ESTIMATING THE NUMBER OF FAILURES AND THE SPARE PARTS DEMAND - INSTALLED BASE APPROACH

by
AYDA AMNIATTALAB

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ABSTRACT

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AYDA AMNIATTALAB

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Dissertation Supervisor: Prof. HANS FRENK

Keywords: Spare parts, Nonstationary demand, Stochastic processes

This thesis investigates the estimation of failures and the corresponding demand for spare parts using the installed base approach. The installed base refers to the total number of units of a product or system that are currently in use by customers. The usage behavior of the customers in conjunction with the failure rate of a product shapes the future demand for spare parts of a product. We include these important factors in our mathematical model to develop a comprehensive framework for accurately predicting the number of failures that can occur and the resultant demand for spare parts.

Accurate estimation of failures and spare parts demand enables better inventory management, optimized maintenance schedules, and improved customer satisfaction. We propose models to estimate the mean, variability, and in general any moments of returned and defective items. In addition, we formulate the relations that give the saturation time of spare parts demand. Knowing when to expect the highest parts demand and how much demand will occur at that time is important for the manufacturer in order to prepare its maintenance facility and staff and stock the necessary spares. We also compute the distribution of spare parts demand by a time point, during a time interval, and after a time till the time that product disappears from the market over the whole life cycle of the product. We can identify the necessary stock level that needs to be kept in the inventory to satisfy the upcoming demand with a certainty level considering the lead time of providing the part. Also, a business can use the information about the distribution of spare parts demand to make a wiser last-time buy order decision. Our models also provide valuable insights by estimating the mean, variance, and covariance of two key factors: the installed base and discarded items. This information is crucial for the manufacturer as it enables them to gauge the success of their product and develop effective strategies to enhance its market performance. Additionally, the data related to discarded items serves as a valuable resource for the manufacturer to meet the demand for spare parts.

ÖZET

ARIZA SAYILARININ VE YEDEK PARÇA TALEPİNİN TAHMİN EDİLMESİ - KURULU TEMEL YAKLAŞIMI

AYDA AMNIATTALAB

ENDÜSTRİ MÜHENDİSLİĞİ DOKTORA TEZİ, ARALIK 2023

Tez Danışmanı: Prof. Dr. HANS FRENK

Anahtar Kelimeler: Yedek parçalar, Durağan olmayan talep, Stokastik süreçler

Bu tez, arızaların tahminini ve karşılık gelen yedek parça talebini aşağıdaki yöntemleri kullanarak araştırmaktadır: kurulu sistem yaklaşımı. Kurulu sistem, bir ürünün toplam ünite sayısını ifade eder veya Şu anda müşterilerin kullandığı sistem. Müşterilerin kullanım davranışları birlikte Bir ürünün arıza oranı, o ürünün yedek parçalarına olan gelecekteki talebi şekillendirir. Biz Kapsamlı bir çerçeve geliştirmek için bu önemli faktörleri matematiksel modelimize dahil edin meydana gelebilecek arızaların sayısını ve bunun sonucunda ortaya çıkan yedek parça talebini doğru bir şekilde tahmin etmek için parçalar.

Arızaların ve yedek parça talebinin doğru tahmin edilmesi, daha iyi envanter yönetimine, optimizasyona olanak tanır. bakım programları ve geliştirilmiş müşteri memnuniyeti. Tahmin etmek için modeller öneriyoruz iade edilen ve kusurlu ürünlerin ortalaması, değişkenliği ve genel olarak herhangi bir anı. Ek olarak, yedek parça talebinin doyum süresini veren ilişkileri formüle ediyoruz. Ne zaman yapılacağını bilmek En yüksek parça talebinin beklenmesi ve o sırada ne kadar talebin oluşacağı, Üreticinin bakım tesisini ve personelini hazırlaması ve gerekli yedek parçaları stoklaması için. Ayrıca yedek parça talebinin belirli bir zaman aralığı boyunca belirli bir zaman noktasına göre dağılımını da hesaplıyoruz ve Bir süre sonra ürünün tüm yaşam döngüsü boyunca piyasadan kaybolduğu ana kadar ürün. Müşteri memnuniyetini sağlamak için envanterde tutulması gereken stok seviyesini belirleyebiliriz. Parçanın tedarik süresi dikkate alınarak yaklaşan talebin kesinlik derecesi. Ayrıca, Bir işletme, yedek parça talebinin dağıtımı hakkındaki bilgileri daha akıllıca bir karar vermek için kullanabilir. son satın alma siparişi kararı. Modellerimiz ayrıca ortalamayı tahmin ederek değerli bilgiler sağlar. iki temel faktörün varyansı ve kovaryansı: kurulu sistem ve atılan öğeler. Bu bilgi Üreticinin, ürününün başarısını ölçmesine ve geliştirmesine olanak sağlaması açısından çok önemlidir. Pazar performansını artırmak için etkili stratejiler. Ek olarak, atılanlarla ilgili veriler öğeler, üreticinin yedek parça talebini karşılamasında değerli bir kaynak görevi görür

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To my family

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1. Introduction

Nowadays, the provision of spare parts has become an essential aspect of manufacturing repairable products. Offering after-sales support not only fosters customer loyalty towards the Original Equipment Manufacturer (OEM), but also provides significant benefits. Customers highly value quality after-sales services, which can help solidify their trust and satisfaction. Moreover, such support minimizes the substantial financial losses incurred due to machine downtime. For instance, a mere two-hour delay in an aircraft's operation can cost an airline up to $150,000 (Pinçe, Turrini & Meissner (2021)).

On the other hand, manufacturers face challenges in stocking large inventories of spare parts. Holding costs and the risk of parts becoming obsolete make it impractical to maintain excessive stock. To illustrate, the commercial aviation industry spends over $10 billion annually on spare part stocks alone (Pinçe et al. (2021)). Consequently, effective spare parts management becomes paramount. However, accurately predicting the demand for spare parts proves challenging due to the inherent variability in timing and magnitude. Therefore, providing a method to tackle this issue would yield considerable benefits.

In this thesis, our objective is to present a comprehensive methodology that addresses the key questions pertaining to spare parts management:

- How to model the spare parts demand more accurately?
- How to obtain specific information from the build model that would be useful for the manufacturer? Examples of this specific information are the saturation time of spare parts demand, the safety stock level needed to satisfy the parts demand during a certain time interval, and the estimation of parts demand that will happen after a particular time till the end of the product life cycle.

Spare parts are needed for products that are sold and consumed. Therefore, the factors that are essential to consider when developing a model are life cycle pattern, lifespan (usage time), and the failure rate of the product. Previous research has primarily relied on time series forecasting and Machine Learning tools which typically overlook the life cycle and life expectancy of the product. Instead, they mainly focused on historical spare parts demand data. While these approaches achieved occasional success, they often struggle to provide accurate estimations of future spare parts demand. To create a model that accurately represents the purchase-use-return system, we employ stochastic point processes allowing for a realistic and explicit representation of the system. A visual explanation of our approach is provided in Figure 1.1. In this example, in the middle box, three products/items

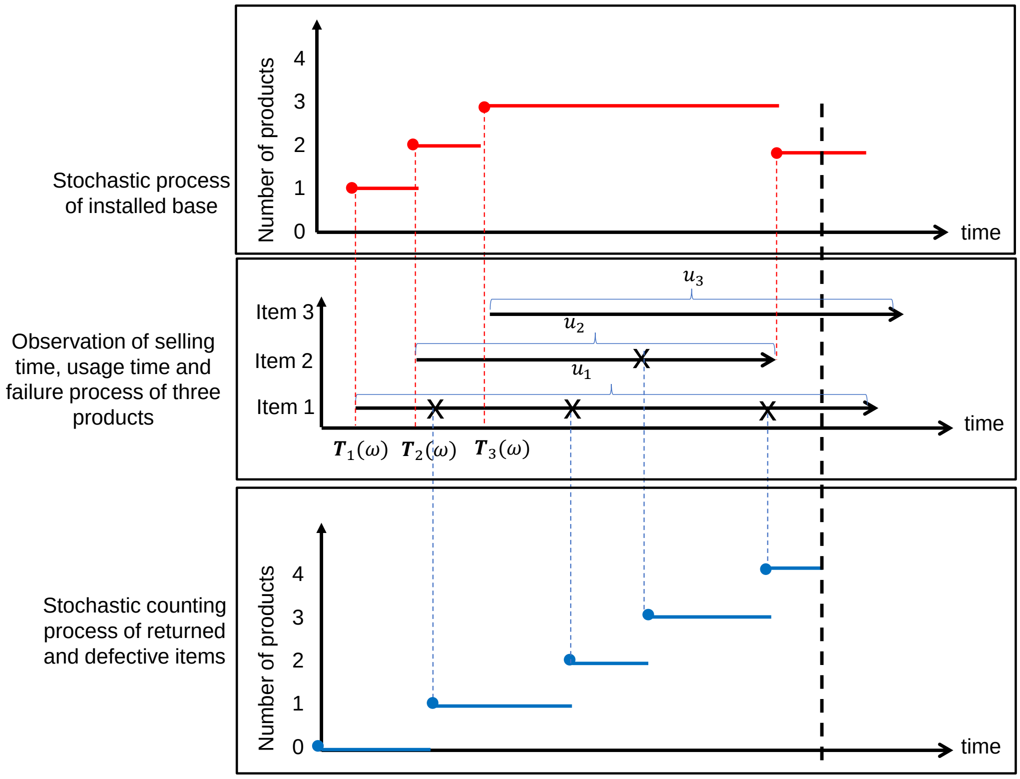

Figure 1.1 Sample illustration Our approach ($\mathbf{T}_n(\omega)$: selling time and $\mathbf{U}_n$: usage time of product $n, n = 1,2,3$)

are sold at times $\mathbf{T}_n, n = 1,2,3$ at which their usage times $\mathbf{U}_n, n = 1,2,3$ start. During their usage time products fail and are returned for repair (cross signs). As a result the stochastic counting process of the returned and defective items forms as in the lower box. When dealing with a straightforward product demand process, a black-box method like time series analysis would suffice for estimating future demand. However, the after-sales business potential of a manufacturer is determined by the size of the product's installed base i.e., the number of products in the market. Hence, we incorporate the installed base size information, which encompasses the life cycle pattern, into our model. The upper box illustrates the changes in this statistic. Estimating the size of an installed base not only optimizes customer support but also provides businesses with valuable insights into customer intentions and growth opportunities in the market. Hence, we model this statistic separately as well.

We divided our work on this topic into six chapters.

Chapter 2 discusses the previous literature and our contribution. In this chapter, one can see the position of our research within the spare parts demand realm clearly.

In Chapter 3 we develop a stochastic model that brings together the installed base, the number of discarded products, and the demand for spare parts in a single model based on three product characteristics: sales rate, life span (usage time), and defect rate. Our model describes the growth and decline of the installed base and spare parts demand over the entire life cycle of the original product using stochastic point processes and computes the expected value of these random variables. By tracking the growth or decline of the installed base, companies can evaluate the success of their products and services over time. Furthermore, monitoring spare parts demand pattern, allows businesses to anticipate future demand, allocate resources effectively, and make well-informed decisions regarding part sourcing. We also propose a simple bisection procedure to calculate the time that the maximum amount of products are at the market and the expected value of the spare parts demand are maximum under very general assumptions about the cdf of usage time and the mean arrival functions of the sales and failure processes. For a manufacturer it would be important to know how high his installed base and parts demand can grow and when he will face the most demand. Lastly, we analyze the effect of the expected usage time (life span) of the product on the expected number of returned defectives during the 1 year after it reaches its maximum. We observe that with the increase in life span, a larger number of parts is demanded. Most of the results in this chapter are published in (Amniattalab, Frenk & Hekimoğlu (2023a)).

In Chapter 4 we continue our analysis in Chapter 3 by investigating the higher moments and the pdf of the (i) spare parts demand by time t, (ii) spare parts demand over the interval $[t, t+\Delta]$ and (iii) spare parts demand after time t till the product is not anymore produced and actively sold. We derive under the only assumption of a nonhomogeneous Poisson sales process the higher moments of the mentioned random variables and under realistic assumptions about the random usage time and the failure behavior of each purchased item the pdf of the demand for spare parts. By utilizing the saturation time of spare parts demand, which can be calculated using the methodology outlined in Chapter 3, we can effectively estimate the expected demand for spare parts during a time period close to the saturation time. Having knowledge of the magnitude of spare part demand and the lead times for providing the requested parts enables the manufacturer to ensure the supply of parts. Furthermore, having an idea about the quantity of spare parts required until the product's end-of-life stage helps the manufacturer to make informed decisions regarding last-time buy size and develop effective strategies to address the demands during the End-of-Life phase. Additionally, we are able to determine safety stock, which represents the necessary inventory level to prevent stock-outs with a high level of certainty, such as 99%. Using this information, businesses can avoid stockout of spare parts due to the uncertainty of the parts demand.

In Chapter 5, we conclude our analysis on spare parts demand estimation by considering the discrete version of our model in Chapter 3 and 4. In real life, sales and repair data are typically discrete time variables. Discrete time views the values of variables as occurring at specific separate points in time. Sales data is often recorded at specific time intervals, such as daily, weekly, or monthly. Each recorded data point represents the sales that occurred within that specific time period. Similarly, repair data is typically logged when repairs are performed, which happens at specific instances or within specific intervals. Discrete time intervals are commonly used for tracking and analyzing sales and repair data because it allows for easier data management, trend analysis, and comparison between different time periods. It provides a structured framework to evaluate performance, identify patterns, and make business decisions based on the recorded data. In our analysis, specifying the time index to be discrete, beside being practical, simplifies the analysis and does not necessitate the restrictive assumption of nonhomogeneous Poisson sales process. In this model, the sales can be anything such as an auto-regressive time series, a random variable coming from any discrete distribution or it can be even deterministic.

In the first half of the chapter, we formulate the expected value and variance of the stochastic processes of installed base, discards, returned, and defective items by time t, and the remaining spare parts demand after time t for general purchase stochastic process. Note that since we assume that every failed and returned item will consume one spare part, we use the terms "returned and defective items" and "spare parts" interchangeably throughout the thesis. Being able to compute the expectation and variance, we can give a tight upper bound on the tail distribution of these stochastic processes by means of the two-tailed Chebyshev's inequality. This chapter in fact completes the work by Minner (2011) by computing the variance, and correlation functions of the related stochastic processes for general purchase stochastic process. Moreover, we have the flexibility of fitting a distribution by the first two moments if we have data about the average and variability of sales.

In the second half of this chapter, we formulate the pmf of the main stochastic processes in the thesis and after this, we work out for some special cases of the sales process these probability mass functions. We consider the cases that (i) the sales process is a deterministic process, (ii) the Sales process follows a Poisson distribution, (iii) the sales process follows a Geometric distribution, and (iv) the sales process follows a Discrete Gamma distribution. Last but not least, we provide an algorithm to compute the pmf of spare parts demand stochastic process which is computationally efficient, unlike the method proposed by Minner (2011).

An overview of our findings and suggestions for future research is presented in Chapter 6.

2. Literature Review

In this chapter we give a short overview of the different streams of literature in spare parts demand and explain the contribution of this thesis to this line of research.

2.1 Literature Review on Spare Parts Demand

Estimating the demand for spare parts is challenging due to its intermittent and most of the times erratic nature. A lot of papers appeared in the literature discussing this difficult problem. The research in this field can be divided into two categories: statistics-oriented approach and stochastic modeling-oriented approach. In the statistics-oriented approach, assuming a simple linear relation between future and past demand for spare parts, one uses time series and regression models to estimate the parameters in this linear relation. This line of research, followed by most papers in the literature, was initiated in Brown (1959) for fast non-trended non-seasonal demands. However, due to the intermittent character of spare part demand, the simple exponential smoothing technique, proposed by Brown, has a bias in estimating the mean and variance of the demand. Croston (Croston (1972)) observed this deficiency and proposed an adaptation of Brown's method. Following this line of research, revised versions of Croston's method were suggested in the literature to overcome bias factors still embedded in Croston's method (Syntetos & Boylan (2001); Snyder (2002); Levén & Segerstedt (2004); Teunter, Syntetos & Babai (2011); Bergman, Noble, McGarvey & Bradley (2017); Pennings, Van Dalen & van der Laan (2017); Dombi, Jónás & Tóth (2018); Zhu, van Jaarsveld & Dekker (2020); Jiang, Tam, Guo & Zhang (2020); Prestwich, Tarim & Rossi (2021); Sanguri, Patra & Punia (2023)). For a comprehensive overview on this line of research explaining the future demand for spare parts from past demand we refer the reader to the book of Boylan & Syntetos (2021).

Besides Croston's method and its adaptations proposed by other authors, some machine learning approaches have been adopted for intermittent demand forecasting. Bootstrapping (Zhou & Viswanathan (2011); Hasni, Aguir, Babai & Jemai (2019)), neural networks (Kourentzes (2013); Babai, Tsadiras & Papadopoulos (2020)) and support vector machines (Boukhtouta & Jentsch (2018); Kaya & Turkyilmaz (2018); Jiang, Huang & Liu (2021)) are examples of these methods.

Recently, the statistics-oriented approach also focused on incorporating in their statistical models the so-called *installed base information* of the product to predict future demand for spare parts. Installed base information includes among other factors the manufacturing rate of the product, the product and part life characteristics, the used replacement and maintenance policy, and the number of products in the market at any particular time.

One of the earliest papers suggesting incorporating installed based information into the statistical models for predicting spare parts demand is Dekker, Pinçe, Zuidwijk & Jalil (2013). Chou, Hsu & Lin (2015) use the installed base concept to forecast the final demand for automobile parts. They showed that a regression on the failure probability of components produces more accurate forecasts than a regression on historical part sales data. Kim et al. (2017) propose a regression model that requires only sales and returns as installed base information. Using real-life data, they show that their method generates rather more accurate estimates than a simple autoregressive model. Stormi, Laine, Suomala & Elomaa (2018) consider the size of the number of products in use as the only driver for demand but also observe in their case study the effect of the failure behavior of the parts. Van der Auweraer & Boute (2019) develop a method to forecast the demand for spare parts by relating it to the service maintenance policy. They forecast the distribution of the future spare parts demand during the upcoming lead time by tracking the active installed base and estimating the part failure behavior. Zhu et al. (2020) propose a model predicting the spare part demand based on the maintenance plan and benchmark this approach to the most up-to-date time series methods used in spare part demand. Van der Auweraer, Zhu & Boute (2021) analyze the different sources of installed base information on the quality of predictions for the future spare parts demand and find out that the number of products in the market at any time is the most valuable information to gather. For an overview of the literature on spare parts demand forecasting discussing the above statistical models and comparing the different approaches (including the installed base information adaptation), we refer to Van der Auweraer, Boute & Syntetos (2019) and Pinçe et al. (2021). Although including installed base information into the prediction for future spare part demand is sensible,these statistical models cannot capture in detail the innate alteration within the installed base or spare parts demand processes. The reason is that they often rely on historical data and patterns. If there are significant shifts in usage patterns, technological advancements, or other unforeseen factors, statistical models may not inherently account for these changes. This is where more advanced predictive modeling could potentially offer better insight.

In the stochastic modeling-oriented approach the different stochastic processes (related to the installed base information of a product) are described in detail and these stochastic processes can be linked in a natural way to the stochastic process of the total number of returned and defective items. In this research stream, one mimics the main causes of spare parts demand and applies a similar reasoning as used in the statistics-based stream using installed base information. In this line of research, the important dynamics of a product such as the life cycle pattern, the usage time of the product, the maintenance policy of the firm, and the failure rate of the product are used to model the spare parts demand process. A minority of papers follow this line of research and the findings in this thesis belong to this category.

One of the earliest papers following this line of research is Ritchie & Wilcox (1977). They state that the decay in the number of machines in the market is too slow to account for the decay in the demand for component replacements. Hence, he proposed a renewal theory forecasting method to be used at the end of the maturity Phase of the product life cycle for forecasting future demand assuming that the demand for spare parts declines eventually since customers decide with a given probability not to replace the broken part. Also, they restrict their analysis to the assumption of constant part failure rates. Hong, Koo, Lee & Ahn (2008) extend the work of Ritchie & Wilcox (1977) by modeling the installed base decline using a more general replacement probability and considering the case of a non-constant failure rate. However, their explicit formulas for point and interval estimation of spare parts demand encounter computational difficulties for general failure distributions. Jin & Liao (2009) consider sales as the installed base and assumes that the installed base grows according to a homogeneous Poisson process. Under the special case of exponential failure time, they were able to obtain closed-form solutions for the mean and variance of the aggregate maintenance demand. Minner (2011) determines in a recursive way the number of products in the market at the beginning of the future period. This is computed using the product sales in the present period and a simple discarding scheme of products in use in that same period. Similarly, Rodrigues & Yoneyama (2020) use remaining useful life predictions of components belonging to non-repairable items that are monitored periodically to forecast future demands of spare parts. Xie, Liao & Zhu (2014) models the sales of a new durable product using the bell-shaped Bass diffusion process and considers the sales and the failed but not reported items as the source of change in the installed base over time. Their model addresses the different issues of determining the gross profit of a product under a given warranty period. In their paper, it is assumed that repair demands for each product in the market are generated according to a nonhomogeneous Poisson failure process. Hekimoğlu & Karlı (2023) consider both growing as well as declining installed base scenarios and determine the mean and the variance of the stochastic process of returned and defective items assuming in both scenarios that sales and repairs are given by a homogeneous Poisson process.

2.2 The Contribution of this Thesis to the Literature

In this section, we position our thesis with respect to the closest studies in the literature given by Hong et al. (2008), Minner (2011), Xie et al. (2014), Van der Auweraer & Boute (2019) and Hekimoğlu & Karlı (2023).

The model proposed by Hong et al. (2008) assumes that the failure process of a product follows a renewal process, specifically referring to perfect repairs that restore the product to an as-good-as-new state. This assumption would mean that the failure process is a renewal process. However, it is important to note that replacing a failed part does not bring the product back to its brand-new condition but rather brings it to an as-bad-as-before state prior to maintenance. Hence, the repairs are considered to be minimal repairs in our work. Hong et al. (2008)'s proposed method is not only ineffective in addressing the essence of repair but also the computation of mean and variance of total spare parts demand using renewal-type arguments proved to be challenging. As a result, the authors approximated their model to facilitate computations to deal with non-constant first failure time distributions. Additionally, their model is applicable to products that are no longer in production while our model has the flexibility of being used even before the product is introduced to the market. Minner (2011)'s model although applicable during the whole life cycle of the product, also assumes the sales are known. Minner suggests a convolutional relation to compute one period ahead spare part demand probability which becomes complex and computationally heavy because of its recursive nature, and therefore hard to apply in practical settings. Our discrete-time model covers the weaknesses in Minner (2011) by suggesting an easy-to-solve method to compute the spare parts demand for a general sales process. Xie et al. (2014) do not consider the usage time effect on the installed base size and obtain the expected number of reported failures of sold units in the warranty period directly as the expectation of the number of sold items times the expected number of failures of each item. Van der Auweraer & Boute (2019) uses real data observations to generate an estimate of the distribution of the demand during the upcoming lead time. Their method uses a statistical approach to analyze historical part failures and continually updates its understanding as more data is gathered which improves its fit over time. Requiring continuous tracking of historical sales and discards and past part failures is a major challenge in their work since collecting accurate data is always a difficult task. Compared to their work, our model necessitates much less data. Hekimoğlu & Karlı (2023) model the growing and declining phases of the installed base using a separate homogeneous Poisson process for each phase and estimate the first three moments of spare parts demand for these phases by including the product death factor. Independent from the moment estimation, they also provide an algorithm to select a parametric distribution for each period's spare parts demand. In general, their model is not flexible in capturing the dynamics of the sales process, the installed base process, as well as the total failure process. These shortcomings are cleared up in our research.

The novelty of our work can be listed as follows:

- The product sales process over the whole life cycle of the product and the failure process of each product are represented by general point processes only knowing the mean arrival functions. Taking into account the random usage time of a product in the market as well, we present the discarded items process, the installed base process, and the returned and defective items stochastic process by general point processes. This implies that we unify the installed base stochastic process and the returned and defective items stochastic process over the entire life cycle of a product. As such our paper seems to be the first one unifying all these processes over the whole life cycle of a product.

- Special cases of these point processes include non-homogeneous and homogeneous Poisson processes. In our continuous time model (Chapters 3 and 4), we assume that the sales process follows a nonhomogeneous Poisson process. However, this limiting assumption is not required to derive closed-form solutions for our discrete time model (Chapter 5). In the discrete time model, the sales process can be deterministic or any other type of stochastic process. This approach is lacking in the literature.

- Using our continuous time model, under fairly general conditions on the cdf of the usage time and the mean arrival functions of sales and failures point processes, we easily compute by applying a bisection procedure the time at which the expected installed base and the increasing rate of the expected number of returned and defective items is maximal. Remember the time at which the expected number of items in the market is maximal is called the saturation time of the installed base stochastic process, while the time at which the increase in the expected number of returned and defective items is maximal is called the saturation time of the spare parts demand stochastic process. As such the analytical characterization of these saturation times seems to be new in the literature. As expected, it follows that the difference between these saturation times is determined by the expected usage time of a product in the market. These results give an indication to the manufacturer at which time after the maturity phase of the product the manufacturer can expect that his market share will decline and at which time he is most busy with aftersales service.

- We can calculate the distribution of spare parts demand within different time intervals $[t_1, t_2]$ or from any time point t until the time the product's life cycle ends. This enables us to determine the necessary safety stock required to prevent stockouts, specifying the lead time duration and a planned service level.

- Next to the expected value, we also compute the variance and the correlation functions of the related stochastic processes which is new in the literature. Using this information we can give a conservative upper bound on the tail distribution of the introduced stochastic processes using Chebyshev's inequality.

- In our proposed discrete time model, we assume the sales process is a versatile stochastic phenomenon. This means that it is not limited to being deterministic but can rather exhibit various distributions, such as a Poisson process, or follow a specific distribution. We explored some of the famous discrete distributions in Chapter 5 showcasing the novelty and breadth of our approach.

- Overall, our model is both comprehensive and computationally efficient. It excels in terms of speed, taking less than a minute to compute the distribution of spare parts demand across various parametric settings. Furthermore, our model stands out from previous literature by requiring the least amount of data.

3. Spare Parts Demand and the Installed Base Concept: A Theoretical Approach

In the first chapter of this thesis we introduce the main continuous time stochastic processes useful in spare parts management and report the results derived for these stochastic processes published in Amniattalab et al. (2023a).

3.1 Introduction

Aftersales services of industrial products constitute an essential business component for manufacturers of durable and capital products. Studies report that aftersales services contribute to the total revenue of manufacturers up to 25% and this contribution could be as high as 40% in terms of profitability (Auramo & Ala-Risku, 2005; Cohen, Agrawal & Agrawal, 2006; Holmström, Cheikhrouhou, Farine & Främling, 2011). Business activities of aftersales services include the supply of spare parts and complementary resources, training of responsible personnel for aftersales services, coordination of secondary markets of products, design and implementation of efficient product recovery processes, such as take-back programs or installation of remanufacturing facilities (Durugbo, 2020).

For each product, these aftersales business activities take place in different amounts. For instance, aircraft need a significant amount of periodic and corrective maintenance by well-trained personnel whereas smartphones have more vibrant aftersales markets. In general, volumes of aftersales operations are driven by three fundamental features of a product that evolves over its life cycle:

- the number of products used by customers, *installed base*;
- the number of returned products due to failure, *demand for spare parts and maintenance*;
- the rate of *discarded products*.

The installed base of durable goods refers to the number of products in use at any point in time. The size of a product's installed base is an important statistic for a manufacturer as it defines the potential volume of manufacturers' aftersales business. Installed base is mainly driven by the sales intensity of a product and the length of a product's economic lifetime.

Each product in an installed base fails occasionally and those products are returned to service centers for maintenance and replacement of their failed components. Therefore, the total amount of returned products is mainly driven by the size of installed base and the failure intensity of that product. This statistic defines the rate of revenue from repair and maintenance services as well as spare parts sales for Original Equipment Manufacturers (OEMs). In addition, OEMs need to design their service network and train their maintenance personnel according to the rate of returned products as shortages in service capacity and potentially-related quality problems can damage their reputation in the market.

Once a product reaches the end of its economic life, it is replaced by a newer model. In many industries, the collection, recovery and remanufacturing of old products is an important business activity for OEMs. In fact, producers of some products, such as household appliances and automobiles, are obligated to collect their used products according to national regulations, such as Extended Producer Responsibility (Atasu & Subramanian, 2012). Therefore, for OEMs it is important to know the rate of discarded products to allocate the necessary amount of resources to these operations. Also, in some industries, discarded products are a viable source of components, which are utilized for the manufacturing of new products or maintenance of the ones in installed base. The rate of discarded products is mainly driven by the sales intensity and the economic lifetime of a product.

In summary, the three fundamental features of durable (consumer and capital) products, which are the size of installed base, the amount of returned products, and discarded products, are driven by three main product characteristics: sales intensity, the economic lifetime, and failure intensity. As revenue streams from aftersales operations depend on the volumes of the three features, OEMs are interested in future projections for different components of aftersales businesses during the design stages of industrial products. The relation between product characteristics, life cycle features, and aftersales business volumes are depicted in Figure 3.1.

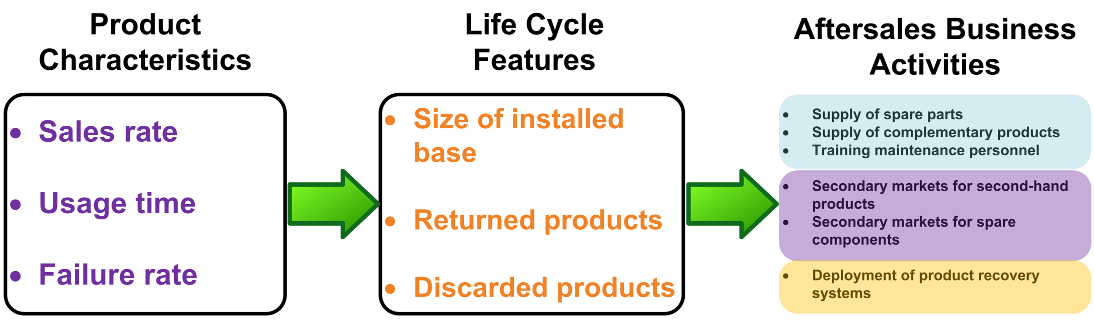

Figure 3.1 Relation between product characteristics and aftersales business activities

The evolution of different life cycle features of a product is usually analyzed by dividing the life cycle into three distinct phases: initial (introduction and growth), maturity, and decline. In each phase, customer demand and after-sales services have different characteristic features. In the initial phase, the sales of the product grow rapidly. Over time the installed base grows and towards the end of the initial phase, the sales intensity decreases and after-sales services and spare parts supply chains become active. Initial phase is followed by the maturity phase in which the size of installed base stays approximately stable. In this phase, customer satisfaction from after-sales services for products becomes an important driver for companies' profitability and financial sustainability in the market. At the end of the maturity phase, sales intensity vanishes and customers start discarding their products. This leads the installed base to decline which reduces the total demand for after-sales services. The evolution of the three fundamental features over different life cycle stages is given in Figure 3.2.

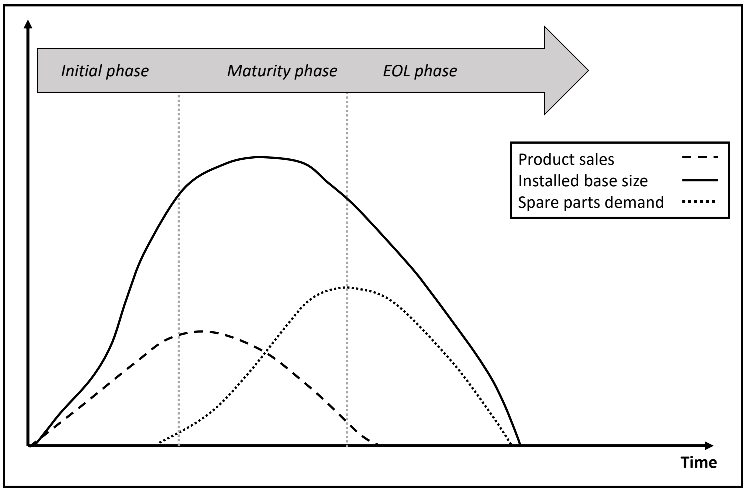

Figure 3.2 Sketch of product sales, installed base size, and spare parts demand over time (Driven from Kim et al. (2017))

In this chapter, we provide a structured stochastic model that can project the size of the expected installed base and the expected number of returned and discarded products over the life cycle of a product within a single modeling framework. Our methodology is particularly suitable for OEMs' products that are at the design phase as it utilizes appropriate stochastic processes instead of empirical observations for making projections for aftersales service demand.

Specifically, we assume that both the sales process of this particular product and the failure process of each individual product in the installed base are given by a point process with a given mean arrival function. Also, each customer uses the product during a random usage time. Based on these processes we introduce the installed base stochastic counting process, the discarded items stochastic counting process, and the stochastic counting process of returned and defective items. We characterize these processes by computing the expectation of all processes at any point in time. In addition, we assume for simplicity that every returned and defective product requires a single spare part. Hence the total number of returned and defective products and the total spare parts demand up to any time are equal to each other. Throughout this chapter, we use these two terms interchangeably.

By using some natural assumptions on the cdf of the random usage time and the mean arrival functions of a product's sales process and failure process, we derive properties of these expectations as a function of time. These properties enable us to propose a bisection procedure to compute the point at which the maximum of the expected installed base process is achieved and the point at which the maximal rate of increase of the expectation of the cumulative returned and defective items process is achieved. We refer to these points as the saturation times. The saturation time of the returned and defective items is especially important since it represents the time that a manufacturer is the busiest with aftersales services. Finally, in the computational section, we present different scenarios characterized by different parameters in which the sales process is given by the well-known

Brockhoff (Brockhoff, 1967) sales model, customers have Weibull distributed usage times and each product in use follows a Weibull minimal repair process. In particular, we show the influence of the expected usage time on the saturation times of the expectation of the installed base process and the rate of increase of the expected cumulative number of returned and defective items. To the best of our knowledge, ours is the first study that unifies all life cycle phases of a product in a continuous-time stochastic process model, which requires very limited historical data. In fact, our model can easily be utilized with time to failure and useful lifetime distributions that can be obtained from engineering studies at the design stage of a product. Also, the computation of the saturation times of installed base, and the number of returned and defective products have never been addressed before.

3.2 On the Installed Base and Related Stochastic Processes

Life cycle features of both durable and capital products are mainly driven by the three characteristics of products, which are sales rate, usage time and failure rate. In this section, we develop a structural model based on these characteristics. In Section 3.2.1 we introduce the total sales process and the stochastic processes that represent the total number of products in the market at any time (installed base stochastic process), the total number of discarded products at any time (discarded items stochastic process) and the total number of returned and defective products at any time (defective items stochastic process). In Section 3.2.2 we derive the expectation of the installed base and discarded items stochastic process at any time and determine under general assumptions on the sales process and the usage time distribution the global behaviour of these functions. At the same time we show how to compute the maximum of the expected installed base and the point in time at which the increase in the expected total discarded items function is maximal. Finally in Section 3.2.3) we derive the expectation of the defective items stochastic process at any time, verify its global properties and determine the point in time at which its increase is maximal. Computing these points in time yields a prediction at which time during the life cycle of the product the expected installed base and the supply of refurbished spare parts is maximal. For the returned and defective items process it indicates at which time during the life cycle the manufacturer is most busy with after sales service.

3.2.1 Fundamental Definitions

To model the sales process of items over the lifetime of a product let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space hosting the point process $(\mathbf{T}_n)_{n\in\mathbb{N}}$ with the positive random variables $\mathbf{T}_n$ denoting the random sales time of the nth item of this product. Hence the total number of sales up to any given time is given by the one-dimensional stochastic counting process $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ with $\mathbf{S}(t)$ denoting the number of sales up to time t given by

$$\mathbf{S}(t) = \sum_{n=1}^{\infty} 1_{\{\mathbf{T}_n \leq t\}}. \tag{3.1}$$

It is assumed that the point process $(\mathbf{T}_n)_{n\in\mathbb{N}}$ has a finite intensity measure Ψ (Brémaud (1981)) given by $\Psi(A) := \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{\{\mathbf{T}_n \in A\}}\right)$ for any Borel set $A \subseteq \mathbb{R}_+$. This measure is uniquely determined by the so-called (right continuous) mean arrival function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Psi(t) := \mathbb{E}\left(\sum_{n=1}^{\infty} 1_{\{\mathbf{T}_n \leq t\}}\right). \tag{3.2}$$

Since the mean arrival function determines the expected total number of sales of some product up to any given time and during its availability on the market only a finite number of products will be sold it is natural to assume that additionally $\Psi(\infty)$ is finite and $\Psi(0) = 0$. In this paper we also assume that the mean arrival function Ψ has a continuous positive derivative ψ and this derivative is called in the marketing literature the product life cycle function (De Kluyver (1977), Rink & Swan (1979)). The maturity time of the product is given by the maximum of the product life cycle function ψ and this represents the time at which most items are sold in the market. Since for most products the product lifetime function is first increasing and then after its maturity time is decreasing it is natural to assume that the function Ψ extended to $\mathbb{R}$ by taking $\Psi[t) = 0$ for every $t < 0$ is unimodal on $\mathbb{R}$ (see Definition 1) with the positive unimodality point given by the maturity time. A well-known example of a product life cycle function is given by $\psi(t) = at^{-b}e^{-ct}, t > 0$ with unknown parameters $a, b, c > 0$ (Brockhoff (1967)). Another often-used representation of a mean arrival function of the sales process is given by the Bass diffusion model (Bass (2004b), Bass (2004a)).

