RANDOM HOLOMORPHIC SECTIONS ASSOCIATED WITH A SEQUENCE OF LINE BUNDLES ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT

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The study of zeros of random polynomials is a fascinating subject due to its numerous connections within mathematics and physics. In particular, the distribution of these zeros is crucial for understanding chaotic dynamics and quantum ergodicity, as it models the behavior of nodal sets of eigenfunctions in chaotic quantum systems. Building upon these ideas, the concepts naturally extend to higher dimensions through random holomorphic sections, which generalize random polynomials, giving rise to the emerging field of stochastic Kähler geometry. This thesis investigates two interconnected problems within the realm of stochastic Kähler geometry, focusing on the equidistribution and statistical fluctuations of zeros of random holomorphic sections associated with Hermitian holomorphic line bundles on compact Kähler manifolds.

In the first part, we establish an equidistribution phenomenon for zeros of systems of random holomorphic sections associated with a sequence of positive Hermitian holomorphic line bundles with $\mathscr{C}^2$ metrics on a compact Kähler manifold X. This is achieved through variance estimates and an analysis of the expected distributions of random zero currents of integration in any codimension k. Our results extend previous findings in the field by encompassing a broader range of probability distributions, including Gaussian, Fubini-Study measures, and probability measures with bounded densities and logarithmically decaying tails.

In the second part, we establish a central limit theorem for random currents of integration along the zero divisors of standard Gaussian holomorphic sections. This theorem, proved within the framework of sequences of holomorphic line bundles, demonstrates the asymptotic normality of smooth linear statistics of random zero divisors. Along the way, using methods from complex differential geometry, such as Demailly's L^2-estimates for the $\bar{\partial}$-operator, we obtain first-order asymptotics and upper decaying estimates for near and off-diagonal Bergman kernels.

ÖZET

KOMPAKT KÄHLER MANIFOLDLAR ÜZERİNDEKİ BİR DİZİ DOĞRU DEMETLERİYLE İLİŞKİLİ RASSAL HOLOMORFİK KESİTLER

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Anahtar Kelimeler: Rassal holomorfik kesitler, sıfırların eşdüzgün dağılımı, varyans, merkezi limit teoremi, Bergman çekirdeği asimptotikleri.

Rassal polinomların sıfırlarının incelenmesi, matematik ve fizik alanlarındaki çeşitli bağlantılar nedeniyle oldukça ilgi çekici bir konudur. Özellikle, bu sıfırların dağılımı, kaotik dinamikler ve kuantum ergodikliğini anlamak için kritik öneme sahiptir, çünkü kaotik kuantum sistemlerdeki özfonksiyonların nodal kümelerinin davranışını modellemektedir. Bu temel fikirler üzerine inşa edilerek, kavramlar rassal holomorfik kesitler aracılığıyla doğal olarak daha yüksek boyutlara genişletilmektedir ve rassal polinomları genelleştirerek, ortaya çıkan stokastik Kähler geometrisi alanının temelini oluşturmaktadır. Bu tezde, stokastik Kähler geometrisi alanında birbirleriyle bağlantılı iki problem ele alınmıştır. Çalışmamız, kompakt Kähler manifoldları üzerinde Hermisyen holomorfik doğru demetleriyle ilişkilendirilen rassal holomorfik kesitlerin sıfırlarının eş dağılımı ve istatistiksel dalgalanmalarına odaklanmıştır.

İlk bölümde, bir kompakt Kähler manifoldu X üzerinde $\mathscr{C}^2$ sınıfı metriklere sahip pozitif Hermisyen holomorfik doğru demetlerinin bir dizisiyle ilişkili rassal holomorfik kesitlerin sistemlerindeki sıfırlar için bir eş dağılım fenomeni kanıtlanmıştır. Bu sonuca, herhangi bir k eşboyutundaki rassal sıfır akışlarının beklenen dağılımlarını ve varyans sınırlamalarını analiz ederek ulaşılmıştır. Bu sonuçlar, Gaussian, Fubini-Study ve sınırlı yoğunluk fonksiyonlara ve logaritmik olarak azalan kuyruklara sahip olasılık dağılımlarını içerecek şekilde önceki sonuçları geliştirmiştir.

İkinci bölümde, standart Gauss holomorfik kesitlerinin sıfırlarıyla ilişkilendirilen rassal sıfır akımları için bir merkezi limit teoremi elde edilmiştir. Bu teorem, holomorfik doğru demetlerinin dizileri çerçevesinde ispatlanmış olup, rassal sıfır kümelerinin lineer istatistiklerinin asimptotik normalitesini göstermektedir. Ek olarak, Demaily'nin $\bar{\partial}$-operatörü için L^2-sınırlamalarından gelen karmaşık diferansiyel geometri tekniklerini kullanarak, diagonale yakın ve diagonal Bergman çekirdeklerinin birinci dereceden asimptotiklerini ve yeterince hızlı azalan üst sınırlar elde edilmiştir.

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To my family

TABLE OF CONTENTS

1. INTRODUCTION

1.1 Literature review

In recent years, equidistribution and statistical properties of zeros of random holomorphic sections have been progressed heavily. There are numerous results as to the distribution of zeros of holomorphic sections in diverse probabilistic frameworks. Amongst these, what has been more largely focused on is the tensor powers of a given positive Hermitian line bundle over a compact Kähler manifold within a Gaussian setting. In this background, [Shiffman & Zelditch (1999)] is one of the very first papers in the mathematics literature considering the equidistribution problem of (Gaussian) random holomorphic sections. In the following years, [Shiffman & Zelditch (2008)] and [Shiffman & Zelditch (2010)] derived asymptotic variance formulas for linear statistics and their smooth analogs, all within the same geometric and probabilistic framework. One of the most recent results, proved via the techniques of [Shiffman & Zelditch (2010)] in [Shiffman (2021)], is the asymptotic expansion of the variance for the codimension 1 case in the aforementioned setting. This asymptotic expansion shows also that the coefficient of the first term in the expansion, which also appeared as the leading-order term in the asymptotic formula obtained in [Shiffman & Zelditch (2010)], is sharp. On the other hand, Dinh and Sibony [Dinh & Sibony (2006)] innovated a method from complex dynamics for analyzing zero distribution, and set convergence speed bounds in the compact case, enhancing Shiffman and Zelditch's initial results, namely [Shiffman & Zelditch (1999)]. Alongside the Gaussian setting, in the papers [Bayraktar (2016), Bayraktar, Coman, Herrmann & Marinescu (2018), Bloom & Levenberg (2015), Coman, Lu, Ma & Marinescu (2023), Coman & Marinescu (2015), Coman, Ma & Marinescu (2017) and Bayraktar, Coman & Marinescu (2020)], more general scenarios are investigated, including the Gaussian case as a particular instance. For example, in [Bloom & Levenberg (2015)], the authors focus on the complex random variables that possesses bounded distribution functions on the whole complex plane $\mathbb{C}$ and outside of a very large disk with radius ρ, its integral with respect to the two-dimensional Lebesgue measure has an upper bound depending on ρ, the latter condition is called the tail-end estimate. Meanwhile, in [Bayraktar et al. (2020); Coman et al. (2017)] the authors expand their research to the equidistribution problem within a wider context, involving a sequence of Hermitian line bundles over a normal reduced complex Kähler space.

It is essential to highlight that global holomorphic sections are natural generalizations of polynomials. In relation to the general setting, there has been a great deal of interest in the statistical problems related to the zero sets of random polynomials of several variables in both real and complex domains. For a comprehensive overview of results in this direction, interested readers can refer to [Bayraktar (2017b); Bloom (2005); Bloom & Dauvergne (2019); Bloom & Levenberg (2015); Bloom & Shiffman (2007); Edelman & Kostlan (1995); Hughes & Nikeghbali (2008); Ibragimov & Zaporozhets (2013); Rojas (1996); Shub & Smale (1993)](and references cited therein). These sources cover a wide range of results encompassing both Gaussian and non-Gaussian cases, along with historical developments of the polynomial theory. Long before these developments, it is important to recognize the pioneering work of mathematicians such as Littlewood-Offord, Kac, Hammersley, and Erdös-Turan, who were among the first to investigate the distribution of roots of algebraic equations both with random and deterministic coefficients in a single real variable. For more insights into these foundational studies, interested readers can consult the papers [Erdös & Turán (1950); Hammersley (1956); Kac (1943); Littlewood & Offord (1943)]. On the other hand, there is also a growing physics literature dealing with the equidistribution and probabilistic problems of zeros of complex random polynomials. For studies of fundamental importance in this direction, see, e.g., [Bogomolny, Bohigas & Leboeuf (1996); Forrester & Honner (1999); Hannay (1996); Nonnenmacher & Voros (1998)].

Alongside these developments, investigation of the central limit theorem in the context of smooth linear statistics, such as integrals of smooth test forms over zero divisors of random holomorphic sections, is another intriguing challenge. In this regard, the work by Sodin and Tsirelson [Sodin & Tsirelson (2004)] holds significant importance. They established an asymptotic normality result for Gaussian random polynomials and analytic functions in the complex plane. This seminal work has been extended in two distinct contexts. The first extension, attributed to the research of Shiffman and Zelditch [Shiffman & Zelditch (2010)], applies within the prequantum line bundle setting. This involves random holomorphic sections of a Hermitian line bundle (with $\mathscr{C}^3$ Hermitian metrics) over a compact Kähler manifold, where the first Chern form and Kähler form satisfy the prequantum line bundle condition.

The second extension, studied in [Bayraktar (2017a)], is attributed to Bayraktar, who expanded this result to encompass general random polynomials in $\mathbb{C}^n$ utilizing techniques from weighted pluripotential theory.

Recent developments have also focused on asymptotic normality in the context of noncompact complex manifolds. In [Bojnik & Gunyüz (2024)], a central limit theorem (proposed in [Drewitz, Liu & Marinescu (2023)]) has been established for currents of integration associated with the zero divisors of standard Gaussian random holomorphic sections of large tensor powers of a positive line bundle over a noncompact Hermitian manifold. Unlike previous cases, this situation involves studying infinite-dimensional separable Hilbert subspaces of the space of holomorphic sections, which is itself a Fréchet space. For further considerations, see [Bojnik & Gunyüz (2024)] and [Drewitz et al. (2023)]. Building on their work in [Drewitz et al. (2023)], in the setting of Berezin-Toeplitz operators, the authors have newly proved a central limit theorem for zero currents of integration related to standard Gaussian holomorphic sections in the closure of the image subspace (which is also a separable Hilbert space) of square-integrable holomorphic sections under a Berezin-Toeplitz operator [Drewitz, Liu & Marinescu (2024)].

In both compact and non-compact cases, the asymptotic behavior of the normalized Bergman kernel, particularly off-diagonal and near-diagonal, is crucial. This kernel, functioning as a covariance function of normalized complex Gaussian processes, is pivotal to the analyses.

Separately, Nazarov and Sodin ([Nazarov & Sodin (2012)]) focused on the asymptotic normality of linear statistics of zeros in Gaussian entire functions on $\mathbb{C}$. They provide a more general approach in terms of test functions, which are typically smooth, by considering measurable bounded test functions and the clustering of k-point correlation functions.

1.2 Motivation

In the standard setup of geometric quantization, we work with a compact Kähler manifold (X,ω), equipped with a Hermitian holomorphic line bundle (L,h), known as the prequantum line bundle, fulfilling the *prequantization* condition given by

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L = c_1(L,h). \tag{1.1}$$

Here R^L is the curvature of the Chern connection on L, and $c_1(L,h)$ is the Chern curvature form of (L,h). The existence of the prequantum line bundle (L,h) allows the investigation into the Hilbert space $H^0(X,L)$ of holomorphic sections and to establish a mapping between classical observables on X and quantum operators on $H^0(X,L)$ in the setting where the Planck constant approaches zero. The modification of Planck's constant corresponds to scaling the Kähler form via tensor powers $L^{\otimes p}$, and the curvature of the line bundle is thus described by $\hbar = \frac{1}{p}$.

The stipulation in (1.1) is recognized as an integrality condition. The existence of a prequantum holomorphic line bundle is strongly connected with the integral nature of the de Rham cohomology class $[\omega]$, i.e., $[\omega] \in H^2(X,\mathbb{Z})$. When dealt with a Kähler form ω that is not integral, one can construct an associated family of positive line bundles (L_p,h_p). The curvatures of these bundles approximate integer multiples of ω, thus serving as a prequantization of the non-integral Kähler form ω. This approach extends the framework of geometric quantization to a broader class of manifolds.

Motivated by this, in a recent work presented by Coman, Lu, Ma, and Marinescu [Coman et al. (2023)], in addition to providing a diagonal Bergman kernel expansion, they establish an equidistribution result for zeros of random holomorphic sections of such a sequence of line bundles (L_p,h_p) by imposing a natural convergence condition on the Chern curvature forms $c_1(L_p,h_p)$. This contrasts with the traditional setting of the tensor powers of a single prequantum line bundle L, i.e., $(L^{\otimes p},h^{\otimes p})$. As a probability measure they consider the Fubini-Study measure and use the standard formalism of meromorphic transforms from complex dynamics, as introduced by Dinh and Sibony in [Dinh & Sibony (2006)].

Building upon the foundations presented in [Coman et al. (2023)] and [Coman et al. (2017)], in the present thesis, we explore two distinct problems. In one direction, we establish an equidistribution phenomenon for zeros of systems of random holomorphic sections associated with a sequence of positive Hermitian holomorphic line bundles with $\mathscr{C}^2$ metrics on a compact Kähler manifold X, by using variance estimates and the expected distribution of random zero currents in any codimension k. This classical yet efficient approach allows for the extension of previous results to a wide spectrum of probability distributions, generalizing Theorem 0.4 considered in [Coman et al. (2023)]. In the other direction, we prove a central limit theorem for the smooth linear statistics of random zero divisors related to the zero sets of standard Gaussian holomorphic sections in a sequence of a positive holomorphic line bundles with Hermitian metrics of class $\mathscr{C}^3$ over a compact Kähler manifold X.

1.3 Statement of main results

In this part, we present our primary results regarding the equidistribution and fluctuations of zero sets of holomorphic sections in the context of compact Kähler manifolds.

I. Equidistribution

Let $(L_p,h_p)_{p\geq 1}$ be a sequence of holomorphic line bundles on a compact Kähler manifold (X,ω) of dimension n with a fixed Kähler form ω and $\mathscr{C}^2$ Hermitian metrics h_p (see Section 2.4 below) such that the curvature forms $c_1(L_p,h_p)$ satisfy the so called "prequantization condition" or "diophantine relation".

$$\left\| \frac{1}{A_p} c_1(L_p,h_p) - \omega \right\|_{\mathscr{C}^0} = O(A_p^{-a}) \tag{1.2}$$

where $A_p > 0,\ a > 0$ and $\lim_{p\to\infty} A_p = +\infty$.

The space of global holomorphic sections of L_p is denoted by $H^0(X,L_p)$, which is a finite-dimensional vector space due to the compactness of X. We consider a Borel probability measure σ_p on $H^0(X,L_p)$, which does not charge pluripolar sets and satisfy the moment condition (3.8) (see Section 3.1 for further elaboration). The current of integration over simultaneous the zero locus of $\Sigma_p^k := (s_p^1,\dots,s_p^k)$, where $s_p^1,\dots,s_p^k$ are k-independent sections of $H^0(X,L_p)$, selected with respect to the probability measure σ_p, is denoted by $[Z_{\Sigma_p^k}]$ and it is almost surely well-defined (see Section 3.5). Additionally, we set $[\widehat{Z}_{\Sigma_p^k}] := \frac{1}{A_p^k}[Z_{\Sigma_p^k}]$ for the normalized current of integration.

Then, we obtain the variance estimate for such zero currents.

Theorem 1.3.1. *Let $(L_p,h_p)_{p\geq 1},\ (X,\omega)$ and σ_p be as defined above. Assume that they satisfy the conditions (3.2), (3.5) and (3.8). Assume further that the sections $s_p^1,s_p^2,\dots,s_p^k$ are independent random holomorphic sections for each p. Then there exists $P\in\mathbb{N}$ such that for all $p\geq P$ and any $\phi\in\mathcal{D}^{n-k,n-k}(X)$, one has the following estimate*

$$\langle \mathrm{Var}[\widehat{Z}_{\Sigma_p^k}],\phi\rangle \leq \frac{(C_p)^{2/\alpha}}{A_p^2}\,(2^{k-1}\mathrm{Vol}(X)B_\phi)^2,$$

where $\alpha \geq 2$, $C_p > 0$, and B_ϕ is a positive constant depending on the form ϕ.

Consequently, using the above variance estimation, we obtain the following equidistribution result.

Theorem 1.3.2. *Let $(L_p, h_p)_{p\geq 1}$, (X,ω) and σ_p be as defined above. Assume that they satisfy the conditions (3.2), (3.5) and (3.8).*

(i) If $\lim_{p\to\infty} \frac{C_p^{1/\alpha}}{A_p} = 0$, *then for* $1 \leq k \leq \dim_{\mathbb{C}} X$

$$\mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right] \longrightarrow \omega^k$$

in the weak topology of currents as* $p \to \infty$.

(ii) If $\sum_{p=1}^{\infty} \frac{C_p^{2/\alpha}}{A_p^2} < \infty$, *then for* σ_∞^k*−almost every sequence* $\{\Sigma_p^k\} \in \mathcal{H}_k^\infty$

$$\left[\widehat{Z}_{\Sigma_p^k}\right] \longrightarrow \omega^k$$

in the weak topology of currents as* $p \to \infty$.

Here, $\mathbb{E}$ and Var denote the expectation and variance of the random current of integration $[\widehat{Z}_{\Sigma_p^k}]$ which are explicitly defined in (3.10) and $\mathcal{D}^{p,q}(X)$ represent the space of test forms of bidegree (p,q) on X.

For codimension one, a method that circumvents the use of variance and expected distribution, such as the approach employed by Marinescu, Coman, and Bayraktar [Bayraktar et al. (2020)] in the setting where the normalized first Chern forms may not converge, works well, however, this method is not applicable for codimensions greater than one. Our approach with one summability condition generalizes their results for equidistribution in various codimensions. One should also emphasize that if the measure σ_p satisfy the moment condition (3.8) with constants $C_p = \Lambda$ independent of p, then the assumption in (i), $\lim_{p\to\infty} C_p^{1/\alpha} A_p^{-1} = 0$, is automatically satisfied. Moreover, the hypothesis (ii) transforms into $\sum_{p=1}^{\infty} A_p^{-2} < \infty$.

At the same time, our principal result, which has been established for codimension k, is applicable to a multitude of frequently investigated probability measures. These include the area measure of spheres, Gaussian, Fubini-Study measures, and measures with bounded density having logarithmic decaying tails. When our main result is applied to these measures in the context of codimension k and tensor powers of a fixed prequantum line bundle, the required summability assumption can be dropped. This reduction indicates that our results are consistent with the existing literature in this particular scenario. In [Bayraktar (2016)], for homogeneous projective manifolds, Bayraktar obtained, within a weighted pluripotential theory setting, an equidistribution result by employing Kolmogorov's law of large numbers in synthesizing the variance estimation with the expected distribution. In doing so, he utilizes the properties of positive closed currents with super-potentials ([Dinh & Sibony (2009)] and [Dinh & Sibony (2010)]), as introduced and studied by Sibony-Dinh in complex dynamics, to demonstrate that the limit of the average sequence of zero currents associated with their super-potentials is, in fact, identical to the limit of the sequence of these zero currents itself. Our methods extend the results of Zelditch-Shiffman. Bayraktar's results can also be regarded as generalizations of Shiffman-Zelditch's if one assumes the projective manifolds considered in [Shiffman & Zelditch (1999)] are homogeneous. Moreover, in [Bayraktar (2016)], when we take the locally regular compact set K to be the whole manifold X and $q=0$, our results in this paper generalize his results in terms of probability distributions and sequences of line bundles. In the same paper, he posed the question whether equidistribution result (Theorem 1.1 of [Bayraktar (2016)]) holds for any projective manifold. We also answer this question affirmatively by Theorem 1.3.2 when $K=X$ is any projective manifold and $q=0$.

Characterizing positive closed currents on complex manifolds that can be approximated by currents of integration along analytic subvarieties, and their local versions, are significant problems in pluripotential theory with numerous applications. For more on this topic, see [Coman & Marinescu (2013)] and [Coman, Marinescu & Nguyên (2018)]. Our result, Theorem 1.3.2, provides insights into the probabilistic version of this problem. Specifically, it demonstrates that a smooth positive closed form ω^k can be approximated by random currents of integration along analytic subsets of X of codimension k, for each integer $k\in\{1,\dots,n\}$.

II. Fluctuations

Building upon the same framework as in the equidistribution setting, with the sole distinction of employing Hermitian metrics of class $\mathscr{C}^3$ (rather than $\mathscr{C}^2$), and utilizing a Gaussian probability measure, we achieve an asymptotic normality result for smooth linear statistics of random zero sets.

Let $s\in H^0(X,L)\setminus\{0\}$. We denote by Z_s the set of zeros of s, and by the symbol $[Z_s]$, we mean the current of integration (with multiplicities) along Z_s. Here and throughout $dd^c=\frac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.

Now we state our main theorem in this direction.

Theorem 1.3.3. *Let $\{(L_p, h_p)\}_{p\geq 1}^{\infty}$ be a sequence of positive holomorphic line bundles over a compact Kähler manifold (X, ω) of dimension n with diophantine condition (4.1) and Hermitian metrics of class $\mathscr{C}^3$ such that $\frac{\|h_p\|_3}{\sqrt{A_p}} \to 0$ as $p \to \infty$. Suppose that $H^0(X, L_p)$ is endowed with the standard Gaussian probability measure for all $p \geq 1$. Let $s_p \in H^0(X, L_p)$ and ϕ be a real valued $(n-1, n-1)$-form on X with $\mathscr{C}^3$-coefficients and $dd^c\phi \not\equiv 0$. Then the distributions of the random variables*

$$\frac{\langle [Z_{s_p}], \phi\rangle - \mathbb{E}\langle [Z_{s_p}], \phi\rangle}{\sqrt{\mathrm{Var}\langle [Z_{s_p}], \phi\rangle}} \tag{1.3}$$

weakly converge towards the standard (real) Gaussian distribution $\mathcal{N}(0,1)$ as $p \to \infty$.

Here, $\mathbb{E}$ and Var denote the expectation and variance of the random variable $\langle [Z_{s_p}], \phi\rangle$, respectively, which are defined in (4.79) and (4.80).

We emphasize that this result is general enough, as we consider a sequence of line bundles instead of powers of a single line bundle, as studied in [Shiffman & Zelditch (2010)]. In particular, if we choose $(L_p, h_p) = (L^{\otimes p}, h^{\otimes p})$ for some fixed prequantum line bundle (L, h), we obtain the result of Shiffman and Zelditch as a special case.

The key to our analysis will be the behaviour of the Bergman kernel. We establish an upper decaying estimate for the off-diagonal Bergman kernel and derive the first order asymptotics of the Bergman kernel function. These results may have further consequences in other contexts as well. Notably, the study of Bergman kernels also plays a crucial role in understanding the existence of Kähler-Einstein metrics, which are special Hermitian metrics with constant scalar curvature.

2. Preliminaries

In this chapter, we provide the essential background required for the thesis. We briefly delve into the basics of complex geometry, define currents on complex manifolds with a main emphasis on positive closed currents, and examine their relationship with intersection theory. The primary reference guiding this chapter is Demaily's book [Demailly (2012)], supplemented by other references such as [Griffiths & Harris (1978)], [Huybrechts (2005)], [Dinh & Sibony (2005)], [Székelyhidi (2014)] and [Ma & Marinescu (2007)].

2.1 Differential calculus on Complex Manifolds

In this section, we lay the groundwork for doing calculus on complex manifolds by introducing essential tools. Specifically, we define the complexified tangent and cotangent spaces, establish consistent notations, and introduce complex differential forms on complex manifolds. These foundational elements will serve as crucial instruments throughout the thesis.

Definition 2.1.1. *A complex manifold X of dimension n is a topological space (that is Hausdorff and separable) that admits an open cover $\{U_\alpha\}_{\alpha\in I}$ and local charts $\phi_\alpha : U_\alpha \to \mathbb{C}^n$, such that for all $\alpha, \beta \in I$, the transition maps*

$$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic maps between open subsets of $\mathbb{C}^n$.

The components $\phi_\alpha(x) = (z_1^\alpha, \dots, z_n^\alpha)$ are called the local coordinates on U_α, defined by the chart ϕ_α, and they are related by means of transition functions $z^\alpha = \phi_{\alpha\beta}(z^\beta)$, where $z^\alpha = (z_1^\alpha, \dots, z_n^\alpha)$.

For $x \in X$, we denote by $T_x X$ the real tangent space of dimension $2n$ at the point x of the underlying smooth manifold X. We also denote by TX (respectively, T^*X) the corresponding real tangent bundle (respectively, cotangent bundle). A differential form of degree k on X is a section of the exterior bundle $\Lambda^k T^*X$. We will use the notation $\Omega^k(X)$ for the space of degree k differential forms on X.

An almost complex structure on X is an endomorphism $J : TX \to TX$ with the property that $J^2 = -\mathrm{Id}$.

We note that when X is a complex manifold, it naturally has a complex structure induced by the coordinate isomorphisms

$$d\phi_\alpha(x) : T_x X \to \mathbb{C}^n$$

which is independent of the coordinate chart U_α, since the transition maps $d\phi_{\alpha\beta}$ are complex linear isomorphisms.

On a complex manifold, it is convenient to work with the complexified tangent bundle $T_{\mathbb{C}}X := TX \otimes \mathbb{C}$, which decomposes as a direct sum:

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X,$$

where

$$T^{1,0}X := \ker(J - i\mathrm{Id}) = \{\xi - iJ\xi : \xi \in TX\}$$

and

$$T^{0,1}X := \ker(J + i\mathrm{Id}) = \{\xi + iJ\xi : \xi \in TX\} = \overline{T^{1,0}X}$$

represent the eigenbundles of the complexified endomorphism $J_{\mathbb{C}} := J \otimes \mathrm{Id}_{\mathbb{C}}$ corresponding to the eigenvalues i and $-i$, respectively. The components $T^{1,0}X$ and $T^{0,1}X$ are called the *holomorphic* and *anti-holomorphic* tangent bundles of X. By duality, a similar decomposition occurs for the complexified cotangent bundle:

$$T^*_{\mathbb{C}}X := T^*X \otimes \mathbb{C} = (T^*X)^{1,0} \oplus (T^*X)^{0,1},$$

where

$$(T^*X)^{1,0} = \{f - if \circ J : f \in T^*X\}, \quad \text{and} \quad (T^*X)^{0,1} = \overline{(T^*X)^{1,0}}.$$

Now, since the exterior algebra of a direct sum is isomorphic to the tensor product of the exterior algebras of the individual spaces, and this isomorphism respects grading, we have

$$\Lambda^k T^*_{\mathbb{C}} X \cong \bigoplus_{j=0}^{k} \left(\Lambda^j (T^*X)^{1,0} \otimes \Lambda^{k-j} (T^*X)^{0,1} \right).$$

Hence, the complexified exterior algebra is given by

$$\Lambda^k_{\mathbb{C}} X \cong \bigoplus_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \Lambda^{p,q} X, \tag{2.1}$$

where $\Lambda^k_{\mathbb{C}} X := \Lambda^k T^*_{\mathbb{C}} X$ and the components $\Lambda^{p,q} X$ are defined by

$$\Lambda^{p,q} X := \Lambda^p (T^*X)^{1,0} \otimes \Lambda^q (T^*X)^{0,1}.$$

A complex differential form of degree k on X is defined to be a smooth section of the exterior bundle $\Lambda^k_{\mathbb{C}} X$. We denote by $\Omega^k_{\mathbb{C}}(X)$ the space of such complex differential forms. Using (2.1), we have a natural decomposition of the space of complex differential forms:

$$\Omega^k_{\mathbb{C}}(X) = \bigoplus_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \Omega^{p,q}(X), \tag{2.2}$$

where $\Omega^{p,q}(X)$ is the space of smooth sections of $\Lambda^{p,q} X$. The elements of $\Omega^{p,q}(X)$ are called complex differential forms of bidegree or type (p,q).

Locally, if $(z_1, \dots, z_n)$ are holomorphic coordinates on an open subset $U \subset X$, where $z_j = x_j + i y_j$ for $j = 1, \dots, n$, then one has real coordinates $(x_1, y_1, \dots, x_n, y_n)$ for the underlying smooth manifold. The natural complex structure on X is given by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

For $j = 1, \dots, n$, define

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}\right)$$

and

$$dz_j := dx_j + i\, dy_j, \quad d\bar{z}_j := dx_j - i\, dy_j.$$

Then, $\left\{\frac{\partial}{\partial z_j}\right\}_{j=1}^{n}$ and $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}_{j=1}^{n}$ form local bases for $T^{1,0}X$ and $T^{0,1}X$, respectively. Similarly, by duality, $\{dz_j\}_{j=1}^n$ and $\{d\bar{z}_j\}_{j=1}^n$ form local bases for $(T^*X)^{1,0}$ and $(T^*X)^{0,1}$, respectively. Hence, $\{dz_j, d\bar{z}_j\}_{j=1}^n$ is a local basis for $T^*_{\mathbb{C}} X$. Therefore, a local basis for $\Lambda^{p,q}X$ is given by

$$\left(dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}\right)_{1\leq i_1<\cdots<i_p,\, j_1<\cdots<j_q\leq n}.$$

In turn, any complex differential k-form ϕ can be locally written as

$$\phi(z) := \sum_{|I|=p,|J|=q} \phi_{I,J}(z)dz_I\wedge d\overline{z}_J, \tag{2.3}$$

where $\phi_{I,J}$ are complex-valued smooth functions, $I=(i_1,\ldots,i_p)$ and $J=(j_1,\ldots,j_q)$ are multiindices with integer components arranged in increasing order, and

$$dz_I = dz_{i_1}\wedge\cdots\wedge dz_{i_p},\quad d\overline{z}_J = d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}.$$

A differential form $\phi\in\Omega^k_{\mathbb{C}}(X)$ is said to be real if $\phi=\bar{\phi}$. In particular, a $(1,1)$-form ϕ is real if and only if, locally,

$$\phi = i\sum_{j,k=1}^n \phi_{j,k}\, dz_j\wedge d\bar{z}_k,$$

where $[\phi_{j,k}]$ is a Hermitian $n\times n$ matrix.

Given the structure of complex differential forms, we can now consider the exterior differential. The $\mathbb{C}$-linear extension of the exterior differential on $\Omega^k_{\mathbb{C}}(X)$ is denoted by

$$d:\Omega^k_{\mathbb{C}}(X)\to\Omega^{k+1}_{\mathbb{C}}(X).$$

The decomposition (2.2) of complex differential forms naturally leads to a corresponding decomposition of the exterior differential. More precisely, if $\phi: X\to\mathbb{C}$ is a complex valued smooth function, then locally

$$\begin{aligned} d\phi &= \sum_{j=1}^n \frac{\partial\phi}{\partial x_j}dx_j + \frac{\partial\phi}{\partial y_j}dy_j \\ &= \sum_{j=1}^n \frac{1}{2}\left(\frac{\partial\phi}{\partial x_j} - i\frac{\partial\phi}{\partial y_j}\right)dz_j + \sum_{j=1}^n \frac{1}{2}\left(\frac{\partial\phi}{\partial x_j} + i\frac{\partial\phi}{\partial y_j}\right)d\overline{z}_j \\ &= \sum_{j=1}^n \frac{\partial\phi}{\partial z_j}dz_j + \sum_{j=1}^n \frac{\partial\phi}{\partial\overline{z}_j}d\overline{z}_j. \end{aligned}$$

More generally, if $\phi\in\Omega^{p,q}(X)$ is a differential (p,q)-form. Then, locally

$$\phi = \sum_{|I|=p,|J|=q}\phi_{I,J}\, dz_I\wedge d\overline{z}_J,$$

where $\phi_{I,J}$ are smooth functions. Since $d(dz_I \wedge d\overline{z}_J) = 0$ for all multiindices I, J, by the Leibniz Rule, we have

$$\begin{aligned}
(2.4)\quad d\phi &= \sum_{|I|=p,|J|=q} d\phi_{I,J} \wedge dz_I \wedge d\overline{z}_J \\
(2.5)\quad &= \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial \phi_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J + \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial \phi_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.
\end{aligned}$$

Hence, the exterior differential d splits into $d = \partial + \overline{\partial}$, where

$$\begin{aligned}
\partial &: \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \\
\overline{\partial} &: \Omega^{p,q}(X) \to \Omega^{p,q+1}(X), \\
\partial\phi &= \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial \phi_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J, \\
\overline{\partial}\phi &= \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial \phi_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J.
\end{aligned}$$

The equality $d^2 = (\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0$, along with the fact that $\partial^2 : \Omega^{p,q}(X) \to \Omega^{p+2,q}(X)$, $\overline{\partial}^2 : \Omega^{p,q}(X) \to \Omega^{p,q+2}(X)$, and $\partial\overline{\partial} + \overline{\partial}\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q+1}(X)$, implies that

$$\partial^2 = 0,\ \partial\overline{\partial} + \overline{\partial}\partial = 0,\ \overline{\partial}^2 = 0.$$

Moreover, ∂ and $\overline{\partial}$ are conjugate, i.e., $\overline{\partial}\phi = \overline{\partial\overline{\phi}}$ for any $\phi \in \Omega^{p,q}(X)$.

