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THE DYNAMIC PROGRAMMING EQUATION FOR THE PROBLEM OF OPTIMAL INVESTMENT UNDER CAPITAL GAINS TAXES*

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Abstract. This paper considers an extension of the Merton optimal investment problem to the case where the risky asset is subject to transaction costs and capital gains taxes. We derive the dynamic programming equation in the sense of constrained viscosity solutions. We next introduce a family of functions $(V_\varepsilon)_{\varepsilon>0}$, which converges to our value function uniformly on compact subsets, and which is characterized as the unique constrained viscosity solution of an approximation of our dynamic programming equation. In particular, this result justifies the numerical results reported in the accompanying paper [I. Ben Tahar, H.M. Soner and N. Touzi (2005), *Modeling Continuous-Time Financial Markets with Capital Gains Taxes*, preprint, http://www.cmap.polytechnique.Fr/∼touzi/bst06.pdf].

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1. Introduction.

The problem of optimal investment and consumption in financial markets has been introduced by Merton [20, 21]. The explicit solution derived in these papers is widely used among fund managers in practical financial markets. Moreover, this problem became very quickly one of the classical examples of application of the verification theorem in stochastic control theory. Indeed, by direct financial considerations, it is easily seen that the value function of the problem satisfies some homogeneity property, which completely determines its dependence on the wealth state variable. Plugging this information into the corresponding dynamic programming equation (DPE) leads to an ordinary differential equation (ODE) which can be solved explicitly, thus providing a candidate smooth solution to the DPE.

In this paper, we consider the extension of the Merton problem to the case where the risky asset is subject to capital gains taxes. For technical reasons, we also assume that the risky asset is subject to proportional transaction costs. This problem is formulated in the accompanying paper [5]. In contrast with the Merton frictionless model, no explicit solution is available in this context. The main result of [5] is the derivation of an explicit first order expansion of the value function for small tax and interest rate parameters. The numerical results reported in [5] show that the relative error induced by this approximation is of the order of 4%. These numerical results are obtained by comparing the explicit first order expansion to the finite differences approximation of the solution of the corresponding DPE.

The literature on the optimal investment problem under capital gains taxes is not very expanded and is mainly developed in discrete-time binomial models; see

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[7, 10, 11, 12, 17, 18, 15, 19].

The main purpose of this paper is to justify the approximation of the value function by means of the finite differences scheme applied to the corresponding DPE. Since our optimal control problem is singular, the DPE takes the form of a variational inequality:

$$\min\left\{-\mathcal{L}v,\ \mathbf{g^b}\cdot Dv,\ \mathbf{g^s}\cdot Dv\right\} \;=\; 0 \text{ on } \bar{\mathcal{S}}, \qquad v=0 \text{ on } \partial^z\mathcal{S},$$

where $\mathcal{L}$ is a second order differential operator defined in (2.13) $\mathbf{g^b}, \mathbf{g^s}$ are two vector fields defined in (2.15) corresponding to the purchase and sale decisions, $\mathcal{S}$ is the state space, and $\partial^z\mathcal{S}$ is part of the boundary of $\mathcal{S}$. The main difficulty comes from the fact that the vector field $\mathbf{g^s}$ is not locally Lipschitz. Then the standard techniques to prove a uniqueness result for the above partial differential equation (PDE) fail. We then introduce a convenient locally Lipschitz approximation $\mathbf{g}^{\mathbf{s}}_{\varepsilon}$ of $\mathbf{g^s}$, and we consider the approximating PDE

$$\min\left\{-\mathcal{L}v,\ \mathbf{g^b}\cdot Dv,\ \mathbf{g}^{\mathbf{s}}_{\varepsilon}\cdot Dv\right\} \;=\; 0 \text{ on } \bar{\mathcal{S}}, \qquad v=0 \text{ on } \partial^z\mathcal{S}.$$

The main result of this paper states that the above approximating PDE has a unique continuous viscosity solution V_ε which converges uniformly on compact subsets towards the value function V of our optimal investment problem under capital gains taxes. Applying the general results of Barles and Souganidis [4], we see that this justifies the convergence of the numerical scheme implemented in the accompanying paper [5] towards this unique solution of the approximating PDE.

The paper is organized as follows. Section 2 provides a quick review of the problem of optimal investment under capital gains taxes. The main approximation result is stated in section 3. In section 4, we prove a comparison result for the approximating PDE, which implies the required uniqueness claim. In section 5 we prove the existence of a solution of the approximating PDE by introducing a family of control problems obtained by modifying conveniently our original problem. Finally, section 6 reports the proof of convergence of V_ε towards V uniformly on compact subsets.

Notation. For a domain $\mathbf{D}$ in $\mathbb{R}^n$, we denote by $\mathrm{USC}(\mathbf{D})$ (resp., $\mathrm{LSC}(\mathbf{D})$) the collection of all upper semicontinuous (resp., lower semicontinuous) functions from $\mathbf{D}$ to $\mathbb{R}$. The set of continuous functions from $\mathbf{D}$ to $\mathbb{R}$ is denoted by $\mathrm{C}^0(\mathbf{D}) := \mathrm{USC}(\mathbf{D})\cap\mathrm{LSC}(\mathbf{D})$. For a parameter $\delta>0$, we say that a function $f\ :\mathbf{D}\longrightarrow\mathbb{R}$ has δ-polynomial growth if

$$\sup_{x\in\mathbf{D}} \frac{|f(x)|}{1+|x|^\delta} < \infty.$$

We finally denote $\mathrm{USC}_\delta(\mathbf{D}) := \{f\in\mathrm{USC}(\mathbf{D})\ :\ f \text{ has } \delta\text{-polynomial growth}\}$. The sets $\mathrm{LSC}_\delta(\mathbf{D})$ and $\mathrm{C}^0_\delta(\mathbf{D})$ are defined similarly.

2. Optimal investment under capital gains taxes.

2.1. Problem formulation. In this section, we quickly review the formulation of the problem of optimal investment under capital gains taxes. We refer the interested reader to the accompanying paper [5] for more details. The financial market consists of a tax-free bank account with constant interest rate $r>0$ and a risky asset subject to proportional transaction costs and to capital gains taxes. The price process of the risky asset evolves according to the Black–Scholes model:

$$dP_t \;=\; P_t\left(\rho dt+\sigma dW_t\right), \qquad t\geq 0, \tag{2.1}$$

where $\rho > 0$ is a constant instantaneous return of the asset, and $\sigma > 0$ is a constant volatility parameter. The process $W = \{W_t, t \geq 0\}$ is a standard Brownian motion with values in $\mathbb{R}^1$ defined on an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall denote by $\mathbb{F}$ the $\mathbb{P}$-completion of the natural filtration of the Brownian motion.

For technical reasons (see section 4), we assume that the risky asset is also subject to proportional transaction costs defined by the coefficients λ, $\mu \in [0,1)$, so that the bid and ask prices at time t of the risky asset are given by $(1-\mu)P_t$ and $(1+\lambda)P_t$.

A control process is a triple of $\mathbb{F}$-adapted processes $\nu = (C, L, M)$, where

$$C \;\geq\; 0 \text{ and } \int_0^T C_t dt \;<\; \infty \;\; \mathbb{P}\text{-a.s. for all } T > 0\,, \tag{2.2}$$

L and M are nondecreasing right-continuous, $L_{0-} = M_{0-} = 0$, and the jumps of M satisfy

$$\Delta M_t \;\leq\; 1 \text{ for } t \geq 0 \;\; \mathbb{P}\text{-a.s.} \tag{2.3}$$

Here C_t is the consumption rate at time t, $dL_t \geq 0$ is the amount invested between times t and $t + dt$ to purchase risky assets, and $dM_t \geq 0$ is the *proportion* of risky assets in the portfolio which is sold between times t and $t + dt$. Then, the amount of wealth $Y = \{Y_t, t \geq 0\}$ on the risky asset account is defined by the dynamics

$$dY_t = Y_t \frac{dP_t}{P_t} + dL_t - Y_{t-} dM_t\,, \quad t \geq 0. \tag{2.4}$$

Since $\Delta M_t \leq 1$, the no short-sales constraint $Y \geq 0$ holds. Capital gains are taxed only when the investor sells the risky asset. The amount of capital gains (or losses) is evaluated by comparing the actual price P_t to a tax basis B_t specified by the taxation code. In our framework the tax basis is defined as the weighted average of past purchase prices,

$$B_t \;:=\; \frac{K_t}{Y_t} P_t \;\text{ if }\; Y_t > 0 \;\text{ and }\; B_t \;:=\; P_t \;\text{ otherwise,} \qquad t \geq 0\,,$$

where

$$dK_t \;=\; dL_t - K_{t-} dM_t\,, \qquad t \geq 0. \tag{2.5}$$

The natural initial condition for the process K is zero, as initially there are no prior stocks bought. However, the method of dynamic programming always forces us to consider all possible initial data. Hence we consider the K-equation with general initial data $K_0 = k$. Also a more detailed derivation of this tax basis and its place in actual tax codes is given in subsection 2.2 of the accompanying paper [5].

Finally, we consider a linear taxation rule, with constant tax rate parameter $\alpha \in [0,1]$, so that the after-tax and after-transaction costs induced by selling the amount $Y_{t-} dM_t$ between times t and $t + dt$ are given by

$$(1-\mu) Y_{t-} dM_t - \alpha(1-\mu) \left[Y_{t-} dM_t - \frac{Y_{t-} dM_t}{P_t} \, B_{t-} \right] = (1-\mu) \left[(1-\alpha) Y_{t-} + \alpha K_{t-} \right] dM_t.$$

This justifies the following dynamics for the nonrisky asset component of wealth process:

$$dX_t \;=\; (rX_t - C_t)dt - (1+\lambda)dL_t + (1-\mu) \left[(1-\alpha) Y_{t-} + \alpha K_{t-} \right] dM_t, \quad t \geq 0. \tag{2.6}$$

We denote by $\mathcal{A}$ the set of all control processes and by $S = (X, Y, K)$ the corresponding state process defined by (2.4), (2.5), (2.6). A control process ν is said to be admissible if the no bankruptcy condition

$$Z_t \;:=\; X_t + (1-\mu)\left[(1-\alpha)Y_t + \alpha K_t\right] \;\geq\; 0, \quad t \geq 0, \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

holds. Here Z_t is the after-tax and after-transaction costs liquidation value of the portfolio at time t. Given an initial condition $S_{0-} = s$, we shall denote by $\mathcal{A}(s)$ the collection of all admissible controls.

The problem of optimal consumption and investment under capital gains taxes is defined by the value function

$$V(s) \;:=\; \sup_{\nu \in \mathcal{A}(s)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t)dt\right], \quad \text{where } U(x) \;:=\; \frac{x^p}{p}, \quad x \geq 0, \tag{2.8}$$

and $\beta > 0$, $p \in (0,1)$ are two given constant parameters.

Throughout this paper, we assume that the coefficients of the model satisfy the condition

$$\frac{\beta}{p} - r - \frac{(\delta - r)^2}{2(1-p)\sigma^2} > 0, \tag{2.9}$$

which ensures that the value function of the Merton optimal consumption-investment problem (the case $\lambda = \mu = \alpha = 0$) is finite. In particular, the value function V is finite under condition (2.9).