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called unimodal on $\mathbb{R}$ if there exists some point $t(f) \in \mathbb{R}$ such that f is convex on $(-\infty, t(f))$ and concave on $(t(f), \infty)$. The point $t(f)$ is called the unimodality point of the function f.*

After item n is sold at time $\mathbf{T}_n$ it is used for a positive random time $\mathbf{U}_n$ before being discarded. We assume in this paper that the random so-called usage times $\mathbf{U}_n, n \in \mathbb{N}$, are independent and identically distributed with continuous cdf $F_{\mathbf{U}}$ satisfying $F_{\mathbf{U}}(0) = 0$ and having a finite expectation. Since the items during their usage time might fail we need spare parts for repair. One source of available spare parts is coming from discarded items and so we are interested in the total number of discarded items up to any time. This is represented by the so-called discarded items stochastic counting process $\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$ given by

$$\mathbf{D}(t) = \sum_{n=1}^{\infty} 1_{\{\mathbf{U}_n \leq t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}. \tag{3.3}$$

The random variable $\mathbf{D}(t)$ denotes the total number of discarded items in the market up to time t. Since the number of spare parts needed at a certain time depends clearly on the number of items available in the market at that time we also introduce the so-called installed base stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ given by

$$\mathbf{B}(t) = \sum_{n=1}^{\infty} 1_{\{\mathbf{U}_n > t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}. \tag{3.4}$$

The random variable $\mathbf{B}(t)$ denotes the random amount of items in the market at time t.

Lastly, we consider the stochastic process representing the number of returned and defective items up to any time. After selling item n at time $\mathbf{T}_n$ it generates during its usage time $\mathbf{U}_n$ failures and these failures need to be repaired using spare parts. Therefore on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we introduce for each sold item n during its usage time $\mathbf{U}_n$ the failure counting process $\mathbf{N}_n = \{\mathbf{N}_n(t) : t \geq 0\}$. Since the sold items are identical we assume that the random vectors $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$ consisting of the failure counting process of item n and its usage time in the market are independent and identically distributed. The mean arrival function of the i.i.d failure processes $\mathbf{N}_n$ is given by Φ satisfying $\mathbb{E}(\Phi(\mathbf{U}_n))$ is finite for every $n \in \mathbb{N}$. This last condition is natural since it means that the expected number of failures of any item during its usage time in the market is finite. Having introduced the counting failure process we consider the stochastic counting process $\mathbf{R} = \{\mathbf{R}(t) : t \geq 0\}$ given by

$$\mathbf{R}(t) := \textstyle\sum_{n=1}^{\infty} \mathbf{N}_n((t - \mathbf{T}_n) \wedge \mathbf{U}_n) \mathbf{1}_{\{\mathbf{T}_n \leq t\}}. \tag{3.5}$$

with $a \wedge b := \min\{a, b\}$ for any $a, b > 0$. The random variable $\mathbf{R}(t)$ represents the number of returned and defective items up to time t. Unfortunately this stochastic process does not have independent increments. In most of the literature, it is assumed that this process is exogenous and has independent increments. In the next subsection, we will derive the expectations of the installed base and discarded items stochastic process

3.2.2 Expectations of the Installed Base and Discarded Items Stochastic Process

Introducing the function $d : (0, \infty) \to (0, \infty)$ given by $d(t) := \mathbb{E}(\mathbf{D}(t))$ it is easy to verify using relation (3.3) and the monotone convergence theorem that

$$d(t) = \textstyle\sum_{n=1}^{\infty} \mathbb{E}(1_{\{\mathbf{U}_n \leq t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}) = \displaystyle\int_0^t F_{\mathbf{U}}(t - y))\psi(y)dy = \int_0^t F_{\mathbf{U}}(y)\psi(t - y)dy. \tag{3.6}$$

Also by relation (3.4) introducing the function $b : (0, \infty) \to (0, \infty)$ given by $b(t) = \mathbb{E}(\mathbf{B}(t))$ we obtain applying the same argument that

$$b(t) = \textstyle\sum_{n=1}^{\infty} \mathbb{E}(1_{\{\mathbf{U}_n > t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}) = \displaystyle\int_0^t (1 - F_{\mathbf{U}}(t - y))\psi(y)dy = \Psi(t) - d(t). \tag{3.7}$$

Next to the maturity time of a product we are interested in the time at which the expected number of items of that product in the market are maximal as well as the time at which the increase in the expected total number of discarded items is maximal. This last time represents the time that the supply of spare parts coming from discarded items is the highest. To solve these optimization problems we need to investigate the global properties of the functions d and b. To relate the properties of the function d and b to properties of convolutions of cdfs known in probability theory we rewrite the functions b and d for mathematical convenience as a convolution of cdfs modulo some constant. An easy way to do this is to introduce the cdf $F_\Psi : \mathbb{R} \to [0,1]$ given by

$$F_\Psi(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{\Psi(t)}{\Psi(\infty)} & \text{if } t \geq 0. \end{cases} \tag{3.8}$$

If the nonnegative random variable $\mathbf{X}$ has cdf F_Ψ and this random variable is independent of the nonnegative random variable $\mathbf{U}$ a more convenient representation of the function d is given by

$$d(t) = \Psi(\infty)\mathbb{P}(\mathbf{X}+\mathbf{U} \leq t), t > 0. \tag{3.9}$$

This shows the function d is increasing on $(0,\infty)$, $d(0) = 0$ and $d(\infty) = \Psi(\infty)$. Also by the continuity of $F_\mathbf{U}$ the increasing positive function d is continuous on $(0,\infty)$. If the cdf $F_\mathbf{U}$ has a continuous density $f_\mathbf{U}$ then the derivative of d exists and since $F_\mathbf{U}(0) = 0$ we obtain by relation (3.6) for every $t > 0$

$$d'(t) = \int_0^t f_\mathbf{U}(t-y)\psi(y)dy \tag{3.10}$$

and $d'(0) = \lim_{t\downarrow 0} d'(t) = 0$. Clearly for $f_\mathbf{U}$ and ψ continuous the derivative function $d' : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and for ψ positive on $(0,\infty)$ and $f_\mathbf{U}$ positive on $\{t > 0 : F_\mathbf{U}(t) < 1\}$ it follows that $d'(t) > 0$ for every $t > 0$. To show some properties of the function b by verifying that the function b can be seen as the derivative function of a convolution we need to rewrite the expression in relation (3.7). Since the expected usage time $\mathbb{E}(\mathbf{U})$ of a product is finite we introduce the so-called equilibrium distribution of the cdf $F_\mathbf{U}$ given by

$$F_{\mathbf{U}_e}(t) := \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\mathbb{E}(\mathbf{U})}\int_0^t (1-F_\mathbf{U}(z))dz & \text{if } t \geq 0 \end{cases} \tag{3.11}$$

and for the nonnegative random variable $\mathbf{U}_e \sim F_{\mathbf{U}_e}$ independent of the random variable $\mathbf{X} \sim F_\Psi$ we introduce the convolution

$$B(t) := \mathbb{P}(\mathbf{U}_e + \mathbf{X} \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\Psi(\infty)}\int_0^t F_{\mathbf{U}_e}(t-y)\psi(y)dy & \text{if } t \geq 0. \end{cases} \tag{3.12}$$

Clearly the cdf $F_{\mathbf{U}_e}$ is unimodal with unimodality point 0. It is easy to verify that the density $B'(t)$ of the cdf B exists for every $t > 0$ and this density is given by

$$B'(t) = \frac{1}{\Psi(\infty)\mathbb{E}(\mathbf{U})}\int_0^t (1-F_\mathbf{U}(t-y))\psi(y)dy = \frac{b(t)}{\Psi(\infty)\mathbb{E}(\mathbf{U})}, t > 0. \tag{3.13}$$

If we introduce $t_* := \sup\{t > 0 : F_\mathbf{U}(t) < 1\}$ and t_* is finite we obtain for every $t > t_*$ that

$$b(t) = \int_{t-t_*}^t (1-F_\mathbf{U}(t-y))\psi(y)dy.$$

Since $1-F_{\mathbf{U}}(y)>0$ for every $y<t_*$ and ψ is positive on $(0,\infty)$ it follows both for t_* finite or $t_*=\infty$ that $b(t)>0$ for every $t>0$. Since the function b is continuous and positive on $(0,\infty)$ and $b(0)=b(\infty)=0$ we obtain by a standard application of Weierstrass theorem (Rudin (1976)) that the function b attains a maximum on $(0,\infty)$. Using the above equivalent representations of the functions b and d we derive in the next subsection under some general conditions on the life cycle function ψ and the cdf $F_{\mathbf{U}}$ some useful properties of the functions b and d.

In many applications of operations management, mathematical properties of installed base processes are crucial for characterizing the dynamics of performance indicators of a system. In our modeling framework, one can now show under certain conditions on the function ψ and the cdf $F_{\mathbf{U}}$ some properties of the functions b and d useful in optimization. Before proving these properties we list the following definition. Observe, if we use a cdf F on $(0,\infty)$, we extend this cdf to $\mathbb{R}$ by defining $F(t)=0$ for every $t<0$.

Definition 2. *Let $\mathbf{Y}$ be a random variable on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ having cdf $F_{\mathbf{Y}}$. The cdf $F_{\mathbf{Y}}$ is called strongly unimodal if for every random variable $\mathbf{Z}$ on $(\Omega,\mathcal{F},\mathbb{P})$ having a unimodal cdf $F_{\mathbf{Z}}$ and the random variable $\mathbf{Z}$ is independent of the random variable $\mathbf{Y}$, the cdf of the random variable $\mathbf{Y}+\mathbf{Z}$ is unimodal.*

It is easy to verify that the cdf of the random variable $\mathbf{Y}+\mathbf{Z}$ is strongly unimodal if the random variables $\mathbf{Y}$ and $\mathbf{Z}$ are independent and both $\mathbf{Y}$ and $\mathbf{Z}$ have strongly unimodal cdfs. For strongly unimodal cdfs the following characterization is proved in Ibragimov (1956). For completeness, we mention this result. (see also page 65 of Keilson (1979)). Observe a nonnegative function $f:\mathbb{R}\to\mathbb{R}$ is called logconcave if the set $D_f:=\{x\in\mathbb{R}:f(x)>0\}$ is convex and the logarithm of the function f is concave on D_f.

Theorem 3. *Let $\mathbf{Y}$ be a randon variable on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ having cdf $F_{\mathbf{Y}}$ and density $f_{\mathbf{Y}}$ and the set $D_{f_{\mathbf{Y}}}:=\{y\in\mathbb{R}:f_{\mathbf{Y}}(y)>0\}$ is a nonempty convex set. The cdf $F_{\mathbf{Y}}$ is strongly unimodal if and only if the density $f_{\mathbf{Y}}$ is logconcave on the set $D_{f_{\mathbf{Y}}}$.*

Before mentioning the next result we introduce for any nonnegative random variable $\mathbf{Y}$ having a cdf $F_{\mathbf{Y}}$ and a continuous density $f_{\mathbf{Y}}$ on the convex set $D_{F_{\mathbf{Y}}}:=\{y>0:F_{\mathbf{Y}}(y)<1\}$ the failure rate function $r_{\mathbf{Y}}:D_{F_{\mathbf{Y}}}\to\mathbb{R}_+$ given by

$$r_{\mathbf{Y}}(y):=\frac{f_{\mathbf{Y}}(y)}{1-F_{\mathbf{Y}}(y)}.$$

It is well known that the failure rate function uniquely determines the cdf and we obtain for $F_{\mathbf{Y}}(0)=0$ that

$$1-F_{\mathbf{Y}}(y)=e^{-\int_0^y r_{\mathbf{Y}}(s)ds}, \tag{3.14}$$

for every $y\in D_{F_{\mathbf{Y}}}$. This shows that the failure rate function $r_{\mathbf{Y}}$ is increasing on $D_{F_{\mathbf{Y}}}$ if and only if the tail function $1-F_{\mathbf{Y}}$ is logconcave on $D_{F_{\mathbf{Y}}}$. Using this result and Theorem 3 one can easily verify the next properties for the expected discarded items up to time t and the expected number of items in the market at time t.

Lemma 1. *If the mean arrival function Ψ of the sales process has a positive and continuous derivative*

ψ on $(0,\infty)$ then the next results hold.

1.1 *If the positive function ψ is logconcave on $(0,\infty)$ and the cdf $F_{\mathbf{U}}$ is unimodal having a density $f_{\mathbf{U}}$ then the function d is unimodal.*

1.2 *If the positive function ψ is logconcave on $(0,\infty)$ and the density $f_{\mathbf{U}}$ is positive and logconcave on the convex set $\{t>0: F_{\mathbf{U}}(t)<1\}$ then the derivative function d' is logconcave on $\{t>0: F_{\mathbf{U}}(t)<1\}$.*

1.3 *If the positive function ψ is logconcave on $(0,\infty)$ then the function b is increasing on $(0,t(b))$ and decreasing on $(t(b),\infty)$ for some $t(b)>0$ given in relation 3.15.*

1.4 *If the positive function ψ is logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has an increasing failure rate on $\{t\geq 0: F_{\mathbf{U}}(t)<1\}$ then the function b is logconcave on $(0,\infty)$.*

Proof. Clearly by Theorem 3 the cdf F_{Ψ} in relation (3.8) is strongly unimodal in all the considered parts. Hence by relation (3.9) the result follows in part 1. To verify part 2 we observe by Theorem 3 that also $F_{\mathbf{U}}$ is strongly unimodal and so by relation (3.9) and applying again Theorem 3 the result in part 2 follows. Part 3 follows due to F_{Ψ} being strongly unimodal and relations (3.12) and (3.13). Finally, in part 4 we obtain by relation (3.14) that the function $1-F_{\mathbf{U}}$ is logconcave on $D_{F_{\mathbf{Y}}}$ and since the density $f_{\mathbf{U}_e}$ of the cdf $F_{\mathbf{U}_e}$ is given by

$$f_{\mathbf{U}_e}(t)=\mathbb{E}(\mathbf{U}_1)^{-1}(1-F_{\mathbf{U}}(t)),$$

and hence $D_{f_{\mathbf{U}_e}}=\{t>0: F_{\mathbf{U}}(t)<1\}$, we obtain by Theorem 3 that both cdfs F_{Ψ} and $F_{\mathbf{U}_e}$ are strongly unimodal. This shows by relation (3.13) and Theorem 3 the desired result. □

For the Brockhoff sales model (Brockhoff (1967)) the function ψ is logconcave on $(0,\infty)$. Also it is well known for $\mathbf{U}$ Weibull distributed with shape parameter $\gamma\geq 1$ or gamma distributed with shape parameter $\alpha\geq 1$ that $\mathbf{U}$ has an increasing failure rate function on $(0,\infty)$ and the cdf $F_{\mathbf{U}}$ is unimodal. These two classes of cdfs are useful for modelling the cdf of the random usage time of a product since for both classes we have a unique two moment fit (Tijms (1994)). In the next subsection we show that the properties for the function b and d shown in Lemma 1 are very useful in constructing an easy algorithm to find the saturation points of the installed base and discarded items stochastic process.

As stated above, OEMs seek to have future projections for the evolution of the installed base of a product over its life cycle to design a proper aftersales service network with attainable cost. To this end, we define the *saturation point* of an installed base as the point at which the function b reaches its maximum. This means the saturation point of the installed base is defined as

$$t(b):=\arg\max_{0<t<\infty}\{b(t)\}. \tag{3.15}$$

and to compute it we need to determine the derivative b' of the function b.

Under some standard regularity conditions the derivative of b on $(0,\infty)$ exists and using relation

(3.7) this derivative is given by

$$b'(t) = \psi(t) - \int_0^t f_{\mathbf{U}}(t-y)\psi(y)dy, t > 0. \tag{3.16}$$

Also $b'(0) := \lim_{t \downarrow 0} b'(t) = \psi(0)$. Using (3.16) and the result is Lemma 1, one can show the following result for the optimization problem $\max_{0<t<\infty} b(t)$ determining the saturation point of the installed base stochastic process. This result shows we can apply an easy bisection procedure to compute this saturation point.

Lemma 2. *If the positive function ψ is logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has cdf $F_{\mathbf{U}}$ satisfying $F_{\mathbf{U}}(0) = 0$ and it has an increasing failure rate on $\{t \geq 0 : F_{\mathbf{U}}(t) < 1\}$ then the point $t(b)$ is an optimal solution if and only if $b'(t(b)) = 0$. Moreover, it follows for any optimal solution $t(b)$ that $b'(t) \leq 0$ for every $t > t(b)$ and $b'(t) \geq 0$ for $0 < t < t(b)$.*

Proof. We already know that the maximum of b is attained in $(0,\infty)$. Clearly the optimization problem $\sup_{0<t<\infty} b(t)$ has the same set of optimal solutions as $\sup_{0<t<\infty} \ln(b(t))$. By part 4 of Lemma 1 the function b is logconcave on $(0,\infty)$ and so the first order conditions for a positive maximum are necessary and sufficient. This means $t(b)$ is an optimal solution of $\sup_{0<t<\infty} \ln(b(t))$ if and only if $\frac{b'(t(b))}{b(t(b))} = 0$. Using $b(t) > 0$ for every $t > 0$ this is the same as $b'(t(b)) = 0$. Applying this equivalence result and the function b is logconcave on $(0,\infty)$ yields the last part and we have completed the proof. $\square$

By the above result the time that the expected installed base is maximal can be easily determined applying a standard bisection procedure on the derivative b' checking in each point the sign of this derivative. Looking at the expected number of discarded items up to time t, we are also interested in the time point at which the rate of increase of d is maximal, which we define as the *saturation point* of the discarded product rate and use the following notation for its mathematical representation:

$$t(d') := \arg\max_{0<t<\infty}\{d'(t)\}. \tag{3.17}$$

The saturation point of discarded products approximately coincides with the time that the most discarded items are available to be used as spare parts. By a similar reasoning as done for the function b it follows under the conditions of part 2 of Lemma 1 that the derivative d' attains a maximum on $(0,\infty)$. For this it follows in case ψ is differentiable with continuous derivative ψ' on $(0,\infty)$ that the second derivative of d exists and using relation (3.10) this second derivative d'' is given by

$$d''(t) = \int_0^t f_{\mathbf{U}}(y)\psi'(t-y)dy + \psi(0) f_{\mathbf{U}}(t), t > 0.$$

Using the same proof as in Lemma 2 one can show the following result.

Lemma 3. *If the positive function ψ is logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has a cdf $F_{\mathbf{U}}$ satisfying $F_{\mathbf{U}}(0) = 0$ and this cdf is continuously differentiable on the convex set $\{t > 0 : F_{\mathbf{U}}(t) < 1\}$ and its density $f_{\mathbf{U}}$ is positive and logconcave on the same convex set then $t(d')$ is an optimal solution* *if and only if* $d''(t(d'))=0$. *Moreover, it follows for any optimal solution* $t(d'')$ *that* $d'(t)\leq 0$ *for every* $t>t(d')$ *and* $d''(t)\geq 0$ *for* $0<t<t(d')$.

Again we can determine $t(d')$ by a standard bisection procedure applied to d'' checking in each point the sign of this second derivative. In general, it seems to be difficult to determine the variance or cdf of the random variable $\mathbf{B}(t)$ and $\mathbf{D}(t)$ for every $t>0$ unless the cumulative sales process $\mathbf{S}$ is given by a nonhomogenous Poisson process with intensity function ψ. For nonhomogeneous Poisson sales processes it is known (Çınlar (2013), Çınlar (2011)) that for every $t>0$ the random variables $\mathbf{B}(t)$ and $\mathbf{D}(t)$ are independent and Poisson distributed with mean $b(t)$, respectively $d(t)$. In the next section, we introduce the cumulative stochastic process of returned and defective items and analyze its expectation as a function of time. Clearly, this stochastic process evaluated at a given time is related to the total demand for spare parts up to that time.

3.2.3 Expectations of the Returned and Defective Items Stochastic Counting Process

Again it is easy to verify introducing the function $r:(0,\infty)\to(0,\infty)$ given by $r(t):=\mathbb{E}(\mathbf{R}(t))$ that the expected number of returned and defective items up to time t is given by

$$r(t)=\sum_{n=1}^{\infty}\mathbb{E}(\mathbf{N}_n((t-\mathbf{T}_n)\wedge\mathbf{U}_n)\mathbf{1}_{\{\mathbf{T}_n\leq t\}})=\int_0^t\mathbb{E}(\mathbf{N}_1((t-y)\wedge\mathbf{U}_1)\psi(y)dy,t>0, \tag{3.18}$$

with $a\wedge b:=\min\{a,b\}$. Due to limited space, this process is analyzed in much more detail in Amniattalab, Frenk & Hekimoğlu (2023b). In particular, in Amniattalab et al. (2023b) we derive under the condition that the sales process is a nonhomogeneous Poisson process the cdf of the random variable $\mathbf{R}(t)$ and $\mathbf{R}(t+\Delta)-\mathbf{R}(t)$ for any $t>0$ and $\Delta>0$. To simplify the above expression we assume for simplicity that the usage time $\mathbf{U}_n$ is independent of the failure process $\mathbf{N}_n$ and under this assumption it follows by conditioning on $\mathbf{U}_1$ in the expression $\mathbb{E}(\mathbf{N}_1((t-y)\wedge\mathbf{U}_1)$ that for every $t>0$

$$r(t)=\int_0^t\mathbb{E}(\Phi((t-y)\wedge\mathbf{U}_1))\psi(y)dy=\int_0^t\mathbb{E}(\Phi(y\wedge\mathbf{U}_1))\psi(t-y)dy.$$

with Φ the mean arrival function of the point failure process of each item in use. As for the function d we are interested in the point in time that the increase in the function r is maximal and this represents roughly the point in time that the manufacturer needs the most spare parts to repair these defective items. To derive some properties of this function r useful in determining this point in time we again rewrite the function r as a convolution modulo some constant. As before we introduce the cdf $F_\Phi:\mathbb{R}_+\to[0,1]$ given by

$$F_\Phi(t)=\frac{\mathbb{E}(\Phi(t\wedge\mathbf{U}_1)}{\mathbb{E}(\Phi(\mathbf{U}_1))},t>0 \tag{3.19}$$

with $F_\Phi(0) = 0$. It now follows using relation (3.18) that an equivalent representation of r is given by

$$r(t) = \mathbb{E}(\Phi(\mathbf{U}_1))\Psi(\infty)\mathbb{P}(\widetilde{\mathbf{U}} + \mathbf{X} \leq t), t > 0 \tag{3.20}$$

with the random variables $\widetilde{\mathbf{U}}$ and $\mathbf{X}$ independent and $\widetilde{\mathbf{U}} \sim F_\Phi$ and $\mathbf{X} \sim F_\Psi$. Clearly the function r has the same structure as the function d. It is now easy to verify for Φ having a continuous derivative φ and $F_{\mathbf{U}}$ denoting the cdf of the random variable $\mathbf{U} \stackrel{d}{=} \mathbf{U}_1$ having a continuous density $f_{\mathbf{U}}$ that the density of the cdf F_Ψ is given by

$$f_{\widetilde{\mathbf{U}}}(t) = \varphi(t)(1 - F_{\mathbf{U}}(t))\mathbb{E}(\Phi(\mathbf{U}))^{-1}.$$

This shows as in relation (3.10) that the derivative r' of the function r exists and for every $t > 0$

$$\begin{aligned} r'(t) &= \textstyle\int_0^t \varphi(t-y)(1 - F_{\mathbf{U}}(t-y))\psi(y)dy \\ &= \textstyle\int_0^t \varphi(y)(1 - F_{\mathbf{U}}(y))\psi(t-y)dy \end{aligned} \tag{3.21}$$

and $r'(0) = 0$. If the continuous functions φ and ψ are positive on $(0, \infty)$ it follows by relation (3.21) that the function r' is positive on $(0, \infty)$. Also, it is easy to verify for ψ continuously differentiable that

$$r''(t) = \int_0^t \varphi(y)(1 - F_{\mathbf{U}}(y))\psi'(t-y)dy + \psi(0)\varphi(t)(1 - F_{\mathbf{U}}(t)), t > 0, \tag{3.22}$$

If the failure processes $\mathbf{N}_n, n \in \mathbb{N}$ are given by a homogeneous Poisson process with arrival rate $\delta > 0$ we obtain $\Phi(t) = \delta t$. This yields $\varphi(t) = \delta$ for every $t > 0$ and by relation (3.21) and (3.6) it follows

$$r'(t) = \delta b(t). \tag{3.23}$$

Hence for a Poisson failure process the time at which the rate of increase of the expected number of returned and defective items up to a given time is maximal occurs at the same time that the expected number of items in the market is maximal. However, the assumption of constant failure rates might not be realistic in practice since in a lot of cases the failure point process seems to have an increasing failure rate. If the failure process is given by a minimal repair process (Block, Borges & Savits (1985)) the failure process $\mathbf{N}_n$ of each sold item is a nonhomogenous Poisson process with the intensity function given by the failure rate function of the cdf $F_{\mathbf{X}_1}$ (having a density $f_{\mathbf{X}_1}$!) with $\mathbf{X}_1$ denoting the time of the first failure of the product in use. This shows for every $t > 0$ that

$$r'(t) = \int_0^t f_{\mathbf{X}_1}(y)\frac{(1 - F_{\mathbf{U}}(y))}{(1 - F_{\mathbf{X}_1}(y))}\psi(t-y)dy. \tag{3.24}$$

Applying relation (3.16) this implies if the random variable $\mathbf{X}_1$ has the same cdf as the random usage time $\mathbf{U}$ that for every $t > 0$

$$r'(t) = \int_0^t f_{\mathbf{U}_1}(y)\psi(t-y)dy = \psi(t) - b'(t) = d'(t). \tag{3.25}$$

Observe we assumed that the failure process is independent of the random usage time and so $\mathbf{X}_1$ is independent of $\mathbf{U}$. Hence the above assumption does not mean that the usage time is the same as the first time to failure. In his particular case, we obtain that the maximum point of the function r' is the same as the maximum point of the function d'. One can now show the following result for r.

Lemma 4. *If the mean arrival function Φ and Ψ of the failure process, respectively the sales process have both a positive continuous derivative at $(0,\infty)$ then the next results hold.*

2.1 If the cdf F_Ψ is unimodal and the function φ is logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has a increasing failure rate on $\{t>0: F_{\mathbf{U}}(t)<1\}$ then the function r is unimodal.

2.2 If the functions φ and ψ are logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has a increasing failure rate on $\{t>0: F_{\mathbf{U}}(t)<1\}$ then the function r' is logconcave on $(0,\infty)$.

Proof. By our assumptions it follows that r' exist and it is given in relation (3.21). Since φ and ψ are positive and continuous on $(0,\infty)$ we obtain that r' is positive on $(0,\infty)$. Also the random variable $\widetilde{\mathbf{U}}$ has cdf F_Φ listed in relation 3.19 and its density is given by

$$f_{\widetilde{\mathbf{U}}}(t) = \varphi(t)(1-F_{\mathbf{U}}(t))\mathbb{E}(\Phi(\mathbf{U}))^{-1}.$$

It follows by our assumptions that the function $f_{\widetilde{\mathbf{U}}}$ is logconcave on the convex set $\{t>0: F_{\mathbf{U}}(t)<1\}$. Since φ is positive and hence $\{t>0: F_{\mathbf{U}}(t)<1\} = \{t>0: f_{\widetilde{\mathbf{U}}}(t)>0\}$ this implies by Theorem 3 that the cdf F_Φ is strongly unimodal and so by relation (3.20) the function r is unimodal. This shows the first part. To verify the second part we observe that both F_Ψ and F_Φ are strongly unimodal and so the cdf $\mathbb{P}(\widetilde{\mathbf{U}}+\mathbf{X}\leq t)$ is strongly unimodal with $\widetilde{\mathbf{U}}$ independent of $\mathbf{X}$ and $\widetilde{\mathbf{U}}\sim F_\Phi$ and $\mathbf{X}\sim F_\Psi$. Applying now relation (3.20) and Theorem 3 yields the second part. □

Using now the above result we can easily determine the saturation point of the returned and defective items stochastic counting process and this is done next. One of the most important features of the installed base stochastic process is the time point at which the rate of increase of the expected number of returned and defective items up to a given time is maximal. We define this time point as the *saturation point* of the rate of returned products, which is denoted as follows:

$$t(r') := \arg\max_{0\leq t<\infty} r'(t). \tag{3.26}$$

The saturation point of the rate of returned products roughly represents the point in time during the life cycle of the product the manufacturer needs to have the highest safety stock of spare parts. To show the next result useful in computing the saturation point of the returned and defective item stochastic process given by the optimization problem in (3.26) we apply a similar proof using Lemma 4 as done in Lemma 3 and 2. So the next result for the function r' is given without proof.

Lemma 5. *If the positive functions φ and ψ are logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has an increasing failure rate on the set $\{t>0: F_{\mathbf{U}}(t)<1\}$ then $t(r')$ is an optimal solution if and only if $r''(t(r'))=0$. Moreover, it follows for any optimal solution $t(r')$ that $r''(t)\leq 0$ for every $t>t(r')$ and $r''(t)\geq 0$ for $0<t<t(r')$.*

Again we can identify $t(r')$ by applying a bisection procedure to r'' checking at each point the sign of r''. If $t(\psi)$ denotes the optimal solution of $\max_{0\leq t<\infty}\psi(t)$ (maturity time of product!) one expects that the maturity point of the installed base stochastic process occurs later. This intuitively clear result is shown under some reasonable assumptions. Unfortunately it is not clear what is the relation between the saturation point of the installed base process and the returned and defective items process.

Lemma 6. *If the function Ψ is twice continuously differentiable and unimodal with derivative ψ and there exists some $\epsilon > 0$ such that $\psi'(t) > 0$ for every $0 < t < \epsilon$ and $F_{\mathbf{U}}(t) < 1$ for every $t > 0$ then $t(b) \geq t(\psi) > 0$.*

Proof. Since $\psi'(t) > 0$ for every $0 < t \leq \epsilon$ it must follow using ψ is nonnegative and strictly increasing in a neighborhood of zero and $\psi(\infty) = 0$ that any optimal solution $t(\psi)$ of $\max_{0\leq t<\infty}\psi(t)$ is strictly positive. Since the mean arrival function Ψ of the sales process is unimodal it follows by the definition of unimodality that Ψ is convex on $(0, t(\psi))$ and concave on $(t(\psi), \infty)$ with $t(\psi)$ an optimal solution of $\max_{0\leq t<\infty}\{\psi(t)\}$. This shows that the function ψ is increasing on $(0, t(\psi))$ and so ψ' is nonnegative on $(0, t(\psi))$. It follows for every $t > 0$ that by partial integration

$$\begin{aligned}\int_0^t f_{\mathbf{U}}(t-y)\psi(y)dy &= \int_0^t f_{\mathbf{U}}(t-y)\int_0^y \psi'(z)dzdy + \psi(0)F_{\mathbf{U}}(t)\\ &= \int_0^t F_{\mathbf{U}}(t-z)\psi'(z)dz + \psi(0)F_{\mathbf{U}}(t).\end{aligned}$$

Hence we obtain for every $t \leq t(\psi)$

$$\begin{aligned}b'(t) &= \psi(t) - \int_0^t f_{\mathbf{U}}(t-y)\psi(y)dy\\ &= \int_0^t (1 - F_{\mathbf{U}}(t-y))\psi'(y)dy + \psi(0)(1 - F_{\mathbf{U}}(t)).\end{aligned} \tag{3.27}$$

Using $\psi(0) \geq 0$ and $\psi'(y) > 0$ in a neighborhood of zero and $F_{\mathbf{U}}(t) < 1$ for every $t > 0$ this implies by relation (3.27) that $b'(t) > 0$ for every $t \leq t(\psi))$. Hence the function b is strictly increasing on $(0, t(\psi))$ and this implies $t(b) \geq t(\psi)$ showing the desired result. □

3.3 Application of the Proposed Model

In this section, we present the application of our model to the particular case of a Weibull distributed usage time cdf. In case we only know from practice the empirical first and second moment of the usage time distribution we can always use the methods of moments estimation approach to fit a unique Weibull distribution to these data. In particular in Section 3.3.1 we derive formulas to measure the sensitivity of the saturation point of the installed base process with respect to the variance of this Weibull distributed usage time cdf for a given first varying moment. In Section 3.3.2 we assume for

Weibull distributed usage times that the failure process of each product in the market is given by a Weibull minimal repair model. A special case is given by a homogeneous Poisson failure process. We have chosen this type of failure process since in maintenance this is an often used model to describe the occurrences of failures over time. Again we measure the sensitivity of the saturation point of the returned and defective items stochastic process with respect to the variance of the Weibull usage time distributıon with a given first moment for different parameter settings of the Weibull minimal repair model.

3.3.1 Weibull Distributed Random Usage Times

We will now apply the above results to a particular usage time distribution and assume that the random variable $\mathbf{U}$ has a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter λ. This means (Tijms (1994)) that the cdf is given by (Tijms (1994)) $F_{\mathbf{U}}(t) = 1 - e^{-(\lambda t)^{\alpha}}, t > 0$ and this cdf has density $f_{\mathbf{U}}(t) = \alpha\lambda^{\alpha}t^{\alpha-1}e^{-(\lambda t)^{\alpha}}$. Hence it follows with $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$ meaning that the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same cdf that

$$\mathbf{U} \stackrel{d}{=} \lambda^{-1}\mathbf{Y}^{\alpha^{-1}}, \tag{3.28}$$

with $\mathbf{Y}$ having an exponential cdf with parameter 1. It is well known (Tijms (1994)) that $\mathbb{E}(\mathbf{X}) = \lambda^{-1}\Gamma(1+\alpha^{-1})$ and $\mathrm{Var}(\mathbf{X}) = \Gamma(1+2\alpha^{-1}) - \Gamma(1+\alpha^{-1})^2$. As already mentioned this class of cdfs is chosen because of its unique two moment fit. To measure for any two Weibull distributed usage random times having the same expectation $\theta > 0$ the influence of the variance on the function b denoting the expectation of the installed base process and the function r representing the expected number of returned and defective items up to any time we introduce the random variables $\mathbf{U}_{\alpha,\theta}, \alpha > 0, \theta > 0$ given by

$$\mathbf{U}_{\alpha,\theta} \stackrel{d}{=} \frac{\theta}{\Gamma(1+\alpha^{-1})}\mathbf{Y}^{\alpha^{-1}}. \tag{3.29}$$

It follows for any $\theta > 0$ and $\alpha > 0$ that $\mathbb{E}(\mathbf{U}_{\alpha,\theta}) = \theta = \frac{\Gamma(1+\alpha^{-1})}{\lambda}$ and this random variable has cdf

$$F_{\mathbf{U}_{\alpha,\theta}}(t) = \mathbb{P}(\mathbf{U}_{\alpha,\theta} \leq t) = \mathbb{P}(\mathbf{Y} \leq (\theta^{-1}\Gamma(1+\alpha^{-1})t)^{\alpha}) = 1 - e^{-\theta^{-\alpha}\Gamma(1+\alpha^{-1})^{\alpha}t^{\alpha}}.$$

For this random usage time $\mathbf{U}_{\alpha,\theta}$ it follows by relations (3.7) and (3.21) that the expected number of items in the market at time t is given by

$$b_{\alpha,\theta}(t) := \int_0^t e^{-\theta^{-\alpha}\Gamma(1+\alpha^{-1})^{\alpha}y^{\alpha}}\psi(t-y)dy, \tag{3.30}$$

and the derivative of the expected number of returned and defective items up to time t

$$r'_{\alpha,\theta}(t) := \int_0^t \varphi(y)e^{-\theta^{-\alpha}\Gamma(1+\alpha^{-1})^{\alpha}y^{\alpha}}\psi(t-y)dy. \tag{3.31}$$

To show the influence of α for fixed θ on the functions b and r' we need the following result.

Lemma 7. *The function* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *given by* $h(\alpha) = \Gamma(1+\alpha^{-1})^\alpha$ *is decreasing on* $(0,\infty)$.

Proof. Introducing the function $h_1 : \mathbb{R}_+ \to \mathbb{R}$ given by

$$h_1(\alpha) = \ln(\Gamma(1+\alpha^{-1})^\alpha) = \alpha \ln(\Gamma(1+\alpha^{-1})).$$

It follows that the function h is decreasing if and only if the function h_1 is decreasing. It is well know (see Theorem 5 of Artin (2015)) that the function $\alpha \to \ln(\Gamma(1+\alpha))$ is convex on $(0,\infty)$ and this implies by the perspective property of convex functions (Boyd & Vandenberghe (2004)) that also the function h_1 is convex on $(0,\infty)$. The derivative h_1' of the function h_1 is therefore increasing and is given by

$$h_1'(\alpha) = \ln(\Gamma(1+\alpha^{-1})) - \alpha^{-1}\psi_1(1+\alpha^{-1}), \alpha > 0,$$

with $\psi_1(\alpha) := \frac{d\ln\Gamma}{d\alpha}(\alpha)$ the so-called digamma function. Since $\psi_1(1) = -\gamma$ (Artin (2015)) with γ representing Eulers constant and $\Gamma(1) = 1$ we obtain

$$h_1'(\infty) = \lim_{\alpha\uparrow\infty} h_1'(\alpha) = 0.$$

Since h_1' is increasing this shows for every $\alpha > 0$ that $h_1'(\alpha) \leq h_1'(\infty) = 0$ and this implies the desired result. □

Applying Lemma 7 and relations (3.30) and (3.31) we immediately obtain the following result.

Lemma 8. *If the random usage time is given by the Weibull distributed random variable* $\mathbf{U}_{\alpha,\theta}, \alpha > 0, \theta > 0$ *satisfying* $\mathbb{E}(\mathbf{U}_{\alpha,\theta}) = \theta$ *then for any sales process and any failure process, the next results hold*

3.1 For every $0 < t \leq \theta$ *the functions* $\alpha \to b_{\alpha,\theta}(t)$ *and* $\alpha \to r'_{\alpha,\theta}(t)$ *are increasing on* $(0,\infty)$.

3.2 For every $t > 0$ *and* $\alpha > 0$ *the functions* $\theta \to b_{\alpha,\theta}(t)$ *and* $\theta \to r'_{\alpha,\theta}(t)$ *are increasing on* $(0,\infty)$.