This shows that for each $p = 0, 1, 2, \ldots, n$ we can define a cohomological complex of $\mathbb{C}$-vector spaces

$$0 \to \Omega^{p,0}(X) \xrightarrow{\overline{\partial}} \Omega^{p,1}(X) \xrightarrow{\overline{\partial}} \cdots\cdots \xrightarrow{\overline{\partial}} \Omega^{p,q}(X) \to \Omega^{p,n}(X) \to 0.$$

Definition 2.1.2. *The (p,q)- Dolbeaut cohomology group of X is the vector space*

$$H^{p,q}(X, \mathbb{C}) = \frac{\ker \overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)}{\operatorname{Im} \overline{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)}.$$

Remark 2.1.1. *It is worth noting that the complex de Rham cohomology group of X is expressed as*

$$H^k_{dR}(X, \mathbb{C}) := \frac{\ker d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)}{\operatorname{Im} d : \Omega^{k-1}_{\mathbb{C}}(X) \to \Omega^k_{\mathbb{C}}(X)} \cong H^k_{dR}(X, \mathbb{R}) \otimes \mathbb{C},$$

where $H^k_{dR}(X,\mathbb{R})$ is the de Rham cohomology group of the underlying (real) smooth manifold X. Moreover, if X is a compact Kähler manifold (our main assumption we maintain throughout the thesis) we have the following Hodge decomposition

$$H^k_{dR}(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}).$$

The following is an analogue of the Poincaré lemma for the exterior derivative.

Lemma 2.1.1 (Dolbeault-Grothendieck Lemma)**.** *A $\overline{\partial}$-closed form is locally $\overline{\partial}$-exact.*

This lemma can be used to establish the local $\partial\overline{\partial}$-Lemma (see 2.3.1), which is a frequently beneficial tool for subsequent analyses.

2.2 Currents on Complex Manifolds

The concept of currents was initially introduced by Georges de Rham and was further developed by other mathematicians such as Federer and Fleming. It is a generalization of the notion of distribution. This section is dedicated to defining currents on complex manifolds and presenting fundamental properties associated with them.

2.2.1 Spaces of Currents

Let X be a complex manifold of dimension n. In this context the space of smooth differential forms of bidegree (p,q) with compact support is denoted by $\mathcal{D}^{p,q}(X)$, often referred to as *test forms*. For a subset U of X, $\mathcal{D}^{p,q}(U)$ represents the space consisting of elements $\phi \in \mathcal{D}^{p,q}(X)$ with compact support contained within U.

Next, we introduce a topology on the space $\mathcal{D}^{p,q}(X)$ of test forms. Given a sequence of relatively compact open subsets $\{U_j\}_{j=1}^{\infty}$ in X such that $\overline{U_j} \subset U_{j+1}$, for each j, and $\bigcup_{j=1}^{\infty} U_j = X$. Associated to each compact subset $K \subset U_j$ contained within a single local coordinate chart $(z_1,\ldots,z_n)$, where $z_j = x_j + iy_j$, and any positive integer

l, we define a semi-norm

$$p_K^l(\phi) := \sup_{z\in K} \max_{|I|=p,|J|=q} \max_{|\alpha|\le l} |\partial^\alpha \phi_{I,J}(z)|,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ runs over $\mathbb{N}^{2n}$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial y_1^{\alpha_{n+1}} \cdots \partial y_n^{\alpha_{2n}}}$ represents mixed partial derivatives of order $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$. The coefficients $\phi_{I,J}$ are the components of the form ϕ in these coordinates.

We endow each of the spaces $\mathcal{D}^{p,q}(\overline{U_j})$ with the topology of local uniform convergence of coefficients and all of their derivatives, this means that a sequence (ϕ_k) converges to ϕ in $\mathcal{D}^{p,q}(\overline{U_j})$ if $p_K^l(\phi_k - \phi) \to 0$ for all compact subsets $K \subset \overline{U_j}$ and for all integers l. This convergence ensures that not only do the forms themselves converge uniformly to ϕ on compact subsets, but also all their derivatives up to any order l converge uniformly. The topology defined in this way is actually given by a countable family of semi-norms p_K^l varying over all compact subsets $K \subset \overline{U_j}$ and all integers l, making $\mathcal{D}^{p,q}(\overline{U_j})$ a Fréchet space, which is a complete, metrizable, and locally convex topological vector space.

Consequently, we furnish $\mathcal{D}^{p,q}(X)$ with the topology of the direct limit (or inductive limit) of the spaces $\mathcal{D}^{p,q}(\overline{U_j})$. This direct limit topology is defined such that a sequence (ϕ_k) converges to ϕ in $\mathcal{D}^{p,q}(X)$ if there exists $N \in \mathbb{N}$ such that for all $k \ge N$, $\phi_k \in \mathcal{D}^{p,q}(\overline{U_m})$ for some m, and $\phi_k \to \phi$ in $\mathcal{D}^{p,q}(\overline{U_m})$. Essentially, beyond a certain point, all forms ϕ_k are supported within a single $\overline{U_j}$ and converge in the local Fréchet space topology of $\mathcal{D}^{p,q}(\overline{U_j})$.

When $p = q = 0$, $\mathcal{D}^{p,q}(X)$ corresponds to the space of test functions studied in distribution theory, denoted as $\mathcal{D}(X)$.

Definition 2.2.1. *A current of bidegree (p,q) and of bidimension $(n-p, n-q)$ on the complex manifold X is a continuous linear form $T : \mathcal{D}^{n-p,n-q}(X) \to \mathbb{C}$. If ϕ is a test form in $\mathcal{D}^{n-p,n-q}(X)$, the pairing between T and ϕ is denoted by $\langle T, \phi \rangle := T(\phi)$.*

The support of a current T, denoted by $\operatorname{supp}(T)$ is the smallest closed subsest of X such that T vanishes outside of it. In other words, $\langle T, \phi \rangle = 0$ for any test form $\phi \in \mathcal{D}^{n-p,n-q}(X \setminus \operatorname{supp}(T))$.

A current of bidegree (p,q) is abbreviated as a (p,q)-current, and the set comprising all (p,q)- currents will be denoted by $\mathcal{D}'_{p,q}(X)$. The principles of complex differential calculus can be easily extended to currents by duality, leading to the following decomposition of the space of test forms and currents, respectively.

$$(2.6) \qquad \mathcal{D}^k(X) = \bigoplus_{\substack{p+q=k \\ p,q\in\mathbb{N}}} \mathcal{D}^{p,q}(X), \;\; \mathcal{D'}_k(X) = \bigoplus_{\substack{p+q=k \\ p,q\in\mathbb{N}}} \mathcal{D'}_{p,q}(X).$$

Here, $\mathcal{D}^k(X)$ and $\mathcal{D}'_k(X)$ denote the complexified spaces of test forms and currents, respectively, of degree k in the underlying (real) smooth manifold X.

Clearly, distributions are currents of maximal bidegree, and we use the notation $\mathcal{D}'(X)$ for them, whereas also $(0,0)$-currents can be thought locally as distributions. For instance, if T is a current of bidegree $(0,0)$, we can associate to T a distribution S just by defining

$$(2.7) \qquad \langle S, \phi\rangle := \langle T, \phi \, dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n\rangle,$$

where $(z_1, \ldots, z_n)$ are local coordinates on an open set U. This is a one-to-one correspondence between distributions and $(0,0)$-currents. Hence, we often use them interchangeably in various contexts. We also remark that distributions are building blocks for (p,q)-currents, since locally any current T can be seen as a differential form with distribution coefficients; see [Demailly (2012), I. Proposition 2.9]. Indeed, if T is a (p,q)-current on X. Then, T has a unique representation

$$(2.8) \qquad T = \sum_{|I|=p, |J|=q} T_{I,J} \, dz_I \wedge d\overline{z}_J,$$

where $T_{I,J}$ are distributions.

Observe that if T is a (p,q)-current, one can form the wedge product with a differential form ψ of bidegree (r,s) to define a new current $T \wedge \psi \in \mathcal{D}'_{p+r,q+s}$ as follows

$$\langle T \wedge \psi, \phi\rangle := \langle T, \phi \wedge \psi\rangle, \quad \text{for all } \phi \in \mathcal{D}^{n-p-r,n-q-s}(X).$$

This operation is one instance for the construction of higher-degree currents from the existing ones. In the next section, we will see other instances as well.

Next, we provide some classical examples of currents.

Example 2.2.1. *Let Z be a closed submanifold of X of dimension p (with no boundary). The current of integration along Z, denoted by $[Z]$, is defined as*

$$\langle [Z], \phi\rangle = \int_Z \phi, \;\; \text{for } \phi \in \mathcal{D}^{p,p}(X).$$

It is clear that $\operatorname{supp}([Z]) = Z$.

Example 2.2.2. *If ψ is a differential form of bidegree (p,q) with coefficients in $L^1_{loc}(X)$, then we can associate to ψ a (p,q)-current T_ψ as follows*

$$\langle T_\psi, \phi\rangle = \int_X \psi \wedge \phi$$

for any $\phi \in \mathcal{D}^{n-p,n-q}(X)$.

Note that the examples above justify the notion of degree and dimension for currents.

Throughout, we employ the *weak* topology* in the space of currents. In this topology, a sequence of (p,q)-currents $\{T_k\}$ converges to a current T if $\langle T_k, \phi\rangle \to \langle T, \phi\rangle$ for every $\phi \in \mathcal{D}^{n-p,n-q}(X)$, and we say that $T_k \to T$ in the weak sense of currents or in the weak* topology of currents.

Finally, by definition, given that a current is a linear form which is continuous in the direct limit topology of $\mathcal{D}^{n-p,n-q}(X)$, we have that for any $T \in \mathcal{D}'_{p,q}(X)$, and any compact set $K \subset X$, there exists $l \in \mathbb{N}$ and a positive constant M such that

$$|\langle T, \phi\rangle| \le M p^l_K(\phi) \tag{2.9}$$

for every $\phi \in \mathcal{D}^{n-p,n-q}(X)$ with $supp(\phi) \subset K$.

A current for which the positive integer l in (2.9) can be chosen independently of K is said to be a current of finite order. In this case, the smallest such integer l is called the *order* of T. Currents of compact support naturally exhibit finite order, and it is easy to see that the examples above have order 0.

2.2.2 Positive closed currents

In this part, we focus on positive closed currents due to [Lelong (1957)]. This notion is a powerful tool for studying complex manifolds and their singularities, particularly in the context of pluripotential theory and geometric measure theory. Additionally, they have applications in other fields, such as algebraic geometry and complex dynamical systems.

Motivated by the classical Stokes Theorem, we can extend the operation of the exterior derivative from differential forms to currents as well. Let T be a current of bidegree (p,q) on X. The exterior derivative of T is defined as

$$dT := \partial T + \overline{\partial} T,$$

where ∂T and $\overline{\partial} T$ are currents of bidegree $(p+1,q)$ and $(p,q+1)$, respectively, and are defined as follows

$$\langle \partial T, \phi \rangle := (-1)^{p+q+1} \langle T, \partial \phi \rangle,$$
$$\langle \overline{\partial} T, \phi \rangle := (-1)^{p+q+1} \langle T, \overline{\partial} \phi \rangle,$$

for any test form ϕ.

Note that $d(dT) = 0$, and thus $\partial(\partial T) = 0$, $\overline{\partial}(\overline{\partial} T) = 0$, and $\partial\overline{\partial} T = -\overline{\partial}\partial T$. Moreover, the maps $T \mapsto dT$, ∂T, and $\overline{\partial} T$ are continuous for the topology of currents defined above.

Definition 2.2.2. *A current T is said to be closed if $dT = 0$.*

Obviously, currents of maximal bidegree (distributions) are always closed.

Example 2.2.3. *If $[Z]$ is the current of integration defined in the example above, by Stokes' Theorem we have*

$$d[Z] = (-1)^{n-p+1}[\partial Z] = 0,$$

where ∂Z is the boundary of the complex submanifold Z.

Next, we define the differential operator $d^c := \frac{1}{2\pi i}(\partial - \overline{\partial})$. Then, $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ is a real operator, and the following relation, which we will consistently use, holds

$$\langle dd^c T, \phi \rangle = \langle T, dd^c \phi \rangle.$$

Additionally, we define the conjugate of a form and a current as

$$\overline{\phi} := \sum_{I,J} \overline{\phi_{I,J}} d\overline{z}_I \wedge dz_J, \quad \text{and} \quad \langle \overline{T}, \phi \rangle := \overline{\langle T, \overline{\phi} \rangle},$$

where $\phi = \sum_{I,J} \phi_{I,J} dz_I \wedge d\overline{z}_J$ is a form of suitable bidegree and T is a current. We say that T (resp. ϕ) is *real* if $T = \overline{T}$ (resp. $\phi = \overline{\phi}$). In particular, a real $(1,1)$-form ϕ locally is equivalent to $\phi = i\sum_{j,k=1}^{n} \phi_{jk}(z) dz_j \wedge d\overline{z}_k$, where $[\phi_{jk}]$ is a Hermitian $n \times n$ matrix.

Definition 2.2.3. *A (p,p)-form ϕ is said to be* positive *if at each point it is equal to a finite combination of forms $(i\alpha_1 \wedge \bar{\alpha_1}) \wedge \cdots \wedge (i\alpha_p \wedge \bar{\alpha_p})$ where α_i are $(1,0)$-forms which might depend on the point. The form ϕ is said to be* weakly positive *if $\phi \wedge \psi$ is positive for any positive $(n-p,n-p)$-form ψ. A (p,p)-current T is called positive (resp. weakly positive) if $\langle T, \phi \rangle \geq 0$ for every weakly positive (resp. positive) test $(n-p,n-p)$-form ϕ.*

The notions of positivity coincide for $p = 0, 1, n-1, n$. It is evident from the definition that positive forms and currents are real. Specifically, if ϕ is a real $(1,1)$-form, then locally $\phi = i\sum_{j,k=1}^{n} \phi_{jk}(z) dz_j \wedge d\bar{z}_k$. In this case, there is a characterization of positivity in terms of its coefficients; see [Demailly (2012), IV. Corollary 1.7], where ϕ is said to be positive if the Hermitian matrix $[\phi_{jk}(z)]$ is positive semidefinite for all z. Throughout this thesis, we further classify such forms as *positive* (resp. *semipositive*) if $[\phi_{jk}(z)]$ is positive definite (resp. semidefinite) for all z. Additionally, when T is a real $(1,1)$-current on X, we call it *strictly positive* if there exists a positive smooth $(1,1)$-form ω on X such that $T - \omega$ is a positive current on X.

In particular, positive currents of maximal bidegree (i.e., distributions) are positive measures. This is because any positive distribution can be extended to a positive linear functional on the space of complex-valued continuous functions. By the Riesz representation theorem, it is then represented as a positive measure. Now, if T is a positive (p,p)-current, then by standard duality arguments (see, e.g., [Demailly (2012), III. Proposition 1.14]), one can show that T is of order zero, meaning that the coefficients $T_{I,J}$ in ϕ are of order zero, i.e., complex measures, and satisfy $\overline{T_{I,J}} = T_{J,I}$. In this case, we define the *mass measure* of T by

$$\|T\| := \sum_{|I|=p, |J|=q} |T_{I,J}|, \tag{2.10}$$

where $|T_{I,J}|$ represents the total variation of the complex measures $T_{I,J}$. Note that $\|T\|$ depends on the coordinate charts, as the expression is locally defined.

Next we present a fundamental result due to Lelong, which demonstrates that we can generalize Example 2.2.1 and define currents of integration on analytic subsets of complex manifolds, providing an important class of positive closed currents.

Recall that a subset $Z \subset X$ is said to be an *analytic subset* of X if Z is closed, and for each point $p \in Z$, there exists a neighborhood U of p and holomorphic functions $f_1, \ldots, f_k$ on U such that

$$Z \cap U = \{f_1 = \cdots = f_k = 0\}.$$

In particular, if $k = 1$, that is, if $Z \cap U = \{f = 0\}$ for some $f \in \mathcal{O}(U)$, we refer to Z as an *analytic hypersurface* of X.

We say that Z has *pure dimension* $n-k$ if $\dim(Z \cap U) = n-k$ for every point $p \in Z$. A point $p \in Z$ is said to be a *regular point* if $Z \cap U$ is a manifold for a sufficiently small neighborhood U of p. The set of all such points of Z is denoted by Z_{reg}.

Theorem 2.2.1 (Lelong (1957)). *Let Z be a pure $(n-k)$-dimensional analytic subset of X. Define the (k,k)-current of integration $[Z]$ by*

$$\langle [Z], \phi \rangle := \int_{Z_{\text{reg}}} \phi, \quad \textit{for all } \phi \in \mathcal{D}^{n-k,n-k}(X).$$

Then $[Z]$ is a well-defined and positive closed current on X.

This theorem illustrates how the concept of an analytic subset in a complex manifold gives an important example of positive closed currents. Furthermore, it indicates that Z_{reg} has finite volume locally near singular points of Z.

2.2.3 Plurisubharmonic functions

Plurisubharmonic functions were introduced by Lelong and Oka in 1942 [Lelong (1942)]. They are a key subject in pluripotential theory and are closely linked to the theory of currents. For instance, positive closed $(1,1)$- currents can be locally analyzed using plurisubharmonic functions. In this part, we recall basic definitions and fundamental properties concerning the local theory of plurisubharmonic functions and their connections with positive $(1,1)$-currents.

Definition 2.2.4. *Let Ω be an open subset of $\mathbb{C}^n$. An upper semi-continuous function $\varphi : \Omega \to [-\infty, +\infty)$ is said to be plurisubharmonic (psh for short) if $\varphi \not\equiv -\infty$ and if for any complex line $L \subset \Omega$, the restriction $\varphi|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \partial\Omega)$, the function φ satisfies the mean value inequality*

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(a + e^{i\theta}\xi\right) d\theta. \tag{2.11}$$

The set of plurisubharmonic functions on Ω is denoted by PSH(Ω).

Observe that every psh function is also subharmonic, meaning it satisfies the mean value property on Euclidean balls or spheres, which follows by integrating (2.11) over $\xi \in S(0,r)$. Consequently, many results for subharmonic functions can be extended to the case of psh functions. As in the subharmonic case, smoothing a psh function u by convolution with a radial regularizing kernel $\rho(z_1, \dots, z_n) = \rho(|z_1|, \dots, |z_n|)$ yields a psh function (on a smaller domain). Thus, given $\varphi \in \text{PSH}(\Omega)$, we can find a decreasing sequence of smooth psh functions $\{\varphi \star \rho_\varepsilon\}$ on $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$, where $\rho_\varepsilon(z) = \varepsilon^{-2n}\rho(z/\varepsilon)$, such that $\lim_{\varepsilon \to 0} \varphi \star \rho_\varepsilon = \varphi$ in Ω. This allows one to verify properties for smooth psh functions and then pass to the limit.

Now we outline some of the properties of psh functions and for the proofs we refer to [Demailly (2012)].

Theorem 2.2.2. *Let Ω be an open subset of $\mathbb{C}^n$.*

(i) The set PSH(Ω) forms a convex cone, i.e., if $\varphi, \psi \in PSH(\Omega)$ and $\alpha, \beta \geq 0$, then $\alpha\varphi + \beta\psi \in PSH(\Omega)$.

(ii) If Ω is connected and $\varphi \in PSH(\Omega)$, then either $\varphi \equiv -\infty$ or $\varphi \in L^1_{loc}(\Omega)$.

(iii) If $\varphi_j \in PSH(\Omega)$ is a decresing sequence of psh functions such that $\varphi_k \to u$, then either $\varphi \in PSH(\Omega)$, or $\varphi = -\infty$.

(iv) Let $\{\varphi_\alpha\} \subset PSH(\Omega)$ be a locally uniformly bounded family and $\varphi = \sup_\alpha \varphi_\alpha$. Then its upper semicontinuous regularization

$$\varphi^*(z) = \limsup_{\zeta \to z} \varphi(\zeta)$$

is also a psh function in Ω and $\varphi^* = \varphi$ almost everywhere.

(v) Let $\varphi_1, \ldots, \varphi_k \in PSH(\Omega)$ and $\chi : \mathbb{R}^k \to \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_k)$ is non-decreasing in each variable t_j. Then $\chi(\varphi_1, \ldots, \varphi_k) \in PSH(\Omega)$. In particular, $\varphi_1 + \ldots + \varphi_k$, $\max\{\varphi_1, \ldots, \varphi_k\}$, and $\log(e^{\varphi_1} + \ldots + e^{\varphi_k})$ are psh functions on Ω.

(vi) Let Ω_1 and Ω_2 be open subsets of $\mathbb{C}^n$ and $\mathbb{C}^d$, respectively. If $\varphi \in PSH(\Omega_2)$, and $f : \Omega_1 \to \Omega_2$ is a holomorphic map, then the composition $\varphi \circ f \in PSH(\Omega_1)$.

Example 2.2.4. *Using the fact that $\log|z|$ is a subharmonic function on $\mathbb{C}$, we have that $\log|f| \in PSH(X)$ for every holomorphic function f on X. More generally, for any $f_j \in \mathcal{O}(X)$ and $\alpha_j \geq 0$ with $1 \leq j \leq k$, we have that*

$$\log\left(|f_1|^{\alpha_1} + \cdots + |f_k|^{\alpha_k}\right) \in PSH(X),$$

which is a simple consequence of Theorem 2.2.2(v) with $\varphi_j = \alpha_j \log|f_j|$.

Note that property (vi) implies that the notion of psh function makes sense on any complex manifold X, unlike subharmonic functions. A function u on X is said to be psh if it is psh on any holomorphic coordinate chart. It is also worth noting that there are no non-constant psh functions on compact complex manifolds (a simple consequence of maximum principle). This observation led to the introduction of the concept of *quasi-plurisubharmonic (quasi-psh)* functions on compact complex manifolds. A function u on X is called quasi-psh if locally it can be written as the sum of a psh function and a smooth function. This generalization allows for the flexibility needed to work with psh-like functions in the compact setting.

If $\varphi \in \mathrm{PSH}(\Omega) \cap \mathscr{C}^2(\Omega)$, by definition, the subharmonicity of the restrictions of u to complex lines implies that u is psh if and only if the complex Hessian $\left[\frac{\partial^2 \varphi(z)}{\partial z_j \partial \bar{z}_k}\right]$ is positive semi-definite on Ω, which means that $dd^c\varphi$ is a semi-positive form. Particularly, if $dd^c\varphi$ is a positive form, we say that φ is *strictly psh (spsh for short).*

For nonsmooth functions, a similar characterization of plurisubharmonicity can be attained through a regularization process. Indeed, if $\varphi \in \mathrm{PSH}(\Omega) \cap L^1_{loc}(\Omega)$, for each $\varepsilon > 0$, define $\varphi_\varepsilon = \varphi \star \rho_\epsilon$ where ρ_ε is the standard smoothing kernel, then $dd^c\varphi_\varepsilon \to dd^c\varphi$ as $\varepsilon \to 0$ in the sense of distributions. However, since φ_ε is a smooth psh function, $dd^c\varphi_\varepsilon$ is semi-positive. Hence, $dd^c\varphi$ is positive in the sense of currents, and it is closed since $d(dd^c\varphi) = 0$. As a result, if $\varphi \in \mathrm{PSH}(\Omega)$, then $dd^c\varphi$ defines a positive closed current. Conversely, if $\varphi \in L^1_{loc}(\Omega)$ such that $dd^c\varphi$ is a positive $(1,1)$-current, one can show that, there exists $\psi \in \mathrm{PSH}(\Omega)$ such that $\varphi = \psi$ almost everywhere (with respect to Lebesgue measure). We note that in this regularity, φ is called *spsh* if $dd^c\varphi$ is a strictly positive current.

More generally, if T is a positive closed $(1,1)$-current, then for any point $x_0 \in X$, there exists a neighborhood Ω of x_0 and $\varphi \in \mathrm{PSH}(\Omega)$ such that $T = dd^c\varphi$, for the proof see [Demailly (2012), III. Proposition 1.19]. The psh function φ with this property is called the *local potential* of the current T.

The following important result known as the *Poincaré-Lelong formula* demonstrates the connection between integration currents over analytic hypersurfaces and their potentials. For the proof, we refer to [Demailly (2012), III. Proposition 2.15].

Theorem 2.2.3 (Poincaré-Lelong formula)**.** *Let f be a holomorphic function on X which does not vanish on any connected component of X . Then* $\log|f|$ *is a psh function, and it satisfies*

$$dd^c \log|f| = \sum_j m_j [Z_j],$$

where Z_j denotes the irreducible components of $f^{-1}(0)$, and m_j represents their respective multiplicities.

Next we introduce the concept of pluripolar sets, which will be useful in our later analysis.

Definition 2.2.5. *A subset E of X is said to be pluripolar if for every point $x \in X$ there exists a neighborhood Ω of x and $\varphi \in PSH(\Omega)$ such that $E \cap \Omega \subset \{\varphi = -\infty\}$.*

Theorem 2.2.2 (ii) implies that pluripolar sets have zero Lebesgue measure.

Example 2.2.5. *Any proper analytic subset of X is pluripolar, and the Hausdorff dimension of a pluripolar set cannot exceed $2n-2$.*

2.2.4 Monge-Ampère operators and Intersection of currents

In this part, we present the intersection theory for analytic cycles from the current point of view. Specifically, we define the wedge product $dd^c\varphi \wedge T$ of a positive closed current T and a "generalized" divisor $dd^c\varphi$ where φ is a psh function on X. In this generality, this is not possible since $dd^c\varphi$ and T have measure coefficients and measures cannot be multiplied. However, if we assume that φ is a locally bounded psh function, then the current φT is well-defined since φ is a locally bounded Borel function and T has measure coefficients. According to [Bedford & Taylor (1982)], we can then proceed to define

$$dd^c\varphi \wedge T = dd^c(\varphi T).$$

An easy consequence of approximation of φ by regularizing kernels implies that $dd^c u \wedge T$ is actually a positive closed current. More generally, when given locally bounded psh functions $\varphi_1, \ldots, \varphi_q$, we define inductively

$$dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_q \wedge T = dd^c(\varphi_1 dd^c\varphi_2 \wedge \cdots \wedge dd^c\varphi_q \wedge T)$$

which is a positive-closed current as well. In particular, when u is a locally bounded psh function, the (n,n)-current $(dd^c\varphi)^n$ is a well-defined positive measure, and the operator $(dd^c)^n$ is called *Monge-Ampère operator.* It is important to highlight that the mapping

$$(\varphi_1, \ldots, \varphi_n) \mapsto dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_n$$

is also commonly referred to as the Monge-Ampère operator.

Next, we introduce the Monge-Ampère operators for unbounded psh functions. This will be of fundamental importance, since we will deal with the products of integration currents along analytic subsets.

Let φ be a psh function on X we define the *unbounded locus* $L(\varphi)$ to be the set of points $x \in X$ such that φ is unbounded in any neighborhood of x. Clearly, $L(\varphi)$ is a closed subset containing the $-\infty$ locus of φ. Now if, $\varphi_1, \ldots, \varphi_q$ are unbounded psh functions, the following result shows that we can define Monge-Ampère operators as long as the intersections of unbounded loci have sufficiently small Hausdorff dimensions with respect to the dimension $n-p$ of T.

Theorem 2.2.4 (Demailly (1993)). *Let T be a (p,p)- current and $\varphi_1, \ldots, \varphi_q$ psh functions on X, such that $q \leq n-p$. If*

$$\mathcal{H}_{2n-2p-2k+1}(L(\varphi_{j_1}) \cap \cdots \cap L(\varphi_{j_k}) \cap \mathrm{supp} T) = 0$$

for all indices $j_1 < \cdots < j_k$ in $\{1, \ldots, q\}$. Then currents $\varphi_1 dd^c \varphi_2 \wedge \cdots \wedge dd^c \varphi_q \wedge T$ and $dd^c \varphi_1 \wedge \cdots \wedge dd^c \varphi_q \wedge T$ are well-defined and have locally finite mass in X.

The proof uses induction on bidegrees, along with an improved version of Chern-Levine-Nirenberg inequalities in this context. For more details, see [Demailly (1993), Theorem 2.5]

Definition 2.2.6. *The analytic subsets $Z_1, \ldots, Z_q$ of X are said to be in general position if* $\mathrm{codim}(Z_{j_1} \cap \cdots \cap Z_{j_k}) \geq k$ *for all indices $j_1 < \cdots < j_k$ in $\{1, \ldots, q\}$.*

If, in particular, T is of bidegree $(0,0)$, and q is arbitrary, the following result holds.

Corollary 2.2.5 (Demailly (1993)). *Let $\varphi_1, \ldots, \varphi_q$ be psh functions on X such that $L(\varphi_j)$ is contained in an analytic subset $Z_j \subset X$ for every j. Then $dd^c \varphi_1 \wedge \ldots \wedge dd^c \varphi_q$ is well-defined provided that $Z_1, \cdots, Z_q$ are in general position.*

When $\varphi_j = \log|f_j|$ for some non-zero holomorphic function f_j on X. Then, $[Z_1] \wedge \cdots \wedge [Z_q] = dd^c \varphi_1 \wedge \cdots \wedge dd^c \varphi_q$ is well-defined provided that the intersection of supports $Z_1 \cap \cdots \cap Z_q$ has pure dimension $n-k$, i.e., $\mathrm{codim}(Z_{j_1} \cap \cdots \cap Z_{j_k}) = k$ for every $j_1 < \cdots < j_k$ in $\{1, \ldots, q\}$. Consequently, we arrive at the following result. For the detailed discussion of the proof we refer to [Demailly (1993), Proposition 2.12].

Proposition 2.2.6 (Demailly (1993)). *Suppose the divisors Z_j satisfy the aforementioned codimension condition. Let $(C_k)_{k \geq 1}$ represent the irreducible components of the intersection of point sets $|Z_1| \cap \cdots \cap |Z_q|$. Then, there exist positive integers m_k such that*

$$[Z_1] \wedge \cdots \wedge [Z_q] = \sum_k m_k [C_k].$$

The term m_k denotes the intersection multiplicity of $Z_1, \ldots, Z_q$ along C_k.

2.3 Kähler Manifolds

In this section, we introduce the definition of Kähler manifolds and provide an overview of some of their key properties, with a particular focus on compact Kähler manifolds.

Let X be a complex manifold of dimension n with complex structure J. We will be interested in Riemannian metrics on X which are compatible with the complex structure in a particularly nice way. Recall that a Riemannian metric g on X is a smooth section of $T^*X \otimes T^*X$ defining a positive definite symmetric bilinear form on T_xX for each $x \in X$.

Definition 2.3.1. *A Hermitian metric on X is a Riemannian metric g on X such that $g(J\xi, J\eta) = g(\xi,\eta)$ for any tangent vectors ξ,η, i.e., J is an orthogonal transformation on each tangent space.*

Any complex manifold admits a Hermitian metric. To elaborate, consider any Riemannian metric g on the manifold X and define

$$h(\xi,\eta) := g(\xi,\eta) + g(J\xi, J\eta)$$

for every $\xi,\eta \in TX$. Then $h(\xi,\eta) = h(J\xi, J\eta)$. Given a Hermitian metric g on X we define its *fundamental form* as

$$\omega(\xi,\eta) := g(J\xi,\eta)$$

for every $\xi,\eta \in TX$. Then one can check that ω defines a real 2-form of type $(1,1)$, i.e., $\omega \in \Omega^{1,1}(X) \cap \Omega^2(X)$. Conversely, we can also retrieve the metric g from ω using the expression

$$g(\xi,\eta) = \omega(\xi, J\eta).$$

Definition 2.3.2. *A Hermitian metric g on a complex manifold X is called a Kähler metric if the associated fundamental form ω is closed, i.e., $d\omega = 0$, and the form ω is called the Kähler form. A Kähler manifold is a complex manifold endowed with a Kähler metric g.*

Remark 2.3.1. *It is a standard notational convention to identify the Kähler metric g with its associated Kähler form ω.*

In a local coordinate system $(z_1,\ldots,z_n)$, a Hermitian metric is determined by components g_{jk} where

$$g_{jk} = g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}),$$

and g is the extended Hermitian metric on the tangent bundle $TX \otimes \mathbb{C}$ by $\mathbb{C}-$bilinearity. The Hermitian condition implies that for any j,k we have

$$g(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}) = g(\frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial \bar{z}_k}) = 0.$$

Therefore, we can write

$$g = \sum_{j,k=1}^{n} g_{jk}(dz_j \otimes d\bar{z}_k + d\bar{z}_k \otimes dz_j).$$

The symmetry property of g implies $\overline{g_{jk}} = g_{kj}$, and the positivity of g ensures that g_{jk} forms a positive definite Hermitian matrix at each point. The corresponding fundamental form ω can be expressed as

$$\omega = i\sum_{j,k} g_{jk} dz_j \wedge d\bar{z}_k,$$

In turn, the Kähler condition implies that g is Kähler if

$$\frac{\partial g_{jk}}{\partial z_i} = \frac{\partial g_{ik}}{\partial z_j}, \text{ for all } i,j,k.$$

Moreover, the Kähler condition gives rise to additional fundamental outcomes in Kähler geometry, which are recognized as the $\partial\bar{\partial}$-lemmas.

