directed edge from the vertex x_i to the vertex x_{i+1} for all $i = 2, \ldots, n-1$. It is not hard to check that F is the Alexander dual of the path ideal generated by all paths of length 2 in the rooted tree T. Now, one can deduce from [33, Theorem 3.2] that $F = I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. This completes the proof. $\square$

Chapter 3

Componetwise Linearity of Dominating Ideals of Path Graphs

Let G be a simple graph. The graph G is called a *path graph* on $\{x_1, x_2, \ldots, x_n\}$ if $E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}\}$. We denote a path graph on n vertices by P_n. In this chapter, we study the dominating ideals of P_n. Our main goal is to prove $DI(P_n)$ is componentwise linear. To do this, first, we show that $DI(P_n)$ has linear quotients by giving a precise order of minimal generators of $DI(P_n)$.

3.1 Linear quotient order of dominating ideals of path graphs

In this section, we will construct a recursive order of $G(DI(P_n))$, which gives linear quotients. To do this, we first give a recursive presentation of dominating sets of P_n. Throughout the following text, for any non-empty set A and for any element x, we set $xA := \{xy|\ y \in A\}$. If A is empty, then we also set xA as an empty set.

Remark 3.1.1. Let $I_n = DI(P_n)$, and set

$$\begin{aligned} A_1 &= \emptyset, & B_1 &= \{x_1\}, \\ A_2 &= \{x_1\}, & B_2 &= \{x_2\}, \\ A_3 &= \{x_2\}, & B_3 &= \{x_1x_3\}, \\ A_4 &= \{x_1x_3, x_2x_3\}, & B_4 &= \{x_1x_4, x_2x_4\}. \end{aligned}$$

It can be readily verified with simple computations that

$$\begin{aligned} G(I_1) &= A_1 \cup B_1 = \{x_1\}; \\ G(I_2) &= A_2 \cup B_2 = \{x_1, x_2\}; \\ G(I_3) &= A_3 \cup B_3 = \{x_2, x_1x_3\}; \\ G(I_4) &= A_4 \cup B_4 = \{x_1x_3, x_2x_3, x_1x_4, x_2x_4\}. \end{aligned}$$

For $n \geq 5$, we set $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$ and $B_n = x_nG(I_{n-2})$. It can be easily verified that

$$G(I_5) = A_5 \cup B_5 = x_4G(I_2) \cup x_4(x_3A_1) \cup x_5G(I_3) = \{x_1x_4, x_2x_4, x_2x_5, x_1x_3x_5\}$$

and similarly,

$$\begin{aligned} G(I_6) &= A_6 \cup B_6 = x_5G(I_3) \cup x_5(x_4A_2) \cup x_6G(I_4) \\ &= \{x_2x_5, x_1x_3x_5, x_1x_4x_5, x_1x_3x_6, x_2x_3x_6, x_1x_4x_6, x_2x_4x_6\}. \end{aligned}$$

Below, we give a recursive way to construct dominating ideals for path graphs.

Theorem 3.1.2. Let $I_n = DI(P_n)$ and $n \geq 5$. Then $G(I_n) = A_n \cup B_n$, where

$$A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}x_{n-2}A_{n-4} \quad \text{and} \quad B_n = x_nG(I_{n-2}).$$

Moreover, the sets $x_{n-1}G(I_{n-3})$, $x_{n-1}x_{n-2}A_{n-4}$, and $x_nG(I_{n-2})$ are pairwise disjoint.

Proof. It immediately follows from the definition of the minimal dominating set and the construction of A_n and B_n that $x_{n-1}G(I_{n-3})$, $x_{n-1}x_{n-2}A_{n-4}$, and $x_nG(I_{n-2})$ are pairwise disjoint. To prove $G(I_n) = A_n \cup B_n$, we apply induction on n. The case

$n = 5$ can be verified from Remark 3.1.1. Assume that $n > 5$. First we show that $A_n \cup B_n \subseteq G(I_n)$. Let $w \in B_n$, then $w = x_n w'$ for some $w' \in G(I_{n-2})$. Since $\text{supp}(w')$ is a minimal dominating set of P_{n-2} and x_n dominates x_{n-1} and itself in P_n, we obtain that $\text{supp}(w)$ is a dominating set of P_n. The minimality of $\text{supp}(w)$ follows from the minimality of $\text{supp}(w')$. This gives $w \in G(I_n)$.

Now, let $w \in A_n$. Then $w = x_{n-1}w''$ for some $w'' \in G(I_{n-3})$ or $w'' \in x_{n-2}A_{n-4}$. If $w'' \in G(I_{n-3})$, then $\text{supp}(w'')$ is a minimal dominating set of P_{n-3}. Furthermore, x_{n-1} dominates x_{n-2}, x_n and itself in P_n. It yields that $\text{supp}(x_{n-1}w'')$ is a dominating set of P_n. The minimality of $\text{supp}(w)$ follows from the minimality of $\text{supp}(w')$. This gives $w \in G(I_n)$. On the other hand, if $w'' \in x_{n-2}A_{n-4}$, then there exists $w''' \in A_{n-4}$ such that $w = x_{n-1}x_{n-2}w'''$. By induction hypothesis, we have $A_{n-4} \subset G(I_{n-4})$, hence $\text{supp}(w''')$ is a minimal dominating set of P_{n-4}. The remaining vertices $x_{n-3}, x_{n-2}, x_{n-1}, x_n$ of P_n are minimally dominated by x_{n-2} and x_{n-1}. Thus, $\text{supp}(x_{n-1}x_{n-2}w''')$ is a minimal dominating set of P_n and $w \in G(I_n)$.

Next we show that $G(I_n) \subseteq A_n \cup B_n$. Since $N[x_n] = \{x_n, x_{n-1}\}$, for any $w \in G(I_n)$, either x_n divides w or x_{n-1} divides w. However x_n and x_{n-1} do not divide w at the same time, by the virtue of minimality of $\text{supp}(w)$ as a dominating set of P_n.

First, assume that $w = x_n w'$. Since x_n dominates itself and x_{n-1}, the set $\text{supp}(w') = \text{supp}(w) \setminus \{x_n\}$ is a dominating set of P_{n-2}. The minimality of $\text{supp}(w')$ follows from the minimality of $\text{supp}(w)$. It gives $w' \in G(I_{n-2})$ and $w \in B_n$.

Now, let $w = x_{n-1}w''$. Below we show that $w \in A_n$. To do this, we consider the following cases:

Case (1): Assume that $x_{n-2} \notin \text{supp}(w'')$. Since x_{n-1} dominates only x_n, x_{n-2} and itself, for each $i = 1, \dots, n-3$, the vertex x_i must be dominated by $\text{supp}(w'')$. This shows that $\text{supp}(w'')$ is a dominating set of P_{n-3}. The minimality of $\text{supp}(w'')$ follows from the minimality of $\text{supp}(w)$, and we obtain $w'' \in G(I_{n-3})$. This shows that $w \in x_{n-1}G(I_{n-3})$.

Case (2): Assume that $x_{n-2} \in \text{supp}(w'')$. Then $w = x_{n-1}w'' = x_{n-1}x_{n-2}u$ for some monomial u. Then $x_{n-3} \notin \text{supp}(u)$, otherwise, $\text{supp}(w) \setminus \{x_{n-2}\}$ is a dominating set of P_n, a contradiction to the minimality of $\text{supp}(w)$. This shows that $\text{supp}(u)$ is a dominating set of P_{n-4}. The minimality of $\text{supp}(u)$ follows from the minimality of $\text{supp}(w)$ and hence $u \in G(I_{n-4})$. By the induction hypothesis we have $G(I_{n-4}) \subseteq$

$A_{n-4} \cup B_{n-4}$. Note that $x_{n-4} \notin \mathrm{supp}(u)$, otherwise, $\mathrm{supp}(w) \setminus \{x_{n-2}\}$ is a dominating set of P_n, a contradiction to the minimality of $\mathrm{supp}(w)$. This shows that $u \notin B_{n-4}$ because every element in B_{n-4} is a multiple of x_{n-4}. Therefore $u \in A_{n-4}$ and $w \in x_{n-1}x_{n-2}A_{n-4} \subset A_n$. This completes the proof. $\square$

Remark 3.1.3. Set $I_n = DI(P_n)$. For each $n \geq 1$, we order the elements of $G(I_n)$ by first listing the elements of A_n and then listing the elements of B_n. In particular, for $1 \leq n \leq 4$, we order the elements in A_n and B_n as given in Remark 3.1.1. For $n \geq 5$, from Theorem 3.1.2 we have

$$\begin{aligned} G(I_n) &= A_n \cup B_n \\ &= [x_{n-1}G(I_{n-3}) \cup x_{n-1}x_{n-2}A_{n-4}] \cup x_nG(I_{n-2}) \end{aligned}$$

Let $G(I_{n-3}) = \{u_1, \ldots, u_t\}$, $A_{n-4} = \{a_1, \ldots, a_k\}$ and $G(I_{n-2}) = \{v_1, \ldots, v_s\}$, then an ordering of $G(I_n)$ for $n \geq 5$ is given below

$$x_{n-1}u_1, \ldots, x_{n-1}u_t, x_{n-1}x_{n-2}a_1, \ldots, x_{n-1}x_{n-2}a_k, x_nv_1, \ldots, x_nv_s.$$

For example, in Remark 3.1.1 the elements of $G(I_5)$ and $G(I_6)$ are listed in the order described above. Throughout the following text, we let $\mathbf{A_n}$ and $\mathbf{B_n}$ be the ideals generated by the elements of A_n and B_n, respectively.

Next, we will show that $DI(P_n)$ has linear quotients with respect to the ordering of the generators given in the remark above. To do this, we first state the following simple observation.

Lemma 3.1.4. Let $I \subset S = K[x_1, \ldots, x_n, x, y]$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$ such that $x, y \notin \mathrm{supp}(u_i)$ for all $i = 1, \ldots, m$. Then the following statements hold.

(i) Let w be a monomial in S with $x \notin \mathrm{supp}(w)$. Then any generator of $xI : (w)$ is divisible by x.

(ii) For any $u_i \in G(I)$, we have $xI : (yu_i) = (x)$.

Proof. (i) It is easy to see that every generator of $xI : (w)$ is of the form $xu_i/\gcd(xu_i, w)$ for some i, for example, see [22, Proposition 1.2.2]. Using the assumption that $x \notin \mathrm{supp}(w)$ gives x does not divide $\gcd(xu_i, w)$, as required.

(ii) It follows from (i) that every generator of $xI : (yu_i)$ is divisible by x. Moreover, we have $(x) = (xu_i) : (yu_i) \subseteq xI : (yu_i)$. This gives $xI : (yu_i) = (x)$. $\square$

Now we give the main theorem of this section.

Theorem 3.1.5. For any $n \geq 1$, $DI(P_n)$ has linear quotients.

Proof. Set $I_n = DI(P_n)$. We show that the order of $G(I_n)$ described in Remark 3.1.3 is a linear quotient order. We proceed by applying induction on n. It is easy to verify the assertion for $1 \leq n \leq 5$ by following straightforward computations. Let $n > 5$ and for all $1 \leq k < n$ assume that I_k has linear quotients with order as in Remark 3.1.3. Let $G(I_{n-3}) = \{u_1, \ldots, u_t\}$ and $G(I_{n-2}) = \{v_1, \ldots, v_s\}$ where the generators are listed in the linear quotient order. Indeed, by inductive hypothesis $\mathbf{A_n}$ has linear quotients as well because $G(I_{n-4}) = A_{n-4} \cup B_{n-4}$. Let $A_{n-4} = \{a_1, \ldots, a_k\}$ where the generators are listed in the linear quotient order.