2.2. The DPE. For an admissible control $\nu \in \mathcal{A}(s)$, the induced state process $S^\nu = (X^\nu, Y^\nu, K^\nu)$ defined by (2.4), (2.5), (2.6) together with some initial data $S_0^\nu = s$ is valued in the state space

$$\bar{\mathcal{S}} := \left\{(x,y,k) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \;:\; z = x + (1-\mu)[(1-\alpha)y + \alpha k] \geq 0\right\}. \tag{2.10}$$

We denote by $\mathcal{S} := \mathrm{int}(\bar{\mathcal{S}})$ the interior of $\bar{\mathcal{S}}$, and we decompose the boundary of this state space into $\partial\mathcal{S} = \partial^y\mathcal{S} \cup \partial^k\mathcal{S} \cup \partial^z\mathcal{S}$, where

$$\partial^y\mathcal{S} = \{s \in \mathcal{S} \;:\; y = 0\},\; \partial^k\mathcal{S} = \{s \in \mathcal{S} \;:\; k = 0\},\; \text{and } \partial^z\mathcal{S} = \{s \in \mathcal{S} \;:\; z = 0\}.$$

Observe that the value function is not known on all of the boundary of the state space $\mathcal{S}$. It is shown in [5] that the only boundary information is

$$V(s) \;=\; 0 \text{ for all } s \in \partial^z\mathcal{S}. \tag{2.11}$$

The main result of this section states that the value function V defined in (2.8) solves the corresponding DPE

$$F(s, v, Dv, D^2v) \;:=\; \min\left\{-\mathcal{L}v,\; \mathbf{g^b}\cdot Dv,\; \mathbf{g^s}\cdot Dv\right\} \;=\; 0 \text{ on } \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}, \tag{2.12}$$

where $\mathcal{L}$ is the second order differential operator

$$\mathcal{L}\varphi(s) := -\beta\varphi(s) + rx\varphi_x(s) + \rho y\varphi_y(s) + \frac{1}{2}\sigma^2y^2\varphi_{yy}(s) + \tilde{U}(\varphi_x(s)), \tag{2.13}$$

$\tilde{U}$ is the Fenchel dual defined by

$$\tilde{U}(\xi) \;:=\; \sup_{c>0}\left(U(c) - c\xi\right) \text{ for all } \xi > 0, \tag{2.14}$$

and $\mathbf{g^b}$, $\mathbf{g^s}$ are the vector fields defined by

$$\mathbf{g^b} := \begin{pmatrix} 1+\lambda \\ -1 \\ -1 \end{pmatrix}, \mathbf{g^s}(s) := \begin{pmatrix} -(1-\mu) \\ \frac{1}{1-\alpha} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\alpha}{1-\alpha} \\ 1 \end{pmatrix} \frac{k\,\mathbf{1}_{(y,k)\neq 0}}{(1-\alpha)\,y+\alpha k}. \tag{2.15}$$

The DPE can be written in different forms by taking other vector fields which are parallel to our choices $\mathbf{g^b}, \mathbf{g^s}$. Since our choice for $\mathbf{g^s}$ is discontinuous and this fact is central to many of the technicalities, one may propose to choose a parallel vector field which is continuous. However, in singular stochastic control, if the vector fields appearing in the equation vanish (which is the case here if we choose continuous vector fields), then the first order part of the equation (i.e., the part $\mathbf{g^s} \cdot Dv$ in the above particular case) becomes degenerate. Indeed, this degeneracy is equivalent to the technical difficulties related to the discontinuity of the vector fields. For this reason, it is standard in singular control to decide that these vector fields are nondegenerate and close to unit vector fields.

Since we have no knowledge of any a priori regularity of the value function V, we will use the theory of viscosity solutions. This notion allows for a weak formulation of solutions to second order parabolic PDEs and boundary conditions; see [23, 9].

In what follows we use the following classical notation from viscosity theory. For a locally bounded function $v\ :\bar{\mathcal{S}} \longrightarrow \mathbb{R}$, we denote the corresponding upper and lower semicontinuous envelopes by

$$v^*(s) \ :=\ \limsup_{\mathcal{S}\ni s'\to s}\ v(s') \text{ and } v_*(s)\ :=\ \liminf_{\mathcal{S}\ni s'\to s}\ v(s').$$

The notation F_* in the subsequent definition is defined similarly. Observe that $F = F_*$ outside the axis $\{(x,0,0):\ x \geq 0\}$.

Definition 2.1. (i) *A locally bounded function v is a constrained viscosity subsolution of* (2.11)–(2.12) *if $v^* \leq 0$ on $\partial^z\mathcal{S}$, and for all $s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$ and $\varphi \in C^2(\bar{\mathcal{S}})$ with $(v^*-\varphi)(s) = \max_{\bar{\mathcal{S}}}(v^*-\varphi)$ we have $F_*\left(s, v(s), D\varphi(s), D^2\varphi(s)\right) \leq 0$.*

(ii) *A locally bounded function v is a viscosity supersolution of* (2.11)–(2.12) *if $v_* \geq 0$ on $\partial^z\mathcal{S}$, and for all $s \in \mathcal{S}$ and $\varphi \in C^2(\mathcal{S})$ with $(v_*-\varphi)(s) = \min_{\mathcal{S}}(v_*-\varphi)$ we have $F\left(s, v(s), D\varphi(s), D^2\varphi(s)\right) \geq 0$.*

(iii) *A locally bounded function v is a constrained viscosity solution of* (2.11)–(2.12) *if it is a constrained viscosity subsolution and supersolution.*

In the above definition, observe that there is no boundary value assigned to the value function on $\partial^y\mathcal{S}\cup\partial^k\mathcal{S}$. Instead, the subsolution property holds on this boundary. Notice that the supersolution property is satisfied only in the interior of the domain $\mathcal{S}$.

Proposition 2.2. *The value function V is a constrained viscosity solution of* (2.11)–(2.12).

The proof is reported in section 5 for the case $\varepsilon = 0$. In the accompanying paper [5] a numerical scheme based on the finite differences approximation of the DPE (2.11)–(2.12) is implemented. In order for us to justify this algorithm, we need a uniqueness result for this DPE. As it is usually the case for parabolic second order equations, uniqueness follows as a consequence of a comparison result. At this point, a chief difficulty is encountered: the vector field $\mathbf{g^s}$ is not locally Lipschitz on the axis $\{(x,0,0),\ x \geq 0\}$. Because of this problem, the standard techniques for the derivation of a comparison result for the DPE (2.11)–(2.12) fail.

Remark 1. Consider the Lipschitz vector field $\mathbf{G^s} := (-(1-\mu)[(1-\alpha)y+\alpha k], y, k) = [(1-\alpha)y+\alpha k]\mathbf{g^s}$. Then, the supersolutions of (2.11)–(2.12) coincide with those of

$$\min\left\{-\mathcal{L}v\ ,\ \mathbf{g^b}\cdot Dv,\ \mathbf{G^s}\cdot Dv\right\} \geq 0 \text{ on } \bar{\mathcal{S}}\setminus\partial^z\mathcal{S} \text{ and } v=0 \text{ on } \partial^z\mathcal{S}. \tag{2.16}$$

However, these two equations do not have the same set of subsolutions. The reason for this is that the subsolution property must hold also on the boundary $\partial^y\mathcal{S}\cup\partial^k\mathcal{S}$. Since $\mathbf{G^s}(x,0,0)=0$ for every $x\geq 0$, (2.16) provides no information on this axis. Notice, however, that $\lim_{n\to\infty}\mathbf{g^s}(s_n)$ exists for some sequences $s_n\to(x,0,0)$, and might be nonzero, so that (2.12) bears more information on this axis.

This remark justifies that the above mentioned difficulty can be avoided if a priori comparison on the axis $\{(x,0,0)\ :\ x\geq 0\}$ is available.

PROPOSITION 2.3. *Let $\lambda+\mu>0$. Let u be an upper semicontinuous constrained viscosity subsolution of* (2.11)–(2.12) *and v be a lower semicontinuous viscosity supersolution of* (2.11)–(2.12) *with $(u-v)^+\in \mathrm{USC}_p(\bar{\mathcal{S}})$. Assume further that $(u-v)(x,0,0)\leq 0$ for all $x\geq 0$. Then $u\leq v$ on $\bar{\mathcal{S}}$.*

The proof of this comparison result is given at the end of section 4. Unfortunately, this result does not provide uniqueness of a constrained viscosity solution for the DPE (2.11)–(2.12), as we have no a priori comparison of two possible solutions on the axis $\{(x,0,0)\ :\ x\geq 0\}$.

The chief goal of this paper is to obtain an alternative characterization of V by considering a convenient approximating PDE which has a unique solution converging to our value function V. Before turning to this issue, we report the following continuity property from [5] which follows from Proposition 2.3.

PROPOSITION 2.4 (see [5]). *Let $\lambda+\mu>0$. For $s=(x,y,k)\in\bar{\mathcal{S}}$ and $z:=x+(1-\mu)[(1-\alpha)y+\alpha k]$, we have $V(s)=z^p\ \mathcal{V}\left(\frac{y}{z},\frac{k}{z}\right)$, where $\mathcal{V}$ is a Lipschitz-continuous function on $\mathbb{R}^2_+$.*

3. The main results. For every $\varepsilon>0$ and $s=(x,y,k)\in\bar{\mathcal{S}}$, we define

$$f^\varepsilon(s)\ :=\ h\left(\frac{k}{\varepsilon z}\right)^+,\ \text{where } z:=x+(1-\mu)\left[(1-\alpha)y+\alpha k\right], \tag{3.1}$$

and h is a nondecreasing $C^2\left(\mathbb{R}_+\right)$-function with

$$h=0\ \text{ on }\ [0,1]\ \text{ and }\ h=1\ \text{ on }\ [2,\infty).$$

For $\varepsilon=0$, we set $f^0(s)=1$.

We next introduce, for all $\varepsilon\geq 0$, the approximation $\mathbf{g}^\mathbf{s}_\varepsilon$ of $\mathbf{g^s}$:

$$\mathbf{g}^\mathbf{s}_\varepsilon(s)\ :=\ \mathbf{g^s}\left(x,y,kf^\varepsilon(s)\right) \text{ for } s=(x,y,k)\in\bar{\mathcal{S}}, \tag{3.2}$$

and the corresponding approximation of the DPE (2.11)–(2.12):

$$\min\left\{-\mathcal{L}v,\ \mathbf{g^b}\cdot Dv,\ \mathbf{g}^\mathbf{s}_\varepsilon\cdot Dv\right\}=0 \text{ on } \bar{\mathcal{S}}\setminus\partial^z\mathcal{S} \text{ and } v=0 \text{ on } \partial^z\mathcal{S}. \tag{3.3}$$

A constrained viscosity solution of this equation is defined exactly as in Definition 2.1, replacing $\mathbf{g^s}$ by $\mathbf{g}^\mathbf{s}_\varepsilon$. For each $\varepsilon>0$ the approximation $\mathbf{g}^\mathbf{s}_\varepsilon$ is Lipschitz-continuous on $\bar{\mathcal{S}}\setminus\partial^z\mathcal{S}$, and this property is sufficient to obtain the following comparison result.

THEOREM 3.1. *Let $\lambda+\mu>0$ and $\varepsilon>0$. Let u be an upper semicontinuous constrained viscosity subsolution of* (3.3) *and v be a lower semicontinuous viscosity* *supersolution of* (3.3) *with* $(u-v)^+ \in \mathrm{USC}_p(\bar{\mathcal{S}})$. *Assume further that* $u \le v$ *on* $\partial^z\mathcal{S}$. *Then* $u \le v$ *on* $\bar{\mathcal{S}}$.

This result is proved in section 4 and implies, as usual, a uniqueness result for the approximating PDE (3.3) for every $\varepsilon > 0$. We can now state our main DPE characterization of the value function V which justifies the numerical scheme implemented in the accompanying paper [5].

THEOREM 3.2. *For every* $\varepsilon > 0$, *there exists a unique constrained viscosity solution* V_ε *for the nonlinear parabolic PDE* (3.3) *in the class* C^0_p. *Moreover, the family* $(V_\varepsilon)_{\varepsilon>0}$ *is nondecreasing and converges to the value function* V *uniformly on compact subsets of* $\bar{\mathcal{S}}$ *as* $\varepsilon \searrow 0$.

The existence of a solution for the approximating PDE (3.3) is proved in section 5 by conveniently modifying the optimal investment problem under capital gains taxes, and showing that the induced value function V_ε is a constrained viscosity solution of (3.3). Moreover, we will prove in Proposition 6.2 that $0 \le V_\varepsilon \le V$, so that V_ε inherits the p-polynomial growth of V stated in [5]. Together with the comparison result of Theorem (3.1), this shows that V^ε is the unique constrained viscosity solution in C^0_p. The convergence result is proved in section 6.

4. The comparison result. We adapt the standard argument based on the Ishii technique; see Theorem 3.2 and Lemma 3.1 in [9]. The subsequent proof is also inspired from [1]. In comparison to the latter paper, we have the additional difficulty implied by the state constraint $(y,k) \in \mathbb{R}^2_+$. We use the idea of Theorem 7.9 in [9] to account for this avoidance of the continuity assumptions of this theorem. We mention that comparison results for second order PDEs with state constraints have been obtained for specific control problems in [2] and [3] but do not apply to our context. In the subsequent analysis, the key result to avoid the continuity is the observation that

$$\begin{aligned} &\text{for each } s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}, \text{ there exists some } \zeta_s > 0 \text{ such that} \\ &\qquad s - \zeta\mathbf{g^b} \in \mathcal{S} \text{ for every } 0 < \zeta < \zeta_s, \end{aligned} \tag{4.1}$$

together with the following.

