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# The edge ideals of t-spread *d*-partite hypergraphs

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**Abstract**
Inspired by the definition of **t**-spread monomial ideals, in this paper, we introduce **t**-spread $d$-partite hypergraph $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ and study its edge ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. We prove that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has linear quotients, all powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution and the Rees algebra of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is a normal Cohen-Macaulay domain. It is also shown that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free and a complete characterization of Cohen-Macaulay $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is given.

**Keywords** Edge ideals of hypergraphs · Cohen-Macaulay edge ideals · Linear quotients · $t$-spread ideals · Strong persistence property · Normally torsion-free ideals

**Mathematics Subject Classification** 05E40 · 13B25 · 13C14 · 13D02

## 1 Introduction

In [6], the third author together with Ene and Herzog introduced the notion of $t$-spread monomials in a polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$ and studied some classes of ideals and $\mathbb{K}$-algebras generated by $t$-spread monomials. Let $u = x_{i_1} \cdots x_{i_d}$ be a monomial in $S$ and $t \geq 0$. The monomial $u$ is called $t$-spread if $i_j - i_{j-1} \geq t$ for all $j = 2, \dots, d$. A monomial ideal $I \subset S$ is called $t$-spread if it is generated by $t$-spread monomials. Any monomial ideal in $S$ can be viewed as 0-spread and any square-free monomial ideal as 1-spread. After their first appearance in 2019, different classes of $t$-spread monomial ideals have been studied by many authors and recently in 2023, Ficarra gave a more generalized

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notion of $t$-spread monomials by replacing the integer $t$ with $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{N}^{d-1}$, (see [7] and the reference therein).

In this paper, we study $\mathbf{t}$-spread monomial ideals which appear as the edge ideals of certain $d$-partite hypergraphs. Let $V = \{V_1, \ldots, V_d\}$ be a partitioning of a finite set $U \subset \mathbb{N}$ such that $p < q$ if $p \in V_i, q \in V_j$ with $i < j$. We call $\{i_1, \ldots, i_d\} \subset U$ a $\mathbf{t}$-spread set if $i_j \in V_j$ for all $j = 1, \ldots, d$ and $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \ldots, d$. We call the hypergraph $\mathrm{K}_\mathrm{V}^\mathbf{t}$ on vertex set $V(\mathrm{K}_\mathrm{V}^\mathbf{t}) = U$, a complete $\mathbf{t}$-spread $d$-partite hypergraph if all $\mathbf{t}$-spread sets of $U$ are the edges of $\mathrm{K}_\mathrm{V}^\mathbf{t}$. For $\mathbf{t} = (1, \ldots, 1)$, the hypergraph $\mathrm{K}_\mathrm{V}^\mathbf{t}$ is a complete $d$-partite hypergraph, see [1, Example 3]. The edge ideal of $\mathrm{K}_\mathrm{V}^\mathbf{t}$, denoted by $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$, is a $t$-spread monomial ideal generated by those monomials whose indices correspond to the edges of $\mathrm{K}_\mathrm{V}^\mathbf{t}$. It turns out that $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ admits many nice algebraic and homological properties. It is shown in Theorem 2.4 that $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ has linear quotients. The ideals with linear quotients were first defined by Herzog and Takayama in [14] and their free resolutions were computed as iterated mapping cones. Using the description of Betti numbers of ideals with linear quotients given in [14], in Proposition 2.5, we provide an intrinsic way to compute Betti numbers of $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$.

In Sect. 3, we study the powers and fiber cone of $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$. One of the main results of Sect. 3 is given in

**Corollary 3.7** *The ideal* $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$*satisfies the strong persistence property and all powers of* $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$*have linear resolution.*

To prove Corollary 3.7, we first show that minimal generating set of $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ is sortable and $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ satisfies the $\ell$-exchange property with respect to sorting order, see Proposition 3.1 and Theorem 3.4. Then it follows from classical results of Fröberg [8], Sturmfels [19] and Hochster [16] that the Rees algebra $\mathcal{R}(I(\mathrm{K}_\mathrm{V}^\mathbf{t}))$ is a normal Cohen-Macaulay domain, see Corollary 3.6. Then Corollary 3.7 is obtained as an application of [15, Corollary 1.6] and [11, Corollary 10.1.8]. We also compute the Krull dimension of fibercone $\mathcal{R}(I(\mathrm{K}_\mathrm{V}^\mathbf{t}))/\mathfrak{m}\mathcal{R}(I(\mathrm{K}_\mathrm{V}^\mathbf{t}))$ which provides the limit depth of $S/I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ in Theorem 3.11.

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$. A set $T \subset V(\mathcal{H})$ is called a *transversal* of $\mathcal{H}$ if it meets all the edges of $\mathcal{H}$ and the family of all minimal transversals of $\mathcal{H}$ is called the *transversal hypergraph* of $\mathcal{H}$, see [1, Chapter 2]. The minimal transversals of a hypergraph $\mathcal{H}$ correspond to the minimal prime ideals of the edge ideal of $\mathcal{H}$. In Sect. 4, we consider $\mathrm{K}_\mathrm{V}^\mathbf{t}$ with $V = \{V_1, \ldots, V_d\}$ such that each $V_i$ is an interval of integers. The description of the minimal primes of $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ is obtained by computing the minimal generating set of Alexander dual of $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ in Theorem 4.1. In Theorem 4.6, we prove that $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ is normally torsion-free which is equivalent to say that $\mathrm{K}_\mathrm{V}^\mathbf{t}$ is a Mengerian hypergraph. A complete characterization of unmixed $I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ is given in Theorem 4.9. With the help of Theorem 4.9, a complete characterization of Cohen-Macaulay $S/I(\mathrm{K}_\mathrm{V}^\mathbf{t})$ is obtained in Theorem 4.11.

## 2 t-spread *d*-partite hypergraphs and their edge ideals

A finite *hypergraph* $\mathcal{H}$ on the vertex set $V(\mathcal{H}) = [n]$ is a collection of edges $E(\mathcal{H}) = \{E_1, \ldots, E_m\}$ with $E_i \subseteq V(\mathcal{H})$ for all $i = 1, \ldots, m$. A hypergraph $\mathcal{H}$ is called *simple*, if $E_i \subseteq E_j$ implies $i = j$. Simple hypergraphs are also known as *clutters*. Moreover, if $|E_i| = d$, for all $i = 1, \ldots, m$, then $\mathcal{H}$ is called a *d-uniform* hypergraph. A 2-uniform hypergraph $\mathcal{H}$ is just a finite simple graph. A vertex of a hypergraph $\mathcal{H}$ is said to be an *isolated vertex* if it is not contained in any edge of $\mathcal{H}$.

A hypergraph $\mathcal{H}$ is a $d$-partite hypergraph if its vertex set $V(\mathcal{H})$ is a disjoint union of sets $V_1, \dots, V_d$ such that if $E$ is an edge of $\mathcal{H}$, then $|E \cap V_i| \leq 1$. In particular, if $\mathcal{H}$ is a $d$-uniform $d$-partite hypergraph with a vertex partition $V_1, \dots, V_d$, then $|E| = d$ and $|E \cap V_i| = 1$ for each $E \in E(\mathcal{H})$. In this paper, all hypergraphs are simple, uniform, and without isolated vertices.

Next, we introduce the definition of $\mathbf{t}$-spread $d$-partite hypergraphs. To do this, we give the following notation. For any integers $i \leq j$, let $[i,j] := \{k : i \leq k \leq j\}$ and for any integer $n$, we set $[n] := \{1, \dots, n\}$.

**Definition 2.1** Let $\mathcal{H}$ be a $d$-partite hypergraph with $V(\mathcal{H}) \subseteq [n]$, and $V = \{V_1, \dots, V_d\}$ be a family defining partitioning of $V(\mathcal{H})$ such that if $p \in V_i$ and $q \in V_j$ with $i < j$, then $p < q$. Let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$. An edge $E$ of $\mathcal{H}$ is called a $\mathbf{t}$-spread edge if

(∗) $E = \{i_1, i_2, \dots, i_d\}$ with $i_j \in V_j$ for all $j = 1, \dots, d$, and $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \dots, d$.

A $d$-partite hypergraph $\mathcal{H}$ is called $\mathbf{t}$-spread if each edge of $\mathcal{H}$ is $\mathbf{t}$-spread. Moreover, $\mathcal{H}$ is called a complete $\mathbf{t}$-spread $d$-partite hypergraph and denoted by $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ if all $E \subseteq V(\mathcal{H})$ satisfying (∗) belong to $E(\mathcal{H})$.