First, we show that $\mathbf{A_n} = x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}$ has linear quotients. We know that $x_{n-1}I_{n-3}$ and $x_{n-1}x_{n-2}\mathbf{A_{n-4}}$ have linear quotients because I_{n-3} and $\mathbf{A_{n-4}}$ have linear quotients by inductive hypothesis as mentioned above. Moreover, for $i = 2, \ldots, k$, we have

$$[x_{n-1}I_{n-3} + (x_{n-1}x_{n-2}a_1, \ldots, x_{n-1}x_{n-2}a_{i-1})] : (x_{n-1}x_{n-2}a_i) =$$
$$x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) + (x_{n-1}x_{n-2}a_1, \ldots, x_{n-1}x_{n-2}a_{i-1}) : (x_{n-1}x_{n-2}a_i)$$

Therefore, we only need to show that $x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i)$ has linear quotients, for all $i = 1, \ldots, k$. We claim that for all $i = 1, \ldots, k$,

$$x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) = (x_{n-3}, x_{n-4}). \tag{3.1}$$

Proof of claim: For $5 < n \leq 9$, the above claim can be verified with straightforward computation. The reason we let $n > 9$ in the following argument is to avoid the negative indices in the following text.

Note that $x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) = I_{n-3} : (x_{n-2}a_i)$. Using Theorem 3.1.2 we obtain

$$I_{n-3} = x_{n-4}I_{n-6} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5}$$

which gives

$$\begin{aligned} I_{n-3} : (x_{n-2}a_i) = x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) \\ + x_{n-3}I_{n-5} : (x_{n-2}a_i) \end{aligned} \tag{3.2}$$

Since $A_{n-4} = x_{n-5}G(I_{n-7}) \cup x_{n-5}x_{n-6}A_{n-8}$, we separate the discussion in the following two cases: $a_i \in x_{n-5}G(I_{n-7})$ or $a_i \in x_{n-5}x_{n-6}A_{n-8}$.

Case 1: Let $a_i \in x_{n-5}G(I_{n-7})$. From Lemma 3.1.4 and Theorem 3.1.2, we obtain

$$\begin{aligned} x_{n-3}I_{n-5} : (x_{n-2}a_i) &= (x_{n-3}\mathbf{A_{n-5}} + x_{n-3}x_{n-5}I_{n-7}) : (x_{n-2}a_i) \\ &= (x_{n-3}). \end{aligned}$$

Note that $x_{n-5}G(I_{n-7}) = x_{n-5}(A_{n-7} \cup B_{n-7})$. If $a_i \in x_{n-5}A_{n-7}$, then again from Lemma 3.1.4, we obtain

$$x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) = (x_{n-4})$$

On the other hand, if $a_i \in x_{n-5}B_{n-7} = x_{n-5}x_{n-7}G(I_{n-9})$, then by using the expansion $I_{n-6} = x_{n-7}I_{n-9} + x_{n-7}x_{n-8}\mathbf{A_{n-10}} + \mathbf{B_{n-6}}$ obtained from Theorem 3.1.2, and as an application of Lemma 3.1.4, we have

$$\begin{aligned} x_{n-4}I_{n-6} : (x_{n-2}a_i) &= (x_{n-4}x_{n-7}I_{n-9} + x_{n-4}x_{n-7}x_{n-8}\mathbf{A_{n-10}} + x_{n-4}\mathbf{B_{n-6}}) : (x_{n-2}a_i) \\ &= (x_{n-4}) \end{aligned}$$

Then, again from Lemma 3.1.4 we obtain

$$x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) = (x_{n-4})$$

Therefore, from (3.2) we conclude that $I_{n-3} : (x_{n-2}a_i) = (x_{n-3}, x_{n-4})$ and the claim holds.

Case 2: Let $a_i \in x_{n-5}x_{n-6}A_{n-8}$. Theorem 3.1.2 gives

$$\begin{aligned} I_{n-6} &= \mathbf{A_{n-6}} + \mathbf{B_{n-6}} = \mathbf{A_{n-6}} + x_{n-6}I_{n-8} \\ &= \mathbf{A_{n-6}} + x_{n-6}\mathbf{A_{n-8}} + x_{n-6}\mathbf{B_{n-8}} \end{aligned}$$

Thanks to Lemma 3.1.4, we obtain

$$\begin{aligned} x_{n-4}I_{n-6} : (x_{n-2}a_i) &= (x_{n-4}\mathbf{A_{n-6}} + x_{n-4}x_{n-6}\mathbf{A_{n-8}} + x_{n-4}x_{n-6}\mathbf{B_{n-8}}) : (x_{n-2}a_i) \\ &= (x_{n-4}) \end{aligned}$$

Hence

$$\begin{aligned} I_{n-3} : (x_{n-2}a_i) &= x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) + x_{n-3}I_{n-5} : (x_{n-2}a_i) \\ &= (x_{n-4}) + x_{n-3}I_{n-5} : (x_{n-2}a_i) \end{aligned}$$

From Theorem 3.1.2, we have the expansion

$$\begin{aligned} I_{n-5} &= \mathbf{A_{n-5}} + \mathbf{B_{n-5}} \\ &= x_{n-6}I_{n-8} + x_{n-6}x_{n-7}\mathbf{A_{n-9}} + x_{n-5}I_{n-7} \\ &= x_{n-6}\mathbf{A_{n-8}} + x_{n-6}\mathbf{B_{n-8}} + x_{n-6}x_{n-7}\mathbf{A_{n-9}} + x_{n-5}I_{n-7} \end{aligned}$$

Once again, as a direct application of Lemma 3.1.4, we obtain

$$x_{n-3}I_{n-5} : (x_{n-2}a_i) = (x_{n-3}).$$

This completes the proof of our claim.

Next, we show that $\mathbf{A_n} + (x_nv_1, \ldots, x_nv_{i-1}) : (x_nv_i)$ has linear quotients for each $i = 2, \ldots, s$. By induction hypothesis, I_{n-2} has linear quotients which is equivalent to $(x_nv_1, \ldots, x_nv_{i-1}) : (x_nv_i)$ has linear quotients for $i = 2, \ldots, r$. Therefore, to complete the proof of the theorem, it only remains to show that $\mathbf{A_n} : (x_nv_i)$ has linear quotients for each $i = 1, \ldots, r$. We claim that

$$\mathbf{A_n} : (x_nv_i) = (x_{n-1}) \tag{3.3}$$

Proof of claim: For $5 < n \leq 7$, the above claim can be verified with straightforward computation. The reason we let $n > 7$ in the following argument is to avoid the negative indices in the following text.

Since $G(I_{n-2}) = A_{n-2} \cup B_{n-2}$, we first consider the case when $v_i \in A_{n-2} = x_{n-3}G(I_{n-5}) \cup x_{n-3}x_{n-4}A_{n-6}$. After a repeated use of Theorem 3.1.2, we obtain

$$\begin{aligned} I_{n-3} &= x_{n-4}I_{n-6} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5} \\ &= x_{n-4}\mathbf{A_{n-6}} + x_{n-4}\mathbf{B_{n-6}} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5} \end{aligned}$$

Using above equality together with Lemma 3.1.4 gives $x_{n-1}I_{n-3} : (x_n v_i) = (x_{n-1})$. Therefore, in this case,

$$\begin{aligned} \mathbf{A_n} : (x_n v_i) &= (x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}) : (x_n v_i) \\ &= x_{n-1}I_{n-3} : (x_n v_i) + x_{n-1}x_{n-2}\mathbf{A_{n-4}} : (x_n v_i) \\ &= (x_{n-1}) \quad \text{ as required.} \end{aligned}$$

Next, let $v_i \in B_{n-2} = x_{n-2}G(I_{n-4}) = x_{n-2}A_{n-4} \cup x_{n-2}B_{n-4}$. If $v_i \in x_{n-2}A_{n-4}$, then using Lemma 3.1.4 gives

$$\mathbf{A_n} : (x_n v_i) = [x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}] : (x_n v_i) = (x_{n-1})$$

If $v_i \in x_{n-2}B_{n-4} = x_{n-2}x_{n-4}G(I_{n-6})$, then using Lemma 3.1.4 gives

$$\begin{aligned} x_{n-1}I_{n-3} : (x_n v_i) &= [x_{n-1}x_{n-4}I_{n-6} + x_{n-1}x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-1}\mathbf{B_{n-3}}] : (x_n v_i) \\ &= (x_{n-1}), \end{aligned}$$

and we again retrieve $\mathbf{A_n} : (x_n v_i) = [x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}] : (x_n v_i) = (x_{n-1})$. This completes the proof. $\square$

Using Theorem 3.1.5, we retrieve the following result from [40, Theorem 2.8].

Corollary 3.1.6. For any $n \geq 1$, $DI(P_n)$ is a componentwise linear ideal.

Proof. By Theorem 3.1.5, $DI(P_n)$ has linear quotients. Thus, by [22, Theorem 8.2.15], $DI(P_n)$ is componentwise linear. $\square$

3.2 Betti numbers of dominating ideals of path graphs

In this section, we give a recursive formula to compute the Betti numbers of dominating ideals of path graphs. To do this, we recall the following result of Sharifan and Varbaro from [49] which gives the Betti numbers, regularity and projective dimension of an ideal with linear quotients.

Theorem 3.2.1. [[49], Corollary 2.7] Let I be a monomial ideal with linear quotients with respect to $u_1,\ldots,u_r$ where $G(I)=\{u_1,\ldots,u_r\}$. Let n_p be the number of minimal generators of $(u_1,\ldots,u_{p-1}):u_p$ for $p=1,\ldots,r$. Then

$$\beta_{i,i+j}(I)=\sum_{1\leq p\leq r,\deg(u_p)=j}\binom{n_p}{i},\qquad \beta_i(I)=\sum_{p=1}^{r}\binom{n_p}{i}$$

$$\operatorname{reg}(I)=\max\{\deg(u_p):p=1,\ldots,r\}$$
$$\operatorname{proj}\dim(I)=\max\{n_p:p=1,\ldots,r\}$$

In [48, Theorem 2.6], authors computed the regularity and projective dimension of $NI(P_n)$. Using $NI(P_n)^\vee=DI(P_n)$ and invoking Terai's well-known result [53, Corollary 0.3] one can formulate the regularity and projective dimension of $DI(P_n)$. However, in the following result, we describe the regularity and projective dimension of $DI(P_n)$ in the terms of n as an application of Theorem 3.2.1 and Theorem 3.1.5.

Theorem 3.2.2. For any $n\geq 2$, the following holds.

(1) $\operatorname{reg}(DI(P_n))=\lceil\frac{n}{2}\rceil=\operatorname{proj}\dim(NI(P_n))+1$,

(2) $\operatorname{proj}\dim(DI(P_n))=\lfloor\frac{n}{2}\rfloor=\operatorname{reg}(NI(P_n))-1$.