The result in part 1 of Lemma 8 does not give any information how for fixed θ the optimal solution of optimization problem $\max_{0\leq t<\infty} b_{\alpha,\theta}(t)$ and $\max_{0\leq t<\infty} r'_{\alpha,\theta}(t)$ behaves as a function of α. Although in part 2 of the same lemma for any shape parameter $\alpha > 0$ the two considered functions are increasing in the expectation θ on $(0,\infty)$ it is again unclear how the maximum point of both functions behaves as a function of θ. This seems to be difficult to verify and will be numerically tested in the computational section for the scenarios $\alpha = 1$ (exponentially distributed usage time) and $\alpha = 2$ (usage time having increasing failure rate) and $3 \leq \theta \leq 9$ using the Brockhoff sales model.

3.3.2 Weibull Minimal Repair Failure Processes

The same experiments will be performed for the maximum of the function r' and the same range of expected usage times in case the failure process is given by a Poisson process having a constant failure rate or a Weibull minimal repair model with an increasing failure rate. Observe a Weibull minimal repair model is a nonhomogenous Poisson process with arrival intensity given by the failure rate function of a Weibull distributed random variable (Block et al. (1985),Aven & Jensen (2000)),Aven (2011)). Since the failure rate function of a Weibull-distributed random variable is given by

$$r_{\mathbf{U}}(t) = \frac{f_{\mathbf{U}}(t)}{1 - F_{\mathbf{U}}(t)} = \alpha\lambda^{\alpha}t^{\alpha-1}, t > 0,$$

it follows that the random variable $\mathbf{U}$ has an increasing failure rate if and only if $\alpha \geq 1$. Also for $\alpha \geq 1$ it follows that the density $f_{\mathbf{U}}$ is logconcave on $(0,\infty)$. Since we believe that failure processes with decreasing failure rates are of limited practical importance we only consider scenarios with $\alpha \geq 1$.

To give a more detailed description of the mean arrival function Ψ of the sales process we already observed that for most products the product life cycle function ψ at first increases and then after the so-called maturity phase of the product decreases and approaches zero. By this observation, it is natural to assume that the function Ψ is unimodal. We now consider an often-used parametric representation of the product life cycle function. A popular representation of this function is given by

$$\psi(t) = at^{b}e^{-ct}, t > 0,$$

for some unknown parameters $a > 0, b > 0$ and $c > 0$. Clearly the function ψ is logconcave on $(0,\infty)$, attains its maximum at $t(\psi) = bc^{-1}$ and

$$\Psi(\infty) = ac^{-(b+1)}\Gamma(b+1) < \infty,$$

with $\Gamma(y) := \int_0^{\infty} x^{y-1}e^{-x}dx$ the well known gamma function evaluated at y. In Brockhoff (Brockhoff (1967)) this representation is proposed and at the same time some data were fitted to determine the unknown parameters. For this so-called Brockhoff model we obtain that relation (3.30) reduces to

$$b_{\alpha,\theta}(t) = a\int_0^t e^{-\theta^{-\alpha}\Gamma(1+\alpha^{-1})^{\alpha}y^{\alpha}}(t-y)^{b}e^{-c(t-y)}dy, t > 0. \tag{3.32}$$

By part 2 and 4 of Lemma 1 we obtain for $\alpha \geq 1$ that both the function b and d' are logconcave on $(0,\infty)$. In case we use for the failure process a minimal repair Weibull model with the cdf $F_{\mathbf{X}_1}$of the first failure time $\mathbf{X}_1$ of an item given by a Weibull cdf with scale parameter $\delta > 0$ and shape parameter $\gamma \geq 1$ it follows that

$$\Phi(t) = -\ln(1 - F_{\mathbf{X}_1}(t)) = (\delta t)^{\gamma}, t > 0,$$

and so its derivative $\varphi(t) = \gamma\delta^{\gamma-1}$ is logconcave on $(0,\infty)$ for $\gamma \geq 1$. Also for this Weibull minimal repair model we obtain by relation (3.28) that

$$\mathbb{E}(\Phi(\mathbf{U}_1)) = \delta^{\gamma}\mathbb{E}(\mathbf{U}_1^{\gamma}) = \delta^{\gamma}\lambda^{-\gamma}\mathbb{E}(\mathbf{Z}^{\gamma\alpha^{-1}}) = \delta^{\gamma}\lambda^{-\gamma}\Gamma(\gamma\alpha^{-1}+1), \tag{3.33}$$

and so $\mathbb{E}(\Phi(\mathbf{U}_1))$ is finite. Applying now part 2 of Lemma 4 it follows that r' is logconcave on $(0,\infty)$

for $\alpha \geq 1$ and $\gamma \geq 1$ and relation (3.31) reduces to

$$r'_{\alpha,\theta}(t) = a\delta^{\gamma}\gamma e^{-ct} \int_0^t y^{\gamma} e^{-\theta^{-\alpha}\Gamma(1+\alpha^{-1})^{\alpha} y^{\alpha}} (t-y)^b e^{-cy} dy. \tag{3.34}$$

This concludes our discussion of Weibull distributed usage times. In the next section, we will perform a numerical experiment and plot the optimal solutions of the optimization problems $\max_{0\leq t<\infty} b_{\alpha,\theta}(t)$ and $\max_{0\leq t<\infty} r'_{\alpha,\theta}(t)$ for the scenarios $\alpha = 1$ and $\alpha = 2$ and varying expected usage times.

3.4 Computational Results

In this section, a numerical example is provided to demonstrate the application of the proposed approach. This example refers to a specific refrigerator model defined by a unique SKU that is expected to be produced and sold for about 3 years before it is redesigned, achieves its maximum sales after about 1 year, and is sold with a warranty of 3 years. Research has shown that a refrigerator can have an annual failure rate of 1.79% which is assumed to be the summation of the failure rates of its important components (Woo & O'Neal, 2016). In our numerical example, we assume that the product's failure rate is equal to 1.79% and sales rate follows the Brockhoff function $\psi(t) = at^b e^{-ct}, t > 0$ (Brockhoff, 1967), where the time t is measured in years. In this setting, the model parameters a, b, and c have the following effects: (i) as a becomes larger, every non-zero point of the function gets magnified; (ii) as b increase, it pushes the ψ function to the right leading to a sharper growth and a later maximum of sales, and (iii) as c becomes larger, the ψ function is pushed to the left; so we would have a faster decline and an earlier maximum point.

3.4.1 Parameter Estimation

To fit the unknown positive parameters in our model considering the above assumptions we first observe the following. Using relations (3.32) and (3.34) it follows for any $a > 0$ that the optimal solution of the optimization problems $\max_{0<t<\infty} b(t)$ and $\max_{0<t<\infty} r'(t)$ does not depend on a. Hence without any loss of generality, we may use any $a > 0$ in our numerical experiments. To fit the other parameters b and c we observe that the maximum $t(\psi)$ of the function ψ occurs at bc^{-1} and since we assume that the sales are maximal after one year we select $c = b$. Our product is sold for approximately 3 years. Based on this premise the unknown parameter b is estimated as follows:

$$\frac{\Psi(\infty) - \Psi(3)}{\Psi(\infty)} \leq \kappa,$$

for some preselected $0 \leq \kappa < 1$. In our numerical experiments $\kappa = 0.01$. It follows for $c = b$ that

$$\kappa \geq \frac{\Psi(\infty) - \Psi(3)}{\Psi(\infty)} = \frac{\int_3^\infty t^b e^{-bt} dt}{\int_0^\infty t^b e^{-bt} dt} = \frac{\int_{3b}^\infty y^b e^{-y} dy}{\Gamma(b+1)} = \mathbb{P}(\mathbf{Z}_{b+1} \geq 3b), \tag{3.35}$$

with $\mathbf{Z}_{b+1}$ gamma distributed with scale parameter 1 and shape parameter $b+1$. It is well known that the moment-generating function of this random variable equals

$$\mathbb{E}(e^{s\mathbf{Z}_{b+1}}) = (1-s)^{-(b+1)}, s < 1.$$

Hence by Markov's inequality, we obtain for any $0 < s < 1$ that

$$\mathbb{P}(\mathbf{Z}_{b+1} \geq 3b) = \mathbb{P}(e^{s\mathbf{Z}_{b+1}} \geq e^{3sb}) \leq \mathbb{E}(e^{s\mathbf{Z}_{b+1}})e^{-3sb} = (1-s)^{-(b+1)}e^{-3sb}$$

This shows by elementary calculations that

$$\mathbb{P}(\mathbf{Z}_{b+1} \geq 3b) \leq \min_{0<s<1} (1-s)^{-(b+1)} e^{-3sb} = (\frac{3b}{b+1})^{b+1} e^{-(2b-1)}, \tag{3.36}$$

and we have constructed an easy upperbound on the given probability. Since $\lim_{b\uparrow\infty}(\frac{3b}{b+1})^{b+1}e^{-(2b-1)} = 0$ it is easy to select some $b > 0$ satisfying

$$\left(\frac{3b}{b+1}\right)^{b+1} e^{-(2b-1)} \leq \kappa$$

and for this b it follows by relation (3.36) that the inequality in relation (3.35) is satisfied. In our particular case, we, therefore, select $b = c = 6.3$. As mentioned earlier, we do not need to specify a in order to calculate $t(b)$ and $t(r')$ but in case we want to see true expected values of the variables in the modeling part such as $\mathbb{E}(\mathbf{R}(t))$ we can easily specify a by deciding on the total market share value we aim to have, i.e., $\Psi(\infty)$ and then using $\Psi(\infty) = ac^{-(b+1)}\Gamma(b+1)$. For example, if we assume that we will sell 200 units of our product in total, we obtain $a = \frac{200c^{b+1}}{\Gamma(b+1)} = 107628.1$.

As random usage times we use the Weibull distributed random variable $\mathbf{U}_{\alpha,\theta}$, $\alpha > 0, \theta > 0$ listed in relation (3.29) and consider the scenarios with $\alpha \geq 1$ as the contrary case $\alpha < 1$ leads to a failure process with a decreasing failure rate, which is an implausible assumption for capital products. Specifically, we set $\alpha = 1$ and $\alpha = 2$ for expected usage times up to three times the warranty period of 3 years. Then the scale parameter λ would be computed as $\lambda = \frac{\Gamma(1+\alpha^{-1})}{\mathbb{E}(\mathbf{U}_{\alpha,\theta})}$.

To compute the maximum of the derivative r' of r we have used the Weibull minimal repair process as a failure process. In this nonhomogenous Poisson process, the first time to failure is Weibull-distributed with shape parameter γ and scale parameter δ. As for the random usage time, we consider the scenarios $\gamma = 1$ or $\gamma = 2$. Knowing that the probability of failure for the first time in one year is equal to 0.0179 it is easy to estimate δ. Let $\mathbf{X}_1$ denote the first failure time of a product in use. Since the failure process is a nonhomogenous Poisson process with mean failure function $\Phi(t) = \delta^\gamma t^\gamma$ it follows that the first failure time $\mathbf{X}_1$ in a nonhomogeneous Poisson process with mean value function Φ can be represented with

$$\mathbf{X}_1 = \Phi^{\leftarrow}(\mathbf{X}_1^P).$$

where $\mathbf{X}_1^P$ follows exponentially distribution with parameter 1. This shows as expected that

$$\pi = \mathbb{P}(\mathbf{X}_1 \leq x) = \mathbb{P}(\Phi^{\leftarrow}(\mathbf{X}_1^P) \leq x) = \mathbb{P}(\mathbf{X}_1^P \leq \Phi(x)) = \mathbb{P}(\mathbf{X}_1^P \leq \delta^{\gamma} x^{\gamma}) = 1 - e^{-(x\delta)^{\gamma}},$$

which implies that $\delta = (-ln(1-0.0179))^{\gamma}$.

In Figure 3.3, we present the expected values of the expressions in the modeling part for a specific set of parameters with all the plots scaled to $1000/a$. In this setting, parameters of the Brockhoff sales are $a = 107628.1, b = c = 6.3$, the random variable $\mathbf{U}$ has a Weibull distribution with $\alpha = 1, \lambda = 0.33$ meaning $\mathbb{E}(\mathbf{U}) = 3$ and the first time to failure $\mathbf{X}_1$ has a Weibull distribution with $\gamma = 2, \delta = 0.13$.

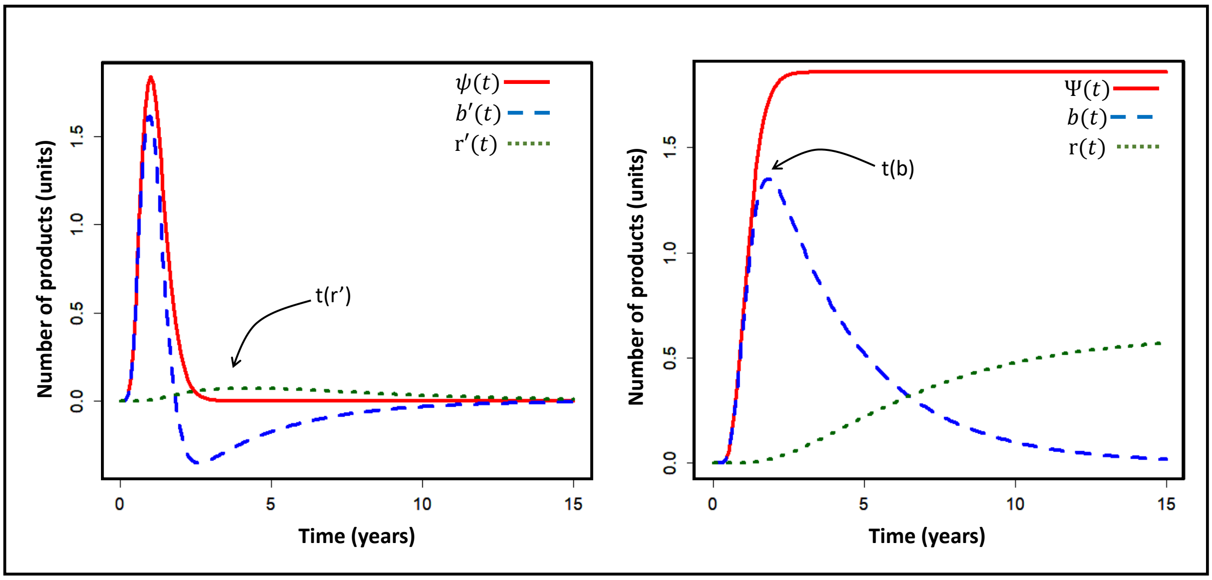

Figure 3.3 The curves of expected sales rates, installed base rate and returned and defectives rate (left panel) and expected total values of the same processes over time (right panel) for Brockhoff sales, $\mathbf{U} \sim Weibull(1, 0.33)$ and $\mathbf{X}_1 \sim Weibull(2, 0.13)$ $(\alpha = 1, \gamma = 2,\ \mathbb{E}(\mathbf{U}) = 3)$. The figure is scaled to $1000/a$.

The maximum points of the functions b and r' are highlighted in the figures. Next, we analyze how the usage behavior of the customer will affect these saturation points.

3.4.2 Sensitivity Analysis

In this section, we analyze the impact of expected usage time on the saturation points of the installed base, $t(b)$, and the number of returned products $t(r')$. For this, we consider four different scenarios related to the shape parameter of the random usage time α and the shape parameter of the random first failure time γ (Figure 3.4). $\alpha = 1$ means the random usage time has exponential distribution (left column) while $\gamma = 1$ means homogeneous Poisson failure process (first row). In the latter case, $r'(t) = \delta b(t)$ so the maximum of $b(t)$ and $r'(t)$ are the same.

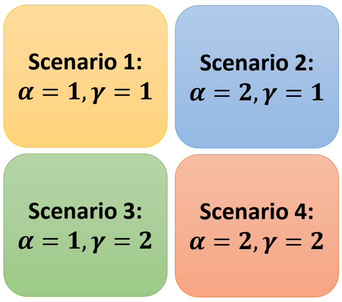

Figure 3.4 The four scenarios to do sensitivity analysis

The four plots in Figure 3.5 refer to the four scenarios. In general, we see in each of these figures that by the increase of the expected usage time of the product by the customer, both the installed base size and the rate of returned and defectives reach their maximum at a later time although this delay is very small for the peak of installed base size. If we examine the effect of α by comparing the left figures with the right ones, we see it has an increasing effect on $t(b)$ and $t(r')$. This increasing effect, which is similar to the effect of expected usage time, is as small as only a few months for $t(b)$. However, by comparing $t(r')$ in the bottom row figures, we see that when $\gamma = 2$, α has a decreasing effect. By analyzing row-wise to see the effect of γ we see that $t(r')$ is higher in the second row.

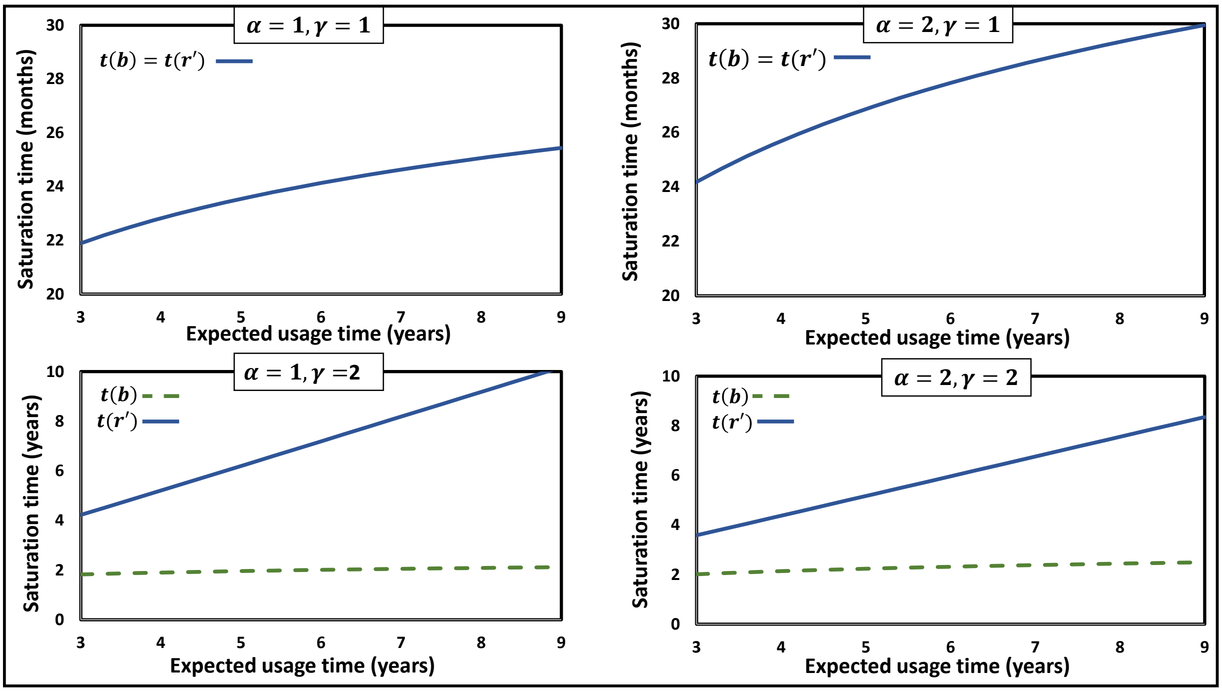

Figure 3.5 Saturation times of installed base ($t(b)$) and the number of returned products ($t(r')$). In scenarios 1 and 2 (Upper panel), the saturation points of both processes coincide. The figure is scaled to $1000/a$.

In Figure 3.6, we see the effect of expected usage time on the expected number of returned defectives during 1 year after it reaches its maximum. i.e., $r(t(r') + 1) - r(t(r'))$. Of course, it is possible to compute this for any period of time but from the OEM point of view, the maximum size of spare parts demand and the time this maximum happens is crucial. We see that both α and γ have positive effects on this expression.

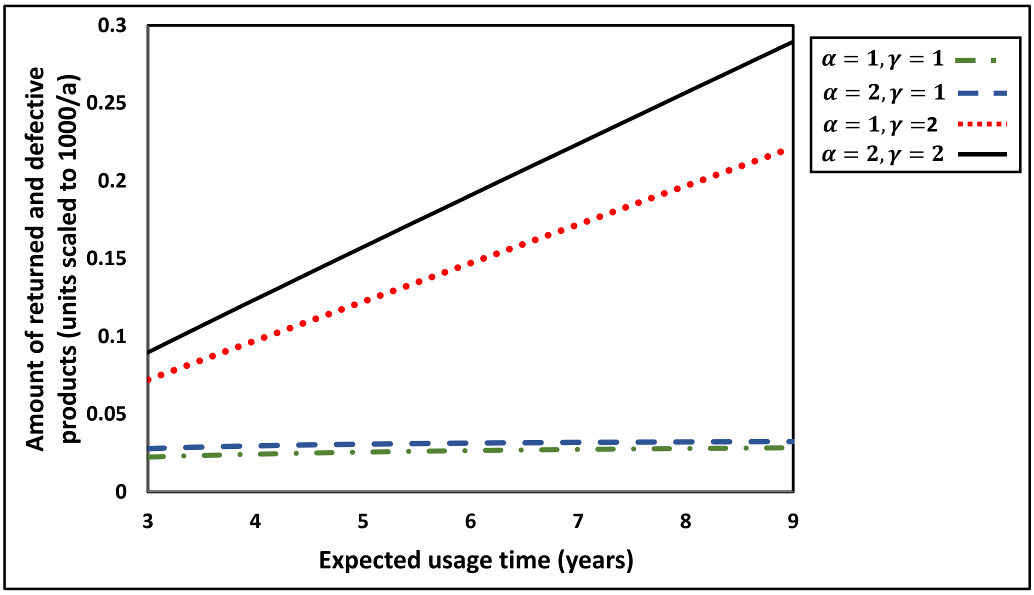

Figure 3.6 Number of the returned and defective items within one year after this expression reaches its maximum for four scenarios. The figure is scaled to $1000/a$.

3.4.3 Managerial Insights

In the previous subsection, we evaluated the sensitivity of the saturation level and time of the number of returned products and the size of installed base. We find that the saturation point of the number of returned products is rather sensitive to the expected usage time when the time to first failure has an increasing failure rate distribution (Weibull). On the other hand, the saturation point of the installed base is driven by the expected usage time when the scale parameter of the usage distribution is larger than 1. These qualitative results and their relation to our experimental scenarios (Figure 3.4) are presented in Table 3.1.

From a managerial point of view, the saturation level and time of returned products indicate the time point when an OEM receives the highest amount of spare parts and maintenance demand for their products. Our results imply that the peak point of aftersales business activities is insensitive to the expected usage time when the first failure time follows an exponential distribution. For increasing failure rate distributions, longer lifetimes lead to higher volumes of aftersales business that arrive in the later stages of the product's life cycle. Furthermore, we find that the saturation point of the installed base is mainly driven by the intensity of the sales process. Increasing expected usage rates by three times shift saturation time of installed base process approximately by 20%.

These results indicate that OEMs should invest in the estimation of the future sales intensity and failure characteristics of their products while designing their aftersales services. Counter-intuitively the impact of usage time on the saturation point of aftersales business activities significantly depends on the failure time distribution of a product. These results also have some implications for the product complexity and business volume of aftersales services. Specifically, for simple products with near-constant failure rate timing and level of aftersales revenues are driven by sales intensity. For more complicated products with increasing failure intensity, both usage time and reliability become important for determining the volume of aftersales services. Hence, these two sets of information are critical for designing maintenance and service networks for OEMs of durable products. Note that our managerial insights derived from our theoretical results and numerical investigations are mostly independent of our selection of parameter values as we utilize normalization procedures given in Section 3.4.1.

Table 3.1 Sensitivity of saturation time and saturation level of the installed base and the number of returned products

Usage Time	First Failure Time	Number of Returned Products		Saturation Time of Installed Base	Experimental Scenario
		Saturation Time	Saturation Level		
Exponential	Exponential	Insensitive	Insensitive	Insensitive	Scenario 1
Weibull	Exponential	Insensitive	Insensitive	Insensitive	Scenario 2
Exponential	Weibull	Sensitive	Sensitive	Insensitive	Scenario 3
Weibull	Weibull	Sensitive	Sensitive	Insensitive	Scenario 4

4. A Continuous Time Stochastic Process for Spare Parts: An Installed Base Approach

To keep this chapter self-contained the cumulative sales process is reintroduced in the first subsection and the installed base and discarded items stochastic process are again presented in the second subsection. In this second subsection we also present without proof for the sake of completeness the properties of the expected behaviour of these processes over time already shown in the first chapter. In the third subsection the cumulative spare parts demand process is reintroduced and again for this process we mention without proof the properties of its expected behaviour over time. Since we address in this chapter different questions related to spare parts demand than the questions presented in the first chapter but we still need the same stochastic processes to answer these different questions this is done to keep this chapter self-contained.

It is assumed in this chapter that each returned and defective item needs repair and it only needs one spare part. Hence the total number of returned and defective items up to any given time completely determines the demand for spare parts up to that time. This assumption can be easily relaxed in our mathematical model and is only done to keep the notation simple. Under the general framework discussed in the first section it is not possible to give an elementary expression for the cumulative distribution function or the higher moments of the total demand for spare parts up to any given time. To achieve this we need to assume that the cumulative sales process is a nonhomogeneous Poisson process. The analysis for such special sales processes and general failure processes and usage times will be presented in the second section. In the second second we also introduce the total demand for spare parts after any given time up to the end of the so-called life cycle of the product and analyse the properties of this related stochastic process. Observe this stochastic process is important in determining the size of the last buy decision. Also we analyse the stochastic process of the total demand for spare parts in a given time interval. Finally in the second subsection we will focus for which special failure counting processes it is possible to evaluate the pmfs and higher moments of the spare parts demand and related stochastic processes. In the third subsection of this chapter we will discuss some computational results and in the last subsection we give some conclusions. This chapter extends the results presented in the first chapter (Amniattalab et al. (2023a)) by focusing on a nonhomogeneos Poisson sales processes. In this particular case contrary to the general point process representation of a cumulative sales process presented in the first chapter one can derive higher moments and the probability distribution of the spare parts demand up to any given time. Also this can be done for related random variables important in determining the safety stock over a given time period.

In this chapter we therefore address two primary research questions: (1) How to compute higher moments of spare parts demand and its probability distribution function, and (2) How to determine the necessary safety stock to prevent stock outs in any given time interval or after some time during the whole life cycle of the product.

4.1 Introduction of the continuous time stochastic processes in spare parts demand

In this section we reintroduce in the first subsection a cumulative sales stochastic process. After the introduction of such a sales process we present in the second subsection the so-called installed base and discarded items stochastic process and analyse the expected behaviour of these general stochastic processes. In the final subsection the cumulative returned and defective items stochastic process is reintroduced and as for the other processes we discuss for this process its expected behaviour over time. As such this section is already discussed in more detail in the first chapter but since this chapter is self contained and addresses other questions in spare parts management we list in the first section the main stochastic processes again.

4.1.1 The Cumulative Sales Process

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space modeling the point process $(\mathbf{T}_n)_{n\in\mathbb{N}}$ with the positive random variables $\mathbf{T}_n$, $n \in \mathbb{N}$ representing the strictly increasing random selling times of the nth item of a specific product. The stochastic process of the total number of sales up to time t is then given by the one-dimensional stochastic counting process $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ with

$$\mathbf{S}(t) = \textstyle\sum_{n=1}^{\infty} 1_{\{\mathbf{T}_n \leq t\}}. \tag{4.1}$$

We assume the cumulative sales point process $(\mathbf{T}_n)_{n\in\mathbb{N}}$ has intensity measure Ψ (see Brémaud (1981) or section 1.4 of Last & Brandt (1995)) defined by

$$\Psi(A) := \mathbb{E}(\textstyle\sum_{n=1}^{\infty} 1_{\{\mathbf{T}_n \in A\}}) = \textstyle\sum_{n=1}^{\infty} \mathbb{P}(\mathbf{T}_n \in A) \tag{4.2}$$

with A any Borel set on $\mathbb{R}_+$. This measure is completely determined by the increasing right-continuous so-called mean arrival function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Psi(t) = \Psi([0,t]) = \sum_{n=1}^{\infty} \mathbb{P}(\mathbf{T}_n \leq t),\ t > 0$ satisfying $\Psi(0) = 0$. Since the total number of items of a product sold during the life cycle of the product is finite we assume that $\Psi(\infty)$ is finite. In the next subsection we present the installed base and discarded items stochastic process.

4.1.2 The Installed Base and the Discarded Items Stochastic Process

Once an item is sold, it will stay in the market for a random amount of time called the usage time or the life span of that item. The usage time of the nth sold item is denoted by the nonnegative random variable $\mathbf{U}_n$ and we assume throughout this paper that these random variables are independent and identically distributed having cdf $F_{\mathbf{U}}$ satisfying $F_{\mathbf{U}}(0) = 0$. Also it is assumed that for every n the usage time $\mathbf{U}_n$ is independent of the time of sales of the nth item at time $\mathbf{T}_n$. Using the definition of the random variable $\mathbf{U}_n$ the number of items of the same product available in the market at time t is represented by the stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ with

$$\mathbf{B}(t) := \sum_{n=1}^{\infty} 1_{\{\mathbf{U}_n > t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}. \tag{4.3}$$

The random variable $\mathbf{B}(t)$ is known in the literature as the *installed base* at time t (see Kim et al. (2017), Minner (2011)) and we refer to the stochastic process $\mathbf{B}$ as the *installed base stochastic process*. By relation (4.3), it is easy to see if the usage times are stochastically increasing, then the random variables $\mathbf{B}(t)$ are stochastically decreasing for every $t > 0$. To compute the expectation of the random variable $\mathbf{B}(t)$, we observe applying the monotone convergence theorem (Çınlar (2011)) and relation (4.3) that

$$b(t) := \mathbb{E}(\mathbf{B}(t)) = \sum_{n=1}^{\infty} \mathbb{E}(1_{\{\mathbf{U}_n > t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}). \tag{4.4}$$

Since the random variables $\mathbf{U}_n, n \in \mathbb{N}$ are identically distributed with cdf $F_{\mathbf{U}}$ and the sequence $(\mathbf{U}_n)_{n\in\mathbb{N}}$ is independent of the sequence $(\mathbf{T}_n)_{n\in\mathbb{N}}$, it follows by conditioning in relation (4.4) for every n on $\mathbf{T}_n$ that for every $t \geq 0$

$$b(t) = \int_0^t (1 - F_{\mathbf{U}}(t-y))\Psi(dy). \tag{4.5}$$

It follows that $b(0) = 0$ and $b(\infty) = 0$ and under some standard regularity conditions on $F_{\mathbf{U}}$ and Ψ, the function b is continuous on $(0,\infty)$ and attains a finite positive maximum. To compute this maximum we need to know its derivative and for simplicity, we assume that the mean arrival function Ψ of the sales process is continuously differentiable with derivative ψ and the cdf $F_{\mathbf{U}}$ has a continuous density $f_{\mathbf{U}}$ on $(0,\infty)$ except at a finite number of points and $F_{\mathbf{U}}(0) = 0$. If this holds, it follows by relation (4.5) that the derivative $b'(t)$ of the function b at time t exists and equals

$$b'(t) = \psi(t) - \int_0^t f_{\mathbf{U}}(t-y)\psi(y)dy. \tag{4.6}$$

Another important stochastic process is given by the so-called discarded items stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$ defined by

$$\mathbf{D}(t) := \sum_{n=1}^{\infty} 1_{\{\mathbf{U}_n \leq t - \mathbf{T}_n\}} 1_{\{\mathbf{T}_n \leq t\}}. \tag{4.7}$$

The random variable $\mathbf{D}(t)$ denotes the total number of discarded items in the market up to time t and this process determines the number of spare parts obtained from discarded items. It is easy to verify that the expected number of discarded items at time $t > 0$ is given by

$$d(t) = \mathbb{E}(\mathbf{D}(t)) = \int_0^t F_{\mathbf{U}}(t-y))\psi(y)dy = \int_0^t F_{\mathbf{U}}(y)\psi(t-y)dy = \Psi(t) - b(t). \tag{4.8}$$

As shown in Amniattalab et al. (2023a), it is possible to show the following result for the functions b and d.

Lemma 9. *If the mean arrival function Ψ of the sales process has a positive and continous derivative ψ on $(0,\infty)$, then the next results hold.*

4.1 If the positive function ψ is logconcave on $(0,\infty)$ and the cdf $F_{\mathbf{U}}$ is unimodal having a density $f_{\mathbf{U}}$, then the function d is unimodal.

4.2 If the positive function ψ is logconcave on $(0,\infty)$ and the density $f_{\mathbf{U}}$ is positive and logconcave on the convex set $\{t>0 : F_{\mathbf{U}}(t)<1\}$, then the derivative function d' is logconcave on $\{t>0 : F_{\mathbf{U}}(t)<1\}$.

4.3 If the positive function ψ is logconcave on $(0,\infty)$, then the function b is increasing on $(0,t(b'))$ and decreasing on $(t(b'),\infty)$ for some $t(b')>0$

4.4 If the positive function ψ is logconcave on $(0,\infty)$ and the random variable $\mathbf{U}$ has an increasing failure rate on $\{t\geq 0 : F_{\mathbf{U}}(t)<1\}$, then the function b is logconcave on $(0,\infty)$.

Using the above properties, special cases of these processes are numerically analyzed in the computational section of Amniattalab et al. (2023a). In particular, the sales process is given by the Brockhoff model (Brockhoff (1967)) and the usage time distribution has a Weibull distribution. We now present in the next subsection the cumulative returned defective items stochastic process.

4.1.3 The Cumulative Returned Defective Items Stochastic Process

After selling item n of the same product at time $\mathbf{T}_n$, it generates during its usage time, a sequence of failures and each failure causes a repair. A failure might occur during the warranty period or outside the warranty period. It is assumed for simplicity that repairs do not take any time and a failed product after repair will always function again. On the same probability space $(\Omega,\mathcal{H},\mathbb{P})$, we therefore introduce the failure counting process $\mathbf{N}_n=\{\mathbf{N}_n(t):t\geq 0\}$ describing the total number of failures of the nth sold item within the interval $[\mathbf{T}_n,\mathbf{T}_n+t)$. Since the items are identical products, it seems reasonable to assume that the stochastic counting failure processes $\mathbf{N}_n, n\in\mathbb{N}$ generated by the sold items are independent and identically distributed having the same mean arrival function Φ. Since we also consider the usage time of an item, we assume throughout this paper that the random vectors $(\mathbf{N}_n,\mathbf{U}_n)$, $n\in\mathbb{N}$ are independent and identically distributed and independent of the random selling times $\mathbf{T}_n$,$n\in\mathbb{N}$. Having introduced the sales and the failure process of each individual sold item, we can now introduce the counting process of the number of returned defective items. This is represented by the stochastic process $\mathbf{R}=\{\mathbf{R}(t):t\geq 0\}$ with $\mathbf{R}(t)$ denoting the total number of returned defective products before time t. Clearly

$$\mathbf{R}(t)=\sum_{n=1}^{\infty}\mathbf{N}_n((t-\mathbf{T}_n)\wedge\mathbf{U}_n)\mathbf{1}_{\{\mathbf{T}_n\leq t\}}, \tag{4.9}$$

with $\mathbf{Z}_1\wedge\mathbf{Z}_2=\min\{\mathbf{Z}_1,\mathbf{Z}_2\}$. By a similar argument as used for the sales process, we obtain using the assumption that the random vectors $(\mathbf{N}_n,\mathbf{U}_n)$,$n\in\mathbb{N}$ are identically distributed and independent of the sequence $(\mathbf{T}_n)_{n\in\mathbb{N}}$ that

$$r(t) := \mathbb{E}(\mathbf{R}(t)) = \int_0^t \mathbb{E}(\mathbf{N}_1((t-y)\wedge \mathbf{U}_1))\Psi(dy). \tag{4.10}$$

If $\mathbf{Y}_1$ is a nonnegative random variable on $(0,t)$ having cdf

$$F_{\mathbf{Y}_1}(y) = \mathbb{P}(\mathbf{Y}_1 \leq y) = \frac{\Psi(y)}{\Psi(t)}, y \leq t \tag{4.11}$$

and the random variable $\mathbf{Y}_1$ is independent of the random vector $(\mathbf{N}_1, \mathbf{U}_1)$, a compact representation of $r(t)$ is given by

$$r(t) = \Psi(t)\mathbb{E}(\mathbf{N}_1((t-\mathbf{Y}_1)\wedge \mathbf{U}_1)). \tag{4.12}$$

To simplify the integral in relation (4.10) or (4.12) and compute its value, we additionally assume that the random usage time $\mathbf{U}_n$ is independent of the failure counting process $\mathbf{N}_n$ for every $n \in \mathbb{N}$. Using this assumption it follows by conditioning on $\mathbf{U}_1$ and using the conditional expectation formula that for every $s \geq 0$

$$\mathbb{E}(\mathbf{N}_1(s\wedge \mathbf{U}_1)) = \mathbb{E}(\mathbb{E}(\mathbf{N}_1(s\wedge \mathbf{U}_1) \mid \mathbf{U}_1)) = \mathbb{E}(\Phi(s\wedge \mathbf{U}_1)), \tag{4.13}$$

with Φ the mean arrival function of the failure point process $\mathbf{N}_1$. This shows by relation (4.10) that

$$r(t) = \int_0^t \mathbb{E}(\Phi((t-y)\wedge \mathbf{U}_1))\Psi(dy). \tag{4.14}$$

Again a compact representation is given by

$$r(t) = \Psi(t)\mathbb{E}(\Phi((t-\mathbf{Y}_1)\wedge \mathbf{U}_1)). \tag{4.15}$$

By relation (4.15), we obtain $r(0) = 0$ and $r(\infty) = \Psi(\infty)\mathbb{E}(\Phi(\mathbf{U}_1)) \leq \infty$. Observe $\mathbb{E}(\Phi(\mathbf{U}_1))$ denotes the expected number of failures during the usage time of the item and $\Psi(\infty)$ is the expected number of sales of the product during its life cycle. This implies $r(\infty)$ is finite if and only if $\mathbb{E}(\Phi(\mathbf{U}_1))$ is finite. If we exclude the usage time from our model and hence the item will be in use an infinite amount of time or equivalently $\mathbf{U}_1$ is equal to ∞ with probability 1, it follows by relation (4.15) that

$$r(t) = \int_0^t \Phi(t-y)\Psi(dy) = \Psi(t)\mathbb{E}(\Phi(t-\mathbf{Y}_1)). \tag{4.16}$$

The assumption that the random usage time is independent of the failure process might in some instances not be realistic but as already observed is related to keeping the model mathematically tractable. A more realistic assumption would be that the usage time $\mathbf{U}_n$ (Çınlar (2011)) is a stopping time with respect to the filtration generated by the failure process $\mathbf{N}_n$. For the definition of a stopping time, the reader is referred to (Çınlar (2011)). It seems reasonable that such a dependence relation exists if the defects occur outside the warranty period and the customer has to pay for repair. We will not pursue this approach since it requires assumptions about the behavior of each customer. However, observe that some results in this paper do not need this independence assumption between usage time and failure process and if this happens it will be explicitly mentioned.