LEMMA 4.1. *Let* $v \in \mathrm{LSC}(\bar{\mathcal{S}})$ *be such that* $v(s_0) = \liminf_{\mathcal{S}\ni s\to s_0} v(s)$ *for* $s_0 \in \partial\mathcal{S}$. *Assume that* $\mathbf{g^b}\cdot Dv \ge 0$ *on* $\mathcal{S}$ *in the viscosity sense. Then*

$$\lim_{\ell\searrow 0} v(s-\ell\mathbf{g^b}) = v(s) \textit{ for any } s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}.$$

Proof. Since v is a viscosity supersolution of $\mathbf{g^b}\cdot Dv \ge 0$ on $\mathcal{S}$ and (4.1) holds, we deduce that, for any $s \in \mathcal{S}$, the function $\ell \longmapsto v(s-\ell\mathbf{g^b})$ is well defined and nonincreasing on a neighborhood of 0. In particular, $v(s-\ell\mathbf{g^b}) \le v(s)$ for any $s \in \mathcal{S}$, and $\ell \ge 0$ sufficiently small. For $s_0 \in \partial\mathcal{S}$, it follows from the assumption of the lemma that $v(s_0) = \liminf_{\mathcal{S}\ni s\to s_0} v(s) \ge \liminf_{\mathcal{S}\ni s\to s_0} v(s'-\ell\mathbf{g^b}) \ge v(s_0-\ell\mathbf{g^b})$. Hence

$$v(s-\ell\mathbf{g^b}) \le v(s) \text{ for any } s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S} \text{ and } \ell \ge 0.$$

This implies that, for any $s \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$,

$$v(s) \ge \limsup_{\ell\searrow 0} v(s-\ell\mathbf{g^b}) \ge \liminf_{\ell\searrow 0} v(s-\ell\mathbf{g^b}) \ge \liminf_{\mathcal{S}\ni s'\to s} v(s') \ge v(s),$$

completing the proof. $\square$

Another important ingredient of our comparison result is the use of a strict supersolution of the equation

$$\min\{\mathbf{g}^{\mathbf{b}} \cdot Dv,\ \mathbf{g}^{\mathbf{s}}_{\varepsilon} \cdot Dv\} \ = \ 0 \text{ on } \bar{\mathcal{S}} \setminus \partial^z \mathcal{S}. \tag{4.2}$$

This is the only place where the presence of transaction costs is crucial.

LEMMA 4.2. *Let $\lambda + \mu > 0$ and assume that condition* (2.9) *holds. Then, there exist two positive parameters*

$$0 \ < \ \bar{\eta} \ < \ \frac{\lambda + \mu}{2} \text{ and } \delta \in (p, 1) \text{ with } \frac{\beta}{\delta} - r - \frac{\theta^2}{2(1-\delta)} \ > \ 0$$

such that the function

$$\Phi(s) \ := \ (x + (1-\mu)\left[(1-\alpha+\bar{\eta})y + (\alpha+\bar{\eta})k\right])^{\delta} \text{ for } s \ \in \ \bar{\mathcal{S}}$$

is a classical strict supersolution of (4.2).

Proof. We show only that $\mathbf{g}^{\mathbf{s}}_{\varepsilon} \cdot D\Phi > 0$, as the other strict are easily seen to hold. Setting $\tilde{z} := x + (1-\mu)\left[(1-\alpha+\bar{\eta})y + (\alpha+\bar{\eta})k\right]$, we directly compute that

$$\left(\mathbf{g}^{\mathbf{s}}_{\varepsilon} \cdot D\Phi\right)(s) = \frac{(1-\mu)\bar{\eta}}{1-\alpha} \tilde{z}^{\delta-1} \left[1 + (1-2\alpha) \frac{k f^{\varepsilon}(s)}{(1-\alpha)y + \alpha k f^{\varepsilon}(s)}\right].$$

If $y = k = 0$ or $1 - 2\alpha \geq 0$, the required inequality is trivial. We next assume that $(y, k) \neq 0$ and $1 - 2\alpha < 0$. Then using the fact that $f^{\varepsilon}(s) \leq 1$, it follows that

$$\begin{aligned}\left(\mathbf{g}^{\mathbf{s}}_{\varepsilon} \cdot D\Phi\right)(s) &\geq \frac{(1-\mu)\bar{\eta}}{1-\alpha} \tilde{z}^{\delta-1} \left[1 + (1-2\alpha) \frac{k}{(1-\alpha)y + \alpha k)}\right] \\ &= \frac{(1-\mu)\bar{\eta}}{1-\alpha} \tilde{z}^{\delta-1} \frac{(1-\alpha)(y+k)}{(1-\alpha)y + \alpha k} \ > \ 0. \qquad \Box\end{aligned}$$

We are now ready for the following proof.

Proof of Theorem 3.1. We start by setting a new notation. We denote by $\tilde{\mathcal{L}}$ the operator

$$\tilde{\mathcal{L}}(s, u, q, Q) \ := \ -\beta u + r x q_1 + \rho y q_2 + \frac{1}{2} \sigma^2 Q_{22}$$

for $s = (x, y, k) \in \bar{\mathcal{S}}$, $u \in \mathbb{R}$, $q = (q_i)_{1 \leq i \leq 3} \in \mathbb{R}^3$, and $Q = (Q_{i,j})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} \in \mathbb{S}(3)$, so that the second order operator $\mathcal{L}$ can be written as

$$\mathcal{L}\varphi(s) \ = \ \tilde{\mathcal{L}}\left(s, \varphi(s), D\varphi(s), D^2\varphi(s)\right) + \tilde{U}(\varphi_x(s)).$$

Let u and v be as in the statement of Theorem 3.1, and let us prove that $u \leq v$ in $\bar{\mathcal{S}}$.

We first observe that we can assume, without loss of generality, that

$$v(s) \ = \ \liminf\left\{v(s') \ : \ s' \in \mathcal{S} \text{ and } s' \neq s\right\} \ \text{ for every } s \in \partial^y \mathcal{S} \cup \partial^k \mathcal{S}. \tag{4.3}$$

Indeed, we may define the function $\underline{v} := v$ on $\mathcal{S} \cup \partial^z \mathcal{S}$ and $\underline{v}(s) := \liminf_{s \neq s' \to s} v(s')$ for $s \in \partial^y \mathcal{S} \cup \partial^k \mathcal{S}$. Then, $\underline{v}$ satisfies the same conditions as v, and if we succeed in proving that $u \leq \underline{v}$, we deduce immediately that $u \leq v$ since the inequality $\underline{v} \leq v$ is trivial.

We now start the proof of the comparison result with the additional condition (4.3). Assume to the contrary that

$$(u - v)\,(s^*) \;>\; 0 \text{ for some } s^* \in \bar{\mathcal{S}}\,. \tag{4.4}$$

Step 1. Let Φ be the strict supersolution of (4.2) defined in Lemma 4.2, and $\eta > 0$, $\zeta > 0$ be some fixed parameters such that

$$m_0 \;:=\; (u - v)(s_0) - 2\eta\Phi(s_0) - \zeta|\mathbf{g^b}|^2 = \max_{s\in\bar{\mathcal{S}}}\,(u - v - 2\eta\Phi) - \zeta|\mathbf{g^b}|^2 \;>\; 0 \tag{4.5}$$

by (4.4), where the maximum is attained thanks to the p-polynomial growth condition on $(u-v)^+$ and the fact that $\delta > p$. In particular, it follows from (4.5), together with $\Phi \geq 0$, $u \leq v$ on $\partial^z\mathcal{S}$ and (4.1), that

$$s_0 \;\in\; \bar{\mathcal{S}} \setminus \partial^z\mathcal{S} \text{ and } s_0 - \zeta\mathbf{g^b} \;\in\; \mathcal{S} \text{ for small } \zeta > 0. \tag{4.6}$$

We next define the mappings on $\bar{\mathcal{S}} \times \bar{\mathcal{S}}$ by

$$\begin{aligned}
\Psi_n(s,s') &:= (u - \eta\Phi)(s) - (v + \eta\Phi)(s') - \psi_n(s,s'),\\
\psi_n(s,s') &:= \left|n(s - s') - \zeta\mathbf{g^b}\right|^2 + \zeta|s - s_0|^2\,.
\end{aligned}$$

Here, $\zeta \in (0,1)$ is some given constant. From the p-polynomial growth condition on $(u - v)^+$ and the fact that $\delta > p$ in the definition of Φ, we see that the upper semicontinuous function Ψ_n attains its maximum at some (s_n, s'_n) in $\bar{\mathcal{S}} \times \bar{\mathcal{S}}$, so that by (4.5),

$$m_n \;:=\; \Psi_n(s_n, s'_n) \;=\; \max_{(s,s')\in\bar{\mathcal{S}}\times\bar{\mathcal{S}}} \Psi_n(s,s') \geq m_0 \;>\; 0.$$

By (4.6) and the definition of Ψ_n, we have the inequality $\Psi_n(s_n, s'_n) \geq \Psi_n(s_0, s_0 - \frac{\zeta}{n}\mathbf{g^b})$ which, together with the p-polynomial growth condition on u and v, provides

$$\begin{aligned}
\left|n(s_n - s'_n) - \zeta\mathbf{g^b}\right|^2 + \zeta\left|s_n - s_0\right|^2 &\leq (u - \eta\Phi)(s_n) - (v + \eta\Phi)(s'_n)\\
&\qquad - (u - \eta\Phi)(s_0) + (v + \eta\Phi)\left(s_0 - \frac{\zeta}{n}\mathbf{g^b}\right)\\
&\leq \tilde{A}\left(1 + |s_n|^p + |s'_n|^p + \eta|s_n|^\delta + \eta|s'_n|^\delta\right)
\end{aligned} \tag{4.7}$$

for some positive constant $\tilde{A}$. We deduce from the last inequality that the sequences $(s_n)_{n\geq1}$ and $(s'_n)_{n\geq1}$ are bounded, and we can assume, without loss of generality, that $s_n, s'_n \longrightarrow \hat{s} \in \bar{\mathcal{S}}$ as $n \to \infty$. We now use Lemma 4.1, together with the upper semicontinuity of u and the lower semicontinuity of v, to pass to the limit as $n \to \infty$ in (4.7). This provides

$$\begin{aligned}
\limsup_{n\to\infty}\left(\left|n(s_n - s'_n) - \zeta\mathbf{g^b}\right|^2 + \zeta\left|s_n - s_0\right|^2\right) &\leq (u - \eta\Phi)(\hat{s}) - (v + \eta\Phi)(\hat{s})\\
&\qquad - \left((u - \eta\Phi)(s_0) - (v - \eta\Phi)(s_0)\right)\\
&\leq 0,
\end{aligned}$$

where the last inequality follows from (4.5). Consequently

$$\left|n(s_n - s'_n) - \zeta\mathbf{g^b}\right|^2 \;\longrightarrow\; 0 \text{ and } s_n, s'_n \;\longrightarrow\; s_0 \text{ as } n \to \infty.$$

In particular, it follows from (4.6) that

$$s'_n \;=\; s_n - \frac{\zeta \mathbf{g^b} + o(1)}{n} \;\in\; \mathcal{S} \text{ and } s_n \;\in\; \bar{\mathcal{S}} \setminus \partial^z \mathcal{S} \text{ for large } n\,. \tag{4.8}$$

Step 2. For each $n \geq 1$, (s_n, s'_n) is a maximum point of

$$\Psi_n \;:\; (s, s') \longmapsto (u - \eta\Phi)(s) - (v + \eta\Phi)(s') - \psi_n(s, s').$$

Then applying Theorem 3.2 in [9] to the upper semicontinuous functions $u - \eta\Phi$ and to the lower semicontinuous function $v + \eta\Phi$, we deduce that there exist 3×3 symmetric matrices Ξ_n and Υ_n, with $\Xi_n \;\leq\; \Upsilon_n$ such that

$$j_n := \left(q_n := D_1\psi(s_n, s'_n) + \eta D\Phi(s_n); Q_n := \Xi_n + \eta D^2\Phi(s_n)\right) \;\in\; \bar{J}^{2,+}_{\bar{\mathcal{S}}\setminus\partial^z\mathcal{S}} u(s_n), \tag{4.9}$$

$$j'_n := \left(q'_n := -D_2\psi(s_n, s'_n) - \eta D\Phi(s'_n); Q'_n := \Upsilon_n - \eta D^2\Phi(s'_n)\right) \;\in\; \bar{J}^{2,-}_{\mathcal{S}} v(s'_n), \tag{4.10}$$

and

$$-\left(2n^2 + ||M_n||\right) I \;\leq\; \begin{pmatrix} \Xi_n & 0 \\ 0 & -\Upsilon_n \end{pmatrix} \;\leq\; M_n + \frac{1}{2n^2} M - n^2, \tag{4.11}$$

where

$$M_n \;:=\; D^2\psi(s_n, s'_n) \;=\; 2n^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\zeta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_1\psi(s, s') = 2n\left(n\,(s - s') - \zeta\mathbf{g^b}\right) + 2\zeta(s - s_0), \quad -D_2\psi(s, s') = 2n\left(n\,(s - s') - \zeta\mathbf{g^b}\right).$$

Here the norm of a symmetric matrix M is defined as $\|M\| = \sup\{M\xi \cdot \xi \;:\; |\xi| \leq 1\,\}$.