Proof. A well known result of Terai [53, Corollary 0.3] states that for any square-free monomial ideal $\operatorname{reg}(I)=\operatorname{proj}\dim(I^\vee)+1$, and from [48, Lemma 2.2], we have $NI(P_n)^\vee=DI(P_n)$. Therefore, to prove the assertion, it is enough to compute the regularity and projective dimension of $DI(P_n)$. Let $n\geq 2$ and $I_n=DI(P_n)$. It follows from Theorem 3.2.1 that the regularity of I_n is

$$\max\{|S|:S\text{ is a minimal dominating set of }P_n\}.$$

Note that a minimal dominating set A of P_n does not contain more than $\lceil \frac{n}{2} \rceil$ vertices of P_n. Otherwise, we readily obtain a contradiction to the minimality of A. Therefore, $\operatorname{reg}(I_n) \leq \lceil \frac{n}{2} \rceil$. On the other hand, it is easy to see that for any n, the set $\{x_i : i \text{ is odd and } i \leq n\}$ is a minimal dominating set of P_n. This gives us $\operatorname{reg}(I_n) = \lceil \frac{n}{2} \rceil$, as required.

To prove $\operatorname{proj\,dim}(I_n) = \lfloor \frac{n}{2} \rfloor$, we apply induction on n. For $2 \leq n \leq 6$, the equality can be verified using Theorem 3.2.1. For $n > 6$, following (3.1), (3.3) and the linear quotient order of $DI(P_n)$ given in Theorem 3.1.5, we obtain

$$\operatorname{proj\,dim}(I_n) = \max\{\operatorname{proj\,dim}(I_{n-3}), \operatorname{proj\,dim}(\mathbf{A_{n-4}}) + 2, \operatorname{proj\,dim}(I_{n-2}) + 1\}$$

Moreover, using $\mathbf{A_{n-4}} \subset I_{n-4}$ and the inductive hypothesis yields $\operatorname{proj\,dim}(\mathbf{A_{n-4}}) \leq \operatorname{proj\,dim}(I_{n-4}) = \lfloor \frac{n-4}{2} \rfloor$, $\operatorname{proj\,dim}(I_{n-3}) = \lfloor \frac{n-3}{2} \rfloor$, and $\operatorname{proj\,dim}(I_{n-2}) = \lfloor \frac{n-2}{2} \rfloor$. This gives us the desired formula. $\square$

Using Theorem 3.2.1, and the linear quotient order of $DI(P_n)$ from Theorem 3.1.5, first we list Betti numbers of $I_n = DI(P_n)$, for $n = 1, \ldots, 6$.

n	$\beta_0(I_n)$	$\beta_1(I_n)$	$\beta_2(I_n)$	$\beta_3(I_n)$
1	1	-	-	-
2	2	1	-	-
3	2	1	-	-
4	4	4	1	-
5	4	4	1	-
6	7	11	6	1

Now, we give recursive formulas for the total and graded Betti numbers of $DI(P_n)$, for $n > 6$. To simplify the notation in the subsequent text, we use the following definition. Let J be a monomial ideal with linear quotients and $u_1, \ldots, u_s$ be the linear quotient order of the generators of J. We call the colon ideal $(u_1, \ldots, u_{k-1}) : u_k$ the *k-th colon* of J. It follows from Theorem 3.1.5 that I_n has linear quotients with respect to the order of generators given in Remark 3.1.3. For each n, we denote by $s_k^{(n)}$, the size of k-th colon of I_n.

Theorem 3.2.3. Let $I_n = DI(P_n)$ with $n > 6$.

$$\begin{aligned}\beta_i(I_n) = \beta_i(I_{n-3})\\ &+ \beta_i(I_{n-2}) + \beta_{i-1}(I_{n-2})\\ &+ \beta_i(I_{n-4}) + 2\beta_{i-1}(I_{n-4}) + \beta_{i-2}(I_{n-4})\\ &- \beta_i(I_{n-6}) - 3\beta_{i-1}(I_{n-6}) - 3\beta_{i-2}(I_{n-6}) - \beta_{i-3}(I_{n-6})\end{aligned}$$

Proof. Recall from Theorem 3.1.2 that $G(I_n) = A_n \cup B_n$, where

$$A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4}), \text{ and } B_n = x_nG(I_{n-2}).$$

Let $|A_n| = t$. We recall the equality in (3.3) from the proof of Theorem 3.1.5 that states $\mathbf{A_n} : (x_n v_i) = (x_{n-1})$. Let $n \geq 3$ and $|G(I_n)| = r$. It follows from Theorem 3.1.5 that I_n has linear quotients with respect to the order of generators given in Remark 3.1.3. For each n, we denote by $s_k^{(n)}$, the size of k-th colon of I_n. Then

$$\begin{aligned}\beta_i(I_n) &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \binom{s_k^{(n)}}{i}\\ &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \binom{s_k^{(n-2)}+1}{i} \quad \text{by using (3.3)}\\ &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \left[\binom{s_k^{(n-2)}}{i} + \binom{s_k^{(n-2)}}{i-1}\right]\\ &= \beta_i(\mathbf{A_n}) + \beta_i(I_{n-2}) + \beta_{i-1}(I_{n-2}).\end{aligned}$$

Therefore

$$\beta_i(\mathbf{A_n}) = \beta_i(I_n) - \beta_i(I_{n-2}) - \beta_{i-1}(I_{n-2}). \tag{3.4}$$

On the other hand, for $n > 4$, using $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$, and the equality (3.1) in Theorem 3.1.5, we obtain the following equations. Below, we let $|G(I_{n-3})| = p$, and denote the size of k-th colon of $\mathbf{A_n}$ by $a_k^{(n)}$.

$$\begin{aligned}\beta_i(\mathbf{A_n}) &= \beta_i(I_{n-3}) + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)}+2}{i} \\ &= \beta_i(I_{n-3}) + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i-2} + 2\sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i-1} + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i}\end{aligned}$$

This gives

$$\beta_i(\mathbf{A_n}) = \beta_i(I_{n-3}) + \beta_{i-2}(\mathbf{A_{n-4}}) + 2\beta_{i-1}(\mathbf{A_{n-4}}) + \beta_i(\mathbf{A_{n-4}}). \tag{3.5}$$

For $n > 6$, using (3.4) together with (3.5) gives us the required recursive formula of total Betti numbers of I_n. □

Next, we give a recursive formula to compute graded Betti numbers of $DI(P_n)$ for $n > 6$.

Theorem 3.2.4. Let $I_n = DI(P_n)$ and $n > 6$.

$$\begin{aligned}\beta_{i,i+j}(I_n) &= \beta_{i,i+j-1}(I_{n-3}) \\ &\quad + \beta_{i,i+j-1}(I_{n-2}) + \beta_{i-1,i+j-2}(I_{n-2}) \\ &\quad + \beta_{i,i+j-2}(I_{n-4}) + 2\beta_{i-1,i+j-3}(I_{n-4}) + \beta_{i-2,i+j-4}(I_{n-4}) \\ &\quad - \beta_{i,i+j-3}(I_{n-6}) - 3\beta_{i-1,i+j-4}(I_{n-6}) - 3\beta_{i-2,i+j-5}(I_{n-6}) - \beta_{i-3,i+j-6}(I_{n-6})\end{aligned}$$

Proof. We proceed as in the case of total Betti numbers and follow the same notations given in Theorem 3.2.3. Let $G(I_n) = \{u_1, \ldots, u_r\}$ and $|A_n| = t$. Using Theorem 3.2.1 together with the linear quotient order given in Remark 3.1.3 gives

$$
\begin{aligned}
\beta_{i,i+j}(I_n) &= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1\\ \deg u_k=j}}^{r} \binom{s_k^{(n)}}{i} \\
&= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1\\ \deg u_k=j-1}}^{r} \binom{s_k^{(n-2)}+1}{i} \quad \text{by using (3.3) and Theorem 3.1.2} \\
&= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1\\ \deg u_k=j-1}}^{r} \left[\binom{s_k^{(n-2)}}{i} + \binom{s_k^{(n-2)}}{i-1}\right] \\
&= \beta_{i,i+j}(\mathbf{A_n}) + \beta_{i,i+j-1}(I_{n-2}) + \beta_{i-1,i+j-2}(I_{n-2}).
\end{aligned}
$$

Therefore

$$
\beta_{i,i+j}(\mathbf{A_n}) = \beta_{i,i+j}(I_n) - \beta_{i,i+j-1}(I_{n-2}) - \beta_{i-1,i-1+j-1}(I_{n-2}). \tag{3.6}
$$

On the other hand, for $n > 4$, using $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$, and the equality (3.1) in Theorem 3.1.5, we obtain the following equations. Below, we let $|G(I_{n-3})| = p$.

$$
\begin{aligned}
\beta_{i,i+j}(\mathbf{A_n}) &= \beta_{i,i+j-1}(I_{n-3}) + \sum_{\substack{k=p+1\\ \deg u_k=j-2}}^{t} \binom{a_k^{(n-4)}+2}{i} \\
&= \beta_{i,i+j-1}(I_{n-3}) + \sum_{\substack{k=p+1\\ \deg u_k=j-2}}^{t} \left[\binom{a_k^{(n-4)}}{i-2} + 2\binom{a_k^{(n-4)}}{i-1} + \binom{a_k^{(n-4)}}{i}\right]
\end{aligned}
$$

This gives

$$
\begin{aligned}
\beta_{i,i+j}(\mathbf{A_n}) = {} & \beta_{i,i+j-1}(I_{n-3}) + \beta_{i-2,i+j-4}(\mathbf{A_{n-4}}) + 2\beta_{i-1,i+j-3}(\mathbf{A_{n-4}}) \\
& + \beta_{i,i+j-2}(\mathbf{A_{n-4}}).
\end{aligned}
$$

For $n > 6$, using (3.6) together with the above equality gives us the following recursive formula of the total Betti numbers of dominating ideals of path graphs. $\square$

Chapter 4

The Edge Ideals of t-Spread d-Partite Hypergraphs

In 2019, Qureshi, Herzog, and Ene introduced the notion of t-spread monomials in a polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$ and studied some classes of ideals and $\mathbb{K}$-algebras generated by t-spread monomials.

Definition 4.0.1. Let $u = x_{i_1} \cdots x_{i_d}$ be a monomial in S and $t \geq 0$. The monomial u is called t-spread if $i_j - i_{j-1} \geq t$ for all $j = 2, \dots, d$. A monomial ideal $I \subset S$ is called t-spread if it is generated by t-spread monomials.

After their first appearance, different classes of t-spread monomial ideals have been studied by many authors (see [5, 42, 3, 9]). In 2023, Ficarra gave a more generalized notion of t-spread monomials by replacing the integer t with $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$.

Definition 4.0.2. [14] Let $\mathbf{t} = (t_1, t_2, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$. A monomial $x_{i_1} x_{i_2} \cdots x_{i_d} \in S = \mathbb{K}[x_1, \dots, x_n]$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called $\mathbf{t}$-*spread* if $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \dots, d$. A monomial ideal in S is called a $\mathbf{t}$-*spread monomial ideal* if it is generated by $\mathbf{t}$-spread monomials.

In this chapter, we study $\mathbf{t}$-spread monomial ideals which appear as the edge ideals of certain d-partite hypergraphs.