Since the manufacturer also likes to know at which time how many resources should be allocated to the repair of defective returned items, the manufacturer is interested in the time that the incremental change of the expected number of returned defective items up to time t is at its highest level. Hence, the manufacturer needs to compute the derivative of the function r and solve the optimization problem $\sup_{0<t<\infty} r'(t)$. It is shown in Amniattalab et al. (2023a) that under some standard regularity conditions

$$r'(t) = \int_0^t \varphi(t-y)(1-F_{\mathbf{U}}(t-y))\psi(y)dy. \tag{4.17}$$

with φ denoting the derivative of the mean arrival function Φ of the failure process $\mathbf{N}_1$. To solve this optimization problem easily, the following result is also shown in Amniattalab et al. (2023a).

Lemma 10. *Let the mean arrival function of the sales process Ψ and the mean arrival function of the failure process Φ have respectively a continuous positive derivative ψ and φ on $(0,\infty)$ and $\mathbb{E}(\Phi(\mathbf{U}_1))$ is finite. Now the following results hold.*

5.1 If the usage time random variable $\mathbf{U}$ has a density and an increasing failure rate, the function φ is logconcave on $(0,\infty)$ and the mean arrival function Ψ of the sales process is unimodal, then the function r given by relation (4.10) is unimodal.

5.2 If the usage time random variable $\mathbf{U}$ has a density and an increasing failure rate and the functions ψ and φ are logconcave on $(0,\infty)$, then the derivative r' of the function r listed in relation (4.17) is logconcave.

Due to cost considerations, the manufacturer selling the item having a warranty period of length w is interested in the total number of defective items up to time t of which the defects occur within the warranty period. To model this, we introduce the stochastic process $\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$ with $\mathbf{R}_w(t)$ the total number of returned defective items up time t under warranty. This is given by

$$\mathbf{R}_w(t) = \sum_{n=1}^{\infty} \mathbf{N}_n((t-\mathbf{T}_n) \wedge (\mathbf{U}_n \wedge w))\mathbf{1}_{\{\mathbf{T}_n \leq t\}}. \tag{4.18}$$

The random variable $\mathbf{R}(t) - \mathbf{R}_w(t)$ denotes the number of returned defective items up to time t which do not fall under the warranty. By similar arguments as done for the counting process of returned and defective items up to time t which covers both items under warranty and not under warranty, it follows that

$$\mathbb{E}(\mathbf{R}_w(t)) = \int_0^t \mathbb{E}(\mathbf{N}_1((t-y) \wedge (\mathbf{U}_1 \wedge w)))\Psi(dy). \tag{4.19}$$

In case we additionally assume that the usage time of the item is independent of the failure process of the same item, this simplifies to

$$r_w(t) := \mathbb{E}(\mathbf{R}_w(t)) = \int_0^t \mathbb{E}(\Phi((t-y) \wedge (\mathbf{U}_1 \wedge w)))\Psi(dy). \tag{4.20}$$

To finalise this section we now summarise the used notation in this section.

Table 4.1 Notations

Symbol	**Meaning**
$\mathbf{T}_n$	random selling time of the nth item of the product
$\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$	stochastic process of total sales up to time t for $t \geq 0$
Ψ	mean arrival function of the total sales process $\mathbf{S}$
$\mathbf{U}_n$	random usage time of the nth sold item having cdf $F_{\mathbf{U}}$
$\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$	stochastic process of total number of items in the market at time t for every $t \geq 0$ and $b(t) = \mathbb{E}(\mathbf{B}(t))$.
$\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$	stochastic process of total number of discarded items up to time t for every $t \geq 0$ and $d(t) = \mathbb{E}(\mathbf{D}(t))$.
$\mathbf{N}_n = \{\mathbf{N}_n(t) : t \geq 0\}$	stochastic process of the total number of failures of the nth sold item if this item after its selling time $\mathbf{T}_n$, is in the market during t units for every $t \geq 0$.
Φ	mean arrival function of the failure process $\mathbf{N}_n$
$\mathbf{R} = \{\mathbf{R}(t) : t \geq 0\}$	stochastic process of the number of returned defective items up to time t and $r(t) = \mathbb{E}(\mathbf{R}(t))$
w	length of the warranty period
$\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$	stochastic process of the total number of returned defective items under warranty up to time t and $r_w(t) = \mathbb{E}(\mathbf{R}_w(t))$

Under this general point process representation, knowing only the intensity measures of the sales and failure processes, it is only possible to compute expectations. If we want to derive more specific properties like higher order moments or the distribution function of the random variable $\mathbf{R}(t)$ and $\mathbf{R}_w(t)$ or related random variables, we need to impose that the cumulative sales process is a nonhomogeneous Poisson process. This will be discussed in the next section.

4.2 The Returned Defective Items Stochastic Process for Poisson Sales

In this second section we assume that the cumulative sales process is given by a nonhomogeneous Poisson process. For such sales processes we derive in the first subsection the two dimensional moment generating function of the random vector $(\mathbf{R}(t), \mathbf{R}(t+\Delta))$ for any $\Delta > 0$ under general conditions on the usage times and failure processes. This result for the two-dimensional moment generating function is used to show that the pmfs of the random variables $\mathbf{R}(t)$, $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ and $\mathbf{R}(\infty) - \mathbf{R}(t)$ have a compound Poisson distribution for any $t > 0$. In the second subsection we analyze the behaviour of the $k$tn moments, $k \in \mathbb{N}$ of the random variable $\mathbf{R}(t)$. To evaluate these moments and the pmf it is still needed to specify in more detail the failure process of each sold item and so in the third subsection, we consider the case that both the cumulative sales and the cumulative failure processes are given by nonhomogeneous Poisson processes.

4.2.1 On the Moment Generating Function for Nonhomogeneous Poisson Sales

To measure the randomness within the returned defective items stochastic process next to expectations one also needs to compute the variance and the pmf of the number of returned defective items before time t and other related random variables. To do so we impose the additional condition that the cumulative sales process is given by a nonhomogeneous Poisson process with a bounded Borel measurable arrival rate function ψ. This means that the mean arrival function of the sales process is given by $\Psi(t) = \int_0^t \psi(s)ds, t \geq 0$. We assume in this paper that the independent and identically distributed counting failure processes $\mathbf{N}_n$ of each individual nth sold item, $n = 1, ..., \mathbf{S}(\infty)$ is a non-explosive point process having mean arrival function Φ satisfying $\Phi(0) = 0$ and $\Phi(\infty) = \infty$ (Brémaud (1981)). This counting failure process additionally satisfies $\mathbb{E}(\mathbf{N}_1(\mathbf{U}_1))$ is finite and this means that the expected number of failures of a product during its usage time is finite.

To derive the pmf of the returned defective items stochastic process a time t and related random variables the following result for the conditional two-dimensional moment generating function of the random vector $(\mathbf{R}(t)), \mathbf{R}(t+\Delta))$ with $t \geq 0$ and $\Delta > 0$ is of importance. Before presenting this result we introduce the notation $(t)^+ := \max(0, t)$.

Theorem 4. *If the cumulative sales process* $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ *is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function* ψ *and the arrival moments* $\mathbf{T}_n, n \in \mathbb{N}$ *of the point process generating the sales process are independent of the independent and identically distributed random vectors* $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, *then for every* $k \in \mathbb{Z}_+, s_2 \geq 0$, $s_1 + s_2 \geq 0$ *and* $t \geq 0$ *and* $\Delta > 0$

$$\mathbb{E}\left(e^{-s_1\mathbf{R}(t) - s_2\mathbf{R}(t+\Delta)} \mid \mathbf{S}(t+\Delta) = k\right) = \mathbb{E}\left(e^{-s_1\mathbf{Z}(t,t+\Delta) - s_2\mathbf{Z}(t+\Delta,t+\Delta)}\right)^k \tag{4.21}$$

with the random variable $\mathbf{Z}(t_1,t_2)$*,* $t_2 \geq t_1$ *defined by*

$$\mathbf{Z}(t_1,t_2) := \mathbf{N}_1((t_1 - \mathbf{Y}_1(t_2))^+ \wedge \mathbf{U}_1). \tag{4.22}$$

The random variable $\mathbf{Y}_1(t_2)$ *listed in relation (4.22) has support* $[0,t_2]$ *and cdf*

$$F_{\mathbf{Y}_1(t_2)}(y) = \frac{\Psi(y)}{\Psi(t_2)}, 0 \leq y \leq t_2 \tag{4.23}$$

with $\Psi(t) = \int_0^t \psi(s)ds, t > 0$ *and this random variable* $\mathbf{Y}_1(t_2)$ *is independent of the stochastic process* $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$*.*

Proof. See Appendix. □

Since for a nonhomogeneous Poisson cumulative sales process $\mathbf{S}$, the random variable $\mathbf{S}(t+\Delta)$ has a Poisson distribution with parameter $\Psi(t+\Delta)$, the following result follows by the tower property for conditional expectations and Theorem 4.

Theorem 5. *If the cumulative sales process* $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ *is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function* ψ *and the arrival moments* $\mathbf{T}_n, n \in \mathbb{N}$ *of the point process generating the sales process are independent of the independent and identically distributed random vectors* $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$*, then the two-dimensional Laplace-Stieltjes transform of the random vector* $(\mathbf{R}(t), \mathbf{R}(t+\Delta))$ *with* $t \geq 0$ *and* $\Delta > 0$ *is given by*

$$\pi_{\mathbf{R}(t),\mathbf{R}(t+\Delta)}(s_1,s_2) = \mathbb{E}\left(e^{-s_1\mathbf{R}(t) - s_2\mathbf{R}(t+\Delta)}\right) = e^{-\Psi(t+\Delta)(1-m_\Delta(s_1,s_2))} \tag{4.24}$$

with

$$m_\Delta(s_1,s_2) := \mathbb{E}\left(e^{-s_1\mathbf{Z}(t,t+\Delta) - s_2\mathbf{Z}(t+\Delta,t+\Delta)}\right). \tag{4.25}$$

with the random vector $(\mathbf{Z}(t,t+\Delta), \mathbf{Z}(t+\Delta,t+\Delta))$ *defined in relation (4.22)*

If we introduce $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ meaning that the random vectors $\mathbf{X}$ and $\mathbf{Y}$ have the same joint cdf then it follows by Theorem 5 that the probabilistic interpretation of the random vector $(\mathbf{R}(t), \mathbf{R}(t+\Delta))$ is given by

$$(\mathbf{R}(t), \mathbf{R}(t+\Delta)) \stackrel{d}{=} \left(\sum_{n=1}^{\mathbf{M}} \mathbf{Z}_n(t,t+\Delta), \sum_{n=1}^{\mathbf{M}} \mathbf{Z}_n(t+\Delta,t+\Delta)\right) \tag{4.26}$$

In relation (4.26) the random variable $\mathbf{M}$ has a Poisson distribution with parameter $\Psi(t+\Delta)$ and is independent of the independent and identically distributed copies $(\mathbf{Z}_n(t,t+\Delta), \mathbf{Z}_n(t+\Delta,t+\Delta))$, $n \in \mathbb{N}$ of the random vector $(\mathbf{Z}(t,t+\Delta), \mathbf{Z}(t+\Delta,t+\Delta))$. Also, the random variable $\mathbf{Y}_1(t+\Delta)$ has the cdf listed in relation (4.23) and this random variable is independent of the stochastic process $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$. The same result as expressed in relation (4.26) also hold for the stochastic process $\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$ defined in relation (4.18) denoting the stochastic process of the total number of defective returns under warranty at each time t replacing the random variable $\mathbf{U}_n$ in relation (4.22) by $\mathbf{U}_n \wedge w$ and substituting this in (4.26). Applying Theorem 5 one can show the following result for Laplace-Stieltjes transform of the random variable $\mathbf{R}(t)$.

Lemma 11. *If the cumulative sales process $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function ψ and the arrival moments $\mathbf{T}_n, n \in \mathbb{N}$ of the point process generating the sales process are independent of the independent and identically distributed random vectors $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, then the random variable $\mathbf{R}(t)$ has a compound Poisson distribution and its Laplace-Stieltjes transform is given by*

$$\pi_{\mathbf{R}(t)}(s) := \mathbb{E}(e^{-s\mathbf{R}(t)}) = e^{-\Psi(t)(1-m_t(0,s))}, m_t(0,s) = \mathbb{E}\left(e^{-s\mathbf{Z}(t,t)}\right). \tag{4.27}$$

with the random variable $\mathbf{Z}(t,t)$ defined in relation (4.22).

Proof. Since $\mathbf{R}(0) = 0$ it follows by Theorem 5 for $t = 0$ and $\Delta > 0$ that for every $s \geq 0$

$$\pi_{\mathbf{R}(\Delta)}(s) := \mathbb{E}(e^{-s\mathbf{R}(\Delta)}) = e^{-\Psi(\Delta)(1-m_\Delta(0,s))}$$

This shows replacing Δ by t the desired result for the Laplace-Stieltjes transform and so (Steutel & Van Harn (2003) we obtain that the random variable $\mathbf{R}(t)$ has a compound Poisson distribution. $\square$

By relation (4.27) it follows that the random variable $\mathbf{R}(t)$ can be represented by

$$\mathbf{R}(t) \stackrel{d}{=} \sum_{n=1}^{\mathbf{M}} \mathbf{Z}_n(t,t) \tag{4.28}$$

with the random variable $\mathbf{M}$ having a Poisson distribution with parameter $\Psi(t)$ independent of the independent and identically distributed copies $\mathbf{Z}_n(t,t)$, $n \in \mathbb{N}$ of the random variable $\mathbf{Z}(t,t)$ defined in relation (4.22). Again the same result as derived for $\mathbf{R}(t)$ in Lemma 11 replacing the random variable $\mathbf{U}_1$ in relation (4.22) by $\mathbf{U}_1 \wedge w$ also holds for the stochastic process $\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$ denoting the stochastic process of the total number of defective returns under warranty up to each time t.

We will now derive some results for the random variable $\mathbf{R}(\infty) - \mathbf{R}(t)$ denoting the total number of returned defective items from time t untill the end of the life cycle. To derive these results we first observe using $\mathbf{S}(\infty)$ is finite with probability 1 that the next result follows from Theorem 4 by letting Δ go to infinity or copy for $\Delta = \infty$ the proof of Theorem 4 in the appendix.

Theorem 6. *If the cumulative sales process $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function ψ and $\Psi(\infty)$ is finite and the arrival moments $\mathbf{T}_n, n \in \mathbb{N}$ of the point process generating the sales process are independent of the independent and identically distributed random vectors $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, then for every $k \in \mathbb{Z}_+, s_2 \geq 0$, $s_1 + s_2 \geq 0$ and $t \geq 0$*

$$\mathbb{E}\left(e^{-s_1\mathbf{R}(t) - s_2\mathbf{R}(\infty)} \mid \mathbf{S}(\infty) = k\right) = \mathbb{E}\left(e^{-s_1\mathbf{Z}(t,\infty) - s_2\mathbf{Z}(\infty,\infty)}\right)^k \tag{4.29}$$

with $\mathbf{Z}(t,\infty) = \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ \wedge \mathbf{U}_1)$ and $\mathbf{Z}(\infty,\infty) = \mathbf{N}_1(\mathbf{U}_1)$. The random variable $\mathbf{Y}_1(\infty)$ in *relation (4.29) has support* $[0,\infty)$ *and cdf* $F_{\mathbf{Y}_1(\infty)}(y) = \frac{\Psi(y)}{\Psi(\infty)}, 0 \leq y < \infty$ *with* $\Psi(t) = \int_0^t \psi(s)ds, t > 0$ *and the random variable* $\mathbf{Y}_1(\infty)$ *is independent of the stochastic process* $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$.

Again by the tower property of conditional expectations and the random variable $\mathbf{S}(\infty)$ has a Poisson distribution with parameter $\Psi(\infty)$ the next result follows from Theorem 6.

Theorem 7. *If the cumulative sales process* $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ *is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function* ψ *and* $\Psi(\infty)$ *is finite and the arrival moments* $\mathbf{T}_n, n \in \mathbb{N}$ *of the point process generating the sales process are independent of the independent and identically distributed random vectors* $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, *then the two-dimensional Laplace-Stieltjes transform of the random vector* $(\mathbf{R}(t), \mathbf{R}(\infty))$ *with* $t \geq 0$ *and* $\Delta > 0$ *is given by*

$$\pi_{\mathbf{R}(t),\mathbf{R}(\infty)}(s_1, s_2) = \mathbb{E}\left(e^{-s_1\mathbf{R}(t) - s_2\mathbf{R}(\infty)}\right) = e^{-\Psi(\infty)(1 - m_\infty(s_1,s_2))} \tag{4.30}$$

with

$$m_\infty(s_1, s_2) = \mathbb{E}\left(e^{-s_1\mathbf{Z}(t,\infty) - s_2\mathbf{Z}(\infty,\infty)}\right) = \mathbb{E}\left(e^{-s_1\mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ \wedge \mathbf{U}_1) - s_2\mathbf{N}_1(\mathbf{U}_1)}\right). \tag{4.31}$$

The random variable $\mathbf{Y}_1(\infty)$ *in relation (4.31) has support* $[0,\infty)$ *and cdf* $F_{\mathbf{Y}_1(\infty)}(y) = \frac{\Psi(y)}{\Psi(\infty)}, 0 \leq y < \infty$ *with* $\Psi(t) = \int_0^t \psi(s)ds, t > 0$ *and the random variable* $\mathbf{Y}_1(\infty)$ *is independent of the stochastic process* $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$.

The same results also hold for the stochastic process $\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$ denoting the stochastic process of the total number of defective returns under warranty at each time t replacing the random variable $\mathbf{U}_1$ in relation (4.22) by $\mathbf{U}_1 \wedge w$. It is now easy to verify the following result.

Lemma 12. *If the cumulative sales process* $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ *is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function* ψ *and* $\Psi(\infty)$ *is finite and the arrival moments* $\mathbf{T}_n, n \in \mathbb{N}$ *of the point process generating the sales process are independent of the independent and identically distributed random vectors* $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, *then the random variable* $\mathbf{R}(\infty) - \mathbf{R}(t)$ *has a compound Poisson distribution and its Laplace-Stieltjes transform is given by*

$$\pi_{\mathbf{R}(\infty)-\mathbf{R}(t)}(s) := \mathbb{E}\left(e^{-s(\mathbf{R}(\infty) - \mathbf{R}(t))}\right) = e^{-\Psi(\infty)(1 - m_\infty(s,-s))}. \tag{4.32}$$

with

$$m_\infty(s, -s) = \mathbb{E}\left(e^{-s(\mathbf{Z}(\infty,\infty) - \mathbf{Z}(t,\infty))}\right) = \mathbb{E}\left(e^{-s(\mathbf{N}_1(\mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ \wedge \mathbf{U}_1))}\right) \tag{4.33}$$

The random variable $\mathbf{Y}_1(\infty)$ *in relation (4.33) has support* $[0,\infty)$ *and cdf* $F_{\mathbf{Y}_1(\infty)}(y) = \frac{\Psi(y)}{\Psi(\infty)}, 0 \leq y < \infty$ *with* $\Psi(t) = \int_0^t \psi(s)ds, t > 0$ *and the random variable* $\mathbf{Y}_1(\infty)$ *is independent of the stochastic process* $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$.

Proof. Apply Theorem 7 with $s_1 = -s$ and $s_2 = s$. □

By a similar argument as used before we obtain from Lemma (12) that the random variable $\mathbf{R}(\infty) -$

$\mathbf{R}(t)$ can be represented by

$$\mathbf{R}(\infty) - \mathbf{R}(t) \stackrel{d}{=} \sum_{n=1}^{\mathbf{M}} (\mathbf{Z}_n(\infty,\infty) - \mathbf{Z}_n(t,\infty)) \tag{4.34}$$

with the random variable $\mathbf{M}$ Poisson distributed with parameter $\Psi(\infty)$ and independent of the independent and identical distributed copies $\mathbf{Z}_n(\infty,\infty) - \mathbf{Z}_n(t,\infty)$ of the random variable $\mathbf{Z}(\infty,\infty) - \mathbf{Z}(t,\infty)$ listed in Theorem 6. As before the same result holds for the stochastic process $\mathbf{R}_w = \{\mathbf{R}_w(t) : t \geq 0\}$ replacing $\mathbf{U}_n$ by $\mathbf{U}_n \wedge w$ in relation (4.34).

Finally we derive in the next result the Laplace-Stieltjes transform of the random variable $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ for $t \geq 0$ and $\Delta > 0$. This result is of importance since the manufacturer is interested in the allocation of resources within a given time interval to handle the repair of defective items within this interval. Clearly, if the product is just released to the market, the number of returned defective items might be less than when the product is in the mature phase of its life cycle and so during different phases of the life cycle of a product, the allocation of resources will differ.

Lemma 13. *If the cumulative sales process $\mathbf{S} = \{\mathbf{S}(t) : t \geq 0\}$ is a nonhomogeneous Poisson process with a bounded Borel measurable intensity function ψ and the arrival moments $\mathbf{T}_n, n \in \mathbb{N}$ of the point process generating the sales process are independent of the independent and identically distributed random vectors $(\mathbf{N}_n, \mathbf{U}_n), n \in \mathbb{N}$, then the random variable $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ has a compound Poisson distribution and its Laplace-Stieltjes transform is given by*

$$\pi_{\mathbf{R}(t+\Delta)-\mathbf{R}(t)}(s) := \mathbb{E}(e^{-s(\mathbf{R}(t+\Delta)-\mathbf{R}(t))}) = e^{-\Psi(t+\Delta)(1-m_\Delta(s,-s))}. \tag{4.35}$$

with

$$\begin{aligned} m_\Delta(s,-s) &= \mathbb{E}\left(e^{-s(\mathbf{Z}(t+\Delta,t+\Delta)-\mathbf{Z}(t,t+\Delta))}\right) \\ &= \mathbb{E}\left(e^{-s(\mathbf{N}_1((t+\Delta-\mathbf{Y}_1(t+\Delta))\wedge \mathbf{U}_1)-\mathbf{N}_1((t-\mathbf{Y}_1(t+\Delta))^+\wedge \mathbf{U}_1))}\right) \end{aligned} \tag{4.36}$$

The random variable $\mathbf{Y}_1(t+\Delta)$ in relation (4.36) has support $[0, t+\Delta)$ and cdf $F_{\mathbf{Y}_1(t+\Delta)}(y) = \frac{\Psi(y)}{\Psi(\infty)}, 0 \leq y < t+\Delta$ with $\Psi(t) = \int_0^t \psi(s)ds, t > 0$ and the random variable $\mathbf{Y}_1(t+\Delta)$ is independent of the stochastic process $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$.

Proof. The result for the Laplace-Stieltjes transform follows from Theorem 5 substituting $s_1 = s$ and $s_2 = -s$. It is clear this is the Laplace-Stieltjes transform of a compound Poisson distribution. $\square$

From lemma 13 it follows for every $t \geq 0$ and $\Delta > 0$ that

$$\mathbf{R}(t+\Delta) - \mathbf{R}(t) \stackrel{d}{=} \sum_{n=1}^{\mathbf{M}} \mathbf{Z}_n(t+\Delta, t+\Delta) - \mathbf{Z}_n(t, t+\Delta) \tag{4.37}$$

In relation (4.37) the random variable $\mathbf{M}$ has a Poisson distribution with parameter $\Psi(t+\Delta)$ independent of the independent and identically distributed copies $\mathbf{Z}_n(t+\Delta, t+\Delta) - \mathbf{Z}_n(t, t+\Delta)$, $n \in \mathbb{N}$ of the random variables $\mathbf{Z}(t+\Delta, t+\Delta) - \mathbf{Z}(t, t+\Delta)$ listed in lemma 13. Again it is easy to adapt the results for $\mathbf{R}(w, t+\Delta) - \mathbf{R}(w, t)$ replacing $\mathbf{U}_n$ by $\mathbf{U}_n \wedge w$ in relation (4.37). In the next subsection, we will consider the moments of the random variable $\mathbf{R}(t)$ for a general failure process and usage times .

4.2.2 Higher Moments of the Total Number of Defective Items for Poisson sales

Using relation (4.27) and the known result that $\mathbb{E}(\mathbf{X}^k) = (-1)^k \pi_{\mathbf{X}}^{(k)}(0^+)$ for any nonnegative random variable $\mathbf{X}$ with $\pi_{\mathbf{X}}(s) := \mathbb{E}(e^{-s\mathbf{X}})$ and $\pi_{\mathbf{X}}(0^+) = \lim_{s\downarrow 0} \pi_{\mathbf{X}}(s)$, the proof of the next recurrence relations for moments of a compound Poisson distribution is standard applying Leibniz rule of the kth derivative of a product of two functions. For an alternative proof of this recurrence relation, see Corollary 2.5.3 of Ross (1983). Hence, we only mention the result.

Lemma 14. *If the functions $\mu_k : (0,\infty) \to (0,\infty)$ and $w_k : (0,\infty) \to (0,\infty)$,$k \in \mathbb{N}$ are given by*

$$\mu_k(t) := \mathbb{E}(\mathbf{R}^k(t)) \text{ and } w_k(t) := \mathbb{E}(\mathbf{N}_1^k((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1)) \tag{4.38}$$

and the conditions of Lemma 11 hold, then

$$\mu_k(t) = \Psi(t) \sum_{n=0}^{k-1} \binom{k-1}{n} \mu_n(t) w_{k-n}(t). \tag{4.39}$$

We need to calculate before applying this recurrence relation for a given t and every $m \leq k-1$ the value $w_m(t) := \mathbb{E}(\mathbf{N}_1^m((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1))$. By similar arguments as used in relation (4.10) using the assumption that the random usage time is independent of the failure process and the mean arrival function Ψ of the sales process has a continuous derivative ψ, we obtain

$$\Psi(t) w_m(t) = \int_0^t \mathbb{E}(\mathbf{N}_1^m(t-y))(1-F_{\mathbf{U}}(t-y))\Psi(dy) + \int_0^t \mathbb{E}(\mathbf{N}_1^m(u))\Psi(t-u)F_{\mathbf{U}}(du). \tag{4.40}$$

We will simplify this expression in the next subsection if we impose the additional assumption that the failure counting process $\mathbf{N}_1$ is a nonhomogeneous Poisson process. An example of such a process in reliability and maintenance is given by the minimal repair model. An immediate consequence of relation (4.39) is given by the next corollary. Again this corollary holds if the random vectors $(\mathbf{N}_n, \mathbf{U}_n)$, $n \in \mathbb{N}$ are independent and identically distributed and independent of the sales time $\mathbf{T}_n$, $n \in \mathbb{N}$.

Corollary 8. *If the conditions of Lemma 11 hold, then for every $t > 0$*

$$Var(\mathbf{R}(t)) = \Psi(t)\mathbb{E}(\mathbf{N}_1^2((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1))) \tag{4.41}$$

Proof. By relations (4.10) and 4.39 it follows that

$$\begin{aligned}
\mu_2(t) &= \Psi(t)\mathbb{E}(\mathbf{N}_1^2((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1)) + \Psi(t)\mathbb{E}(\mathbf{N}_1((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1))\mu_1(t) \\
&= \Psi(t)\mathbb{E}(\mathbf{N}_1^2((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1)) + \mu_1^2(t).
\end{aligned}$$

Since $\mathrm{Var}(\mathbf{R}(t)) = \mu_2(t) - \mu_1^2(t)$ the result follows. $\square$

One might also be interested in the dependence of the random variable $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ on the random variable $\mathbf{R}(t)$ by evaluating $\mathrm{Cov}(\mathbf{R}(t)), \mathbf{R}(t+\Delta) - \mathbf{R}(t))$ for $t > 0$ and $\Delta > 0$. It follows using Corollary 8 that

$$\begin{aligned}\mathrm{Cov}(\mathbf{R}(t), \mathbf{R}(t+\Delta) - \mathbf{R}(t)) &= \mathrm{Cov}(\mathbf{R}(t), \mathbf{R}(t+\Delta)) - \mathrm{Var}(\mathbf{R}(t)) \\ &= \mathrm{Cov}(\mathbf{R}(t), \mathbf{R}(t+\Delta)) - \Psi(t)\mathbb{E}(\mathbf{N}_1^2((t - \mathbf{Y}_1) \wedge \mathbf{U}_1)\end{aligned} \tag{4.42}$$

Applying Theorem 5 it is possible to compute $\mathrm{Cov}(\mathbf{R}(t)), \mathbf{R}(t+\Delta))$. By similar arguments relating joint moments to joint moment generating functions, we can show the following result.

Lemma 15. *It follows for every $t > 0$ and $\Delta > 0$ that*

$$\begin{aligned}\mathit{Cov}(\mathbf{R}(t), \mathbf{R}(t+\Delta)) &= \Psi(t+\Delta)\mathbb{E}(\mathbf{N}_1((t - \mathbf{Y}_1(t+\Delta))^+ \wedge \mathbf{U}_1)\mathbf{N}_1((t+\Delta - \mathbf{Y}_1(t+\Delta)) \wedge \mathbf{U}_1)) \\ &= \Psi(t+\Delta)\mathbb{E}(\mathbf{N}_1((t - \mathbf{Y}_1(t+\Delta)) \wedge \mathbf{U}_1)\mathbf{N}_1((t+\Delta - \mathbf{Y}_1(t+\Delta)) \wedge \mathbf{U}_1)\mathbf{1}_{\{\mathbf{Y}_1(t+\Delta) \leq t\}})\end{aligned}$$

with $\mathbf{Y}_1(t+\Delta)$ independent of the failure counting process $\mathbf{N}_1$ having cdf

$$\mathbb{P}(\mathbf{Y}_1(t+\Delta) \leq y) = \frac{\Psi(y)}{\Psi(t+\Delta)}, 0 \leq y \leq t+\Delta.$$

Lemma 15 and relation (4.42) imply that in general $\mathrm{Cov}(\mathbf{R}(t), \mathbf{R}(t+\Delta) - \mathbf{R}(t))$ is not equal to zero and so the stochastic process $\mathbf{R}$ does not have independent increments. This is expressed in the following corollary.

Corollary 9. *The stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \geq 0\}$ does not have independent increments.*

This is to be expected since within the number of defective items returning within the time interval $[t, t+\Delta]$, there are also items being sold before time t and are still in use and break down in $[t, t+\Delta]$. Hence, the random variable $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ also depends on the installed base process $\mathbf{B}(t)$ at time t, and this information is hidden in the random variable $\mathbf{R}(t)$. Therefore, the random variable $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ is not independent of the random variable $\mathbf{R}(t)$. Depending on the used filtration and the properties of the failure processes, we might be able to derive some more structural properties of the stochastic process of returned and defective items. This will be the topic of future research. Next we will derive recurrence relations for evaluating the pmfs of $\mathbf{R}(t)$, $\mathbf{R}(\infty) - \mathbf{R}(t)$ and $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ still for general failure processes usage times.

4.2.3 On the pmf of the Total Number of Defective Items for Poisson Sales

In this subsection we will derive recurrence relations for evaluating the pmf of the random variables $\mathbf{R}(t)$, $\mathbf{R}(\infty) - \mathbf{R}(t)$ and $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ for any $\Delta > 0$. We start at first with the easiest case of evaluating the pmf of the random variable $\mathbf{R}(t)$. Since by Lemma 11 the random variable $\mathbf{R}(t)$ has a compound Poisson distribution, we can derive a recurrence relation for the pmf of this random variable. This information is more detailed than only moment information. This recurrence relation is called Adelson's recursion relation (see Theorem 1.2.6 of Tijms (1994)) and is listed in the next lemma. An alternative proof of Adelsons recurrence relation is given in Corollary 2.5.4 of Ross (1983).

Lemma 16. *If the conditions of Lemma 11 hold and for every $k \in \mathbb{Z}_+$ the functions $p_k : (0,\infty) \to [0,1]$ are defined by*

$$p_k(t) := \mathbb{P}(\mathbf{Z}(t,t) = k) = \mathbb{P}(\mathbf{N}_1((t - \mathbf{Y}_1(t))^+ \wedge \mathbf{U}_1) = k), \tag{4.43}$$

with the random variable $\mathbf{Y}$ concentrated on $(0,t)$ having cdf $F(y) = \frac{\Psi(y)}{\Psi(t)}, 0 \leq y < t$ and independent of the stochastic process $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$ then it follows that $\mathbb{P}(\mathbf{R}(t) = 0) = e^{-\Psi(t)(1-p_0(t))}$ and for every $k \in \mathbb{N}$

$$k\mathbb{P}(\mathbf{R}(t) = k) = \Psi(t)\sum_{j=0}^{k-1}(k-j)\mathbb{P}(\mathbf{R}(t) = j)p_{k-j}(t). \tag{4.44}$$

In relation (4.44) we need to compute the pmf of the random variable $\mathbf{N}_1((t - \mathbf{Y}_1)^+ \wedge \mathbf{U}_1)$. This is possible for a minimal repair failure counting process and the analysis will be presented in section 4.2.4. Notice, if we consider the return process $\mathbf{R}_w$ of defective items under warranty, the same recurrence relation as presented in Lemma 18 holds with the random variable $\mathbf{U}_1$ replaced by the random variable $\mathbf{U}_1 \wedge w$. Using a similar approach we derive a recurrence relation for the pmf of the random variable $\mathbf{R}(\infty) - \mathbf{R}(t)$. To compute the cdf of the number of returned and defective items in the period of $[t,\infty)$ for any $t > 0$ we again use applying Lemma 12 that the random variable $\mathbf{R}(\infty) - \mathbf{R}(t)$ has a compound Poisson cdf As in the previous subsection one can apply Adelsons recurrence relation and so the next result holds.

Lemma 17. *If the conditions of Lemma 12 hold and for every $k \in \mathbb{Z}_+$ the functions $v_k : (0,\infty) \to [0,1]$ are given by*

$$v_k(t) := \mathbb{P}(\mathbf{Z}(\infty,\infty) - \mathbf{Z}(t,\infty) = k) = \mathbb{P}(\mathbf{N}_1(\mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ \wedge \mathbf{U}_1) = k) \tag{4.45}$$

then $\mathbb{P}(\mathbf{R}(t) = 0) = e^{-\Psi(\infty)(1-v_0(t))}$ and for every $k \in \mathbb{N}$

$$k\mathbb{P}(\mathbf{R}(\infty) - \mathbf{R}(t) = k) = \Psi(\infty)\sum_{j=0}^{k-1}(k-j)\mathbb{P}(\mathbf{R}(\infty) - \mathbf{R}(t) = j)v_{k-j}(t). \tag{4.46}$$

In relation (4.46) we need to compute the pmf of the random variable $\mathbf{N}_1(\mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ \wedge \mathbf{U}_1)$. This will be done for minimal repair failure counting processes and its analysis is presented in Subsection 4.2.4. Finally we list in the following result the recurrence relation for the pmf of the random variable $\mathbf{R}(t + \Delta) - \mathbf{R}(t)$.

To compute the cdf of the number of returned and defective items in the period of $[t, t + \Delta]$ we again use applying Lemma 13 that the random variable $\mathbf{R}(t + \Delta) - \mathbf{R}(t)$ for $t > 0$ and $\Delta > 0$ has a compound Poisson distribution.

Lemma 18. *If the conditions of Lemma 13 hold and for every $k \in \mathbb{Z}_+$ the functions $v_k : (0,\infty) \times (0,\infty) \to [0,1]$ are given by*

$$v_k(\Delta,t) = \mathbb{P}(\mathbf{N}_1((t+\Delta - \mathbf{Y}_1(t+\Delta)) \wedge \mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(t+\Delta))^+ \wedge \mathbf{U}_1) = k) \tag{4.47}$$

with the random variable $\mathbf{Y}_1(t+\Delta)$ having the cdf

$$\mathbb{P}(\mathbf{Y}_1(t+\Delta) \leq y) = \frac{\Psi(y)}{\Psi(t+\Delta)}, 0 \leq y \leq t+\Delta$$

then $\mathbb{P}(\mathbf{R}(t) = 0) = e^{-\Psi(t+\Delta)(1-w_0(\Delta,t))}$ and for every $k \in \mathbb{N}$

$$k\mathbb{P}(\mathbf{R}(t+\Delta) - \mathbf{R}(t) = k) = \Psi(t+\Delta)\sum_{j=0}^{k-1}(k-j)\mathbb{P}(\mathbf{R}(t+\Delta) - \mathbf{R}(t) = j)v_{k-j}(\Delta,t). \tag{4.48}$$

To apply the above recurrence relation, we need to be able to compute $w_k(\Delta,t)$ for every $k \in \mathbb{Z}_+$. This depends on the relation between $\mathbf{U}_1$ and the failure counting process $\mathbf{N}_1$ and the specific cdf of the random usage time variable. Observe we already know that $\mathbf{U}_1$ is independent of $\mathbf{Y}_1(t+\Delta)$. In the next subsection we will consider these special cases.