By (4.8) and for large $n \geq 1$, the subsolution property of u holds at j_n and the supersolution property of v holds at j'_n, i.e.,

$$\min\left\{\beta u(s_n) - \tilde{\mathcal{L}}(s_n, q_n, Q_n) - \tilde{U}(q_{n\,1}),\; \mathbf{g^b} \cdot q_n,\; \mathbf{g^s_\varepsilon}(s_n) \cdot q_n\right\} \leq 0, \tag{4.12}$$

$$\min\left\{\beta v(s'_n) - \tilde{\mathcal{L}}(s'_n, q'_n, Q'_n) - \tilde{U}(q'_{n\,1}),\; \mathbf{g^b} \cdot q'_n,\; \mathbf{g^s_\varepsilon}(s'_n) \cdot q'_n\right\} \geq 0. \tag{4.13}$$

Step 3. For each $n \geq 1$,

$$\mathbf{g^b} \cdot q_n - \mathbf{g^b} \cdot q'_n = \eta \mathbf{g^b} \cdot (D\Phi(s_n) + D\Phi(s'_n)) + 2\zeta \mathbf{g^b} \cdot (s_n - s_0)\,.$$

Recall that $s_n, s'_n \longrightarrow s_0 \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$, and $\mathbf{g^b} \cdot \Phi > 0$ on $\bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$; then

$$\lim_{n\to\infty} \left(\mathbf{g^b} \cdot q_n - \mathbf{g^b} \cdot q'_n\right) = 2\eta \mathbf{g^b} \cdot D\Phi(s_0) \;>\; 0. \tag{4.14}$$

We also compute for all $n \geq 1$ that

$$\begin{aligned}\mathbf{g^s_\varepsilon}(s_n) \cdot q_n - \mathbf{g^s_\varepsilon}(s'_n) \cdot q'_n = \eta\left(\mathbf{g^s_\varepsilon}(s_n) \cdot D\Phi(s_n) + \mathbf{g^s_\varepsilon}(s'_n) \cdot D\Phi(s'_n)\right) + 2\zeta \mathbf{g^s_\varepsilon}(s_n) \cdot (s_n - s_0) \\ + \left(\mathbf{g^s_\varepsilon}(s_n) - \mathbf{g^s_\varepsilon}(s'_n)\right) \cdot 2n\left[n(s_n - s'_n) - \zeta\mathbf{g^b}\right].\end{aligned}$$

By the local Lipschitz continuity of the function $\mathbf{g^s_\varepsilon}$ at s_0, there exists some positive constant C_0 such that for large n,

$$\begin{aligned}&|\mathbf{g^s_\varepsilon}(s_n) \cdot q_n - \mathbf{g^s_\varepsilon}(s'_n) \cdot q'_n - \eta\left(\mathbf{g^s_\varepsilon}(s_n) \cdot D\Phi(s_n) + \mathbf{g^s_\varepsilon}(s'_n) \cdot D\Phi(s'_n)\right)| \\ &\quad\leq\; 2\zeta\,|\mathbf{g^s_\varepsilon}(s_n)|\,|s_n - s_0|\,C_0\,|s_n - s'_n|\,2n\left|n\,(s_n - s'_n) - \zeta\mathbf{g^b}\right| \\ &\quad\leq\; 2\zeta\,|\mathbf{g^s_\varepsilon}(s_n)|\,|s_n - s_0|\,2C_0\left|n\,(s_n - s'_n) - \zeta\mathbf{g^b}\right|^2 + 2C_0\zeta|\mathbf{g^b}|\left|n\,(s_n - s'_n) - \zeta\mathbf{g^b}\right|.\end{aligned}$$

Since $s_n \longrightarrow s_0$ and $|n(s_n - s_n') - \zeta \mathbf{g^b}| \longrightarrow 0$, we get

$$\lim_{n\to\infty} (\mathbf{g}^{\mathbf{s}}_{\varepsilon}(s_n)\cdot q_n - \mathbf{g}^{\mathbf{s}}_{\varepsilon}(s_n')\cdot q_n') = 2\eta \mathbf{g}^{\mathbf{s}}_{\varepsilon}\cdot D\Phi(s_0) \;>\; 0. \tag{4.15}$$

We deduce from (4.13), (4.14), and (4.15), together with Lemma 4.2, that for large n,

$$\min\left\{\mathbf{g^b}\cdot q_n,\ \mathbf{g}^{\mathbf{s}}_{\varepsilon}(s_n)\cdot q_n\right\} \geq 2\min\left\{\mathbf{g^b}\cdot D\Phi(s_0),\ \mathbf{g}^{\mathbf{s}}_{\varepsilon}(s_n)\cdot D\Phi(s_0)\right] + o(1) \;>\; 0.$$

Consequently (4.12) implies that for large n,

$$\beta u(s_n) - \tilde{\mathcal{L}}(s_n, q_n, Q_n) - \tilde{U}(q_{n\,1}) \leq 0. \tag{4.16}$$

Step 4. From (4.13) and (4.16), it follows that for large n,

$$\beta u(s_n) - \tilde{\mathcal{L}}(s_n, q_n, Q_n) - \tilde{U}(q_{n\,1}) \leq\; 0 \;\leq \beta v(s_n') - \tilde{\mathcal{L}}(s_n', q_n', Q_n') - \tilde{U}(q_{n\,1}').$$

Using the local Lipschitz continuity property of the function $\tilde{U}$, a direct calculation shows that for some positive constant C and for large n,

$$\begin{aligned}\beta(u(s_n) - v(s_n')) &\leq \tilde{\mathcal{L}}(s_n, q_n, Q_n) - \tilde{\mathcal{L}}(s_n', q_n', Q_n') + \tilde{U}(q_{n\,1}) - \tilde{U}(q_{n\,1}')\\ &\leq C\left(|s_n|\zeta|s_n - s_0| + |n(s_n - s_n') - \zeta\mathbf{g^b}|^2 + |D\Phi(s_n) - D\Phi(s_n')|\right)\\ &\quad + \frac{\sigma^2}{2}\left(y_n^2(Q_n)_{22} - (y_n')^2(Q_n')_{22}\right)\\ &\quad + \eta\left\{\tilde{\mathcal{L}}(s_n, D\Phi(s_n), D^2\Phi(s_n)) + \tilde{\mathcal{L}}(s_n', D\Phi(s_n'), D^2\Phi(s_n'))\right\}.\end{aligned}$$

From (4.11), we have that

$$\left(y_n^2(Q_n)_{22} - (y_n')^2(Q_n')_{22}\right) \leq 4\zeta y_n(u_n - y_n') + \frac{\zeta^2}{n^2} y_n.$$

Moreover, the mapping Φ satisfies $\beta\Phi(.) - \tilde{\mathcal{L}}(., D\Phi, D^2\Phi)$ on $\bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$, and hence for some positive constant $\tilde{C}$ and for large n,

$$\begin{aligned}&\beta[u(s_n) - v(s_n')] - \eta\Phi(s_n) - \eta\Phi(s_n')\\ &\quad\leq \tilde{\mathcal{L}}(s_n, q_n, Q_n) - \tilde{\mathcal{L}}(s_n', q_n', Q_n') + \tilde{U}(q_{n\,1}) - \tilde{U}(q_{n\,1}')\\ &\quad\leq \tilde{C}\left\{\frac{1}{n^2} + |s_n|\zeta|s_n - s_0| + |n(s_n - s_n') - \zeta\mathbf{g^b}|^2 + |D\Phi(s_n) - D\Phi(s_n')|\right\},\end{aligned}$$

where the right-hand side of the inequality goes to zero as $n \to \infty$. This implies

$$\beta[u(s_0) - v(s_0)] - 2\eta\Phi(s_0) = \limsup_{n\to\infty}\ (\beta[u(s_n) - v(s_n')] - \eta\Phi(s_n) - \eta\Phi(s_n')) \;\leq\; 0,$$

contradicting (4.5). ∎

We conclude this section with the following proof.

Proof of Proposition 2.3. We use the same arguments as in the proof of Theorem 3.1, but this time substituting $\mathbf{g^s}$ for $\mathbf{g}^{\mathbf{s}}_{\varepsilon}$. The only difference is the following. The maximizer s_0 in (4.5) is now known to be in $\bar{\mathcal{S}} \setminus (\partial^z\mathcal{S} \cup \{(x,0,0):\ x \geq 0\})$, as it is assumed in the statement of the proposition that $u \leq v$ on $\partial^z\mathcal{S} \cup \{(x,0,0):\ x \geq 0\}$. Then, the sequences $(s_n)_n$ and $(s_n')_n$, defined in Step 1, are valued in a ball around s_0 which does not intersect the axis $\{(x,0,0):\ x \geq 0\}$. Since $\mathbf{g^s}$ is locally Lipschitz on $\bar{\mathcal{S}} \setminus \{(x,0,0):\ x \geq 0\}$, we just follow along the lines of the previous proof. ∎

5. An approximating control problem. Let $s = (x, y, k)$ be an initial condition in the state space $\bar{\mathcal{S}}$, and consider a control process $\nu \in \mathcal{A}$, i.e., a triple of $\mathbb{F}$-adapted processes $\nu = (C, L, M)$, with nondecreasing right-continuous processes L, M, $L_{0-} = M_{0-} = 0$ and satisfying conditions (2.2) and (2.3). For every parameter $\varepsilon \geq 0$, we denote by $S^{\varepsilon,s,\nu} = (X^{\varepsilon,s,\nu}, Y^{\varepsilon,s,\nu}, K^{\varepsilon,s,\nu})$ the unique strong solution of

(5.1) $dX_t^\varepsilon = (rX_t^\varepsilon - C_t)dt - (1+\lambda)dL_t + (1-\mu)\left[(1-\alpha)Y_{t-}^\varepsilon + \alpha f^\varepsilon(S_{t-}^\varepsilon)K_{t-}^\varepsilon\right] dM_t,$

(5.2) $dY_t^\varepsilon = Y_t^\varepsilon\left[\rho dt + \sigma dW_t\right] + dL_t - Y_{t-}^\varepsilon dM_t,$

(5.3) $dK_t^\varepsilon = dL_t - f^\varepsilon(S_{t-}^\varepsilon)K_{t-}^\varepsilon dM_t$

with initial condition $S_{0-}^{\varepsilon,s,\nu} = s$. With this definition, observe that the jumps of the state processes $S^{\varepsilon,s,\nu}$ are given by

$$\Delta S_t^{\varepsilon,s,\nu} = -\Delta L_t\, \mathbf{g^b} - \Delta M_t\, \left[(1-\alpha)Y_{t-}^{\varepsilon,s,\nu} + \alpha f^\varepsilon(S_{t-}^{\varepsilon,s,\nu})K_{t-}^{\varepsilon,s,\nu}\right]\, \mathbf{g}_\varepsilon^\mathbf{s}\left(S_{t-}^{\varepsilon,s,\nu}\right),$$

where the vector fields $\mathbf{g^b}$ and $\mathbf{g}_\varepsilon^\mathbf{s}$ are defined as in (2.15) and (3.2).

A control process $\nu = (C, L, M)$ is said to be (s, ε)-admissible if the corresponding state process $S^{\varepsilon,s,\nu}$ is valued in $\bar{\mathcal{S}}$. We shall denote by $\mathcal{A}^\varepsilon(s)$ the collection of all (s, ε)-admissible controls.