4.1 t-spread d-partite hypergraphs and their edge ideals

Let $V = \{V_1, \ldots, V_d\}$ be a partitioning of a finite set $U \subset \mathbb{N}$ such that $p < q$ if $p \in V_i, q \in V_j$ with $i < j$. We call $\{i_1, \ldots, i_d\} \subset U$ a $\mathbf{t}$-spread set if $i_j \in V_j$ for all $j = 1, \ldots, d$ and $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \ldots, d$. We call the hypergraph $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ on vertex set $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) = U$, a complete $\mathbf{t}$-spread d-partite hypergraph if all $\mathbf{t}$-spread sets of U are the edges of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. For $\mathbf{t} = (1, \ldots, 1)$, the hypergraph $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ is a complete d-partite hypergraph, see [6, Example 3]. The edge ideal of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$, denoted by $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$, is a t-spread monomial ideal generated by those monomials whose indices correspond to the edges of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. It turns out that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ admits many nice algebraic and homological properties.

Now, we introduce the definition of $\mathbf{t}$-spread d-partite hypergraphs. To do this, we give the following notation. For any integers $i \leq j$, let $[i,j] := \{k : i \leq k \leq j\}$ and for any integer n, we set $[n] := \{1, \ldots, n\}$.

Definition 4.1.1. Let $\mathcal{H}$ be a d-partite hypergraph with $V(\mathcal{H}) \subseteq [n]$, and $V = \{V_1, \ldots, V_d\}$ be a family defining partitioning of $V(\mathcal{H})$ such that if $p \in V_i$ and $q \in V_j$ with $i < j$, then $p < q$. Let $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{N}^{d-1}$. An edge E of $\mathcal{H}$ is called a $\mathbf{t}$-spread edge if

($*$) $E = \{i_1, i_2, \ldots, i_d\}$ with $i_j \in V_j$ for all $j = 1, \ldots, d$, and $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \ldots, d$.

A d-partite hypergraph $\mathcal{H}$ is called $\mathbf{t}$-spread if each edge of $\mathcal{H}$ is $\mathbf{t}$-spread. Moreover, $\mathcal{H}$ is called a complete $\mathbf{t}$-spread d-partite hypergraph and denoted by $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ if all $E \subseteq V(\mathcal{H})$ satisfying ($*$) belong to $E(\mathcal{H})$.

Let $\mathbf{1} = (1, \ldots, 1)$. A complete $\mathbf{1}$-spread d-partite hypergraph is just a complete d-partite hypergraph as studied in [6]. The class of complete d-partite hypergraphs has many nice combinatorial properties. We refer the reader to [6] for more information.

Let $\mathcal{H}$ be a hypergraph on $V(\mathcal{H}) = [n]$. The *edge ideal* of $\mathcal{H}$ is given by

$$I(\mathcal{H}) = (\prod_{j \in E_i} x_j : E_i \in E(\mathcal{H})).$$

Note that a **0**-spread monomial ideal is just an ordinary monomial ideal, while a **1**-spread monomial ideal is just a square-free monomial ideal. When $\mathbf{t} = (t,\ldots,t)$ for some fixed integer $t \geq 0$, then $\mathbf{t}$-spread monomial ideal is t-spread introduced in [12]. In the following text, we will assume that $t_i \geq 1$ for all $1 \leq i \leq d-1$. It follows from the above definitions that the edge ideal of a $\mathbf{t}$-spread d-partite hypergraph is a $\mathbf{t}$-spread monomial ideal. To illuminate these definitions, we provide the following example.

Example 4.1.2. Let $\mathbf{t} = (3,2,4)$ and $\mathrm{V} = \{V_1,V_2,V_3,V_4\}$ with $V_1 = \{1,2,3\}, V_2 = \{5,7\}, V_3 = \{8,9,11\}$ and $V_4 = \{12,13\}$. Then the minimal generators of the edge ideal of $\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}$ are as follows:

$$\begin{array}{lll} x_1x_5x_8x_{12} & x_2x_5x_8x_{12} & \\ x_1x_5x_8x_{13} & x_2x_5x_8x_{13} & \\ x_1x_5x_9x_{13} & x_2x_5x_9x_{13} & \\ x_1x_7x_9x_{13} & x_2x_7x_9x_{13} & x_3x_7x_9x_{13} \end{array}$$

The ambient ring of $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ in this case is $S = \mathbb{K}[x_1,x_2,x_3,x_5,x_7,x_8,x_9,x_{12},x_{13}]$. Indeed, we can remove 11 from V_3 to exclude the isolated vertices.

The edge ideals of $\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}$ have many nice algebraic and combinatorial properties.

We first prove that $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ has a linear resolution. To do this, we show that $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ has linear quotients. Recall that an ideal $I \subset S = \mathbb{K}[x_1,\ldots,x_n]$ is said to have *linear quotients* if $\mathrm{G}(I)$ admits an ordering $u_1,\ldots,u_r$ such that the colon ideal $(u_1,\ldots,u_{i-1}):(u_i)$ is generated by variables for all $i = 2,\ldots,r$. It is known from [26, Theorem 1.12] or [22, Propositon 8.2.1] that an ideal generated in a single degree has linear resolution if it admits linear quotients.

Theorem 4.1.3. The ideal $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ has linear quotients.

Proof. Let $>_{\mathrm{lex}}$ denote the lexicographical order induced by the total order $x_1 > x_2 > \cdots > x_n$. Furthermore, let $\mathbf{t} = (t_1,\ldots,t_{d-1}) \in \mathbb{N}^{d-1}$ and set $I = I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ and let $\mathrm{G}(I) = \{u_1,\ldots,u_r\}$ ordered such that $u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} u_r$. We need to show that $(u_1,\ldots,u_{i-1}):(u_i)$ is generated by variables for all $i = 2,\ldots,r$. To do this, it is enough to show that for all $1 \leq j \leq i-1$, there exists $x_p \in (u_1,\ldots,u_{i-1}):(u_i)$ such that x_p divides $u_j/\gcd(u_j,u_i)$.

Let $j < i$ and $u_i = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $u_j = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $i_1 < i_2 < \cdots < i_d$ and $j_1 < j_2 < \cdots < j_d$. On account of $u_j >_{\text{lex}} u_i$, there exists some ℓ such that $j_1 = i_1, j_2 = i_2, \ldots, j_{\ell-1} = i_{\ell-1}$ and $j_\ell < i_\ell$. Note that $j_\ell, i_\ell \in V_\ell$. Let $v = x_{j_\ell}(u_i/x_{i_\ell}) = x_{i_1}x_{i_2}\cdots x_{i_{\ell-1}}x_{j_\ell}x_{i_{\ell+1}}\cdots x_{i_d}$. We have $j_\ell - i_{\ell-1} = j_\ell - j_{\ell-1} \geq t_{\ell-1}$ and $i_{\ell+1} - j_\ell \geq i_{\ell+1} - i_\ell \geq t_\ell$. This shows that v corresponds to a $\mathbf{t}$-spread edge of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Hence, $v \in \mathrm{G}(I)$ and $v = u_k$ for some $k < i$. This completes the proof because $x_{j_\ell} \in (u_1, \ldots, u_{i-1}) : (u_i)$ and x_{j_ℓ} divides $u_j/\gcd(u_j, u_i)$. $\square$

Let I be a monomial ideal with linear quotients with respect to the ordering $u_1, \ldots, u_r$ of $\mathrm{G}(I)$. If I is generated in a single degree d, then I has linear resolution as shown in [26]. Following [26], we define

$$\mathrm{set}(u_k) = \{i : x_i \in (u_1, \ldots, u_{k-1}) : (u_k)\} \text{ for } k = 2, \ldots, r.$$

Using [26, Lemma 1.5], we can conclude that

$$\beta_{i,i+d}(I) = |\{\alpha \subseteq \mathrm{set}(u) : u \in \mathrm{G}(I) \text{ and } |\alpha| = i\}|.$$

In the following proposition, we give a description of $\mathrm{set}(u)$ when $u \in \mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$. For any $S \subseteq [n]$, we set $\min S$ to be the smallest integer in S, and $\max S$ to be the largest integer in S.

Proposition 4.1.4. Let $u = x_{k_1}x_{k_2}\cdots x_{k_d} \in \mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ with $\mathbf{t} = (t_1, t_2, \ldots, t_{d-1})$ and $i_1 = \min V_1$. With the notations introduced above, $\mathrm{set}(u)$ is the union of $[i_1, k_1 - 1] \cap V_1$ and $[k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$ for $j = 2, \ldots, d$.

Proof. Let $\ell \in \mathrm{set}(u)$. Following Theorem 4.1.3, there exists $v \in \mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ such that $v >_{\text{lex}} u$ and $(v) : (u) = (x_\ell)$. This gives $v = (u/x_{k_j})x_\ell$ for some $1 \leq j \leq d$ and $x_{k_j}, x_\ell \in V_j$. Since $v >_{\text{lex}} u$, we must have $\ell \leq k_j - 1$. If $j = 1$, then $\ell \in [i_1, k_1 - 1]$. Moreover, if $2 \leq j \leq d$, then $k_{j-1} + t_{j-1} \leq \ell$ because v is a $\mathbf{t}$-spread monomial, and hence $\ell \in [k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$.

On the other hand, if $\ell \in [i_1, k_1 - 1] \cap V_1$ or $\ell \in [k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$ for any $j = 2, \ldots, d$, then set $v = (u/x_{k_j})x_\ell$ for all $j = 1, \ldots, d$. In both cases, $v \in \mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ and $v >_{\text{lex}} u$. Therefore, $x_\ell \in (v) : (u)$, and hence $\ell \in \mathrm{set}(u)$, as required. $\square$

4.2 The powers and the fiber cone of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$

Let $\mathbb{K}$ be a field and S_d be the $\mathbb{K}$-vector space generated by all monomials of degree d in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. Let $u, v \in S_d$ and $uv = x_{i_1} x_{i_2} \cdots x_{i_{2d}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{2d-1} \leq i_{2d}$. Set $u' = x_{i_1} x_{i_3} \cdots x_{i_{2d-1}}$ and $v' = x_{i_2} x_{i_4} \cdots x_{i_{2d}}$. The map

$$\mathrm{sort} : S_d \times S_d \to S_d \times S_d \text{ which maps } (u, v) \mapsto (u', v'),$$

is called the *sorting operator*. A pair $(u, v) \in S_d \times S_d$ is called *sorted* if $\mathrm{sort}(u, v) = (u', v')$. A subset $A \subset S_d$ is called *sortable* if $\mathrm{sort}(A \times A) \subseteq A \times A$. Furthermore, an r-tuple of monomials $(u_1, \dots, u_r) \in S_d^r$ is called sorted if for any $1 \leq i < j \leq n$, the pair (u_i, u_j) is sorted. In other words, if we write the monomials $(u_1, \dots, u_r)$ as $u_1 = x_{i_1} \cdots x_{i_d}$, $u_2 = x_{j_1} \cdots x_{j_d}$, $\dots$ $, u_r = x_{l_1} \cdots x_{l_d}$, then $(u_1, \dots, u_r)$ is sorted if and only if

$$i_1 \leq j_1 \leq \cdots \leq l_1 \leq i_2 \leq j_2 \leq \cdots \leq l_2 \leq \cdots \leq i_d \leq j_d \leq \cdots \leq l_d. \tag{4.1}$$

Proposition 4.2.1. The set $\mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ is sortable.