Lemma 2.3.1 (Local $\partial\overline{\partial}$-lemma)**.** *Let X be a Kähler manifold with Kähler form ω. Then there exists an open neighborhood U such that*

$$\omega = i\partial\overline{\partial}\varphi \tag{2.12}$$

for some $\varphi \in \mathscr{C}^\infty(U,\mathbb{R})$ strictly plurisubharmonic function.

The local real-valued smooth function φ is called the *Kähler potential.* If the manifold X is compact, then we have the global version of $\partial\overline{\partial}$-lemma.

Lemma 2.3.2 (Global $\partial\overline{\partial}$-lemma)**.** *Let X be a compact Kähler manifold. If ω and ω' are two real $(1,1)$- forms in the same cohomology class, then there is a function $\varphi : X \to \mathbb{R}$ such that*

$$\omega' = \omega + i\partial\overline{\partial}\varphi.$$

As the Kähler form ω is a closed real form, it defines a cohomology class

$[\omega] \in H^2(X,\mathbb{R})$. The global $\partial\overline{\partial}$ lemma, implies that on a compact manifold, Kähler metrics in a fixed cohomology class can be parameterized using real-valued functions. The proof of this lemma relies on the Hodge theory of compact Kähler manifolds; for details, see [Huybrechts (2005), Corollary 3.2.10].

Example 2.3.1. *$\mathbb{C}^n$ with the standard Kähler form*

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = i\sum_{j=1}^{n} dx_j \wedge dy_j,$$

is a Kähler manifold with the global Kähler potential $\varphi = \frac{1}{2}\sum_{j=1}^{n}|z_j|^2$.

Example 2.3.2. *The complex projective space $\mathbb{P}^n$ is a Kähler manifold. Let $\mathbb{P}^n = \bigcup_{j=0}^{n} U_j$ be the standard open cover where $U_j = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n : z_j \neq 0\}$ with the charts $\phi_j : U_j \to \mathbb{C}^n$, given by*

$$\phi_j([z_0 : \ldots : z_n]) = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right).$$

On each U_j, define the function

$$\varphi_j = \log\left(1 + \sum_{l=1}^{n} |\xi_l|^2\right),$$

where $\xi_l = \frac{z_l}{z_j}$ for $l \neq j$, and the form

$$\omega_j := \frac{i}{\pi}\partial\overline{\partial}\varphi_j,$$

which is a real, closed $(1,1)$-form. We now show that ω_j defines a global element $\omega \in \Omega^{1,1}(\mathbb{P}^n) \cap \Omega^2(\mathbb{P}^n)$, meaning that $\omega_j|_{U_j \cap U_k} = \omega_k|_{U_j \cap U_k}$. On $U_j \cap U_k$, we have

$$\varphi_j = \log\left(\left|\frac{z_k}{z_j}\right|^2 \sum_{l=0}^{n}\left|\frac{z_l}{z_k}\right|^2\right) = \log\left(\left|\frac{z_k}{z_j}\right|^2\right) + \varphi_k.$$

Since $\partial\overline{\partial}\log|\xi|^2 = \partial\left(\frac{\xi d\overline{\xi}}{|\xi|^2}\right) = \partial\left(\frac{d\overline{\xi}}{\overline{\xi}}\right) = 0$ on $\mathbb{C}$, we have $\partial\overline{\partial}\left(\left|\frac{z_k}{z_j}\right|^2\right) = 0$, which implies that ω_j is globally well-defined.

Now, a straightforward computation yields

$$\omega_j = \frac{i}{\pi}\partial\overline{\partial}\varphi_j = \frac{i}{\pi}\sum_{k,l=1}^{n} g_{kl}\, d\xi_k \wedge d\overline{\xi}_l,$$

where

$$G = (g_{kl})_{kl} = \left(\frac{(1+\|\xi\|^2)\delta_{kl} - \overline{\xi}_k \xi_l}{(1+\|\xi\|^2)^2} \right)_{1 \leq k,l \leq n}.$$

The matrix G is positive definite because, for any $v \neq 0 \in \mathbb{C}^n$, the Cauchy-Schwarz inequality for the standard Hermitian product on $\mathbb{C}^n$ yields

$$\langle Gv, v \rangle = \frac{1}{(1+\|\xi\|^2)^2} \left(\|v\|^2 + \|\xi\|^2 \|v\|^2 - |\langle \xi, v \rangle|^2 \right) \geq \frac{\|v\|^2}{(1+\|\xi\|^2)^2} > 0.$$

As a result, ω is a Kähler form, referred to as the Fubini-Study *form, denoted by ω_{FS}, and the functions φ_j are its local Kähler potentials.*

The next result demonstrates that the above examples are not a coincidence, that is, in the presence of a Kähler metric, one can choose holomorphic coordinates near any point with particularly favorable properties. Specifically, it indicates that locally, a Kähler metric approximates the Euclidean metric up to second order. This fact will prove to be highly beneficial for computational purposes.

Theorem 2.3.3. *Let X be a complex manifold and g a Kähler metric on X. Then, given $x \in X$ there exists holomorphic coordinates $(z_1, \ldots, z_n)$ around x such that*

$$\omega = i \sum_{j,k} \left(\delta_{jk} + O(|z|^2) \right) dz_j \wedge d\overline{z}_k,$$

where δ_{jk} is the identity matrix and $O(|z|^2)$ denotes terms which are at least quadratic in $z^i, \overline{z}^i$.

Such coordinates are known as *normal coordinates.* We will use a slightly modified version of these coordinates in Chapter 4.

2.4 Holomorphic Line Bundles

In this section, we present a comprehensive overview of the theory of holomorphic line bundles. We delve into their definition, construction via trivializations, and their interplay with algebraic operations like tensor products and duality. Additionally, we discuss hermitian metrics on line bundles, their role in defining positive and ample line bundles, and their connection to curvature and Chern classes.

2.4.1 Holomorphic line bundles and their curvature

Let X be a complex manifold of dimension n. A holomorphic line bundle on X consists of a family $\{L_x\}_{x\in X}$ of one dimensional complex vector spaces parametrized by X, together with a structure of a complex manifold on $L = \cup_{x\in X} L_x$ such that

(i) The projection map $\pi : L \to X$ taking L_x to x is holomorphic.

(ii) There exists an open cover $\{U_\alpha\}$ of X and biholomorphisms

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$$

taking the vector space $L_x = \pi^{-1}(x)$ isomorphically onto $\{x\} \times \mathbb{C}$ for each $x \in U_\alpha$. The map ϕ_α is called a *trivialization* of L over U_α. We define the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ for L relative to the trivializations ϕ_α by

$$g_{\alpha\beta}(x) = (\phi_\alpha \circ \phi_\beta^{-1}) \restriction_{\{x\}\times\mathbb{C}} \in \mathbb{C}^*$$

Clearly, the functions $g_{\alpha\beta}$ are non-vanishing holomorphic functions and satisfy the cocycle condition

$$\begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \end{cases} \tag{2.13}$$

Conversely, given a collection of holomorphic functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ satisfying the cocycle conditions. We can construct a unique (i.e. up to isomorphism) holomorphic line bundle $L \to X$ with transition functions $\{g_{\alpha\beta}\}$ basically by defining

$$L := \Big(\bigsqcup_\alpha U_\alpha \times \mathbb{C}\Big) / \sim$$

where $(x, v) \sim (x, g_{\alpha\beta}(x)v)$ whenever $x \in U_\alpha \cap U_\beta$. Any linear algebraic operation on fibers induces operation on line bundles and this is easily described by the transition functions, e.g. if L and L' are line bundles on X with transition functions $g^L_{\alpha\beta}$ and $g^{L'}_{\alpha\beta}$, respectively. Then $L \otimes L' \to X$ is a line bundle with transition functions $g^{L\otimes L'}_{\alpha\beta} = g^L_{\alpha\beta} \cdot g^{L'}_{\alpha\beta}$. Similarly, the dual of L is the line bundle $L^* \to X$ with transition functions $g^{L^*}_{\alpha\beta} = (g^L_{\alpha\beta})^{-1}$ etc.

Example 2.4.1 (Line bundles on $\mathbb{P}^n$)**.** *Consider the set*

$$\mathcal{O}(-1) := \{([z], \xi) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : \xi \in [z]\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$$

and the projection map $\pi : \mathcal{O}(-1) \to \mathbb{P}^n$ *defined by* $\pi([z],\xi) = [z]$. *Then* $\mathcal{O}(-1)$ *is a holomorphic line bundle on* $\mathbb{P}^n$, *called* tautological line bundle. *Indeed, if* $\{U_j\}_{j=0}^n$ *is an open cover of* $\mathbb{P}^n$, *where* $U_j = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n : z_j \neq 0\}$. *Define the trivializations* $\phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}$ *by* $\phi_j([z],\xi) := \xi_j$, *then*

$$\phi_i \circ \phi_j^{-1}([z],1) = \phi_i([z],(z_0/z_j,\ldots,\underbrace{1}_{j\text{-}th},\ldots,z_n/z_j)) = ([z],z_i/z_j).$$

Hence, transition functions

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^* \text{ are given by } g_{ij}([z]) = \frac{z_i}{z_j} \in \mathcal{O}^*(U_i \cap U_j).$$

The dual of $\mathcal{O}(-1)$ *is called the* hyperplane line bundle *denoted by* $\mathcal{O}(1)$ *and for* $p > 0$ *we define* $\mathcal{O}(p) := \mathcal{O}(1)^{\otimes p}$ *whose transition functions are simply obtained by inversion and multiplication of transition functions of the tautological line bundle* $\mathcal{O}(-1)$, *that is* $g_{ij}^p([z]) = \left(\frac{z_j}{z_i}\right)^p$.

The cocycle condition for the transition functions yields that they define a cohomology class, denoted as $[g_{\alpha\beta}] \in H^1(X,\mathcal{O}^*)$. Here, $H^1(X,\mathcal{O}^*)$ is the first sheaf cohomology group of the manifold X with coefficients in the sheaf of non-zero holomorphic functions, denoted by $\mathcal{O}^*$. The exponential short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

produces a mapping $c_1 : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z})$, and the *first Chern class* $c_1(L)$ is defined by the image of $[g_{\alpha\beta}]$ under this mapping.

A holomorphic section of $L \to X$ over $U_\alpha \subset X$ is a holomorphic map $s : U_\alpha \to L$ such that $s(x) \in L_x$ i.e., $\pi \circ s = id_{U_\alpha}$. A holomorphic frame for L over U_α is a non-zero holomorphic section of L over U_α, and we denote it by e_α. In turn, any section s of L over U_α can be written as $s = s_\alpha e_\alpha$ where $s_\alpha \in \mathcal{O}(U_\alpha)$ and satisfies the compatibility condition $s_\alpha = g_{\alpha\beta} e_\beta$ on $U_\alpha \cap U_\beta$. We denote by $H^0(X,L)$ the space of global holomorphic sections of L. By the observation above, we can think of a global section $s \in H^0(X,L)$ as a collection of holomorphic functions $s = \{s_\alpha\}$ satisfying the compatibility condition on the overlap.

Example 2.4.2. *For* $p > 0$, $H^0(\mathcal{O}(p),\mathbb{P}^n) \cong \mathbb{C}_p^{hom}[z_0,\ldots,z_n]$. *For a detailed proof see [Huybrechts (2005), Proposition 2.4.1]*

A Hermitian metric h on a holomorphic line bundle L is a choice of the Hermitian inner product $h_x : L_x \times L_x \to \mathbb{C} \cup \{\infty\}$ on each fiber L_x varying smoothly with $x \in X$.

If e_α is a holomorphic frame over U_α, then we define its hermitian norm

$$|e_\alpha|_h^2 := h(e_\alpha, e_\alpha) = e^{-2\varphi_\alpha} \in [0, \infty], \tag{2.14}$$

where $\varphi_\alpha \in \mathscr{C}^\infty(U_\alpha)$. The local functions φ_α are called local weights of the hermitian metric h and satisfy the relation

$$\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|, \ \text{ on } U_\alpha \cap U_\beta. \tag{2.15}$$

Consequently, a hermitian metric is a collection $h = \{e^{-\varphi_\alpha}\}$ of functions $\varphi_\alpha \in \mathscr{C}^\infty(U_\alpha)$ satisfying the above compatibility condition (2.15).

We say that the metric h is of class $\mathscr{C}^k$ if the local weight functions $\varphi_\alpha \in \mathscr{C}^k(U_\alpha)$. Moreover, h is said to be *positive, semi-positive metric of class $\mathscr{C}^k$* if φ_α is strictly plurisubharmonic and plurisubharmonic, respectively. The metrics that we will consider in this thesis will be mainly positive of class $\mathscr{C}^k$ where $k = 2, 3$, (i.e. $\varphi_\alpha \in \mathscr{C}^k$, and $dd^c\varphi_\alpha > 0$ holds pointwise). Note that if s is an arbitrary holomorphic section of L over U_α then locally $s = s_\alpha e_\alpha$ with $s_\alpha \in \mathcal{O}(U_\alpha)$ and $|s|_h^2 = |s_\alpha|^2 |e_\alpha|_h^2$ almost everywhere (unless $s(x) = 0$ and $|e_\alpha|_h(x) = \infty$).

A holomorphic line bundle (L, h) equipped with a $\mathscr{C}^k$ metric h is called *positive* if h is a positive metric of class $\mathscr{C}^k$. Note that unless we specify the regularity, the term positive holomorphic line bundle implies that we are considering a positive smooth Hermitian metric.

Given such a metric h, its *curvature form* defined by

$$c_1(L, h) := -dd^c \log|e_\alpha|_h = dd^c\varphi_\alpha \text{ in } U_\alpha$$

is a globally well-defined real, closed $(1,1)$ form due to the relation $dd^c \log|g_{\alpha\beta}| = 0$ on $U_\alpha \cap U_\beta$. By de Rham's isomorphism theorem, it represents $c_1(L)$ where $c_1(L)$ is the image of the first Chern class of L under the mapping $i : H^2(X, \mathbb{Z}) \to H^{1,1}(X, \mathbb{R})$ induced by the inclusion $i : \mathbb{Z} \to \mathbb{R}$.

Definition 2.4.1 (Canonical line bundle)**.** *The top exterior power of the holomorphic cotangent bundle $(T^*X)^{(1,0)}$ is called the* canonical line bundle *of X and is denoted by K_X, i.e.,*

$$K_X := \bigwedge^n (T^*X)^{(1,0)} \equiv \det((T^*X)^{(1,0)}).$$

Its dual is called the anti-canonical line bundle *and is denoted by K_X^*.*

A local holomorphic frame for K_X on a coordinate neighborhood (U, z) is given by $dz_1 \wedge \cdots \wedge dz_n$ and the transition functions are given by jacobians of coordinate changes, that is if $\{(U_\alpha, z^\alpha)\}_\alpha$ are holomorphic coordinates then on $U_\alpha \cap U_\beta \neq \emptyset$

$$dz_1^\alpha \wedge \cdots \wedge dz_n^\alpha = \det\left[\frac{\partial z_j^\alpha}{\partial z_k^\beta}\right] dz_1^\beta \wedge \cdots \wedge dz_n^\beta$$

If g is a Hermitian metric on X, then it induces a natural hermitian metric on K_X by $h^{K_X} = (\det g)^{-1}$

Definition 2.4.2. *The* first Chern class *of X is defined by*

$$c_1(X) := c_1(K_X^*, h^{K_X^*}) = -c_1(K_X, h^{K_X}).$$

2.4.2 Poincaré Lelong Formula for Holomorphic Sections

Our goal here is to generalize the Poincaré-Lelong formula from Section 2.2.2 to holomorphic sections of holomorphic line bundles. Let L be a holomorphic line bundle, and $s \in H^0(X, L)$ a non-trivial holomorphic section of L. Recall that *zero divisor* of s is defined as the formal sum

$$Z_s = \sum_V \mathrm{ord}_V(s) \cdot V,$$

where the sum ranges over all irreducible analytic hypersurfaces within the zero set of s, and $\mathrm{ord}_V(s) \in \mathbb{N} \setminus \{0\}$ represents the vanishing order of s along V. The current of integration over the zero divisor of s is given by

$$[Z_s] = \sum_V \mathrm{ord}_V(s)[V],$$

where $[V]$ denotes the current of integration over V.

In a trivializing neighborhood U_α, we express $s = s_\alpha e_\alpha$, where $s_\alpha \in \mathcal{O}(U_\alpha)$. Utilizing the Poincaré-Lelong formula locally, given by Theorem 2.2.3, we have

$$[Z_s] = dd^c \log |s_\alpha|.$$

Now, employing the fact that $|s|_h = |s_\alpha| e^{-\varphi_\alpha}$ and $c_1(L, h) = dd^c \varphi_\alpha$ on U_α, along with compatibility conditions, we deduce

$$[Z_s] = c_1(L,h) + dd^c \log|s|_h, \tag{2.16}$$

which represents the Poincaré-Lelong formula for holomorphic sections of line bundles. This formula will serve as an indispensable tool in the subsequent analysis.

Remark 2.4.1. $[Z_s]$ *is a positive-closed* $(1,1)$ *current with local psh potentials of the form* $\log|s_\alpha|$.

In this framework, we treat the concepts of zero set and zero divisor interchangeably, denoting both by the same symbol for simplicity. Additionally, by $[Z_s]$, we indicate the current of integration along the zero divisor Z_s or equivalently along the zero set of s with multiplicities. Similarly, in higher codimensions, if $\Sigma = (s_1, \dots, s_k) \in H^0(X,L)^k$, Z_Σ is used to represent the zero cycle of Σ and the zero set of Σ interchangeably. Furthermore, $[Z_\Sigma]$ denotes the current of integration over the zero cycle Z_Σ or equivalently over the analytic subvariety Z_Σ, considering multiplicities.

2.4.3 Projectivity Criterion

In this part, we introduce the classical Kodaira embedding theorem, which serves as a criterion for determining whether a compact Kähler manifold is projective. Additionally, we present a theorem by Grauert, which also addresses the projectivity of a compact Kähler manifold under different sufficient conditions.

Let X be a compact Kähler manifold and L be a holomorphic line bundle over X. A point $x \in X$ is said to be a *base point* of L if $s(x) = 0$ for all $s \in H^0(X,L)$. The *base locus* of L, denoted as $\mathrm{Bs}(L)$ (sometimes also denoted as $\mathrm{Bs}(H^0(X,L))$), is the set of all such base points of L. According to Hodge Theory ([Huybrechts (2005), Theorem 4.1.3]), it is known that as X is compact $H^0(X,L)$ forms a finite-dimensional vector space and say $d = \dim H^0(X,L)$. If $S_1, \dots, S_d$ constitute a basis for $H^0(X,L)$, then

$$\mathrm{Bs}(L) = S_1^{-1}(0) \cap \cdots \cap S_d^{-1}(0)$$

is an analytic subvariety.

Definition 2.4.3. *The Kodaira map associated with* L *is defined as*

$$\Phi : X \to \mathbb{P}(H^0(X,L)^\star)$$

$$\Phi(x) := \{s \in H^0(X,L) : s(x) = 0\}.$$

Fix a basis $S_1,\dots,S_d$ for $H^0(X,L)$ we can then identify $H^0(X,L) \cong \mathbb{C}^d$ and hence $\mathbb{P}(H^0(X,L)^\star) \cong \mathbb{P}^{d-1}$. Locally, when we trivialize L over an open cover $\{U_\alpha\}$ of X using the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, the Kodaira map over the open set U_α is given by

$$\Phi : X \dashrightarrow \mathbb{P}^{d-1},\ \Phi(x) = [S_{1,\alpha}(x) : \dots : S_{d,\alpha}(x)].$$

Here, $S_{j,\alpha} \in \mathcal{O}(U_\alpha)$ are local holomorphic functions representing S_j over U_α. Note that the map is well-defined, i.e., independent of trivializations, since for $x \in U_\alpha \cap U_\beta$ as $S_{j,\alpha} = g_{\alpha\beta} S_{j,\beta}$, we have $[S_{1,\alpha}(x) : \dots : S_{d,\alpha}(x)] = [g_{\alpha\beta}(x)S_{1,\alpha}(x) : \dots : g_{\alpha\beta}(x)S_{d,\alpha}(x)]$. Moreover, this local description shows that Φ is meromorphic on X and holomorphic on $X \setminus \mathrm{Bs}(L)$. Note that since $\mathrm{Bs}(L)$ is a closed subset of X, then $X \setminus \mathrm{Bs}(L)$ is an open submanifold of X, so the notion of holomorphic map is meaningful.

Remark 2.4.2. *Evidently, this map does depend on the choice of basis. However, for two different basis choices, the induced maps differ by a linear transformation of $\mathbb{P}^{d-1}$.*

Definition 2.4.4. *A holomorphic line bundle L is called ample if there exists a positive integer p such that the Kodaira map associated with $L^{\otimes p}$ is an embedding.*

By definition, a compact Kähler manifold is projective if and only if it possesses an ample line bundle. The characterization of ampleness in this context is given by the following well-known theorem due to Kuhiniko Kodaira.

Theorem 2.4.1 (Kodaira Embedding Theorem)**.** *A holomorphic line bundle L over a compact Kähler manifold X is ample if and only if it is positive.*

Next, we recall an important criterion for projectivity due to Grauert

Theorem 2.4.2 (Grauert (1962))**.** *Let (X,ω) be a compact Kähler manifold equipped with a $\mathscr{C}^2$- Hermitian holomorphic line bundle (L,h) such that $c_1(L,h) \geq \epsilon\omega$ for some positive constant ϵ. Then the line bundle L is ample and X is projective.*

We note that this theorem was originally given in a more general setting, where X is a reduced Hermitian space. However, this version is sufficient for our needs.

Therefore, in this thesis, the line bundles considered will always be ample, as assured by Grauert's projectivity criterion mentioned above.

2.4.4 Ricci Curvature

In this part, we introduce the Ricci curvature of compact Kähler manifolds, which will be utilized in Chapter 4 for L^2-estimations of the $\bar{\partial}$- operator.

Let (X,ω) be a compact Kähler manifold with natural complex structure J. It is well-known that the Kähler form ω and the complex structure J, compatible with ω, determine a Riemannian metric g on X by $g(\xi,\eta) := \omega(\xi, J\eta)$ for all $\xi,\eta \in TX$. Let *Ric* be the Ricci curvature of g which is defined as $Ric(\eta,\zeta) := tr(\xi \to R(\xi,\eta)\zeta)$ for all $\xi,\eta,\zeta \in TX$, where $R(\xi,\eta)\zeta := \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta$ is the Riemannian curvature of X and ∇ represents the Levi-Civita connection. The *Ricci form*, denoted by Ric_ω is defined on X as the $(1,1)$-form associated with *Ric*, given by

$$\mathrm{Ric}_\omega(\xi,\eta) = \mathrm{Ric}(J\xi,\eta), \text{ for any } \xi,\eta \in TX.$$

The volume form ω^n induces a metric, denoted by $h^{K_X^*}$, on the anti-canonical line bundle K_X^*. By the result from [Ma & Marinescu (2007), Problem 1.7], since the metric is Kähler, we have

$$\mathrm{Ric}_\omega = iR^{K_X^*} = -iR^{K_X}$$

where $R^{K_X^*}$ (resp. R^{K_X}) is the curvature of the holomorphic Hermitian connection on K_X^*(resp. K_X). For more details on connections see [Ma & Marinescu (2007)]. Let $[\mathrm{Ric}_\omega]$ be the cohomology class of Ric_ω, then we have

$$[\mathrm{Ric}_\omega] = 2\pi c_1(X) \in H^2(X,\mathbb{R}).$$

Locally, let $\omega = i\sum_{j,k} g_{jk} dz_j \wedge d\bar{z}_k$ be the Kähler form. Then *Ricci form of* ω can be written as

$$\mathrm{Ric}_\omega = -i\sum_{j,k} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(\det(g_{jk})) dz_j \wedge d\bar{z}_k. \tag{2.17}$$

Example 2.4.3. *Consider the complex projective space $\mathbb{P}^n$ with the Fubini-Study metric ω_{FS} defined as in the Example 2.3.2 with*

$$g_{jk} = \frac{\delta_{jk}}{(1+\|z\|^2)} - \frac{\bar{z}_j z_k}{(1+\|z\|^2)^2}.$$

A straightforward computation shows that

$$\log(\det(g_{jk})) = \log \frac{1}{(1+\|z\|^2)^{n+1}} = -(n+1)\log(1+\|z\|^2)$$

and hence its Ricci form is given by

$$\mathrm{Ric}_{\omega_{FS}} = (n+1)\,\omega_{FS}.$$

3. Equidistribution of zeros of random systems of holomorphic sections

In this chapter, we establish an equidistribution phenomenon related to the simultaneous zeros of random holomorphic sections arising from sequences of positive holomorphic line bundles. Our investigation include a diverse range of probability measures, including classical ones. The analytical tools utilized include variance estimation and the study of the expected distribution of random zeros. Additionally, to illustrate the results, we present explicit examples such as the Gaussian and Fubini-Study measures, among many others.

3.1 Geometric Framework and Randomization

Let $(L_p,h_p)_{p\geq 1}$ be a sequence of holomorphic line bundles on a compact Kähler manifold (X,ω) of $\dim_{\mathbb{C}} X=n$ with a fixed Kähler form ω and $\mathscr{C}^2$ Hermitian metrics h_p (see Section 2.4) such that the curvature forms $c_1(L_p,h_p)$ satisfy the following so called diophantine approximation of ω or sometimes we refer as "prequantization condition":

$$\frac{1}{A_p}c_1(L_p,h_p)=\omega+O(A_p^{-a}) \text{ in the } \mathscr{C}^0\text{-topology as } p\to\infty, \tag{3.1}$$

where $a>0$, $A_p>0$ and $\lim_{p\to\infty}A_p=\infty$. This means that

$$\left\|\frac{1}{A_p}c_1(L_p,h_p)-\omega\right\|_{\mathscr{C}^0}=O(A_p^{-a}), \tag{3.2}$$

where $A_p>0,\ a>0$ and $\lim_{p\to\infty}A_p=+\infty$.

The approximation condition (3.2) is derived naturally as follows: Starting with a Kähler form ω, one may initially approximate the associated cohomology class

$[\omega] \in H^2(X,\mathbb{R})$ with integral classes in $H^2(X,\mathbb{Z})$ through diophantine approximation, as described by Kronecker's lemma. Subsequently, one can construct smooth forms corresponding to these integral approximations.

This was first considered in [Coman et al. (2023)] in the $\mathscr{C}^\infty$-norm topology induced by the Levi-Civita connection ∇^{TX} because the authors deal with the complete asymptotic expansion of the Bergman kernel restricted to the diagonal. We do not need such a strong topology, in fact, only the $\mathscr{C}^0$-norm (or continuous norm) topology will be sufficient for us.

It is also important to highlight that, within the examples of sequences of line bundles (L_p, h_p) fulfilling the condition (3.2), one natural instance is $(L_p, h_p) = (L^{\otimes p}, h^{\otimes p})$ for some fixed prequantum line bundle (L, h). Other examples include cases where $(L_p, h_p) = (L^{\otimes p}, h_p)$ but here, h_p is not necessarily the product metric h^p, e.g., one can consider the twisted metrics $h_p = h^p e^{-\varphi_p}$, with appropriate weights φ_p. Additionally, there are examples involving tensor powers of several line bundles, for more details see [Coman et al. (2023)].

In this context, the k-volume is expressed as $\mathrm{Vol}_k(A) = \int_A \omega^{n-k}$. We suppress the subindex because it will be clear from the context which codimension is meant. We also remark that, unlike [Coman et al. (2023)], for the sake of simplifying our notation, we will not utilize additional volume form on the base manifold X besides ω. Although employing a different form than ω might alter the notation, it will not affect the equidistribution results of this paper. The adjustment mainly involves substituting the appropriate powers of ω with another form ϑ in the relevant parts, such as basic cohomology arguments and the total variation of the signed measure $dd^c\phi$ for a test form ϕ.

We denote the vector space of global holomorphic sections of L_p by $H^0(X, L_p)$. Take into consideration the following inner product on the space of smooth sections $\mathscr{C}^\infty(X, L_p)$ with respect to the metric h_p and the volume form ω^n on X:

$$\langle s_1, s_2 \rangle_p := \int_X \left\langle s_1(x), s_2(x) \right\rangle_{h_p} \omega^n \text{ and } \|s\|_p^2 := \langle s, s \rangle_p.$$

By virtue of Cartan-Serre finiteness theorem (see, e.g., Chapter 6, [Grauert & Remmert (2004)]), the space $H^0(X, L_p)$ is finite dimensional (since every line bundle can be seen as a coherent sheaf) and we will write $d_p := \dim H^0(X, L_p)$. The Associated Bergman kernel will be denoted by $K_p(x,y)$ and its restriction to the diagonal is denoted by, $K_p(x) := K_p(x,x)$ which is called the *Bergman function.* If $\{S_1^p, \ldots, S_{d_p}^p\}$ is an orthonormal basis for $H^0(X, L_p)$, the Bergman kernel function has the following representation

$$K_p(x) = \sum_{j=1}^{d_p} |S_j^p(x)|^2_{h_p}, \ x \in X. \tag{3.3}$$

Note that K_p is independent of the chosen basis $\{S_1^p, \ldots, S_{d_p}^p\}$, (see [Coman & Marinescu (2015)], Section 3), and

$$\log K_p \in L^1(X, \omega^n), \tag{3.4}$$

as it is locally difference of psh functions. Moreover, it has the dimensional density property, that is

$$\int_X K_p(x)\,\omega^n = d_p.$$

Next, we make the following assumption about the behaviour of $K_p(x)$: There exists a constant $M_0 > 1$ and $p_0 \in \mathbb{N}$ such that

$$\frac{A_p^n}{M_0} \le K_p(x) \le M_0 A_p^n \tag{3.5}$$

holds for every $x \in X$ and $p \ge p_0$. Consequently, this leads to the following estimates on the dimension d_p which will be useful in the subsequent analysis:

$$M_0{}^{-1}\mathrm{Vol}(X) A_p^n \le d_p \le M_0 \mathrm{Vol}(X) A_p^n \tag{3.6}$$

for all $p > p_0$.

Randomization

Let us fix an orthonormal basis $\{S_j^p\}_{j=1}^{d_p}$ of $H^0(X, L_p)$. Then, every $s \in H^0(X, L_p)$ is written in a unique way

$$s_p = \sum_{j=1}^{d_p} a_j^p S_j^p. \tag{3.7}$$

Using this representation, we identify the spaces $H^0(X, L_p)$ with $\mathbb{C}^{d_p}$ and equip them with the d_p-fold probability measures σ_p which does not put any mass on pluripolar sets and satisfy the following moment condition:

There exists a constant $\alpha \geq 2$, and for every p, a constant C_p such that

$$\int_{\mathbb{C}^{d_p}} \left|\log|\langle a, v\rangle|\right|^{\alpha} d\sigma_p(a) \leq C_p \tag{3.8}$$

for every $v \in \mathbb{C}^{d_p}$ with $\|v\| = 1$. The probability space $(H^0(X, L_p), \sigma_p)$ depends, as is seen above, on the choice of the orthonormal basis used in the identification of $H^0(X, L_p)$ (unless σ_p is unitary invariant).
Additionally, we consider the product probability space

$$(\mathcal{H}_\infty, \sigma_\infty) := \Big(\prod_{p=1}^{\infty} H^0(X, L_p), \prod_{p=1}^{\infty} \sigma_p\Big)$$

consisting of independent random sequences of holomorphic sections of L_p with increasing values of p.

3.1.1 Intersection of random zero currents

Before we proceed further, it is essential to define the random integration currents over the simultaneous zero loci of $k-$ independent holomorphic sections, chosen with respect to the probability measure σ_p as defined above. Furthermore, we define their variance and expectation, which will be consistently employed throughout.

Let σ_p be the aforementioned probability measure. For $1 \leq k \leq \dim_{\mathbb{C}} X$, consider the following probability spaces

$$(\mathcal{H}_p^k, \sigma_p^k) = \Big(\prod_{j=1}^{k} H^0(X, L_p), \prod_{j=1}^{k} \sigma_p\Big) \text{ and } (\mathcal{H}_\infty^k, \sigma_\infty^k) = \Big(\prod_{p=1}^{\infty} \mathcal{H}_p^k, \prod_{p=1}^{\infty} \sigma_p^k\Big).$$

These probability spaces consist of random k-systems of independent holomorphic sections of L_p and sequences of k-systems of holomorphic sections with increasing values of p, respectively.