Let $\mathbf{1} = (1, \dots, 1)$. A complete $\mathbf{1}$-spread $d$-partite hypergraph is just a complete $d$-partite hypergraph as studied in [1]. The class of complete $d$-partite hypergraphs have many nice combinatorial properties. We refer reader to [1] for more information.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and $I$ be a monomial ideal in $S$. Throughout the following text, the unique minimal generating set of a monomial ideal $I$ will be denoted by $\mathcal{G}(I)$. The *support* of a monomial $u$, denoted by $\mathrm{supp}(u)$, is the set of variables that divide $u$. Moreover, we set $\mathrm{supp}(I) = \bigcup_{u \in \mathcal{G}(I)} \mathrm{supp}(u)$. Let $\mathcal{H}$ be a hypergraph on $V(\mathcal{H}) = [n]$. The *edge ideal* of $\mathcal{H}$ is given by

$$I(\mathcal{H}) = (\prod_{j \in E_i} x_j : E_i \in E(\mathcal{H})).$$

**Definition 2.2** [7] Let $\mathbf{t} = (t_1, t_2, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$. A monomial $x_{i_1} x_{i_2} \cdots x_{i_d} \in S = \mathbb{K}[x_1, \dots, x_n]$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called $\mathbf{t}$-*spread* if $i_j - i_{j-1} \geq t_{j-1}$ for all $j = 2, \dots, d$. A monomial ideal in $S$ is called a $\mathbf{t}$-*spread monomial ideal* if it is generated by $\mathbf{t}$-spread monomials.

Note that a $\mathbf{0}$-spread monomial ideal is just an ordinary monomial ideal, while a $\mathbf{1}$-spread monomial ideal is just a square-free monomial ideal. When $\mathbf{t} = (t, \dots, t)$ for some fixed integer $t \geq 0$, then $\mathbf{t}$-spread monomial ideal is $t$-spread introduced in [6]. In the following text, we will assume that $t_i \geq 1$ for all $1 \leq i \leq d-1$. It follows from the above definitions that the edge ideal of a $\mathbf{t}$-spread $d$-partite hypergraph is a $\mathbf{t}$-spread monomial ideal. To illuminate these definitions, we provide the following example.

***Example* 2.3** Let $\mathbf{t} = (3, 2, 4)$ and $\mathrm{V} = \{V_1, V_2, V_3, V_4\}$ with $V_1 = \{1, 2, 3\}, V_2 = \{5, 7\}, V_3 = \{8, 9, 11\}$ and $V_4 = \{12, 13\}$. Then the minimal generators of the edge ideal of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ are as follows:

$x_1x_5x_8x_{12}$ $x_2x_5x_8x_{12}$
$x_1x_5x_8x_{13}$ $x_2x_5x_8x_{13}$
$x_1x_5x_9x_{13}$ $x_2x_5x_9x_{13}$
$x_1x_7x_9x_{13}$ $x_2x_7x_9x_{13}$ $x_3x_7x_9x_{13}$

The ambient ring of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ in this case is $S = \mathbb{K}[x_1, x_2, x_3, x_5, x_7, x_8, x_9, x_{12}, x_{13}]$. Indeed, we can remove 11 from $V_3$ to exclude the isolated vertices.

The edge ideals of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ have many nice algebraic and combinatorial properties. Let $I$ be a homogenous ideal in $S = \mathbb{K}[x_1, \dots, x_n]$ with graded minimal free resolution

$$0 \to \mathbb{F}_p \xrightarrow{\phi_p} \mathbb{F}_{p-1} \to \cdots \to \mathbb{F}_1 \xrightarrow{\phi_1} \mathbb{F}_0 \xrightarrow{\phi_0} I \to 0, \tag{1}$$

where for all $i = 0, \dots, p$, the free $S$-module $\mathbb{F}_i$ is equal to $\bigoplus_j S(-j)^{\beta_{i,j}(I)}$. Recall that $\beta_{i,j}(I)$ is the $(i, j)$*-th graded Betti number of* $I$ and the rank of $\mathbb{F}_i$ is called the *i-th Betti number of I* and denoted by $\beta_i(I)$. Then the ideal $I$ is said to have *d-linear* resolution if $\beta_{i,j}(I) = 0$ for all $i$ and all $j \neq d$.

We first prove that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has linear resolution. To do this, we show that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has linear quotients. Recall that an ideal $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ is said to have *linear quotients* if $\mathcal{G}(I)$ admits an ordering $u_1, \dots, u_r$ such that the colon ideal $(u_1, \dots, u_{i-1}) : (u_i)$ is generated by variables for all $i = 2, \dots, r$. It is known from [14, Theorem 1.12] or [11, Propositon 8.2.1] that an ideal generated in a single degree has linear resolution if it admits linear quotients.

**Theorem 2.4** *The ideal* $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ *has linear quotients.*

***Proof*** Let $>_{\mathrm{lex}}$ denote the lexicographical order induced by the total order $x_1 > x_2 > \cdots > x_n$. Furthermore, let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$ and set $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and let $\mathcal{G}(I) = \{u_1, \dots u_r\}$ ordered such that $u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} u_r$. We need to show that $(u_1, \dots, u_{i-1}) : (u_i)$ is generated by variables for all $i = 2, \dots, r$. To do this, it is enough to show that for all $1 \le j \le i-1$, there exists $x_p \in (u_1, \dots, u_{i-1}) : (u_i)$ such that $x_p$ divides $u_j / \gcd(u_j, u_i)$.

Let $j < i$ and $u_i = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $u_j = x_{j_1}x_{j_2}\cdots x_{j_d}$ with $i_1 < i_2 < \cdots < i_d$ and $j_1 < j_2 < \cdots < j_d$. On account of $u_j >_{\mathrm{lex}} u_i$, there exists some $\ell$ such that $j_1 = i_1, j_2 = i_2, \dots, j_{\ell-1} = i_{\ell-1}$ and $j_\ell < i_\ell$. Note that $j_\ell, i_\ell \in V_\ell$. Let $v = x_{j_\ell}(u_i/x_{i_\ell}) = x_{i_1}x_{i_2}\cdots x_{i_{\ell-1}}x_{j_\ell}x_{i_{\ell+1}}\cdots x_{i_d}$. We have $j_\ell - i_{\ell-1} = j_\ell - j_{\ell-1} \ge t_{\ell-1}$ and $i_{\ell+1} - j_\ell \ge i_{\ell+1} - i_\ell \ge t_\ell$. This shows that $v$ corresponds to a $\mathbf{t}$-spread edge of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Hence, $v \in \mathcal{G}(I)$ and $v = u_k$ for some $k < i$. This completes the proof because $x_{j_\ell} \in (u_1, \dots, u_{i-1}) : (u_i)$ and $x_{j_\ell}$ divides $u_j / \gcd(u_j, u_i)$. □

Let $I$ be a monomial ideal with linear quotients with respect to the ordering $u_1, \dots, u_r$ of $\mathcal{G}(I)$. If $I$ is generated in a single degree $d$, then $I$ has linear resolution as shown in [14]. Following [14], we define

$$\mathrm{set}(u_k) = \{i : x_i \in (u_1, \dots, u_{k-1}) : (u_k)\} \text{ for } k = 2, \dots, r.$$

Using [14, Lemma 1.5], we can conclude that

$$\beta_{i,i+d}(I) = |\{\alpha \subseteq \mathrm{set}(u) : u \in \mathcal{G}(I) \text{ and } |\alpha| = i\}|.$$

In the following proposition, we give a description of $\mathrm{set}(u)$ when $u \in \mathcal{G}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$. For any $S \subseteq [n]$, we set $\min S$ to be the smallest integer in $S$, and $\max S$ to be the largest integer in $S$.

**Proposition 2.5** *Let* $u = x_{k_1}x_{k_2}\cdots x_{k_d} \in \mathcal{G}(I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}))$ *with* $\mathbf{t} = (t_1, t_2, \dots, t_{d-1})$ *and* $i_1 = \min V_1$. *With the notations introduced above,* $\mathrm{set}(u)$ *is the union of* $[i_1, k_1 - 1] \cap V_1$ *and* $[k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$ for $j = 2, \dots, d$.

***Proof*** Let $\ell \in \mathrm{set}(u)$. Following Theorem 2.4, there exists $v \in \mathcal{G}(I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}))$ such that $v >_{\mathrm{lex}} u$ and $(v) : (u) = (x_\ell)$. This gives $v = (u/x_{k_j})x_\ell$ for some $1 \le j \le d$ and $x_{k_j}, x_\ell \in V_j$. Since $v >_{\mathrm{lex}} u$, we must have $\ell \le k_j - 1$. If $j = 1$, then $\ell \in [i_1, k_1 - 1]$. Moreover, if $2 \le j \le d$, then $k_{j-1} + t_{j-1} \le \ell$ because $v$ is a **t**-spread monomial, and hence $\ell \in [k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$.