Proof. Assume that $u, v \in \mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ and $uv = x_{i_1} x_{i_2} x_{i_3} x_{i_4} \cdots x_{i_{2d-1}} x_{i_{2d}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{2d}$. Since $\mathrm{supp}(u)$ and $\mathrm{supp}(v)$ correspond to the edges of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$, it follows that $i_1, i_2 \in V_1, i_3, i_4 \in V_2, \dots, i_{2d-1}, i_{2d} \in V_d$. Consequently, $u' = x_{i_1} x_{i_3} \cdots x_{i_{2d-1}}$ and $v' = x_{i_2} x_{i_4} \cdots x_{i_{2d}}$ are monomials associated to the edges of a complete d-partite hypergraph. It only remains to show that u' and v' are $\mathbf{t}$-spread. We show that u' is a $\mathbf{t}$-spread monomial and the argument for v' follows in a similar fashion. For any $1 \leq l \leq d-1$, we have $i_{2l-1} \leq i_{2l} \leq i_{2l+1}$ and at least two of the variables among $x_{i_{2l-1}}, x_{i_{2l}}, x_{i_{2l+1}}$ belong to either $\mathrm{supp}(u)$ or $\mathrm{supp}(v)$. Using the fact that u and v are $\mathbf{t}$-spread monomials, this implies that $i_{2l+1} - i_{2l-1} \geq i_{2l+1} - i_{2l}$ and $i_{2l+1} - i_{2l-1} \geq i_{2l} - i_{2l-1}$, we obtain the desired conclusion. $\square$

Let $I \subset S$ be an ideal generated by the monomials of the same degree. Here, set $T = \mathbb{K}[\{t_u : u \in \mathrm{G}(I)\}]$ and $\mathbb{K}[I] = \mathbb{K}[u : u \in \mathrm{G}(I)]$. Consider the $\mathbb{K}$-algebra homomorphism

$$\phi : T \to \mathbb{K}[I] \text{ defined by } t_u \mapsto u \text{ for } u \in \mathrm{G}(I).$$

The kernel of ϕ is called the *defining ideal* of $\mathbb{K}[I]$. If $\mathrm{G}(I)$ is a sortable set, then it follows from [52] or [11, Theorems 6.15 and 6.16] that there exists a monomial order $<_{\text{sort}}$ such that the defining ideal of $\mathbb{K}[I]$ admits the reduced Gröbner basis consisting of binomials of the form $t_u t_v - t_{u'} t_{v'}$, where $\operatorname{sort}(u,v) = (u',v')$.

Corollary 4.2.2. The $\mathbb{K}$-algebra $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is a Koszul and Cohen-Macaulay normal domain.

Proof. As discussed above, with respect to $>_{\text{sort}}$, the Gröbner basis of the defining ideal of $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ contains quadratic binomials. Due to Fröberg [16], we conclude that $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is Koszul and due to a theorem of Sturmfels [52] we obtain $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is normal, see also [11, Theorem 5.16]. Therefore, $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is Cohen-Macaulay domain by [29, Theorem 1]. $\square$

Our next goal is to establish that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has strong persistence property and its powers have linear resolution.

To achieve our goal, we first recall the definition of l-exchange property, see [25] or [11, Sec 6.4] for more details. Let T and ϕ be the same as above and $<$ be a monomial order defined on T. A monomial $t_{u_1} t_{u_2} \cdots t_{u_N} \in T$ is called a *standard monomial* of $\ker\phi$ with respect to $<$, if $t_{u_1} t_{u_2} \cdots t_{u_N} \notin \operatorname{in}_<(\ker\phi)$.

Definition 4.2.3. The monomial ideal $I \subset S$ is said to satisfy the l-exchange property with respect to the monomial order $<$ on T if the following two conditions hold: let $t_{u_1} t_{u_2} \cdots t_{u_N}$ and $t_{v_1} t_{v_2} \cdots t_{v_N}$ be two standard monomials of $\ker\phi$ with respect to $<$ such that

(i) $\deg_{x_i} u_1 u_2 \cdots u_N = \deg_{x_i} v_1 v_2 \cdots v_N$, for $i = 1, \ldots, q-1$ and $q \le n-1$,

(ii) $\deg_{x_q} u_1 u_2 \cdots u_N < \deg_{x_q} v_1 v_2 \cdots v_N$.

Then there exist some j and α with $q < j \le n$ such that $x_q u_\alpha / x_j \in I$.

Theorem 4.2.4. The ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the l-exchange property with respect to the sorting order $<_{\text{sort}}$.

Proof. Let $t_{u_1} t_{u_2} \cdots t_{u_N}$ and $t_{v_1} t_{v_2} \cdots t_{v_N}$ be two standard monomials of $\ker\phi$ with respect to $<_{\text{sort}}$ and $\mathbf{t} = (t_1, t_2, \ldots, t_{d-1})$. It can be seen from Proposition 4.2.1 together with (4.1) that the N-tuples with $\mathbf{t}$-spread monomials $(u_1, u_2, \ldots, u_N)$ and

$(v_1, v_2, \ldots, v_N)$ are sorted. Assume that the products $u_1u_2\cdots u_N$ and $v_1v_2\cdots v_N$ satisfy both conditions in Definition 4.2.3. The condition (i) together with (4.1) gives

$$\deg_{x_i} u_\gamma = \deg_{x_i} v_\gamma, \text{ for } 1 \le i \le q-1 \text{ and for all } 1 \le \gamma \le N, \tag{4.2}$$

and the condition (ii) of Definition 4.2.3 implies that there exists α with $1 \le \alpha \le N$ such that

$$\deg_{x_q} u_\alpha < \deg_{x_q} v_\alpha. \tag{4.3}$$

Following (4.2) and (4.3), we can write

$$u_\alpha = x_{j_1}x_{j_2}\cdots x_{j_p}\cdots x_{j_d} \quad \text{and} \quad v_\alpha = x_{j_1}x_{j_2}\cdots x_{j_{p-1}}x_q x_{k_{p+1}}\cdots x_{k_d},$$

with $j_p > q$. To complete the proof, it is enough to show that $w = x_q u_\alpha / x_{j_p} \in I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Note that q and j_p belong to V_p. Moreover, $q - j_{p-1} \ge t_{p-1}$ because v_α is $\mathbf{t}$-spread and $j_{p+1} - q \ge j_{p+1} - j_p \ge t_p$ because $j_p > q$. This yields that w is a $\mathbf{t}$-spread monomial, as desired. $\square$

Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $R = S[\{t_u : u \in \mathrm{G}(I)\}]$. We define a monomial order on R as following: if $u_1, u_2 \in S$ and $v_1, v_2 \in T$, then $u_1v_1 > u_2v_2$ if and only if $u_1 >_{\mathrm{lex}} u_2$ or $u_1 = u_2$ and $v_1 >_{\mathrm{sort}} v_2$, where $>_{\mathrm{lex}}$ denotes the lexicographical order on S induced by $x_1 > \cdots > x_n$. Let $\mathcal{R}(I) = \oplus_{j \ge 0} I^j t^j \subseteq S[t]$ be the Rees ring of I. The Rees ring $\mathcal{R}(I)$ has the following presentation

$$\psi : R = S[\{t_u : u \in \mathrm{G}(I)\}] \to \mathcal{R}(I),$$

with $x_i \mapsto x_i$ for $1 \le i \le n$ and $t_u \mapsto ut$ for $u \in \mathrm{G}(I)$. Let $P = \ker \psi$. Then we have the next result.

Corollary 4.2.5. Let $>$ be the monomial order on R as defined above. The reduced Gröbner basis of P consists of the binomials of the following form:

5.1 $t_u t_v - t_{u'} t_{v'}$, where $sort(u, v) = (u', v')$;

5.2 $x_i t_u - x_j t_v$, where $i < j$, $x_i u = x_j v$, and j is the largest integer for which $x_i v / x_j \in \mathrm{G}(I)$.

Proof. According to [25, Theorem 5.1] (or see [11, Theorem 6.24]), it is enough to show that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is sortable and satisfies the l-exchange property with respect to $>_{\mathrm{sort}}$ as noted in Proposition 4.2.1 and Theorem 4.2.4. $\square$

Following the similar argument as in the proof of Corollary 4.2.2, we obtain the following corollary.

Corollary 4.2.6. The Rees algebra $\mathcal{R}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ is a normal Cohen-Macaulay domain.

We are in a position to state the main result of this section in the next corollary.

Corollary 4.2.7. The ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the strong persistence property and all powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution.

Proof. The strong persistence property of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ can be deduced from [27, Corollary 1.6] and Corollary 4.2.6. Moreover, Corollary 4.2.5 together with [22, Corollary 10.1.8] provides that all the powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution, as claimed. $\square$

Here, we determine the limit depth of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. By a theorem of Brodmann [7], $\operatorname{depth} S/I^k$ is constant for large enough k. This constant value is known as the limit depth of I and is denoted by $\lim_{k\to\infty} \operatorname{depth} S/I^k$. The minimum value of k for which $\operatorname{depth} S/I^k = \operatorname{depth} S/I^{k+t}$ for all $t > 0$ is called the *index of depth stability* and denoted by $\operatorname{dstab}(I)$. Let $\mathfrak{m}$ be the graded maximal ideal of S. The analytic spread of an ideal $I \subset S$ is the Krull dimension of the fiber cone $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ and denoted by $\ell(I)$.

Definition 4.2.8. [[27], Definition 3.1] Let $I \subset S$ be a monomial ideal in $S = K[x_1,\ldots,x_n]$ and $\mathrm{G}(I) = \{u_1,\ldots,u_r\}$. Then the linear relation graph Γ of I is the graph with the edge set

$$E(\Gamma) = \{\{i,j\} : \textit{ there exist } u_t, u_m \in \mathrm{G}(I) \textit{ such that } x_i u_t = x_j u_m\},$$

and the vertex set $V(\Gamma) = \bigcup_{\{i,j\}\in E(\Gamma)} \{i,j\}$.

An ideal $I \subset S$ is said to have *linear relations* if I is generated in degree d and $\beta_{1,j}(I) = 0$ for all $j \neq d+1$. We employ the following lemma to compute $\ell(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$.

Lemma 4.2.9. ([9, Lemma 5.2]) Let I be a monomial ideal with linear relations generated in a single degree whose linear relation graph Γ has r vertices and s connected components. Then $\ell(I) = r - s + 1$.

We are now ready to determine the analytic spread of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ in the following lemma.

Lemma 4.2.10. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph and $|V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| = r$. Then $\ell(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = r - d + 1$.

Proof. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $V = \{V_1, \ldots, V_d\}$. Using Theorem 4.1.3 and [9, Lemma 5.2], it is enough to show that $\Gamma(I)$ has r vertices and d connected components. Let $a_i = \min V_i$ and $b_i = \max V_i$, for all $i = 1, \ldots, d$. Let $h, k \in V_i$ for some i. Since $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ does not have isolated vertices, this implies that the sets $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ are $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Then $u = x_{a_1} \cdots x_{a_{i-1}} x_h x_{b_{i+1}} \cdots x_{b_d}$ and $v = x_{a_1} \cdots x_{a_{i-1}} x_k x_{b_{i+1}} \cdots x_{b_d}$ are also $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. This shows that $x_k u = x_h v$; hence, $\{h, k\} \in E(\Gamma)$ and $V(\Gamma) = r$. Moreover, it follows from the definition of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ that for $i \neq j$ and $h \in V_i$ and $k \in V_j$, we have the edge $\{h, k\} \notin E(\Gamma)$. Therefore, Γ has exactly d connected components, as required. $\square$

We now give the last result of this section in the following theorem.

Theorem 4.2.11. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph and $|V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| = r$, and S be the ambient ring of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then

$$\lim_{k \to \infty} \operatorname{depth}(S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^k) = d - 1,$$

and $\operatorname{dstab}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) \leq r - d$.