4.2.4 Nonhomogeneous Poisson Failure Processes

If the failure process of each sold item is given by a minimal repair model it is well known (Aven & Jensen (2000)) that the stochastic failure counting process $\mathbf{N}_1$ is a nonhomogeneous Poisson process with arrival intensity function given by the failure rate function $r(t) = \frac{f(t)}{1-F(t)}$ with F the cdf of the random time $\mathbf{X}_1$ to the first failure and f its density. We assume for simplicity that $F_{\mathbf{X}}(0) = 0, F_{\mathbf{X}}(\infty) = 1$ and $F_{\mathbf{X}}(t) < 1$ for every $t > 0$. This means with $\Phi(t) := \int_0^t r(s)ds$ that

$$\mathbb{P}(\mathbf{N}_1(t) = k) = e^{-\Phi(t)}\frac{\Phi(t)^k}{k!}, k \in \mathbb{Z}_+ \tag{4.49}$$

By the above representation of Φ for the minimal repair model, it follows for $F(0) = 0, F(\infty) = 1$ and $F(t) < 1$ for every $t > 0$ that

$$\Phi(t) = -\ln(1 - F_{\mathbf{X}}(t)). \tag{4.50}$$

To compute the expression $\mathbb{E}(\mathbf{N}_1^k((t - \mathbf{Y}_1(t)) \wedge \mathbf{U}_1))$ used in Lemma 14 and to evaluate the recurrence relation for higher moments of the defective items return process, we first introduce the Stirling numbers $S(k,j)$, $k \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$ of the second kind (Kesidis, Konstantopoulos & Zazanis (2018)). These numbers represent the number of different ways to partition a set of k objects into j nonempty sets. By this interpretation, we obtain $S(k,j) = 0$ for $j > k, S(0,j) = 0$ for $j \in \mathbb{N}$, $S(k,0) = 0$ for $k \in \mathbb{N}$

and $S(0,0)=1$. Also by its interpretation, we obtain the recurrence relation

$$S(k,j)=S(k-1,j-1)+jS(k-1,j) \tag{4.51}$$

It is known (see formula (1.246) of Johnson, Kemp & Kotz (2005) or Riordan (1937) that

$$\mathbb{E}(\mathbf{X}^k)=\sum_{j=1}^k S(k,j)\beta^k$$

for any Poisson distributed random variable having parameter β and $k\in\mathbb{N}$. This shows for the minimal repair failure stochastic counting process $\mathbf{N}_1$ that its kth moment, $k\in\mathbb{N}$ equals

$$\mathbb{E}(\mathbf{N}_1^k(u))=\sum_{j=1}^k S(k,j)\Phi(u)^k \tag{4.52}$$

with Φ listed in relation (4.50). Using relation (4.50) and (4.52) one can easily show the following result.

Lemma 19. *If for every $n\in\mathbb{N}$ the random usage time $\mathbf{U}_n$ is independent of the minimal failure process $\mathbf{N}_n$ having mean arrival function Φ, then it follows for every $k\in\mathbb{N}$ that*

$$\begin{aligned} w_k(t):=\mathbb{E}(\mathbf{N}_1^k((t-\mathbf{Y}_1)\wedge\mathbf{U}_1) &= \sum_{j=1}^k S(k,j)\mathbb{E}(\Phi^j((t-\mathbf{Y}_1(t))\wedge\mathbf{U}_1) \\ &= -\sum_{j=1}^k S(k,j)\mathbb{E}(\ln(1-F_{\mathbf{X}}((t-\mathbf{Y}_1(t))\wedge\mathbf{U}_1))^j) \end{aligned} \tag{4.53}$$

Observe the value $\Psi(t)w_k(t)$ for $k=1$ yields by relation (4.15) the expected number of returned items up to time t, while for $k=2$ we obtain by Corollary 8 the variance of the random variable $\mathbf{R}(t)$. To write out $\mathbb{E}(\Phi^j((t-\mathbf{Y}_1(t))\wedge\mathbf{U}_1)$ as an integral, we observe introducing the random variable $\mathbf{Z}_t:=(t-\mathbf{Y}_1(t))\wedge\mathbf{U}_1$ that by the independence of $\mathbf{U}_1$ and $\mathbf{Y}_1(t)$ it follows that

$$1-F_t(z):=\mathbb{P}(\mathbf{Z}_t>z)=\frac{\Psi(t-z)}{\Psi(t)}(1-F_{\mathbf{U}}(z)),$$

for every $z\leq t$ and $1-F_t(z)=0$ for every $z>t$. This shows for Φ having the failure rate function $r_{\mathbf{X}}$ as a derivative, and using $\Phi^j(z)=j\int_0^z r_{\mathbf{X}}(u)\Phi^{j-1}(u)du$ that by partial integration

$$\mathbb{E}(\Phi^j((t-\mathbf{Y}_1(t))\wedge\mathbf{U}_1)=\int_0^t\Phi^j(z)F_t(dz)=\frac{j}{\Psi(t)}\int_0^t r_{\mathbf{X}}(u)\Phi^{j-1}(u)(1-F_{\mathbf{U}}(u))\Psi(t-u)du. \tag{4.54}$$

This shows by Lemma 19 that

$$\Psi(t)w_k(t)=\sum_{j=1}^k jS(k,j)\int_0^t r_{\mathbf{X}}(u)\Phi^{j-1}(u)(1-F_{\mathbf{U}}(u))\Psi(t-u)du. \tag{4.55}$$

If we have a black box to evaluate the function φ, Ψ and $F_{\mathbf{U}}$, we can numerically compute the above expression. In the special case that the item will be used for an infinite amount of time in the market and so the random usage time $\mathbf{U}_1$ equals infinity with probability 1, the expression in relation (4.55) reduces to

$$\Psi(t)w_k(t) = \sum_{j=1}^{k} S(k,j) \int_0^t \Phi^j(t-y)\Psi(dy). \tag{4.56}$$

If in a minimal repair model, the random variable $\mathbf{X}_1$ until the first failure has a Weibull distribution with shape parameter $\gamma > 0$ and scale parameter $\delta > 0$ then the cdf $F_{\mathbf{X}}$ is given by $F_{\mathbf{X}}(x) = 1 - e^{-(\delta x)^\gamma}$. This shows by relation (4.50) that for every $t > 0$

$$\Phi(t) = \delta^\gamma t^\gamma. \tag{4.57}$$

Hence, it follows by relation (4.55) for a Weibull minimal repair model with shape parameter α and scale parameter $\delta > 0$ that

$$\Psi(t)w_k(t) = \alpha \sum_{j=1}^{k} jS(k,j)\delta^{\gamma j} \int_0^t u^{\alpha j-1}\Psi(t-u)(1-F_{\mathbf{U}}(u))du. \tag{4.58}$$

This shows that for a Brockhoff or Bass sales model with Ψ given by a particular function, random usage time having a gamma distribution and the failure process given by a Weibull minimal repair model that we can evaluate by a numerical integration technique the kth moment of the returned and defective items process $\mathbf{R}$.

To evaluate the pmf of the random variable $\mathbf{R}(t)$ we need to compute before applying the recurrence relation in Lemma 16 for every $k \in \mathbb{Z}_+$ the probability

$$p_k(t) := \mathbb{P}(\mathbf{Z}(t,t) = k) = \mathbb{P}(\mathbf{N}_1((t-\mathbf{Y}_1(t))^+ \wedge \mathbf{U}_1) = k) = \mathbb{P}(\mathbf{N}_1((t-\mathbf{Y}_1(t)) \wedge \mathbf{U}_1) = k) \tag{4.59}$$

with the random variable $\mathbf{Y}_1(t)$ concentrated on $(0,t)$ having cdf $F_{\mathbf{Y}_1(t)}(y) = \frac{\Psi(y)}{\Psi(t)}, 0 \leq y < t$ and this random variable is independent of the stochastic process $\{\mathbf{N}_1(s \wedge \mathbf{U}_1) : s \geq 0\}$. This is presented in the next lemma.

Lemma 20. *If the conditions of Lemma 13 are satisfied and the failure process is a minimal repair process with mean arrival rate function $\Phi(t) = -\ln(1 - F_{\mathbf{X}}(t))$ and $F_{\mathbf{X}}$ denotes the cdf of the first failure time in this minimal repair process and the usage time is independent of the failure process, then for every $k \in \mathbb{Z}_+$ and $t \geq 0$, the function p_k defined in relation (4.43) is given by*

$$p_k(t) = I_k(t) + J_k(t) \tag{4.60}$$

with

$$I_k(t) = \frac{1}{\Psi(t)k!} \int_0^t e^{-\Phi(t-y)}\Phi^k(t-y)(1-F_{\mathbf{U}}(t-y))\Psi(dy) \tag{4.61}$$

and

$$J_k(t) = \frac{1}{\Psi(t)k!} \int_0^t e^{-\Phi(u)}\Phi^k(u)\Psi(t-u)F_{\mathbf{U}}(du). \tag{4.62}$$

Proof. Introducing for every $t \geq 0$ the random variable $\mathbf{Z}_t = (t - \mathbf{Y}_1(t)) \wedge \mathbf{U}_1$ we obtain by the independence of the random vector $(\mathbf{Y}_1(t), \mathbf{U}_1)$ and the failure process $\mathbf{N}_1$ and $\mathbf{N}_1(u)$ is a Poisson distributed random variable with parameter $\Phi(u)$ that by conditioning on $\mathbf{Z}_t$

(4.63)

$$\begin{aligned} p_k(t) &= \tfrac{1}{k!}\mathbb{E}(e^{-\Phi(\mathbf{Z}_t)}\Phi^k(\mathbf{Z}_t)) \\ &= \tfrac{1}{k!}\mathbb{E}(e^{-\Phi(t-\mathbf{Y}_1(t))}\Phi^k(t-\mathbf{Y}_1(t))1_{\{\mathbf{U}_1+\mathbf{Y}_1(t)>t\}}) + \tfrac{1}{k!}\mathbb{E}(e^{-\Phi(\mathbf{U}_1)}\Phi^k(\mathbf{U}_1)1_{\{\mathbf{U}_1+\mathbf{Y}_1(t)\le t\}}). \end{aligned}$$

Considering the first term in relation (4.63), it follows by conditioning on $\mathbf{Y}_1(t)$ having cdf $\frac{\Psi(y)}{\Psi(t)}, y \le t$ and $\mathbf{U}_1$ independent of $\mathbf{Y}_1(t)$ that

(4.64)

$$\begin{aligned} \mathbb{E}(e^{-\Phi(t-\mathbf{Y}_1(t))}\Phi^k(t-\mathbf{Y}_1(t))\mathbf{1}_{\{\mathbf{U}_1+\mathbf{Y}_1(t)>t\}}) &= \tfrac{1}{\Psi(t)}\int_0^t \mathbb{E}(e^{-\Phi(t-y)}\Phi^k(t-y)1_{\{\mathbf{U}_1+y>t\}})\Psi(dy) \\ &= \tfrac{1}{\Psi(t)}\int_0^t e^{-\Phi(t-y)}\Phi^k(t-y)(1-F_{\mathbf{U}}(t-y))\Psi(dy). \end{aligned}$$

Also, for the second term in relation (4.63), we obtain conditioning on $\mathbf{U}_1$ and $\mathbf{U}_1$ is independent of $\mathbf{Y}_1(t)$ that

$$\mathbb{E}(e^{-\Phi(\mathbf{U}_1)}\Phi^k(\mathbf{U}_1)\mathbf{1}_{\{\mathbf{U}_1+\mathbf{Y}_1(t)\le t\}}) = \frac{1}{\Psi(t)}\int_0^t e^{-\Phi(u)}\Phi^k(u)\Psi(t-u)F_{\mathbf{U}}(du), \tag{4.65}$$

and substituting relations (4.64) and (4.65) in relation (4.63) shows the desired result. □

Despite the simplicity of the above formulas for $p_k(t)$, it seems difficult to simplify these formulas for important classes of usage time cdfs such as gamma distributions. Due to this, we use in our computational section the trapezoidal rule (Davis & Rabinowitz (1984)) to evaluate the integrals representing $I_k(t)$ and $J_k(t)$. If we do not include the usage time of an item in our model and hence set $\mathbf{U}_1$ equal to infinity with probability 1, we obtain for every $k \in \mathbb{Z}_+$ the simpler expression

$$\mathbb{P}(\mathbf{N}_1(t-\mathbf{Y}_1) = k) = \frac{1}{\Psi(t)k!}\int_0^t e^{-\Phi(t-y)}\Phi^k(t-y)\Psi(dy). \tag{4.66}$$

If the function Ψ has a derivative ψ and the cdf $F_{\mathbf{U}}$ of the usage time random variable a density $f_{\mathbf{U}}$, then by Lemma 20 we obtain

$$p_k(t) = \frac{1}{\Psi(t)k!}\int_0^t e^{-\Phi(u)}\Phi^k(u)((1-F_{\mathbf{U}}(u))\psi(t-u) + \Psi(t-u)f_{\mathbf{U}}(u))du. \tag{4.67}$$

To evaluate the pmf of $\mathbf{R}(\infty) - \mathbf{R}(t)$ we need to compute before applying the recurrence relation in Lemma 17 for every $k \in \mathbb{Z}_+$ the probability $v_k(t)$ defined in relation (4.43). Knowing the pmf of the random variable $\mathbf{R}(\infty) - \mathbf{R}(t)$ helps the manufacturer decide on the last-time buy size.

Lemma 21. *If the conditions of Lemma 12 are satisfied and the failure process is a minimal repair process with mean arrival rate function $\Phi(t) = -\ln(1-F_{\mathbf{X}}(t))$ and $F_{\mathbf{X}}$ denotes the cdf of the first failure time in this minimal repair process, then for every $k \in \mathbb{Z}_+$ and $t \ge 0$ it follows that the function $v_k(t)$ defined in relation (4.45) is given by*

$$v_k(t) = I_k(t) + J_k(t) + 1_{\{0\}}(k)\mathbb{P}(\mathbf{U}_1 + \mathbf{Y}_1(\infty) \le t) \tag{4.68}$$

with

$$I_k(t) = \frac{\Psi(\infty) - \Psi(t)}{k!\Psi(\infty)} \int_0^\infty e^{-\Phi(u)} \Phi(u)^k f_{\mathbf{U}}(u) du \tag{4.69}$$

and

$$J_k(t) = \frac{1}{\Psi(\infty)} \int_0^\infty \int_{t-u}^t e^{-(\Phi(u)-\Phi(t-y))} \frac{(\Phi(u) - \Phi(t-y))^k}{k!} \psi(y) dy f_{\mathbf{U}}(u) du \tag{4.70}$$

Proof. It follows by the definition of $v_k(t)$ in relation (4.45) that

$$v_k(t) = \mathbb{P}(\mathbf{N}_1(\mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+) = k, \mathbf{U}_1 + \mathbf{Y}_1(\infty) > t) + 1_{\{0\}}(k)\mathbb{P}(\mathbf{U}_1 + \mathbf{Y}_1(\infty) \leq t) \tag{4.71}$$

To analyse the first term in relation (4.71) we observe by conditioning on the independent random variables $\mathbf{U}_1$ and $\mathbf{Y}_1(\infty)$ and using that the failure counting process $\mathbf{N}_1$ is independent of these random variables that

$$\begin{aligned}
& \mathbb{P}(\mathbf{N}_1(\mathbf{U}_1) - \mathbf{N}_1((t - \mathbf{Y}_1(\infty))^+ = k, \mathbf{U}_1 + \mathbf{Y}_1(\infty) > t) \\
= & \tfrac{1}{\Psi(\infty)} \int_0^\infty \int_{t-u}^\infty \mathbb{P}(\mathbf{N}_1(u) - \mathbf{N}_1((t-y)^+) = k) \psi(y) f_{\mathbf{U}}(u) dy du \\
= & \begin{cases} \tfrac{1}{\Psi(\infty)} \int_0^\infty \int_{t-u}^t \mathbb{P}(\mathbf{N}_1(u) - \mathbf{N}_1((t-y)^+) = k) \psi(y) f_{\mathbf{U}}(u) dy du \\ + \tfrac{1}{\Psi(\infty)} \int_0^\infty \int_t^\infty \mathbb{P}(\mathbf{N}_1(u) - \mathbf{N}_1((t-y)^+) = k) \psi(y) f_{\mathbf{U}}(u) dy du \end{cases} \\
= & \begin{cases} \tfrac{1}{\Psi(\infty)k!} \int_0^\infty \int_{t-u}^t e^{-(\Phi(u)-\Phi(t-y))} (\Phi(u) - \Phi(t-y))^k \psi(y) f_{\mathbf{U}}(u) dy du \\ + \tfrac{1}{\Psi(\infty)} \int_0^\infty \int_t^\infty \mathbb{P}(\mathbf{N}_1(u) = k) \psi(y) f_{\mathbf{U}}(u) dy du \end{cases} \\
= & \begin{cases} \tfrac{1}{\Psi(\infty)k!} \int_0^\infty \int_{t-u}^t e^{-(\Phi(u)-\Phi(t-y))} (\Phi(u) - \Phi(t-y))^k \psi(y) f_{\mathbf{U}}(u) dy du \\ + \tfrac{\Psi(\infty)-\Psi(t)}{k!\Psi(\infty)} \int_0^\infty e^{-\Phi(u)} \Phi(u)^k f_{\mathbf{U}}(u) du \end{cases}
\end{aligned} \tag{4.72}$$

Combing the above relations shows the desired result. □

Using the two-dimensional trapezoidal rule one can evaluate the above integrals. Finally we show in the next subsection how to numerical evaluate the value $v_k(\Delta, t)$ defined in relation (4.47).

If we also like to evaluate the pmf of $\mathbf{R}(t+\Delta) - \mathbf{R}(t)$ for $t \geq 0$ and $\Delta > 0$ given by the recurrence relation given in relation (4.48), we need the following generalisation of Lemma 20 and show a similar result for $v_k(\Delta, t)$. Before mentioning this result let us introduce the function $\Phi_\Delta : (0, \infty) \to (0, \infty)$ given by

$$\Phi_\Delta(t) = \Phi(t + \Delta) - \Phi(t), \tag{4.73}$$

and for every set $A \subseteq \mathbb{R}$ the indicator function $1_A : \mathbb{Z}_+ \to \{0,1\}$ of the set A given by

$$1_A(t) = \begin{cases} 1 & \text{if } k \in A \\ \\ 0 & \text{if } k \notin A. \end{cases} \tag{4.74}$$

One can now show the following result.

Lemma 22. *If the failure process is a minimal repair process with mean arrival rate function* $\Phi(t) = -\ln(1 - F_{\mathbf{X}}(t))$ *and* $F_{\mathbf{X}}$ *denotes the cdf of the first failure time in this minimal repair process, then for every* $k \in \mathbb{Z}_+$ *and* $t \geq 0$ *it follows that the function* $v_k(\Delta, .)$ *defined in relation (4.47) equals*

$$v_k(\Delta,t) = I_k^{(1)}(\Delta,t) + I_k^{(2)}(\Delta,t) + J_k^{(1)}(\Delta,t) + J_k^{(2)}(\Delta,t) + 1_{\{0\}}(k)\mathbb{P}(\mathbf{U}_1 + \mathbf{Y}_1(t+\Delta) \leq t),$$

with

$$I_k^{(1)}(\Delta,t) = \frac{1}{\Psi(t+\Delta)k!} \int_0^t e^{-\Phi_\Delta(t-y)} \Phi_\Delta^k(t-y)(1 - F_{\mathbf{U}}(t+\Delta-y))\Psi(dy),$$

and

$$I_k^{(2)}(\Delta,t) = \frac{1}{\Psi(t+\Delta)k!} \int_t^{t+\Delta} e^{-\Phi(t+\Delta-y)} \Phi^k(t+\Delta-y)(1 - F_{\mathbf{U}}(t+\Delta-y))\Psi(dy),$$

and

$$J_k^{(1)}(\Delta,t) = \frac{1}{\Psi(t+\Delta)k!} \int_0^t \int_{t-y}^{t+\Delta-y} e^{-(\Phi(u)-\Phi(t-y))}(\Phi(u) - \Phi(t-y))^k F_{\mathbf{U}}(du)\Psi(dy),$$

and

$$J_k^{(2)}(\Delta,t) = \frac{1}{\Psi(t+\Delta)k!} \int_t^{t+\Delta} \int_{t-y}^{t+\Delta-y} e^{-\Phi(u)} \Phi^k(u) F_{\mathbf{U}}(du)\Psi(dy).$$

Proof. Introducing for every $t \geq 0$ the random variables

$$\mathbf{V}_t = \Phi((t+\Delta-\mathbf{Y}_1) \wedge \mathbf{U}_1) - \Phi((t-\mathbf{Y}_1)^+ \wedge \mathbf{U}_1)$$

it follows by the independence of the random vector $(\mathbf{Y}_1, \mathbf{U}_1)$ and the failure process $\mathbf{N}_1$ and $\mathbf{N}_1(s \wedge u) - \mathbf{N}_1(s^+ \wedge u)$ has a Poisson cdf with parameter $\Phi(s \wedge u) - \Phi(s^+ \wedge u)$ that

$$p_k(\Delta,t) = \frac{1}{k!}\mathbb{E}(e^{-\mathbf{V}_t}\mathbf{V}_t^k).$$

This shows introducing for every $t \geq 0$ the random variables

$$\mathbf{A}_t := \Phi((t+\Delta-\mathbf{Y}_1)) - \Phi((t-\mathbf{Y}_1)^+)$$

and

$$\mathbf{B}_t := \Phi(\mathbf{U}_1) - \Phi((t-\mathbf{Y}_1)^+)$$

that

$$\mathbb{E}(e^{-\mathbf{V}_t}\mathbf{V}_t^k) = \mathbb{E}(e^{-\mathbf{A}_t}\mathbf{A}_t^k 1_{\{\mathbf{U}_1+\mathbf{Y}_1 > t+\Delta\}}) + \mathbb{E}(e^{-\mathbf{B}_t}\mathbf{B}_t^k 1_{\{t < \mathbf{U}_1+\mathbf{Y}_1 \leq t+\Delta\}}) + 1_{\{0\}}(k)\mathbb{P}(\mathbf{U}_1+\mathbf{Y}_1 \leq t) \tag{4.75}$$

By conditioning on $\mathbf{Y}_1$, the first term in relation (4.75) reduces to

$$(4.76)\quad \mathbb{E}(e^{-\mathbf{A}_t}\mathbf{A}_t^k 1_{\{\mathbf{U}_1+\mathbf{Y}_1>t+\Delta\}}) = \begin{cases} \int_0^t e^{-\Phi_\Delta(t-y)}\Phi_\Delta^k(t-y)(1-F_{\mathbf{U}}(t+\Delta-y))\Psi(dy) \\ +\int_t^{t+\Delta} e^{-\Phi(t+\Delta-y)}\Phi^k(t+\Delta-y)(1-F_U(t+\Delta-y))\Psi(dy). \end{cases}$$

Also, for the second term in relation (4.75), we obtain again conditioning on $\mathbf{Y}_1$ that
(4.77)

$$\mathbb{E}(e^{-\mathbf{B}_t}\mathbf{B}_t^k 1_{\{t<\mathbf{U}_1+\mathbf{Y}_1\leq t+\Delta\}}) = \begin{cases} \int_0^t \mathbb{E}(e^{-\Phi(\mathbf{U}_1)-\Phi(t-y))}(\Phi(\mathbf{U}_1)-\Phi(t-y))^k 1_{\{t-y<\mathbf{U}_1\leq t+\Delta-y\}})\Psi(dy) \\ +\int_t^{t+\Delta} \mathbb{E}(e^{-\Phi(\mathbf{U}_1)}\Phi^k(\mathbf{U}_1)^k 1_{\{t-y<\mathbf{U}_1\leq t+\Delta-y\}})\Psi(dy) \end{cases}$$

It also follows that

$$\mathbb{E}(e^{-\Phi(\mathbf{U}_1)-\Phi(t-y))}(\Phi(\mathbf{U}_1)-\Phi(t-y))^k 1_{\{t-y<\mathbf{U}_1\leq t+\Delta-y\}}) = \int_{t-y}^{t-y+\Delta} e^{-(\Phi(u)-\Phi(t-y))}(\Phi(u)-\Phi(t-y))^k dF_{\mathbf{U}}(u)$$

and

$$\mathbb{E}(e^{-\Phi(\mathbf{U}_1)}\Phi^k(\mathbf{U}_1)1_{\{t-y<\mathbf{U}_1\leq t+\Delta-y\}}) = \int_{t-y}^{t+\Delta-y} e^{-\Phi(u)}\Phi^k(u)F_{\mathbf{U}}(du)$$

and substituting this in relation (4.77) and using relations (4.75) and (4.76) shows the desired result.
□

Again we evaluate in our computational section the above integrals by the two-dimensional trapezoidal rule. Unfortunately, even for the most simplest cases, it seems that the two-dimensional integrals cannot be simplified. As before, if the usage time $\mathbf{U}_1$ is excluded from the model and hence set to ∞ with probability 1, these integrals simplify. In particular, we obtain
(4.78)

$$\mathbb{P}(\mathbf{N}_1(t+\Delta-\mathbf{Y}_1(t+\Delta))-\mathbf{N}_1((t-\mathbf{Y}_1(t+\Delta))^+)=k) = \int_0^{t+\Delta}\mathbb{P}(\mathbf{N}_1(t+\Delta-y)-\mathbf{N}_1((t-y)^+)=k)\Psi(dy)$$

and

$$(4.79)\quad \mathbb{P}(\mathbf{N}_1(t+\Delta-y)-\mathbf{N}_1((t-y)^+)=k) = e^{-(\Phi(t+\Delta-y)-\Phi((t-y)^+))}\frac{(\Phi(t+\Delta-y)-\Phi((t-y)^+))^k}{k!}$$

4.3 Computational Results

Knowing when the number of items in the market and the number of returned defective items are the highest and how many spare parts should be stored in the warehouse to prevent an out-of-stock situation with a high certainty, i.e., the safety stock considering the lead time to deliver the parts by the supplier are the values we find in this section using our proposed theoretical model. This numerical analysis refers to a specific model refrigerator that is expected to be produced and sold for about 3 years before it is redesigned, achieves its maximum sales after 1 year, and is sold with a warranty of 3 years. The yearly failure rate of a refrigerator is reported to be 1.79% including the failure rates of its important components (Woo & O'Neal (2016)).

4.3.1 Parameter Selection

The sales stochastic model which was modeled using the Brokhoff model in Amniattalab et al. (2023a) is modeled by the Bass diffusion model Bass (2004b) in our paper. Within our stochastic framework, the Bass diffusion model can be represented by a nonhomogeneous Poisson process with mean arrival function Ψ satisfying $\Psi(\infty) = m$ and

$$\Psi(t) = \frac{m(1 - e^{-(\alpha_1+\alpha_2)t})}{1 + \frac{\alpha_2}{\alpha_1} e^{-(\alpha_1+\alpha_2)t}} \tag{4.80}$$

and intensity function ψ

$$\psi(t) = \frac{m(\alpha_1 + \alpha_2)(1 + \frac{\alpha_2}{\alpha_1})e^{-(\alpha_1+\alpha_2)t}}{(1 + \frac{\alpha_2}{\alpha_1} e^{-(\alpha_1+\alpha_2)t})^2} \tag{4.81}$$

with $\alpha_2 > \alpha_1$ (Xie et al. (2014)). The constant $\alpha_1 > 0$ denotes the innovation factor, $\alpha_2 > 0$ denotes the imitator factor, and m represents the size of the new product's potential market. We select the parameters of this model taking the product specifications mentioned at the beginning of this section into consideration. In the Bass model, the saturation point of the sales process happens at $t^* = \frac{ln(\alpha_2) - ln(\alpha_1)}{\alpha_2 - \alpha_1}$. The initial market size parameter m does not play a changing role in our results. So we optionally select $m = 100$ since we plan to sell a hundred units of our product. Our time index is in years. We have assumed that the sales of this specific product will stop in 3 years with the introduction of a newer model. This time value is approximate because usually sales of products still continue in the retail shops although the producer does not produce that model anymore. Here we impose the condition that the tail probability, i.e., the probability of not selling the item after 3 years, is 0.01 meaning $\frac{\Psi(\infty) - \Psi(3)}{\Psi(\infty)} = 1\%$. From this, we get $\frac{\alpha_2}{\alpha_1} = 10.42$ and $\alpha_1 + \alpha_2 = 2.34$. By substituting $\alpha_2 = 10.42\alpha_1$, we obtain $\hat{\alpha_1} = 0.20$ and $\hat{\alpha_2} = 2.13$. Slight changes in the tail probability did not affect our results. Figure 4.1 shows the yearly sales amount.

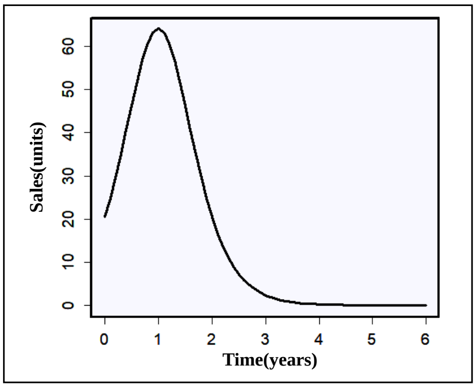

Figure 4.1 Yearly sales of a hypothetical product modeled by Bass diffusion model with $\alpha_1 = 0.20$, $\alpha_2 = 2.13$ and $m = 100$

The random usage time U follows the gamma distribution with the probability density function in the shape-rate parameterization as

$$f(u;\alpha,\lambda) = \frac{u^{\alpha-1}e^{-\lambda u}\lambda^{\alpha}}{\Gamma(\alpha)}, u > 0, \alpha, \lambda > 0$$

and $\mathbb{E}(U) = \frac{\alpha}{\lambda}$. In the sensitivity analysis subsection, we want to observe the effect of the change of expected usage time on the saturation times of the installed base and spare parts demand, and on the safety stock size during the lead time. For this, we look at $\mathbb{E}(U) \in \{w,...,2w,...,3w\}$ with $w = 3$ referring to the warranty length. We select this range because we do not expect that users will discard their products before their warranty expires. Also, customers tend to keep their durable household products for nearly 9 years (Cooper (2004)). Fixing α and $\mathbb{E}(U)$, one can obtain the rate parameter λ. In the sensitivity analysis, we select α from the set $\{1,2\}$. Obviously, with $\alpha = 1$, gamma distributed usage time becomes exponentially distributed with parameter λ (Scenarios in the upper row of Figure 4.2).

The failure process is a nonhomogeneous Poisson process with mean value function Φ. We assume repairs are minimal and the first time to failure $\mathbf{X}_1$ has a Weibull cdf $F_{\mathbf{X}_1}$ with shape parameter γ and rate parameter δ. Therefore,

$$\Phi(t) = -\ln(1 - F_{\mathbf{X}_1}(t)) = (\delta t)^{\gamma}, t > 0.$$

Observe that the arrival intensity of a nonhomogenous Poisson process under the Weibull minimal repair assumption is given by the failure rate function of a Weibull distributed random variable (Block et al. (1985); Aven & Jensen (2000)). Since the failure rate function of a Weibull-distributed random variable is given by

$$r_{\mathbf{X_1}}(t) = \frac{f_{\mathbf{X_1}}(t)}{1 - F_{\mathbf{X_1}}(t)} = \gamma\delta^{\gamma}t^{\gamma-1}, t > 0,$$

it follows that the random variable $\mathbf{X_1}$ has an increasing failure rate if and only if $\gamma \geq 1$. In the sensitivity analysis subsection, we make scenarios with $\gamma \in \{1, 1.3, 1.6, 2\}$. Now $\mathbf{X}_1$ can be represented with

$$\mathbf{X}_1 = \Phi^{\leftarrow}(\mathbf{X}_1^P).$$

where $\mathbf{X}_1^P$ follows exponential distribution with parameter 1. This shows as expected that

$$\pi = \mathbb{P}(\mathbf{X}_1 \leq x) = \mathbb{P}(\Phi^{\leftarrow}(\mathbf{X}_1^P) \leq x) = \mathbb{P}(\mathbf{X}_1^P \leq \Phi(x)) = \mathbb{P}(\mathbf{X}_1^P \leq \delta^{\gamma} x^{\gamma}) = 1 - e^{-(x\delta)^{\gamma}}$$

which implies that $\delta = \frac{(-ln(1-\pi))^{1/\gamma}}{x}$ with $\pi = 0.0179$, $x = 1$ year and γ fixed.

Accordingly, the combination of the selected values for $\alpha \in \{1,2\}$ and $\gamma \in \{1, 1.3, 1.6, 2\}$ provides us with eight scenarios to perform sensitivity analysis (Figure 4.2).

Scenario 1 $\alpha = 1$ $\gamma = 1$ ($U \sim Exp(\lambda)$, Failure process is HPP)	**Scenario 2** $\alpha = 1$ $\gamma = 1.3$ ($U \sim Exp(\lambda)$)	**Scenario 3** $\alpha = 1$ $\gamma = 1.6$ ($U \sim Exp(\lambda)$)	**Scenario 4** $\alpha = 1$ $\gamma = 2$ ($U \sim Exp(\lambda)$)
Scenario 5 $\alpha = 2$ $\gamma = 1$ (Failure process is HPP)	**Scenario 6** $\alpha = 2$ $\gamma = 1.3$ (Increasing failure rate)	**Scenario 7** $\alpha = 2$ $\gamma = 1.6$ (Increasing failure rate)	**Scenario 8** $\alpha = 2$ $\gamma = 2$ (Increasing failure rate)

Figure 4.2 Scenarios to perform sensitivity analysis

4.3.2 Sensitivity Analysis

In our analysis, we compute the saturation times of the installed base size $t(b)$ and the spare parts demand $t(r')$. Next to this, we compute the stock level k with which we can be at least 95(99) percent confident that we can cover the demand during the lead time. That is we find k such that $Pr\{R(t+\Delta) - R(t) \leq k\} \geq 0.95(0.99)$ with Δ representing the lead time. We illustrate how the safety stock k is changing with respect to the expected usage time and different supplier lead times at the saturation times $t(r')$.

In the next four figures, we present the analysis of the eight scenarios in Figure 4.2. The left panels depict the saturation times of the installed base process and the spare part demand rate against the expected usage time and the right panels depict the safety stock level against the expected usage time for 1 month and 5 month lead time and for 95% and 99% demand coverage (Figures 4.3, 4.4, 4.5 and 4.6). Note that the estimated safety stock levels are very small numbers because in our analysis the market size m is equal to 100 and as it can be anticipated, for different values of m, we will need different safety stock levels. The effect of m on the safety stock level is shown in the last figure in this section (Figure 4.7).