On the other hand, if $\ell \in [i_1, k_1 - 1] \cap V_1$ or $\ell \in [k_{j-1} + t_{j-1}, k_j - 1] \cap V_j$ for any $j = 2, \dots, d$, then set $v = (u/x_{k_j})x_\ell$ for all $j = 1, \dots, d$. In both cases, $v \in \mathcal{G}(I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}))$ and $v >_{\mathrm{lex}} u$. Therefore, $x_\ell \in (v) : (u)$, and hence $\ell \in \mathrm{set}(u)$, as required. □

## 3 The powers and the fiber cone of $I(\mathrm{K}^{\mathbf{t}}_{V})$

Let $\mathbb{K}$ be a field and $S_d$ be the $\mathbb{K}$-vector space generated by all monomials of degree $d$ in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. Let $u, v \in S_d$ and $uv = x_{i_1}x_{i_2}\cdots x_{i_{2d}}$ with $i_1 \le i_2 \le \cdots \le i_{2d-1} \le i_{2d}$. Set $u' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}}$ and $v' = x_{i_2}x_{i_4}\cdots x_{i_{2d}}$. The map

$$\mathrm{sort} : S_d \times S_d \to S_d \times S_d \text{ which maps } (u, v) \mapsto (u', v'),$$

is called the *sorting operator*. A pair $(u, v) \in S_d \times S_d$ is called *sorted* if $\mathrm{sort}(u, v) = (u', v')$. A subset $A \subset S_d$ is called *sortable* if $\mathrm{sort}(A \times A) \subseteq A \times A$. Furthermore, an $r$-tuple of monomials $(u_1, \dots, u_r) \in S^r_d$ is called sorted if for any $1 \le i < j \le n$, the pair $(u_i, u_j)$ is sorted. In other words, if we write the monomials $(u_1, \dots, u_r)$ as $u_1 = x_{i_1}\cdots x_{i_d}$, $u_2 = x_{j_1}\cdots x_{j_d}$, … , $u_r = x_{l_1}\cdots x_{l_d}$, then $(u_1, \dots, u_r)$ is sorted if and only if

$$i_1 \le j_1 \le \cdots \le l_1 \le i_2 \le j_2 \le \cdots \le l_2 \le \cdots \le i_d \le j_d \le \cdots \le l_d. \tag{2}$$

**Proposition 3.1** *The set* $\mathcal{G}(I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}))$ *is sortable.*

***Proof*** Assume that $u, v \in \mathcal{G}(I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}))$ and $uv = x_{i_1}x_{i_2}x_{i_3}x_{i_4}\cdots x_{i_{2d-1}}x_{i_{2d}}$ with $i_1 \le i_2 \le \cdots \le i_{2d}$. Since $\mathrm{supp}(u)$ and $\mathrm{supp}(v)$ correspond to the edges of $\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}$, it follows that $i_1, i_2 \in V_1, i_3, i_4 \in V_2, \dots, i_{2d-1}, i_{2d} \in V_d$. Consequently, $u' = x_{i_1}x_{i_3}\cdots x_{i_{2d-1}}$ and $v' = x_{i_2}x_{i_4}\cdots x_{i_{2d}}$ are monomials associated to the edges of a complete $d$-partite hypergraph. It only remains to show that $u'$ and $v'$ are **t**-spread. We show that $u'$ is a **t**-spread monomial and the argument for $v'$ follows in a similar fashion. For any $1 \le l \le d - 1$, we have $i_{2l-1} \le i_{2l} \le i_{2l+1}$ and at least two of the variables among $x_{i_{2l-1}}, x_{i_{2l}}, x_{i_{2l+1}}$ belong to either $\mathrm{supp}(u)$ or $\mathrm{supp}(v)$. Using the fact that $u$ and $v$ are **t**-spread monomials, this implies that $i_{2l+1} - i_{2l-1} \ge i_{2l+1} - i_{2l}$ and $i_{2l+1} - i_{2l-1} \ge i_{2l} - i_{2l-1}$, we obtain the desired conclusion. □

Let $I \subset S$ be an ideal generated by the monomials of same degree. Here, set $T = \mathbb{K}[\{t_u : u \in \mathcal{G}(I)\}]$ and $\mathbb{K}[I] = \mathbb{K}[u : u \in \mathcal{G}(I)]$. Consider the $\mathbb{K}$-algebra homomorphism

$$\phi : T \to \mathbb{K}[I] \text{ defined by } t_u \mapsto u \text{ for } u \in \mathcal{G}(I).$$

The kernel of $\phi$ is called the *defining ideal* of $\mathbb{K}[I]$. If $\mathcal{G}(I)$ is a sortable set, then it follows from [19] or [5, Theorems 6.15 and 6.16] that there exists a monomial order $<_{\mathrm{sort}}$ such that the defining ideal of $\mathbb{K}[I]$ admits the reduced Gröbner basis consisting of binomials of the form $t_u t_v - t_{u'} t_{v'}$, where $\mathrm{sort}(u, v) = (u', v')$.

**Corollary 3.2** *The $\mathbb{K}$-algebra $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is a Koszul and Cohen-Macaulay normal domain.*

***Proof*** As discussed above, with respect to $>_{\mathrm{sort}}$, the Gröbner basis of the defining ideal of $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ contains quadratic binomials. Due to Fröberg [8], we conclude that $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is Koszul and due to a theorem of Sturmfels [19] we obtain $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is normal, see also [5, Theorem 5.16]. Therefore, $\mathbb{K}[I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})]$ is Cohen-Macaulay domain by [16, Theorem 1]. □

Our next goal is to establish $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ has the strong persistence property and its powers have linear resolution. Remember an ideal $I$ is said to satisfy the *strong persistence* property if $(I^{k+1} : I) = I^k$ for all $k \geq 1$, see [15] for more information. In addition, an ideal $I$ is said to satisfy the *persistence property* if:

$$\mathrm{Ass}(I) \subseteq \mathrm{Ass}(I^2) \subseteq \cdots \subset \mathrm{Ass}(I^k) \subseteq \cdots .$$

In [15], it is proved that an ideal with strong persistence property has the persistence property.

To achieve our goal, we first recall the definition of $l$-exchange property, see [13] or [5, Sec 6.4] for more details. Let $T$ and $\phi$ be the same as above and $<$ be a monomial order defined on $T$. A monomial $t_{u_1} t_{u_2} \cdots t_{u_N} \in T$ is called a *standard monomial* of $\ker \phi$ with respect to $<$, if $t_{u_1} t_{u_2} \cdots t_{u_N} \notin \mathrm{in}_<(\ker \phi)$.

**Definition 3.3** The monomial ideal $I \subset S$ is said to satisfy the $l$-exchange property with respect to the monomial order $<$ on $T$ if the following two conditions hold: let $t_{u_1} t_{u_2} \cdots t_{u_N}$ and $t_{v_1} t_{v_2} \cdots t_{v_N}$ be two standard monomials of $\ker \phi$ with respect to $<$ such that

(i) $\deg_{x_i} u_1 u_2 \cdots u_N = \deg_{x_i} v_1 v_2 \cdots v_N$, for $i = 1, \dots, q-1$ and $q \leq n-1$,
(ii) $\deg_{x_q} u_1 u_2 \cdots u_N < \deg_{x_q} v_1 v_2 \cdots v_N$.

Then there exist some $j$ and $\alpha$ with $q < j \leq n$ such that $x_q u_\alpha / x_j \in I$.