For every initial condition $s \in \bar{\mathcal{S}}$, $\varepsilon \geq 0$, and (ε, s)-admissible control $\nu = (C, L, M)$, we introduce the criterion

$$J_T^\varepsilon(s,\nu) := \mathbb{E}\left[\int_0^T e^{-\beta t}U(C_t)dt + e^{-\beta T}U(Z_T^{\varepsilon,s,\nu})\mathbf{1}_{T<\infty}\right], \qquad T \in \mathbb{R}_+ \cup \{\infty\}, \tag{5.4}$$

where U is the power utility function defined in (2.8). The value function V_ε is then defined by

$$V_\varepsilon(s) := \sup_{\nu \in \mathcal{A}^\varepsilon(s)} J_\infty^\varepsilon(s,\nu). \tag{5.5}$$

Remark 2. When $\varepsilon = 0$, the above problem reduces to the optimal investment problem under capital gains taxes reviewed in section 2, in particular $V_0 = V$. For positive ε, the control problem (5.5) can be interpreted as a utility maximization problem with a modified taxation rule. Under this new taxation rule, the tax basis used to evaluate the capital gains is equal to the relative weighted average purchase price as long as the ratio K/Z is larger than 2ε, but it is set to zero when $K/Z < \varepsilon$. Roughly speaking, for $\varepsilon > 0$, the investor pays more taxes than in the original market when the ratio $K/Z < \varepsilon$. Consequently, we expect that V_ε increases towards V as ε goes to zero. This will be proved in Proposition 6.2 below.

The main objective of this section is to prove that the function V_ε is a constrained viscosity solution of the approximating PDE (3.3), thus proving the existence statement in Theorem 3.2. The arguments of this section hold for every $\varepsilon \geq 0$. In particular, the proof of Proposition 2.2 corresponds to the special case $\varepsilon = 0$.

As usual, the key ingredient for deriving the DPE is a dynamic programming principle. We state it here without proof, and we refer the reader to [6, 13, 14].

THEOREM 5.1. *Let $\varepsilon \geq 0$, $s \in \bar{\mathcal{S}}$, and let τ be some $\mathbb{P}$-a.s. finite $\mathbb{F}$-stopping time. Then*

$$V_\varepsilon(s) = \sup_{\nu=(C,L,M)\in\mathcal{A}^\varepsilon(s)} \mathbb{E}\left[\int_0^\tau e^{-\beta t}U(C_t)dt + e^{-\beta\tau}V_\varepsilon\left(S_\tau^{\varepsilon,s,\nu}\right)\right].$$

Before turning to the derivation of the DPE for the problem V_ε, we introduce a notation which will be used frequently in what follows. Let $\varepsilon \geq 0$, $s \in \bar{\mathcal{S}}$, $\nu = (C, L, M) \in \mathcal{A}(s)$, and consider some stopping time τ such that $S^{\varepsilon,s,\nu}_{\tau-} \in \bar{\mathcal{S}}$. Then, it is easy to verify that the strategy $\nu(\tau)$ defined by

$$\nu(\tau)_t \ := \ \left(\bar{C}, \bar{L}, \bar{M}\right) := \nu_t \mathbf{1}_{[0,\tau[}(t) + (0, L_{\tau-}, M_{\tau-} + (1 - \Delta M_\tau))\, \mathbf{1}_{[\tau,\infty)}(t) \tag{5.6}$$

is in $\mathcal{A}^\varepsilon(s)$, and that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} U\left(\bar{C}_t\right) dt\right] = \mathbb{E}\left[\int_0^\tau e^{-\beta t} U\left(\bar{C}_t\right) dt\right]. \tag{5.7}$$

5.1. Supersolution property. In this section, we prove that the value function V_ε is a viscosity supersolution of (3.3) on $\mathcal{S}$ for every $\varepsilon \geq 0$.

Step 1. Fix some $\varepsilon \geq 0$. Recall that $V_\varepsilon \geq 0$ by definition, and in particular $(V_\varepsilon)_*(0) \geq 0$. So it remains to show that, for s_0 in $\mathcal{S}$ and φ in $C^2(\bar{\mathcal{S}})$ such that

$$0 \ = \ \left((V_\varepsilon)_* - \varphi\right)(s_0) = \min_{\mathcal{S}} \left((V_\varepsilon)_* - \varphi\right),$$

the test function φ must satisfy, at the point s_0,

$$\min\left\{-\mathcal{L}\varphi,\ \mathbf{g^b} \cdot D\varphi,\ \mathbf{g^s_\varepsilon} \cdot D\varphi\right\}(s_0) \geq 0.$$

Step 2.1. Let $\eta > 0$ be such that $B(s_0, \eta) \subset \mathcal{S}$, and consider some sequence $(s_n)_{n\geq 1}$ satisfying

$$\begin{aligned} &\text{(i) } B(s_0,\eta) \ni s_n \underset{n\to\infty}{\longrightarrow} s_0, \\ &\text{(ii) } \xi_n \ := \ V_\varepsilon(s_n) - \varphi(s_n) \ \longrightarrow \ 0 \text{ as } n \to 0. \end{aligned}$$

Fix some (c, ℓ, m) in $(0,\infty)^3$, define the strategy $\nu \in \mathcal{A}$ by

$$\nu_t \ = \ \left(C_t = c, L_t = \ell\, t, M_t = m\, t\right),$$

and let $(\tau^n)_{n\geq 0}$ be the stopping times

$$\tau^n \ := \ \inf\left\{t \geq 0 \ : \ S^{\varepsilon,s_n,\nu}_t \notin \mathcal{S}\right\} n \geq 0.$$

Given that for each $n \geq 0$, $s_n \notin \partial^z \mathcal{S}$, and that the strategy ν is continuous, we have

$$\tau^n \ > \ 0 \text{ for all } n \geq 0 \ \text{ and } \ \tau^n \underset{n\to\infty}{\longrightarrow} \tau^0 \mathbb{P}\text{-a.s.} \tag{5.8}$$

Step 2.2. To each $n \geq 1$ we associate the (ε, s_n)-admissible strategy $\nu(\tau^n) = (C^n, L^n, M^n) \in \mathcal{A}^\varepsilon(s_n)$ defined in (5.6). To simplify the notation, we set $S^n := S^{\varepsilon,s_n,\nu(\varepsilon,s_n)}$. For any $\mathbb{P}$-a.s. finite stopping time θ^n, the dynamic programming principle of Theorem 5.1 provides

$$V_\varepsilon(s_n) \geq \mathbb{E}\left[\int_0^{\theta^n \wedge \tau^n/2} e^{-\beta t} U(C^n_t) dt \ + \ e^{-\beta\, \theta^n \wedge \tau^n/2} V_\varepsilon\left(S^n_{\theta^n \wedge \tau^n/2}\right)\right].$$

Notice that $S^n_{\theta^n \wedge \tau^n/2} \in \mathcal{S}$; we then deduce from the inequalities $\varphi \leq (V_\varepsilon)_* \leq V_\varepsilon$ on $\mathcal{S}$ that

$$\xi_n \ + \ \varphi(s_n) \geq \mathbb{E}\left[\int_0^{\theta^n \wedge \tau^n/2} e^{-\beta t} U(C^n_t) dt \ + \ e^{-\beta \theta^n \wedge \tau^n/2} \varphi\left(S^n_{\theta^n \wedge \tau^n/2}\right)\right].$$

By the definition of the strategy $\nu(\tau^n)$, jumps of the process S^n may occur only at the stopping time τ^n, and by definition of the stopping time τ^n, the process $\left\{S_t^n \mathbf{1}_{[0,\tau^n]}(t), t \geq 0\right\}$ is uniformly bounded. Hence, using the Itô formula we get

$$
\begin{aligned}
-\xi_n \leq \mathbb{E}&\left[\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left\{-\mathcal{L}\varphi+\tilde{U}(\varphi_x)-(U(C_t^n)-C_t^n\varphi_x)\right\}(S_t^n)dt\right]\\
&+\ell\,\mathbb{E}\left[\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\mathbf{g^b}\cdot D\varphi(S_t^n)dt\right]\\
&+m\,\mathbb{E}\left[\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left[(1-\alpha)Y_t^n+\alpha f^\varepsilon(S_t^n)K_t^n\right]\left(\mathbf{g}_\varepsilon^\mathbf{s}\cdot D\varphi\right)(S_t^n)dt\right].
\end{aligned}
\tag{5.9}
$$

Step 2.3. Set

$$
\theta_n=\begin{cases}\sqrt{\xi_n} & \text{if } \xi_n>0,\\ n^{-1} & \text{if } \xi_n=0.\end{cases}
$$

Since $\theta^n \longrightarrow 0$ and $\tau^n \longrightarrow \tau^0 > 0$ $\mathbb{P}$-a.s. as $n\to\infty$, it follows that for $\mathbb{P}$-a.e. ω, $\theta^n \wedge \tau^n/2 = \theta^n$ for large n. Rewriting (5.9), and taking the limits as $n\to\infty$, we obtain

$$
\begin{aligned}
0 &= \lim_{n\to\infty} -\frac{\xi_n}{\theta_n},\\
&\leq \liminf_{n\to\infty}\mathbb{E}\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left\{-\mathcal{L}\varphi+\tilde{U}(\varphi_x)-(U(C_t^n)-C_t^n\varphi_x)\right\}(S_t^n)dt\right]\\
&\qquad+\ell\,\mathbb{E}\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\mathbf{g^b}\cdot D\varphi(S_t^n)dt\right]\\
&\qquad+m\,\mathbb{E}\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left[(1-\alpha)Y_t^n+\alpha f^\varepsilon(S_t^n)K_t^n\right]\left(\mathbf{g}_\varepsilon^\mathbf{s}\cdot D\varphi\right)(S_t^n)dt\right].
\end{aligned}
\tag{5.10}
$$

Since $\varphi\in C^2\left(\bar{\mathcal{S}}\right)$, and the process $\left\{S_t^n\mathbf{1}_{[0,\tau^n/2]}(t), t\geq 0\right\}$ is continuous and uniformly bounded, we get by dominated convergence

$$
\begin{aligned}
\liminf_{n\to\infty}\mathbb{E}&\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left\{-\mathcal{L}\varphi+\tilde{U}(\varphi_x)-(U(C_t^n)-C_t^n\varphi_x)\right\}(S_t^n)dt\right]\\
&+\ell\,\mathbb{E}\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\mathbf{g^b}\cdot D\varphi(S_t^n)dt\right]\\
&+m\,\mathbb{E}\left[\frac{1}{\theta_n}\int_0^{\theta^n\wedge\tau^n/2} e^{-\beta t}\left[(1-\alpha)Y_t^n+\alpha f^\varepsilon(S_t^n)K_t^n\right]\left(\mathbf{g}_\varepsilon^\mathbf{s}\cdot D\varphi\right)(S_t^n)dt\right]\\
&= -\mathcal{L}\varphi(s_0)+\tilde{U}(\varphi_x(s_0)-(U(c)-c\,\varphi_x(s_0))\\
&\quad+\ell\,\mathbf{g^b}\cdot D\varphi(s_0)\;+\;m\;\left[(1-\alpha)y_0+\alpha f^\varepsilon(s_0)k_0\right]\mathbf{g}_\varepsilon^\mathbf{s}(s_0)\cdot D\varphi(s_0).
\end{aligned}
$$

Recall (5.10); then

$$
\begin{aligned}
0 \leq &-\mathcal{L}\varphi(s_0)+\tilde{U}(\varphi_x(s_0)-(U(c)-c\,\varphi_x(s_0))\\
&+\ell\,\mathbf{g^b}\cdot D\varphi(s_0)+m\;\left[(1-\alpha)y_0+\alpha f^\varepsilon(s_0)k_0\right]\mathbf{g}_\varepsilon^\mathbf{s}(s_0)\cdot D\varphi(s_0).
\end{aligned}
\tag{5.11}
$$

Step 2.4. Observe that $s_0 \in \mathcal{S}$ implies that $[(1-\alpha)y_0 + \alpha f^\varepsilon(s_0)k_0] > 0$. Since $(c, \ell, m) \in (0,\infty)^3$, (5.11) provides

$$0 \le \min\left\{-\mathcal{L}\varphi,\ \mathbf{g^b} \cdot D\varphi,\ \mathbf{g^s_\varepsilon} \cdot D\varphi\right\}(s_0).$$

5.2. Subsolution property. In this section, we prove that the value function V_ε is a constrained viscosity subsolution of (3.3) for every $\varepsilon \ge 0$. In preparation for this proof, we state some intermediate results.