Proof. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then it follows from Corollary 4.2.6 and a result of Eisenbud and Huneke [10] that $\lim_{k \to \infty} \operatorname{depth}(S/I^k) = r - \ell(I)$. From Lemma 4.2.10, we have $r - \ell(I) = r - (r - d + 1) = d - 1$ as required. In addition, using [27, Theorem 3.3] and Lemma 4.2.10, we see that $\operatorname{depth}(S/I^{r-d}) = d - 1$. It is shown in [24, Proposition 2.1] that if all powers of an ideal have linear resolution, then $\operatorname{depth} S/I^k \leq \operatorname{depth} S/I^t$ for all $k < t$. It follows now from Corollary 4.2.7 that $\operatorname{dstab}(I) \leq r - d$. This completes the proof. $\square$

4.3 Normally torsion-free and Cohen-Macaulay $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$

In this section, our main goal is to show that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free and give a complete characterization of Cohen-Macaulay $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ for $V=\{V_1,\ldots,V_d\}$ such that each V_i is of the form $[a_i,b_i]$ for some integers $a_i,b_i\in\mathbb{Z}^+$. To this aim, we begin with the description of minimal prime ideals of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and view $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ as a simplicial complex. For more details on simplicial complexes, we refer the reader to [22].

Given a square-free monomial ideal $I\subset R$, the *Alexander dual* of I, denoted by $I^\vee$ is given by $I^\vee=\bigcap_{u\in \mathrm{G}(I)}(x_i\ :\ x_i\in \mathrm{supp}(u))$. The minimal generators of $I^\vee$ correspond to the minimal prime ideals of I. Below we give a description of $\mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^\vee)$.

Theorem 4.3.1. *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})\subseteq [n]$ and $V=\{V_1,\ldots,V_d\}$. Furthermore, let $|V_j|=n_j$ with $V_j=[i_j,i_j+n_j-1]$ for all $j=1,\ldots,d$. Then $\mathrm{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^\vee)$ consists of the following monomials:*

(i) $\prod\limits_{k\in V_i} x_k$ *for all $i=1,\ldots,d$; and,*

(ii) $(\prod_{i=j}^{p}\prod_{k\in V_i}x_k)/(\prod_{i=j}^{p-1}v_{q_i}\prod_{i=j+1}^{p}v_{q_i'})$, *for all $1\le j<p\le d$ and for each sequence of nonnegative integers $q_j,\ldots,q_{p-1}$ satisfying*

$$i_\ell+q_\ell' < i_\ell+n_\ell-1-q_\ell \quad \text{for } j+1\le \ell\le p-1, \tag{4.4}$$

$$i_\ell+q_\ell'-(i_{\ell-1}+n_{\ell-1}-1-q_{\ell-1}) = t_{\ell-1}-1 \quad \text{for } \ell=j+1,\ldots,p, \tag{4.5}$$

where $v_{q_\ell}=\prod_{r=1}^{1+q_\ell}x_{i_\ell+n_\ell-r}$, for $\ell=j,\ldots,p-1$ and $v_{q_\ell'}=\prod_{r=0}^{q_\ell'}x_{i_\ell+r}$, for $\ell=j+1,\ldots,p$.

Proof. Let Δ be the simplicial complex on $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ such that $I_\Delta=I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ be the Stanley-Reisner ideal of Δ. Let $\mathcal{F}(\Delta)$ be the set of facets of Δ. For any $F\in\Delta$, we set $x_F=\prod_{i\in F}x_i$. It follows from [22, Lemma 1.5.4] that the standard primary decomposition of I_Δ is given by

$$I_\Delta=\bigcap_{F\in\mathcal{F}(\Delta)}P_{\bar{F}},$$

where $P_{\bar{F}}$ is the monomial prime ideal generated by the variables x_i with $i \in \bar{F} = V(\mathrm{K}_{\mathrm{v}}^{\mathbf{t}}) \setminus F$. Therefore, using [22, Corollary 1.5.5], it is enough to show that $\mathcal{F}(\Delta)$ is the disjoint union of $\mathcal{F}_1$ and $\mathcal{F}_2$, defined below:

(i) $\mathcal{F}_1 = \{F_1, \dots, F_d\}$, where $F_i = \bigcup_{j \neq i,\, j=1}^{d} V_j$ for all $i = 1, \dots, d$,

(ii) For all $1 \leq j < p \leq d$, set $A_{j,p} := \bigcup_{i \notin \{j,\dots,p\},\, i=1}^{d} V_j$. For each sequence of non-negative integers $q_j, \dots, q_{p-1}$ satisfying conditions (4.4) and (4.5), we set

$$B_{q_\ell} := \{i_\ell + n_\ell - 1 - q_\ell, \dots, i_\ell + n_\ell - 1\} \subsetneq V_\ell \text{ for } \ell = j, \dots, p-1,$$

and

$$B_{q'_\ell} = \{i_\ell, \dots, i_\ell + q'_\ell\} \subsetneq V_\ell \text{ for } \ell = j+1, \dots, p.$$

Then we get

$$\mathcal{F}_2 = \{A_{j,p} \cup (\bigcup_{\ell=j}^{p-1} B_{q_\ell}) \cup (\bigcup_{\ell=j+1}^{p} B_{q'_\ell}) \colon \text{ for all } 1 \leq j < p \leq d \text{ and } q_j, \dots, q_{p-1}\}.$$

The condition (4.5) translates into the following: for each $\ell = j, \dots, p-1$ we have $\max B_{q'_{\ell+1}} - \min B_{q_\ell} = t_\ell - 1$. In the construction of elements in $\mathcal{F}_2$, it is enough to determine the integers $q_j, \dots, q_{p-1}$, because q'_ℓ is uniquely determined from $q_{\ell-1}$, for all $\ell = j+1, \dots, p$, by using the equality in (4.5).

First, we show that $\mathcal{F}_1 \subseteq \mathcal{F}(\Delta)$. For any $F_i \in \mathcal{F}_1$, we have $F_i \cap V_i = \emptyset$. Therefore, $x_{F_i} \notin I_\Delta$. Moreover, for any $k \in V_i$, using the assumption that $\mathrm{K}_{\mathrm{v}}^{\mathbf{t}}$ does not contain any isolated vertices, we obtain that $F_i \cup \{k\}$ contains a $\mathbf{t}$-spread edge, and hence $x_{F_i} x_k \in I_\Delta$ and $F_i \in \mathcal{F}(\Delta)$.

Now, assume that $F \in \mathcal{F}_2$, where $F = A_{j,p} \cup (\bigcup_{\ell=j}^{p-1} B_{q_\ell}) \cup (\bigcup_{\ell=j+1}^{p} B_{q'_\ell})$ for some $1 \leq j < p \leq d$ and $q_j, \dots, q_{p-1}$. We here show that $F \in \Delta$. On contrary, if $x_F \in I_\Delta$, then F contains a $\mathbf{t}$-spread edge, say $G = \{k_1, \dots, k_d\}$. Then $k_j \in B_{q_j}$ because $G \cap V_j \subseteq F \cap V_j = B_{q_j}$. If $p = j+1$, then by using the condition (4.5), it immediately follows that for any choice of $k_j \in B_{q_j}$, there is no suitable $k_{j+1} \in B_{q'_{j+1}}$ such that $k_{j+1} - k_j \geq t_{j-1}$. If $p > j+1$, then the condition (4.5) gives that $k_{j+1} \in B_{q_{j+1}}$. Using the condition (4.5) repeatedly in a similar way, we obtain $k_{p-1} \in B_{q_{p-1}}$. However, there is no suitable $k_p \in B_{q'_p}$ such that $k_p - k_{p-1} \geq t_{p-1}$, a contradiction. Consequently, we get $F \in \Delta$.

In what follows, we demonstrate that $F \in \mathcal{F}(\Delta)$. Note that

$$V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \setminus F = (V_j \setminus B_{q_j}) \cup (\bigcup_{l=j+1}^{p-1} (V_l \setminus (B_{q'_\ell} \cup B_{q_\ell})) \cup (V_p \setminus B_{q'_p}).$$

Let $a \in V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \setminus F$. Then $a \in V_s$ for some $j \leq s \leq p$. Set

$$k_r = \begin{cases} i_r, & \text{if } r = 1, \dots, j-1, \\ i_r + n_r - 1 - q_r, & \text{if } r = j, \dots, s-1, \\ a, & \text{if } r = s, \\ i_r + q'_r, & \text{if } r = s+1, \dots, p, \\ i_r + n_r - 1, & \text{if } r = p+1, \dots, d. \end{cases}$$

When $s = j$, then we remove the condition on k_r for $r = j, \dots, s-1$, and similarly, when $s = p$, then we remove the condition on k_r for $r = s+1, \dots, p$. Using conditions (4.4) and (4.5) together with the assumption that Δ has no isolated vertices, we obtain that $k_r - k_{r-1} \geq t_{r-1}$ for all $r = 2, \dots, d$. Therefore, $G = \{k_1, \dots, k_d\} \subseteq F \cup \{a\}$ is a $\mathbf{t}$-spread edge, and hence $x_G \in I_\Delta$, as required.

It remains to check that $\mathcal{F}(\Delta) \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$. This is equivalent to show that for every face G of Δ there exists a facet $F \in \mathcal{F}_1 \cup \mathcal{F}_2$ such that $G \subseteq F$. Let $G \in \Delta$ such that $G \cap V_k = U_k$ for all $k = 1, \dots, d$. If $U_k = \emptyset$ for some k, then $G \subseteq F_k \in \mathcal{F}_1$. Now, assume that $U_k \neq \emptyset$ for all $k = 1, \dots, d$. Set $a_k = \min U_k$ and $b_k = \max U_k$ for all $k = 1, \dots, d$. In the rest of the proof, we will use the following fact repeatedly:

($*$) If there exist $a \in V_\ell$ and $b \in V_{\ell+1}$ such that $b - a < t_\ell$ and $a + t_\ell - 1 < i_{\ell+1} + n_{\ell+1} - 1$, then by letting $q_\ell = i_\ell + n_\ell - 1 - a$, and using the condition (4.5), there is a unique $q'_{\ell+1}$ such that $b < i_{\ell+1} + q'_{\ell+1}$.

Case(1): If there exists some k with $b_{k+1} - a_k < t_k$, then it follows from the statement ($*$) that for a suitable choice of q_k we have $U_k \subseteq B_{q_k}$ and $U_{k+1} \subseteq B_{q'_{k+1}}$. Since $U_i \subseteq V_i \subset A_{k,k+1}$ for all $i = 1, \dots, k-1, k+2, \dots, d$, we can deduce that $G \subseteq A_{k,k+1} \cup B_{q_k} \cup B_{q'_{k+1}} \in \mathcal{F}_2$, as desired.

Case(2): Assume that $b_{k+1} - a_k \geq t_k$ for all $k = 1, \dots, d-1$. Since $G \in \Delta$, we know that G does not contain any $\mathbf{t}$-spread edge. In particular, $\{a_1,\dots,a_d\} \subseteq G$ is not a $\mathbf{t}$-spread edge. This yields that there exists some $k \in \{2,\dots,d\}$ for which $a_{k+1} - a_k < t_k$. We choose minimum $j \geq 1$ for which $a_{j+1} - a_j < t_j$. Note that $M = \{a_1, a_2, \dots, a_j\} \subset G$ such that, $a_{i+1} - a_i \geq t_i$, for all $i = 1, \dots, j-1$. In the discussion below, we aim to construct a suitable $F \in \mathcal{F}_2$ such that $G \subset F$. To this aim, we perform the Step j as introduced below.