**Theorem 3.4** *The ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the l-exchange property with respect to the sorting order $<_{\mathrm{sort}}$.*

***Proof*** Let $t_{u_1} t_{u_2} \cdots t_{u_N}$ and $t_{v_1} t_{v_2} \cdots t_{v_N}$ be two standard monomials of $\ker \phi$ with respect to $<_{\mathrm{sort}}$ and $\mathbf{t} = (t_1, t_2, \dots, t_{d-1})$. It can be seen from Proposition 3.1 together with (2) that the $N$-tuples with $\mathbf{t}$-spread monomials $(u_1, u_2, \dots, u_N)$ and $(v_1, v_2, \dots, v_N)$ are sorted. Assume that the products $u_1 u_2 \cdots u_N$ and $v_1 v_2 \cdots v_N$ satisfy both conditions in Definition 3.3. The condition (i) together with (2) gives

$$\deg_{x_i} u_\gamma = \deg_{x_i} v_\gamma, \text{ for } 1 \leq i \leq q-1 \text{ and for all } 1 \leq \gamma \leq N, \tag{3}$$

and the condition (ii) of Definition 3.3 implies that there exists $\alpha$ with $1 \leq \alpha \leq N$ such that

$$\deg_{x_q} u_\alpha < \deg_{x_q} v_\alpha. \tag{4}$$

Following (3) and (4), we can write

$$u_\alpha = x_{j_1}x_{j_2}\cdots x_{j_p}\cdots x_{j_d} \text{ and } v_\alpha = x_{j_1}x_{j_2}\cdots x_{j_{p-1}}x_q x_{k_{p+1}}\cdots x_{k_d},$$

with $j_p > q$. To complete the proof, it is enough to show that $w = x_q u_\alpha / x_{j_p} \in I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Note that $q$ and $j_p$ belong to $V_p$. Moreover, $q - j_{p-1} \geq t_{p-1}$ because $v_\alpha$ is **t**-spread and $j_{p+1} - q \geq j_{p+1} - j_p \geq t_p$ because $j_p > q$. This yields that $w$ is a **t**-spread monomial, as desired. □

Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $R = S[\{t_u : u \in \mathcal{G}(I)\}]$. We define a monomial order on $R$ as following: if $u_1, u_2 \in S$ and $v_1, v_2 \in T$, then $u_1v_1 > u_2v_2$ if and only if $u_1 >_{\mathrm{lex}} u_2$ or $u_1 = u_2$ and $v_1 >_{\mathrm{sort}} v_2$, where $>_{\mathrm{lex}}$ denotes the lexicographical order on $S$ induced by $x_1 > \cdots > x_n$. Let $\mathcal{R}(I) = \bigoplus_{j\geq 0} I^j t^j \subseteq S[t]$ be the Rees ring of $I$. The Rees ring $\mathcal{R}(I)$ has the following presentation

$$\psi : R = S[\{t_u : u \in \mathcal{G}(I)\}] \to \mathcal{R}(I),$$

with $x_i \mapsto x_i$ for $1 \leq i \leq n$ and $t_u \mapsto ut$ for $u \in \mathcal{G}(I)$. Let $P = \ker \psi$. Then we have the next result.

**Corollary 3.5** *Let $>$ be the monomial order on $R$ as defined above. The reduced Gröbner basis of $P$ consists of the binomials of the following form*:

(1) $t_u t_v - t_{u'} t_{v'}$, *where* $sort(u, v) = (u', v')$;
(2) $x_i t_u - x_j t_v$, *where* $i < j$, $x_i u = x_j v$, *and $j$ is the largest integer for which* $x_i v / x_j \in \mathcal{G}(I)$.

***Proof*** According to [13, Theorem 5.1] (or see [5, Theorem 6.24]), it is enough to show that $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is sortable and satisfies the $l$-exchange property with respect to $>_{\mathrm{sort}}$ as noted in Proposition 3.1 and Theorem 3.4. □

Following the similar argument as in the proof of Corollary 3.2, we obtain the following corollary.

**Corollary 3.6** *The Rees algebra $\mathcal{R}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ is a normal Cohen-Macaulay domain.*

We are in a position to state the main result of this section in the next corollary.

**Corollary 3.7** *The ideal $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the strong persistence property and all powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution.*

***Proof*** The strong persistence property of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ can be deduced from [15, Corollary 1.6] and Corollary 3.6. Moreover, Corollary 3.5 together with [11, Corollary 10.1.8] provides that all the powers of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ have linear resolution, as claimed. □

Here, we determine the limit depth of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. By a theorem of Brodmann [2], $\operatorname{depth} S/I^k$ is constant for large enough $k$. This constant value is known as the limit depth of $I$, and denoted by $\lim_{k\to\infty} \operatorname{depth} S/I^k$. The minimum value of $k$ for which

$\operatorname{depth} S/I^k = \operatorname{depth} S/I^{k+t}$ for all $t > 0$ is called the *index of depth stability* and denoted by $\operatorname{dstab}(I)$. Let $\mathfrak{m}$ be the graded maximal ideal of $S$. The analytic spread of an ideal $I \subset S$ is the Krull dimension of the fiber cone $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ and denoted by $\ell(I)$.

**Definition 3.8** ( [15], Definition 3.1) Let $I \subset S$ be a monomial ideal in $S = K[x_1, \dots, x_n]$ and $\mathcal{G}(I) = \{u_1, \dots, u_r\}$. Then the linear relation graph $\Gamma$ of $I$ is the graph with the edge set

$$E(\Gamma) = \{\{i,j\} \ : \ \textit{there exist } u_t, u_m \in \mathcal{G}(I) \textit{ such that } x_i u_t = x_j u_m\},$$

and the vertex set $V(\Gamma) = \bigcup_{\{i,j\} \in E(\Gamma)} \{i,j\}$.

An ideal $I \subset S$ is said to have *linear relations* if $I$ is generated in degree $d$ and $\beta_{1,j}(I) = 0$ for all $j \neq d + 1$. We employ the following lemma to compute $\ell(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$.

**Lemma 3.9** ( [3, Lemma 5.2]) *Let $I$ be a monomial ideal with linear relations generated in a single degree whose linear relation graph $\Gamma$ has $r$ vertices and $s$ connected components. Then $\ell(I) = r - s + 1$.*

We are now ready to determine the analytic spread of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ in the following lemma.

**Lemma 3.10** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph and $|V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| = r$. Then $\ell(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = r - d + 1$.*

***Proof*** Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $V = \{V_1, \dots, V_d\}$. Using Theorem 2.4 and [3, Lemma 5.2], it is enough to show that $\Gamma(I)$ has $r$ vertices and $d$ connected components. Let $a_i = \min V_i$ and $b_i = \max V_i$, for all $i = 1, \dots, d$. Let $h, k \in V_i$ for some $i$. Since $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ does not have isolated vertices, this implies that the sets $\{a_1, \dots, a_d\}$ and $\{b_1, \dots, b_d\}$ are $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Then $u = x_{a_1} \cdots x_{a_{i-1}} x_h x_{b_{i+1}} \cdots x_{b_d}$ and $v = x_{a_1} \cdots x_{a_{i-1}} x_k x_{b_{i+1}} \cdots x_{b_d}$ are also $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. This shows that $x_k u = x_h v$; hence, $\{h, k\} \in E(\Gamma)$ and $V(\Gamma) = r$. Moreover, it follows from the definition of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ that for $i \neq j$ and $h \in V_i$ and $k \in V_j$, we have the edge $\{h, k\} \notin E(\Gamma)$. Therefore, $\Gamma$ has exactly $d$ connected components, as required. □

We now give the last result of this section in the following theorem.

**Theorem 3.11** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph and $|V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| = r$, and $S$ be the ambient ring of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then*

$$\lim_{k \to \infty} \operatorname{depth}(S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^k) = d - 1,$$

*and* $\operatorname{dstab}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) \leq r - d$.

***Proof*** Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then it follows from Corollary 3.6 and a result of Eisenbud and Huneke [4] that $\lim_{k \to \infty} \operatorname{depth}(S/I^k) = r - \ell(I)$. From Lemma 3.10, we have $r - \ell(I) = r - (r - d + 1) = d - 1$ as required. In addition, using [15, Theorem 3.3] and Lemma 3.10, we see that $\operatorname{depth}(S/I^{r-d}) = d - 1$. It is shown in [12, Proposition 2.1] that if all powers of an ideal have linear resolution, then $\operatorname{depth} S/I^k \leq \operatorname{depth} S/I^t$ for all $k < t$. It follows now from Corollary 3.7 that $\operatorname{dstab}(I) \leq r - d$. This completes the proof. □

## 4 Normally torsion-free and Cohen-Macaulay $I(\mathrm{K}_V^{\mathbf{t}})$

In this section, our main goal is to show that $I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})$ is normally torsion-free and give a complete characterization of Cohen-Macaulay $I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})$ for $V = \{V_1, \dots, V_d\}$ such that each $V_i$ is of the form $[a_i, b_i]$ for some integers $a_i, b_i \in \mathbb{Z}^+$. To this aim, we begin with the description of minimal prime ideals of $I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})$ and view $\mathrm{K}_\mathrm{V}^{\mathbf{t}}$ as a simplicial complex. For more details on simplicial complexes, we refer the reader to [11].

Given a square-free monomial ideal $I \subset R$, the *Alexander dual* of $I$, denoted by $I^\vee$ is given by $I^\vee = \bigcap_{u\in\mathcal{G}(I)}(x_i \ : \ x_i \in \mathrm{supp}(u))$. The minimal generators of $I^\vee$ correspond to the minimal prime ideals of $I$. Below we give a description of $\mathcal{G}(I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})^\vee)$.