Let $\Sigma_p^k := (s_p^1, \ldots, s_p^k) \in \mathcal{H}_p^k$ be such a random system of k-sections. We denote its simultaneous zero locus by

$$Z_{\Sigma_p^k} := \{x \in X : s_p^1(x) = \cdots = s_p^k(x) = 0\}.$$

The current of integration (with multiplicities, whenever well-defined) along the analytic subvariety $Z_{\Sigma_p^k}$ is defined as follows, given $\phi \in \mathcal{D}^{n-k,n-k}(X)$

$$\langle [Z_{\Sigma_p^k}], \phi \rangle := \int_{Z_{\Sigma_p^k}} \phi.$$

Now, since the base locus $Bs(H^0(X, L_p)) = \emptyset$ for $p \geq p_0$ (by 3.5), and our probability measure does not charge pluripolar subsets, Bertini's theorem (see Section 3.5) implies that with probability one the zero sets $Z_{s_p^j}, j = 1, \ldots, k$ are in general position. In particular, for σ_p^k-almost every system $\Sigma_p^k \in \mathcal{H}_p^k$ the common zero set $Z_{\Sigma_p^k}$ is a complex submanifold of pure dimension $n-k$ and the current $[Z_{\Sigma_p^k}]$ is well-defined. Moreover, the current of integration along $Z_{\Sigma_p^k}$ is represented by

$$[Z_{\Sigma_p^k}] = [Z_{s_p^1}] \wedge \cdots \wedge [Z_{s_p^k}]. \tag{3.9}$$

The expectation and the variance of the current-valued random variable $(H^0(X, L_p)^k, \sigma_p^k) \ni \Sigma_p^k \longmapsto [Z_{\Sigma_p^k}]$ are defined as follows:

$$\Big\langle \mathbb{E}[Z_{\Sigma_p^k}], \phi \Big\rangle := \mathbb{E}\langle [Z_{\Sigma_p^k}], \phi \rangle = \int_{H^0(X,L_p)^k} \langle [Z_{\Sigma_p^k}], \phi \rangle \, d\sigma_p^k(\Sigma_p^k) \tag{3.10}$$

$$\Big\langle \mathrm{Var}[Z_{\Sigma_p^k}], \phi \boxtimes \phi \Big\rangle := \mathrm{Var}\langle [Z_{\Sigma_p^k}], \phi \rangle = \mathbb{E}\langle [Z_{\Sigma_p^k}], \phi \rangle^2 - (\mathbb{E}\langle [Z_{\Sigma_p^k}], \phi \rangle)^2, \tag{3.11}$$

where $\phi \in \mathcal{D}^{n-k,n-k}(X)$. Here, $\phi \boxtimes \phi := \pi_1^* \phi \wedge \pi_2^* \phi$, where $\pi_1, \pi_2 : X \times X \to X$ are projections onto the first and the second factors, respectively.

By definition, the expectation and variance can be regarded as currents as well. Specifically, we have $\mathbb{E}[Z_{\Sigma_p^k}] \in \mathcal{D}'_{k,k}(X)$ whereas $\mathrm{Var}[Z_{\Sigma_p^k}] \in \mathcal{D}'_{2k,2k}(X \times X)$

3.2 Variance Estimate

In this section, we delve into the proof of Theorem 1.3.1 using an inductive approach based on the codimension k. We start with the case of codimension 1, serving as the initial step in our induction process. In order to initiate our analysis, we first make some preliminary observations and recall the facts needed for the proof.

We start by proving the following useful lemma, which will be used in multiple calculations in this chapter.

Lemma 3.2.1. *There exists a constant $b > 0$ such that*

$$-b\|\phi\|_{\mathscr{C}^2}\omega^n \leq dd^c\phi \leq b\|\phi\|_{\mathscr{C}^2}\omega^n \tag{3.12}$$

for any real-valued $\phi \in \mathcal{D}^{n-1,n-1}(X)$.

Proof. Let $x \in X$ and (U,z) be a local holomorphic coordinate system centered at x. In U, we write

$$\phi(z) = \sum_{|I|=n-1,|J|=n-1} \phi_{I,J}(z)\, dz_I \wedge d\overline{z}_J, \quad \phi_{I,J} \in \mathscr{C}^\infty(U).$$

The operator dd^c applied to ϕ gives

$$dd^c\phi = \frac{i}{\pi} \sum_{|I|=n-1,|J|=n-1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_{I,J}(z)}{\partial z_k \partial \overline{z}_l}\, dz_k \wedge d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J.$$

For any relatively compact subset $G \subset\subset U$, we have

$$\left| \frac{\partial^2 \phi_{I,J}(z)}{\partial z_k \partial \overline{z}_l} \right| \leq \|\phi\|_{\mathscr{C}^2(G)} \text{ for all } I, J, k, l.$$

To bound $dd^c\phi$ by ω^n, we consider the Kähler form in local coordinates,

$$\omega = i \sum_{k,l=1}^{n} g_{kl}\, dz_k \wedge d\overline{z}_l,$$

where g_{kl} are the components of the Hermitian metric. The volume form is then given by

$$\omega^n = i^n \det(g_{kl})\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.$$

Thus, noting that there are n^2 terms in the double sum below, and comparing the terms with ω^n, we have

$$\begin{aligned}
|dd^c\phi| &\leq \frac{1}{\pi} \sum_{|I|=n-1,|J|=n-1} \sum_{k,l=1}^{n} \left| \frac{\partial^2 \phi_{I,J}(z)}{\partial z_k \partial \overline{z}_l} \right| |dz_k \wedge d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J| \\
&\leq \Big(\frac{n^2}{\pi \det(g_{kl})}\Big) \|\phi\|_{\mathscr{C}^2(G)}\ |i^n \det(g_{kl})\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n| \\
&= \frac{n^2}{\pi \det(g_{kl})} \|\phi\|_{\mathscr{C}^2(G)}\ \omega^n.
\end{aligned}$$

Now, since X is compact, we can cover X by finitely many such charts U_i. For each chart, consider a relatively compact subset $G_i \subset\subset U_i$. Let $C_i = \frac{n^2}{\pi \det(g_{kl})}$ be the constants corresponding to these subsets. Taking $b = \max_i C_i$, we have

$$|dd^c\phi| \leq b\|\phi\|_{\mathscr{C}^2}\omega^n \quad \text{on } X.$$

Therefore,

$$-b\|\phi\|_{\mathscr{C}^2}\omega^n \leq dd^c\phi \leq b\|\phi\|_{\mathscr{C}^2}\omega^n.$$

where $b > 0$ is a universal constant inherently depending on the dimension n and the Kähler metric. □

Let $s_p \in H^0(X, L_p)$, then we can write it as

$$s_p = \sum_{j=1}^{d_p} a_j^p S_j^p = \langle a, \Gamma_p \rangle,$$

where $\Gamma_p = (S_1^p, \ldots, S_{d_p}^p)$, $a = (a_1^p, \ldots, a_{d_p}^p) \in \mathbb{C}^{d_p}$ and $\{S_j^p\}_{j=1}^{d_p}$ is an orthonormal basis of $H^0(X, L_p)$.

Let $x \in X$. $U \subseteq X$ be an open trivializing neighborhood of x and e_p be a holomorphic frame of L_p in U. Then locally $S_j^p = f_j^p e_p$, where f_j^p are holomorphic functions in U and so, by writing $f = (f_1^p, \ldots, f_{d_p}^p)$,

$$s_p = \sum_{j=1}^{d_p} a_j^p f_j^p e_p = \langle a, f \rangle e_p.$$

By Poincaré-Lelong formula (2.16), on the neighborhood U, we have

$$[Z_{s_p}] = dd^c \log|\langle a, f \rangle| = dd^c \log|\langle a, \Gamma_p \rangle|_{h_p} + c_1(L_p, h_p).$$

Now, for any $\phi \in \mathcal{D}^{n-1,n-1}(X)$, we define the following random variable

$$W_{s_p} := [Z_{s_p}] - c_1(L_p, h_p) = dd^c \log|\langle a, \Gamma_p \rangle|_{h_p}. \tag{3.13}$$

By the invariance property of the variance under translations with deterministic constants, we get

$$\mathrm{Var}\langle [Z_{s_p}], \phi \rangle = \mathrm{Var}\langle W_{s_p}, \phi \rangle. \tag{3.14}$$

Therefore, in the light of (3.14) it is enough to estimate $\mathrm{Var}\langle W_{s_p}, \phi \rangle$. Employing certain methods from [Shiffman & Zelditch (1999)] and [Shiffman & Zelditch (2008)] in our setting, we obtain the following theorem for codimension one.

Theorem 3.2.2. *Under the hypotheses of Theorem 1.3.1, if $s_p \in H^0(X, L_p)$, then for all $p \geq 1$ and any $(n-1,n-1)$-form ϕ of class $\mathscr{C}^2$ on X, the following variance estimate holds true*

$$\langle \mathrm{Var}[\widehat{Z}_{s_p}], \phi\rangle = \frac{(C_p)^{2/\alpha}}{A_p^2}(B_\phi \mathrm{Vol}(X))^2,$$

where B_ϕ is a constant depending on the form ϕ.

Proof. First, note that

$$\mathrm{Var}\langle W_{s_p}, \phi\rangle = \mathbb{E}\langle W_{s_p}, \phi\rangle^2 - \Big(\mathbb{E}\langle W_{s_p}, \phi\rangle\Big)^2. \tag{3.15}$$

By the relation (3.13), we have

$$\mathbb{E}\langle W_{s_p}, \phi\rangle^2 = \int_{H^0(X,L_p)}\int_X\int_X \log|\langle a, \Gamma_p(x)\rangle|_{h_p} \log|\langle a, \Gamma_p(y)\rangle|_{h_p} dd^c\phi(x) dd^c\phi(y) d\sigma_p(s). \tag{3.16}$$

Writing

$$|\Gamma_p(x)|_{h_p} := \Big(\sum_{j=1}^{d_p} |S_j^p(x)|^2_{h_p}\Big)^{1/2} = \sqrt{K_p(x)}$$

gives $\Gamma_p(x) = |\Gamma_p(x)|_{h_p} u_p(x)$ (so that $|u_p|_{h_p} = 1$) and we insert $\Gamma_p = |\Gamma_p|_{h_p} u_p$ into the integrand in (3.16), which breaks it into four terms:

$$\begin{aligned}\log|\Gamma_p(x)|_{h_p} \log|\Gamma_p(y)|_{h_p} + \log|\Gamma_p(x)|_{h_p} \log|\langle a, u_p(y)\rangle|_{h_p} + \log|\Gamma_p(y)|_{h_p} \log|\langle a, u_p(x)\rangle|_{h_p} \\ + \log|\langle a, u_p(x)\rangle|_{h_p} \log|\langle a, u_p(y)\rangle|_{h_p}.\end{aligned} \tag{3.17}$$

Before continuing with the variance estimate, we will see that $\mathbb{E}\langle W_{s_p}, \phi\rangle$ is bounded. To do this, we establish an auxiliary inequality to begin with. First, by the relation (3.12), we have

$$\int_X \Big|\log|\Gamma_p(x)|_{h_p}\Big| |dd^c\phi(x)| \leq b\,\|\phi\|_{\mathscr{C}^2} \int_X \Big|\log\Big(|\Gamma_p(x)|_{h_p}\Big)\Big|\,\omega^n < \infty \tag{3.18}$$

since $\log K_p \in L^1(X, \omega^n)$ for every $p \geq 1$. Now, it is evident that

$$\begin{aligned}\Big|\mathbb{E}\langle W_{s_p}, \phi\rangle\Big| &= \left|\int_{H^0(X,L_p)}\int_X \Big(\log|\Gamma_p(x)|_{h_p} + \log|\langle a, u_p(x)\rangle|_{h_p}\Big) dd^c\phi(x)\, d\sigma_p(s_p)\right| \\ &\leq \int_{H^0(X,L_p)}\int_X \Big|\log|\Gamma_p(x)|_{h_p}\Big| |dd^c\phi(x)|\, d\sigma_p(s_p) \\ &\quad + \int_{H^0(X,L_p)}\int_X \Big|\log|\langle a, u_p(x)\rangle|_{h_p}\Big| |dd^c\phi(x)|\, d\sigma_p(s_p)\end{aligned}$$

The first integral has an upper bound by (3.18) and σ_p being a probability measure on $H^0(X,L_p)$. The second double integral is also bounded from above. To elaborate on this, by the identification $H^0(X,L_p)\simeq\mathbb{C}^{d_p}$ and the moment condition (3.8) combined with Hölder's inequality and the relation (3.12), we get

$$\int_X\int_{\mathbb{C}^{d_p}}\Big|\log|\langle a,\rho_p(x)\rangle|\Big|d\sigma_p(a)|dd^c\phi(x)|\le(C_p)^{\frac{1}{\alpha}}\,b\,\|\phi\|_{\mathscr{C}^2}\mathrm{Vol}(X), \tag{3.19}$$

where

$$\rho_p(x)=\Big(\frac{f_1(x)}{\sqrt{\sum_{j=1}^{d_p}|f_j(x)|^2}},\dots,\frac{f_{d_p}(x)}{\sqrt{\sum_{j=1}^{d_p}|f_j(x)|^2}}\Big).$$

It follows from Fubini-Tonelli's theorem that

$$\int_{H^0(X,L_p)}\int_X\Big|\log|\langle a,u_p(x)\rangle|_{h_p}\Big|\,|dd^c\phi(x)|d\sigma_p(s_p)=\int_X\int_{\mathbb{C}^{d_p}}\Big|\log|\langle a,\rho_p(x)\rangle|\Big|d\sigma_p(a)|dd^c\phi(x)|. \tag{3.20}$$

Hence, we are done.

We now return to the variance estimate of W_{s_p}. In order to do this, we also expand the second term, $\Big(\mathbb{E}\langle W_{s_p},\phi\rangle\Big)^2$, in the variance expression (3.15) by using the aforementioned expression for expected distribution:

$$\mathbb{E}\langle W_{s_p},\phi\rangle=\int_{H^0(X,L_p)}\int_X(\log|\Gamma_p(x)|_{h_p}+\log|\langle a,u_p(x)\rangle|_{h_p})dd^c\phi(x)d\sigma_p(s_p),$$

which gives the following

$$\Big(\mathbb{E}\langle W_{s_p},\phi\rangle\Big)^2=J_1+2J_2+J_3,$$

where

$$J_1=\Big(\int_{H^0(X,L_p)}\int_X\log|\Gamma_p(x)|_{h_p}dd^c\phi(x)d\sigma_p(s_p)\Big)^2 \tag{3.21}$$

$$\begin{aligned}J_2=&\Bigg(\int_{H^0(X,L_p)}\int_X\log|\Gamma_p(x)|_{h_p}\,dd^c\phi(x)\,d\sigma_p(s_p)\Bigg)\\&\times\Bigg(\int_{H^0(X,L_p)}\int_X\log|\langle a,u_p(x)\rangle|_{h_p}\,dd^c\phi(x)\,d\sigma_p(s_p)\Bigg)\end{aligned} \tag{3.22}$$

and lastly,

$$J_3=\Big(\int_{H^0(X,L_p)}\int_X\log|\langle a,u_p(x)\rangle|_{h_p}dd^c\phi(x)d\sigma_p(s_p)\Big)^2. \tag{3.23}$$

Note that all of the integrals J_1,J_2 and J_3 are finite since $\mathbb{E}\langle W_{s_p},\phi\rangle$ is bounded.

According to the four terms given in (3.17), we have

$$\mathbb{E}\langle W_{s_p}, \phi\rangle^2 = B_1 + 2B_2 + B_3,$$

where

$$B_1 := \int_{H^0(X,L_p)} \int_X \int_X \log|\Gamma_p(x)|_{h_p} \log|\Gamma_p(y)|_{h_p} dd^c\phi(x) dd^c\phi(y) d\sigma_p(s), \tag{3.24}$$

$$B_2 := \int_{H^0(X,L_p)} \int_X \int_X \log|\Gamma_p(x)|_{h_p} \log|\langle a, u_p(y)\rangle|_{h_p} dd^c\phi(x) dd^c\phi(y) d\sigma_p(s). \tag{3.25}$$

(The second term, $\log|\Gamma_p(x)|_{h_p} \log|\langle a, u_p(y)\rangle|_{h_p}$, and the third one, $\log|\Gamma_p(y)|_{h_p} \log|\langle a, u_p(x)\rangle|_{h_p}$, in (3.17) are actually the integrands that yield the same result) and finally,

$$B_3 := \int_X \int_X dd^c\phi(y) dd^c\phi(x) \int_{\mathbb{C}^{d_p}} \log|\langle a, \rho_p(x)\rangle| \log|\langle a, \rho_p(y)\rangle| d\sigma_p(a). \tag{3.26}$$

From (3.18), the moment assumption (3.8) and the Fubini-Tonelli's theorem, we see that B_1, B_2 and B_3 are all finite, moreover, we have that $B_1 = J_1$ and $B_2 = J_2$. Therefore, the only integrals that survive are J_3 and B_3, which are not always equal to each other, thus we obtain

$$\mathrm{Var}\langle W_{s_p}, \phi\rangle = B_3 - J_3, \tag{3.27}$$

so it will suffice to estimate the term B_3 from above to complete the variance estimation. To this end, by Tonelli's theorem and Hölder's inequality with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, where $\alpha \geq 2$ is the constant satisfying the moment condition (3.8), we obtain

$$\begin{aligned}
B_3 &\leq \int_X \int_X dd^c\phi(y) dd^c\phi(x) \int_{\mathbb{C}^{d_p}} \Big|\log|\langle a, \rho_p(x)\rangle|\Big| \Big|\log|\langle a, \rho_p(y)\rangle|\Big| d\sigma_p(a) \\
&\leq \int_X \int_X dd^c\phi(y) dd^c\phi(x) \Big\{ \int_{\mathbb{C}^{d_p}} \Big|\log|\langle a, \rho_p(x)\rangle|\Big|^\alpha d\sigma_p(a) \Big\}^{\frac{1}{\alpha}} \Big\{ \int_{\mathbb{C}^{d_p}} \Big|\log|\langle a, \rho_p(y)\rangle|\Big|^\beta d\sigma_p(a) \Big\}^{\frac{1}{\beta}} \\
&\leq \int_X \int_X dd^c\phi(y) dd^c\phi(x) (C_p)^{\frac{1}{\alpha}} \Big\{ \int_{\mathbb{C}^{d_p}} \Big|\log|\langle a, \rho_p(y)\rangle|\Big|^\beta d\sigma_p(a) \Big\}^{\frac{1}{\beta}}.
\end{aligned}$$

If we again apply Hölder's inequality to the innermost integral in the last line (since $\alpha \geq 2 \geq \beta$ allows us to do so), we get

$$B_3 \leq \int_X \int_X dd^c\phi(y) dd^c\phi(x) (C_p)^{\frac{2}{\alpha}}. \tag{3.28}$$

Consequently, using the total variation inequality (3.12) of dd^c twice in (3.28) leads to the following

$$B_3 \le (C_p)^{2/\alpha} b^2 \|\phi\|^2_{\mathscr{C}^2} \mathrm{Vol}(X)^2, \tag{3.29}$$

which, by normalization, gives the desired variance estimate. □

Before proceeding to the proof of our main theorems, we present a lemma concerning cohomology classes of integration currents that will be instrumental in the sequel. This lemma has been previously proven in [Shiffman & Zelditch (2008)](as part of proposition 2.2), and for the sake of completeness, we include its proof here.

Lemma 3.2.3. *Let (X,ω) be a compact Kähler manifold with the fixed Kähler form ω. If $Z_{s_p^1},\dots,Z_{s_p^k}$ are smooth and intersect transversally, we have*

$$\left\langle \left[Z_{\Sigma_p^k}\right], \omega^{n-k} \right\rangle = \int_X c_1(L_p,h_p)^k \wedge \omega^{n-k}. \tag{3.30}$$

Proof. For $k=1$, this is just a consequence of Poincaré-Lelong formula and the fact that ω is a closed form. Indeed,

$$\left\langle \left[Z_{s_p}\right], \omega^{n-1} \right\rangle = \int_X c_1(L_p,h_p)\wedge\omega^{n-1} + \int_X dd^c \log|s_p|_{h_p} \wedge \omega^{n-1} = \int_X c_1(L_p,h_p)\wedge\omega^{n-1}.$$

Let us now suppose that the assertion (3.30) is true for $k-1$ sections $\Sigma_p^{k-1} = (s_p^2,\dots,s_p^k)$. Then by the induction hypothesis and the base step of induction, it yields that

$$\begin{aligned}\left\langle \left[Z_{s_p^1}\cap Z_{\Sigma_p^{k-1}}\right], \omega^{n-k} \right\rangle &= \int_{Z_{s_p^1}} c_1(L_p,h_p)^{k-1}\wedge\omega^{n-k} \\ &= \int_X c_1(L_p,h_p)\wedge c_1(L_p,h_p)^{k-1}\wedge\omega^{n-k} = \int_X c_1(L_p,h_p)^k\wedge\omega^{n-k}.\end{aligned}$$

Finally, based on the observation that $\left\langle \left[Z_{\Sigma_p^k}\right], \omega^{n-k} \right\rangle = \left\langle \left[Z_{s_p^1}\cap Z_{\Sigma_p^{k-1}}\right], \omega^{n-k} \right\rangle$, where $\Sigma_p^k = \left(s_p^1, \Sigma_p^{k-1}\right)$, we complete the proof. □

Next we proceed with the proof of the variance estimate in higher codimensions. We adapt some of the methods in [Shiffman (2008)] into our setting.

Proof of Theorem 1.3.1. Theorem 1.3.1 provides the case $k=1$ of induction on the codimension k. Now let $n\ge k\ge 2$ and we suppose that the variance estimate of Theorem 1.3.1 holds true for $k-1$ sections. We pick a system of k independent random holomorphic sections $\Sigma_p^k = (s_p^1,\dots,s_p^k)\in H^0(X,L_p)^k$ and write $\Sigma_p^k = (\Sigma_p^{k-1}, s_p^k)$, where $\Sigma_p^{k-1} = (s_p^1,\dots,s_p^{k-1})$. Thus, we may write $[Z_{\Sigma_p^k}] = [Z_{\Sigma_p^{k-1}}]\wedge[Z_{s_p^k}]$ for almost all Σ_p^k and p large enough (by Bertini's Theorem). Let $\phi\in\mathcal{D}^{n-k,n-k}(X)$ be a test form. Since

$\mathbb{E}[Z_{\Sigma_p^k}] = \mathbb{E}[Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}]$ in view of the independence of the random holomorphic sections $s_p^1, \ldots, s_p^k$, we first have

$$\begin{aligned}
\text{Var}\langle [Z_{\Sigma_p^k}], \phi\rangle &= \mathbb{E}\langle [Z_{\Sigma_p^k}], \phi\rangle^2 - (\mathbb{E}\langle [Z_{\Sigma_p^k}], \phi\rangle)^2 \\
&= \mathbb{E}\langle [Z_{\Sigma_p^{k-1}}] \wedge [Z_{s_p^k}], \phi\rangle^2 - (\langle \mathbb{E}[Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}], \phi\rangle)^2.
\end{aligned} \tag{3.31}$$

We shall use the following

$$\langle \Big\{ [Z_{\Sigma_p^{k-1}}] \wedge [Z_{s_p^k}] \Big\}, \phi\rangle^2 - \langle \mathbb{E}[Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}], \phi\rangle^2 = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= I_1(\Sigma_p^{k-1}, s_p^k) \\
&= \langle [Z_{\Sigma_p^{k-1}}] \wedge [Z_{s_p^k}], \phi\rangle^2 - \langle [Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}], \phi\rangle^2
\end{aligned}$$

and

$$I_2 := I_2(\Sigma_p^{k-1}) = \langle [Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}], \phi\rangle^2 - (\langle \mathbb{E}[Z_{\Sigma_p^{k-1}}] \wedge \mathbb{E}[Z_{s_p^k}], \phi\rangle)^2$$

for generic choice of Σ_p^{k-1} and s_p^k.

We observe that

$$\text{Var}\langle [Z_{\Sigma_p^k}], \phi\rangle = \mathbb{E}[I_1] + \mathbb{E}[I_2]. \tag{3.32}$$

Now let $Y := \Big\{ x \in X : \Sigma_p^{k-1}(x) = 0 \Big\}$. Notice that $\langle [Z_{\Sigma_p^{k-1}}] \wedge [Z_{s_p^k}], \phi\rangle = \langle [Z_{s_p^k}]\big|_Y, \phi|_Y\rangle$. Initially, we will estimate $\mathbb{E}[I_1]$. To do so, the first step is to integrate I_1 over $H^0(X, L_p)$ then use the observation above and apply Theorem 3.2.2:

$$\begin{aligned}
\int_{H^0(X,L_p)} I_1(\Sigma_p^{k-1}, s_p^k) d\sigma_p(s_p^k) &= \int_{H^0(X,L_p)} \langle [Z_{s_p^k}]\big|_Y, \phi|_Y\rangle^2 - (\langle \mathbb{E}\langle [Z_{s_p^k}]\big|_Y, \phi|_Y\rangle)^2 \\
&= \text{Var}\langle [Z_{s_p^k}]\big|_Y, \phi|_Y\rangle \\
&\leq (C_p)^{\frac{2}{\alpha}} \Big(B_\phi\big|_Y \int_Y \omega^{n-k+1} \Big)^2 \\
&\leq (C_p)^{\frac{2}{\alpha}} \Big(B_\phi \int_Y \omega^{n-k+1} \wedge c_1(L_p, h_p)^{k-1} \Big)^2 \\
&\leq (C_p)^{2/\alpha} \Big(B_\phi Vol(X) (2A_p)^{k-1} \Big)^2,
\end{aligned}$$

where the last inequality is obtained by using the fact that

$$\int_X \omega^{n-k+1} \wedge c_1(L_p, h_p)^{k-1} \leq (2A_p)^{k-1} Vol(X) \text{ for } p \geq p_1,$$

which is a simple consequence of the diophantine approximation condition (3.2).

Now integrating over $H^0(X,L_p)^{k-1}$ and taking the last inequality just above into account, it yields that

$$\begin{aligned}\mathbb{E}[I_1] &= \int_{H^0(X,L_p)^{k-1}}\int_{H^0(X,L_p)} I_1(\Sigma_p^{k-1},s_p^k)d\sigma_p(s_p^k)d\sigma_p^{k-1}(\Sigma_p^{k-1})\\ &\le (C_p)^{2/\alpha}\Big(B_\phi Vol(X)(2A_p)^{k-1}\Big)^2,\end{aligned}$$

since σ_p^{k-1} is a product probability measure on $H^0(X,L_p)^{k-1}$. This finishes the estimation of $\mathbb{E}[I_1]$.

In order to get the upper bound for $\mathbb{E}[I_2]$, first observe that

$$\mathbb{E}[I_2] = \mathbb{E}\Big\langle\Big([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}]\Big)\wedge\mathbb{E}[Z_{s_p^k}],\phi\Big\rangle^2. \tag{3.33}$$

Also we see that

$$\begin{aligned}\Big\langle\Big([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}]\Big)\wedge\mathbb{E}[Z_{s_p^k}],\phi\Big\rangle^2 &= \Big\{\int_{H^0(X,L_p)}\Big\langle\Big([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}]\Big)\wedge[Z_{s_p^k}],\phi\Big\rangle d\sigma_p(s_p^k)\Big\}^2\\ &\le \int_{H^0(X,L_p)}\Big\{\Big\langle([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}])\wedge[Z_{s_p^k}],\phi\Big\rangle\Big\}^2 d\sigma_p(s_p^k),\end{aligned}$$

where, in the second line, we have used Cauchy-Schwarz inequality. Analogous to the case for $\mathbb{E}[I_1]$, this time we consider the zero set of single s_p^k, namely, the set $Y := \Big\{x\in X : s_p^k(x)=0\Big\}$. We first get

$$\mathbb{E}[I_2] \le \int_{H^0(X,L_p)^{k-1}}\int_{H^0(X,L_p)}\Big\{\Big\langle([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}])\wedge[Z_{s_p^k}],\phi\Big\rangle\Big\}^2 d\sigma_p(s_p^k)d\sigma_p^{k-1}(\Sigma_p^{k-1}). \tag{3.34}$$

As has been argued for $\mathbb{E}[I_1]$ above, since

$$\langle([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}])\wedge[Z_{s_p^k}],\phi\rangle = \langle([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}])\Big|_Y,\phi\Big|_Y\rangle,$$

by invoking Fubini-Tonelli's theorem, (3.34) becomes

$$\mathbb{E}[I_2] \le \int_{H^0(X,L_p)}\int_{H^0(X,L_p)^{k-1}}\Big\{\Big\langle([Z_{\Sigma_p^{k-1}}] - \mathbb{E}[Z_{\Sigma_p^{k-1}}])\Big|_Y,\phi\Big|_Y\Big\rangle\Big\}^2 d\sigma_p^{k-1}(\Sigma_p^{k-1})d\sigma_p(s_p^k). \tag{3.35}$$

The inner integral is, by definition, the variance of $Z_{\Sigma_p^{k-1}}\Big|_Y$, so (3.35) takes the following form:

$$\mathbb{E}[I_2] \le \int_{H^0(X,L_p)}\mathrm{Var}\Big\langle[Z_{\Sigma_p^{k-1}}]\Big|_Y,\phi|_Y\Big\rangle d\sigma_p(s_p^k). \tag{3.36}$$

By using the induction hypothesis we have

$$\begin{aligned}\mathrm{Var}\Big\langle [Z_{\Sigma_p^{k-1}}]\Big|_Y, \phi|_Y\Big\rangle &\le (C_p)^{2/\alpha}\Big(B_\phi\Big|_Y (2A_p)^{k-2}\int_Y \omega^{n-1}\Big)^2\\ &\le (C_p)^{2/\alpha}\Big(B_\phi\ (2A_p)^{k-2}\int_X \omega^{n-1}\wedge c_1(L_p,h_p)\Big)^2\\ &\le (C_p)^{2/\alpha}\Big(B_\phi Vol(X)(2A_p)^{k-1}\Big)^2,\end{aligned}$$

where the last inequality is obtained by using the fact that

$$\int_X \omega^{n-1}\wedge c_1(L_p,h_p) \le 2A_p Vol(X) \text{ for } p\ge p_2,$$

for some $p_2\in\mathbb{N}$. Now by integrating over $H^0(X,L_p)$ and taking the last inequality just above into account, we obtain

$$\mathbb{E}[I_2] \le (C_p)^{2/\alpha}\Big(B_\phi Vol(X)(2A_p)^{k-1}\Big)^2$$

since σ_p^1 is a product probability measure on $H^0(X,L_p)$. Lastly, using the relation (3.32) and applying normalization ends the proof of Theorem 1.3.1.

□

3.3 Equidistribution of Zeros of Random Sections

In this section, we provide the proof for Theorem 1.3.1. We begin by first establishing the asymptotic behaviour of the expected zero distribution. Subsequently, by utilizing the variance estimate from the previous section in conjunction with the expected distribution, we prove that, subject to one summability condition, the normalized zero currents equidistribute with respect to ω^k.

3.3.1 Expected Distribution of Zeros

Here, we embark on proving the first assertion in Theorem 1.3.1. To achieve this, we employ an inductive argument. Initially, we establish the expected distribution for the case of codimension one, and then extend our proof to cover higher codimensions.