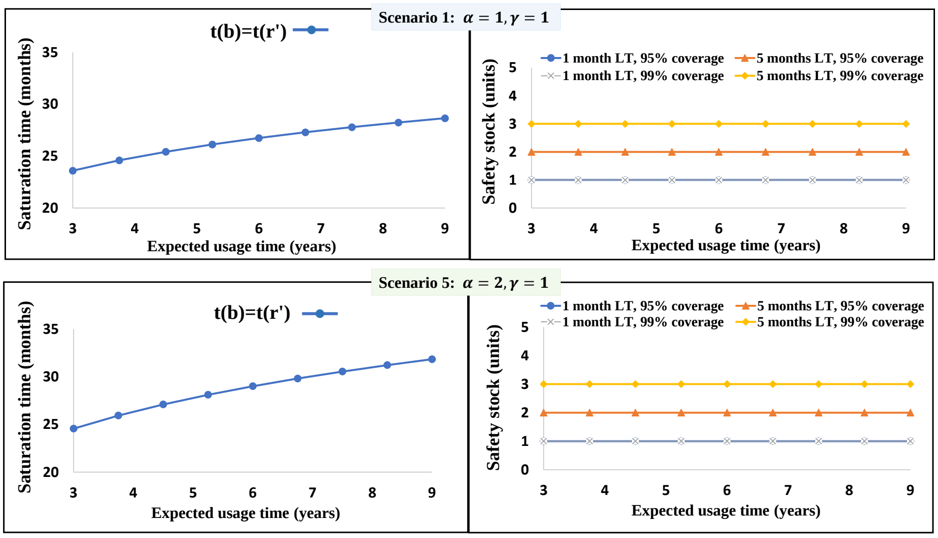

Figure 4.3 Times of maximum installed base and maximum spare parts demand (left figures) and the safety stock at these points with lead times $\Delta = 1$ and 5 months (right figures) against expected usage time for Scenario 1 ($\alpha = 1, \gamma = 1$) and Scenario 5 ($\alpha = 2, \gamma = 1$) with 95% and 99% lead time demand coverage

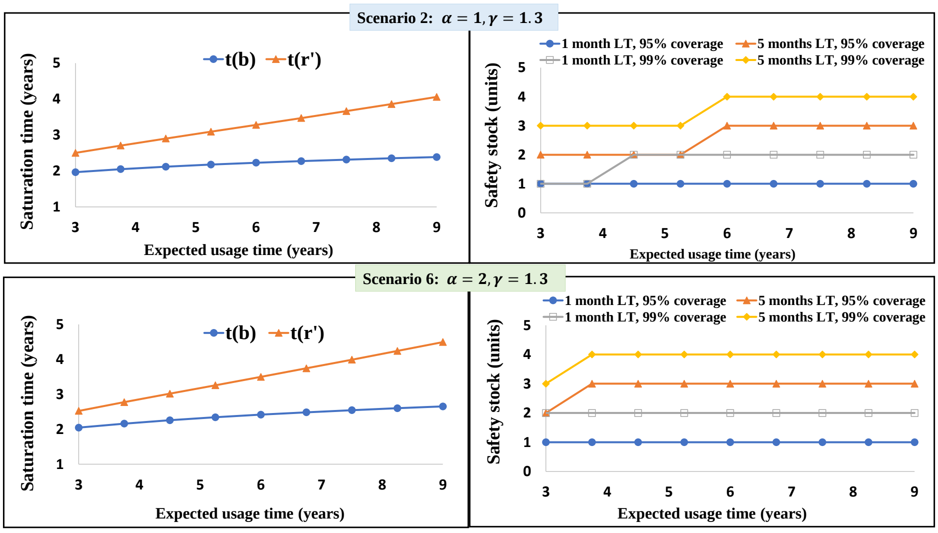

Figure 4.4 Times of maximum installed base and maximum spare parts demand (left figures) and the safety stock at these points with lead times $\Delta = 1$ and 5 months (right figures) against expected usage time for Scenario 2 ($\alpha = 1, \gamma = 1.3$) and Scenario 6 ($\alpha = 2, \gamma = 1.3$) with 95% and 99% lead time demand coverage

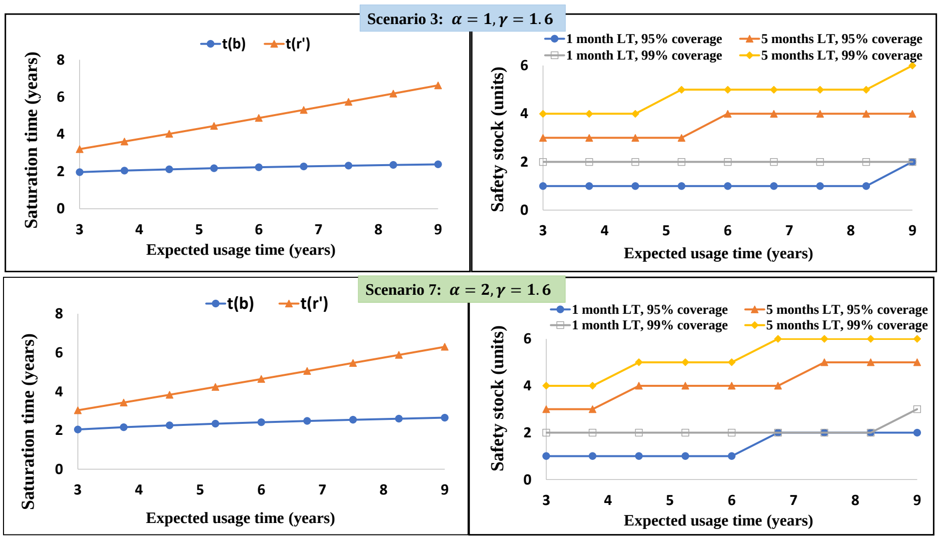

Figure 4.5 Times of maximum installed base and maximum spare parts demand (left figures) and the safety stock at these points with lead times $\Delta = 1$ and 5 months (right figures) against expected usage time for Scenario 3 ($\alpha = 1, \gamma = 1.6$) and Scenario 7 ($\alpha = 2, \gamma = 1.6$) with 95% and 99% lead time demand coverage

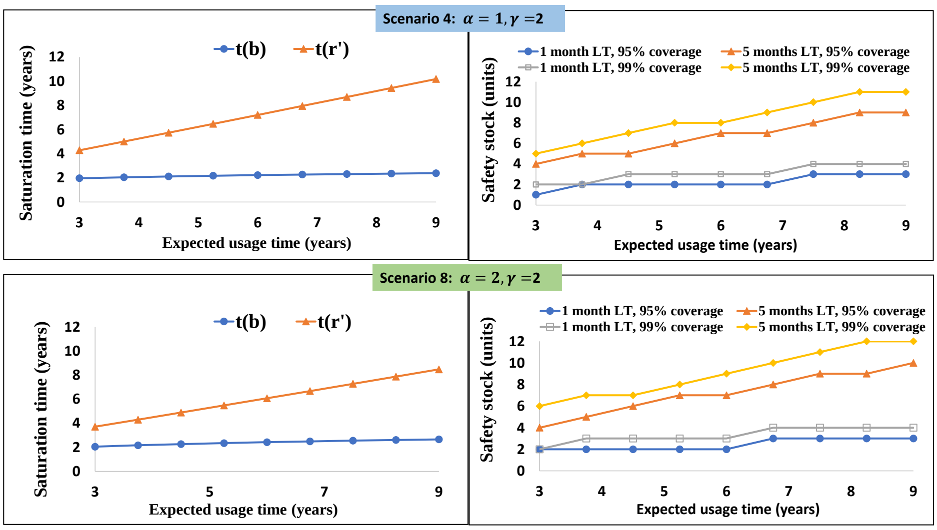

Figure 4.6 Times of maximum installed base and maximum spare parts demand (left figures) and the safety stock at these points with lead times $\Delta = 1$ and 5 months (right figures) against expected usage time for Scenario 4 ($\alpha = 1, \gamma = 2$) and Scenario 8 ($\alpha = 2, \gamma = 2$) with 95% and 99% lead time demand coverage

Considering that we only sell 100 items, observing a small number of spare parts demand during the lead time is not unexpected. In real cases where thousands or millions of a product are delivered to the market, we will naturally face higher spare parts demand over time. We mentioned that the total number of products being sold $\Psi(\infty)$ is equal to the market size parameter m in the Bass diffusion model. In the following figure, we see that there is a positive linear relationship between the initial market size and the safety stock. This relation is however not definable with a clear mathematical formula.

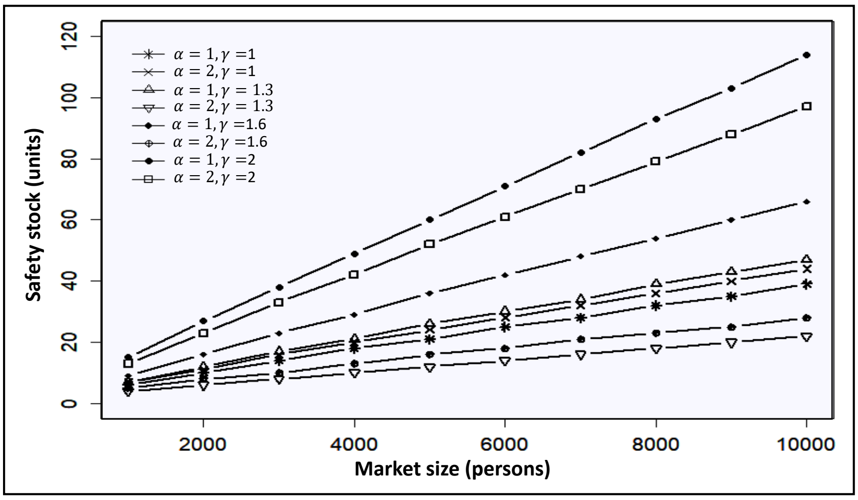

Figure 4.7 Effect of market size parameter m on the safety stock k for all the scenarios when $\mathbb{E}(U) = 3$ years, lead time $\Delta = 3$ months and the demand coverage percentage is 95%

In all, we can clearly see the increasing effect of $\mathbb{E}(\mathbf{U})$, the lead time duration, and the coverage percentage on the safety stock level (right panel of each figure). Also, we see that the maxima of the installed base and the spare parts demanded will be reached in a belated manner if customers use the product for a longer time (left panels of each figure).

Next, we analyze the effect of the γ parameter of the Weibull distributed first failure time on the safety stock level. In Figure 4.8 we see that with the increase of γ which refers to a higher failure rate, the safety stock level increases too.

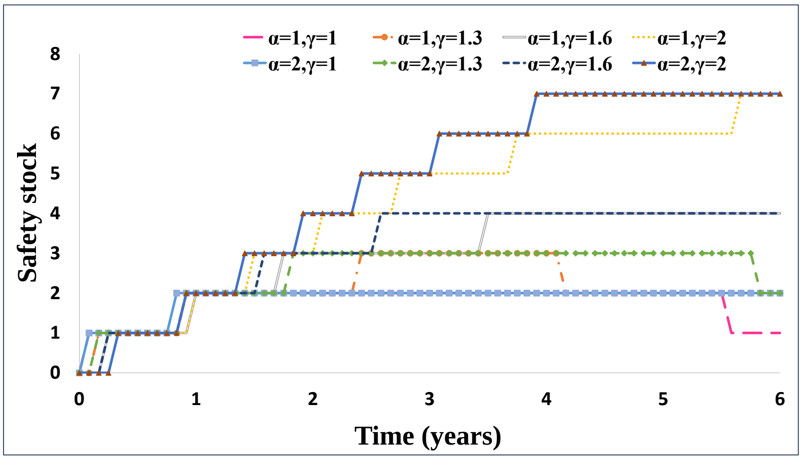

Figure 4.8 Safety stock level for different scenarios when lead time is $\Delta = 5$ months, expected usage time is 6 years and demand coverage percentage is 95%

As a final piece of information, we compute the number of returned and defective items or spare parts demanded during the usage time for one product within our setting (Figure 4.9).

$$\mathbb{E}(\Phi(\mathbf{U})) = \mathbb{E}(\delta^{\gamma} U^{\gamma}) = \delta^{\gamma} \int_0^{+\infty} u^{\gamma} f_U(u) du = \frac{\delta^{\gamma} \lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{+\infty} u^{\gamma+\alpha-1} e^{-\lambda u} du.$$

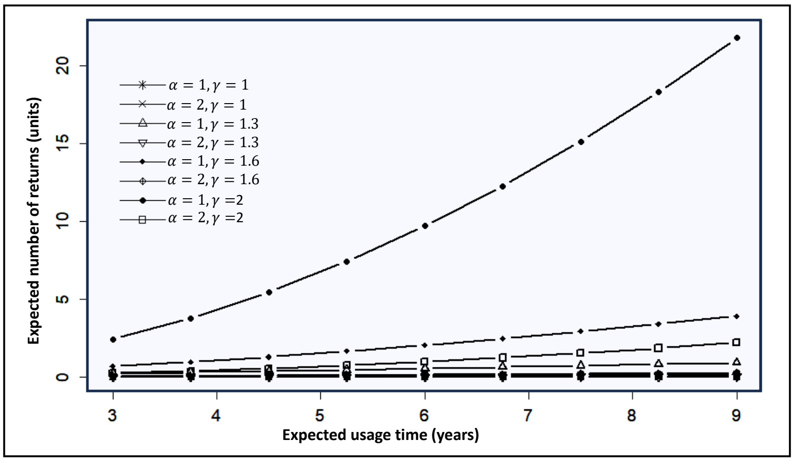

Figure 4.9 Expected number of returned and defectives during the usage time for a product

4.4 Discussion

The study used the life cycle pattern, life span, and reliability specifications of simple one-door refrigerators obtained from literature to fit stochastic process models to sales, installed base, and failure process of a product. In order to attain solutions to the mathematics of probability distribution of spare parts demand, we made the NHPP assumption on the sales and the failure processes. We built eight scenarios to confirm that our results are not accidental.

In all the scenarios, we make the following observations:

- **With respect to saturation times $t(b)$ and $t(r')$:**
  - The longer we use the product, the slower installed base size and spare part demand reach their maximum. This is expected and conforming to the results by Amniattalab et al. (2023a). This delaying effect is higher on spare parts demand saturation $t(r')$ than on the installed base size saturation $t(b)$ (Diverging lines in the left panels of Figures 4.3 to 4.6). The intuitive reason is that even when we use the product longer, the installed base is limited to the items that are being sold and so it is not as affected by the usage time as the failures process.
  - Increase of γ - the shape parameter of the Weibull-distributed first failure time - meaning a higher failure rate in the NHPP failure process, delays the saturations $t(r')$ in a non-monotonically increasing way. We make this comment by observing that the slope of the line $t(r')$ in the left panels becomes larger as γ increases.
  - Increase of α - the shape parameter of the Gamma distributed usage time - from 1 to 2 grasps the delay caused by the increase of γ. One can observe this by comparing the top subfigure to the bottom subfigure in Figures 4.3,...,4.6. The reason is that with $\alpha = 1$,

we have a sharper cdf of the random usage time and this reduces the effect of failure rate with the factor $(1-F)$.

- **With respect to the safety stock level:**
  - As expected 99% coverage demands more safety stock than 95% coverage and 5 months lead time leads to higher safety stock than 1 month lead time
  - The longer the expected usage time, the higher the safety stock level should be.
  - γ, in general, increases the safety stock level as it increases but increasing the shape parameter of the Gamma distributed usage time from $\alpha = 1$ to $\alpha = 2$ (from Exponential to Erlang-2 usage time) reduces the safety stock level. However, when $\gamma = 1$ (Homogeneous Poisson failure process), changing the usage time distribution from exponential to Erlang-2 and the expected usage time do not change the safety stock level.

A similar type of analysis is done in Amniattalab et al. (2023a) with the difference that the random usage time follows Gamma distribution.

4.5 Appendix

In this Appendix we give the proof of the main result listed in Theorem 4 of this paper.

Proof. Since the sales process is a nonhomogeneous Poisson process with mean arrival function function Ψ it is well known (cf.Ross (2013)) that for every $y_1 \leq y_2 \leq ... \leq y_k$ and $k \in \mathbb{N}$

$$\mathbb{P}(\mathbf{T}_1 \leq y_1, ..., \mathbf{T}_k \leq y_k \mid \mathbf{S}(t+\Delta) = k) = \mathbb{P}(\mathbf{Y}_{1:k} \leq y_1, ..., \mathbf{Y}_{k:k} \leq y_k)$$

with $(\mathbf{Y}_{1:k}, ..., \mathbf{Y}_{k:k})$ the random vector of order statistic arising from the independent and identically distributed random variable $\mathbf{Y}_k, 1 \leq k \leq n$ having continuous cdf

$$\mathbb{P}(\mathbf{Y} \leq y) = \frac{\Psi(y)}{\Psi(t+\Delta)}, y \leq t+\Delta$$

This shows using relation (4.9) and introducing $(x)^+ := \max(0,x)$ that for every $s_1, s_2 \geq 0$

(4.82)

$$\begin{aligned}\mathbb{E}(e^{-s_1\mathbf{R}(t)-s_2\mathbf{R}(t+\boldsymbol{\Delta})} \mid \mathbf{S}(t+\Delta) = k) &= \mathbb{E}\left(e^{-s_1\sum_{n=1}^{k}\mathbf{N}_n((t-\mathbf{Y}_{n:k})^+\wedge\mathbf{U}_n)-s_2\sum_{n=1}^{k}\mathbf{N}_n((t+\Delta-\mathbf{Y}_{n:k})\wedge\mathbf{U}_n)}\right)\\ &= \mathbb{E}\left(e^{-s_1\sum_{n=1}^{k}\mathbf{N}_n^*((t-\mathbf{Y}_{n:k})\wedge\mathbf{U}_n)-s_2\sum_{n=1}^{k}\mathbf{N}_n^*((t+\Delta-\mathbf{Y}_{n:k})\wedge\mathbf{U}_n)}\right)\end{aligned}$$

with $\mathbf{N}_n^*(t) = \mathbf{N}_n(t^+)$ for every $t \geq 0$. Introducing the set Π_k of all permutations on the set $\{1, ..., k\}$

and for every $\pi \in \Pi_k$ the event

$$E_\pi = \{(\mathbf{Y}_1, ..., \mathbf{Y}_k) : \mathbf{Y}_{\pi(1)} \leq .... \leq \mathbf{Y}_{\pi(n)}\}$$

we obtain using the cdf $\frac{\Psi(t)}{\Psi(t_2)}$ is continuous on $(0, t_2)$ and setting $t_1 = t$ and $t_2 = t + \Delta$ that

$$\mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - \mathbf{Y}_{n:k}) \wedge \mathbf{U}_n)}\right) = \sum_{\pi \in \Pi_k} \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - \mathbf{Y}_{\pi(n)}) \wedge \mathbf{U}_n)} 1_{E_\pi}\right). \tag{4.83}$$

We will now verify that

$$\mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - \mathbf{Y}_{\pi(n)}) \wedge \mathbf{U}_n)} 1_{E_\pi}\right) = \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - \mathbf{Y}_n) \wedge \mathbf{U}_n)} 1_{E_\pi}\right). \tag{4.84}$$

To show relation (4.84) we observe using the random vector $(\mathbf{Y}_1, ..., \mathbf{Y}_k)$ is independent of the failure point processes $\mathbf{N}_n, n \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - \mathbf{Y}_{\pi(n)}) \wedge \mathbf{U}_n)} 1_{E_\pi}\right) \\
= \;& \int ... \int_{[0,t_2]^k} \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_n)} \mid \mathbf{Y}_{\pi(i)} = y_{\pi(i)}, i = 1, ..., k\right) 1_{\overline{E}_\pi}(\mathbf{y}_\pi) dF(\mathbf{y}) \\
= \;& \int ... \int_{[0,t_2]^k} \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_n)}\right) 1_{\overline{E}_\pi}(y_1, ..., y_k) dF(y_1, ..., y_k)
\end{aligned} \tag{4.85}$$

with

$$\overline{E}_\pi = \{(y_1, ..., y_k) : y_{\pi(1)} \leq .... \leq y_{\pi(n)}\}$$

and F the joint cdf of the random vector $(\mathbf{Y}_1, ..., \mathbf{Y}_k) \in [0, t_2]^k$ given by

$$F(y_1, ..., y_k) = \Pi_{n=1}^k \frac{\Psi(y_n)}{\Psi(t_2)}.$$

Observe for every $\pi \in \Pi_k$ that

$$\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_{\pi(n)}^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_{\pi(n)}) = \sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - y_n) \wedge \mathbf{U}_n)$$

and since the failure processes $\mathbf{N}_n^*(. \wedge \mathbf{U}_n), n \in \mathbb{N}$ are independent and identically distributed this implies for every $\pi \in \Pi_k$ that the integrand in relation (4.85) satisfies

$$\begin{aligned}
\mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_n)}\right) &= \Pi_{n=1}^k \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \mathbf{N}_n^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_n)}\right) \\
&= \Pi_{n=1}^k \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \mathbf{N}_{\pi(n)}^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_{\pi(n)})}\right) \\
&= \mathbb{E}\left(e^{-\sum_{i=1}^2 \sum_{n=1}^k s_i \mathbf{N}_{\pi(n)}^*((t_i - y_{\pi(n)}) \wedge \mathbf{U}_{\pi(n)})}\right) \\
&= \mathbb{E}\left(e^{-\sum_{i=1}^2 s_i \sum_{n=1}^k \mathbf{N}_n^*((t_i - y_n) \wedge \mathbf{U}_n)}\right)
\end{aligned}$$

Applying relation (4.85) it follows that relation (4.84) holds. Using relations (4.83) and (4.84) we obtain

$$
\begin{aligned}
\mathbb{E}\left(e^{-\sum_{i=1}^{2} s_i \sum_{n=1}^{k} \mathbf{N}_n^*((t_i - \mathbf{Y}_{n:k})\wedge \mathbf{U}_n)}\right) &= \sum_{\pi\in\Pi_k} \mathbb{E}\left(e^{-\sum_{i=1}^{2} s_i \sum_{n=1}^{k} \mathbf{N}_n^*((t_i - \mathbf{Y}_{\pi(n)})\wedge \mathbf{U}_n)} 1_{E_\pi}\right) \\
&= \sum_{\pi\in\Pi_k} \mathbb{E}\left(e^{-\sum_{i=1}^{2} s_i \sum_{n=1}^{k} \mathbf{N}_n^*((t_i - \mathbf{Y}_n)\wedge \mathbf{U}_n)} 1_{E_\pi}\right) \\
&= \mathbb{E}\left(e^{-\sum_{i=1}^{2} s_i \sum_{n=1}^{k} \mathbf{N}_n^*((t_i - \mathbf{Y}_n)\wedge \mathbf{U}_n)} \sum_{\pi\in\Pi_k} 1_{E_\pi}\right) \\
&= \mathbb{E}\left(e^{-\sum_{i=1}^{2} s_i \sum_{n=1}^{k} \mathbf{N}_n^*((t_i - \mathbf{Y}_n)\wedge \mathbf{U}_n)}\right).
\end{aligned}
\tag{4.86}
$$

By the independence of the identically distributed random vector $(\mathbf{N}_n^*((t_i - \mathbf{Y}_n)\wedge \mathbf{U}_n), n = 1, ..., k$ the desired result follow from relation (4.86). $\square$

5. A Discrete Time Stochastic Process for Spare Parts: An Installed Base Approach

In this chapter, we will introduce a discrete time stochastic process for spare parts demand using the installed base approach taking into account the random usage times for any item purchased during the life cycle of a particular product and the failure processes for each purchased item. We denote by $n \in \mathbb{N}$ the index of the item and by $t \in \mathbb{Z}_+$ the time. Contrary to the continuous time approach presented in the previous chapters a discrete time model is more suitable for statistical analysis. This will be the topic of future research and is not presented in this thesis.

As for the continuous time model we introduce for an arbitrary discrete time counting sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ in the first section of this chapter the main definitions and representations of the installed base stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$, the discarded items stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \in \mathbb{Z}_+\}$, the returned defective items stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$ and the total remaining returned defective items stochastic process $\mathbf{V} = \{\mathbf{V}(t) : t \in \mathbf{Z}_+\}$. The analysis in this chapter extends the model proposed by Minner (2011) to cumulative sales stochastic processes having a general covariance structure. These processes are called in the literature second order stochastic processes. Notice in Minner (2011) only deterministic sales processes using a different approach were considered and under this assumption the pmf of the returned defective items process at any time t was derived. Since deterministic sales processes are special instances of second order stochastic sales processes this simplifies the analysis. In the second section of this chapter, we compute for all these stochastic processes the expectation and variance at any time t and the so-called covariance function. In the third section of this chapter, we derive for the installed base stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$ and the returned defective items stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$ under very general conditions on the cumulative sales stochastic process and the failure process of each purchased item that the expected installed base and the expected returned and defective items process at time t shows a unimodal behaviour over time. These results are the discrete time counterpart of the results derived in Chapter 3 for the continuous time model and enable us to give easy algorithms to compute the period within the life cycle of the product at which the expected installed base and the expected number of returned and defective items are maximal. This means we can determine in which period within the life cycle of the product the manufacturer is most busy with after-sales services. Finally, in the last section of this chapter, we restrict ourselves to purchase processes having independent compound Poisson distributed purchases in any period and show that the pmfs of these stochastic processes at any time t are compound Poisson distributed. Since this result is too general to evaluate these pmfs numerically we identify subclass of the compound Poisson distributed purchases for which it is relatively easy to evaluate the pmf numerically at any time t of each of the above considered stochastic processes. In particular, we consider a deterministic purchase process and random purchase processes having independent Poisson, geometric, or discrete gamma distributed purchases. At the same time, we give in that section easy numerical procedures to compute these pmfs for these special cases. Finally, in the last section, we list future research which involves implementing these algorithms and showing the behaviour of these pmfs over time. In the next section, we start as already mentioned with the basic definitions.

5.1 Introduction of the main discrete time stochastic processes in spare parts demand

In this section, we will introduce the main discrete time stochastic processes occurring in spare parts demand. This model is a discrete time version of the continuous time model discussed in Chapter 4 and is more suitable for statistical analysis. It is assumed that the sales of the product start at time 0 and last for a certain amount of time called the product life cycle (see page 247 of Rogers, Singhal & Quinlan (2014), Kim et al. (2017), Brockhoff (1967)). We introduce on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the stochastic purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ with $\mathbf{P}_k$ given by

$$\mathbf{P}_k := \text{total random number of purchases in period } k \tag{5.1}$$

The discrete time cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ is defined by

$$\mathbf{S}(t) := \textstyle\sum_{k=0}^{t} \mathbf{P}_k. \tag{5.2}$$

Clearly the random variable $\mathbf{S}(\infty) := \lim_{t\uparrow\infty} \mathbf{S}(t)$ denotes the total random number of sales of the product during its life cycle. Since the total expected number of sold items during the life cycle of the product is finite, it is reasonable to impose the additional condition that $S(\infty) := \mathbb{E}(\mathbf{S}(\infty)$ is finite. Using the monotone convergence theorem (Çınlar (2011)) and relation (5.2) this means that

$$S(\infty) = \textstyle\sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k) < \infty. \tag{5.3}$$

Since $S(\infty)$ is finite this shows applying the Markov inequality that the random variable $\mathbf{S}(\infty)$ is finite with probability 1. Clearly the finite constant S_∞ represents the expected market potential of the considered product.

Every customer buying this product will use it for a random amount of time and to model this we introduce the positive integer valued random variables $\mathbf{U}_n$, $n = 1, ..., \mathbf{S}(\infty)$, denoting the discrete random usage times of the nth purchased product in the market. Observe, if $1 \leq n \leq \mathbf{S}(0)$, the product n is purchased at time 0 and if $\mathbf{S}(k-1)+1 \leq n \leq \mathbf{S}(k)$ and $\mathbf{S}(k) > \mathbf{S}(k-1)$ it is purchased at time k, $k \in \mathbb{N}$. The random variables $\mathbf{B}_k(t), 0 \leq k \leq t$, $k \in \mathbf{Z}_+$ represent now the total number of products purchased at time $k \leq t$ which are still in the market at time t. Using this definition it is clear that the non-negative integer valued random variable $\mathbf{B}_k(t)$, $0 \leq k \leq t$, $k \in \mathbb{Z}_+$ has for every

$0 \leq k \leq t$ the representation

$$\mathbf{B}_k(t) = \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n > t-k\}} \tag{5.4}$$

with 1_A denoting the Bernoulli random variable of the event A. Observe if for some $k \in \mathbb{Z}_+$ it follows that $\mathbf{P}(k) = 0$ or equivalently $\mathbf{S}(k) = \mathbf{S}(k-1)$, then the series in relation (5.4) is given by an empty series and we always use the convention that an empty series equals 0. Also, we use for convenience the convention that $\mathbf{S}(-1) = 0$. Since products will be discarded after their usage time, the random variable $\mathbf{D}_k(t)$ denotes the total number of products purchased at time k and discarded before or at time t. Using this definition it is clear for every $0 \leq k \leq t$, $k \in \mathbb{Z}_+$ that the non-negative integer valued random variable $\mathbf{D}_k(t)$ has the representation

$$\mathbf{D}_k(t) = \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n \leq t-k\}}. \tag{5.5}$$

Obviously it holds for every $k \leq t$ that $\mathbf{B}_k(t) + \mathbf{D}_k(t) = \mathbf{P}_k$. Since the demand for spare parts after time t clearly depends both on the total number of products available in the market at time t and future sales after this time, we introduce the discrete time non-negative integer valued stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$ denoting the total number of products in the market at time $t \in \mathbb{Z}_+$. This stochastic process is called the installed base stochastic process. By relation (5.4) it follows for every $t \in \mathbb{Z}_+$ that

$$\mathbf{B}(t) = \sum_{k=0}^{t} \mathbf{B}_k(t) = \sum_{k=0}^{t} \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n > t-k\}}. \tag{5.6}$$

Since one might also use refurbished spare parts from discarded items, the supply of spare parts also depends on the total number of discarded items $\mathbf{D}(t)$ up to any time t. This is represented by the discrete time counting stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \in \mathbb{Z}_+\}$ and this stochastic process is called the discarded items stochastic process. By its definition and relation (5.5), the non-negative integer valued random variable $\mathbf{D}(t)$ for every $t \in \mathbb{Z}_+$ has the representation

$$\mathbf{D}(t) = \sum_{k=0}^{t} \mathbf{D}_k(t) = \sum_{k=0}^{t} \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n \leq t-k\}}. \tag{5.7}$$

Each purchased item n of the same product generates a counting failure process of defective returns on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is denoted by the stochastic process $\mathbf{N}_n$. This stochastic process $\mathbf{N}_n$ is determined by the increasing sequence of non-negative integer valued stochastic variables $\mathbf{T}_m^{(n)}, m \in \mathbb{Z}_+$ satisfying

$$0 = \mathbf{T}_0^{(n)} \leq \mathbf{T}_1^{(n)} \leq ..... \tag{5.8}$$

This means if item n is purchased at time k and its usage time is infinity that $k + \mathbf{T}_m^{(n)}$ denotes the time of the mth failure. Hence the random variables

$$\mathbf{T}_{m+1}^{(n)} - \mathbf{T}_m^{(n)}, m \in \mathbb{Z}_+$$

denote the random time elapsed between the $(m+1)$th and mth failure of item n. It is assumed that any repair takes a negligible amount of time and that after this repair the product is operable again and can be used for some positive time. Hence we assume that $\mathbf{T}_{m+1}^{(n)} - \mathbf{T}_m^{(n)} \geq 1$ almost surely. Now the number of failures up to age t of this particular item n is given by the discrete time non-negative counting random variable

$$\mathbf{N}_n(t) = \sum_{m=1}^{\infty} 1_{\{\mathbf{T}_m^{(n)} \leq t\}} \tag{5.9}$$

and so the stochastic counting process $\mathbf{N}_n$ has the representation $\mathbf{N}_n = \{\mathbf{N}_n(t) : t \in \mathbb{Z}_+\}$. We are now interested in the number of returned defective products at time t and therefore we consider the random variables $\mathbf{R}_k(t), k = 1, ..., t$ denoting the total number of products in the market at time t which were purchased at time $0 \leq k \leq t$ and returned defective at time t. Clearly by its definition we obtain for every $t \in \mathbb{N}$ and $0 \leq k \leq t$, $k \in \mathbb{Z}_+$

$$\mathbf{R}_k(t) = \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n > t-k, \cup_{m=1}^{\infty}(\mathbf{T}_m^{(n)} = t-k)\}}. \tag{5.10}$$

Since defective items returning at time t can be any items purchased before time t, we introduce the discrete time counting stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$ with $\mathbf{R}(t)$ denoting the total number of returned defective products at time $t \in \mathbb{N}$. This stochastic process is called the returned defective items stochastic process and by relation (5.10) it follows for every $t \in \mathbb{Z}_+$ that

$$\mathbf{R}(t) = \sum_{k=0}^{t} \mathbf{R}_k(t) = \sum_{k=0}^{t} \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} 1_{\{\mathbf{U}_n > t-k, \cup_{m=1}^{\infty}(\mathbf{T}_m^{(n)} = t)\}}. \tag{5.11}$$

Finally, we consider the stochastic process $\mathbf{V} = \{\mathbf{V}(t) : t \in \mathbb{Z}_+\}$ with $\mathbf{V}(t)$ denoting the number of returned defective items from time t until the end of the life cycle of the product. Clearly for every $t \in \mathbb{Z}_+$ we obtain

$$\mathbf{V}(t) = \sum_{m=t}^{\infty} \mathbf{R}(m). \tag{5.12}$$

This stochastic process plays a key role in determining the size of the so-called last buy decision and is called the total remaining returned defective items stochastic process. To give a more detailed expression of the random variable $\mathbf{V}(t)$ suitable for analysis, we first observe that any item n of the same product purchased at time $k \geq t$ and so $\mathbf{S}(k-1)+1 \leq n \leq \mathbf{S}(\infty)$ will return defective $\mathbf{N}_n(\mathbf{U}_n - 1)$ times within the discrete time interval $\{t, t+1, ...\}$. Since at time $k + \mathbf{U}_n$ this item having age $\mathbf{U}_n$ is taken out of the market and so at that time it is clearly not counted as being a defective item, the total number of times this item returns defective equals $\mathbf{N}_n(\mathbf{U}_n - 1)$. Notice $\mathbf{N}_n(t)$ denotes the total number of times an item being in the market for t time units or having age t returns defective. We introduce now the random variable $\mathbf{A}(t)$, $t \in \mathbb{Z}_+$ with $\mathbf{A}(t)$ denoting the number of times that items purchased at times $k = t, t+1, ...$ will return defective within the time interval $\{t, t+1, ...\}$. Clearly the random variable $\mathbf{A}(0)$ denotes the total number of times that items purchased during the life cycle will return defective. Using the above observation about the number of times an item purchased at time $k = t, t+1, ...$ returns defective within the interval $\{t, t+1, ...\}$, it is clear that for every $t \in \mathbb{Z}_+$

$$\mathbf{A}(t) = \sum_{n=\mathbf{S}(t-1)+1}^{\mathbf{S}(\infty)} \mathbf{N}_n(\mathbf{U}_n - 1). \tag{5.13}$$

Observe we use in formula (5.13) the convention that $\mathbf{S}(-1) := 0$. Also for any item n purchased at time $0 \leq k \leq t-1$, $k \in \mathbb{Z}_+$ the total number of times this item will return defective within the discrete time interval $\{t, t+1, ...\}$ is given by

$$\mathbf{N}_n(\mathbf{U}_n - 1) - \mathbf{N}_n((\mathbf{U}_n - 1) \wedge (t-k-1)) \tag{5.14}$$

with $a \wedge b := \min\{a, b\}$ for any numbers a and b. Notice that the random expression in relation (5.14) equals zero if the item purchased at time k is discarded before or at time t. Since this item is discarded before or at time t, there will be no defective returns of this particular item after or at time t. In this particular case it follows that $\mathbf{U}_n \leq t-k$ and it is easy to check that the expression in relation (5.14) equals zero. To model the total number of times an item purchased at time $k \leq t-1$ returns defective within the interval $\{t, t+1, ...\}$, we introduce for every $t \in \mathbb{Z}_+$ the random variable $\mathbf{A}_k(t)$,$0 \leq k \leq t-1$. Clearly this random variable $\mathbf{A}_k(t)$ has the representation

$$\mathbf{A}_k(t) = \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} \mathbf{N}_n(\mathbf{U}_n - 1) - \mathbf{N}_n((\mathbf{U}_n - 1) \wedge (t-k-1)). \tag{5.15}$$

In the above formula, we use the convention that $\mathbf{S}(-1) = 0$. For $k = t-1$ using $\mathbf{N}(0) = 0$ the above formula reduces to

$$\mathbf{A}_{t-1}(t) = \sum_{n=\mathbf{S}(t-2)+1}^{\mathbf{S}(t-1)} \mathbf{N}_n(\mathbf{U}_n - 1). \tag{5.16}$$

Hence for every $t \in \mathbb{Z}_+$ the total number $\mathbf{V}_t$ of times that items return defective from time t until the end of the life cycle is given by

$$\mathbf{V}(t) = \sum_{k=0}^{t-1} \mathbf{A}_k(t) + \mathbf{A}(t) \tag{5.17}$$

with $\mathbf{A}(t)$ and $\mathbf{A}_k(t)$, $0 \leq k \leq t-1$ defined in relation (5.13), respectively relation (5.15). It is assumed for simplicity in this paper that any defective item cannot be repaired and needs only one spare part. This assumption can be easily removed without encountering any mathematical difficulty. In the case of using multiple spare parts, we impose given probabilities on which spare parts are needed for an arbitrary defective item and the same applies to a given probability of being able to repair or not an arbitrary returned defective item. To conclude this section we summarise the used notation.

- The discrete time non-negative integer valued stochastic process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ denotes the stochastic process of purchases with $\mathbf{P}_k$ denoting the number of purchases in period k for any $n \in \mathbb{Z}_+$.

- The discrete time non-negative integer valued stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ denotes the counting stochastic process of sales up to time t for any $t \in \mathbb{Z}_+$.

- The positive integer valued stochastic variables $\mathbf{U}_n$, $n \in \mathbb{N}$ denote the integer time the nth purchased item stays in the market.

- The discrete time non-negative integer valued stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$ denotes the stochastic process of the total number of items in the market at time t for every $t \in \mathbb{Z}_+$.
- The discrete time non-negative integer valued stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \in \mathbb{Z}_+\}$ denotes the counting process of the total number of discarded items up to time t for every $t \in \mathbb{Z}_+$.
- The discrete time non-negative integer valued stochastic process $\mathbf{N}_n = \{\mathbf{N}_n(t) : t \in \mathbb{Z}_+\}$ denotes counting process of the total number of failures of the nth purchased item up to its age t if this item after its purchase time stays forever in the market.
- The discrete time non-negative integer valued stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$ denotes the stochastic process of the number of returned defective items at time t for any $t \in \mathbb{Z}_+$.
- The discrete time non-negative integer valued stochastic process $\mathbf{V} = \{\mathbf{V}(t) : t \in \mathbb{Z}_+\}$ denotes the stochastic process of the total number of returned defective items from time t until the end of the life cycle for any $t \in \mathbb{Z}_+$.

In the next section, we will discuss under some conditions on the random usage times and failure counting process of each purchased item how to compute both expectations, variances, and the covariance function of the discarded items, installed base, number of returned defective items stochastic process, and the total remaining returned defective items stochastic process.

5.2 The Expectation and Covariance Function of the Different Stochastic Processes

In this section we will first compute in different subsections for an arbitrary cumulative sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ the expectation and variance of the installed base stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$, the discarded items stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \in \mathbb{Z}_+\}$, the returned defective items stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$, and the total remaining returned defective items stochastic process $\mathbf{V} = \{\mathbf{V}(t) : t \in \mathbb{Z}_+\}$ at time $t \in \mathbb{Z}_+$. To keep this section readable all the different results are listed in a series of lemmas. Computing the expectation and variance is not as detailed as computing the pmf of the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$, $\mathbf{R}(t)$ and $\mathbf{V}(t)$ but applying the well-known Chebyshev's inequality (Ross (2023)) a possible application of these calculations is to give an upper bound on the two-sided tail of the pmf of the random variable $\mathbf{X} - \mathbb{E}(\mathbf{X})$ with the random variable $\mathbf{X}$ given by either the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$, $\mathbf{R}(t)$ or $\mathbf{V}(t)$. We do not impose any conditions on the cumulative sales stochastic process except that the expected number of items sold during the life cycle of the product is finite. For the variance computation of $\mathbf{V}(t)$ we assume that the variance of the total random sales during the life cycle is also finite. The analysis in Minner (2011) is different from our analysis and at the same time we extend the analysis of Minner (2011) to a much more general setting. For example in Minner (2011) no variance and covariance function is computed for the returned and defective items process $\mathbf{R}$ and the remaining total number of returned defective items process $\mathbf{V}$ for an arbitrary cumulative sales process $\mathbf{S}$. However, as in Minner (2011) we need to assume in this paper that the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are independent of the cumulative sales process $\mathbf{S}$. Since we are dealing with items of the same product it is also assumed as in Minner (2011) that the sequence of random vectors

$$(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...), n \in \mathbb{N} \tag{5.18}$$

describing the usage time and the failure counting process of each purchased item are independent and identically distributed. This last assumption seems to be reasonable since each item purchased is the same product and is subject to a failure process having the same probability law. This implies that the failure counting process of each purchased item has the same function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, given by

$$\Phi(t) := \mathbb{E}(\mathbf{N}(t)). \tag{5.19}$$

The value $\Phi(t)$ is known as the mean value function of the point failure process on $[0,t]$ and this function uniquely determines the so-called intensity measure of the counting failure process (Brémaud (1981)). If the counting failure process is a renewal process this function is called in the literature (Ross (2023)) the renewal function. Since arrivals only happen at integer points this is a stepwise constant right continuous increasing function. Observe by statistical techniques the shape and representation of this function could be determined by extensive durability tests of the product in a laboratory environment in the development phase of the product. Another consequence of our assumption is that the positive integer valued random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ are independent and identically distributed with a given discrete cdf $F_{\mathbf{U}}$ on $\mathbb{Z}_+$ satisfying $F_{\mathbf{U}}(0) = 0$. It might be difficult to obtain reliable data about usage times of the product except for data covering the average and variability of the usage time. If this happens we can select the pmf of the usage time from a parametric family of pmfs having an exact two moment fit (Adan, van Eenige & Resing (1995)). If these data are not available one could fit the pmf of the usage time from the same parametric family of pmfs as suggested in Adan et al. (1995) based on judgments of experts giving an estimation of the average usage time being a multiple of the warranty period of the purchased item and having a given variance. Since all failure processes $\mathbf{N}_n$, $n \in \mathbb{N}$ and usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ follow the same probability law, we delete for convenience in some parts of this paper the index n. The same applies to the arrival times $\mathbf{T}_m^{(n)}$ listed in relation (5.18). In reality, the expected usage time of any product is finite and so we impose additionally that the random usage time $\mathbf{U}$ satisfies $\mathbb{E}(\mathbf{U})$ is finite. Possible candidates for the cdf $F_{\mathbf{U}}$ are given by discrete Weibull pmfs (Almalki & Nadarajah (2014)). However, in our analysis the cdf $F_{\mathbf{U}}$ is taken arbitrary unless otherwise specified. Since it is realistic to assume that the expected number of times an item returns defective during its usage time is finite, we also impose the condition on the failure process and usage time that $\mathbb{E}(\mathbf{N}((\mathbf{U})$ is finite. Observe, if we additionally assume that the usage time is independent of the failure counting process, this means conditioning on random variable $\mathbf{U}$ that

$$\mathbb{E}(\mathbf{N}(\mathbf{U}) = \mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U})) \tag{5.20}$$

with $\mathbb{E}_{\mathbf{U}}$ denoting the expectation taken with respect to the non-negative integer valued random variable $\mathbf{U}$ and Φ the mean value function of the counting failure process.