**Theorem 4.1** *Let* $\mathrm{K}_\mathrm{V}^{\mathbf{t}}$ *be a complete* $\mathbf{t}$*-spread* $d$*-partite hypergraph with* $V(\mathrm{K}_\mathrm{V}^{\mathbf{t}}) \subseteq [n]$ *and* $V = \{V_1, \dots, V_d\}$. *Furthermore, let* $|V_j| = n_j$ *with* $V_j = [i_j, i_j + n_j - 1]$ *for all* $j = 1, \dots, d$. *Then* $\mathcal{G}(I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})^\vee)$ *consists of the following monomials*:

(i) $\prod_{k\in V_i} x_k$ *for all* $i = 1, \dots, d$; *and,*

(ii) $(\prod_{i=j}^{p}\prod_{k\in V_i} x_k)/(\prod_{i=j}^{p-1} v_{q_i}\prod_{i=j+1}^{p} v_{q_i'})$, *for all* $1 \le j < p \le d$ *and for each sequence of nonnegative integers* $q_j, \dots, q_{p-1}$ *satisfying*

$$i_\ell + q_\ell' < i_\ell + n_\ell - 1 - q_\ell \quad \text{for}\ j+1 \le \ell \le p-1, \tag{5}$$

$$i_\ell + q_\ell' - (i_{\ell-1} + n_{\ell-1} - 1 - q_{\ell-1}) = t_{\ell-1} - 1 \ \text{for}\ \ell = j+1, \dots, p, \tag{6}$$

where $v_{q_\ell} = \prod_{r=1}^{1+q_\ell} x_{i_\ell+n_\ell-r}$, for $\ell = j, \dots, p-1$ and $v_{q_\ell'} = \prod_{r=0}^{q_\ell'} x_{i_\ell+r}$, for $\ell = j+1, \dots, p$.

***Proof*** Let $\Delta$ be the simplicial complex on $V(\mathrm{K}_\mathrm{V}^{\mathbf{t}})$ such that $I_\Delta = I(\mathrm{K}_\mathrm{V}^{\mathbf{t}})$ be the Stanley-Reisner ideal of $\Delta$. Let $\mathcal{F}(\Delta)$ be the set of facets of $\Delta$. For any $F \in \Delta$, we set $x_F = \prod_{i\in F} x_i$. It follows from [11, Lemma 1.5.4] that the standard primary decomposition of $I_\Delta$ is given by

$$I_\Delta = \bigcap_{F\in\mathcal{F}(\Delta)} P_{\bar{F}},$$

where $P_{\bar{F}}$ is the monomial prime ideal generated by the variables $x_i$ with $i \in \bar{F} = V(\mathrm{K}_\mathrm{V}^{\mathbf{t}})\setminus F$. Therefore, using [11, Corollary 1.5.5], it is enough to show that $\mathcal{F}(\Delta)$ is the disjoint union of $\mathcal{F}_1$ and $\mathcal{F}_2$, defined below:

(i) $\mathcal{F}_1 = \{F_1, \dots, F_d\}$, where $F_i = \bigcup_{j\ne i,\, j=1}^{d} V_j$ for all $i = 1, \dots, d$,

(ii) For all $1 \le j < p \le d$, set $A_{j,p} := \bigcup_{i\notin\{j,\dots,p\},\, i=1}^{d} V_j$. For each sequence of nonnegative integers $q_j, \dots, q_{p-1}$ satisfying conditions (5) and (6), we set

$$B_{q_\ell} := \{i_\ell + n_\ell - 1 - q_\ell, \dots, i_\ell + n_\ell - 1\} \subsetneq V_\ell \ \text{for}\ \ell = j, \dots, p-1,$$

and

$$B_{q_\ell'} = \{i_\ell, \dots, i_\ell + q_\ell'\} \subsetneq V_\ell \ \text{for}\ \ell = j+1, \dots, p.$$

Then we get

$$\mathcal{F}_2 = \{A_{j,p} \cup (\bigcup_{\ell=j}^{p-1} B_{q_\ell}) \cup (\bigcup_{\ell=j+1}^{p} B_{q'_\ell}) \text{ for all } 1 \le j < p \le d \text{ and } q_j, \dots, q_{p-1}\}.$$

The condition (6) translates into the following: for each $\ell = j, \dots, p-1$ we have $\max B_{q'_{\ell+1}} - \min B_{q_\ell} = t_\ell - 1$. In the construction of elements in $\mathcal{F}_2$, it is enough to determine the integers $q_j, \dots, q_{p-1}$, because $q'_\ell$ is uniquely determined from $q_{\ell-1}$, for all $\ell = j+1, \dots, p$, by using the equality in (6).

First, we show that $\mathcal{F}_1 \subseteq \mathcal{F}(\Delta)$. For any $F_i \in \mathcal{F}_1$, we have $F_i \cap V_i = \emptyset$. Therefore, $x_{F_i} \notin I_\Delta$. Moreover, for any $k \in V_i$, using the assumption that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ does not contain any isolated vertices, we obtain that $F_i \cup \{k\}$ contains a $\mathbf{t}$-spread edge, and hence $x_{F_i} x_k \in I_\Delta$ and $F_i \in \mathcal{F}(\Delta)$.

Now, assume that $F \in \mathcal{F}_2$, where $F = A_{j,p} \cup (\bigcup_{\ell=j}^{p-1} B_{q_\ell}) \cup (\bigcup_{\ell=j+1}^{p} B_{q'_\ell})$ for some $1 \le j < p \le d$ and $q_j, \dots, q_{p-1}$. We here show that $F \in \Delta$. On contrary, if $x_F \in I_\Delta$, then $F$ contains a $\mathbf{t}$-spread edge, say $G = \{k_1, \dots, k_d\}$. Then $k_j \in B_{q_j}$ because $G \cap V_j \subseteq F \cap V_j = B_{q_j}$. If $p = j+1$, then by using the condition (6), it immediately follows that for any choice of $k_j \in B_{q_j}$, there is no suitable $k_{j+1} \in B_{q'_{j+1}}$ such that $k_{j+1} - k_j \ge t_{j-1}$. If $p > j+1$, then the condition (6) gives that $k_{j+1} \in B_{q_{j+1}}$. Using the condition (6) repeatedly in a similar way, we obtain $k_{p-1} \in B_{q_{p-1}}$. However, there is no suitable $k_p \in B_{q'_p}$ such that $k_p - k_{p-1} \ge t_{p-1}$, a contradiction. Consequently, we get $F \in \Delta$.

In what follows, we demonstrate that $F \in \mathcal{F}(\Delta)$. Note that

$$V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \setminus F = (V_j \setminus B_{q_j}) \cup \left( \bigcup_{l=j+1}^{p-1} \left( V_l \setminus \left( B_{q'_\ell} \cup B_{q_\ell} \right) \right) \cup \left( V_p \setminus B_{q'_p} \right) \right).$$

Let $a \in V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \setminus F$. Then $a \in V_s$ for some $j \le s \le p$. Set

$$k_r = \begin{cases} i_r, & \text{if } r = 1, \dots, j-1, \\ i_r + n_r - 1 - q_r, & \text{if } r = j, \dots, s-1, \\ a, & \text{if } r = s, \\ i_r + q'_r, & \text{if } r = s+1, \dots, p, \\ i_r + n_r - 1, & \text{if } r = p+1, \dots, d. \end{cases}$$

When $s = j$, then we remove the condition on $k_r$ for $r = j, \dots, s-1$, and similarly, when $s = p$, then we remove the condition on $k_r$ for $r = s+1, \dots, p$. Using conditions (5) and (6) together with the assumption that $\Delta$ has no isolated vertices, we obtain that $k_r - k_{r-1} \ge t_{r-1}$ for all $r = 2, \dots, d$. Therefore, $G = \{k_1, \dots, k_d\} \subseteq F \cup \{a\}$ is a $\mathbf{t}$-spread edge, and hence $x_G \in I_\Delta$, as required.

It remains to check that $\mathcal{F}(\Delta) \subseteq \mathcal{F}_1 \cup \mathcal{F}_2$. This is equivalent to show that for every face $G$ of $\Delta$ there exists a facet $F \in \mathcal{F}_1 \cup \mathcal{F}_2$ such that $G \subseteq F$. Let $G \in \Delta$ such that $G \cap V_k = U_k$ for all $k = 1, \dots, d$. If $U_k = \emptyset$ for some $k$, then $G \subseteq F_k \in \mathcal{F}_1$. Now, assume that $U_k \neq \emptyset$ for all $k = 1, \dots, d$. Set $a_k = \min U_k$ and $b_k = \max U_k$ for all $k = 1, \dots, d$. In the rest of the proof, we will use the following fact repeatedly:

$(*)$ If there exist $a \in V_\ell$ and $b \in V_{\ell+1}$ such that $b - a < t_\ell$ and $a + t_\ell - 1 < i_{\ell+1} + n_{\ell+1} - 1$, then by letting $q_\ell = i_\ell + n_\ell - 1 - a$, and using the condition (6), there is a unique $q'_{\ell+1}$ such that $b < i_{\ell+1} + q'_{\ell+1}$.