Theorem 3.3.1. *Let (X,ω) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ and let $(L_p,h_p)_{p\geq 1}$,be a sequence of Hermitian holomorphic line bundles on X with $\mathscr{C}^2$ metrics h_p. Assume that the conditions (3.2) and (3.5) hold. Then*

$$\frac{1}{A_p}\mathbb{E}[Z_{s_p}] \longrightarrow \omega \tag{3.37}$$

in the weak topology of currents as $p\to\infty$.*

Proof. Let $s_p \in H^0(X,L_p)$, then

$$s_p = \sum_{j=1}^{d_p} a_j^p S_j^p = \langle a, \Gamma_p\rangle,$$

where $\Gamma_p = (S_1^p,\dots,S_{d_p}^p)$, and $a = (a_1,\dots,a_{d_p}) \in \mathbb{C}^{d_p}$. Let $x \in X$, $U \subseteq X$ be an open neighborhood of x and e_p be a holomorphic frame of L_p in U. Then locally $S_j^p = f_j e_p$, where f_j^p are holomorphic functions in U and so, by writing $f = (f_1^p,\dots,f_{d_p}^p)$, we have

$$s_p = \sum_{j=1}^{d_p} a_j^p f_j^p e_p = \langle a, f\rangle e_p.$$

By Poincaré-Lelong formula (2.16), on the neighborhood U, we have

$$[Z_{s_p}] = dd^c \log|\langle a,f\rangle| = dd^c \log|\langle a,\Gamma_p\rangle|_{h_p} + c_1(L_p,h_p). \tag{3.38}$$

Let us now fix $\phi \in \mathcal{D}^{n-1,n-1}(X)$, without lost of generality we may assume that $\text{supp}(\phi)\subset U$ as the general case follows by covering $\text{supp}(\phi)$ by U_α's and using the compatibility conditions. Using the definition of expectation and incorporating (3.38) along with the observation that $c_1(L_p,h_p)$ is independent of s_p we have

(3.39)
$$\frac{1}{A_p}\langle \mathbb{E}[Z_{s_p}],\phi\rangle = \frac{1}{A_p}\langle c_1(L_p,h_p),\phi\rangle + \frac{1}{A_p}\int_{H^0(X,L_p)}\int_X \log|\langle a,\Gamma_p(x)\rangle|_{h_p} dd^c\phi(x)d\sigma_p(s_p)$$

Let us denote the second term above by $I(p)$, then by exploiting the fact that $\Gamma_p(x) = |\Gamma_p(x)|_{h_p} u_p(x)$ (so that $|u_p|_{h_p} = 1$) one has

$$\begin{aligned} I(p) \leq{}& \frac{1}{A_p}\int_{H^0(X,L_p)}\int_X \Big|\log|\Gamma_p(x)|_{h_p}\Big|\,|dd^c\phi(x)|\,d\sigma_p \\ &+ \frac{1}{A_p}\int_{H^0(X,L_p)}\int_X \Big|\log|\langle a,u_p(x)\rangle|_{h_p}\Big|\,|dd^c\phi(x)|\,d\sigma_p. \end{aligned} \tag{3.40}$$

Utilizing (3.12) and the fact that $\frac{1}{A_p}\log K_p(x)\to 0$ as $p\to\infty$, in $L^1(X,\omega)$ we obtain the following

$$\frac{1}{A_p}\int_{H^0(X,L_p)}\int_X \Big|\log|\Gamma_p(x)|_{h_p}\Big|\,|dd^c\phi(x)|\,d\sigma_p(s_p) \le \frac{B_\phi}{2A_p}\int_X |\log K_p(x)|\,\omega^n(x) \to 0 \tag{3.41}$$

as $p\to\infty$. Additionally, using the identification $H^0(X,L_p)\simeq\mathbb{C}^{d_p}$, the moment condition (3.8) along with Hölder's inequality and the relation (3.12), we get

$$\int_X\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a,\rho_p(x)\rangle|\Big| d\sigma_p(a)\,|dd^c\phi(x)| \le (C_p)^{\frac{1}{\alpha}} B_\phi\, Vol(X), \tag{3.42}$$

where

$$\rho_p(x) = \Big(\frac{f_1(x)}{\sqrt{\sum_{j=1}^{d_p}|f_j(x)|^2}},\dots,\frac{f_{d_p}(x)}{\sqrt{\sum_{j=1}^{d_p}|f_j(x)|^2}}\Big).$$

It follows from Fubini-Tonelli's theorem that

$$\int_{H^0(X,L_p)}\int_X \Big|\log|\langle a,u_p(x)\rangle|_{h_p}\Big|\,|dd^c\phi(x)|d\sigma_p(s_p) = \int_X\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a,\rho_p(x)\rangle|\Big|\,d\sigma_p(a)|dd^c\phi(x)|. \tag{3.43}$$

Consequently, employing the given hypothesis, we deduce that

$$\frac{1}{A_p}\int_{H^0(X,L_p)}\int_X \Big|\log|\langle a,u_p(x)\rangle|_{h_p}\Big|\,|dd^c\phi(x)|d\sigma_p(s_p) \le \frac{C_p^{1/\alpha}}{A_p} B_\phi\, Vol(X) \to 0, \tag{3.44}$$

as $p\to\infty$. In turn, (3.41) and (3.44) imply that $I_p\to 0$, as $p\to\infty$. Finally, by using (3.2) we obtain that, $\frac{1}{A_p}\langle c_1(L_p,h_p),\phi\rangle \to \langle\omega,\phi\rangle$, thus concluding the proof. □

Theorem 3.3.2. *Let (X,ω) be a compact Kähler manifold of $\dim_{\mathbb{C}}X=n$ and let $(L_p,h_p)_{p\ge1}$, be a sequence of Hermitian holomorphic line bundles on X with $\mathscr{C}^2$ metrics h_p. Assume that the conditions (3.2) and (3.5) hold. Then for $1\le k\le\dim_{\mathbb{C}}X$*

$$\mathbb{E}\Big[Z_{\Sigma_p^k}\Big] = \mathbb{E}\Big[Z_{s_p^1}\Big]\wedge\cdots\wedge\mathbb{E}\Big[Z_{s_p^k}\Big].$$

Moreover, if $\lim_{p\to\infty}\frac{C_p^{1/\alpha}}{A_p}=0$, *then*

$$\mathbb{E}\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k$$

in the weak topology of currents as $p\to\infty$.*

Proof. Let $\phi\in\mathcal{D}^{n-k,n-k}(X)$, then

$$\Big|\Big\langle\Big[\widehat{Z}_{\Sigma_p^k}\Big],\phi\Big\rangle\Big| = \Big|\frac{1}{A_p^k}\int_{Z_{\Sigma_p^k}}\phi\Big| \le \frac{1}{A_p^k}\sup\|\phi\|\,\mathrm{Vol}(Z_{\Sigma_p^k}). \tag{3.45}$$

Using Lemma 3.2.3 above, and the prequantization condition (3.2)

$$\mathrm{Vol}(Z_{\Sigma_p^k}) = \left\langle \left[Z_{\Sigma_p^k}\right], \omega^{n-k} \right\rangle = \int_X c_1(L_p, h_p)^k \wedge \omega^{n-k} \leq (2A_p)^k \,\mathrm{Vol}(X) \tag{3.46}$$

for all $p \geq p_1$. Thus, combining both of the inequalities above, we obtain that for $p \geq p_1$,

$$\left| \left\langle \left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle \right| \leq 2^k \;\mathrm{Vol}(X) \sup \|\phi\| < \infty \tag{3.47}$$

which means that $\left\langle \left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle$ is bounded for almost all Σ_p^k. Consequently, $\mathbb{E}[\widehat{Z}_{\Sigma_p^k}]$ is a well-defined current of bidegree (k,k).

Let us now prove the first assertion. To accomplish this, we induct on the codimension k. The base step, $k = 1$ is obvious. Suppose that the claim holds for $k-1$ sections, say $\Sigma_p^{k-1} = (s_p^2, \dots, s_p^k)$. Fix $s_p^1 \in H^0(X, L_p)$ so that $Y := Z_{s_p^1}$ is a complex submanifold. For almost all s_p^j, we write $(s_p^j)' = s_p^j|_Y$ and $H^0(X, L_p)' := H^0(X, L_p)\Big|_Y$, which in turn for almost all $\Sigma_p^k = (s_p^1, \dots, s_p^k) \in \mathcal{H}_p^k$ gives rise to the notation $(\Sigma_p^k)' = \Sigma_p^k\Big|_Y$, and $(\mathcal{H}_p^k)' = \mathcal{H}_p^k\Big|_Y$, where $\Sigma_p^k\Big|_Y = (s_p^1|_Y, \dots, s_p^k|_Y)$, and $\mathcal{H}_p^k = H^0(X, L_p)^k$ respectively. We endow $H^0(X, L_p)'$ with the push-forward probability measure $\sigma_p' := \theta_* \sigma_p$, where $\theta : H^0(X, L_p) \to H^0(X, L_p)'$ is the restriction map. The induced measure on $(\mathcal{H}_p^k)'$ will be denoted by $(\sigma_p')^k$. By the independence and induction hypothesis applied to $Y = Z_{s_p^1}$ and $(\mathcal{H}_p^{k-1})'$, and observation that $Z_{\Sigma_p^k} = Z_{(\Sigma_p^{k-1})'}$ we have, for $\phi \in \mathcal{D}^{n-k,n-k}(X)$,

$$\begin{aligned}
\int_{\mathcal{H}_p^{k-1}} \left\langle \left[Z_{\Sigma_p^k}\right], \phi \right\rangle d\sigma_p^{k-1}(\Sigma_p^{k-1}) &= \int_{(\mathcal{H}_p^{k-1})'} \left\langle \left[Z_{(\Sigma_p^{k-1})'}\right], \phi\Big|_Y \right\rangle d(\sigma_p')^{k-1}((\Sigma_p^{k-1})') \\
&= \left\langle \mathbb{E}[Z_{(s_p^2)'}] \wedge \dots \wedge \mathbb{E}[Z_{(s_p^k)'}], \phi\Big|_Y \right\rangle \\
&= \int_{Z_{s_p^1}} \mathbb{E}[Z_{s_p^2}] \wedge \dots \wedge \mathbb{E}[Z_{s_p^k}] \wedge \phi.
\end{aligned}$$

Finally, integrating over all s_p^1 in the last expression, leads us to the desired result.

We will now prove, using an inductive approach based on codimensions, that for $\phi \in \mathcal{D}^{n-k,n-k}(X)$,

$$\left\langle \mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle = \left\langle \frac{1}{A_p^k} c_1(L_p, h_p)^k, \phi \right\rangle + R_{X,p}(\phi). \tag{3.48}$$

Here, $\left| R_{X,p}(\phi) \right| \leq 2^{n-1} \frac{\mathrm{Vol}(X)}{(2A_p)^{\mathrm{codim}(X)}} B_\phi \left[(n+1) \frac{\log A_p}{2A_p} + \frac{C_p^{1/\alpha}}{A_p} \right]$. We refer to $R_{X,p}$ as a remainder current. Similar to the previous part, by writing $Y = Z_{s_p^1}$ and using the induction assumption on Y, one finds, for $\phi \in \mathcal{D}^{n-k,n-k}(X)$,

$$\begin{aligned}\int_{\mathcal{H}_p^{k-1}} \langle \big[\widehat{Z}_{\Sigma_p^k}\big], \phi\rangle d\sigma_p^{k-1}(\Sigma_p^{k-1}) &= \langle \mathbb{E}\big[\widehat{Z}_{(\Sigma_p^{k-1})'}\big], \phi\big|_Y\rangle \\ &= \int_{Z_{s_p^1}} \frac{1}{A_p^{k-1}} c_1(L_p,h_p)^{k-1}\wedge\phi + R_{Y,p}(\phi) \\ &= \Big\langle \big[Z_{s_p^1}\big], \frac{1}{A_p^{k-1}} c_1(L_p,h_p)^{k-1}\wedge\phi\Big\rangle + R_{Y,p}(\phi),\end{aligned}$$

where

$$|R_{Y,p}(\phi)| \le 2^{n-1} B_\phi\big|_Y \Big[(n+1)\frac{\log A_p}{2A_p} + \frac{C_p^{1/\alpha}}{A_p}\Big] \int_Y \frac{\omega^{n-1}}{2A_p}. \tag{3.49}$$

Using Lemma 3.2.3 and the fact that $\int_X \frac{c_1(L_p,h_p)}{2A_p}\wedge\omega^{n-1} \le Vol(X)$ for sufficiently large p, we have

$$|R_{Y,p}(\phi)| \le 2^{n-1} B_\phi \Big[(n+1)\frac{\log A_p}{2A_p} + \frac{C_p^{1/\alpha}}{A_p}\Big] \int_X \frac{c_1(L_p,h_p)}{2A_p}\wedge\omega^{n-1} \tag{3.50}$$

$$\le 2^{n-1} \frac{Vol(X)}{(2A_p)^{\operatorname{codim}(X)}} B_\phi \Big[(n+1)\frac{\log A_p}{2A_p} + \frac{C_p^{1/\alpha}}{A_p}\Big]. \tag{3.51}$$

By taking the average over $s_p^1 \in H^0(X,L_p)$ and using the information about codimension one, specifically

$$\mathbb{E}[\widehat{Z_{s_p^1}}] = \frac{1}{A_p} c_1(L_p,h_p) + \frac{1}{2A_p} dd^c \log K_p(x) + \frac{1}{A_p} dd^c \log\big|\langle a, u_p(x)\rangle\big|_{h_p} \tag{3.52}$$

one finds

$$\begin{aligned}\Big\langle \mathbb{E}\big[\widehat{Z}_{\Sigma_p^k}\big], \phi\Big\rangle &= \int_{\mathcal{H}_p^1}\int_{\mathcal{H}_p^{k-1}} \langle \big[\widehat{Z}_{\Sigma_p^k}\big], \phi\rangle d\sigma_p^{k-1}(\Sigma_p^{k-1}) d\sigma_p^1(s_p^1) \\ &= \frac{1}{A_p}\int_{\mathcal{H}_p^1} \Big\langle \big[Z_{s_p^1}\big], \frac{1}{A_p^{k-1}} c_1(L_p,h_p)^{k-1}\wedge\phi\Big\rangle d\sigma_p^1(s_p^1) + \int_{\mathcal{H}_p^1} R_{Y,p}(\phi) d\sigma_p^1(s_p^1) \\ &= \Big\langle \mathbb{E}[\widehat{Z}_{s_p^1}], \frac{1}{A_p^{k-1}} c_1(L_p,h_p)^{k-1}\wedge\phi\Big\rangle + \int_{\mathcal{H}_p^1} R_{Y,p}(\phi) d\sigma_p^1(s_p^1) \\ &= \langle \frac{1}{A_p^k} c_1(L_p,h_p)^k, \phi\rangle + \int_{\mathcal{H}_p^1}\int_X \frac{1}{2A_p}\log K_p(x) \frac{c_1(L_p,h_p)^{k-1}}{A_p^{k-1}} dd^c\phi(x) d\sigma_p(s_p^1) \\ &\quad + \int_{\mathcal{H}_p^1}\int_X \frac{1}{A_p}\log\big|\langle a, u_p(x)\rangle\big|_{h_p} \frac{c_1(L_p,h_p)^{k-1}}{A_p^{k-1}} dd^c\phi(x) d\sigma_p(s_p^1) \\ &\quad + \int_{\mathcal{H}_p^1} R_{Y,p}(\phi) d\sigma_p^1(s_p^1)\end{aligned}$$

Choosing $p \in \mathbb{N}$ sufficiently large so that $A_p \ge M_0$ and the assumption (3.5) is satisfied, we obtain that

$$A_p^{n-1} \leq K_p(x) \leq A_p^{n+1}$$

for all such p. In turn, we have $\frac{1}{2A_p}\log K_p(x) \leq (n+1)\frac{\log A_p}{2A_p}$. Now, using the moment condition (3.8), relation (3.2), total variation inequality (3.12), the previous estimate for $\left|R_{Y,p}(\phi)\right|$, and the induction hypothesis

(3.53)
$$\left\langle \mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle = \langle \frac{1}{A_p^k} c_1(L_p, h_p)^k, \phi \rangle + 2^{k-1} Vol(X) B_\phi \left[(n+1)\frac{\log A_p}{A_p} + \frac{C_p^{1/\alpha}}{A_p}\right] + R_{Y,p}(\phi)$$

Thus we have

$$\left\langle \mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle = \left\langle \frac{1}{A_p^k} c_1(L_p, h_p)^k, \phi \right\rangle + R_{X,p}(\phi) \tag{3.54}$$

where

$$\begin{aligned}\left|R_{X,p}(\phi)\right| &\leq \left|R_{Y,p}(\phi)\right| + 2^{k-1} Vol(X) B_\phi \left[(n+1)\frac{\log A_p}{A_p} + \frac{C_p^{1/\alpha}}{A_p}\right] \\ &\leq 2^{n-1} Vol(X) B_\phi \left[(n+1)\frac{\log A_p}{A_p} + \frac{C_p^{1/\alpha}}{A_p}\right]\end{aligned}$$

Finally, passing to the limit as $p \to \infty$ we obtain the result. □

3.3.2 Almost Sure Distribution of Zeros

In this subsection, we will delve into the proof of the second assertion from Theorem 1.3.2, which deals with the almost sure behaviour of random zeros.

Theorem 3.3.3. *Let (X, ω) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ and let $(L_p, h_p)_{p\geq 1}$, be a sequence of Hermitian holomorphic line bundles on X with $\mathscr{C}^2$ metrics h_p. Assume that the conditions (3.2) and (3.5) hold. If $\sum_{p=1}^{\infty} \frac{C_p^{2/\alpha}}{A_p^2} < \infty$, then for σ_∞^k- almost every sequence $\mathbf{\Sigma_k} = \left\{\Sigma_p^k\right\}_{p\geq 1} \in \mathcal{H}_\infty$,*

$$\left[\widehat{Z}_{\Sigma_p^k}\right] \longrightarrow \omega^k$$

in the weak topology of currents as $p \to \infty$.*

Proof. Fix $\phi \in \mathcal{D}^{n-k,n-k}(X)$, and pick $\mathbf{\Sigma_k} = \left\{\Sigma_p^k\right\}_{p\geq 1} \in \mathcal{H}_\infty$. Let us examine the non-negative current valued random variables

$$\mathcal{X}_p(\mathbf{\Sigma_k}) := \left\langle \left[\widehat{Z}_{\Sigma_p^k}\right] - \mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle^2 \geq 0. \tag{3.55}$$

By appealing to the equivalent characterization of variance, notice that

$$\int_{\mathcal{H}_\infty} \mathcal{X}_p(\mathbf{\Sigma_k}) d\sigma_\infty^k(\mathbf{\Sigma_k}) = \left\langle \mathrm{Var}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle. \tag{3.56}$$

Using Theorem 1.3.1 along with the summability condition given by the hypothesis, we obtain

$$\sum_{p=1}^{\infty} \int_{\mathcal{H}_\infty} \mathcal{X}_p(\mathbf{\Sigma_k}) d\sigma_\infty^k(\mathbf{\Sigma_k}) = \sum_{p=1}^{\infty} \left\langle \mathrm{Var}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle < \infty. \tag{3.57}$$

By the relation (3.56) above and invoking Beppo-Levi Theorem from the standard measure theory, we get

$$\int_{\mathcal{H}_\infty} \sum_{p=1}^{\infty} \mathcal{X}_p(\mathbf{\Sigma_k}) d\sigma_\infty^k(\mathbf{\Sigma_k}) = \sum_{p=1}^{\infty} \left\langle \mathrm{Var}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle < \infty. \tag{3.58}$$

This implies that, for σ_∞^k- almost every sequence of $k-$systems $\mathbf{\Sigma_k} \in \mathcal{H}_\infty$, the series $\sum_{p=1}^{\infty} \mathcal{X}_p(\mathbf{\Sigma_k})$ converges, leading to the conclusion that $\mathcal{X}_p \to 0$, σ_∞^k- almost surely. By definition (3.55) of random variables $\mathcal{X}_p$ this also indicates that

$$\left\langle \left[\widehat{Z}_{\Sigma_p^k}\right] - \mathbb{E}\left[\widehat{Z}_{\Sigma_p^k}\right], \phi \right\rangle \to 0 \tag{3.59}$$

σ_∞^k-almost surely. Combining this last information with Theorem 3.3.2, we conclude that for σ_∞^k- almost every sequence,

$$\left[\widehat{Z}_{\Sigma_p^k}\right] \longrightarrow \omega^k \tag{3.60}$$

in the weak* topology of currents as $p \to \infty$. $\square$

3.4 Some Special Cases

In [Bayraktar et al. (2020)], certain types of measures which satisfy the assumption (3.8) have been investigated as special cases. We will now provide some insights regarding a few of these measures in connection with Theorem 1.3.1 and 1.3.2. The first two measures to be considered here will be the Gaussian and the Fubini-Study measures, both of which are unitary invariant measures that come with certain advantages in estimations.

3.4.1 Gaussian and Fubini-Study

In what follows, λ_n represents the Lebesgue measure on $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$). We will present the variance estimate simultaneously for both Gaussian and Fubini-Study cases, with detailed explanations provided for the Gaussian case, as the computations are exactly the same. It turns out that, in these cases, the constants C_p reduce to the ones independent of p and Theorem 1.3.1 remains valid for every $\alpha \geq 1$. The standard Gaussian measure is precisely defined as follows, for $a = (a_1, \dots, a_n) \in \mathbb{C}^n$,

$$d\sigma_n(a) = \frac{1}{\pi^n} e^{-||a||^2} d\lambda_n(a), \tag{3.61}$$

and the Fubini-Study measure on $\mathbb{CP}^n \supset \mathbb{C}^n$ is defined as:

$$d\sigma_n(a) = \frac{n!}{\pi^n} \frac{1}{(1+||a||^2)^{n+1}} d\lambda_n(a). \tag{3.62}$$

As for these two measures, we record two facts (Lemma 4.8, Lemma 4.10) from [Bayraktar et al. (2020)]: Given that σ_n is the Gaussian measure, for every integer $n \geq 1$ and every $\alpha \geq 1$, we have

$$\int_{\mathbb{C}^n} \Big|\log|\langle a, v\rangle|\Big|^\alpha d\sigma_n(a) = 2\int_0^\infty r|\log r|^\alpha e^{-r^2} dr, \ \forall v \in \mathbb{C}^n, \ ||v|| = 1; \tag{3.63}$$

if σ_n is the Fubini-Study, then for every integer $n \geq 1$ and every $\alpha \geq 1$

$$\int_{\mathbb{C}^n} \Big|\log|\langle a, v\rangle|\Big|^\alpha d\sigma_n(a) = 2\int_0^\infty \frac{r|\log r|^\alpha}{(1+r^2)^2} dr, \ \forall v \in \mathbb{C}^n, \ ||v|| = 1. \tag{3.64}$$

It is evident that they are indeed independent of the dimension n.

Let us first show that,

$$\mathbb{E}[W_{s_p}] = dd^c \log |\Gamma_p|_{h_p}. \tag{3.65}$$

Since

$$\mathbb{E}\langle W_{s_p},\phi\rangle = \int_{H^0(X,L_p)}\int_X \log|\Gamma_p(x)|_{h_p} dd^c\phi(x)d\sigma_{d_p} + \int_{H^0(X,L_p)}\int_X \log|\langle a,u_p(x)\rangle|_{h_p} dd^c\phi(x)d\sigma_{d_p},$$

we need to show that the second double integral is zero. By (3.43),

$$\int_{H^0(X,L_p)}\int_X \log|\langle a,u_p(x)\rangle|_{h_p} dd^c\phi(x)d\sigma_{d_p}(s_p) = \int_X\int_{\mathbb{C}^{d_p}} \log|\langle a,\rho_p(x)\rangle| d\sigma_{d_p}(a)dd^c\phi(x).$$

Hence, in the form of currents, this gives

$$\Big\langle dd^c\Big\{\int_{\mathbb{C}^{d_p}} \log|\langle a,\rho_p(x)\rangle| d\sigma_{d_p}(a)\Big\},\phi\Big\rangle = 0$$

since the integral acted by dd^c is a constant independent of x due to the relation (3.63), which gives (3.65).

Now, the first term is $\mathbb{E}\langle W_{s_p},\phi\rangle^2$ owing to the relation (3.65), which cancels out the second term of the variance. Also, similar to the above reasoning, we see that the integrals of the second and third terms become zero by using the fact (3.63). Therefore, we estimate only the fourth term, which results in the variance of W_{s_p}, that is

(3.66)

$$\mathrm{Var}\langle W_{s_p},\phi\rangle = \int_X\int_X dd^c\phi(y)dd^c\phi(x)\int_{\mathbb{C}^{d_p}} \log|\langle a,\rho_p(x)\rangle|\log|\langle a,\rho_p(y)\rangle| d\sigma_{d_p}(a).$$

By Cauchy-Schwarz inequality, the relations (3.63) and (3.12), we get

$$\begin{aligned}\mathrm{Var}\langle W_{s_p},\phi\rangle &\le \int_X\int_X dd^c\phi(y)dd^c\phi(x)\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a,\rho_p(x)\rangle|\Big|\Big|\log|\langle a,\rho_p(y)\rangle|\Big| d\sigma_{d_p}(a)\\ &\le \int_X\int_X dd^c\phi(y)dd^c\phi(x)\Big\{\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a,\rho_p(x)\rangle|\Big|^2 d\sigma_{d_p}\Big\}^{\frac{1}{2}}\Big\{\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a,\rho_p(y)\rangle|\Big|^2 d\sigma_{d_p}\Big\}^{\frac{1}{2}}\\ &\le \Lambda\|\phi\|^2_{\mathscr{C}^2}\, b^2\,\mathrm{Vol}^2(X),\end{aligned}$$

where $\Lambda := 2\int_0^\infty r|\log r|^2 e^{-r^2}dr$, which we have obtained the Gaussian (Fubini-Study) version of Theorem 3.2.2. Thus, by carrying out the proof of this theorem in the same way, what we have is the next theorem:

Theorem 3.4.1. *Under the same assumptions of Theorem 1.3.1, let $\sigma^k_{d_p}$ be the product Gaussian (Fubini-Study) measure on $H^0(X,L_p)^k$ given by (3.61) (by (3.62)). Then for any $\phi\in\mathcal{D}^{n-k,n-k}(X)$, one gets*

$$\mathrm{Var}\Big\langle\Big[\widehat{Z}_{\Sigma^k_p}\Big],\phi\Big\rangle \le \frac{1}{A_p^2}\Lambda_k\, B_\phi^2\,\mathrm{Vol}^2(X),$$

where $\Lambda_k = 2^{k-1}\int_0^\infty r|\log r|^\alpha e^{-r^2}dr$ *(*$\Lambda_k = 2^{k-1}\int_0^\infty \frac{r|\log r|^\alpha}{(1+r^2)^2}dr$*).*

We infer from Theorem 3.3.2 and (3.65) the following theorem

Theorem 3.4.2. *With the same assumptions of Theorem 1.3.2, let* σ_p *be the Gaussian (Fubini-Study) measure on* $H^0(X,L_p)\simeq\mathbb{C}^{d_p}$ *given by (3.61) (by 3.62). Then, for* $1\le k\le \dim_{\mathbb{C}}X$

$$\mathbb{E}\Big[\widehat{Z}_{\Sigma_p^k}\Big]\longrightarrow\omega^k \tag{3.67}$$

in the weak topology of currents as* $p\to\infty$*. In addition, if* $\sum_{p=1}^\infty \frac{1}{A_p^2}<\infty$*, then for* σ_∞^k-*almost every sequence* $\{\Sigma_p^k\}\in\mathcal{H}_\infty^k$ *we have*

$$\Big[\widehat{Z}_{\Sigma_p^k}\Big]\longrightarrow\omega^k \tag{3.68}$$

in the weak topology of currents as* $p\to\infty$*.*

When we consider the prequantum line bundle setting, where $(L_p,h_p)=(L^p,h^p)$ and $c_1(L,h)=\omega$ in Theorem 3.4.1 and Theorem 3.4.2, we recover the results of Shiffman-Zelditch ([Shiffman & Zelditch (1999)], [Shiffman & Zelditch (2008)]).

3.4.2 Area Measure of Spheres

Let $\mathcal{A}_n$ be the surface area measure on the unit sphere S^{2n-1} in $\mathbb{C}^n$, given by $\mathcal{A}_n=\frac{2\pi^n}{(n-1)!}$. Let us consider the following probability measure on S^{2n-1}

$$\sigma_n=\frac{1}{\mathcal{A}_n(S^{2n-1})}\mathcal{A}_n. \tag{3.69}$$

Given that σ_n is the normalized area measure on the unit sphere, in accordance with Lemma 4.11 from [Bayraktar et al. (2020)], for every $\alpha\ge1$, there exists a constant $C_\alpha>0$ such that for every integer $n\ge2$, we have:

$$\int_{\mathbb{C}^n}\Big|\log|\langle a,v\rangle|\Big|^\alpha d\sigma_n(a)\le C_\alpha(\log n)^\alpha,\ \ \forall v\in\mathbb{C}^n,\ ||v||=1. \tag{3.70}$$

One should remark that, in this specific case, even though the measure is unitary invariant, the aforementioned upper bound is not a universal constant.

Now, due to the fact that $C_\alpha(\log d_p)^\alpha\le C_\alpha((n+2)\log A_p)^\alpha$ for sufficiently large p, using Theorem 1.3.1 with $C_p=C_\alpha((n+2)\log A_p)^\alpha$ leads to the following estimate.

Theorem 3.4.3. *Under the same assumptions of Theorem 1.3.1, let $\sigma_p := \sigma_{d_p}$ be the normalized area measure on the unit sphere of $H^0(X, L_p) \simeq \mathbb{C}^{d_p}$ given by (3.69). Then for any $\phi \in \mathcal{D}^{n-k,n-k}(X)$ and sufficiently large p, one has*

$$\Big\langle \mathrm{Var}\Big[\widehat{Z}_{\Sigma_p^k}\Big], \phi \Big\rangle \leq \Big(\frac{\log A_p}{A_p}\Big)^2 \Lambda_{k,n,\alpha} B_\phi^2,$$

where $\Lambda_{k,n,\alpha} = (2^{k-1}(n+2)\mathrm{Vol}(X)\ C_\alpha^{1/\alpha})^2$ is a positive constant.

Consequently, we have the equidistribtion theorem.

Theorem 3.4.4. *Under the same assumptions of Theorem 1.3.2, let σ_p be the normalized area measure on the unit sphere of $H^0(X, L_p) \simeq \mathbb{C}^{d_p}$ given by (3.69). Then, for $1 \leq k \leq \dim_{\mathbb{C}} X$*

$$\mathbb{E}\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k \tag{3.71}$$

in the weak topology of currents as $p \to \infty$. In addition, if $\sum_{p=1}^{\infty} \big(\frac{\log A_p}{A_p}\big)^2 < \infty$, then for σ_∞^k-almost every sequence $\{\Sigma_p^k\}_{p\geq 1} \in \mathcal{H}_k^\infty$ we have*

$$\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k \tag{3.72}$$

in the weak topology of currents as $p \to \infty$.*

3.4.3 Random Holomorphic Sections with i.i.d. coefficients

In this context, we examine the probability space $(H^0(X, L_p), \sigma_p)$ where σ_p is the product probability measure induced by the probability distribution law $\mathbb{P}$ governing the i.i.d. random coefficients a_j^p in the representation (3.7). This distribution possesses a bounded density $\psi : \mathbb{C} \to [0, M]$, and satisfies the property that there exist constants $\epsilon > 0$ and $\delta > 1$, such that

$$\mathbb{P}\Big(\{z \in \mathbb{C} : \log|z| > R\}\Big) \leq \frac{\epsilon}{R^\delta}, \text{ for all } R \geq 1. \tag{3.73}$$

This particular density type has been investigated in [Bayraktar (2016)] and [Bayraktar et al. (2020)], and it encompasses distributions such as the real or complex Gaussian distributions. Given such a measure σ_p on $H^0(X, L_p)$, according to Lemma 4.15 of [Bayraktar et al. (2020)] we have the following for any $1 \le \alpha < \delta$:

$$\int_{\mathbb{C}^{d_p}} \Big| \log |\langle a, v \rangle| \Big|^{\alpha} d\sigma_p(a) \le B\, d_p^{\alpha/\delta}, \ \forall v \in \mathbb{C}^{d_p}, \ ||v|| = 1; \tag{3.74}$$

where $B = B(M, \epsilon, \delta, \alpha) > 0$. In our present setting, for p sufficiently large, $d_p \le M_0 Vol(X) A_p^n$. Using Theorem 1.3.1 with $C_p = D A_p^{n\frac{\alpha}{\delta}}$, where $D = B\, (M_0 Vol(X))^{\alpha/\delta}$ we obtain

Theorem 3.4.5. *Under the same assumptions of Theorem 1.3.1, if σ_p is the probability measure on $H^0(X, L_p) \simeq \mathbb{C}^{d_p}$ defined as above. Then for any $\phi \in \mathcal{D}^{n-k,n-k}(X)$ and sufficiently large p, one has*

$$\Big\langle \mathrm{Var}\Big[\widehat{Z}_{\Sigma_p^k}\Big], \phi \Big\rangle \le \Big(\frac{1}{A_p^{1-n/\delta}} \Big)^2 \Big(2^{k-1} D^{1/\alpha} \mathrm{Vol}(X) B_\phi \Big)^2,$$

where $D = (M_0 \mathrm{Vol}(X))^{\alpha/\delta} B$ is a positive constant.

As a consequence, we have the following equidistribution result;

Theorem 3.4.6. *Let $(L_p, h_p)_{p \ge 1}$, (X, ω) be as in Theorem 1.3.2. Assume that σ_p is the probability measure on $H^0(X, L_p)$ defined as above. If $\delta > n$ then for $1 \le k \le \dim_{\mathbb{C}} X$*

$$\mathbb{E}\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k \tag{3.75}$$

in the weak topology of currents as $p \to \infty$. In addition, if $\sum_{p=1}^{\infty} \frac{1}{A_p^{2-2n/\delta}} < \infty$, where $\delta > 2n$, then almost surely*

$$\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k \tag{3.76}$$

in the weak topology of currents as $p \to \infty$.*

3.4.4 Locally moderate measures

Consider a complex manifold X and a positive measure σ on X. In accordance with [Dinh, Nguyen & Sibony (2010)], we define σ as a locally moderate measure if, for any open set $U \subset X$, any compact set $K \subset U$, and any compact family $\mathcal{F}$ of plurisubharmonic functions on U, there exist positive constants M and β such that

$$\int_K e^{-\beta\varphi} d\sigma \le M, \quad \text{for all } \varphi \in \mathcal{F}. \tag{3.77}$$

It is evident that σ does not charge pluripolar sets. Furthermore, it is noteworthy to mention that significant examples of such measures arise from the Monge-Ampère measures associated with Hölder continuous plurisubharmonic functions, for more details in this direction check [Dinh et al. (2010)]. According to [Bayraktar et al. (2020), Lemma 4.16], if σ_p is a locally moderate probability measure with compact support in $\mathbb{C}^{d_p} \cong H^0(X, L_p)$, then for every $\alpha \geq 1$

$$\int_{\mathbb{C}^{d_p}} \Big|\log|\langle a, v\rangle|\Big|^{\alpha} d\sigma_p(a) \leq \Lambda_p R_p^{2\beta_p}, \ \ \forall v \in \mathbb{C}^{d_p}, \ ||v|| = 1; \tag{3.78}$$

where $\Lambda_p, \beta_p > 0$ are positive constants and $R_p \geq 1$ such that $\|a\| \leq R_p$ for all $a \in \operatorname{supp} \sigma_p$. Continuing in the same manner as the previous examples, we deduce the following results.