Step j: We set $e_j := a_j$ and $e_{j+1} := \min\{a \in U_{j+1} : a - e_j \geq t_j\}$. Note that $\{a \in U_{j+1} : a - e_j \geq t_j\} \neq \emptyset$ because $b_{j+1} - a_j \geq t_j$. We define e_{j+r} recursively as $e_{j+r} = \min\{a \in U_{j+r} : a - e_{j+r-1} \geq t_{j+r-1}\}$ such that

$$\{a \in U_{j+r} : a - e_{j+r-1} \geq t_{j+r-1}\} \neq \emptyset \text{ for some } 1 < r < d - j.$$

There exists some $p > j+1$ for which $\{a \in U_{j+r} : a - e_{j+r-1} \geq t\} = \emptyset$, that is, for some $p > j+1$ we have $b_p - e_{p-1} < t_{p-1}$, otherwise, $M \cup \{e_{j+1}, \dots, e_d\} \subseteq G$ is a $\mathbf{t}$-spread edge in G, a contradiction. Choose minimum $p > j+1$ such that $b_p - e_{p-1} < t_{p-1}$.

Subcase(2.1): If for all $j+1 \leq l \leq p-1$ we have $i_{\ell+1} - e_\ell < t_\ell$, then take $c_{\ell+1} \in V_{\ell+1}$ such that $c_{\ell+1} - e_\ell = t_\ell - 1$ for $\ell = j, \dots, p-1$. This gives us j, p and $q_j, \dots, q_p$ as described in statement $(*)$ for which $e_\ell \in V_\ell$ and $c_{\ell+1} \in V_{\ell+1}$ with $c_{\ell+1} - e_\ell < t_\ell$. Moreover, $U_i \subseteq A_{j,p}$ for all $i \notin \{j, \dots, p\}$, and $U_j \subseteq B_{q_j}$, $U_p \subseteq B_{q'_p}$, and $U_\ell \subseteq B_{q_\ell} \cup B_{q'_\ell}$ for all $\ell = j+1, \dots, p-1$. Hence, this implies that

$$G \subseteq A_{j,p} \cup \left(\bigcup_{\ell=j}^{p-1} B_{q_\ell}\right) \cup \left(\bigcup_{\ell=j+1}^{p} B_{q'_\ell}\right),$$

and we are done.

Subcase(2.2): If for some $j+1 \leq l \leq p-1$, $i_{\ell+1} - e_\ell \geq t_\ell$, then replace M with $M \cup \{e_{j+1}, \dots, e_\ell, a_{\ell+1}\} \subset G$. In this case, there exists a minimum $j' \geq \ell + 1$ such that $a_{j'+1} - a_{j'} < t_{j'}$. Otherwise, $M \cup \{a_{\ell+2}, \dots, a_d\} \subseteq G$ is a $\mathbf{t}$-spread edge, a contradiction. Repeat Step j by replacing j with j'.

Thanks to we have a finite number of partitions, this process must be terminated after a finite number of steps. If the desired j and p are obtained, then we construct a suitable $F \in \mathcal{F}_2$ with $G \subset F$ as described in Case(2.1). If the desired j and p are not obtained, then G contains a $\mathbf{t}$-spread edge in G, a contradiction. $\square$

We illustrate the construction of monomials of the forms (i) and (ii) in Theorem 4.3.1 in the following example.

Example 4.3.2. Let $V = \{V_1, V_2, V_3, V_4\}$ with $V_1 = [1,2]$, $V_2 = [4,6]$, $V_3 = [8,10]$, $V_4 = [12,13]$, and $\mathbf{t} = (3,4,3)$. One can easily see that the minimal generators of the edge ideal of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ are as follows:

$$
\begin{array}{ll}
x_1x_4x_8x_{12} & \\
x_1x_4x_8x_{13} & \\
x_1x_4x_9x_{12} & \\
x_1x_4x_9x_{13} & \\
x_1x_4x_{10}x_{13} & \\
x_1x_5x_9x_{12} & x_2x_5x_9x_{12} \\
x_1x_5x_9x_{13} & x_2x_5x_9x_{13} \\
x_1x_5x_{10}x_{13} & x_2x_5x_{10}x_{13} \\
x_1x_6x_{10}x_{13} & x_2x_6x_{10}x_{13}
\end{array}
$$

Following Theorem 4.3.1, the minimal generators of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^{\vee}$ are given as follows:

(i) The monomials of the form (i) described in Theorem 4.3.1 are $x_1x_2, x_4x_5x_6$, $x_8x_9x_{10}$, and $x_{12}x_{13}$.

(ii) The construction of monomials of the form (ii) described in Theorem 4.3.1 is given in the following table.

j	p	$q_j,\dots,q_{p-1},\ q'_{j+1},\dots,q'_p$	u
1	2	$q_1 = 0,\ q'_2 = 0$	$x_1x_5x_6$
1	3	$q_1 = 0,\ q'_2 = 0,\ q_2 = 0,\ q'_3 = 1$	$x_1x_5x_{10}$
		$q_1 = 0,\ q'_2 = 0,\ q_2 = 1,\ q'_3 = 0$	$x_1x_9x_{10}$
1	4	$q_1 = 0,\ q'_2 = 0,\ q_2 = 0,\ q'_3 = 1,\ q_3 = 0\ ,\ q'_4 = 0$	$x_1x_5x_{13}$
		$q_1 = 0,\ q'_2 = 0,\ q_2 = 1,\ q'_3 = 0,\ q_3 = 0,\ q'_4 = 0$	$x_1x_9x_{13}$
2	3	$q_2 = 0,\ q'_3 = 1$	$x_4x_5x_{10}$
		$q_2 = 1,\ q'_3 = 0$	$x_4x_9x_{10}$
2	4	$q_2 = 0,\ q'_3 = 1,\ q_3 = 0,\ q'_4 = 0$	$x_4x_5x_{13}$
		$q_2 = 1,\ q'_3 = 0,\ q_3 = 0,\ q'_4 = 0$	$x_4x_9x_{13}$
3	4	$q_3 = 0,\ q'_4 = 0$	$x_8x_9x_{13}$

Accordingly, we get

$$\begin{aligned}\mathrm{Ass}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = \{&(x_1,x_2),(x_4,x_5,x_6),(x_8,x_9,x_{10}),(x_{12},x_{13}),(x_1,x_5,x_6),(x_1,x_5,x_{10}),\\ &(x_1,x_9,x_{10}),(x_1,x_5,x_{13}),(x_1,x_9,x_{13}),(x_4,x_5,x_{10}),(x_4,x_9,x_{10}),\\ &(x_4,x_5,x_{13}),(x_4,x_9,x_{13}),(x_8,x_9,x_{13})\}.\end{aligned}$$

As an immediate consequence of Theorem 4.3.1, we obtain the following corollary, which will be used to prove the normally torsion-freeness of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

Corollary 4.3.3. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1,\ldots,V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1,\ldots,d$. If $v := \prod_{j=1}^{d} x_{i_j}$, then $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for all $\mathfrak{p} \in \mathrm{Min}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$.

Proof. Let $v = \prod_{j=1}^{d} x_{i_j}$. The minimal prime ideals of $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ correspond to the minimal generators of $I^{\vee}$ described in statements (i) and (ii) of Theorem 4.3.1. The minimal primes corresponding to the generators of the form (i) are $\mathfrak{p}_i = (x_k : k \in V_i)$ and $v \notin \mathfrak{p}_i^2$ for all $i = 1,\ldots,d$. Moreover, each generator of $I^{\vee}$ of the form (ii) is constructed by fixing j, p and $q_j,\ldots,q_p$. Let $\mathfrak{q}$ be a minimal prime of I corresponding to a generator of the form (ii). Then $x_{i_k} \in \mathfrak{q}$ if and only if $k = j$, as required. $\square$

We recollect the following lemma, which will be used repeatedly in the next proposition and Theorem 4.3.6.

Lemma 4.3.4. ([46, Lemma 3.12]) Let I be a monomial ideal in a polynomial ring $S = \mathbb{K}[x_1,\ldots,x_n]$ with $\mathrm{G}(I) = \{u_1,\ldots,u_m\}$, and $h = x_{j_1}^{b_1}\cdots x_{j_s}^{b_s}$ with $j_1,\ldots,j_s \in \{1,\ldots,n\}$ be a monomial in S. Then I is normally torsion-free if and only if hI is normally torsion-free.

In order to establish Theorem 4.3.6, we require the following auxiliary proposition. For a given square-free monomial ideal $I \subset \mathbb{K}[x_1,\ldots,x_n]$, we denote by $I \setminus x_i$ the ideal generated by those elements in $\mathrm{G}(I)$ that does not contain x_i in their support.

Proposition 4.3.5. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1,\ldots,V_d\}$. Furthermore, let $|V_j| = 2$ with $V_j = \{i_j, i_j + 1\}$ for all $j = 1,\ldots,d$. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free.

Proof. To simplify the notation, set $I := I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. We proceed by induction on d. If $d = 1$, then there is nothing to show. Hence, assume that $d > 1$ and that the result holds for any complete $\mathbf{t}$-spread $(d-1)$-partite hypergraph. Choose an arbitrary element $\mathfrak{p} \in \mathrm{Min}(I)$ and set $v := \prod_{j=1}^{d} x_{i_j}$. It follows at once from Corollary 4.3.3 that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$. We show that $I \setminus x_r$ is normally torsion-free for each $x_r \in \mathrm{supp}(v)$. Without loss of generality, we let $V_1 = \{1,2\}$ and we prove that $I \setminus x_1$ is normally torsion-free. It is not hard to check that $I \setminus x_1 = x_2 L$ where L is the edge ideal of $\mathbf{t}$-spread d-partite hypergraph with vertex partition $\mathrm{V}' = \{V_2', \ldots, V_d'\}$ such that, for all $i = 2, \ldots, d$, the set V_i' is obtained from V_i after removing the isolated vertices, if any. One can conclude from the inductive hypothesis that L is normally torsion-free. Here, using Lemma 4.3.4 implies that $I \setminus x_1$ is normally torsion-free. It follows now from [47, Theorem 3.7] that I is normally torsion-free, as claimed. □

Theorem 4.3.6. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \ldots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \ldots, d$. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free. In particular, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normal.

Proof. We first assume that $|V_j| = 1$ for some $1 \leq j \leq d$, say $V_j = \{z\}$. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then we can write $I = x_z L$ such that L can be viewed as the edge ideal associated to a complete $\mathbf{t}$-spread $(d-1)$-partite hypergraph. According to Lemma 4.3.4, I is normally torsion-free if and only if L is normally torsion-free. Thus, we reduce to the case $|V_j| \geq 2$ for all $j = 1, \ldots, d$. Set $v := \prod_{j=1}^{d} x_{i_j}$. Pick an arbitrary element $\mathfrak{p} \in \mathrm{Min}(I)$. One can derive from Corollary 4.3.3 that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$. To complete the proof, it is sufficient to establish $I \setminus x_s$ in normally torsion-free for each $x_s \in \mathrm{supp}(v)$. To accomplish this, we use the induction on $n := |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})|$. On account of $|V_j| \geq 2$ for all $j = 1, \ldots, d$, this implies that $n \geq 2d$. The case in which $n = 2d$ can be deduced according to Proposition 4.3.5. Now, suppose that $n > 2d$. It is not hard to see that $I \setminus x_s$ is again the edge ideal of the $\mathbf{t}$-spread d-partite hypergraph obtained from $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ by removing all the edges that contain s. One can deduce from the inductive hypothesis that $I \setminus x_s$ is normally torsion-free. Here, in view of [47, Theorem 3.7], we conclude that I is normally torsion-free, as desired.