**Case(1):** If there exists some $k$ with $b_{k+1} - a_k < t_k$, then it follows from the statement $(*)$ that for a suitable choice of $q_k$ we have $U_k \subseteq B_{q_k}$ and $U_{k+1} \subseteq B_{q'_{k+1}}$. Since $U_i \subseteq V_i \subset A_{k,k+1}$ for all $i = 1, \dots, k-1, k+2, \dots, d$, we can deduce that $G \subseteq A_{k,k+1} \cup B_{q_k} \cup B_{q'_{k+1}} \in \mathcal{F}_2$, as desired.

**Case(2):** Assume that $b_{k+1} - a_k \geq t_k$ for all $k = 1, \dots, d-1$. Since $G \in \Delta$, we know that $G$ does not contain any **t**-spread edge. In particular, $\{a_1, \dots, a_d\} \subseteq G$ is not a **t**-spread edge. This yields that there exists some $k \in \{2, \dots, d\}$ for which $a_{k+1} - a_k < t_k$. We choose minimum $j \geq 1$ for which $a_{j+1} - a_j < t_j$. Note that $M = \{a_1, a_2, \dots, a_j\} \subset G$ such that, $a_{i+1} - a_i \geq t_i$, for all $i = 1, \dots, j-1$. In the discussion below, we aim to construct a suitable $F \in \mathcal{F}_2$ such that $G \subset F$. To this aim, we perform the Step $j$ as introduced below.

Step $j$: We set $e_j := a_j$ and $e_{j+1} := \min\{a \in U_{j+1} : a - e_j \geq t_j\}$. Note that $\{a \in U_{j+1} : a - e_j \geq t_j\} \neq \emptyset$ because $b_{j+1} - a_j \geq t_j$. We define $e_{j+r}$ recursively as $e_{j+r} = \min\{a \in U_{j+r} : a - e_{j+r-1} \geq t_{j+r-1}\}$ such that

$$\{a \in U_{j+r} : a - e_{j+r-1} \geq t_{j+r-1}\} \neq \emptyset \text{ for some } 1 < r < d-j.$$

There exists some $p > j+1$ for which $\{a \in U_{j+r} : a - e_{j+r-1} \geq t\} = \emptyset$, that is, for some $p > j+1$ we have $b_p - e_{p-1} < t_{p-1}$, otherwise, $M \cup \{e_{j+1}, \dots, e_d\} \subseteq G$ is a **t**-spread edge in $G$, a contradiction. Choose minimum $p > j+1$ such that $b_p - e_{p-1} < t_{p-1}$.

**Subcase(2.1):** If for all $j+1 \leq l \leq p-1$ we have $i_{\ell+1} - e_\ell < t_\ell$, then take $c_{\ell+1} \in V_{\ell+1}$ such that $c_{\ell+1} - e_\ell = t_\ell - 1$ for $\ell = j, \dots, p-1$. This gives us $j$, $p$ and $q_j, \dots, q_p$ as described in statement $(*)$ for which $e_\ell \in V_\ell$ and $c_{\ell+1} \in V_{\ell+1}$ with $c_{\ell+1} - e_\ell < t_\ell$. Moreover, $U_i \subseteq A_{j,p}$ for all $i \notin \{j, \dots, p\}$, and $U_j \subseteq B_{q_j}$, $U_p \subseteq B_{q'_p}$, and $U_\ell \subseteq B_{q_\ell} \cup B_{q'_\ell}$ for all $\ell = j+1, \dots, p-1$. Hence, this implies that

$$G \subseteq A_{j,p} \cup \Big(\bigcup_{\ell=j}^{p-1} B_{q_\ell}\Big) \cup \Big(\bigcup_{\ell=j+1}^{p} B_{q'_\ell}\Big),$$

and we are done.

**Subcase(2.2):** If for some $j+1 \leq l \leq p-1$, $i_{\ell+1} - e_\ell \geq t_\ell$, then replace $M$ with $M \cup \{e_{j+1}, \dots, e_\ell, a_{\ell+1}\} \subset G$. In this case, there exists a minimum $j' \geq \ell+1$ such that $a_{j'+1} - a_{j'} < t_{j'}$. Otherwise, $M \cup \{a_{\ell+2}, \dots, a_d\} \subseteq G$ is a **t**-spread edge, a contradiction. Repeat Step $j$ by replacing $j$ with $j'$.

Thanks to we have finite number of partitions, this process must be terminated after a finite number of steps. If the desired $j$ and $p$ are obtained, then we construct a suitable $F \in \mathcal{F}_2$ with $G \subset F$ as described in Case(2.1). If the desired $j$ and $p$ are not obtained, then $G$ contains a **t**-spread edge in $G$, a contradiction. □

We illustrate the construction of monomials of the forms (i) and (ii) in Theorem 4.1 in the following example.

***Example 4.2*** Let $V = \{V_1, V_2, V_3, V_4\}$ with $V_1 = [1,2]$, $V_2 = [4,6]$, $V_3 = [8,10]$, $V_4 = [12,13]$, and $\mathbf{t} = (3,4,3)$. One can easily see that the minimal generators of the edge ideal of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ are as follows:

| | |
|---|---|
| $x_1x_4x_8x_{12}$ | |
| $x_1x_4x_8x_{13}$ | |
| $x_1x_4x_9x_{12}$ | |
| $x_1x_4x_9x_{13}$ | |
| $x_1x_4x_{10}x_{13}$ | |
| $x_1x_5x_9x_{12}$ | $x_2x_5x_9x_{12}$ |
| $x_1x_5x_9x_{13}$ | $x_2x_5x_9x_{13}$ |
| $x_1x_5x_{10}x_{13}$ | $x_2x_5x_{10}x_{13}$ |
| $x_1x_6x_{10}x_{13}$ | $x_2x_6x_{10}x_{13}$ |

Following Theorem 4.1, the minimal generators of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^{\vee}$ are given as follows:

(i) The monomials of the form (i) described in Theorem 4.1 are $x_1x_2, x_4x_5x_6, x_8x_9x_{10}$, and $x_{12}x_{13}$.

(ii) The construction of monomials of the form (ii) described in Theorem 4.1 is given in the following table.

Accordingly, we get

$$\begin{aligned}\mathrm{Ass}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = \{&(x_1,x_2),(x_4,x_5,x_6),(x_8,x_9,x_{10}),(x_{12},x_{13}),(x_1,x_5,x_6),(x_1,x_5,x_{10}),\\ &(x_1,x_9,x_{10}),(x_1,x_5,x_{13}),(x_1,x_9,x_{13}),(x_4,x_5,x_{10}),(x_4,x_9,x_{10}),\\ &(x_4,x_5,x_{13}),(x_4,x_9,x_{13}),(x_8,x_9,x_{13})\}.\end{aligned}$$

| $j$ | $p$ | $q_j,\dots,q_{p-1},q'_{j+1},\dots,q'_p$ | $u$ |
|---|---|---|---|
| 1 | 2 | $q_1=0, q'_2=0$ | $x_1x_5x_6$ |
| 1 | 3 | $q_1=0, q'_2=0, q_2=0, q'_3=1$ | $x_1x_5x_{10}$ |
| | | $q_1=0, q'_2=0, q_2=1, q'_3=0$ | $x_1x_9x_{10}$ |
| 1 | 4 | $q_1=0, q'_2=0, q_2=0, q'_3=1, q_3=0, q'_4=0$ | $x_1x_5x_{13}$ |
| | | $q_1=0, q'_2=0, q_2=1, q'_3=0, q_3=0, q'_4=0$ | $x_1x_9x_{13}$ |
| 2 | 3 | $q_2=0, q'_3=1$ | $x_4x_5x_{10}$ |
| | | $q_2=1, q'_3=0$ | $x_4x_9x_{10}$ |
| 2 | 4 | $q_2=0, q'_3=1, q_3=0, q'_4=0$ | $x_4x_5x_{13}$ |
| | | $q_2=1, q'_3=0, q_3=0, q'_4=0$ | $x_4x_9x_{13}$ |
| 3 | 4 | $q_3=0, q'_4=0$ | $x_8x_9x_{13}$ |