LEMMA 5.2. *Let φ be a mapping in $C^2(\bar{\mathcal{S}})$, and let $s_0 \in \bar{\mathcal{S}}$ such that $\varphi_x(s_0) > 0$. Then there exists $\eta > 0$, $\gamma > 0$, and $c_0 > 0$ such that*

$$\tilde{U}(\varphi_x(s)) - [U(c) - c\varphi_x(s)] \ \ge\ \gamma (c - c_0)^+ \text{for all } c \ge 0 \text{ and } s \in B(s_0,\eta) \cap \bar{\mathcal{S}}.$$

Proof. Since $\varphi_x(s_0) > 0$, we can find some $\eta, \delta > 0$ such that $\varphi_x > \delta$ on $B(s_0,\eta) \cap \bar{\mathcal{S}}$. The mapping $s \longmapsto \mathcal{I}(\varphi_x(s)) := (U')^{-1}(\varphi_x(s))$ is then bounded on $B(s_0,\eta) \cap \bar{\mathcal{S}}$, and since U' is a decreasing function, we can find $c_0 > 0$ such that

$$c_0 > \max_{B(s_0,\eta)\cap\bar{\mathcal{S}}} \mathcal{I}(\varphi_x) \text{ and } \gamma := \min_{B(s_0,\eta)\cap\bar{\mathcal{S}}} (\varphi_x - U'(c_0)) > 0.$$

For all $s \in B(s_0,\eta) \cap \bar{\mathcal{S}}$, using the nonnegativity and the convexity of the function $c \in \mathbb{R}_+ \longmapsto \tilde{U}(\varphi_x(s)) - (U(c) - c\,\varphi_x(s))$, we get

$$\begin{aligned}
\tilde{U}(\varphi_x(s)) - (U(c) - c\,\varphi_x(s)) &\ge \tilde{U}(\varphi_x(s)) - (U(c) - c\,\varphi_x(s)) \\
&\quad - \tilde{U}(\varphi_x(s)) + (U(c_0) - c_0\,\varphi_x(s)) \\
&\ge (\varphi_x(s) - U'(c_0))(c - c_0)^+ \\
&\ge \gamma\ (c - c_0)^+. \qquad \square
\end{aligned}$$

LEMMA 5.3. *Let $\varphi \in \mathrm{C}^1(\bar{\mathcal{S}})$ and $s_0 \in \bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$. Assume that*

$$\min\left\{\mathbf{g^b} \cdot D\varphi,\ \mathbf{g^s_\varepsilon} \cdot D\varphi\right\}(s_0) > 0.$$

Then, there exist $\eta, \gamma > 0$ such that for $s = (x,y,k) \in B(s_0,\eta) \cap \bar{\mathcal{S}}$ and $s' := s - \ell\mathbf{g^b}\eta - m[(1-\alpha)y + \alpha f^\varepsilon(s)k]\mathbf{g^s_\varepsilon} \in B(s_0,\eta) \cap \bar{\mathcal{S}}$ with $\ell, m \ge 0$,

$$\varphi(s) - \varphi(s') \ \ge\ \gamma\ell + \gamma m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k].$$

Proof. We first observe that $\|\mathbf{g^s_\varepsilon}\|_\infty < \infty$. In view of the definition of $\mathbf{g^s_\varepsilon}$, this follows from

$$0 \ \le\ \frac{kf^\varepsilon(s)}{(1-\alpha)y + \alpha k f^\varepsilon(s)} \le \frac{k}{(1-\alpha)y + \alpha k} \ \le\ \frac{1}{\alpha},$$

where we used the inequality $f^\varepsilon \le 1$. Set

$$4\gamma := \min\left\{\mathbf{g^b} \cdot D\varphi\,;\ \mathbf{g^s_\varepsilon} \cdot D\varphi\right\}(s_0) > 0.$$

Since $\mathbf{g^s_\varepsilon}$ and $D\varphi$ are continuous on $\bar{\mathcal{S}} \setminus \partial^z\mathcal{S}$, there exists some $\eta > 0$ such that for all $s, s' \in B(s_0,\eta) \cap \bar{\mathcal{S}}$,

(i) $\min\left\{\mathbf{g^b} \cdot D\varphi,\ \mathbf{g^s_\varepsilon} \cdot D\varphi\right\}(s) > 2\gamma$,

(ii) $|D\varphi(s) - D\varphi(s')| \ \le\ \dfrac{\gamma}{\|\mathbf{g^s_\varepsilon}\|_\infty}$.

Let s and s' be as in the statement of the lemma. By the mean value theorem, there exists some $s^* \in [s, s'] \subset B(s_0, \eta) \cap \bar{\mathcal{S}}$ such that

$$\begin{aligned}
\varphi(s) - \varphi(s') &= (s - s') \cdot D\varphi(s^*) \\
&= \ell\, \mathbf{g^b} \cdot D\varphi(s^*) + m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\ \mathbf{g}^{\mathbf{s}}_\varepsilon(s) \cdot D\varphi(s^*) \\
&= \ell\, \mathbf{g^b} \cdot D\varphi(s^*) + m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\ \mathbf{g}^{\mathbf{s}}_\varepsilon(s) \cdot D\varphi(s) \\
&\qquad - m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\ \mathbf{g}^{\mathbf{s}}_\varepsilon(s) \cdot [D\varphi(s) - D\varphi(s^*)] \\
&\geq \ell\, \mathbf{g^b} \cdot D\varphi(s^*) + m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\ \mathbf{g}^{\mathbf{s}}_\varepsilon(s) \cdot D\varphi(s) \\
&\qquad - m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\, \|\mathbf{g}^{\mathbf{s}}_\varepsilon\|_\infty |D\varphi(s) - D\varphi(s^*)|. \\
&\geq \ell\, 2\gamma + m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\,(2\gamma - \gamma) \\
&\geq \gamma\,\ell + \gamma\, m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k]\,. \qquad \square
\end{aligned}$$

Proof of the subsolution property.

Step 1. For each $\varepsilon \geq 0$, the value function V_ε is bounded from above by V, see Proposition 6.2 below. We also recall from Proposition 4.5 in [5] that for every $s = (x, y, k) \in \bar{\mathcal{S}}$,

$$V(s) \leq V^0(x + (1-\mu)\alpha k, (1-\alpha)y),$$

where the function V^0, defined in [5], is continuous and satisfies

$$V^0(\bar{x}, \bar{y}) \ = \ 0 \text{ for all } (\bar{x}, \bar{y}) \in \mathbb{R}^2 \text{ such that } \bar{x} + (1-\mu)\bar{y} = 0.$$

It then follows that for each $\varepsilon \geq 0$, the lower semicontinuous envelope of V_ε satisfies $(V_\varepsilon)_* \leq 0$ on $\partial^z \mathcal{S}$.

Let $s_0 \in \bar{\mathcal{S}} \setminus \partial^z \mathcal{S}$ and $\varphi \in C^2\left(\bar{\mathcal{S}}\right)$ be such that

$$0 \ = \ (V_\varepsilon^* - \varphi)(s_0) = \max_{\bar{\mathcal{S}}}\ (V_\varepsilon^* - \varphi)\,,$$

and assume to the contrary that

$$F_*\left(s_0, \varphi(s_0), D\varphi(s_0), D^2\varphi(s_0)\right) > 0.$$

Observe that the last inequality implies that $\tilde{U}\left(\varphi_x(s_0)\right) < \infty$ and therefore $\varphi_x(s_0) > 0$. Since $\varphi \in C^2(\bar{\mathcal{S}})$, we deduce from Lemmas 5.2 and 5.3 the existence of $\eta, \gamma, c_0 > 0$, with $B(s_0, \eta) \subset \bar{\mathcal{S}} \setminus \partial^z \mathcal{S}$, such that

$$\min\left\{-\mathcal{L}\varphi,\ \mathbf{g^b} \cdot D\varphi,\ \mathbf{g}^{\mathbf{s}}_\varepsilon \cdot D\varphi\right\}(s) \wedge \varphi_x(s) \ > \ 0, \tag{5.12}$$

$$\tilde{U}\left(\varphi_x(s)\right) - \left(U(c) - c\varphi_x(s)\right) \geq \gamma\,(c - c_0)\,, \tag{5.13}$$

$$\varphi(s) - \varphi(s') \ \geq \ \gamma l + \gamma m\,[(1-\alpha)y + \alpha f^\varepsilon(s)k] \tag{5.14}$$

for all $s \in B(s_0, \eta) \cap \bar{\mathcal{S}}$, and $s' = s - \ell \mathbf{g^b}\eta - m\mathbf{g}^{\mathbf{s}}_\varepsilon \in B(s_0, \eta) \cap \bar{\mathcal{S}}$ for some $\ell, m \geq 0$.

Step 2. Let $(s_n = (x_n, y_n, k_n))_{n \geq 1}$ be some sequence such that

(i) $s_n \ \in \ B\left(s_0, \dfrac{\eta}{2}\right)$,

(ii) $s_n \ \longrightarrow_{n\to\infty} \ s_0$,

(iii) $\xi_n \ := \ |V_\varepsilon(s_n) - V_\varepsilon^*(s_0)| \ \longrightarrow_{n\to\infty} \ 0.$

For each $n \geq 1$, there exists a strategy $\nu^n = (C^n, L^n, M^n) \in \mathcal{A}^\varepsilon(s_n)$ such that

$$V_\varepsilon(s_n) \leq \xi_n + \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t^n) dt\right].$$

Set $S^n = (X^n, Y^n, K^n) := S^{\varepsilon, s_n, \nu^n}$ for $n \geq 1$, and fix some finite positive time horizon $T > 0$. By the dynamic programming principle of Theorem 5.1,

$$V_\varepsilon(s_n) \leq \xi_n + \mathbb{E}\left[\int_0^{T\wedge\theta^n} e^{-\beta t} U(C_t^n) dt\right] + \mathbb{E}\left[e^{-\beta T\wedge\theta^n} V_\varepsilon\left(S^n_{T\wedge\theta^n}\right)\right],$$

where $\theta^n := \inf\{t \geq 0 : S_t^n \notin B(s_0, \eta)\}$. Since $V_\varepsilon \leq V_\varepsilon^* \leq \varphi$ on $\bar{\mathcal{S}} \setminus \partial^z \mathcal{S}$, and $\xi_n = |V_\varepsilon(s_n) - V_\varepsilon^*(s_0)| = |V_\varepsilon(s_n) - \varphi(s_0)|$, it follows that for all $n \geq 1$,

$$\varphi(s_0) - \mathbb{E}\left[e^{-\beta T\wedge\theta^n} \varphi\left(S^n_{T\wedge\theta^n}\right)\right] \leq 2\xi_n + \mathbb{E}\left[\int_0^{T\wedge\theta^n} e^{-\beta t} U(C_t^n) dt\right].$$