Theorem 3.4.7. *Under the same assumptions of Theorem 1.3.1, if σ_p is a locally moderate probability measure with compact support in $\mathbb{C}^{d_p} \cong H^0(X, L_p)$. Then for any $\phi \in \mathcal{D}^{n-k,n-k}(X)$ and sufficiently large p, one has*

$$\Big\langle \mathrm{Var}\Big[\widehat{Z}_{\Sigma_p^k}\Big], \phi \Big\rangle \leq \frac{(\Lambda_p R_p^{2\beta_p})^{2/\alpha}}{A_p^2} (2^{k-1}\mathrm{Vol}(X) B_\phi)^2,$$

where $\Lambda_p, \beta_p > 0$ are positive constants and $R_p \geq 1$.

Theorem 3.4.8. *Let $(L_p, h_p)_{p\geq 1}$, (X,ω) be as in Theorem 1.3.2. Assume that σ_p is the locally moderate probability measure on $H^0(X, L_p)$ defined as above.*

(i) If $\lim_{p\to\infty} \frac{(\Lambda_p R_p^{2\beta_p})^{1/\alpha}}{A_p} = 0$ then for $1 \leq k \leq \dim_{\mathbb{C}} X$

$$\mathbb{E}\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k$$

in the weak topology of currents as $p \to \infty$.*

(ii) If $\sum_{p=1}^{\infty} \frac{(\Lambda_p R_p^{2\beta_p})^{2/\alpha}}{A_p^2} < \infty$, then for σ_∞^k-almost all $\{\Sigma_p^k\}_{p\geq 1} \in \mathcal{H}_k^\infty$

$$\Big[\widehat{Z}_{\Sigma_p^k}\Big] \longrightarrow \omega^k$$

in the weak topology of currents as $p \to \infty$.*

In [Bayraktar et al. (2020)], various significant probability measures are thoroughly examined, including small ball probability measures, among others. For a more in-depth exploration of such measures, refer to [Section 4, [Bayraktar et al. (2020)]].

3.5 Probabilistic Bertini Theorem

In this section, by adapting the proof of Proposition 3.2 in [Coman, Marinescu & Nguyen (2016)], we present a general version of Bertini's theorem applicable to any probability measure σ_d defined on $\mathbb{C}^d$ that assigns zero mass to pluripolar subsets of $\mathbb{C}^d$. As a result, we demonstrate that the intersection current is almost surely well-defined with respect to the product measure induced by σ_d.

Recall that the analytic subsets $A_1, \ldots, A_k,\ k \leq n$ of a compact complex manifold X of dimension n are said to be in general position if $\operatorname{codim} A_{i_1} \cap \ldots \cap A_{i_m} \geq m$ for every $1 \leq m \leq k$ and $1 \leq i_1 < \ldots < i_m \leq k$.

Proposition 3.5.1. *Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. Suppose the following:*

(i) V is a subspace of $H^0(X,L)$ with a basis $\{S_1, \ldots, S_d\}$, and the base locus $Bs(V) = \{x \in X : S_1(x) = \cdots = S_d(x) = 0\} \subseteq X$ such that $\dim Bs(V) \leq n-k$.

(ii) $Z(t) = \{x \in X : \sum_{j=1}^{d} t_j S_j(x) = 0\}$, where $t = (t_1, \ldots, t_d) \in \mathbb{C}^d$.

If σ_d^k is the product measure on $(\mathbb{C}^d)^k$ induced by the probability measure σ_d on $\mathbb{C}^d$ coming from the identification $V \simeq \mathbb{C}^d$, then analytic sets $Z(t^1), \ldots, Z(t^k)$ are in general position for σ_d^k- almost every $(t^1, \ldots, t^k) \in (\mathbb{C}^d)^k$.

Proof. The proof will be based on induction on k. Let

$$H_k := \{(t^1, \ldots, t^k) \in (\mathbb{C}^d)^k : \dim Z(t^1) \cap \ldots \cap Z(t^k) \cap \mathrm{Bs}(V) \leq n-k\}. \tag{3.79}$$

We start with the case $k = 1$. If $Z(t^1) \cap \mathrm{Bs}(V) = \emptyset$, then $Z(t^1)$ is a complex submanifold of dimension $n-1$ whatever $t^1 \in \mathbb{C}^d$ is chosen, that is $\{t^1 \in \mathbb{C}^d : \dim Z(t^1) \leq n-1\} = \mathbb{C}^d$. If $Z(t^1) \cap \mathrm{Bs}(V) \neq \emptyset$, corresponding to the set $H_1 = \{t^1 \in \mathbb{C}^d : \dim Z(t^1) \cap \mathrm{Bs}(V) \leq n-1\}$, we consider the decomposition of the base locus

$$\mathrm{Bs}(V) = \bigcup_{l=1}^{N_0} E_l \cup Y, \tag{3.80}$$

where E_l are the irreducible components of $\mathrm{Bs}(V)$ with $\dim E_l = n-1$ and $\dim Y \leq n-2$. The set $\{t^1 \in \mathbb{C}^d : E_l \subset Z(t^1)\}$ has to be a proper algebraic subvariety of $\mathbb{C}^d$, because, if $E_l \subset Z(t^1)$ for all $t^1 \in \mathbb{C}^d$, then since $E_l \subset \mathrm{Bs}(V)$, we must have that $\dim Z(t^1) \geq \dim Z(t^1) \cap \mathrm{Bs}(V) \geq n-1$, which is a contradiction since any single analytic variety is always in general position. Therefore, for any point $t^1 \in \mathbb{C}^d \backslash H_1$, we get $\dim Z(t^1) \cap \mathrm{Bs}(V) \geq n-1$. Since $Z(t^1) \cap \mathrm{Bs}(V)$ is an analytic subvariety of $\mathrm{Bs}(V)$, there exists $l \in \{1, \ldots, N_0\}$ such that $E_l \subset Z(t^1) \cap \mathrm{Bs}(V)$, so we can write

$$\mathbb{C}^d \backslash H_1 = \bigcup_{l=1}^{N_0} \{t^1 \in \mathbb{C}^d : E_l \subset Z(t^1)\}.$$

Since σ_d puts no mass on pluripolar sets, it follows that $\sigma_d(\mathbb{C}^d \backslash H_1) = 0$, which completes the initial step for induction. Suppose that $\sigma_d^k(H_k) = 1$ for all H_k defined as in (3.79). Let

(3.81)
$$H_{k+1} = \{(t^1, \ldots, t^{k+1}) \in (\mathbb{C}^d)^{k+1} : \dim Z(t^1) \cap \cdots \cap Z(t^{k+1}) \cap \mathrm{Bs}(V) \leq n-k-1\}.$$

We need to show that $\sigma_d^{k+1}(H_{k+1}) = 1$. For this purpose, we show that the σ_d^{k+1}-measure of the complement set H_{k+1}^c is zero. First, let us fix $t = (t^1, \ldots, t^k) \in H_k$. Define, $Z(t) := Z(t^1) \cap \ldots \cap Z(t^k)$ and

(3.82) $$G(t) := \{t^{k+1} \in \mathbb{C}^d : \dim Z(t) \cap \mathrm{Bs}(V) \cap Z(t^{k+1}) \geq n-k-1\}.$$

It will suffice to prove that $\sigma_d(G(t)) = 0$. These sets $G(t)$ are called the slices of the set H_{k+1}^c. Let

(3.83) $$Z(t) \cap \mathrm{Bs}(V) = \bigcup_{k=1}^{N_0} E_l \cup Y,$$

where E_l are, as in the case $k=1$, the irreducible components of $Z(t) \cap \mathrm{Bs}(V)$ with $\dim E_l = n-k$ and $\dim Y = n-k-1$. If $t^{k+1} \in G(t)$, then $Z(t) \cap Z(t_{k+1}) \cap \mathrm{Bs}(V)$ is an analytic subset of $Z(t) \cap \mathrm{Bs}(V)$ with $\dim Z(t) \cap Z(t_{k+1}) \cap \mathrm{Bs}(V) = n-k$, and this gives that there is some $l \in \{1, \ldots, N_0\}$ such that

(3.84) $$E_l \subset Z(t) \cap Z(t_{k+1}) \cap \mathrm{Bs}(V).$$

Hence, we have

(3.85) $$G(t) = \bigcup_{l=1}^{N_0} A_l(t),\ A_l(t) := \{t^{k+1} \in \mathbb{C}^d : E_l \subset Z(t^{k+1})\}.$$

Now we see that not all sections become zero on E_l. Indeed, if it were not so, by arguing as in the case $k=1$ above, $E_l \subset \mathrm{Bs}(V)$ would imply that $\dim Z(t) \cap \mathrm{Bs}(V) \geq n-k$, contradicting $t \in H_k$. We may then assume that $S_d \not\equiv 0$ on E_l. As before, by examining, this time, the slices of $A_l(t)$, for any $(t_1^{k+1}, \ldots, t_{d-1}^{k+1}) \in \mathbb{C}^{d-1}$, there exist at most one $h \in \mathbb{C}$ such that

$(t_1^{k+1},\dots,t_{d-1}^{k+1},h) \in A_l(t)$, otherwise, if there exist two different elements $h,h' \in \mathbb{C}$ with this property, we have

$$t_1^{k+1}S_1+\ldots+t_{d-1}^{k+1}S_{d-1}+hS_d=0$$
$$t_1^{k+1}S_1+\ldots+t_{d-1}^{k+1}S_{d-1}+h'S_d=0,$$

which immediately gives that $S_d \equiv 0$ on E_l, which is a contradiction. Thus, $\sigma_d(A_l(t))=0$. This implies that $\sigma_d(G(t))=0$, which finishes the proof. □

In our setting, because of the relation (3.5), there exists $p_0 \in \mathbb{N}$ such that $\mathrm{Bs}(H^0(X,L_p))=\emptyset$ for all $p>p_0$. Therefore, by using the arguments from Lemma 3.1 in [Coman et al. (2023)] which is based on the results of Demailly (Corollary 2.11 and Proposition 2.12 in [Demailly (1993)], as a result of Proposition 3.5.1, we arrive at the following proposition

Proposition 3.5.2. *There exists $p_0 \in \mathbb{N}$ such that for all $p>p_0$,*

(i) The analytic subvarietes $Z_{s_p^1},\dots,Z_{s_p^k}$ are all in general position for σ_p^k-almost all $(s_p^1,\dots,s_p^k) \in H^0(X,L_p)^k$.

(ii) For σ_p^k-almost every $\Sigma_p^k=(s_p^1,\dots,s_p^k) \in H^0(X,L_p)^k$, the analytic subvariety $Z_{s_p^{j_1}} \cap \dots \cap Z_{s_p^{j_l}}$ is of pure dimension $n-l$ for each $1 \le l \le k$ and $1 \le j_1 < \dots < j_l \le k$.

(iii) The intersection current $[Z_{\Sigma_p^k}] := [Z_{s_p^1}] \wedge \dots \wedge [Z_{s_p^k}]$ is well-defined and is equal to the current of integration with multiplicities over the common zero set $Z_{\Sigma_p^k}$.

Proof. By (3.5) there exists $p_0 \in \mathbb{N}$ such that $K_p(x)>0$ for all $x \in X$ and $p>p_0$, hence $Bs(H^0(X,L_p))=\emptyset$ for all $p>p_0$. Now using Proposition 3.5.1, by taking $V=H^0(X,L_p) \cong \mathbb{C}^{d_p}$ with the measure $\sigma_p := \sigma_{d_p}$, and fixing an orthonormal basis $\{S_1^p,\dots,S_{d_p}^p\}$, we have that for σ_p^k- almost every $(s_p^1,\dots,s_p^k) \in H^0(X,L_p)^k$, the analytic hypersurfaces $Z_{s_p^1},\dots,Z_{s_p^k}$ are in general position. Thus, $Z_{s_p^{j_1}} \cap \dots \cap Z_{s_p^{j_l}}$ has dimension at most $n-l$ for each $1 \le l \le k$ and $1 \le j_1 < \dots < j_l \le k$, which proves (i). Now let

$$T := [Z_{s_p^{j_1}}] \wedge \dots \wedge [Z_{s_p^{j_l}}]$$

then by the part (i) and [Demailly (1993), Corollary 2.11], T is a well-defined positive-closed current of bidegree (l,l), supported in the set $Z_{s_p^{j_1}} \cap \dots \cap Z_{s_p^{j_l}}$. Moreover, by Poincaré-Lelong formula we know that for each $s_p \in H^0(X,L_p)$, the cohomology class of $[Z_{s_p}]$ is the same as $c_1(L_p,h_p)$. Then using the diophantine relation we obtain

$$\int_X T \wedge \omega^{n-l} = \int_X c_1(L_p, h_p)^l \wedge \omega^{n-l} > 0.$$

Hence, $Z_{s_p^{j_1}} \cap \cdots \cap Z_{s_p^{j_l}} \neq \emptyset$ and has pure dimension $n-l$, which shows (ii). Finally, the last assertion follows from [Demailly (1993), Corollary 2.11, Proposition 2.12]. $\square$

4. Central limit theorem for random zero divisors

In this chapter, we prove a central limit theorem for random zero currents related to the zero divisors of standard Gaussian holomorphic sections in a sequence of holomorphic line bundles with Hermitian metrics of class $\mathscr{C}^3$ over a compact Kähler manifold.

4.1 Reference Covers

Let (X,ω) be a compact Kähler manifold, with $\dim_{\mathbb{C}} X = n$, and let $\{(L_p,h_p)\}_{p\geq 1}^{\infty}$ be a sequence of positive line bundles with Hermitian $\mathscr{C}^3$-metrics whose curvatures satisfy the following diophantine approximation relation:

$$\frac{1}{A_p} c_1(L_p,h_p) = \omega + O(A_p^{-a}) \text{ in the } \mathscr{C}^0\text{-topology as } p \to \infty, \tag{4.1}$$

where $a>0$, $A_p>0$ and $\lim_{p\to\infty} A_p = \infty$.

In order to measure the distance between any two points x,y on the compact Kähler manifold (X,ω), we use the Riemannian distance, which is defined as follows: As it is well-known, the Kähler form ω and the complex structure J on X compatible with ω determine a Riemannian metric g on X by $g(u,v) := \omega(u,Jv)$ for all $u,v \in TX$. Given a piecewise $\mathscr{C}^1$ curve $\gamma : [a,b] \to X$ with $\gamma(a) = x$ and $\gamma(b) = y$, the length $L(\gamma)$ of the curve γ is given by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}dt$$

and the Riemannian distance d is defined by

$$d(x,y) = \inf\{L(\gamma) : \gamma(a) = x,\ \gamma(b) = y\}.$$

Throughout, (U,z), $z=(z_1,\dots,z_n)$, will indicate local coordinates centered at a point $x\in X$. The closed polydisk around $y\in U$ of equilateral radius $(r,\dots,r), r>0$, is given by

$$P^n(y,r):=\{z\in U:|z_j-y_j|\le r,\ j=1,2,\dots,m\}.$$

The coordinates (U,z) are said to be Kähler at $y\in U$ in case

$$\omega_z=\frac{i}{2}\sum_{j=1}^{m}dz_j\wedge\overline{dz_j}+O(|z-y|^2)\sum_{j,k}dz_j\wedge d\overline{z_k}\text{ on }U. \tag{4.2}$$

Definition 4.1.1. *A reference cover of X is defined as follows: for $j=1,2,\dots,N$, a set of points $x_j\in X$ and*

(a) Stein open simply connected coordinate neighborhoods $(U_j,w^{(j)})$ centered at $x_j\equiv 0$.

(b) $R_j>0$ such that $P^n(x_j,2R_j)\Subset U_j$ and for every $y\in P^n(x_j,2R_j)$ there exist coordinates on U_j which are Kähler at y.

(c) $X=\bigcup_{j=1}^N P^n(x_j,R_j)$.

We will write $R=\min R_j$ once a reference cover is provided.

It is not difficult to see how one can construct a reference cover. Indeed, first, for $x\in X$, take a Stein open simply connected neighborhood (for instance, a round ball in $\mathbb{C}^n$) U of $0\in\mathbb{C}^n$, where $x\equiv 0$ under a determined chart. Choose some $R>0$ so that $P^n(x,R)\Subset U$ and for every $y\in P^n(x,R)$ there exist Kähler coordinates (U,z) at y. The compactness of X implies that there exist finitely many points $\{x_j\}_{j=1}^N$ such that the three conditions above are satisfied.

We take into consideration the differential operators D^α_w, $\alpha\in\mathbb{N}^{2n}$ on U_j, corresponding to the real coordinates associated to $w=w^j$. For $\varphi\in\mathscr{C}^k(U_j)$, we define

$$\|\varphi\|_k=\|\varphi\|_{k,w}=\sup\{|D^\alpha_w\varphi(w)|:w\in P^n(x_j,2R_j),\,|\alpha|\le k\}.$$

Let (L,h) be a Hermitian holomorphic line bundle on X, i.e., the metric h is smooth. For $k\le l$, write

$$\|h\|_{k,U_j}=\inf\{\|\varphi_j\|_k:\varphi_j\in\mathscr{C}^l(U_j)\text{ is a weight of }h\text{ on }U_j\},$$

and

$$\|h\|_k=\max\{1,\|h\|_{k,U_j}:1\le j\le N\}.$$

φ_j is said to be a weight of h on U_j if there exists a holomorphic frame e_j of L on

U_j such that $|e_j|_h = e^{-\varphi_j}$.

Lemma 4.1.1. *Let a reference cover of X be given. Then there exists a constant $D > 1$ relying on the reference cover with the following property: When provided with any Hermitian line bundle (L,h) on X, any $j \in \{1,\dots,N\}$ and any $x \in P^n(x_j,R_j)$, there exist coordinates $z = (z_1,\dots,z_n)$ on $P^n(x,R)$ which are centered at $x \equiv 0$ and Kähler coordinates for x such that*

(i) $dV \le (1+Dr^2)\frac{\omega^n}{n!}$ *and* $\frac{\omega^n}{n!} \le (1+Dr^2)dV$ *hold on $P^n(x,r)$ for any $r < R$ where $dV = dV(z)$ is the Euclidean volume relative to the coordinates z,*

(ii) (L,h) *has a weight φ on $P^n(x,R)$ with $\varphi(z) = \Re t(z) + \sum_{j=1}^n \lambda_j |z_j|^2 + \widetilde{\varphi}(z)$, where t is a holomorphic polynomial of degree at most 2, $\lambda_j \in \mathbb{R}$ and $|\widetilde{\varphi}(z)| \le D'\|h\|_3|z|^3$ for $z \in P^n(x,R)$.*

Proof. By the definition of a reference cover, there exist coordinates z on U_j which are Kähler for $x \in P^n(x_j,R_j)$. Then, $\omega = \sum_{l=1}^n dz_l \wedge d\overline{z}_l + O(|z-x|^2)\sum_{j,k} dz_j \wedge d\overline{z_k}$ and (i) holds with a constant D_j uniform for $x \in P^n(x_j,R_j)$. Let e_j be a frame of L on U_j, φ a weight of h on U_j with $|e_j|_h = e^{-\varphi}$ and $||\varphi||_{3,z} \le 2||h||_3$. By translation, we may assume $x = 0$ and write $\varphi(z) = \Re t(z) + \varphi_2(z) + \varphi_3(z)$, where $t(z)$ is a holomorphic polynomial of degree ≤ 1 in z, $\varphi_2(z) = \sum_{k,l=1}^n \mu_{kl} z_k \overline{z}_l$ and $\Re f(z) + \varphi_2(z)$ is the Taylor polynomial of order 2 of φ at 0. In order to estimate $\varphi_3(z)$, let $\|\varphi\|_{3,z}$ be the supremum norm of the derivatives of φ of order 3 on $P^n(x_j,R_j)$ in the z-coordinates. Then, by (4.1), there exists a constant D'_j being uniform on $P^n(x_j,R_j)$ such that $||\varphi||_{3,z} \le D'_j ||\varphi||_{3,w} \le 2D'_j ||h||_3$, which also gives that $|\varphi_3(z)| \le 2D'_j \|h\|_3 |z|^3$ for all $z \in P^n(x,R)$.

Applying a unitary change of coordinates, we may suppose that $\varphi(\zeta) = \Re t(\zeta) + \sum_{j=1}^n \lambda_j^p |\zeta_j|^2 + \widetilde{\varphi}(\zeta)$. Under these coordinates, $\frac{w^n}{n!}$ and $\widetilde{\varphi}(\zeta)$ verify the required estimates with a uniform constant D_j for $x \in P^n(x_j,R_j)$, as unitary transformations preserve distances. Finally putting $D' = \max_{1 \le j \le N} D'_j$ finishes the proof. □

Now, we make the following observation, which will play a crucial role in the forthcoming theorems.

Let $\{U_j\}_{j=1}^N$ be a finite subcover of X. Locally, on each U_j, we have the following representations

$$\frac{1}{A_p} c_1(L_p,h_p)(z) = i \sum_{k,j} \frac{1}{\pi} \frac{1}{A_p} \alpha_{kj}(z) dz_k \wedge d\overline{z_j} \tag{4.3}$$

and

$$\omega(z) = i\sum_{k,j} \chi_{kj}(z) dz_k \wedge d\overline{z_j}. \tag{4.4}$$

Here, $[\chi_{kj}(z)]_{kj}$ is a positive definite Hermitian matrix because, on a Kähler manifold, by the local $\partial\bar{\partial}$- lemma, we always have a strictly plurisubharmonic local potential function ψ so that

$$\chi_{kj}(z) = \frac{\partial^2 \psi(z)}{\partial z_j \partial \overline{z_k}} \in \mathbb{R}, \text{ for each } 1 \leq k, j \leq n.$$

Similarly, since line bundles L_p are positive, by the definition of positivity, $[\alpha_{kj}(z)]_{kj} = \left[\frac{\partial^2 \varphi_p}{\partial z_j \partial \overline{z_k}}(z)\right]_{kj}$ is a positive definite Hermitian matrix , where φ_p is the corresponding local weight function for h_p. Note that in particular $\alpha_{kj}(z) \in \mathbb{R}$ for every $1 \leq k, j \leq n$.

Let us fix some U_l taken from the subcover. By the diophantine approximation condition (4.1) on $\overline{U_l}$, for any $\epsilon > 0$, there exists some $p_0 = p_0(\epsilon) \in \mathbb{N}$ such that, for all $p \geq p_0$,

$$-\epsilon \leq \frac{1}{\pi A_p} \alpha_{kj}^{(p)}(z) - \chi_{kj}(z) \leq \epsilon \tag{4.5}$$

for all $z \in \overline{U_l}$. Take, for example,

$$\epsilon = \frac{1}{4}(\min_{j,k} \min_{z \in \overline{U_j}} \chi_{kj}(z)).$$

Then (4.5) gives

$$\frac{3}{4}\chi_{kj}(z) \leq \frac{1}{\pi A_p} \alpha_{kj}^{(p)}(z) \leq \frac{5}{4}\chi_{kj}(z).$$

Summing this last inequality over $i dz_k \wedge d\overline{z_j}$, we have, for all $z \in \overline{U_l} \subset X$ and for all $p \geq p_0$

$$i\sum_{k,j} \frac{3\chi_{kj}(z)}{4} dz_k \wedge d\overline{z_j} \leq i\sum_{k,j} \frac{1}{\pi A_p} \alpha_{kj}^{(p)}(z) dz_k \wedge d\overline{z_j} \leq i\sum_{k,j} \frac{5\chi_{kj}(z)}{4} dz_k \wedge d\overline{z_j},$$

which concludes that

$$\frac{3\omega}{4} \leq \frac{1}{A_p} c_1(L_p, h_p) \leq \frac{5\omega}{4}, \tag{4.6}$$

for $p \geq p_0$. This will be useful in the proof of Theorem 4.3.1 and Theorem 4.3.2.

We also observe that, at the point $x \equiv 0$ where we have the Kähler coordinates by

(4.2), we have

$$\omega_x = i\sum_{j=1}^{n}\frac{1}{2}dz_j\wedge d\overline{z_j}.$$

Also, by using the local representation of $c_1(L_p,h_p)$ and Lemma 4.1.1,

$$c_1(L_p,h_p)_x = dd^c\varphi_p(0) = i\sum_{j=1}^{n}\frac{\lambda_j^p}{\pi}dz_j\wedge d\overline{z_j}. \tag{4.7}$$

Diophantine approximation (4.1) implies

$$\lim_{p\to\infty}\frac{\lambda_j^p}{\pi A_p} = \frac{1}{2} \text{ for } j = 1,2,\ldots,n, \tag{4.8}$$

which in turn gives

$$\lim_{p\to\infty}\frac{\lambda_1^p\ldots\lambda_n^p}{A_p^n} = (\frac{\pi}{2})^n. \tag{4.9}$$

4.2 Demailly's L^2-estimations for $\overline{\partial}$ -operator

Essential for proving both the upper decay estimate of the Bergman Kernel and the first order asymptotics of the Bergman kernel function in our current diophantine setting, we follow the approaches in [Coman et al. (2017)] and [Bayraktar et al. (2020)] to provide first certain L^2- estimations for solutions of the $\overline{\partial}$-equation, and then derive a weighted estimate for these solutions.

Theorem 4.2.1. *(Demailly (1982), Théorème 5.1) Let (X,ω) be a Kähler manifold with $\dim_{\mathbb{C}} X = n$ having a complete Kähler metric. Let (L,h) be a singular Hermitian holomorphic line bundles such that $c_1(L,h)\geq 0$. Then for any form $g\in L^2_{n,1}(X,L,loc)$ verifying*

$$\overline{\partial}g = 0,\ \int_X |g|_h^2\frac{\omega^n}{n!} < \infty, \tag{4.10}$$

there is $u\in L^2_{n,0}(X,L,loc)$ with $\overline{\partial}u = g$ such that

$$\int_X |u|_h^2\frac{\omega^n}{n!} \leq \int_X |g|_h^2\frac{\omega^n}{n!}. \tag{4.11}$$

Theorem 4.2.2. *Let X be a complete Kähler manifold with $\dim_{\mathbb{C}} X = n$ and let ω be a Kähler form (not necessarily complete) on X such that its Ricci form $\mathrm{Ric}_\omega \geq -2\pi T_0\omega$ on X for some constant $T_0 > 0$. Let (L_p, h_p) be a sequence of holomorphic line bundles on X with Hermitian metrics h_p of class $\mathscr{C}^3$ such that (4.1) holds and there is a $p_0 \in \mathbb{N}$ such that $A_p \geq 4T_0$ for all $p > p_0$. If $p > p_0$ and $f \in L^2_{0,1}(X, L_p, loc)$ satisfies $\overline{\partial} f = 0$ and $\int_X |f|^2_{h_p} \frac{\omega^n}{n!} < \infty$, then there exists $u \in L^2_{0,0}(X, L_p, loc)$ such that $\overline{\partial} u = f$ and $\int_X |u|^2_{h_p} \frac{\omega^n}{n!} \leq \frac{2}{A_p} \int_X |f|^2_{h_p} \frac{\omega^n}{n!}$.*

Proof. By the diophantine approximation relation (4.1), fix some $p_0 \in \mathbb{N}$ so that the assertions in the theorem are satisfied and also for all $p > p_0$, $\frac{3\omega}{2} \geq \frac{1}{A_p} c_1(L_p, h_p) \geq \frac{3\omega}{4}$. Let $L_p = F_p \otimes K_X$, where $F_p = L_p \otimes K_X^{-1}$. The canonical line bundle K_X is endowed with the metric h^{K_X} induced by ω. If $g_p = h_p \otimes h^{K_X^{-1}}$ is the induced metric on F_p, then, since $c_1(K_X, h_{K_X}) = -\frac{1}{2\pi}\mathrm{Ric}_\omega$ and $A_p \geq 4T_0$ for all $p > p_0$,

$$\begin{aligned}
\text{(4.12)} \qquad \frac{1}{A_p} c_1(F_p, g_p) &= \frac{1}{A_p} c_1(L_p, h_p) - \frac{1}{A_p} c_1(K_X, h^{K_X}) \\
&= \frac{1}{A_p} c_1(L_p, h_p) + \frac{1}{2\pi A_p} Ric_\omega \geq \frac{3\omega}{4} - \frac{\omega}{4} = \frac{\omega}{2} \geq 0
\end{aligned}$$

for all $p > p_0$. On the other hand, there exists a natural isometry,

$$\Psi = \sim : \Lambda^{0,q}(T^*(X)) \otimes L_p \to \Lambda^{n,q}(T^*(X)) \otimes F_p$$

by

$$\text{(4.13)} \qquad \Psi(s) = \tilde{s} = (w^1 \wedge \cdots \wedge w^n \wedge s) \otimes (w_1 \wedge \cdots \wedge w_n),$$

where $w_1, \dots, w_n$ is a local orthonormal frame of $T^{(1,0)}(X)$ and $\{w^1, \dots, w^n\}$ is the dual frame. This operator Ψ commutes with the action of $\overline{\partial}$. Now for a form $f \in L^2_{0,1}(X, L_p, loc)$ satisfying $\overline{\partial} f = 0$ and $\int_X |f|^2_{h_p} \frac{\omega^n}{n!} < \infty$, obviously, we have $\int_X \frac{2}{A_p} |f|^2_{h_p} \frac{\omega^n}{n!} < \infty$. By using the isometry Ψ, we can find $\Psi(f) = F \in L^2_{n,1}(X, F_p, loc)$ with $\overline{\partial} F = \overline{\partial}\Psi(f) = \Psi\overline{\partial} f = 0$ and $\int_X \frac{2}{A_p} |F|^2_{g_p} \frac{\omega^n}{n!} < \infty$ since isometries preserve the L^2-norm. By Theorem 4.2.1, there exists $\tilde{f} \in L^2_{n,0}(X, F_p, loc)$ such that $\overline{\partial}\tilde{f} = \frac{\sqrt{2}}{\sqrt{A_p}} F$ and $\int_X |\tilde{f}|^2_{g_p} \frac{\omega^n}{n!} \leq \int_X \frac{2}{A_p} |F|^2_{g_p} \frac{\omega^n}{n!}$. Taking $u := \Psi^{-1}\tilde{f}$ and $f = \Psi^{-1}(F)$ finishes the proof since Ψ^{-1} is an isometry as well. $\square$

Theorem 4.2.3. *Let (X,ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} X = n$ and let $\{(L_p,h_p)\}_{p\geq 1}$ be a sequence of holomorphic line bundles on X with $\mathscr{C}^3$ Hermitian metrics as before, such that the diophantine approximation condition (4.1) holds. Then there exists $p_0 \in \mathbb{N}$ such that if u_p are real-valued functions of class $\mathscr{C}^2$ on X such that*

$$\|\overline{\partial} u_p\|_{L^\infty(X)} \leq \frac{\sqrt{A_p}}{8},\ dd^c u_p \geq -\frac{A_p}{4}\omega, \tag{4.14}$$

then

$$\int_X |v|^2_{h_p} e^{2u_p}\frac{\omega^n}{n!} \leq \frac{16}{3A_p}\int_X |\overline{\partial} v|^2_{h_p} e^{2u_p}\frac{\omega^n}{n!} \tag{4.15}$$

holds for $p > p_0$ and for every $\mathscr{C}^1$-smooth section v of L_p which is orthogonal to $H^0(X,L_p)$ with respect to the inner product induced by h_p and ω^n.

Proof. As in the proof of Theorem 4.2.2, via the diophantine convergence assumption (4.1), we first fix some $p_0 \in \mathbb{N}$ so that for any (fixed) $p > p_0$, one can get $\frac{3\omega}{4} \leq \frac{1}{A_p}c_1(L_p,h_p) \leq \frac{3\omega}{2}$ and $A_p \geq 4T_0$. The main idea is to use Theorem 4.2.2. To this end, let us fix a constant $T_0 > 0$ so that $\mathrm{Ric}_\omega \geq -2\pi T_0\omega$ on X. Using the real-valued functions u_p given in the assumptions of theorem, we consider the metrics $g_p := e^{-2u_p}h_p$ on L_p. From (2.3.5) in [Ma & Marinescu (2007), (p. 98)] and the second relation in (4.14) yield the following

$$c_1(L_p,g_p) = c_1(L_p,h_p) + dd^c u_p \geq \frac{3A_p\omega}{4} - \frac{A_p\omega}{4} = \frac{A_p\omega}{2}.$$

If we define an inner product by using g_p in $L^2(X,L_p)$ as $(s_1,s_2)_{g_p} := \int_X \langle s_1,s_2\rangle_{g_p}\frac{\omega^n}{n!}$, we see, by the relation $g_p = e^{-2u_p}h_p$, for every $s \in H^0(X,L_p)$,

$$(e^{u_p}v,s)_{g_p} = \int_X \langle e^{2u_p}v,s\rangle_{g_p}\frac{\omega^n}{n!} = \int_X \langle v,s\rangle_{h_p}\frac{\omega^n}{n!} = 0$$

for every $\mathscr{C}^1$-smooth section v of L_p.