To unify our analysis of the installed base, discarded items, and returned defective items stochastic process, we introduce for any integer valued non-negative random variable $\mathbf{X}$ and every $0 \leq \alpha \leq 1$ the so-called Steutel van Harn thinning operator $\circ$ given by (Steutel & van Harn (1979))

$$\alpha \circ \mathbf{X} = \sum_{j=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha\}} \tag{5.21}$$

with $\mathbf{Z}_j, j \in \mathbb{N}$ a sequence of independent and standard uniform distributed random variables independent of the integer valued non-negative random variable $\mathbf{X}$. This thinning operator $\circ$ was introduced in Steutel & van Harn (1979) to define the class of self-decomposable pmfs on $\mathbb{Z}_+$. For $\mathbf{X} = 0$ the random sum in relation (5.21) equals zero or equivalently the empty sum in relation (5.21) equals zero. By our assumptions, it follows using relations (5.4) and (5.5) that for every non-negative integer $k \leq t$ and $t \in \mathbb{Z}_+$

$$\mathbf{B}_k(t) \stackrel{d}{=} a_{t-k} \circ \mathbf{P}_k, \mathbf{D}_k(t) \stackrel{d}{=} (1 - a_{t-k}) \circ \mathbf{P}_k \tag{5.22}$$

with the non-negative sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ given by

$$a_i := 1 - F_{\mathbf{U}}(i) \tag{5.23}$$

and $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$ meaning that the random variables $\mathbf{X}$ and $\mathbf{Y}$ have the same cdf. Also, it follows by relation (5.10) that for every non-negative integer $k \leq t$ and $t \in \mathbb{Z}_+$

$$\mathbf{R}_k(t) \stackrel{d}{=} q_{t-k} \circ \mathbf{P}_k \tag{5.24}$$

with the non-negative sequence $\mathbf{q} = (q_i)_{i \in \mathbb{Z}_+}$ given by

$$q_i := \mathbb{P}(\mathbf{U} > i, \cup_{m=1}^{\infty} (\mathbf{T}_m = i)). \tag{5.25}$$

As observed in Minner (2011), conditional on the random purchases $\mathbf{P}_k$ at time k, the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ are binomial distributed with the number of trials given by $\mathbf{P}_k$ and the properly selected probability of success. It now follows by relations (5.6), (5.7) and (5.11) that for every $t \in \mathbb{Z}_+$

$$\mathbf{B}(t) \stackrel{d}{=} \sum_{k=0}^{t} a_{t-k} \circ \mathbf{P}_k, \mathbf{D}(t) \stackrel{d}{=} \sum_{k=0}^{t} (1 - a_{t-k}) \circ \mathbf{P}_k, \mathbf{R}(t) \stackrel{d}{=} \sum_{k=0}^{t} q_{t-k} \circ \mathbf{P}_k. \tag{5.26}$$

In the next subsection, we compute the expectation, variance, and covariance function of the installed base stochastic process for a second order cumulative sales stochastic process $\mathbf{S}$.

5.2.1 Expectation and Covariance Function of the Installed Base Stochastic Process

In this subsection, we first compute the expected value of the installed base process $\mathbf{B}$ at any time $t \in \mathbb{Z}_+$. It is easy to show the following result.

Lemma 23. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2), then the expectation $\mathbb{E}(\mathbf{B}(t))$ is finite and this expectation is given by*

$$\mathbb{E}(\mathbf{B}(t) = \sum\nolimits_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k} \tag{5.27}$$

with the value a_i,$i \in \mathbb{Z}_+$ defined in relation (5.23).

Proof. By relation (5.26) and (5.196) it follows for every $t \in \mathbb{Z}_+$

$$\mathbb{E}(\mathbf{B}(t) = \sum\nolimits_{k=0}^{t} \mathbb{E}(a_{t-k} \circ \mathbf{P}_k) = \sum\nolimits_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k} \tag{5.28}$$

and we have verified the result. □

To give a more abstract representation of relation (5.28) useful in computations, we first introduce for any sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ the so-called L_1-norm of the sequence $\mathbf{a}$ defined by

$$\|\mathbf{a}\|_1 := \sum\nolimits_{i=0}^{\infty} \mid a_i \mid . \tag{5.29}$$

We introduce now for any sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ and $\mathbf{b} = (b_i)_{i \in \mathbb{Z}_+}$ the so-called convolution $\mathbf{a} * \mathbf{b} = ((a * b)_i)_{i \in \mathbb{Z}_+}$ of the sequence $\mathbf{a}$ and $\mathbf{b}$ defined by

$$(a * b)_i = \sum\nolimits_{j=0}^{i} a_j b_{i-j}. \tag{5.30}$$

It is easy to verify by reversing the order of summation for any sequences $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ and $\mathbf{b} = (b_i)_{i \in \mathbb{Z}_+}$ that

$$\|\mathbf{a} * \mathbf{b}\|_1 \leq \|\mathbf{a}\|_1 \|\mathbf{b}\|_1 \leq \infty \tag{5.31}$$

while for non-negative sequences $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ and $\mathbf{b} = (b_i)_{i \in \mathbb{Z}_+}$

$$\|\mathbf{a} * \mathbf{b}\|_1 = \|\mathbf{a}\|_1 \|\mathbf{b}\|_1 \leq \infty. \tag{5.32}$$

It now follows by relation (5.27) introducing the non-negative sequences $\mathbf{b} = (b_i)_{i \in \mathbb{Z}_+}$ and $\mathbf{p} = (p_i)_{i \in \mathbb{Z}_+}$ given by

$$b_i := \mathbb{E}(\mathbf{B}(i)), p_i := \mathbb{E}(\mathbf{P}_i) \tag{5.33}$$

that a more abstract representation of the non-negative sequence $\mathbf{b} = (\mathbb{E}(\mathbf{B}(i)))_{i\in\mathbb{Z}_+}$ is given by

$$\mathbf{b} = \mathbf{p} * \mathbf{a} \tag{5.34}$$

with the non-negative sequence $\mathbf{a} = (a_i)_{i\in\mathbb{Z}_+}$ defined in relation (5.23). By relations (5.32) and (5.34), the total number of items (not necessarily different items!) in the market during the life cycle of a product is given by

$$\sum_{t=0}^{\infty} \mathbb{E}(\mathbf{B}(t) = \|\mathbf{b}\|_1 = \|\mathbf{p}\|_1 \|\mathbf{a}\|_1 = S_\infty \|\mathbf{a}\|_1 \tag{5.35}$$

with the expected market potential $S(\infty)$ defined in relation (5.3). Since the expected usage time $\mathbb{E}(\mathbf{U})$ is finite and $\mathbf{U} = \sum_{t=0}^{\infty} 1_{\{\mathbf{U}>t\}}$ we obtain

$$\infty > \mathbb{E}(\mathbf{U}) = \mathbb{E}\left(\sum_{t=0}^{\infty} 1_{\{\mathbf{U}>t\}}\right) = \sum_{t=0}^{\infty} 1 - F_{\mathbf{U}}(t) = \|\mathbf{a}\|_1 \tag{5.36}$$

Hence by relation (5.35) it follows that

$$\sum_{t=0}^{\infty} \mathbb{E}(\mathbf{B}(t) = \mathbf{S}(\infty)\mathbb{E}(\mathbf{U}) < \infty. \tag{5.37}$$

In Minner (2011) the expected sales $\mathbb{E}(\mathbf{P}_k)$ in period k are given by the Brockhoff model (Brockhoff (1967)), while for the cumulative sales the logistic growth model is suggested. In the Brockhoff sales model, it is assumed that

$$\mathbb{E}(\mathbf{P}_k) = ak^b e^{-ck} \tag{5.38}$$

for some $a > 0$, $b > 0$ and $c > 0$ and so for the Brockhof sales model we obtain $S(\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k)$ is finite. The logistics growth model is given by

$$\mathbb{E}(\mathbf{S}(k)) = \frac{a}{1 + e^{b-ck}} \tag{5.39}$$

Also for the logistic model, we obtain $S(\infty) = a$ is finite. Notice Minner (Minner (2011)) only mentioned these two different sales processes and did not consider the variance and covariance structure of the sales process and the installed base process. In his computational section, Minner assumed that the random demands are known by simulation and so the purchase process is a deterministic process. To add more structure to the purchase process, we assume as observed in automobile sales (Stephan, Gschwind & Minner (2010)), that the purchase process shows an autoregressive structure. In particular, as observed in Storey (2006), the purchase process of some well-known automobile brands resembles an autoregressive process of order 1. This means that the purchases in a given period also depend on the previous period's purchases. Since these purchase data are integer valued non-negative counting data collected sequentially in time, it seems odd to model this integer valued purchase process as a classical non-negative autoregressive time series model of order 1 (Chatfield & Xing (2019)). Instead, it seems more appropriate to model this by the non-negative integer valued time series model (Al-Osh & Alzaid (1987))

$$\mathbf{P}_0 = \epsilon_0, \mathbf{P}_i = \alpha \circ \mathbf{P}_{i-1} + \epsilon_i, i \in \mathbb{N} \tag{5.40}$$

with ϵ_i, $i \in \mathbb{N}$ a sequence of independent non-negative integer valued random variables having different means and variances and $0 < \alpha < 1$. This means using relation (5.196) iteratively that for every $i \in \mathbb{Z}_+$ and $0 < \alpha < 1$

$$\mathbb{E}(\mathbf{P}_i) = \alpha \mathbb{E}(\mathbf{P}_{i-1}) + \mathbb{E}(\epsilon_i) = \ldots = \sum_{j=0}^{i} \alpha^{i-j} \mathbb{E}(\epsilon_j) = (g * e)_i \tag{5.41}$$

with the non-negative sequences $\mathbf{e} = (e_i)_{i \in \mathbb{Z}_+}$ and $\mathbf{g} = (g_i)_{i \in \mathbb{Z}_+}$ given by

$$g_i := \alpha^i, e_i := \mathbb{E}(\epsilon_i). \tag{5.42}$$

This shows using relation (5.32) and (5.41) that

$$S(\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k) = \|\mathbf{g}\|_1 \|\mathbf{e}\|_1 = (1-\alpha)^{-1} \parallel \mathbf{e} \parallel_1 \tag{5.43}$$

Since the expected market potential $S(\infty)$ defined in relation (5.3) is finite, it follows by relation (5.43) that we need to assume additionally for an integer valued autoregressive purchase process $\mathbf{P}$ of order 1 listed in relation (5.40) that the sequence of integer valued non-negative random variables $\epsilon_t, t \in \mathbb{Z}_+$ must satisfy

$$E(\infty) := \|\mathbf{e}\|_1 = \sum_{i=0}^{\infty} \mathbb{E}(\epsilon_i) < \infty. \tag{5.44}$$

One can now show the following result for an non-negative integer valued autoregressive purchase process $\mathbf{P}$ of order 1.

Lemma 24. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ is given by a non-negative integer valued autoregressive time series model of order 1 introduced in relation (5.40) and satisfying $E(\infty) := \sum_{i=0}^{\infty} \mathbb{E}(\epsilon_i)$ is finite then the expectation $\mathbb{E}(\mathbf{B}(t))$ is finite and this expectation is given by*

$$\mathbb{E}(\mathbf{B}(t)) = \sum_{k=0}^{t} (g * e)_k a_{t-k} \tag{5.45}$$

with the sequence $\mathbf{g} = (g_i)_{i \in \mathbf{Z}_+}$ and $\mathbf{e} = (e_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.42)

Proof. It follows for a non-negative integer valued autoregressive purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ of order 1 applying relation (5.28) and (5.41) that

$$\mathbb{E}(\mathbf{B}(t)) = \sum_{k=0}^{t} \sum_{j=0}^{k} \alpha^{k-j} \mathbb{E}(\epsilon_j) a_{t-k} = \sum_{k=0}^{t} (g * e)_k a_{t-k}. \tag{5.46}$$

This shows the desired result. ☐

We will now compute the variance of the installed base process at time t. Applying Theorem 19 and relation (5.26), the following result for the variance of the installed base process at time t is easy to derive.

Lemma 25. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ introduced in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances then for every $t \in \mathbb{Z}_+$ the variance $Var(\mathbf{B}(t))$ is finite and this variance is given by*

$$Var(\mathbf{B}(t)) = \sum_{k=0}^{t} \sum_{i=0}^{t} a_{t-k} a_{t-i} Cov(\mathbf{P}_k, \mathbf{P}_i) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1 - a_{t-k}). \tag{5.47}$$

with the sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.23).

Proof. It follows by relation (5.26) and the well known rules for covariances (Ross (2023)) that

$$\begin{aligned} \mathrm{Var}(\mathbf{B}(t)) &= \mathrm{Cov}\left(\textstyle\sum_{k=0}^{t} a_{t-k} \circ \mathbf{P}_k, \sum_{i=0}^{t} a_{t-i} \circ \mathbf{P}_i\right) \\ &= \textstyle\sum_{k=0}^{t} \sum_{i=0}^{t} \mathrm{Cov}\left(a_{t-k} \circ \mathbf{P}_k, a_{t-i} \circ \mathbf{P}_i\right) \end{aligned} \tag{5.48}$$

This shows applying relation (5.196) for $k = i$ and relation (5.199) for $k \neq i$ the desired result. $\square$

The above result indicates that we can evaluate numerically $\mathbb{E}(\mathbf{B}(t))$ and $\mathrm{Var}(\mathbf{B}(t))$ for a purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ having finite variances and known correlated demands. Such a stochastic process is called in the literature a second-order stochastic process. We will again consider some special second order purchase processes. If the random purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$ at time k are not correlated meaning $\mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_i) = 0$ for every $k \neq i$ then the formula in relation (5.47) simplifies to

$$\mathrm{Var}(\mathbf{B}(t)) = \sum_{k=0}^{t} a_{t-k}^2 \mathrm{Var}(\mathbf{P}_k) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1 - a_{t-k}). \tag{5.49}$$

Also, if the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ is given by a non-negative integer valued autoregressive model of order 1 introduced in relation (5.40), we obtain applying relation (5.199) and for every m the random variable ϵ_m is independent of the random variables $\mathbf{P}_0,...,\mathbf{P}_{m-1}$ that for every $i > k$

$$\mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_i) = \mathrm{Cov}(\mathbf{P}_k, \alpha \circ \mathbf{P}_{i-1} + \epsilon_i) = \alpha \mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_{i-1}) = ... = \alpha^{i-k} \mathrm{Var}(\mathbf{P}_k). \tag{5.50}$$

Similarly reversing the indices i and k we obtain for $k > i$ that

$$\mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_i) = \alpha^{k-i} \mathrm{Var}(\mathbf{P}_i). \tag{5.51}$$

Applying relations (5.50) and (5.51), we obtain by relation (5.47) and introducing $a \wedge b := \min\{a, b\}$ for any numbers a, b that for any non-negative integer valued autoregressive purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ of order 1

$$\mathrm{Var}(\mathbf{B}(t)) = \sum_{k=0}^{t} \sum_{i=0}^{t} a_{t-k} a_{t-i} \alpha^{|k-i|} \mathrm{Var}(\mathbf{P}_{k \wedge i}) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1 - a_{t-k}). \tag{5.52}$$

Since the random variable ϵ_i is independent of $\mathbf{P}_{i-1}$, it also follows by relation (5.196) that

$$\operatorname{Var}(\mathbf{P}_i) = \operatorname{Var}(\alpha \circ \mathbf{P}_{i-1} + \epsilon_i) = \alpha^2 \operatorname{Var}(\mathbf{P}_{i-1}) + \alpha(1-\alpha)\mathbb{E}(\mathbf{P}_{i-1}) + \operatorname{Var}(\epsilon_i). \tag{5.53}$$

Applying the above formulas, we list the following algorithm to calculate $\operatorname{Var}(\mathbf{B}(t))$ for a non-negative integer valued autoregressive purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ of order 1.

Algorithm 1 Calculation of $\operatorname{Var}(\mathbf{B}(t))$ for a non-negative integer valued autoregressive purchase process $\mathbf{P}$ of order 1.

1. Set $\mathbb{E}(\mathbf{P}_0) = \mathbb{E}(\epsilon_0)$ and for $i = 1$ up to t evaluate

$$\mathbb{E}(\mathbf{P}_i) = \alpha \mathbb{E}(\mathbf{P}_{i-1}) + \mathbb{E}(\epsilon_i).$$

2. Set $\operatorname{Var}(\mathbf{P}_0) = \operatorname{Var}(\epsilon_0)$ and for $i = 1$ up to t evaluate

$$\operatorname{Var}(\mathbf{P}_i) = \alpha^2 \operatorname{Var}(\mathbf{P}_{i-1}) + \alpha(1-\alpha)\mathbb{E}(\mathbf{P}_{i-1}) + \operatorname{Var}(\epsilon_i)$$

3. Evaluate $Var(\mathbf{B}(t))$ applying relation (5.52).

Finally, we compute the covariance function of the installed base stochastic process $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$.

Lemma 26. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances then for every $t \in \mathbb{Z}_+$ and $s > t$ the covariance $Cov(\mathbf{B}(t), \mathbf{B}(s))$ is finite and this covariance function is given by*

$$Cov(\mathbf{B}(t), \mathbf{B}(s)) = \sum_{k=0}^{t} \sum_{i=0}^{s} a_{t-k} a_{s-i} Cov(\mathbf{P}_k, \mathbf{P}_i) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{s-k}(1 - a_{t-k}) \tag{5.54}$$

with the sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ defined in (5.23).

Proof. It follows by relation (5.26) using $a_i = 1 - F_{\mathbf{U}}(i)$ and the well known properties of covariances that

$$\begin{aligned}
\operatorname{Cov}(\mathbf{B}(t), \mathbf{B}(s)) &= \operatorname{Cov}\left(\sum_{k=0}^{t} a_{t-k} \circ \mathbf{P}_k, \sum_{i=0}^{s} a_{s-i} \circ \mathbf{P}_i\right) \\
&= \begin{cases} \operatorname{Cov}\left(\sum_{k=0}^{t} a_{t-k} \circ \mathbf{P}_k, \sum_{i=0}^{t} a_{s-i} \circ \mathbf{P}_i\right) + \\ \operatorname{Cov}\left(\sum_{k=0}^{t} a_{t-k} \circ \mathbf{P}_k, \sum_{i=t+1}^{s} a_{s-i} \circ \mathbf{P}_i\right) \end{cases} \\
&= \begin{cases} \sum_{k=0}^{t} \sum_{i=0}^{t} \operatorname{Cov}(a_{t-k} \circ \mathbf{P}_k, a_{s-i} \circ \mathbf{P}_i) + \\ \sum_{k=0}^{t} \sum_{i=t+1}^{s} \operatorname{Cov}(a_{t-k} \circ \mathbf{P}_k, a_{s-i} \circ \mathbf{P}_i) \end{cases}
\end{aligned} \tag{5.55}$$

Applying relation (5.199) we obtain for $k \leq t$ and $t+1 \leq i \leq s$ that

$$\operatorname{Cov}(a_{t-k} \circ \mathbf{P}_k, a_{s-i} \circ \mathbf{P}_i) = a_{t-k} a_{s-i} \operatorname{Cov}(\mathbf{P}_k, \mathbf{P}_i)$$

and this shows that the second double summation in relation (5.55) equals

$$\sum_{k=0}^{t}\sum_{i=t+1}^{s}\text{Cov}(a_{t-k}\circ\mathbf{P}_k, a_{s-i}\circ\mathbf{P}_i) = \sum_{k=0}^{t}\sum_{i=t+1}^{s} a_{t-k}a_{s-i}\text{Cov}(\mathbf{P}_k, \mathbf{P}_i) \tag{5.56}$$

Also by relation (5.199) we obtain that

$$\sum_{k=0}^{t}\sum_{i=0}^{t}\text{Cov}(a_{t-k}\circ\mathbf{P}_k, a_{s-i}\circ\mathbf{P}_i) = \begin{cases} \sum_{k=0}^{t}\sum_{i=0,i\neq k}^{t} a_{t-k}a_{t-i}\text{Cov}(\mathbf{P}_k,\mathbf{P}_i) \\ +\sum_{k=0}^{t}\text{Cov}(a_{t-k}\circ\mathbf{P}_k, a_{s-k}\circ\mathbf{P}_k) \end{cases} \tag{5.57}$$

Applying relation (5.197) to the summand in the second summation in relation (5.57) and using a_i is decreasing it follows for $s > t$

$$\begin{aligned} \text{Cov}(a_{t-k}\circ\mathbf{P}_k, a_{s-k}\circ\mathbf{P}_k) &= a_{t-k}a_{s-k}\text{Var}(\mathbf{P}_k) + ((a_{t-k}\wedge a_{s-k}) - a_{t-k}a_{s-k})\mathbb{E}(\mathbf{P}_k) \\ &= a_{t-k}a_{s-k}\text{Var}(\mathbf{P}_k) + (a_{s-k} - a_{t-k}a_{s-k})\mathbb{E}(\mathbf{P}_k). \end{aligned} \tag{5.58}$$

Combining relations (5.55), (5.56), (5.57) and (5.58) shows the desired result. □

If the random purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$ at time k are uncorrelated then as for the variance the formula in relation (5.54) simplifies for every $s > t$ to

$$\text{Cov}(\mathbf{B}(t), \mathbf{B}(s)) = \sum_{k=0}^{t} a_{t-k}a_{s-k}\text{Var}(\mathbf{P}_k) + \sum_{k=0}^{t}\mathbb{E}(\mathbf{P}_k)a_{s-k}(1-a_{t-k}) \tag{5.59}$$

In the next subsection, we will compute the expectation, variance and covariance function of the discarded items stochastic process $\mathbf{D} = \{\mathbf{D}(t) : t \in \mathbb{Z}_+\}$.

5.2.2 Expectation and Covariance Function of the Discarded Items Stochastic Process

In this subsection, we first compute the expectation and variance of the discarded items stochastic process $\mathbf{D}$ at any time t. By similar arguments as used in Lemma 23 for the installed base stochastic process and applying relation (5.26) we obtain replacing a_{t-k} by $1 - a_{t-k}$ the following result.

Lemma 27. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2), then for every $t \in \mathbb{Z}_+$ the expectation $\mathbb{E}(\mathbf{D}(t))$ is finite and this expectation is given by*

$$\mathbb{E}(\mathbf{D}(t)) = \sum_{k=0}^{t}\mathbb{E}(\mathbf{P}_k)(1-a_{t-k}). \tag{5.60}$$

Again by similar arguments as used in Lemma 25 for the installed base process and applying relation (5.26), we obtain replacing a_{t-k} by $1 - a_{t-k}$ the following result for the variance of the discarded items process at time t.

Lemma 28. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances, then for every $t \in \mathbb{Z}_+$ the variance $Var(\mathbf{D}_t)$ is finite and this variance is given by*

$$
Var(\mathbf{D}(t)) = \begin{cases} \sum_{k=0}^{t} \sum_{i=0}^{t} (1-a_{t-k})(1-a_{t-i})\, Cov(\mathbf{P}_k, \mathbf{P}_i) \\ + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1-a_{t-k}) \end{cases} \tag{5.61}
$$

If the random purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$, at time k are uncorrelated and so the random variables $\mathbf{P}_k, k \in \mathbb{N}$ satisfy $\mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_j) = 0$ for any $k \neq j$, we obtain from relation (5.61) that

$$
\mathrm{Var}(\mathbf{D}(t)) = \sum_{k=0}^{t} (1-a_{t-k})^2 \mathrm{Var}(\mathbf{P}_k) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1-a_{t-k}) \tag{5.62}
$$

Again, if the purchase process is an integer valued autoregressive process of order 1, one can simplify the computation of $\mathrm{Var}(\mathbf{D}(t))$ by using relations (5.50), (5.51) and (5.53). As for the installed base stochastic process, we obtain

$$
\mathrm{Var}(\mathbf{D}(t)) = \begin{cases} \sum_{k=0}^{t} \sum_{i=0}^{t} (1-a_{t-k})(1-a_{t-i}) \alpha^{|k-i|} \mathrm{Var}(\mathbf{P}_{k \wedge i}) \\ + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{t-k}(1-a_{t-k}) \end{cases} \tag{5.63}
$$

Computing $\mathrm{Var}(\mathbf{D}(t))$, we again use relation (5.53) given by

$$
\mathrm{Var}(\mathbf{P}_j) = \alpha^2 \mathrm{Var}(\mathbf{P}_{j-1}) + \alpha(1-\alpha)\mathbb{E}(\mathbf{P}_{j-1}) + \mathrm{Var}(\epsilon_j). \tag{5.64}
$$

and apply a similar algorithm as done for the installed base process. Finally by similar arguments as used in Lemma 26 and applying relation (5.26), we obtain replacing a_{t-k} by $1 - a_{t-k}$ the following result for the covariance function of the discarded items stochastic process.

Lemma 29. *If the random usage times $\mathbf{U}_n$, $n \in \mathbb{N}$ of the nth purchased item of the same product are independent and identically distributed and these random usage times are independent of the counting sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances then for every $t \in \mathbb{Z}_+$ and $s > t$ the covariance $Cov(\mathbf{D}(t), \mathbf{D}(s))$ is finite and the covariance function is given by*

$$
Cov(\mathbf{D}(t), \mathbf{D}(s)) = \sum_{k=0}^{t} \sum_{i=0}^{s} (1-a_{t-k})(1-a_{s-i})\, Cov(\mathbf{P}_k, \mathbf{P}_i) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) a_{s-k}(1-a_{t-k}) \tag{5.65}
$$

with the sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ defined in (5.23).

The above results hold for a second order discrete time purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$. To add more structure to the purchase process we first assume that the random purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$, at time k are not correlated. If this holds the covariance function for $s > t$ in relation (5.65) simplifies to

$$\text{(5.66)} \qquad \operatorname{Cov}(\mathbf{D}(t),\mathbf{D}(s)) = \sum\nolimits_{k=0}^{t}(1-a_{t-k})(1-a_{s-k})\operatorname{Var}(\mathbf{P}_k) + \sum\nolimits_{k=0}^{t}\mathbb{E}(\mathbf{P}_k)a_{s-k}(1-a_{t-k})$$

We might also use as a possible parametric model for the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ the so-called integer valued autoregressive time series model of order q. Since the random variables $\mathbf{P}_k$ count the number of sales in period k and are taken sequentially in time, it seems not appropriate to use the standard autoregressive model in classical time series analysis (Chatfield & Xing (2019), Franses, van Dijk & Opschoor (2014), Brockwell & Davis (1991), Brockwell & Davis (2002)) to model the sales process. An integer valued autoregressive model of order q for non-negative count data is given by (Jin-Guan & Yuan (1991))

$$\text{(5.67)} \qquad \mathbf{P}_i = \sum\nolimits_{j=1}^{q}\alpha_j \circ \mathbf{P}_{i-j} + \epsilon_j, j \in \mathbb{Z}_+$$

with $\mathbf{P}_i = 0$, $i < 0$ and ϵ_j, $j \in \mathbb{Z}_+$ a sequence of independent and non-negative integer valued random variables have a finite possible different mean and common variance σ and $0 \leq \alpha_i \leq 1$ for every $i = 1,...,q$. Also these random variables are independent of the so-called counting series $\alpha_j \circ \mathbf{P}_{i-j}$, $0 < \alpha_j < 1$, $j = 1,...,q$. The Steutel van Harn thinning operator is used in integer valued time series and replaces in classical time series models the multiplication $\alpha_j\mathbf{P}_{i-j}$. Observe for $q = 0$ the sales in each period are independent and hence uncorrelated and are represented by independent random variables having different means. Observe the mean of the random variable ϵ_j represents the seasonal effect. In this model the parameters $0 < \alpha_j < 1$ need to be estimated. For more details on integer valued non-negative time series models and how to estimate the unknown parameters the reader is referred to Fokianos (2012) and Scotto, Weiss & Gouveia (2015). If the sales process is a integer valued non-negative autoregressive model of order q then

$$\text{(5.68)} \qquad \mathbb{E}(\mathbf{P}_i) = \sum\nolimits_{j=1}^{q}\alpha_j\mathbb{E}(\mathbf{P}_{i-j}) + \mathbb{E}(\epsilon_j)$$

(Use $\mathbf{P}_m = 0$ for $m < 0$ in the above formula!). Applying this expression in relations (5.28), (5.60) and (5.75), we can numerically evaluate the expectations of the installed base, the discarded items, and the returned and defective items at time t. For $q = 0$ the demands are uncorrelated and we obtain $\mathbb{E}(\mathbf{P}_i) = \mathbb{E}(\epsilon_i)$. For $q = 1$ (a Markovian integer valued time series sales model) it follows by iterating relation (5.28) that

$$\text{(5.69)} \qquad \mathbb{E}(\mathbf{P}_i) = \sum\nolimits_{j=0}^{i}\alpha_1^j\mathbb{E}(\epsilon_{i-j}).$$

For any integer valued non-negative autoregressive purchase process of order $q \geq 2$, it is in theory possible to compute the variances of the installed base, discarded items and returned and defective items process using relations (5.47), (5.61) and (5.78) respectively by evaluating first the covariance function of the integer-valued autoregressive sales. However, to do a proper and mathematical sound analysis regarding the finiteness of these covariances of the sales process and how to compute them efficiently requires next to some tedious calculations also results regarding invertible elements in commutative Banach algebras (Frenk (1987)). Since this is too advanced this will not be part of this paper. For related simpler questions in classical time series, the reader is referred to Section 9.2 of Karlin (2014). In the next subsection, we derive the expectation and covariance function of the returned and defective items stochastic process.

5.2.3 Expectation and Covariance Function of the Defective Items Stochastic Process

We now consider similar questions as answered in the previous subsection for the returned and defective items stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$. Since by assumption the random vectors

$$(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...), n \in \mathbb{N} \tag{5.70}$$

are identically distributed and independent and these random vectors are independent of the sales process $\mathbf{S}$, we introduced in relation (5.25) the non-negative sequence $\mathbf{q} = (q_i)_{n\in\mathbb{Z}_+}$ with

$$q_i := \mathbb{P}(\mathbf{U} > i, \cup_{m=1}^{\infty}(\mathbf{T}_m = i)) \tag{5.71}$$

It is not clear in general whether q_i is increasing or decreasing in i. It should be clear that for every i, the above probability might be difficult to compute since the random usage time of each purchased item might depend on the number of failures occurring to this particular item. Hence the usage time might be a so-called stopping time (Çınlar (2011)) of the failure counting process $\mathbf{N}$. We will not continue in the next subsections this generality within the model since then we need to take into account the behaviour of customers. However, in this section, we do not impose any assumptions about the relation between usage times and failure counting processes except that the random vectors $(\mathbf{U}_n, \mathbf{N}_n)$, $n \in \mathbb{N}$ have the same probability law for each purchased item. To approximate this more difficult problem that the usage time depends on the failure counting process we might take the random usage time equal to a random variable having a mean equal to a given multiple of the warranty period of the product with some given variance but still assume that this random variable is independent of the failure process. This case is covered in our analysis. In the next subsection, we will therefore assume the simplifying assumption that the random usage time of the item is independent of the failure process of that item. The same assumption was also imposed in Minner (2011). If we impose this condition, then by relation (5.25)

$$q_i = a_i u_i, u_i := \mathbb{P}(\cup_{m=1}^{\infty}(\mathbf{T}_m = i)). \tag{5.72}$$

with the sequence $\mathbf{a} = (a_i)_{i\in\mathbb{Z}_+}$ given in relation (5.23). Clearly u_i represents the probability that an item in use at age i breaks down. If this multiplicative property holds for q_i, then we obtain by relations (5.22), (5.24) and (5.201) that for every $k \leq t$, $t \in \mathbb{Z}_+$

$$\mathbf{R}_k(t) \stackrel{d}{=} q_{t-k} \circ \mathbf{P}_k \stackrel{d}{=} u_{t-k} \circ \mathbf{B}_k(t). \tag{5.73}$$

To simplify relation (5.73), we may assume that the failure process of each item is a homogeneous Bernoulli counting process (see definition in next section) meaning $u_i = u$ with $0 < u < 1$ the probability of success of a Bernoulli variable. Differently, it represents the success probability in a geometric distribution representing the pmf of the interarrival failure times within this failure process (Çınlar (2013)). If this holds, we obtain by the same arguments that

$$\mathbf{R}(t) = \sum\nolimits_{k=0}^{t} \mathbf{R}_k(t) = \sum\nolimits_{k=0}^{t} (ua_{t-k}) \circ \mathbf{P}_k \stackrel{d}{=} u \circ \mathbf{B}(t). \tag{5.74}$$

Since with probability u every item in the market at time t will fail irrespective of the time it is already in the market and independent of the number of repairs already done on this item, relation (5.74) has an obvious interpretation. However, the assumption of a constant failure rate over time might not be realistic for most products. It seems more realistic to assume that a lot of products sold at the market have an increasing failure rate. Therefore, we can represent the failure process by a nonhomogeneous Bernoulli counting process or a renewal process. If the failure process is represented by a renewal process governed by independent and positive integer identically distributed interarrival times this means that repairs restore the item back to the new state. Assuming only independence of the failure point process and the usage time from the sales process, it follows by the same arguments as used before and applying the relation (5.11) that for every $t \in \mathbb{Z}_+$ the next result holds.

Lemma 30. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$, then for every $t \in \mathbb{Z}_+$ the expectation $\mathbb{E}(\mathbf{R}(t))$ is finite and this expectation is given by*

$$\mathbb{E}(\mathbf{R}(t)) = \sum\nolimits_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) q_{t-k}. \tag{5.75}$$

with the sequence the sequence $\mathbf{q} = (q_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.71).

As done for the expected installed base $\mathbb{E}(\mathbf{B}(t))$ at time t in relation (5.34), a more abstract representation for $\mathbb{E}(\mathbf{R}(t))$, $t \in \mathbb{Z}_+$ is given by

$$\mathbf{r} = \mathbf{p} * \mathbf{q} \tag{5.76}$$

with the sequences $\mathbf{p}$ and $\mathbf{q}$ listed in relation (5.33), respectively (5.25) and the sequence $\mathbf{r} = (r_i)_{i \in \mathbb{Z}+}$ defined by

$$r_i := \mathbb{E}(\mathbf{R}(i)) \tag{5.77}$$

By the same arguments as used before, one can now show the following expression for the variance $\text{Var}(\mathbf{R}_t)$ of the returned and defective items process at time t.

Lemma 31. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances, then for every*

$t \in \mathbb{Z}_+$ the variance $Var(\mathbf{R}(t))$ is finite and given by

$$Var(\mathbf{R}(t)) = \sum_{k=0}^{t} \sum_{i=0}^{t} q_{t-k} q_{t-i} Cov(\mathbf{P}_k, \mathbf{P}_i) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) q_{t-k}(1 - q_{t-k}) \tag{5.78}$$

with $\mathbf{q} = (q_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.71).

Again for the random variables $\mathbf{P}_k, k \in \mathbb{N}$ uncorrelated it follows from relation (5.78) that

$$\mathrm{Var}(\mathbf{R}(t)) = \sum_{k=0}^{t} q_{t-k}^2 \mathrm{Var}(\mathbf{P}_k) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k) q_{t-k}(1 - q_{t-k}) \tag{5.79}$$

The above result shows that the returned and defective items process is not a process with independent increments as often assumed in the literature. Finally we evaluate the covariance function of the returned and defective items process and as before this result is easy to prove using the same arguments as in Lemma 26 and applying relation (5.11) replacing a_t by q_t.