Notice that for all $n \geq 1$, the process $\left\{S_t^n \mathbf{1}_{[0, T\wedge\theta^n)}(t), t \geq 0\right\}$ is uniformly bounded; then the Itô formula provides

$$\begin{aligned}
2\,\xi_n \geq\ & \mathbb{E}\left[\int_0^{T\wedge\theta^n} e^{-\beta t}\left[-\mathcal{L}\varphi + \tilde{U}(\varphi_x) - (U(C_t^n) - C_t^n \varphi_x)\right](S_t^n) dt\right] \\
& + \mathbb{E}\left[\int_0^{T\wedge\theta^n} e^{-\beta t} \mathbf{g}^{\mathbf{b}} \cdot D\varphi(S_t^n) dL_t^{nc}\right] \\
& + \mathbb{E}\left[\int_0^{T\wedge\theta^n} e^{-\beta t}\left[(1-\alpha)Y_t^n + \alpha f^\varepsilon(S_t^n) K_t^n\right]\left(\mathbf{g}_\varepsilon^{\mathbf{s}} \cdot D\varphi\right)(S_t^n) dM_t^{nc}\right] \\
& + \mathbb{E}\left[\sum_{0 \leq t < T\wedge\theta^n} e^{-\beta t}\left(\varphi(S_{t-}^n) - \varphi(S_t^n)\right)\right],
\end{aligned}$$

where L^{nc} and M^{nc} denote the continuous part of L^n and M^n. Recall that φ satisfies (5.12), (5.13) and (5.14); then it follows from the previous inequality that

$$\begin{aligned}
2\,\xi_n \geq\ & \gamma\, e^{-\beta T}\, \mathbb{E}\left[(T\wedge\theta^n) + L^{nc}_{T\wedge\theta^n} + \int_0^{T\wedge\theta^n} \left[(1-\alpha)Y_t^n + \alpha f^\varepsilon\left(S_t^n\right) K_t^n\right] dM_t^{nc}\right] \\
& + \gamma e^{-\beta T}\, \mathbb{E}\left[\sum_{0 \leq t < T\wedge\theta^n} \Delta L_t^n + \left[(1-\alpha)Y_{t-}^n + \alpha f^\varepsilon\left(S_{t-}^n\right) K_{t-}^n\right] \Delta M_t^n\right] \\
& + e^{-\beta T} \gamma \mathbb{E}\left[\int_0^{T\wedge\theta^n} \left(C_t^n - c_0\right)^+ dt\right], \\
\geq\ & \mathbb{E}[h^n(T\wedge\theta^n)],
\end{aligned}$$

where

$$\begin{aligned}
h^n(T\wedge\theta^n) = \gamma e^{-\beta T} \Bigg\{ & (T\wedge\theta^n) + L^n_{T\wedge\theta^n} + \int_0^{T\wedge\theta^n} \left[(1-\alpha)Y_{t-}^n + \alpha f^\varepsilon\left(S_{t-}^n\right) K_{t-}^n\right] dM_t^n \\
& + \int_0^{T\wedge\theta^n} \left(C_t^n - c_0\right)^+ dt \Bigg\}.
\end{aligned}$$

Step 3. To obtain a contradiction, we show that for a sufficiently small T, there is some constant m_* such that for large $n \geq 1$, $\mathbb{E}\left[h^n(T \wedge \theta^n)\right] \geq m_*$. The following argument is largely inspired from [22].

Step 3.1. We start by providing estimates for $|X^n - x_0|$, $|Y^n - y_0|$, and $|K^n - k_0|$. Fix some $n \geq 1$, and assume that n is sufficiently large so that $\xi_n \leq \eta/2$ holds. Let Λ be the process defined by: $\Lambda_t := (\rho - \frac{\sigma^2}{2})t + \sigma W_t$, and set

$$\Lambda_t^\star := |\rho - \frac{\sigma^2}{2}|t + \sigma\left(W_t^\star - W_{\star t}\right), \text{ where} W_t^\star := \max_{u\in[0,t]} W_u \text{ and } W_{\star t} := \min_{u\in[0,t]} W_u.$$

Since $d\left[Y_t^n e^{-\Lambda_t}\right] = e^{-\Lambda_t} dL_t^n - e^{-\Lambda_t} Y_{t-}^n dM_t^n$, we deduce by a direct calculation that

$$|Y_t^n - y_0| \leq |y_0 - y_n| + y_n|1 - e^{\Lambda_t}| + e^{\Lambda_t^\star} L_t^n + e^{\Lambda_t^\star}\int_0^t Y_{u-}^n dM_u^n. \tag{5.15}$$

The dynamics of the processes K^n and X^n are such that

$$|K_t^n - k_0| \leq |k_0 - k_n| + L_t^n + \int_0^t f^\varepsilon(S_{u-}^n) K_{u-}^n dM_u^n, \tag{5.16}$$

$$\begin{aligned}|X_t^n - x_0| \leq |x_n - x_0| + |x_n|\left(e^{rt} - 1\right) + e^{rt}\int_0^t e^{-ru} C_u^n du + e^{rt}(1+\lambda)\int_0^t e^{-ru} dL_u^n \\ + e^{rt}\int_0^t e^{-ru}(1-\mu)\left[(1-\alpha)Y_{u-}^n + \alpha f^\varepsilon(S_{u-}^n)K_{u-}^n\right] dM_u^n.\end{aligned} \tag{5.17}$$

Step 3.2. We have $|1 - e^{\Lambda_T}| \leq \max\left[e^{\Lambda_T^\star} - 1; 1 - e^{-\Lambda_T^\star}\right]$. Define the set

$$F_T := \left\{\omega \in \Omega \ : \ \max\left[e^{\Lambda_T^\star} - 1; 1 - e^{-\Lambda_T^\star}\right] \leq \min\left[1, \frac{\eta}{4(y_0+1)}\right]\right\}.$$

We claim that it is possible to choose the parameter $T > 0$ such that

$$\mathbb{P}(F_T) \geq \frac{1}{2}, \quad e^{rT} - 1 \leq \frac{\eta}{4(1+|x_0|)}, \quad \text{and} \quad e^{rT} \leq 2. \tag{5.18}$$

Indeed, Doob's Maximal martingale inequalities provide, for $\delta > 0$,

$$\mathbb{P}\{W_T^\star \geq \delta\} \leq \frac{1}{\delta^2}\mathbb{E}[W_T^\star]^2 \leq \frac{4}{\delta^2}\mathbb{E}[W_T]^2 = \frac{4\,T}{\delta^2}; \text{ similarly } \mathbb{P}\{W_{\star T} \leq \delta\} \leq \frac{4\,T}{\delta^2}.$$

Hence for all $\delta > 0$,

$$\mathbb{P}\{W_T^\star - W_{\star T} \geq \delta\} \leq \mathbb{P}\{W_T^\star \geq \delta/2\} + \mathbb{P}\{W_{\star T} \leq \delta/2\} \leq \frac{32\,T}{\delta^2}.$$

We now return to the estimates (5.15), (5.16), (5.17) and recall that $\xi_n \leq \eta/2$. Since T satisfies (5.18), the following inequalities (where A denotes some positive constant depending on (x_0, y_0, k_0)) hold $\mathbb{P}$-a.s. on the set F_T:

$$|X_T^n - x_0| \leq \eta/2 + \eta/4 + 2\int_0^T C_t^n dt + AL_T^n + A\int_0^T G^\varepsilon(S_{t-}^n) dM_t^n, \tag{5.19}$$

$$|Y_T^n - y_0| \leq \eta/2 + \eta/4 + AL_T^n + A\int_0^T G^\varepsilon(S_{t-}^n) dM_t^n, \tag{5.20}$$

$$|K_T^n - k_0| \leq \eta/2 + AL_T^n + A\int_0^T G^\varepsilon(S_{t-}^n) dM_t^n, \tag{5.21}$$

where

$$G^{\varepsilon}(s) \; := \; (1-\alpha)y + \alpha f^{\varepsilon}(s)k \text{ for } s=(x,y,k)\in \bar{\mathcal{S}}\,.$$

Step 3.3. For ω in F_T, we consider the following cases.

Case 1: $\theta^n(\omega) \geq T$. Then, by the definition of $h^n(T\wedge\theta^n)$, we have $h^n(T\wedge\theta^n) \;\geq\; \gamma\; e^{-\beta T}T$.

Case 2: $\theta^n(\omega) < T$. Recall that S^n is càdlağ, then, by the definition of the stopping time θ^n, this happens when $S^n_{\theta^n}(w) \notin B(s_0,\eta]$, i.e.,

$$\max\Big[|X^n_{\theta^n(\omega)}(\omega)-x_0|; \quad |Y^n_{\theta^n(\omega)}(\omega)-y_0|; \quad |K^n_{\theta^n(\omega)}(\omega)-k_0|\Big] \;\geq\; \eta.$$

Subcase 2.1: $|X^n_{\theta^n(\omega)}(\omega)-x_0| \geq \eta$. It follows from (5.19) that at least one of the following inequalities holds:

$$\text{(i)} \;\; \int_0^{\theta^n(\omega)} C^n_t dt \;\geq\; \eta/16 \;\; \text{or} \;\; \text{(ii)} \;\; L^n_{\theta^n} + \int_0^{\theta^n(\omega)} G^{\varepsilon}(S^n_{t-})dM^n_t \;\geq\; \frac{\eta}{8\,A}.$$

In inequality (i),

$$\frac{\eta}{16} \leq \int_0^{\theta^n(\omega)} C^n_t dt \;\leq\; c_0 T \;+\; \int_0^{\theta^n(\omega)} (C^n_t - c_0)\,dt.$$

Since it is possible to choose T such that $c_0 T \;\leq\; \frac{\eta}{32}$, it follows that

$$\frac{\eta}{16} \leq \frac{\eta}{32} \;+\; \int_0^{\theta^n(\omega)} (C^n_t - c_0)^+\,dt;$$

then $\eta/32 \;\leq\; \int_0^{\theta^n(\omega)} (C^n_t - c_0)^+\,dt$, and it follows that

$$h^n(T\wedge\theta^n) \geq \gamma e^{-\beta T} \int_0^{\theta^n(\omega)} (C^n_t - c_0)^+\,dt \;\geq\; \gamma e^{-\beta T}\frac{\eta}{32}.$$

In inequality (ii), it immediately follows that $h^n(T\wedge\theta^n) \;\geq\; \gamma e^{-\beta T}\frac{\eta}{8\,A}$.

Subcase 2.2: $|Y^n_{\theta^n(\omega)} - y_0| \;\geq\; \eta$. Then, it follows from inequality (5.20) that

$$\frac{\eta}{4} \leq A\;\left(L^n_{\theta^n(\omega)} + \int_0^{\theta^n} G^{\varepsilon}(S^n_{t-})dM^n_t\right),$$

and hence, $h^n(T\wedge\theta^n(\omega)) \;\geq\; \gamma e^{-\beta T}\frac{\eta}{4\,A}$.

Subcase 2.3: $|K^n_{\theta^n(\omega)} - k_0| \;\geq\; \eta$. By inequality (5.21) we see that in this case,

$$\frac{\eta}{2} \leq A\;\left(L^n_{\theta^n(\omega)} + \int_0^{\theta^n} G^{\varepsilon}(S^n_{t-})dM^n_t\right),$$

and hence, $h^n(T\wedge\theta^n(\omega)) \;\geq\; \gamma e^{-\beta T}\frac{\eta}{2\,A}$.

From the several cases discussed above, it follows that for $\mathbb{P}$-a.e. ω in F_T,

$$h^n(T\wedge\theta^n(\omega)) \geq m_\star \;:=\; \gamma\min\left[T, \frac{\eta}{32}, \frac{\eta}{8A}\right],$$

and therefore, for T sufficiently small and large n,

$$\mathbb{E}\left[h^n(T\wedge\theta^n)\right] \geq \mathbb{E}\left[\mathbf{1}_{F_T} h^n(T\wedge\theta^n)\right] \;\geq\; m_\star \mathbb{P}(F_T) \;=\; \frac{m_\star}{2}. \qquad \square$$

Remark 3. Let $\mathcal{A}_0(s)$ be the subset of $\mathcal{A}(s)$ consisting of all controls $\nu=(C,L,M)$ with a Lebesgue absolutely continuous component M. Then, it is clear that the above derivation of the DPE is not altered by this additional restriction. Hence, the value problem of this new control problem coincides with V_ε by the comparison result of Theorem 3.1. The same comment holds if the component L is, or both components L and M are, restricted to be Lebesgue absolutely continuous.

6. The convergence result. We first derive a useful estimate.

LEMMA 6.1. *Let s be in $\bar{\mathcal{S}}$. Then for any $\varepsilon\geq 0$, $\mathcal{A}^\varepsilon(s)\subset\mathcal{A}(s)$, and for all $\nu\in\mathcal{A}(s)$ and $t\geq 0$,*

$$0 \;\leq\; Z_t^{0,s,\nu} - Z_t^{\varepsilon,s,\nu} \;\leq\; 4\varepsilon r\, Z_T^{0,s,\nu\,\star} e^{rt}, \quad \textit{where} \quad Z_t^{0,s,\nu\,\star} \;:=\; \sup_{u\in[0,t]} |Z_u^{0,s,\nu}|.$$

Proof. Clearly the inclusion $\mathcal{A}^\varepsilon(s)\subset\mathcal{A}(s)$ follows from the inequality $Z^{0,s,\nu}\geq Z^{\varepsilon,s,\nu}$.