Write

$$\beta = \overline{\partial}(e^{2u_p}v) = e^{2u_p}(2\overline{\partial}u_p \wedge v + \overline{\partial} v). \tag{4.16}$$

Since $\overline{\partial}\beta = 0$ and by assumptions on u_p and v, it follows immediately that $\beta \in L^2_{0,1}(X,L_p,loc)$, so by Theorem 4.2.2, there exists $\tilde{v} \in L^2_{0,0}(X,L_p,loc)$ such that

$\overline{\partial}\tilde{v} = \beta$ and

$$\int_X |\tilde{v}|^2_{g_p} \frac{\omega^n}{n!} \leq \frac{2}{A_p} \int_X |\beta|^2_{g_p} \frac{\omega^n}{n!}. \tag{4.17}$$

Since $e^{2u_p}v$ is orthogonal to $H^0(X, L_p)$ for every $v \in \mathscr{C}^1(X, L_p)$, by writing $\tilde{v} = e^{2u_p}v + s$ for some $s \in H^0(X, L_p)$, one can observe

$$(e^{2u_p}v + s, e^{2u_p}v + s)_{g_p} = \int_X |\tilde{v}|^2_{g_p} \frac{\omega^n}{n!} = \int_X (|e^{2u_p}v|^2_{g_p} + |s|^2_{g_p}) \frac{\omega^n}{n!} \geq \int_X |e^{2u_p}v|^2_{g_p} \frac{\omega^n}{n!}. \tag{4.18}$$

From (4.17) and (4.18), we have

$$\int_X |e^{2u_p}v|^2_{g_p} \frac{\omega^n}{n!} \leq \int_X |\tilde{v}|^2_{g_p} \frac{\omega^n}{n!} \leq \frac{2}{A_p} \int_X |\beta|^2_{g_p} \frac{\omega^n}{n!}. \tag{4.19}$$

Let us now estimate $|\beta|^2_{g_p}$ from above. By (4.16) and the first upper bound in (4.14), we obtain the following

$$|\beta|^2_{g_p} = e^{2u_p}|2\overline{\partial}u_p \wedge v + \overline{\partial}v|^2_{h_p} \leq 2e^{2u_p}(4|\overline{\partial}u_p \wedge v|^2_{h_p} + |\overline{\partial}v|^2_{h_p} \leq 2e^{2u_p}(\frac{A_p}{16}|v|^2_{h_p} + |\overline{\partial}v|^2_{h_p}), \tag{4.20}$$

where, in the first estimation, we use an elementary inequality for norms: $|x+y|^2 \leq 2(|x|^2 + |y^2|)$. Finally, putting (4.20) into (4.19) finishes the proof. □

4.3 Bergman Kernel Estimations

In this section, we establish a first-order asymptotic behavior for Bergman kernels when restricted to the diagonal for sequences of positive line bundles. Additionally, we provide an exponential off-diagonal decay for $K_p(x, y)$ in the given context. Our proofs rely on papers [Coman et al. (2017)] and [Bayraktar et al. (2020)].

To initiate our analysis, we start by recalling fundamental properties of Bergman kernels.

Let $H^0(X, L_p)$ be the space of global holomorphic sections of L_p. In this context, unlike the equidistribution setting we consider an inner product on the space of smooth sections $\mathscr{C}^\infty(X, L_p)$, using the metric h_p and the Riemannian volume form

$\frac{\omega^n}{n!}$ on X (instead of ω^n). More precisely,

$$\langle s_1, s_2\rangle_p := \int_X \langle s_1(x), s_2(x)\rangle_{h_p} \frac{\omega^n}{n!}, \tag{4.21}$$

and the norm of a section s is given by $\|s\|_p^2 := \langle s, s\rangle_p$. We denote the dimension of this space as $d_p := \dim H^0(X, L_p)$ and consider $\mathcal{L}^2(X, L_p)$, which is the completion of the space of smooth sections $\mathscr{C}^\infty(X, L_p)$ under this norm, forming a Hilbert space of square-integrable sections of L_p. A normal family argument shows that $H^0(X, L_p)$ is a closed subspace of $\mathcal{L}^2(X, L_p)$

Next, we introduce the orthogonal projection operator $K_p : \mathcal{L}^2(X, L_p) \to H^0(X, L_p)$. The Bergman kernel, $K_p(x, y)$, turns out to be the integral kernel of this projection. If $\{S_j^p\}_{j=1}^{d_p}$ is an orthonormal basis for $H^0(X, L_p)$, by using the reproducing property of $K_p(x, y)$, we express $K_p(x, y)$ in terms of this basis as follows:

$$K_p(x, y) = \sum_{j=1}^{d_p} S_j^p(x) \otimes S_j^p(y)^* \in L_{p,x} \otimes L_{p,y}^*, \tag{4.22}$$

where $S_j^p(y)^* = \langle\, .\, , S_j^p(y)\rangle_{h_p} \in L_{p,y}^*$ is the metric dual of $S_j^p(y)$ with respect to h_p. As in the previous chapter, the restriction of the Bergman kernel to the diagonal of X is called the Bergman kernel function of $H^0(X, L_p)$, which we denote by $K_p(x) := K_p(x, x)$, and (4.22) becomes

$$K_p(x) = \sum_{j=1}^{d_p} |S_j^p(x)|_{h_p}^2. \tag{4.23}$$

The Bergman kernel function has the dimensional density property, namely

$$\int_X K_p(x) \frac{\omega^n}{n!} = d_p.$$

In addition, it satisfies the following variational principle

$$K_p(x) = \max\{|S(x)|_{h_p}^2 : S \in H^0(X, L_p),\ ||S||_p = 1\}. \tag{4.24}$$

This holds for every $x \in X$ for which $\varphi_p(x) > -\infty$, with φ_p denoting a local weight, for the metric h_p in the vicinity of x.

We also defined the normalized Bergman kernel

$$\widehat{\mathcal{K}}_p(x, y) := \frac{|K_p(x, y)|_{h_{p,x} \otimes h_{p,y}}}{K_p(x)^{1/2} K_p(y)^{1/2}},$$

which will be important throughout this chapter.

Theorem 4.3.1. *Let (X,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Let $\{(L_p,h_p)\}_{p\geq 1}^{\infty}$ be a sequence of holomorphic line bundles with Hermitian metrics h_p of class $\mathscr{C}^3$ such that (4.1) holds. Assume that $\eta_p = \frac{\|h_p\|_3}{\sqrt{A_p}} \to 0$ as $p\to\infty$. Then we have*

$$\lim_{p\to\infty} \frac{K_p(x)}{A_p^n} = 1. \tag{4.25}$$

Proof. We begin by taking a reference cover of the Kähler manifold X, as in Definition 4.1.1. Selecting $x \in X$ and a corresponding z-coordinate system based on Lemma 4.1.1 at $x \in X$. Then

$$\varphi_p(z) = \Re t_p(z) + \varphi'_p(z) + \widetilde{\varphi_p}(z), \quad \varphi'_p(z) = \sum_{j=1}^{n} \lambda_j^p |z_j|^2, \tag{4.26}$$

φ_p is the weight for the Hermitian metric h_p on $P^n(x,R)$ satisfying the condition (ii) in Lemma 4.1.1 and t_p is the polynomial of degree at most 2. Let e_p be a local frame of L_p on U_j with the norm $|e_p|_{h_p} = e^{-\varphi_p}$. Next, we choose $R_p \in (0,R/2)$, which we will determine later.

To estimate the norm of a section $S \in H^0(X,L_p)$ at the point $x \equiv 0$, we consider $S = f\,e_p$, where f is a holomorphic function on $P^n(x,R)$. Utilizing the sub-averaging property for plurisubharmonic functions, we obtain:

(4.27)
$$|S(x)|^2_{h_p} = |f(0)e^{-t_p(0)}|^2 = |f(0)|^2 e^{-2\Re t_p(0)} \leq \frac{\int_{P^n(0,R_p)} |f|^2 e^{-2\Re t_p} e^{-2\varphi'_p} e^{-2\widetilde{\varphi_p}} \frac{\omega^n}{n!}}{\int_{P^n(0,R_p)} e^{-2\varphi'_p} e^{-2\widetilde{\varphi_p}} \frac{\omega^n}{n!}}.$$

For the right-hand side of (4.27), by Lemma 4.1.1, there exists a constant $D>0$ such that $-D\|h_p\|_3|z|^3 \leq \widetilde{\varphi_p}(z) \leq D\|h_p\|_3|z|^3$, and by considering (4.21) and (4.26) we have,

$$\begin{aligned}
\frac{\int_{P^n(0,R_p)} |f|^2 e^{-2\Re t_p} e^{-2\varphi'_p} e^{-2\widetilde{\varphi_p}} \frac{\omega^n}{n!}}{\int_{P^n(0,R_p)} e^{-2\varphi'_p} e^{-2\widetilde{\varphi_p}} \frac{\omega^n}{n!}} &\leq \frac{\int_{P^n(0,R_p)} |f|^2 e^{-2\varphi_p} \frac{\omega^n}{n!}}{\int_{P^n(0,R_p)} e^{-2\varphi'_p} e^{-2\widetilde{\varphi_p}} \frac{dV}{1+D\,R_p^2}} \\
&\leq \frac{(1+D\,R_p^2) e^{2D\|h_p\|_3 R_p^3} \|S\|_p^2}{\int_{P^n(0,R_p)} e^{-2\varphi'_p} dV}.
\end{aligned}$$

Combining the above inequality with (4.27) yields

$$(4.28)\quad |S(x)|^2_{h_p} = |f(0)e^{-t_p(0)}|^2 = |f(0)|^2 e^{-2\Re t_p(0)} \le \frac{(1+D\,R_p^2)e^{2D\,||h_p||_3 R_p^3}\,\|S\|_p^2}{\int_{P^n(0,R_p)} e^{-2\varphi'_p} dV}.$$

Let us now estimate the integral in the denominator of (4.28). To do this, we consider the Gaussian-type integrals of finite radius,

$$(4.29)\qquad F(\rho) := \int_{|\xi|\le\rho} e^{-2|\xi|^2} dm(\xi) = \frac{\pi}{2}(1-e^{-2\rho^2}),$$

where dm is the Lebesgue measure on $\mathbb{C}$. It is easy to see that F is an increasing function of ρ. We also write

$$(4.30)\qquad F(\infty) := \lim_{\rho\to\infty}\int_{|\xi|\le\rho} e^{-2|\xi|^2} dm(\xi) := \int_{\mathbb{C}} e^{-2|\xi|^2} dm(\xi) = \frac{\pi}{2}.$$

Since

$$(4.31)\qquad \int_{P^n(0,R_p)} e^{-2\varphi'_p} dV = \int_{P^n(0,R_p)} e^{-2(\lambda_1^p|z_1|^2+\ldots+\lambda_n^p|z_n|^2)} dV(z),$$

it is enough to treat the integral

$$(4.32)\qquad \int_{\Delta(0,R_p)} e^{-2\lambda_j^p|z_j|^2} dm(z_j)$$

in order to get a lower bound for the integral (4.31), where $\Delta(0,R_p)$ is the unit closed disk in $\mathbb{C}$. By the relation (4.6), there exists $p_1\in\mathbb{N}$ such that, for all $p>p_1$,

$$(4.33)\qquad \frac{3A_p}{4}\omega_x \le c_1(L_p,h_p)_x \le \frac{5A_p}{4}\omega_x,$$

which, on account of (4.1) and (4.7), leads to

$$(4.34)\qquad \frac{3\pi A_p}{8} \le \lambda_j^p \le \frac{5\pi A_p}{8}.$$

Let us go back to the integral (4.31),

$$\int_{\Delta(0,R_p)=\{|z_j|\le R_p\}} e^{-2\lambda_j^p|z_j|^2} dm(z_j),$$

which, by a change of variable $\left(\sqrt{\lambda_j^p}z_j = w_j\right)$, equals the following

$$\frac{1}{\lambda_j^p}\int_{\{|w_j|\le R_p\sqrt{\lambda_j^p}\}} e^{-2|w_j|^2} dm(w_j).$$

Now, (4.34) gives $\sqrt{\frac{\lambda_j^p}{A_p}} > \sqrt{\frac{3\pi}{8}} > 1$, which gives $R_p\sqrt{A_p} \le R_p\sqrt{\lambda_j^p}$. Combining this

with the fact that F is increasing, we get
(4.35)
$$\int_{\Delta(0,R_p)=\{|z_j|\le R_p\}} e^{-2\lambda_j^p|z_j|^2}dm(z_j)=\frac{1}{\lambda_j^p}\int_{\left\{|w_j|\le R_p\sqrt{\lambda_j^p}\right\}} e^{-2|w_j|^2}dm(w_j)\ge\frac{F(R_p\sqrt{A_p})}{\lambda_j^p}.$$

Consequently, from (4.31), we have

$$\int_{P^n(0,R_p)} e^{-2\varphi_p'}dV\ge\frac{F(R_p\sqrt{A_p})^n}{\lambda_1^p\dots\lambda_n^p}.$$

Inserting this last inequality in (4.28) give

$$|S(x)|_{h_p}^2\le\frac{(1+DR_p^2)e^{2D\,\|h_p\|_3\,R_p^3}}{F(R_p\sqrt{A_p})^n}\lambda_1^p\dots\lambda_n^p\,\|S\|_p^2. \tag{4.36}$$

If we take the supremum in (4.36) for all $S\in H^0(X,L_p)$ with $\|S\|_p=1$ and use the variational principle (4.24) for K_p, we get

$$K_p(x)\le\frac{(1+DR_p^2)e^{2D\,\|h_p\|_3\,R_p^3}}{F(R_p\sqrt{A_p})^n}\lambda_1^p\dots\lambda_n^p \tag{4.37}$$

for any $R_p\in(0,\frac{R}{2})$.

We will now determine a lower bound for K_p by employing L^2 estimations obtained earlier as Theorem 4.2.2. Let $\kappa:\mathbb{C}^n\to[0,1]$ be a cut-off function with a compact support in $P^n(0,2)$, $\kappa=1$ on $P^n(0,1)$. By defining $\kappa_p(z):=\kappa(\frac{z}{R_p})$, we consider $H=\kappa_p e^{t_p}e_p$, which is a (smooth) section of L_p, and $|H(x)|_{h_p}^2=|\kappa_p(x)|^2e^{2\Re t_p(x)}\,e^{-\varphi_p(x)}$. We estimate $\|H\|_p$ from above as follows:

$$\|H\|_p^2\le\int_{P^n(0,2R_p)} e^{2\Re t_p(x)}e^{-2\varphi_p(x)}\frac{\omega_x^n}{n!}=\int_{P^n(0,2R_p)} e^{-2\varphi_p'(x)}e^{-2\widetilde{\varphi}_p(x)}\frac{\omega_x^n}{n!}. \tag{4.38}$$

By using Lemma 4.1.1 (i) along with the relations (4.30) and (4.35) on the integral at the very right end of the inequality (4.38), we get the following

$$\begin{aligned}\|H\|_p^2&\le(1+4DR_p^2)e^{16D\|h_p\|_3\,R_p^3}\int_{P^n(0,2R_p)} e^{-2\varphi_p'}dV\\&\le(1+4D\,R_p^2)e^{16\,D\,\|h_p\|_3\,R_p^3}\left(\frac{\pi}{2}\right)^n\frac{1}{\lambda_1^p\cdots\lambda_n^p}.\end{aligned} \tag{4.39}$$

Let us define $\Phi=\overline{\partial}H$. Noting that $\|\overline{\partial}\kappa_p\|^2=\|\overline{\partial}\kappa\|^2/R_p^2$, where $\|\overline{\partial}\kappa\|$ is the supremum of $|\overline{\partial}\kappa|$, we deduce the following inequality:

$$\|\Phi\|_p^2 \le \int_{P^n(0,2R_p)} |\overline{\partial}\kappa_p|^2 e^{-2\varphi_p'} e^{-2\widetilde{\varphi}_p} \frac{\omega^n}{n!} \le \frac{\|\overline{\partial}\kappa\|^2}{R_p^2} (\frac{\pi}{2})^n \frac{(1+4D\,R_p^2)e^{16\,D\,\|h_p\|_3\,R_p^3}}{\lambda_1^p \dots \lambda_n^p}.$$

As $A_p \to \infty$, by using Theorem 4.2.2, there exists $p_0 \in \mathbf{N}$ such that, for all $p > p_0$, we can find a smooth section Γ of L_p as a solution to the $\overline{\partial}$-equation for Φ such that $\overline{\partial}\Gamma = \Phi = \overline{\partial}H$ and

$$\|\Gamma\|_p^2 \le \frac{2}{A_p}\|\Phi\|_p^2 \le \frac{2\|\overline{\partial}\kappa\|}{A_p\,R_p^2} (\frac{\pi}{2})^n \frac{(1+4D\,R_p^2)\,e^{(16D\,\|h_p\|_3 R_p^3)}}{\lambda_1^p \dots \lambda_n^p}. \tag{4.40}$$

Given that $H = e_p$ is holomorphic on $P^n(0,R_p)$, Γ is holomorphic on $P^n(0,R_p)$ as well since $\overline{\partial}\Gamma = \overline{\partial}H = 0$ on $P^n(0,R_p)$. Applying estimate (4.36) to Γ on $P^n(0,R_p)$ leads us to the following inequality

$$\begin{aligned} |\Gamma(x)|_{h_p}^2 &\le \frac{(1+D\,R_p^2)e^{2D\|h_p\|_3 R_p^3}}{F(R_p\sqrt{A_p})^n} \lambda_1^p \dots \lambda_n^p \|G\|_p^2 \\ &\le \frac{2\|\overline{\partial}\kappa\|^2}{A_p R_p^2 F(R_p\sqrt{A_p})^n} (\frac{\pi}{2})^n\,(1+4\,D\,R_p^2)^2\,e^{18\,D\,\|h_p\|_3\,R_p^3}. \end{aligned} \tag{4.41}$$

Now we will construct a new section $\Lambda := H - \Gamma \in H^0(X, L_p)$. Then, by a basic inequality $|S_1 - S_2|_{h_p}^2 \ge (|S_1|_{h_p} - |S_2|_{h_p})^2$ for norms applied to Λ, combined with (4.41) and the observation $|F(x)|_{h_p} = 1$, we get

$$\begin{aligned} |\Lambda(x)|_{h_p}^2 &\ge (|H(x)|_{h_p} - |\Gamma(x)|_{h_p})^2 \\ &\ge \Big(1 - (\frac{\pi}{2})^{n/2} \frac{\sqrt{2}\|\overline{\partial}\kappa\|(1+4\,D\,R_p^2)}{R_p\sqrt{A_p}F(R_p\sqrt{A_p})^{n/2}} e^{9\,D\,\|h_p\|_3\,R_p^3}\Big)^2. \end{aligned} \tag{4.42}$$

On the other hand, by (4.39) and (4.40) together with the triangle inequality, we obtain

$$\|\Lambda\|_p^2 \le (\|H\|_p + \|\Gamma\|_p)^2 \le (\frac{\pi}{2})^n \frac{1}{\lambda_1^p \dots \lambda_n^p} (1+4\,D\,R_p^2)\,e^{16\,D\,\|h_p\|_3\,R_p^3} \Big(1 + \frac{\sqrt{2}\|\overline{\partial}\kappa\|}{R_p\sqrt{A_p}}\Big)^2. \tag{4.43}$$

To simplify what we have done so far, we write

$$B_1(R_p) := \Big(1 - (\frac{\pi}{2})^{n/2} \frac{\sqrt{2}\|\overline{\partial}\kappa\|(1+4\,D\,R_p^2)}{R_p\sqrt{A_p}F(R_p\sqrt{A_p})^{n/2}} e^{9\,D\,\|h_p\|_3\,R_p^3}\Big)^2 \tag{4.44}$$

and

$$B_2(R_p) := (1+4\,D\,R_p^2)\,e^{16\,D\,\|h_p\|_3\,R_p^3}\Big(1+\frac{\sqrt{2}\|\overline{\partial}\kappa\|}{R_p\sqrt{A_p}}\Big)^2. \tag{4.45}$$

The variational property (4.24) combined with (4.42) and (4.43) implies

$$K_p(x) \geq \frac{|\Lambda(x)|^2_{h_p}}{\|\Lambda\|^2_p} \geq \frac{\lambda_1^p \dots \lambda_n^p}{(\frac{\pi}{2})^n}.\frac{B_1(R_p)}{B_2(R_p)}. \tag{4.46}$$

For the upper bound (4.37), as above, we put

$$B_3(R_p) := (\frac{\pi/2}{F(R_p\sqrt{A_p})})^n(1+D\,R_p^2)\,e^{2\,D\,\|h_p\|_3\,R_p^3}. \tag{4.47}$$

Observe that

$$(\frac{\pi}{2})^n K_p(x) \leq B_3(R_p)\,\lambda_1^p \dots \lambda_n^p. \tag{4.48}$$

By our hypothesis $\eta_p = \frac{\|h_p\|_3}{\sqrt{A_p}} \to 0$, and now we determine R_p in the following way

$$R_p := \eta_p^{1/3}\|h_p\|_3^{-1/3} = \frac{\eta_p^{-2/3}}{\sqrt{A_p}},$$

which means

$$\eta_p = \|h_p\|_3 R_p^3,\ \eta_p^{-2/3} = R_p\sqrt{A_p}.$$

Since $\|h_p\|_3 \geq 1$ from Subsection 4.1, we have $R_p \leq \eta_p^{1/3}$, and so $R_p \to 0$ when $p \to \infty$. All in all, based on R_p, it follows from the quantities $B_1(R_p)$, $B_2(R_p)$ and $B_3(R_p)$ that we find uniform upper and lower bounds for K_p depending only on η_p

$$\frac{B_1(R_p)}{B_2(R_p)} \geq 1 - D'\eta_p^{2/3} \text{ and } B_3(R_p) \leq 1 + D'\eta_p^{2/3}. \tag{4.49}$$

Here $D' > 0$ denotes a constant that merely depends on the reference cover. We finally consider the following inequality that holds for all $p > p_0$

$$\frac{1}{A_p^n}(\frac{\pi}{2})^n\frac{\lambda_1^p \dots \lambda_n^p}{(\frac{\pi}{2})^n}.\frac{B_1(R_p)}{B_2(R_p)} \leq (\frac{\pi}{2})^n\frac{K_p(x)}{A_p^n} \leq \frac{1}{A_p^n}B_3(R_p)\lambda_1^p \dots \lambda_n^p, \tag{4.50}$$

which, in light of the findings (4.9), (4.46), (4.48) and (4.49), finishes the proof.

□

Relying on the proof presented in [Bayraktar et al. (2020)], which incorporates methods from [Berndtsson (2003)], [Coman et al. (2017)] and [Lindholm (2001)], we provide a proof for the off-diagonal decay estimate of the Bergman kernels $K_p(x,y)$ associated with the corresponding line bundles (L_p,h_p) in the current setting of diophantine approximation.

Theorem 4.3.2. *Let (X,ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Let $\{(L_p,h_p)\}_{p\geq 1}$ be a sequence of holomorphic line bundles with Hermitian metrics h_p of class $\mathscr{C}^3$ such that (4.1) is satisfied. Write $\eta_p = \frac{\|h\|_3}{\sqrt{A_p}} \to 0$ when $p\to\infty$. Then there exist constants $G,B>0$, $p_0\geq 1$ such that for every $x,y\in X$ and $p>p_0$, the following estimation holds true*

$$|K_p(x,y)|^2_{h_p} \leq Ge^{-B\sqrt{A_p}d(x,y)}A_p^{2n}. \tag{4.51}$$

Proof. Initially, we select a reference cover for X in accordance with the earlier definition above and choose a large enough $p_0\in\mathbb{N}$ such that

$$R_p := \frac{1}{\sqrt{A_p}} < \frac{R}{2}$$

and Theorem 4.2.2 and Theorem 4.2.3 are valid for all $p>p_0$.

Let $y\in X$ and $r>0$. Write

$$B(y,r) := \{x\in X : d(y,x)<r\},$$

which is the ball of radius $r>0$ centered at y. Choose a constant $\theta\geq 1$ so that for any $y\in X$,

$$P^n(y,R_p)\subseteq B(y,\theta R_p),$$

where $P^n(y,R_p)$ is the (closed) polydisk centered at y given by the coordinates centered at y in view of Lemma 4.1.1.

<u>*Claim:*</u> There exists a constant $D'>1$ such that if $y\in X$, so $y\in P^n(x_j,R_j)$ for some j and z-coordinates centered at y are due to Lemma 4.1.1, then

$$|S(y)|^2_{h_p} \leq D'A_p^n\int_{P^n(y,R_p)}|S|^2_{h_p}\frac{\omega^n}{n!}, \tag{4.52}$$

where, as above, $P^n(y,R_p)$ is the (closed) polydisk centered at $y\equiv 0$ in the z-coordinates and S is an arbitrary continuous section of L_p on X which is holomorphic on $P^n(y,R_p)$.

Proof of the Claim: By Lemma 4.1.1(ii), (L_p, h_p) has a weight φ_p on $P^n(y,R)$ such that

$$\varphi_p(z) = \Re t_p(z) + \varphi_p'(z) + \widetilde{\varphi_p}(z), \tag{4.53}$$

where $t_p(z)$, $\varphi_p'(z) = \sum_{l=1}^{n} \lambda_l^p |z_l|^2$ and $\widetilde{\varphi_p}(z)$ satisfies the inequality

$$-D\|h_p\|_3 |z|^3 \le \widetilde{\varphi_p}(z) \le D\|h_p\|_3 |z|^3 \tag{4.54}$$

for $z \in P^n(y,R)$ (Recall that $R = \min R_j$). Let e_p be a frame of L_p on U_j so that $S = f\,e_p$, where f is a holomorphic function on $P^n(y,R_p)$ and $|e_p|_{h_p} = e^{-\varphi_p}$. As in the beginning of the proof of Theorem 4.3.1, we have first the relation, which is nothing but (4.28)

$$|S(y)|^2_{h_p} = |f(0)e^{-t_p(0)}|^2 = |f(0)|^2 e^{-2\Re t_p(0)} \le \frac{(1+D\,R_p^2)e^{2D\|h_p\|_3 R_p^3}\,\|S\|_p^2}{\int_{P^n(0,R_p)} e^{-2\varphi_p'} dV}. \tag{4.55}$$

Since

$$\int_{P^n(0,R_p)} e^{-2\varphi_p'} dV = \int_{P^n(0,R_p)} e^{-2(\lambda_1^p|z_1|^2+\ldots+\lambda_n^p|z_n|^2)} dV(z), \tag{4.56}$$

as we have done in the proof of Theorem 4.3.1, it will be sufficient for us to find a lower bound for the integral

$$\int_{\Delta(0,R_p)} e^{-2\lambda_j^p|z_j|^2} dm(z_j) \tag{4.57}$$

in order to get a lower bound for the whole integral (4.56), where $\Delta(0,R_p)$ is the unit closed disk in $\mathbb{C}$. By the relation (4.34), there exists $p_1 \in \mathbb{N}$ such that, for all $p > p_1$, $\frac{3\pi A_p}{8} \le \lambda_j^p \le \frac{5\pi A_p}{8}$, and as before, $\sqrt{\frac{\lambda_j^p}{A_p}} > \sqrt{\frac{3\pi}{8}} > 1$. We also observe that

$$F(1) = \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) > \frac{\pi}{2}\frac{21}{25} > 1 \tag{4.58}$$

since $\frac{5}{2} < e < 3$. By the same argument used in the proof of Theorem 4.3.1, we get

$$\int_{\Delta(0,R_p)=\{|z_j|\le \frac{1}{\sqrt{A_p}}\}} e^{-2\lambda_j^p|z_j|^2} dm(z_j) = \frac{1}{\lambda_j^p}\int_{\left\{|w_j|\le \sqrt{\frac{\lambda_j^p}{A_p}}\right\}} e^{-2|w_j|^2} dm(w_j) \ge \frac{1}{\lambda_j^p},$$

since $F(1) > 1$ by (4.58) and F is increasing. Consequently, from (4.56), we have

$$\int_{P^n(0,R_p)} e^{-2\varphi_p'} dV \ge \frac{1}{\lambda_1^p \ldots \lambda_n^p}. \tag{4.59}$$

Inserting (4.59) into the sub-mean estimation (4.55) and using (4.34), one has

$$
\begin{aligned}
|S(y)|^2_{h_p} &\le (1+DR_p^2)e^{2D\|h_p\|_3 R_p^3}\lambda_1^p\dots\lambda_n^p \int_{P^n(y,R_p)} |S|^2_{h_p}\frac{\omega^n}{n!} \\
&\le (1+DR_p^2)e^{2D\|h_p\|_3 R_p^3}(\frac{5\pi}{8})^n A_p^n \int_{P^n(y,R_p)} |S|^2_{h_p}\frac{\omega^n}{n!}.
\end{aligned}
\tag{4.60}
$$

As $R_p = \frac{1}{\sqrt{A_p}} \to 0$ and by our assumption that $\eta_p = \frac{\|h_p\|_3}{\sqrt{A_p}} \to 0$ when $p \to \infty$, one can find a constant $D' > 1$ such that, for a large enough $p_2 \in \mathbb{N}$,

$$
(1+DR_p^2)e^{2D\|h_p\|_3 R_p^3}(\frac{5\pi}{8})^n \le D'
$$

for all $p > \max\{p_0, p_1, p_2\}$. Hence,

$$
|S(y)|^2_{h_p} \le D'\, A_p^n \int_{P^n(y,R_p)} |S|^2_{h_p}\frac{\omega^n}{n!},
\tag{4.61}
$$

which completes the proof of the claim. Let us fix $x \in X$. Then there exists $S_p = S_{p,x} \in H^0(X, L_p)$ such that

$$
|S_p(y)|^2_{h_p} = |K_p(x,y)|^2_{h_p}
$$

for all $y \in X$. By Theorem 4.3.1, there exists a constant $D'' > 1$ and $p_3 \in \mathbb{N}$ such that, for all $p > p_3$,

$$
K_p(x) \le D'' A_p^n,
\tag{4.62}
$$

where D'' is some constant that depends only on the reference cover. On the other hand,

$$
\|S_p\|_p^2 = \int_X |S_p(y)|^2_{h_p}\frac{\omega^n(y)}{n!} = \int_X |K_p(x,y)|^2_{h_p}\frac{\omega^n(y)}{n!} = K_p(x).
\tag{4.63}
$$

In the rest of the proof, we proceed with the near-diagonal and off-diagonal estimations of $K_p(x,y)$.

For the near-diagonal estimation, let $y \in X$ and $d(x,y) \le \frac{4\theta}{\sqrt{A_p}}$. By the variational property (4.24) of $K_p(x)$, the inequality (4.62) and (4.92), we have

$$
\begin{aligned}
|K_p(x,y)|^2_{h_p} &= |S_p(y)|^2_{h_p} \le K_p(y)\|S_p\|_p^2 \\
&\le K_p(x)K_p(y) \le (D'')^2 A_p^{2n} \\
&\le e^{4\theta}(D'')^2 A_p^{2n} e^{-\sqrt{A_p}d(x,y)}.
\end{aligned}
\tag{4.64}
$$

We go on with the far off-diagonal estimation. Let $y \in X$, and this time, consider

$$\delta := d(x,y) > 4\theta \frac{1}{\sqrt{A_p}} = 4\theta R_p.$$

By the choice of S_p and the claim in the beginning of the proof, we get

$$|S_p(y)|^2_{h_p} = |K_p(x,y)|^2_{h_p} \leq A_p^n \int_{P^n(y,R_p)} |K_p(x,\zeta)|^2_{h_p} \frac{\omega_\zeta^n}{n!}. \tag{4.65}$$

We observe that the inclusions

$$P^n(x,R_p) \subset B(x,\frac{\delta}{4}) \text{ and } P^n(y,R_p) \subset \{\zeta \in X : d(x,\zeta) > \frac{3\delta}{4}\} \tag{4.66}$$

hold.