As an immediate consequence of Theorem 4.1, we obtain the following corollary, which will be used to prove the normally torsion-freeness of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

**Corollary 4.3** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V=\{V_1,\dots,V_d\}$. Furthermore, let $|V_j|=n_j$ with $V_j=[i_j, i_j+n_j-1]$ for all $j=1,\dots,d$. If $v := \prod_{j=1}^{d} x_{i_j}$, then $v \in \mathfrak{p}\backslash\mathfrak{p}^2$ for all $\mathfrak{p} \in \mathrm{Min}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$.*

***Proof*** Let $v=\prod_{j=1}^{d} x_{i_j}$. The minimal prime ideals of $I=I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ correspond to the minimal generators of $I^{\vee}$ described in statements (i) and (ii) of Theorem 4.1. The minimal primes corresponding to the generators of the form (i) are $\mathfrak{p}_i=(x_k : k \in V_i)$ and $v \notin \mathfrak{p}_i^2$ for all $i=1,\dots,d$. Moreover, each generator of $I^{\vee}$ of the form (ii) is constructed by fixing $j$, $p$ and

$q_j, \dots, q_p$. Let $\mathfrak{q}$ be a minimal prime of $I$ corresponding to a generator of the form (ii). Then $x_{i_k} \in \mathfrak{q}$ if and only if $k = j$, as required. ☐

We recollect the following lemma, which will be used repeatedly in the next proposition and Theorem 4.6.

**Lemma 4.4** ( [17, Lemma 3.12]) *Let $I$ be a monomial ideal in a polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $h = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ with $j_1, \dots, j_s \in \{1, \dots, n\}$ be a monomial in $S$. Then $I$ is normally torsion-free if and only if $hI$ is normally torsion-free.*

In order to establish Theorem 4.6, we require the following auxiliary proposition. For a given square-free monomial ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$, we denote by $I \setminus x_i$ the ideal generated by those elements in $\mathcal{G}(I)$ that do not contain $x_i$ in their support.

**Proposition 4.5** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = 2$ with $V_j = \{i_j, i_j + 1\}$ for all $j = 1, \dots, d$. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free.*

***Proof*** To simplify the notation, set $I := I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. We proceed by induction on $d$. If $d = 1$, then there is nothing to show. Hence, assume that $d > 1$ and that the result holds for any complete $\mathbf{t}$-spread $(d-1)$-partite hypergraph. Choose an arbitrary element $\mathfrak{p} \in \mathrm{Min}(I)$ and set $v := \prod_{j=1}^{d} x_{i_j}$. It follows at once from Corollary 4.3 that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$. We show that $I \setminus x_r$ is normally torsion-free for each $x_r \in \mathrm{supp}(v)$. Without loss of generality, we let $V_1 = \{1, 2\}$ and we prove that $I \setminus x_1$ is normally torsion-free. It is not hard to check that $I \setminus x_1 = x_2 L$ where $L$ is the edge ideal of $\mathbf{t}$-spread $d$-partite hypergraph with vertex partition $\mathrm{V}' = \{V_2', \dots, V_d'\}$ such that, for all $i = 2, \dots, d$, the set $V_i'$ is obtained from $V_i$ after removing the isolated vertices, if any. One can conclude from the inductive hypothesis that $L$ is normally torsion-free. Here, using Lemma 4.4 implies that $I \setminus x_1$ is normally torsion-free. It follows now from [18, Theorem 3.7] that $I$ is normally torsion-free, as claimed. ☐

**Theorem 4.6** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \dots, d$. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free. In particular, $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normal.*

***Proof*** We first assume that $|V_j| = 1$ for some $1 \leq j \leq d$, say $V_j = \{z\}$. Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. Then we can write $I = x_z L$ such that $L$ can be viewed as the edge ideal associated to a complete $\mathbf{t}$-spread $(d-1)$-partite hypergraph. According to Lemma 4.4, $I$ is normally torsion-free if and only if $L$ is normally torsion-free. Thus, we reduce to the case $|V_j| \geq 2$ for all $j = 1, \dots, d$. Set $v := \prod_{j=1}^{d} x_{i_j}$. Pick an arbitrary element $\mathfrak{p} \in \mathrm{Min}(I)$. One can derive from Corollary 4.3 that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$. To complete the proof, it is sufficient to establish $I \setminus x_s$ in normally torsion-free for each $x_s \in \mathrm{supp}(v)$. To accomplish this, we use the induction on $n := |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})|$. On account of $|V_j| \geq 2$ for all $j = 1, \dots, d$, this implies that $n \geq 2d$. The case in which $n = 2d$ can be deduced according to Proposition 4.5. Now, suppose that $n > 2d$. It is not hard to see that $I \setminus x_s$ is again the edge ideal of the $\mathbf{t}$-spread $d$-partite hypergraph obtained from $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ by removing all the edges that contain $s$. One can deduce from the inductive hypothesis that $I \setminus x_s$ is normally torsion-free. Here, in view of [18, Theorem 3.7], we conclude that $I$ is normally torsion-free, as desired.

The last assertion can be deduced according to [11, Theorem 1.4.6]. □

We can readily provide the following corollary inspired by Theorem 4.6. A *matching* in a hypergraph $\mathcal{H}$ is a family of pairwise disjoint edges, and the maximum cardinality of a matching is denoted by $\nu(\mathcal{H})$. The transversal number of a hypergraph $\mathcal{H}$, denoted by $\tau(\mathcal{H})$ is the minimal cardinality of a transversal of $\mathcal{H}$. A hypergraph $\mathcal{H}$ is said to satisfy the König property if $\nu(\mathcal{H}) = \tau(\mathcal{H})$, see [1, Chapter 2, Section 4].

**Corollary 4.7** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph. Then $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the König property.*

***Proof*** Based on Theorem 4.6, we get $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is normally torsion-free. In addition, by virtue of [20, Theorem 14.3.6], one can deduce that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ has the max-flow min-cut property. It follows now from [20, Corollary 14.3.18] that $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ has the packing property. On the other hand, by virtue of [10, Definition 2.3], we obtain $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ satisfies the König property. This completes the proof. □

Next, we give a characterization of Cohen-Macaulay $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$. To do this, we first determine the height of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.

**Proposition 4.8** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \dots, d$. Then $\mathrm{ht}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})) = \min\{n_1, \dots, n_d\}$, where $\mathrm{ht}(I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}))$ denotes the height of $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$.*

***Proof*** Let $I := I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $n_k = \min\{n_1, \dots, n_d\}$. Since $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ does not contain any isolated vertices, this yields that

$$\{i_1, \dots, i_d\}, \{i_1 + 1, \dots, i_d + 1\}, \dots, \{i_1 + n_k - 1, \dots, i_d + n_k - 1\}, \tag{7}$$

are pairwise disjoint $\mathbf{t}$-spread edges in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Hence, we obtain the following monomials

$$x_{i_1}x_{i_2}\cdots x_{i_d}, x_{i_1+1}x_{i_2+1}\cdots x_{i_d+1}, \dots, x_{i_1+n_k-1}x_{i_2+n_k-1}\cdots x_{i_d+n_k-1}$$

belong to $\mathcal{G}(I)$. This gives that $\mathrm{ht}(I) \geq n_k$. It follows also from Theorem 4.1 that $(x_i : i \in V_k)$ is a minimal prime of $I$ with height $n_k$. This finishes our proof. □

Note that the König property of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ can be also observed from the proof of above proposition. Indeed, the inequality $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ holds for any hypergraph $\mathcal{H}$ and the $\mathbf{t}$-spread edges given in (7) give a maximal matching in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$.

Under the assumptions of Theorem 4.1, one can compute the degree of generators of $I^\vee = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})^\vee$. It is easy to see that $\deg \prod_{k\in V_i} x_k = n_i$ for all $i = 1, \dots, d$. Now, let $u \in \mathcal{G}(I^\vee)$ of the form (ii) for some $1 \leq j < p \leq d$ and $q_j, \dots, q_p$. Then $u = (\prod_{i=j}^{p} \prod_{k\in V_i} x_k)/(\prod_{i=j}^{p-1} v_{q_i} \prod_{i=j+1}^{p} v_{q_i'})$. Let $h$ be the product of variables with indices in $[i_j, i_p + n_p - 1] \setminus (V_j \cup \cdots \cup V_p)$ and $w = (uh)/h$. Then $\deg w = \deg u$.