The last assertion can be deduced according to [22, Theorem 1.4.6]. □

We can readily provide the following corollary inspired by Theorem 4.3.6. A *matching* in a hypergraph $\mathcal{H}$ is a family of pairwise disjoint edges, and the maximum cardinality of a matching is denoted by $\nu(\mathcal{H})$. The transversal number of a hypergraph $\mathcal{H}$, denoted by $\tau(\mathcal{H})$ is the minimal cardinality of a transversal of $\mathcal{H}$. A hypergraph $\mathcal{H}$ is said to satisfy the König property if $\nu(\mathcal{H}) = \tau(\mathcal{H})$, see [6, Chapter 2, Section 4].

Corollary 4.3.7. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the König property.

Proof. Based on Theorem 4.3.6, we get $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free. In addition, by virtue of [55, Theorem 14.3.6], one can deduce that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ has the max-flow min-cut property. It follows now from [55, Corollary 14.3.18] that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ has the packing property. On the other hand, by virtue of [20, Definition 2.3], we obtain $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the König property. This completes the proof. □

Next, we give a characterization of Cohen-Macaulay $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. To do this, we first determine the height of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

Proposition 4.3.8. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \dots, d$. Then $\mathrm{ht}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = \min\{n_1, \dots, n_d\}$, where $\mathrm{ht}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ denotes the height of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

Proof. Let $I := I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $n_k = \min\{n_1, \dots, n_d\}$. Since $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ does not contain any isolated vertices, this yields that

$$\{i_1, \dots, i_d\}, \{i_1 + 1, \dots, i_d + 1\}, \dots, \{i_1 + n_k - 1, \dots, i_d + n_k - 1\}, \tag{4.6}$$

are pairwise disjoint $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Hence, we obtain the following monomials

$$x_{i_1} x_{i_2} \dots x_{i_d}, x_{i_1+1} x_{i_2+1} \dots x_{i_d+1}, \dots, x_{i_1+n_k-1} x_{i_2+n_k-1} \dots x_{i_d+n_k-1}$$

belong to $\mathrm{G}(I)$. This gives that $\mathrm{ht}(I) \geq n_k$. It follows also from Theorem 4.3.1 that $(x_i : i \in V_k)$ is a minimal prime of I with height n_k. This finishes our proof. □

Note that the König property of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ can be also observed from the proof of above proposition. Indeed, the inequality $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ holds for any hypergraph $\mathcal{H}$ and the $\mathbf{t}$-spread edges given in (4.6) give a maximal matching in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$.

Under the assumptions of Theorem 4.3.1, one can compute the degree of generators of $I^\vee = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^\vee$. It is easy to see that $\deg \prod_{k\in V_i} x_k = n_i$ for all $i = 1,\dots,d$. Now, let $u \in \mathrm{G}(I^\vee)$ of the form (ii) for some $1 \le j < p \le d$ and $q_j,\dots,q_p$. Then $u = (\prod_{i=j}^{p}\prod_{k\in V_i} x_k)/(\prod_{i=j}^{p-1} v_{q_i} \prod_{i=j+1}^{p} v_{q_i'})$. Let h be the product of variables with indices in $[i_j, i_p + n_p - 1] \setminus (V_j \cup \cdots \cup V_p)$ and $w = (uh)/h$. Then $\deg w = \deg u$.

We have $\deg h(\prod_{i=j}^{p}\prod_{k\in V_i} x_k) = (i_p + n_p - 1) - i_j + 1$. Moreover, it follows from the condition (4.5) that $\deg(h\prod_{i=j}^{p-1} v_{q_i} \prod_{i=j+1}^{p} v_{q_i'}) = \sum_{i=j}^{p-1} t_i$. We thus get

$$\deg w = (i_p + n_p - 1) - i_j + 1 - \sum_{i=j}^{p-1} t_i = i_p - i_j + n_p - \sum_{i=j}^{p-1} t_i.$$

Hence, we obtain

$$\deg u = i_p - i_j + n_p - \sum_{i=j}^{p-1} t_i. \tag{4.7}$$

A square-free monomial ideal is said to be *unmixed* if its minimal prime ideals are of the same height. Using the description of generators of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^\vee$ and their degrees, we obtain the following characterization for unmixedness of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

Theorem 4.3.9. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1,\dots,V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1,\dots,d$. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is unmixed if and only if $n_1 = \cdots = n_d = s$, and for each $j = 1,\dots,d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$.

Proof. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ be unmixed. Then every minimal prime of I has the same height; equivalently, $I^\vee$ is generated in the same degree by Theorem 4.3.1, we know that every V_j corresponds to a minimal generator in $I^\vee$, and this yields $n_1 = \cdots = n_d$. Let $n_1 = \cdots = n_d = s$. We only need to show that for each $j = 1,\dots,d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Indeed, if $i_{j+1} - (i_j + s - 1) \le t_j - 1$ for some j, then we obtain $u \in \mathrm{G}(I^\vee)$ of the form (ii) with $p = j+1$ and a suitable choice of q_j and q_{j+1}' as described in statement $(*)$ in the proof of Theorem 4.3.1. It follows from (4.7) that $\deg u = i_{j+1} - i_j + s - t_j$. Since $\deg u = s$, we obtain $i_{j+1} - i_j = t_j$.

Now, assume that for all $j = 1,\dots,d$ we have $n_j = s$ and for each $j = 1,\dots,d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Then all generators of $I^\vee$ of the form

(i) have the same degree s. If $I^\vee$ has no generator of the form (ii), then the proof is complete. Otherwise, let $u \in G(I^\vee)$ of the form (ii) for some j, p and $q_j \dots, q_{p-1}$. Then, for all $\ell = j, \dots, p-1$, we have $i_{\ell+1} - i_\ell = t_\ell$, because if $i_{\ell+1} - (i_\ell + s - 1) > t_\ell - 1$ for some ℓ, then q_ℓ and $q'_{\ell+1}$ do not satisfy the condition (4.5). This gives that $i_p = i_j + \sum_{i=j}^{p-1} t_i$. Using (4.7), we obtain

$$\deg u = i_p - i_j + s - \sum_{i=j}^{p-1} t_i = i_j + \sum_{i=j}^{p-1} t_i - i_j + s - \sum_{i=j}^{p-1} t_i = s,$$

and the proof is done. $\square$

Remark 4.3.10. Let $V = \{V_1, V_2, V_3, V_4\}$ with $V_1 = [2,4]$, $V_2 = [6,8]$, $V_3 = [9,11]$, $V_4 = [13,15]$, and $\mathbf{t} = (2,3,4)$. By virtue of Theorem 4.3.9, the edge ideal $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is unmixed. In fact, by using Theorem 4.3.1, the minimal primes of I are as follows:

$$\begin{aligned}\mathrm{Ass}(I) = \{&(x_2,x_3,x_4),(x_6,x_7,x_8),(x_9,x_{10},x_{11}),(x_{13},x_{14},x_{15}),(x_6,x_7,x_{11}),\\ &(x_6,x_7,x_{15}),(x_6,x_{10},x_{11}),(x_6,x_{10},x_{15}),(x_6,x_{14},x_{15}),(x_9,x_{10},x_{15}),\\ &(x_9,x_{14},x_{15})\}.\end{aligned}$$

However, one can verify with *Macaulay2* [18] that S/I is not Cohen-Macaulay.

The above remark states that unmixedness is not sufficient for the edge ideal of $\mathbf{t}$-spread d-partite hypergraphs being Cohen-Macaulay. In what follows, we give a characterization of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ with Cohen-Macaulay edge ideals. To do this, we introduce the following notations, that is, $q(u_k) := |\mathrm{set}(u_k)|$ and $q(I) := \max\{q(u_1), \dots, q(u_r)\}$.

We are in a position to state the last result of this section in the subsequent theorem.

Theorem 4.3.11. Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread d-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \dots, d$. Then $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is Cohen-Macaulay if and only if either $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is a principal ideal, or $n_1 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1, \dots, d-1$.

Proof. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and S be the ambient ring of I. Since I has linear quotients, thanks to Theorem 4.1.3, it follows from [26, Corollary 1.6] that the length of the minimal free resolution of S/I over S is equal to $q(I)+1$. This implies that $\operatorname{depth}(S/I) = |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| - q(I) - 1$. Moreover, $\dim(S/I) = |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| - \operatorname{ht}(I)$, where $\operatorname{ht}(I)$ denotes the height of I. This summarizes to S/I is Cohen–Macaulay if and only if $\operatorname{ht}(I) = q(I)+1$. Therefore, it is enough to show that $\operatorname{ht}(I) = q(I)+1$ if and only if $n_1 = n_2 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1,\ldots,d-1$.

If I is a principal ideal then S/I is Cohen Macaulay. Now, assume $n_1 = n_2 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1,\ldots,d-1$. Let $u = x_{k_1}\cdots x_{k_d} \in \mathrm{G}(I)$, where $k_i \in V_i$ for all $i = 1,\ldots,d$. Since $[i_1, k_1 - 1] \subseteq V_1$ and $[k_{j-1} + t_{j-1}, k_j - 1] \subseteq V_j$ for all $j = 2,\ldots,d$, by Proposition 4.1.4, we obtain $q(u) = k_d - i_1 - \sum_{j=1}^{d-1} t_j$. This shows that the maximum value of $q(u)$ is obtained when k_d takes the maximum possible value which is $\max V_d = i_d + s - 1$. Furthermore, using $i_{j+1} - i_j = t_j$ for all $j = 1,\ldots,d-1$, this gives that $i_d = i_1 + \sum_{j=1}^{d-1} t_j$. Hence, we have $q(I) = s-1$, as required.

Conversely, suppose S/I is Cohen-Macaulay, that is, $\operatorname{ht}(I) = q(I)+1$. It follows from $\operatorname{ht}(I) = q(I)+1$ that I is unmixed and by using Proposition 4.3.9, this yields that, for all $j = 1,\ldots,d$, we have $n_j = s$, and for each $j = 1,\ldots,d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Then $\operatorname{ht}(I) = s$ thanks to Proposition 4.3.8. If $s = 1$, then I is a principal ideal. Now, let $s > 1$. We only need to show that, for each $j = 1,\ldots,d-1$, we have $i_{j+1} - i_j = t_j$. Suppose that for some j we have $i_{j+1} - (i_j + s - 1) > t_j - 1$. Let $v = x_{i_1+s-1}x_{i_2+s-1}\cdots x_{i_d+s-1}$. Then $v \in \mathrm{G}(I)$ because $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ do not contain isolated vertices and $\{i_1 + s - 1, i_2 + s - 1, \ldots, i_d + s - 1\}$ is a $\mathbf{t}$-spread edge in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Now, Proposition 4.1.4 gives that $\operatorname{set}(v) \cap V_1 = [i_1, i_1 + s - 2]$ and $\operatorname{set}(v) \cap V_{j+1} = \{i_{j+1}, \ldots, i_{j+1} + s - 2\}$. This shows that $q(v) > 2(s-1)$ and $q(I) + 1 > \operatorname{ht}(I) = s$, a contradiction. $\square$

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