Step 1. We first prove that $Z^{\varepsilon,s,\nu}\leq Z^{0,\varepsilon,\nu}$ $\mathbb{P}$-a.s.. To see this, we consider a sequence of stopping times $(\tau_n)_{n\geq 0}$ exhausting the jumps of the càdlàg process M, with $\tau_0=0$. The dynamics of the processes $K^{\varepsilon,s,\nu}$ and $K^{0,s,\nu}$ are such that

$$d\left(K^{\varepsilon,s,\nu}-K^{0,s,\nu}\right)_t = -\left(K^{\varepsilon,s,\nu}-K^{0,s,\nu}\right)_{t-} dM_t + \left[1-f^\varepsilon\left(S_{t-}^{\varepsilon,s,\nu}\right)\right] K_{t-}^{\varepsilon,s,\nu} dM_t.$$

Then, for all $n\geq 0$, we have $\mathbb{P}$-a.s. for $t\in[\tau_n,\tau_{n+1})$,

$$\begin{aligned} &K_t^{\varepsilon,s,\nu}-K_t^{0,s,\nu} \\ &= e^{-(M_t^c-M_{\tau_n}^c)}\left(K_{\tau_n}^{\varepsilon,s,\nu}-K_{\tau_n}^{0,s,\nu}+\int_{\tau_n}^t e^{M_u^c-M_{\tau_n}^c}\left[1-f^\varepsilon(S_{u-}^{\varepsilon,s,\nu})\right] K_{u-}^{\varepsilon,s,\nu} dM_u\right). \end{aligned} \tag{6.1}$$

Since $1-f^\varepsilon\geq 0$, this implies that

$$\begin{aligned} K_t^{\varepsilon,s,\nu}-K_t^{0,s,\nu} &\geq e^{-(M_t^c-M_{\tau_n}^c)}(K_{\tau_n}^{\varepsilon,s,\nu}-K_{\tau_n}^{0,s,\nu}) \\ &= e^{-(M_t^c-M_{\tau_n}^c)}((K_{\tau_n-}^{\varepsilon,s,\nu}-K_{\tau_n-}^{0,s,\nu})(1-\Delta M_{\tau_n}) \\ &\qquad +[1-f^\varepsilon(S_{\tau_n-}^{\varepsilon,s,\nu})]K_{\tau_n-}^{\varepsilon,s,\nu}\Delta M_{\tau_n})\geq 0. \end{aligned} \tag{6.2}$$

Clearly, $Y^{\varepsilon,s,\nu}=Y^{0,s,\nu}$. Then

$$d\left(Z^{\varepsilon,s,\nu}-Z^{0,s,\nu}\right)_t = r\left(Z^{\varepsilon,s,\nu}-Z^{0,s,\nu}\right)_t dt - r(1-\mu)\alpha\left(K^{\varepsilon,s,\nu}-K^{0,s,\nu}\right)dt.$$

Since $Z_0^{\varepsilon,s,\nu}-Z_0^{0,s,\nu}=0$ and $K^{\varepsilon,s,\nu}\geq K^{0,s,\nu}$, this implies that

$$Z_t^{\varepsilon,s,\nu}-Z_t^{0,s,\nu} = -r(1-\mu)\alpha e^{rt}\int_0^t e^{-ru}\left(K_u^{\varepsilon,s,\nu}-K_u^{0,s,\nu}\right)du \;\leq\; 0. \tag{6.3}$$

Step 2. We next prove the second inequality. Observe that $[1-f_\varepsilon(s)]\,k\leq 2\varepsilon z$ for $s=(x,y,k)\in\bar{\mathcal{S}}$, where $z:=x+(1-\mu)\left[(1-\alpha)y+\alpha k\right]$. Together with (6.1) and (6.2) this shows that, for all $n\geq 0$ and $t\in[\tau_n,\tau_{n+1})$,

$$K_t^{\varepsilon,s,\nu}-K_t^{0,s,\nu}\leq 2\varepsilon\, e^{-(M_t^c-M_{\tau_n}^c)}\left(Z_{\tau_n-}^{\varepsilon,s,nu}+\int_{\tau_n}^t e^{M_u^c-M_{\tau_n}^c} Z_{u-}^{\varepsilon,s,\nu} dM_u\right).$$

Using the increase of M together with the fact that $Z^{\varepsilon,s,\nu} \le Z^{0,s,\nu}$, as shown in the first step of this proof, this provides

$$K_t^{\varepsilon,s,\nu} - K_t^{0,s,\nu} \le 2\varepsilon Z_t^{0,s,\nu\star}\, e^{-(M_t^c - M_{\tau_n}^c)} \left(1 + \int_{\tau_n}^t e^{M_u^c - M_{\tau_n}^c} dM_u\right) \;\le\; 4\varepsilon Z_t^{0,s,\nu\star}.$$

The required inequality is obtained by plugging this estimate into (6.3). ∎

PROPOSITION 6.2. *The sequence $(V_\varepsilon)_{\varepsilon>0}$ is nonincreasing and $V_\varepsilon \le V$.*

Proof. The inequality $V_\varepsilon \le V$ follows immediately from the fact that $\mathcal{A}^\varepsilon(s) \subset \mathcal{A}(s)$, as stated in Lemma 6.1. To prove that the sequence $(V_\varepsilon)_{\varepsilon>0}$ is nonincreasing, we shall prove that $\mathcal{A}^{\varepsilon_1}(s) \subset \mathcal{A}^{\varepsilon_2}(s)$ whenever $\varepsilon_1 \ge \varepsilon_2$. To do this, it is sufficient to prove that for any control $\nu = (C, L, M) \in \mathcal{A}_\varepsilon(s)$, the associated process $Z^\varepsilon := X^{\varepsilon,s,\nu} + (1-\mu)\left[(1-\alpha)Y^{\varepsilon,s,\nu} + \alpha K^{\varepsilon,s,\nu}\right]$ is nonincreasing with respect to ε. Recall that

$$f^\varepsilon(s) \;=\; h\left(\frac{k}{\varepsilon z}\right), \qquad \text{where } z = x + (1-\mu)[(1-\alpha y + \alpha z],$$

and h is a smooth function. From Remark 3, we may restrict the process M to be absolutely continuous with respect to the Lebesgue measure, i.e., $M_t = \int_0^t m_u du$ for some $\mathbb{F}$-adapted process $\{m_t,\ t \ge 0\}$, as the restriction of the control M to this class produces the same value function V_ε.

Then, by classical results on the regularity of flows of stochastic differential equations (see, e.g., [16]), the processes Z^ε, $Y^\varepsilon := Y^{\varepsilon,s,\nu}$ and $K^\varepsilon := K^{\varepsilon,s,\nu}$ are differentiable in ε, and the processes

$$z_t^\varepsilon \;:=\; e^{-rt}\frac{\partial Z_t^\varepsilon}{\partial \varepsilon}, \quad y_t^\varepsilon \;:=\; \frac{\partial Y_t^\varepsilon}{\partial \varepsilon}, \quad k_t^\varepsilon \;:=\; e^{-rt}\frac{\partial K_t^\varepsilon}{\partial \varepsilon}$$

satisfy $y_t^\varepsilon = 0$ for all $t \ge 0$, $z_0^\varepsilon = k_0^\varepsilon = 0$, and solve the system of ODEs

$$\dot z_t^\varepsilon \;=\; -r\alpha k_t \quad \text{and} \quad \dot k_t^\varepsilon \;=\; a_t + b_t z_t - c_t k_t,$$

where

$$a_t \;:=\; \frac{(K_t^\varepsilon)^2}{\varepsilon Z_t^\varepsilon}\, h'\left(\frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon}\right), \qquad b_t \;:=\; \frac{(K_t^\varepsilon)^2}{\varepsilon (Z_t^\varepsilon)^2}\, h'\left(\frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon}\right),$$

and

$$e^{-rt}c_t := r + m_t\left[h\left(\frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon}\right) + \frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon)}\, h'\left(\frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon}\right)\right].$$

Differentiating once more with respect to the t-variable, we obtain the following second order differential equation for z^ε:

$$-\ddot z_t^\varepsilon \;-\; c_t\, \dot z_t^\varepsilon \;-\; r\alpha b_t\, z_t^\varepsilon \;-\; r\alpha a_t \;=\; 0 \quad \text{and} \quad \dot z_0^\varepsilon = z_0 = 0. \tag{6.4}$$

We now consider the function

$$\hat z_t \;:=\; -\frac{r\alpha}{\varepsilon}\int_0^t \int_0^u \frac{(K_t^\varepsilon)^2}{\varepsilon Z_t^\varepsilon}\, h'\left(\frac{K_t^\varepsilon}{\varepsilon Z_t^\varepsilon}\right) du\, dt \quad \text{for} \quad t \ge 0.$$

Since $\hat z_t \le 0$, $\dot{\hat z}_t \le 0$, $b_t \ge 0$ and $c \ge 0$, it follows that $\hat z_t$ is a supersolution of (6.4). By a standard comparison result, we deduce that $z_t^\varepsilon \le \hat z_t$, and therefore $z_t^\varepsilon \le 0$ for all $t \ge 0$. This completes the proof. ∎

Our final result states the convergence of V_ε towards V.

Proposition 6.3. *The sequence $(V_\varepsilon)_{\varepsilon>0}$ is nonincreasing and converges towards V, as $\varepsilon \searrow 0$, uniformly on compact subsets of $\bar{\mathcal{S}}$.*

Proof. Let $(\nu^n = (C^n, L^n, M^n))_{n\geq 1}$ be a maximizing sequence of controls for $V(s)$:

$$V(s) - \frac{1}{n} \;\leq\; \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t^n) dt\right] \quad \text{for all} \quad n \;\geq\; 1.$$

By the monotone convergence theorem, we verify that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t^n) dt\right] = \lim_{T\to\infty} \;\mathbb{E}\left[\int_0^T e^{-\beta t} U(C_t^n) dt\right].$$

Then $V(s) - \frac{1}{2n} \;\leq\; \mathbb{E}[\int_0^{T^n} e^{-\beta t} U(C_t^n) dt]$ for some $T^n \;>\; 0$. By Lemma 6.1 we have $Z_{t\wedge T^n}^{0,s,\nu} \geq Z_{t\wedge T^n}^{\varepsilon,s,\nu} \geq Z_{t\wedge T^n}^{0,s,\nu} - 4r\varepsilon Z_{T^n}^{0,s,\nu\star}$ $\mathbb{P}$-a.s. for all $t \geq 0$. Then, the stopping times $\tau(\varepsilon, s, n) \;:=\; \inf\{t \geq 0 \;:\; Z_t^{\varepsilon,s,\nu} \leq 0\}$, $\varepsilon \geq 0$, satisfy

$$\tau(0, s, n) \wedge T^n \geq \tau(\varepsilon, s, n) \wedge T^n \text{ and } \lim_{\varepsilon\to 0} \tau(\varepsilon, s, n) \wedge T^n = \tau(0, s, n) \wedge T^n \;\; \mathbb{P}\text{-a.s.}$$

Hence, by the monotone convergence theorem,

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\int_0^{\tau(\varepsilon,s,n)\wedge T^n} e^{-\beta t} U(C_t^n) dt\right] = \mathbb{E}\left[\int_0^{\tau(0,s,n)\wedge T^n} e^{-\beta u} U(C_t^n) dt\right].$$

Recall from (5.6) and (5.7) that

$$V_\varepsilon(s) \geq \mathbb{E}\left[\int_0^{\tau(\varepsilon,s,n)\wedge T^n} e^{-\beta t} U(C_t^n) dt\right] \quad \text{and}$$

$$\mathbb{E}\left[\int_0^{\tau(0,s,n)\wedge T^n} e^{-\beta t} U(C_t^n) dt\right] = \mathbb{E}\left[\int_0^{T^n} e^{-\beta u} U(C_u^n) du\right].$$

Then

$$\liminf_{\varepsilon\to 0} V_\varepsilon(s) \geq \mathbb{E}\left[\int_0^{T^n} e^{-\beta t} U(C_t^n) dt\right] \;\geq\; V(s) - \frac{1}{2n}.$$

By arbitrariness of $n \geq 1$, this provides $\liminf_{\varepsilon\to 0} V_\varepsilon(s) \;\geq\; V(s)$. Together with Proposition 6.2, this shows that $V_\varepsilon(s) \longrightarrow V(s)$ as $\varepsilon \searrow 0$ for every $s \in \bar{\mathcal{S}}$.

We finally recall from Proposition 2.4 that the limit function V is continuous. Since $(V_\varepsilon)_{\varepsilon>0}$ is a monotonic sequence of continuous functions, it follows from the Dini theorem that the convergence holds uniformly on compact subsets of $\bar{\mathcal{S}}$. □

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