Let β be a non-negative smooth function on X with the following properties:

$$\begin{aligned} \beta(\zeta) = 1 \text{ if } d(x,\zeta) \geq \frac{3\delta}{4} \\ \beta(\zeta) = 0 \text{ if } d(x,\zeta) \leq \frac{\delta}{2} \\ |\overline{\partial}\beta(\zeta)|^2 \leq \frac{c}{\delta^2}\beta(\zeta) \text{ for some } c > 0. \end{aligned}$$

According to these data, we first have

$$\int_{P^n(y,R_p)} |K_p(x,\zeta)|^2_{h_p} \frac{\omega_\zeta^n}{n!} \leq \int_X |K_p(x,\zeta)|^2_{h_p} \beta(\zeta) \frac{\omega_\zeta^n}{n!}. \tag{4.67}$$

Using the variational property for $K_p(x)$, the right-hand side of the inequality (4.67) takes the following form

$$\max\{|K_p(\beta S)|^2_{h_p} : S \in H^0(X,L_p), \int_X |S|^2_{h_p} \beta \frac{\omega^n}{n!} = 1\},$$

where

$$K_p(\beta S)(x) = \int_X K_p(x,\zeta)(\beta(\zeta)\, S(\zeta)) \frac{\omega_\zeta^n}{n!}$$

is the Bergman projection of the smooth section βS to $H^0(X,L_p)$. Note that when $\beta \equiv 1$, the usual variational formula (4.24) is obtained. Therefore, if we manage to estimate $|K_p(\beta S)|^2_{h_p}$, then we will be done. To this end, to find an upper bound for $|K_p(\beta S)|^2_{h_p}$, we use the decomposition of the space $L^2(X,L_p)$ as below

$$L^2(X,L_p) = H^0(X,L_p) \oplus Y. \tag{4.68}$$

(Since $L^2(X,L_p)$ is a Hilbert space and $H^0(X,L_p)$ is a closed subspace of it, such an orthogonal complementary subspace Y always exists). Since $\beta S \in \mathscr{C}^\infty(X, L_p) \hookrightarrow L^2(X, L_p)$ it follows from the decomposition (4.68) that there exists an element $u \in Y$ such that

$$u = \beta S - K_p(\beta S).$$

Owing to the inclusion $P^n(x, R_p) \subset B(x, \frac{\delta}{4})$ from (4.66) and $\beta(\zeta) = 0$ for $\zeta \in B(x, \frac{\delta}{2})$ given previously in the proof, we readily have $\beta = 0$ on $P^n(x, R_p)$, so $u = \beta S - K_p(\beta S) = -K_p(\beta S)$, which is holomorphic on $P^n(x, R_p)$ (defined by the coordinates centered at x provided by Lemma 4.1.1). Therefore, by using the claim in the beginning, one has

$$|K_p(\beta S)|^2_{h_p} = |u(x)|^2_{h_p} \leq D' A_p^n \int_{P^n(x,R_p)} |u|^2_{h_p} \frac{\omega^n}{n!}. \tag{4.69}$$

We provide an upper bound for the integral on the right-hand side of (4.69) by Theorem 4.2.3. For this purpose, let $\tau : [0, \infty) \to (-\infty, 0]$ be a smooth function defined as follows

$$\tau(x) := \begin{cases} 0, & \text{if } x \leq \frac{1}{4} \\ -x, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Write $\phi_\delta(x) = \delta\tau(\frac{x}{\delta})$. Observe that τ' and τ'' have compact supports within the set $[\frac{1}{4}, \frac{1}{2}]$, and so are ϕ'_δ and ϕ''_δ. This means that there exists a constant $M_0 > 0$ such that $|\phi'_\delta(x)| \leq M_0$ and $|\phi''_\delta(x)| \leq \frac{M_0}{\delta}$ for all $x \geq 0$. Define the function

$$v_p(\zeta) := \epsilon\sqrt{A_p}\phi_\delta(d(x, \zeta)).$$

Since ϕ'_δ and ϕ''_δ are smooth and have compact supports, we can find a constant $M_1 > 0$ such that

$$\|\overline{\partial} v_p\|_{L^\infty(X)} \leq M_1 \epsilon\sqrt{A_p} \tag{4.70}$$

$$dd^c v_p \geq -\frac{M_1}{\delta}\epsilon\sqrt{A_p}\omega \geq -\frac{M_1}{4\theta}\epsilon A_p \omega \tag{4.71}$$

because of the inequality $\delta > 4\theta\frac{1}{\sqrt{A_p}}$. Now we can choose $\epsilon = \frac{1}{8M_1}$ for the conditions of Theorem 4.2.3 to hold. Since $\tau(x) = 0$ for $x \leq \frac{1}{4}$, we have that

$$v_p(\zeta) = \epsilon\sqrt{A_p}\phi_\delta(d(x, \zeta)) = \epsilon\sqrt{A_p}\delta\tau(\frac{d(x, \zeta)}{\delta}) = 0$$

for $d(x, \zeta) \leq \frac{\delta}{4}$. Also, by the inclusion $P^n(x, R_p) \subset B(x, \frac{\delta}{4})$ given in (4.66), we get $v_p(\zeta) = 0$ on $P^n(x, R_p)$. By the definition of β, it is seen that $\overline{\partial} u = \overline{\partial}(\beta S) = \overline{\partial}\beta \wedge S$

(because S is holomorphic) has the following (compact) support

$$U_\delta = \{\zeta \in X : \frac{\delta}{2} \leq d(x,\zeta) \leq \frac{3\delta}{4}\},$$

so, for $\zeta \in U_\delta$, by the definitions of U_δ, τ and ϕ_δ we obtain

$$v_p(\zeta) = \epsilon\sqrt{A_p}\,\phi_\delta(d(x,\zeta)) = \epsilon\sqrt{A_p}\,\delta\,\tau(\frac{d(x,\zeta)}{\delta}) = -\epsilon\sqrt{A_p}\,d(x,\zeta) \leq -\epsilon\sqrt{A_p}\,\frac{\delta}{2}.$$

By Theorem 4.2.3 and the definition of β, we have

$$\begin{aligned}
\int_{P^n(x,R_p)} |u|^2_{h_p}\frac{\omega^n}{n!} &\leq \int_X |u|^2_{h_p} e^{v_p}\frac{\omega^n}{n!} \\
&\leq \frac{16}{3A_p}\int_{U_\delta} |\overline{\partial}(\beta S)|^2_{h_p} e^{v_p}\frac{\omega^n}{n!} \\
&\leq \frac{16c}{3A_p\delta^2} e^{-\epsilon\sqrt{A_p}\delta}\int_{U_\delta} |S|^2_{h_p}\beta\frac{\omega^n}{n!} \\
&\leq \frac{c}{3}e^{-\sqrt{A_p}\delta} \leq ce^{-\sqrt{A_p}\delta}.
\end{aligned}$$

Plugging this last inequality into (4.69) gives

$$|K_p(\beta S)|^2_{h_p} \leq D'\,A_p^n\,c\,e^{-\epsilon\sqrt{A_p}\delta},$$

which gives, by using the inequality (4.67),

$$\int_{P^n(y,R_p)} |K_p(x,\zeta)|^2_{h_p}\frac{\omega^n_\zeta}{n!} \leq D'\,A_p^n\,c\,e^{-\epsilon\sqrt{A_p}d(x,y)}.$$

From the inequality (4.65), we infer

$$|K_p(x,y)|^2_{h_p} \leq c\,(D')^2 A_p^{2n}\,e^{-\epsilon\sqrt{A_p}d(x,y)}, \tag{4.72}$$

which finalizes the proof. $\square$

4.3.1 Linearization and near diagonal asymptotics

Let $V \subset X$, $U \subset \mathbb{C}^n$ be open subsets and $x_0 \in V$, $0 \in U$. Let us take a (Kähler) coordinate chart as follows $\gamma : (V, x_0) \to (U, 0)$, $\gamma(x_0) = 0$. We will use the following notation, so-called linearization of the coordinates on the Kähler manifold X: For any $u,v\in\mathbb{C}^n$, we write $\gamma^{-1}(u)=x_0+u$ and $\gamma^{-1}(v)=x_0+v$, and

$$K_p(\gamma^{-1}(u),\gamma^{-1}(v)) := K_p(x_0+u,x_0+v). \tag{4.73}$$

Since $0\in\mathbb{C}^n$ and $\mathbb{C}^n$ is a complex vector space, we can write $v=0+v$ and $u=0+u$. Linearization means that when we translate $0\in\mathbb{C}^n$ by u (or by v for that matter), by thinking of $0\in\mathbb{C}^n$ as $x_0\in X$, we can also write $\gamma^{-1}(u)=x_0+u$, so in local coordinates we express the difference between $\gamma^{-1}(u)$ and x_0 (not meaningful in X) by the difference $u-0$ (meaningful in $\mathbb{C}^n$ because $\mathbb{C}^n$ is a complex vector space). This is also called the abuse of notation in, for instance, [Shiffman & Zelditch (2008)] and [Shiffman & Zelditch (2010)].

Near diagonal asymptotics

Modifying the argument in [Bayraktar (2017a), Theorem 2.3], we consider the following holomorphic functions

$$\begin{aligned}\Gamma_p(u,v) &= \frac{K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)e^{-t_p\left(\frac{u}{\sqrt{A_p}}\right)-\overline{t_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}}{A_p^n e^{\frac{2}{A_p}\langle\Lambda_p u,\overline{v}\rangle}}\\ &= \frac{K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)e^{-t_p\left(\frac{u}{\sqrt{A_p}}\right)-\overline{t_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}e^{-\widetilde{\varphi_p}\left(\frac{u}{\sqrt{A_p}}\right)}e^{-\widetilde{\varphi_p}\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^n e^{\frac{2}{A_p}\langle\Lambda_p u,\overline{v}\rangle}}\\ &\quad\times e^{\widetilde{\varphi_p}\left(\frac{u}{\sqrt{A_p}}\right)}e^{\widetilde{\varphi_p}\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}.\end{aligned}$$

on $\Omega=\{(u,v): u,\, v\in P^n(0,R)\}$, where $\Lambda_p := \mathrm{Diag}[\lambda_1^p,\dots,\lambda_n^p]$, which is a diagonal matrix whose diagonal entries λ_j^p are positive from the discussions in Section 4.1. Let $\Omega_0 := \{(u,\overline{u}): u\in P^n(0,R)\}$. It follows from Theorem 4.3.1 and Lemma 4.1.1 (ii) that $\Gamma_p\to 1$ on Ω_0. Observe that since Γ_p is uniformly bounded on Ω, there is a subsequence $\{\Gamma_{p_d}\}$ such that $\Gamma_{p_d}\to\Gamma_0$ uniformly on Ω, where we must have that $\Gamma_0\equiv 1$ on Ω_0. Since Ω_0 is a maximally totally real submanifold, we get $\Gamma_0=1$ on the whole Ω. Since this argument can be applied to any subsequence of Γ_p, we see that $\Gamma_p\to 1$. We make now an observation for $|\Gamma_p(u,v)|^2$ that will be used in our main theorem. Since Λ_p has positive diagonal entries, its square root $\Lambda_p^{1/2}$ is defined, so we have

$$\langle\Lambda_p u,\overline{v}\rangle = \langle\Lambda_p^{1/2}u,\Lambda_p^{1/2}\overline{v}\rangle. \tag{4.74}$$

(4.75)

$$
\begin{aligned}
|\Gamma_p(u,v)|^2 &= \frac{\left|K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)\right|^2 e^{-2\Re t_p\left(\frac{u}{\sqrt{A_p}}\right)} e^{-2\Re t_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^{2n} e^{\frac{4}{A_p}\Re\langle\Lambda_p u,\overline{v}\rangle}} \\
&= \frac{\left|K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)\right|^2 e^{-2\Re t_p\left(\frac{u}{\sqrt{A_p}}\right)} e^{-2\Re t_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^{2n} e^{2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|u_j|^2} e^{2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2} e^{-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|u_j-\overline{v_j}|^2}} \\
&= \frac{\left|K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)\right|^2 e^{-2\varphi_p\left(\frac{u}{\sqrt{A_p}}\right)} e^{-2\varphi_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^{2n} e^{-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|u_j-\overline{v_j}|^2}} e^{\widetilde{\varphi_p}\left(\frac{u}{\sqrt{A_p}}\right)} e^{\widetilde{\varphi_p}\left(\frac{\overline{v}}{\sqrt{A_p}}\right)},
\end{aligned}
$$

where, in the second equality, we have used (4.74) and the polarization identity

$$\Re(\langle \mathbf{x},\mathbf{y}\rangle) = \frac{1}{2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-\|\mathbf{x}-\mathbf{y}\|^2)$$

for the vectors $\mathbf{x}=\Lambda_p^{1/2}u$ and $\mathbf{y}=\Lambda_p^{1/2}\overline{v}$, and in the third equality, we take into account the representation $\varphi_p(z)=\Re t_p(z)+\sum_{j=1}^m \lambda_j^p|z_j|^2+\widetilde{\varphi}_p(z)$ from Lemma 4.1.1(ii). By the limit argument made above regarding the holomorphic functions Γ_p, the expression (4.75) for $|\Gamma_p(u,v)|^2$ and Lemma 4.1.1(ii), we have

$$\frac{\left|K_p\left(\frac{u}{\sqrt{A_p}},\frac{\overline{v}}{\sqrt{A_p}}\right)\right|^2 e^{-2\varphi_p\left(\frac{u}{\sqrt{A_p}}\right)} e^{-2\varphi_p\left(\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^{2n} e^{-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|u_j-\overline{v_j}|^2}} \to 1 \tag{4.76}$$

as $p\to\infty$. By linearization (4.73) on the coordinate polydisk $P^n(x,R)\subset U_j$, where we have the Kähler coordinates at the point $x\equiv 0$ provided by Lemma 4.1.1 for (4.76), we obtain

$$\frac{\left|K_p\left(x+\frac{u}{\sqrt{A_p}},x+\frac{\overline{v}}{\sqrt{A_p}}\right)\right|^2 e^{-2\varphi_p\left(x+\frac{u}{\sqrt{A_p}}\right)} e^{-2\varphi_p\left(x+\frac{\overline{v}}{\sqrt{A_p}}\right)}}{A_p^{2n} e^{-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|u_j-\overline{v_j}|^2}} \to 1 \quad \text{as} \quad p\to\infty. \tag{4.77}$$

4.4 Asymptotic Normality

4.4.1 Gaussian holomorphic sections and random zero currents of integration

A complex random variable W is said to be standard Gaussian in case $W = X + \sqrt{-1}Y$, where X and Y are i.i.d. centered Gaussian distributions of variance $1/2$.

Given an orthonormal basis $\{S_j^p\}_{j=1}^{d_p}$ of $H^0(X, L_p)$ with respect to the inner product (4.21), a *Gaussian holomrophic section* of L_p is a linear combination

$$s_p = \sum_{j=1}^{d_p} \xi_j S_j^p,$$

where ξ_j are i.i.d. real or complex Gaussian random variables of mean zero and variance one. For any such Gaussian holomorphic section s_p its zero locus Z_{s_p} is a purely 1-codimensional analytic subvariety of X, and the current of integration (with multiplicities) along Z_{s_p} is defined as in the previous chapter, that is for $\phi \in \mathcal{D}^{n-1,n-1}(X)$

$$\langle [Z_{s_p}], \phi \rangle = \int_{Z_{s_p}} \phi.$$

We remark that the random variables

$$s_p \longmapsto \langle [Z_{s_p}], \phi \rangle \tag{4.78}$$

on the probability space $(H^0(X, L_p), \gamma_p)$, where γ_p is the d_p-fold Gaussian product measure on $H^0(X, L_p)$ are called *smooth linear statistics of zeros* for Gaussian holmorphic sections $s_p \in H^0(X, L_p)$.

The expectation and variance of the current valued random variable $s_p \mapsto [Z_{s_p}]$ are defined by their action on $\phi \in \mathcal{D}^{n-1,n-1}(X)$, i.e.

$$\phi \longmapsto \langle \mathbb{E}[Z_{s_p}], \phi \rangle := \mathbb{E}\langle [Z_{s_p}], \phi \rangle, \tag{4.79}$$

and

$$\phi \longmapsto \langle \mathrm{Var}[Z_{s_p}], \phi \boxtimes \phi \rangle := \mathrm{Var}\langle [Z_{s_p}], \phi \rangle = \mathbb{E}\langle [Z_{s_p}], \phi] \rangle^2 - (\mathbb{E}\langle [Z_{s_p}], \phi \rangle)^2. \tag{4.80}$$

Note that the expectation here is taken with respect to the Gaussian measure γ_p on $H^0(X, L_p) \cong \mathbb{C}^{d_p}$, and the external product $\phi \boxtimes \phi$ is defined as in Section 3.1.1.

4.4.2 Asymptotic normality of random zero currents

In this section, we delve into the proof of Theorem 1.3.3. We begin our analysis by recalling basic facts needed for the proof and stating a fundamental theorem by Sodin and Tsirelson [Sodin & Tsirelson (2004)], which provides sufficient conditions for proving an asymptotic normality result.

Theorem of Sodin and Tsirelson:

Given a sequence $\{\gamma_j\}_{j=1}^{\infty}$ of complex-valued measurable functions on a measure space (G, σ) such that

$$\sum_{j=1}^{\infty} |\gamma_j(x)|^2 = 1 \text{ for any } x \in G, \tag{4.81}$$

following [Sodin & Tsirelson (2004)] (and also [Shiffman & Zelditch (2010)]), a *normalized complex Gaussian process* is defined to be a complex-valued random function $\alpha(x)$ on a measure space (G, σ) in the following form

$$\alpha(x) = \sum_{j=1}^{\infty} b_j \gamma_j(x), \tag{4.82}$$

where the coefficients b_j are i.i.d. centered complex Gaussian random variables with variance one. The covariance function of $\alpha(x)$ is defined by

$$\mathcal{C}(x, y) = \mathbb{E}[\alpha(x)\overline{\alpha(y)}] = \sum_{j=1}^{\infty} \gamma_j(x)\overline{\gamma_j(y)}. \tag{4.83}$$

A simple observation gives that $|\mathcal{C}(x, y)| \leq 1$ and $\beta(x, x) = 1$.

Consider a sequence $\{\alpha_j\}_{j=1}^{\infty}$ of normalized complex Gaussian processes on a finite measure space (G,σ), and let $\lambda(\rho) \in L^2(\mathbb{R}^+, e^{\frac{-\rho^2}{2}}\rho d\rho)$. Suppose $\phi : G \to \mathbb{R}$ is a bounded and measurable function, we will focus on the following non-linear functionals that also serve as random variables in this context.

$$\mathcal{F}_p^{\phi}(\alpha_p) = \int_G \lambda(|\alpha_p(x)|)\phi(x)d\sigma(x). \tag{4.84}$$

The next theorem (Theorem 2.2 of [Sodin & Tsirelson (2004)]) was proved by Sodin and Tsirelson.

Theorem 4.4.1. *For each $n = 1,2,\ldots$, let $\beta_p(r,s)$ be the covariance functions for the complex Gaussian processes. Assume that the two conditions below hold for all $\nu \in \mathbb{N}$:*

(i)
$$\liminf_{n\to\infty} \frac{\int_G\int_G |\mathcal{C}_p(r,s)|^{2\nu}\phi(r)\phi(s)d\sigma(r)d\sigma(s)}{\sup_{r\in G}\int_G |\mathcal{C}_p(r,s)|d\sigma(s)} > 0.$$

(ii)
$$\lim_{n\to\infty}\sup_{r\in G}\int_G |\mathcal{C}_p(r,s)|d\sigma(s) = 0.$$

Then the distributions of the random variables

$$\frac{\mathcal{F}_p^{\phi}(\alpha_p) - \mathbb{E}[\mathcal{F}_p^{\phi}(\alpha_p)]}{\sqrt{\mathrm{Var}[\mathcal{F}_p^{\phi}(\alpha_p)]}}$$

converges weakly to the normal distribution $\mathcal{N}(0,1)$ as $p\to\infty$. If λ is increasing, then it is sufficient for (i) to hold only for $\nu = 1$.

The proof is based on applying the method of moments, a fundamental tool in probability theory, coupled with the use of the diagram technique. This approach facilitates the computation of moments for non-linear functionals, which are then compared to the moments found in a standard Gaussian distribution. Such an approach is a classical one in establishing the central limit theorem for non-linear functionals within Gaussian fields. We also remark that the condition (ii) ensures that $\mathrm{Var}[\mathcal{F}_p^{\phi}(\alpha_p)] \to 0$ as $p\to\infty$.

Now we are ready to establish our main theorem. For the proof, we use the arguments from [Shiffman & Zelditch (2010)] and the primary objective will be to apply Theorem 4.4.1 to our setting, specifically to Gaussian holomorphic sections in a sequence of positive holomorphic line bundles with class $\mathscr{C}^3$ Hermitian metrics on a compact Kähler manifold. To accomplish this, we will make use of the Bergman kernel estimates from the previous sections.

Proof of Theorem 1.3.3. To begin with, we modify the information about analytic functions from Theorem 4.4.1 to suit our present setting. The normalized Gaussian processes α_p on X will be constructed as follows: We take a measurable section e_{L_p} of L_p such that $e_{L_p} : X \to L_p$ with $|e_{L_p}(x)|_{h_p} = 1$ for any $x \in X$. We pick now an orthonormal basis $\{s_p^j\}_{j=1}^{d_p}$ of $H^0(X, L_p)$, where $s_p^j = \varphi_p^j e_{L_p}$. Let us write

$$f_p^j(x) = \frac{\varphi_p^j(x)}{\sqrt{K_p(x,x)}}, \; j = 1, 2, \ldots, d_p. \tag{4.85}$$

Notice that $|\varphi_p^j| = |s_p^j|_{h_p}$ and $\sum_{j=1}^{d_p} |f_p^j(x)|^2 = 1$ by the relation (4.23). Therefore, we can express a normalized complex Gaussian process on X for each $p \in \mathbb{N}$ as follows:

$$\alpha_p = \sum_{j=1}^{d_p} b_j f_p^j, \tag{4.86}$$

where the coefficients b_j are i.i.d. complex Gaussian centered random variables with variance one. We observe that a random holomorphic section $s_p = \sum_{j=1}^{d_p} b_j s_p^j$ can be represented as

$$s_p = \sum_{j=1}^{d_p} b_j s_p^j = \sqrt{K_p(x,x)}\alpha_p e_{L_p}, \tag{4.87}$$

which indicates the presence of the normalized complex Gaussian process. The relation (4.87) gives us that

$$|\alpha_p(x)| = \frac{|s_p^j(x)|_{h_p}}{\sqrt{K_p(x,x)}}. \tag{4.88}$$

We proceed to compute the covariance functions β_p of the complex Gaussian processes α_p. We observe from the fact that the complex Gaussian random coefficients b_j in (4.86) are centered, i.i.d., and have variance one

$$\mathrm{Var}[b_j] = \mathbb{E}[|b_j|^2] = 1, \; \mathbb{E}[b_k \bar{b_l}] = 0 \text{ if } k \neq l. \tag{4.89}$$

By the relation (4.83), (4.86) and (4.89), we have

$$\mathcal{C}_p(x,y) = \mathbb{E}[\sum_{j=1}^{d_p} b_j f_p^j(x) \overline{\sum_{j=1}^{d_p} b_j f_p^j(y)}] = \sum_{j=1}^{d_p} f_p^j(x)\overline{f_p^j(y)}. \tag{4.90}$$

Since $\{s_p^j\}_{j=1}^{d_p}$ is an orthonormal basis, due to the representation $s_p^j = \varphi_p^j e_{L_p}$ and the relation (4.85), we have $\varphi_p^k(x)\overline{\varphi_p^l(y)} = 0$ whenever $k \neq l$. Combining this with (4.90) yields

$$|\mathcal{C}_p(x,y)| = \sqrt{\sum_{j=1}^{d_p} |f_p^j(x)|^2 |f_p^j(y)|^2}. \tag{4.91}$$

Next, after a series of computations, we arrive at

$$|K_p(x,y)|_{h_{p,x}\otimes h_{p,y}} = \sqrt{\sum_{j=1}^{d_p} |s_p^j(x)|^2_{h_{p,x}} |s_p^j(y)|^2_{h_{p,y}}}\ . \tag{4.92}$$

By putting together (4.91), (4.92) and (4.85), we find

$$\widehat{\mathcal{K}}_p(x,y) = |\mathcal{C}_p(x,y)| \tag{4.93}$$

because $|s_p^j|_{h_p} = |\varphi_p^j|$.

Take $\lambda(\rho) = \log\rho$ and $(G,\sigma) = (X, \frac{\omega^n}{n!})$. In the rest of the proof, to make the notation lighter, we will write $d\vartheta_X := \frac{\omega^n}{n!}$ for the (Riemannian) volume form on X. Let us fix a $(n-1,n-1)$ real-valued form ϕ with $\mathscr{C}^3$ coefficients. Then,

$$dd^c\phi = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\phi = \psi\, d\vartheta_X, \tag{4.94}$$

where the function ψ is a real-valued $\mathscr{C}^1$ function on X. By invoking the Poincaré-Lelong formula (2.16), the non-linear random functional given by (4.84) assumes the subsequent form in our case,

$$\mathcal{F}_p^{\psi}(\alpha_p) = \int_X \Big(\log|s_p|_{h_p} - \log\sqrt{K_p(x,x)}\Big)\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\phi(x) = \langle [Z_{s_p}],\phi\rangle + \zeta_{p,L_p}, \tag{4.95}$$

where $\zeta_{p,L_p} := \Big\langle -c_1(L_p,h_p) - \log\sqrt{K_p(x,x)},\phi\Big\rangle$, namely, ζ_{p,L_p} is some constant depending only on the line bundle L_p and the dimension of the Kähler manifold X. Thanks to a standard property of variance, the expression (4.95) shows that $\mathcal{F}_n^{\psi}(\alpha_p)$ and $\langle [Z_{s_p}],\phi\rangle$ have the equal variances.

For the remaining part of the proof, our goal is to validate the fulfillment of the requirements (i) and (ii) of Theorem 4.4.1 for the current setting. First, with $\lambda(\rho) = \log\rho$ being increasing, we only focus on the case where $\nu = 1$. To use both far-off-diagonal and near-diagonal asymptotics, we split the integration regions accordingly: $d(x,y) \le \frac{\log A_p}{\sqrt{A_p}}$ and $d(x,y) \ge \frac{\log A_p}{\sqrt{A_p}}$. Let us start with the simpler condition (ii). For the integral on the far off-diagonal set where $d(x,y) \ge \frac{\log A_p}{\sqrt{A_p}}$, by Theorem 4.3.2, we have

$$\lim_{n\to\infty} \sup_{x\in X} \int_{d(x,y)\ge \frac{\log A_p}{\sqrt{A_p}}} \widehat{\mathcal{K}}_p(x,y) d\vartheta_X(y) \le \lim_{p\to\infty} \sup_{x\in X} \int_{d(x,y)\ge \frac{\log A_p}{\sqrt{A_p}}} Ge^{-B\sqrt{A_p}d(x,y)} d\vartheta_X(y) = 0.$$

For the integral over the near-diagonal set where $d(x,y) \le \frac{\log A_p}{\sqrt{A_p}}$, due to the relations (4.93) and (4.83), we get

$$\lim_{n\to\infty} \sup_{x\in X} \int_{d(x,y)\le \frac{\log A_p}{\sqrt{A_p}}} \widehat{\mathcal{K}}_p(x,y) d\vartheta_X(y) \le \lim_{n\to\infty} \sup_{x\in X} \int_{d(x,y)\le \frac{\log A_p}{\sqrt{A_p}}} 1 \, d\vartheta_X(y) = 0.$$

Our next step is to affirm condition (i). For the integral on the far off-diagonal set where $d(x,y) \ge \frac{\log A_p}{\sqrt{A_p}}$, as $p \to \infty$, the integrand of the numerator approaches zero more rapidly compared to that of the denominator because, by Theorem 4.3.2, the corresponding decaying orders (to zero) for numerator and denominator are $O(A_p^{-\epsilon})$ and $O(A_p^{-\epsilon/2})$, respectively.

Finally, we verify that the lower limit below will be strictly positive on the near-diagonal set, where $\{|v| \le \frac{\log A_p}{\sqrt{A_p}}\}$:

$$\liminf_{p\to\infty} \frac{\int_X \int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p^2(x, x+\frac{v}{\sqrt{A_p}})\psi(x)\psi(x+\frac{v}{\sqrt{A_p}})\, dv\, d\vartheta_X(x)}{\int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p(x, x+\frac{v}{\sqrt{A_p}})\, dv} > 0. \tag{4.96}$$

Let

$$J(p) := \frac{\int_X \int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p^2(x, x+\frac{v}{\sqrt{A_p}})\psi(x)\psi(x+\frac{v}{\sqrt{A_p}})\, dv\, d\vartheta_X(x)}{\int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p(x, x+\frac{v}{\sqrt{A_p}})\, dv}. \tag{4.97}$$

Let us examine the numerator and the denominator separately. Using the left part of the inequality (4.50) for the denominator and the right part of the same inequality for the numerator, the linearization (4.73) in the neighborhood U_j, where we have the Kähler coordinates at the point x on the polydisk $P^n(x,R) \subset U_j$ provided by Lemma 4.1.1, we get the following.

(4.98)
$$\int_X \int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p^2(x,x+\frac{v}{\sqrt{A_p}})\psi(x)\psi(x+\frac{v}{\sqrt{A_p}})\,dv\,d\vartheta_X(x)$$
$$\ge \int_X \int_{|v|\le \log A_p} \left(\frac{4}{5}\right)^{2n} \frac{|K_p(x,x+\frac{v}{\sqrt{A_p}})|^2_{h_p}\psi(x)\psi(x+\frac{v}{\sqrt{A_p}})}{A_p^{2n}(1+D'\eta_p^{2/3})^2}\,dv\,d\vartheta_X(x) := I_1(p).$$

and

$$\int_{|v|\le \log A_p} \widehat{\mathcal{K}}_p(x,x+\frac{v}{\sqrt{A_p}})\,dv$$
$$\le \int_{|v|\le \log A_p} \left(\frac{4}{3}\right)^{n} \frac{|K_p(x,x+\frac{v}{\sqrt{A_p}})|_{h_p}}{A_p^{n}(1-D'\eta_p^{2/3})}\,dv := I_2(p), \tag{4.99}$$

which implies, by extracting the weight functions and multiplying and dividing both the integrand of $I_1(p)$ and $I_2(p)$ by $e^{-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2}$ and $e^{-\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2}$, respectively. We obtain the following expression for $I_1(p)$

(4.100)
$$\int_X \int_{|v|\le \log A_p} \left(\frac{4}{5}\right)^{2n} \frac{|K_p(x,x+\frac{v}{\sqrt{A_p}})|^2 \exp\left(-2\varphi_p(x)-2\varphi_p\left(x+\frac{v}{\sqrt{A_p}}\right)-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2\right)}{A_p^{2n}(1+D'\eta_p^{2/3})^2\exp\left(-2\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2\right)}$$
$$\times \psi(x)\psi\left(x+\frac{v}{\sqrt{A_p}}\right)dv\,d\vartheta_X(x).$$

and

(4.101)
$$I_2(p) = \int_{|v|\le \log A_p} \left(\frac{4}{3}\right)^{n} \frac{|K_p(x,x+\frac{v}{\sqrt{A_p}})|e^{-\varphi_p(x)}e^{-\varphi_p\left(x+\frac{v}{\sqrt{A_p}}\right)}}{A_p^{n}(1-D'\eta_p^{2/3})e^{-\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2}} e^{-\sum_{j=1}^n \frac{\lambda_j^p}{A_p}|v_j|^2}\,dv.$$

Now, as $p\to\infty$, and utilizing (4.77) and (4.8), along with the fact that for $\psi \in \mathscr{C}^1$, we have $\psi(x+\frac{v}{\sqrt{A_p}}) = \psi(x)+O\left(\frac{v}{\sqrt{A_p}}\right)$, it follows that

$$I_1(p) \to \int_X \psi^2(x)\,d\vartheta_X(x)\int_{\mathbb{C}^n}\left(\frac{4}{5}\right)^{2n} e^{-\pi\sum_{j=1}^n |v_j|^2}\,dv, \tag{4.102}$$

and

$$I_2(p) \to \int_{\mathbb{C}^n}\left(\frac{4}{3}\right)^{n} e^{-\frac{\pi}{2}\sum_{j=1}^n |v_j|^2}\,dv. \tag{4.103}$$

By (4.98) and (4.99), we obtain

$$J(p) \geq \frac{I_1(p)}{I_2(p)}. \tag{4.104}$$

Hence, employing (4.102) and (4.103), one gets

$$\liminf_{p\to\infty} J(p) \geq \frac{\int_X \psi^2(x) d\vartheta_X(x) \int_{\mathbb{C}^n} (\frac{4}{5})^{2n} \, e^{-\pi \Sigma_{j=1}^n |v_j|^2} dv}{\int_{\mathbb{C}^n} (\frac{4}{3})^n e^{-\frac{\pi}{2}\Sigma_{j=1}^n |v_j|^2} \, dv} = \frac{6^n}{5^{2n}} \int_X \psi^2(x) d\vartheta_X(x) > 0,$$

which ends the proof. □

In comparison, the authors in [Coman et al. (2017)] and [Bayraktar et al. (2020)] consider a more general hypothesis between the first Chern forms and the Kähler form. More precisely, they assume

$$c_1(L_p, h_p) \geq a_p \omega \text{ for all } p \geq 1 \text{ and } a_p > 0, \text{ such that } a_p \to \infty. \tag{4.105}$$

In our setting, the role of a_p is played by A_p (which was defined as $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$ in [Coman et al. (2017)] and [Bayraktar et al. (2020)]), and there are two different limits because of the diophantine approximation relation (4.1) between λ_j^p and A_p. In the case of (4.105), we do not have (4.8) (and consequently (4.9)). However, as Theorem 1.3 in [Coman et al. (2017)] shows, there still exists a limit: $\lim_{p\to\infty} \frac{K_p(x)}{\lambda_1^p \dots \lambda_n^p} = (\frac{2}{\pi})^n$. Despite the existence of the limit in terms of λ_j^p, we do not know whether the limit $\lim_{p\to\infty} \frac{\lambda_j^p}{A_p}$ exists, which is crucial in the proof of Theorem 1.3.3. Therefore, the arguments followed in this paper cannot be used to prove a central limit theorem in this framework.

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