We have $\deg h(\prod_{i=j}^{p} \prod_{k\in V_i} x_k) = (i_p + n_p - 1) - i_j + 1$. Moreover, it follows from the condition (6) that $\deg(h \prod_{i=j}^{p-1} v_{q_i} \prod_{i=j+1}^{p} v_{q_i'}) = \sum_{i=j}^{p-1} t_i$. We thus get

$$\deg w = (i_p + n_p - 1) - i_j + 1 - \sum_{i=j}^{p-1} t_i = i_p - i_j + n_p - \sum_{i=j}^{p-1} t_i.$$

Hence, we obtain

$$\deg u = i_p - i_j + n_p - \sum_{i=j}^{p-1} t_i. \tag{8}$$

A square-free monomial ideal is said to be *unmixed* if its minimal prime ideals are of the same height. Using the description of generators of $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})^\vee$ and their degrees, we obtain the following characterization for unmixedness of $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$.

**Theorem 4.9** *Let $\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}}) \subseteq [n]$ and $V = \{V_1, \dots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \dots, d$. Then $I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ is unmixed if and only if $n_1 = \cdots = n_d = s$, and for each $j = 1, \dots, d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$.*

***Proof*** Let $I = I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ be unmixed. Then every minimal prime of $I$ has the same height, equivalently, $I^\vee$ is generated in the same degree. By Theorem 4.1, we know that every $V_j$ corresponds to a minimal generator in $I^\vee$, and this yields $n_1 = \cdots = n_d$. Let $n_1 = \cdots = n_d = s$. We only need to show that for each $j = 1, \dots, d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Indeed, if $i_{j+1} - (i_j + s - 1) \le t_j - 1$ for some $j$, then we obtain $u \in \mathcal{G}(I^\vee)$ of the form (ii) with $p = j + 1$ and a suitable choice of $q_j$ and $q'_{j+1}$ as described in statement $(*)$ in the proof of Theorem 4.1. It follows from (8) that $\deg u = i_{j+1} - i_j + s - t_j$. Since $\deg u = s$, we obtain $i_{j+1} - i_j = t_j$.

Now, assume that for all $j = 1, \dots, d$ we have $n_j = s$ and for each $j = 1, \dots, d-1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Then all generators of $I^\vee$ of the form (i) have same degree $s$. If $I^\vee$ has no generator of the form (ii), then the proof is complete. Otherwise, let $u \in \mathcal{G}(I^\vee)$ of the form (ii) for some $j$, $p$ and $q_j \dots, q_{p-1}$. Then, for all $\ell = j, \dots, p-1$, we have $i_{\ell+1} - i_\ell = t_\ell$, because if $i_{\ell+1} - (i_\ell + s - 1) > t_\ell - 1$ for some $\ell$, then $q_\ell$ and $q'_{\ell+1}$ do not satisfy the condition (6). This gives that $i_p = i_j + \sum_{i=j}^{p-1} t_i$. Using (8), we obtain

$$\deg u = i_p - i_j + s - \sum_{i=j}^{p-1} t_i = i_j + \sum_{i=j}^{p-1} t_i - i_j + s - \sum_{i=j}^{p-1} t_i = s,$$

and the proof is done.

***Remark 4.10*** Let $V = \{V_1, V_2, V_3, V_4\}$ with $V_1 = [2, 4]$, $V_2 = [6, 8]$, $V_3 = [9, 11]$, $V_4 = [13, 15]$, and $\mathbf{t} = (2, 3, 4)$. By virtue of Theorem 4.9, the edge ideal $I = I(\mathrm{K}^{\mathbf{t}}_{\mathrm{V}})$ is unmixed. In fact, by using Theorem 4.1, the minimal primes of $I$ are as follows:

$$\begin{aligned}\mathrm{Ass}(I) = \{&(x_2, x_3, x_4), (x_6, x_7, x_8), (x_9, x_{10}, x_{11}), (x_{13}, x_{14}, x_{15}), (x_6, x_7, x_{11}),\\ &(x_6, x_7, x_{15}), (x_6, x_{10}, x_{11}), (x_6, x_{10}, x_{15}), (x_6, x_{14}, x_{15}), (x_9, x_{10}, x_{15}),\\ &(x_9, x_{14}, x_{15})\}.\end{aligned}$$

However, one can verify with *Macaulay2* [9] that $S/I$ is not Cohen-Macaulay.

The above remark states that unmixedness is not a sufficient for the edge ideal of $\mathbf{t}$ -spread $d$-partite hypergraphs being Cohen-Macaulay. In what follows, we give a characterization of $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ with Cohen-Macaulay edge ideals. To do this, we introduce the following notations, that is, $q(u_k) := |\mathrm{set}(u_k)|$ and $q(I) := \max\{q(u_1), \ldots, q(u_r)\}$.

We are in a position to state the last result of this section in the subsequent theorem.

**Theorem 4.11** *Let $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ be a complete $\mathbf{t}$-spread $d$-partite hypergraph with $V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}) \subseteq [n]$ and $V = \{V_1, \ldots, V_d\}$. Furthermore, let $|V_j| = n_j$ with $V_j = [i_j, i_j + n_j - 1]$ for all $j = 1, \ldots, d$. Then $S/I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is Cohen-Macaulay if and only if either $I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ is a principal ideal, or $n_1 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1, \ldots, d - 1$.*

***Proof*** Let $I = I(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})$ and $S$ be the ambient ring of $I$. Since $I$ has linear quotients, thanks to Theorem 2.4, it follows from [14, Corollary 1.6] that the length of the minimal free resolution of $S/I$ over $S$ is equal to $q(I) + 1$. This implies that $\mathrm{depth}(S/I) = |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| - q(I) - 1$. Moreover, $\dim(S/I) = |V(\mathrm{K}_{\mathrm{V}}^{\mathbf{t}})| - \mathrm{ht}(I)$, where $\mathrm{ht}(I)$ denotes the height of $I$. This summarizes to $S/I$ is Cohen-Macaulay if and only if $\mathrm{ht}(I) = q(I) + 1$. Therefore, it is enough to show that $\mathrm{ht}(I) = q(I) + 1$ if and only if $n_1 = n_2 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1, \ldots, d - 1$.

If $I$ is a principal ideal then $S/I$ is Cohen Macaulay. Now, assume $n_1 = n_2 = \cdots = n_d = s$ and $i_{j+1} - i_j = t_j$ for each $j = 1, \ldots, d - 1$. Let $u = x_{k_1} \cdots x_{k_d} \in \mathcal{G}(I)$, where $k_i \in V_i$ for all $i = 1, \ldots, d$. Since $[i_1, k_1 - 1] \subseteq V_1$ and $[k_{j-1} + t_{j-1}, k_j - 1] \subseteq V_j$ for all $j = 2, \ldots, d$, by Proposition 2.5, we obtain $q(u) = k_d - i_1 - \sum_{j=1}^{d-1} t_j$. This shows that the maximum value of $q(u)$ is obtained when $k_d$ takes the maximum possible value which is $\max V_d = i_d + s - 1$. Furthermore, using $i_{j+1} - i_j = t_j$ for all $j = 1, \ldots, d - 1$, this gives that $i_d = i_1 + \sum_{j=1}^{d-1} t_j$. Hence, we have $q(I) = s - 1$, as required.

Conversely, suppose $S/I$ is Cohen-Macaulay, that is, $\mathrm{ht}(I) = q(I) + 1$. It follows from $\mathrm{ht}(I) = q(I) + 1$ that $I$ is unmixed and by using Proposition 4.9, this yields that, for all $j = 1, \ldots, d$, we have $n_j = s$, and for each $j = 1, \ldots, d - 1$ either $i_{j+1} - (i_j + s - 1) > t_j - 1$ or $i_{j+1} - i_j = t_j$. Then $\mathrm{ht}(I) = s$ thanks to Proposition 4.8. If $s = 1$, then $I$ is a principal ideal. Now, let $s > 1$. We only need to show that, for each $j = 1, \ldots, d - 1$, we have $i_{j+1} - i_j = t_j$. Suppose that for some $j$ we have $i_{j+1} - (i_j + s - 1) > t_j - 1$. Let $v = x_{i_1+s-1} x_{i_2+s-1} \cdots x_{i_d+s-1}$. Then $v \in \mathcal{G}(I)$ because $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$ do not contain isolated vertices and $\{i_1 + s - 1, i_2 + s - 1, \ldots, i_d + s - 1\}$ is a $\mathbf{t}$-spread edge in $\mathrm{K}_{\mathrm{V}}^{\mathbf{t}}$. Now, Proposition 2.5 gives that $\mathrm{set}(v) \cap V_1 = [i_1, i_1 + s - 2]$ and $\mathrm{set}(v) \cap V_{j+1} = \{i_{j+1}, \ldots, i_{j+1} + s - 2\}$. This shows that $q(v) > 2(s - 1)$ and $q(I) + 1 > \mathrm{ht}(I) = s$, a contradiction. □

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