Lemma 32. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ defined in relation (5.2) and the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbf{Z}_+\}$ has finite variances, then for every $t \in \mathbb{Z}_+$ and $s > t$ the covariance $Cov(\mathbf{R}(t), \mathbf{R}(s))$ is finite and given by*

$$Cov(\mathbf{R}(t), \mathbf{R}(s)) = \sum_{k=0}^{t} \sum_{i=0}^{s} q_{t-k} q_{s-i} Cov(\mathbf{P}_k, \mathbf{P}_i) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k)((q_{s-k} \wedge q_{t-k}) - q_{t-k} q_{s-k}). \tag{5.80}$$

with the sequence $\mathbf{q} = (q_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.71)

Again if the purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$ are uncorrelated then by Lemma 32 we obtain

$$\mathrm{Cov}(\mathbf{R}(t), \mathbf{R}(s)) = \sum_{k=0}^{t} q_{t-k} q_{s-k} \mathrm{Var}(\mathbf{P}_k) + \sum_{k=0}^{t} \mathbb{E}(\mathbf{P}_k)((q_{s-k} \wedge q_{t-k}) - q_{t-k} q_{s-k}). \tag{5.81}$$

In the next subsection, we will evaluate the expectation and variance of the remaining returned and defective items stochastic process at any time t. Clearly, if the manufacturer has to make a last buy decision at a given time t it is important to know the expected demand for spare parts needed after time t and the variance of this random variable. This knowledge will determine the size of the last buy decision.

5.2.4 Expectation and Variance of the Remaining Defective Items Stochastic Process

Finally, we will analyze the stochastic process $\mathbf{V} = \{\mathbf{V}(t) : t \in \mathbb{Z}_+\}$ of the total number of returned defective items from time t until the end of the life cycle of the product. In the next lemma, we obtain an expression for $\mathbb{E}(\mathbf{V}(t))$ again assuming that the usage times and failure point processes of each purchased item are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$. Notice we always assume that the expected market potential $S(\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k)$ is finite and the expected number of failures $\mathbb{E}(\mathbf{N}(\mathbf{U}))$ of an item during its usage time is finite. This implies that the next expectation is finite.

Lemma 33. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ...\mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ *are identically distributed and independent of the cumulative sales stochastic process* $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$, *and the failure process and the usage time of each purchased item satisfies* $\mathbb{E}(\mathbf{N}(\mathbf{U}))$ *is finite, then* $\mathbb{E}(\mathbf{V}(t))$ *is finite for every* $t \in \mathbb{Z}_+$ *and this expectation is given by*

$$\mathbb{E}(\mathbf{V}(t)) = S(\infty)\mathbb{E}(\mathbf{N}(\mathbf{U}-1)) - \sum_{k=0}^{t-1}\mathbb{E}(\mathbf{P}_k)\mathbb{E}(\mathbf{N}((\mathbf{U}-1)\wedge(t-k-1))) \tag{5.82}$$

with $S(\infty)$ *defined in relation (5.3).*

Proof. Since the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ...\mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent of the sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the failure process and usage time of each purchased item satisfies $\mathbb{E}(\mathbf{N}(\mathbf{U}))$ is finite, it follows by relation (5.15) and Lemma 61

$$\mathbb{E}(\mathbf{A}(t)) = \mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))\mathbb{E}(\mathbf{N}(\mathbf{U}-1)) = \mathbb{E}(\mathbf{N}(\mathbf{U}-1))\sum_{k=t}^{\infty}\mathbb{E}(\mathbf{P}_k) \tag{5.83}$$

and for every $k \leq t-1$

$$\begin{aligned}\mathbb{E}(\mathbf{A}_k(t)) &= \mathbb{E}(\mathbf{S}(k) - \mathbf{S}(k-1))\mathbb{E}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1)\wedge(t-k-1))) \\ &= \mathbb{E}(\mathbf{P}_k)\mathbb{E}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1)\wedge(t-k-1)))\end{aligned}$$

This shows applying relation (5.17) the desired result. □

If we assume additionally in Lemma 33 that also the usage time is independent of the failure process then by relation (5.20) it follows that

$$\mathbb{E}(\mathbf{V}(t)) = \mathbf{S}(\infty)\mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U}-1)) - \sum_{k=0}^{t-1}\mathbb{E}(\mathbf{P}_k)\mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U}-1)\wedge(t-k-1))) \tag{5.84}$$

with Φ the mean value function of the failure counting stochastic process defined in relation (5.19). In our computational section we will only deal with the case that the random usage time is independent of the failure counting process and so we need to evaluate the simplified formula in relation (5.84) for the expected number of returned and defective items after time t. We will now continue to compute for an arbitrary cumulative sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ the variance $\mathrm{Var}(\mathbf{V}(t))$ for any $t \in \mathbb{Z}_+$. To do so we first simplify the notation introducing for every $t \in \mathbb{Z}_+$ the (increasing) sequence $(\alpha_{i,t})_{i=0}^{t-1}$ and the sequence $(\beta_{i,t-1})_{i=0}^{t-1}$ given by

$$\alpha_{i,t} := \mathbb{E}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}(\mathbf{U}-1)\wedge(t-i-1))) \tag{5.85}$$

and

$$\beta_{i,t} := \mathrm{Var}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))). \tag{5.86}$$

Since $\mathbf{N}(0) \overset{a.s}{=} 0$ we obtain for every $t \in \mathbb{N}$

$$\alpha_{t-1,t} = \mathbb{E}(\mathbf{N}(\mathbf{U}-1)), \beta_{t-1,t} = \mathrm{Var}(\mathbf{N}(\mathbf{U}-1)). \tag{5.87}$$

Applying Lemma 64 we obtain for $S(\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k)$ is finite that

$$\mathrm{Var}(\mathbf{S}(\infty) - \mathbf{S}(t-1)) = \lim_{k \uparrow \infty} \mathrm{Var}(\mathbf{S}(k) - \mathbf{S}(t)) = \sum_{k=t}^{\infty} \sum_{j=t}^{\infty} \mathrm{Cov}(\mathbf{P}_k, \mathbf{P}_j) \tag{5.88}$$

This shows that $\mathrm{Var}(\mathbf{S}(\infty))$ is finite if and only if for every $\mathrm{Var}(\mathbf{S}(\infty) - \mathbf{S}(t-1))$ is finite for every $t \in \mathbb{Z}_+$. Also it follows for any sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ using Liapounovs inequality that $\mathrm{Var}(\mathbf{S}(\infty))$ is finite implies $S(\infty) = \mathbb{E}(\mathbf{S}(\infty))$ is finite. Before computing the variance $\mathrm{Var}(\mathbf{V}(t))$ for every $t \in \mathbb{Z}_+$ we need the following result.

Lemma 34. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ...\mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are independent and identically distributed satisfying $\mathbb{E}(\mathbf{N}^2(\mathbf{U}))$ is finite and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ satisfying $Var(\mathbf{S}(\infty))$ is finite then the next results hold.*

6.1 The variance of the random variable $\mathbf{A}(t)$ listed in relation (5.13) is finite for every $t \in \mathbb{Z}_+$ and given by

$$\begin{aligned} Var(\mathbf{A}(t)) &= \mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1)\beta_{t-1,t} + Var(\mathbf{S}(\infty) - \mathbf{S}(t-1))\alpha_{t-1,t}^2 \\ &= \textstyle\sum_{i=t}^{\infty} \mathbb{E}(\mathbf{P}_i)\beta_{t-1,t} + Var(\mathbf{S}(\infty) - \mathbf{S}(t-1))\alpha_{t-1,t}^2 \end{aligned} \tag{5.89}$$

6.2 For any $0 \leq k \leq t-1$ the variance of the random variable $\mathbf{A}_k(t)$ listed in relation (5.15) is finite and given by

$$Var(\mathbf{A}_k(t)) = \mathbb{E}(\mathbf{P}_k)\beta_{k,t} + Var(\mathbf{P}_k)\alpha_{k,t}^2 \tag{5.90}$$

6.3 For any $0 \leq k \leq t-1$ the covariance of the random variables $\mathbf{A}_k(t)$ and $\mathbf{A}(t)$ is finite and given by

$$Cov(\mathbf{A}_k(t), \mathbf{A}(t)) = \alpha_{t-1,t}\alpha_{k,t} Cov(\mathbf{P}_k, \mathbf{S}(\infty) - \mathbf{S}(t-1)) \tag{5.91}$$

6.4 For any $0 \leq k \neq i \leq t-1$ the covariance of the random variables $\mathbf{A}_k(t)$ and $\mathbf{A}_i(t)$ is finite and given by

$$Cov(\mathbf{A}_k(t), \mathbf{A}_i(t) = \alpha_{k,t}\alpha_{i,t} Cov(\mathbf{P}_k, \mathbf{P}_i). \tag{5.92}$$

Proof. Since $\mathrm{Var}(\mathbf{S}(\infty))$ and $\mathbb{E}(\mathbf{N}^2(\mathbf{U}))$ is finite we obtain by the observations before this lemma that $\mathrm{Var}(\mathbf{S}(\infty) - \mathbf{S}(t-1))$ and the constants $\alpha_{i,t}$ and $\beta_{i,t}$, $i = 0, ..., t-1$ are finite for every t. Since the random vectors

$$(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ...\mathbf{T}_m^{(n)}, ...), n \in \mathbb{N}$$

are independent and identically distributed and independent of the sales process $\mathbf{S}$ we obtain by relations (5.13) and (5.15) and applying Lemma 61 the desired results. □

Using Lemma 34 and the rules for covariances one can easily verify the next result for $\mathrm{Var}(\mathbf{V}(t))$ for any t.

Lemma 35. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ...\mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ *are independent and identically distributed satisfying* $\mathbb{E}(\mathbf{N}^2(\mathbf{U}))$ *is finite and these random vectors are independent of the cumulative sales stochastic process* $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ *satisfying* $Var(\mathbf{S}(\infty))$ *is finite then it follows for every* $t \in \mathbb{Z}_+$ *that the variance of the random variable* $\mathbf{V}(t)$ *listed in relation (5.17) is finite and given by*

$$Var(\mathbf{V}(t)) = \sum_{k=0}^{t-1}\sum_{i=0}^{t-1} Cov(\mathbf{A}_k(t), \mathbf{A}_i(t)) + Var(\mathbf{A}(t)) + 2\sum_{k=0}^{t-1} Cov(\mathbf{A}_k(t)), \mathbf{A}(t)). \tag{5.93}$$

with $Var(\mathbf{A}(t))$ *given in relation (5.89),* $Var(\mathbf{A}_k(t))$ *given in relation (5.90) and* $Cov(\mathbf{A}_k(t), \mathbf{A}_i(t))$ *given in relation (5.92).*

Proof. By relation (5.17) we know that $\mathbf{V}(t) = \sum_{k=0}^{t-1} \mathbf{A}_k(t) + \mathbf{A}(t)$. This implies by the standard rules for covariances that

$$\begin{aligned} \mathrm{Var}(\mathbf{V}(t)) &= \mathrm{Cov}(\textstyle\sum_{k=0}^{t-1} \mathbf{A}_k(t) + \mathbf{A}(t), \sum_{i=0}^{t-1} \mathbf{A}_i(t) + \mathbf{A}(t)) \\ &= \mathrm{Var}(\textstyle\sum_{k=0}^{t-1} \mathbf{A}_k(t)) + \mathrm{Var}(\mathbf{A}(t)) + 2\sum_{i=0}^{t-1} \mathrm{Cov}(\mathbf{A}_i(t), \mathbf{A}(t)). \end{aligned} \tag{5.94}$$

Also it follows by the definition of a covariance that

$$\mathrm{Var}(\sum_{k=0}^{t-1} \mathbf{A}_k(t)) = \sum_{k=0}^{t-1}\sum_{i=0}^{t-1} \mathrm{Cov}(\mathbf{A}_k(t), \mathbf{A}_i(t)). \tag{5.95}$$

Combining relations (5.94) and (5.95) yields the desired result. □

Clearly it might be difficult to evaluate the above formula for a second order stochastic purchase process satisfying $\mathrm{Var}(\mathbf{S}(\infty))$ is finite. Hence we need to impose additional conditions. In case the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ satisfying $\mathrm{Var}(\mathbf{S}(\infty))$ is finite consists of uncorrelated purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ the formula in relation (5.88) simplifies to

$$Var(\mathbf{S}(\infty) - \mathbf{S}(t-1)) = \sum_{k=t}^{\infty} \mathrm{Var}(\mathbf{P}_k). \tag{5.96}$$

and so $\mathrm{Var}(\mathbf{S}(\infty)) = \sum_{k=0}^{\infty} \mathrm{Var}(\mathbf{P}_k)$. If this holds we also obtain by relation (5.91) and (5.92) that for every $0 \leq k \leq t-1$ and $k \neq i$

$$\mathrm{Cov}(\mathbf{A}_k(t), \mathbf{A}(t)) = \mathrm{Cov}(\mathbf{A}_k(t), \mathbf{A}_i(t)) = 0 \tag{5.97}$$

and so we obtain by relation (5.93)

$$\mathrm{Var}(\mathbf{V}(t)) = \sum_{k=0}^{t-1} \mathrm{Var}(\mathbf{A}_k(t)) + \mathrm{Var}(\mathbf{A}(t)) \tag{5.98}$$

Although we have derived a simplified formula for $\mathrm{Var}(\mathbf{V}(t))$ for uncorrelated purchases we still need to numerically evaluate in this formula the constants $\alpha_{i,t}$ and $\beta_{i,t}$ listed in relation for every $0 \leq i \leq t-1$. For $\alpha_{i,t}$ this is relatively easy using the mean value function of the failure point process and the pmf of the usage time. For the constants $\beta_{i,t}$ this is more difficult and we need to introduce some special class of nonhomogeneous failure counting processes. Without any conditions except knowing the mean vale function of the failure counting process and assuming in Lemma 34 that the usage time is independent of the failure process we obtain applying relation (5.20) that for every $0 \le i \le t-1$

$$\begin{aligned} \alpha_{i,t} &= \mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U}-1)-\Phi((\mathbf{U}-1)\wedge(t-i-1)) \\ &= \mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U}-1)-\Phi(t-i-1))\mathbf{1}_{\{\mathbf{U}\ge t-i+1\}}) \qquad (5.99) \\ &= \mathbb{E}_{\mathbf{U}}(\Phi(\mathbf{U}-1)\mathbf{1}_{\{\mathbf{U}\ge t-i+1\}})-\Phi(t-i-1)\mathbb{P}(\mathbf{U}\ge t-i+1) \end{aligned}$$

This formula can easily be computed if the random usage time is bounded from above with probability 1. For the constants $\beta_{i,t}$ this is more complicated and we can only derive for every $0 \le i \le t-1$

$$\begin{aligned} \beta_{i,t} &= \mathrm{Var}((\mathbf{N}(\mathbf{U}-1)-\mathbf{N}(t-i-1))\mathbf{1}_{\{\mathbf{U}\ge t-i+1\}}) \qquad (5.100) \\ &= \mathbb{E}((\mathbf{N}(\mathbf{U}-1)-\mathbf{N}(t-i-1))^2\mathbf{1}_{\{\mathbf{U}\ge t-i+1\}})-\alpha_{i,t}^2 \end{aligned}$$

To overcome this difficulty we will now introduce the following class of stochastic processes (Çınlar (2013)) and their associated counting processes for which these constants can be evaluated numerically.

Definition 10. *The stochastic process $\mathbf{Z}=\{\mathbf{Z}_i : i\in\mathbb{N}\}$ is called a Bernoulli process if the random variables $\mathbf{Z}_i, i\in\mathbb{N}$ are independent and Bernoulli distributed. It is called a homogeneous Bernoulli process if the random variables $\mathbf{Z}_i$, $i\in\mathbb{N}$ are identically distributed and a nonhomogeneous Bernoulli process if the random variables $\mathbf{Z}_i, i\in\mathbb{N}$ are not identically distributed. If $\mathbf{Z}=\{\mathbf{Z}_i : i\in\mathbb{N}\}$ is a Bernoulli process then the counting process $\mathbf{N}=\{\mathbf{N}(k) : k\in\mathbb{N}\}$ given by*

$$\mathbf{N}(k) := \sum_{i=1}^{k}\mathbf{1}_{\{\mathbf{Z}_i=1\}}$$

is called a Bernoulli counting process. It is called a (nonhomogeneous) homogeneous Bernoulli counting process if the Bernoulli process $\mathbf{Z}$ is (nonhomogeneous) homogeneous. We say it is a Bernoulli counting process with parameters $0<\delta i<1$, $i\in\mathbb{N}$ if

$$\delta_i = \mathbb{P}(\mathbf{Z}_i=1)$$

If the failure process of an item is a so-called minimal repair failure process that this counting failure process is a a Bernoulli counting process with parameters

$$\delta_i = \mathbf{P}(\mathbf{Z}_i=1) = \mathbb{P}(\mathbf{L}=i|\mathbf{L}\ge i), i\in\mathbb{N} \qquad (5.101)$$

with $\mathbf{L}$ denoting the random time of the first failure of the item. In this particular process any repairs will not change the residual life time of the item. If the failure process is given by a nonhomogeneous Bernoulli counting process it is possible to compute the constants $\alpha_{i,t}$ and $\beta_{i,t}$, $0\le i\le t-1$ listed in relation (5.85) and (5.86) needed in the formula for $\mathrm{Var}(\mathbf{A}(t))$ once we know the pmf of the usage time $\mathbf{U}$ and this random usage time is independent of the Bernoulli counting process.

Lemma 36. *If the usage time $\mathbf{U}$ is independent of the nonhomogeneous Bernoulli counting process*

$\mathbf{N}=\{\mathbf{N}(k):k\in\mathbb{N}\}$ *having parameters* $\delta_k,k\in\mathbb{N}$ *then for every* $0\leq i\leq t-1$

$$\alpha_{i,t}=\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right) \tag{5.102}$$

Proof. It follows for a nonhomogeneous Bernoulli counting process with parameters $\delta_i,i\in\mathbb{N}$ that

$$\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))=\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\mathbf{1}_{\{\mathbf{Z}_k=1\}} \tag{5.103}$$

and by the tower property of conditional expectations we obtain

$$\alpha_{i,t}:=\mathbb{E}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}(\mathbf{U}-1)\wedge(t-i-1)))=\mathbb{E}\left(\mathbb{E}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U})\right). \tag{5.104}$$

Since the random usage time $\mathbf{U}$ is independent of the Bernoulli counting process $\mathbf{Z}=\{\mathbf{Z}_k,k\in\mathbb{N}\}$ having parameters $\delta_k,k\in\mathbb{N}$ we obtain by relation (5.103) that this conditional expectation equals

$$\mathbb{E}\left(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U}\right)=\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\beta_k \tag{5.105}$$

and this shows by relation (5.104) the desired result. □

Hence knowing the pmf of the random usage time $\mathbf{U}$ and the parameters δ_k, $k\in\mathbb{Z}_+$ we can numerically evaluate $\alpha_{i,t}$ for every $i\leq t-1$. This is especially easy if we assume that the pmf of the ramdom usage time is only nonzero on a bounded set. If $i=t-1$ we obtain using $\mathbf{N}_0=0$ that

$$\alpha_{t-1,t}=\mathbb{E}(\mathbf{N}_{\mathbf{U}-1})=\mathbb{E}\left(\sum_{k=1}^{\mathbf{U}-1}\beta_k\mathbf{1}_{\{\mathbf{U}\geq 2\}}\right)=\mathbb{E}(\Phi(\mathbf{U}-1))$$

with $\Phi(k)=\mathbb{E}(\mathbf{N}_k)=\sum_{i=1}^k\beta_i,k\geq 1,\Phi(0)=0$ the mean value function of the Bernoulli counting process. For the constants $\beta_{i,t},0\leq i\leq t-1$ one can show the following result.

Lemma 37. *If the random usage time* $\mathbf{U}$ *is independent of the Bernoulli counting process* $\mathbf{N}=\{\mathbf{N}(k):k\in\mathbb{N}\}$ *with parameters* $0<\delta_k<1,k\in\mathbb{N}$ *then for every* $0\leq i\leq t-1$

$$\beta_{i,t}=\left\{\begin{array}{l}\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)+\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\left(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)^2\right)\\ \\ -\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)^2-\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k^2\right)\end{array}\right.$$

Proof. It follows as in the previous lemma for a nonhomogeneous Bernoulli counting process that

$$\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))=\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\mathbf{1}_{\{\mathbf{Z}_k=1\}} \tag{5.106}$$

and so

$$\mathrm{Var}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1)))=\mathrm{Var}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\mathbf{1}_{\{\mathbf{Z}_k=1\}}\right).$$

By the independence of the usage time $\mathbf{U}$ and the Bernoulli process $\mathbf{Z}_k,k\in\mathbb{N}$ we obtain by relation

(5.106) that the conditional variance $\mathrm{Var}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U})$ equals

$$\mathrm{Var}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U})=\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k(1-\delta_k)$$

and the conditional expectation $\mathbb{E}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U})$ equals

$$\mathbb{E}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U})=\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\beta_k.$$

This implies by the conditional variance formula that

$$\begin{aligned}\mathrm{Var}\left(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\right) &= \begin{cases}\mathbb{E}(\mathrm{Var}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U}))\\ +\mathrm{Var}(\mathbb{E}(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1))\mid\mathbf{U}))\end{cases}\\ &= \begin{cases}\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k(1-\delta_k)\right)\\ +\mathrm{Var}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right).\end{cases}\end{aligned}$$

It follows by the definition of the variance of a random variable that

$$\mathrm{Var}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)=\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\left(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)^2\right)-\mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)^2$$

and this shows after some calculation that

$$\beta_{i,n}=\mathrm{Var}\left(\mathbf{N}(\mathbf{U}-1)-\mathbf{N}((\mathbf{U}-1)\wedge(t-i-1)\right)=\begin{cases}\mathbb{E}(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}})\\ +\mathbb{E}\left(\left(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\right)^2\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\right)\\ -\mathbb{E}\left(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\right)^2\\ -\mathbb{E}\left(\sum_{k=t-i}^{\mathbf{U}-1}\delta_k^2\mathbf{1}_{\{\mathbf{U}\geq t-i+1\}}\right).\end{cases}$$

This verifies the result. □

Since next to knowing how many spare parts are needed to obtain a certain service level and to avoid an underestimation or overestimation of requested spare parts at a given time it is also important to know for the manufacturer at which time during the life cycle of a product most resources should be allocated to spare parts management. Therefore in the next section we derive global properties of the functions expressing over time the size of the expected installed base and the expected number of returned and defective item at a given time. It is to be expected that at the period that the expected returned and defective items are maximal the manufacture will be most busy with after-sales services.

To determine the global behaviour of the sequence $(B_t)_{t\in\mathbb{Z}_+}$ with $B_t := \mathbb{E}(\mathbf{B}(t))$, it is convenient to give the sequence $(B_t)_{t\in\mathbb{Z}_+}$ the following probabilistic interpretation. Since by assumption

$$S(\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k) = \|\mathbf{p}\|_1$$

is finite with the sequence $\mathbf{p} = (p_i)_{i\in\mathbb{Z}_+}$ introduced in relation (5.33), we obtain that the non-negative sequence $\tilde{\mathbf{p}} = (\tilde{p}_i)_{i\in\mathbb{Z}_+}$ given by

(5.107) $$\tilde{p}_i := \frac{p_i}{S_\infty} = \frac{p_i}{\|\mathbf{p}\|_1}, i \in \mathbb{Z}_+$$

is a pmf on $\mathbb{Z}_+$. Also, since $\mathbb{E}(\mathbf{U})$ is finite, it follows by relation (5.36) and the sequence $\mathbf{a} = (a_i)_{i\in\mathbb{Z}_+}$ introduced in relation (5.23) that the non-negative sequence $\tilde{\mathbf{a}} = (\tilde{a}_i)_{i\in\mathbb{Z}_+}$ given by

(5.108) $$\tilde{a}_i := \frac{a_i}{\mathbb{E}(\mathbf{U})} = \frac{a_i}{\|\mathbf{a}\|_1}, i \in \mathbb{Z}_+$$

is a pmf on $\mathbb{Z}_+$. If the non-negative integer valued random variable $\mathbf{X}$ has pmf $\tilde{\mathbf{a}} = (\tilde{a}_i)_{i\in\mathbb{Z}_+}$ denoted by $\mathbf{X} \sim \tilde{\mathbf{a}}$ and the non-negative integer valued random variable $\mathbf{Y}$ has pmf $\tilde{\mathbf{p}} = (\tilde{p}_i)_{i\in\mathbb{Z}_+}$ and the random variables $\mathbf{X}$ and $\mathbf{Y}$ are independent, then it follows by relation (5.28) that for every $t \in \mathbb{Z}_+$

(5.109) $$\mathbb{E}(\mathbf{B}(t)) = \mathbf{S}(\infty)\mathbb{E}(\mathbf{U})(\tilde{a} * \tilde{p})_t = \mathbf{S}(\infty)\mathbb{E}(\mathbf{U})\mathbb{P}(\mathbf{X}+\mathbf{Y} = t).$$

We now introduce the next definition (Keilson (1979)). Observe any sequence $(b_i)_{i\in\mathbb{Z}_+}$ on $\mathbb{Z}_+$ is always extended to a sequence on $\mathbb{Z}$ by setting $b_i = 0$ for $i < 0$.

Definition 11. *A non-negative sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ is called logconcave if $b_i^2 \geq b_{i+1}b_{i-1}$ for every $i \in \mathbb{Z}$. Moreover, a non-negative sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ is called unimodal if there exists some $u \in \mathbb{Z}$ such that $b_{i-1} \leq b_i$ for every $i \leq u$ and $b_{i+1} \leq b_i$ for every $i \geq u$. The point u is called the unimodality point of the unimodal non-negative sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$. The pmf $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ is called strongly unimodal if every convolution of this sequence with a unimodal pmf is unimodal.*

To compute the unimodality point of a unimodal non-negative sequence $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$ we observe that this point is given by $i^\star := \text{argmax}\{b_i : i \in \mathbb{Z}\}$. This shows using the definition of a unimodal non-negative sequence and introducing the first order difference $\Delta(b)_i := b_i - b_{i-1}$ of this sequence that this unimodality point is given by

(5.110) $$i^\star = \sup\{i \in \mathbb{Z} : \Delta(b)_i \geq 0\}.$$

This shows it is easy to compute for any unimodal sequence its unimodality point. Notice that a unimodal non-negative sequence might have multiple unimodality points. Since in the introduction phase of the lifecycle, one expects that the expected purchases per time period go up over time and after the maturity phase of the lifecycle the expected purchases per time period go down over time, it seems natural to assume that the expected sales $\mathbb{E}(\mathbf{P}_k)$, $k \in \mathbb{Z}_+$ satisfy the unimodality property. If the positive sequence $\mathbb{E}(\mathbf{P}_k)$, $k \in \mathbb{Z}_+$ is log concave meaning the ratios $\frac{\mathbb{E}(\mathbf{P}_{k+1})}{\mathbb{E}(\mathbf{P}_k)}$ are decreasing and next to the increasing decreasing property of the expected purchase per time period, this implies for such a sales process that the percentage increase of the expected purchases per time period in the introduction phase of the life cycle becomes smaller over time before reaching the maturity phase while after the maturity phase, the percentage decrease in expected purchases per time period declines over time. Also, the logconcavity of the sequence q_n, $n \in \mathbb{Z}_+$ listed in relation means that the pmf of the random usage time has an increasing discrete failure rate function. This implies the longer one uses a product, the more likely it becomes that one discards the product in the next time period. This also seems a reasonable probabilistic assumption in reality. For the sequence $\mathbb{E}(\mathbf{B}(n))$, $n \in \mathbb{Z}_+$ one can now show the following result. A similar result as the next one for the continuous time version of the installed base process is shown in Lemma 1 of Amniattalab et al. (2023a).

Lemma 38. *The next results hold for the installed base stochastic process* $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$.

7.1 If the non-negative sequence $\mathbf{p} = (p_i)_{i \in \mathbb{Z}_+}$ *listed in relation (5.33) is unimodal and the non-negative sequence* $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ *listed in relation (5.23) is logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t))_{t \in \mathbb{Z}_+}$ *is unimodal.*

7.2 If the non-negative sequence $\mathbf{p} = (p_i)_{i \in \mathbb{Z}_+}$ *listed in relation (5.33) is logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t)))_{t \in \mathbb{Z}_+}$ *is unimodal.*

7.3 If the non-negative sequences $\mathbf{p} = (p_i)_{i \in \mathbb{Z}_+}$ *listed in relation (5.33) and* $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ *listed in relation (5.23) are logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t)))_{t \in \mathbb{Z}_+}$ *is logconcave.*

Proof. To prove part 1, we observe by Theorem 3 of Keilson & Gerber (1971) that the pmf $\tilde{\mathbf{a}} = (\tilde{a}_i)_{i \in \mathbb{Z}_+}$ listed in relation (5.25) is strongly unimodal and the pmf $\tilde{\mathbf{p}} = (\tilde{p}_i)_{i \in \mathbb{Z}_+}$ listed in relation (5.107) is unimodal. This shows by relation (5.109) that the sequence $(\mathbb{E}(\mathbf{B}(i)))_{i \in \mathbb{Z}_+}$ is unimodal. As in part 1, we observe again by Theorem 3 of Keilson & Gerber (1971) that the pmf $\tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_i)_{i \in \mathbb{Z}_+}$ is strongly unimodal. Since the pmf $\tilde{\mathbf{a}} = (\tilde{a}_t)_{t \in \mathbb{Z}_+}$ defined in relation (5.25) is clearly unimodal with unimodality point 0, it follows applying relation (5.109) that the sequence $(\mathbb{E}(\mathbf{B}(t)))_{t \in \mathbb{Z}_+}$ is unimodal. In part 3, both pmfs $\tilde{\mathbf{p}} = (\tilde{p}_i)_{i \in \mathbb{Z}_+}$ and $\tilde{\mathbf{a}} = (\tilde{a}_i)_{i \in \mathbb{Z}_+}$ are strongly unimodal and hence by Theorem 2 of Keilson & Gerber (1971) the pmf $\tilde{\mathbf{a}} * \tilde{\mathbf{p}}$ is strongly unimodal. This shows by Theorem 3 of Keilson & Gerber (1971) that the pmf $\tilde{\mathbf{a}} * \tilde{\mathbf{p}}$ is logconcave and hence by relation (5.109) the sequence $(\mathbb{E}(\mathbf{B}(t)))_{t \in \mathbb{Z}_+}$ is logconcave. $\square$

It is easy to check for a sales process satisfying $\mathbb{E}(\mathbf{P}_k) = ak^b e^{-ck}$ for any $a, b, c > 0$ that the sequence $ak^b e^{-ck}$, $k \in \mathbb{Z}$ is logconcave. We call such a purchase process a Brockhoff purchase process (Brockhoff (1967)). By part 2 of Lemma 38 we obtain that for a Brockhoff purchase process the sequence $\mathbb{E}(\mathbf{B}(t))$, $t \in \mathbb{Z}$ is unimodal. Checking the conditions of Lemma 38, one can show the following result for an autoregressive purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ of order 1.

Lemma 39. *The next results hold for the installed base stochastic process* $\mathbf{B} = \{\mathbf{B}(t) : t \in \mathbb{Z}_+\}$.

8.1 If the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ *is an autoregressive process of order* 1 *listed in* *relation (5.40) and the non-negative sequence* $\mathbf{e} = (\mathbb{E}(\epsilon_i))_{i\in\mathbb{Z}_+}$ *is unimodal and the non-negative sequence* $\mathbf{a} = (a_i)_{i\in\mathbb{Z}_+}$ *listed in relation (5.23) is logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t)))_{t\in\mathbb{Z}_+}$ *is unimodal.*

8.2 If the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ *is an autoregressive process of order* 1 *listed in relation (5.40) and the non-negative sequence* $\mathbf{e} = (\mathbb{E}(\epsilon_i))_{i\in\mathbb{Z}_+}$ *is logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t)))_{t\in\mathbb{Z}_+}$ *is unimodal.*

8.3 If the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ *is an autoregressive process of order* 1 *listed in relation (5.40) and the non-negative sequences* $\mathbf{e} = (\mathbb{E}(\epsilon_i))_{i\in\mathbb{Z}_+}$ *and* $\mathbf{a} = (a_i)_{i\in\mathbb{Z}_+}$ *listed in relation (5.23) are logconcave, then the sequence* $(\mathbb{E}(\mathbf{B}(t)))_{t\in\mathbb{Z}_+}$ *is logconcave.*

Proof. To show part 1 we observe that the geometric pmf $\mathbf{g} = (g_i)_{n\in\mathbb{Z}_+}$ given by

$$g_i := (1-\alpha)\alpha^i$$

is logconcave. This shows by relation (5.41) and applying similar arguments as in the proof of Lemma 38 that the sequence $(\mathbb{E}(\mathbf{P}_k))_{k\in\mathbb{Z}_+}$ is unimodal. Applying part 1 of Lemma 38 it follows that the sequence $(\mathbb{E}(\mathbf{B}(t)))_{t\in\mathbb{Z}_+}$ is unimodal. Since the geometric pmf is logconcave it follows by similar arguments as used in the proof of part 3 of Lemma 38 using relation (5.41) that the sequence $(\mathbb{E}(\mathbf{P}_k))_{k\in\mathbb{Z}_+}$ is logconcave. Applying now part 2 of Lemma 38 yields the result. To show part 3 we observe using relation (5.41) and similar arguments as in the proof of part 3 of Lemma 38 that $(\mathbb{E}(\mathbf{P}_k))_{k\in\mathbb{Z}_+}$ is logconcave and this shows the desired result applying part 2 of Lemma 38. □

Clearly the unimodality point of a unimodal sequence $\mathbb{E}(\mathbf{B}(t))$, $t \in \mathbb{Z}_+$ represents the time that the expected number of products in the market is maximal and before that time the expected installed base is increasing and after that time it is decreasing. We will now consider the global behaviour of the sequence $\mathbb{E}(\mathbf{R}(t))$. Since

$$\|\mathbf{q}\|_1 = \sum_{i=0}^{\infty} q_i \le \sum_{i=0}^{\infty} (1 - F_{\mathbf{U}}(i)) = \mathbb{E}(\mathbf{U}) < \infty \tag{5.111}$$

we obtain by $q_i > 0$ in relation (5.71) that the sequence $\widetilde{\mathbf{q}} = (\widetilde{q}_i)_{i\in\mathbb{Z}_+}$ given by

$$\widetilde{q}_i = \frac{q_i}{\|\mathbf{q}\|_1}, n \in \mathbb{Z}_+ \tag{5.112}$$

is a pmf. Also one can give a probabilistic interpretation for $\mathbb{E}(\mathbf{R}(t))$. It follows with $\widetilde{\mathbf{X}}$ a non-negative integer valued random variable having pmf $\widetilde{\mathbf{q}} = (\widetilde{q}_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.112) and $\mathbf{Y}$ a non-negative integer valued random variable having pmf $\widetilde{\mathbf{p}} = (\widetilde{p}_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.25) and the random variables $\widetilde{\mathbf{X}}$ and $\mathbf{U}_e$ are independent then as in relation (5.109) it follows for every $t \in \mathbb{Z}_+$

$$\mathbb{E}(\mathbf{R}(t)) = \mathbf{S}(\infty) \parallel \mathbf{q}\|_1 \mathbb{P}(\widetilde{\mathbf{X}} + \mathbf{Y} = t). \tag{5.113}$$

Using relation (5.76) one can show similarly as done in Lemma 38 applying relation (5.113) the following result. Again this result is also shown in Amniattalab et al. (2023a) for the continuous time version of the returned and defective items process.

Lemma 40. *The next results hold for any returned and defective items stochastic process $\mathbf{R} = \{\mathbf{R}(t) : t \in \mathbb{Z}_+\}$.*

9.1 If the non-negative sequence $(\mathbb{E}(\mathbf{P}_k))_{k\in\mathbb{Z}_+}$ is unimodal and the non-negative sequence $\mathbf{q} = (q_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.25) is logconcave, then the sequence $(\mathbb{E}(\mathbf{R}(t)))_{t\in\mathbb{Z}_+}$ is unimodal.

9.2 If the non-negative sequence $\mathbf{p} = (p_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.33) is logconcave and the sequence $\mathbf{q} = (q_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.25) is unimodal, then the sequence $(\mathbb{E}(\mathbf{R}(t)))_{t\in\mathbb{Z}_+}$ is unimodal.

9.3 If the non-negative sequences $(\mathbb{E}(\mathbf{P}_k))_{k\in\mathbb{Z}_+}$ and $\mathbf{q} = (q_i)_{i\in\mathbb{Z}_+}$ listed in relation (5.25) are logconcave, then the sequence $(\mathbb{E}(\mathbf{R}(t)))_{t\in\mathbb{Z}_+}$ is logconcave.

In the next section we will focus on computing the pmf of the installed base, the discarded items, the returned and defective items and remaining spare parts demand stochastic process at any given time. Clearly this is too complicated for an arbitrary purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$. To obtain relatively simple formulas which can be numerically evaluated we need to assume that the purchases in different time periods are independent and the pmfs of these purchases belong to certain families of pmfs. This will be investigated in the next section.

5.4 On the pmfs of the main discrete time stochastic processes in spare parts demand

In this section we will analyse for a purchase process consisting of independent purchases the pmf of the sales process of the installed base, discarded items, returned and defective items and remaining returned and defective items stochastic process at any time t during the life cycle. In section 5.1 we give a unified analysis of the pmf of the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$, $\mathbf{R}(t)$ and $\mathbf{V}(t)$ and determine for general usage times and failure counting processes being independent of the sales process that all the considered pmfs of $\mathbf{B}(t)$, $\mathbf{D}(t)$, $\mathbf{R}(t)$ and $\mathbf{V}(t)$ belong to the class of compound Poisson distributions for independent and compound Poisson distributed purchases. Due to the generality of the failure counting process and usage times it is not possible to numerically evaluate this pmf and so we consider in the remaining sections special cases of purchase processes and failure counting processes and usage times for which one can construct an easy algorithm to numerically evaluate these pmfs.

In Section 5.2 we consider a deterministic sales process and give an easy algorithm to compute the pmf of $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ together with recurrence relations to evaluate these pmfs. These pmfs are actually convolutions of independent Poisson binomial distributed random variables.

In section 5.3 we consider a purchase process consisting of independent purchases having a Poisson pmf. For this case it is also possible to derive simple algorithms to numerically evaluate the different pmfs in case the usage time are independent of a the failure counting processes. In this case our failure counting process is given by a so-called nonhomogeneous Bernoulli failure counting process and includes as a special case so-called minimal repair failure processes. In section 5.4 we consider a purchase process consisting of independent purchases having a geometric pmf. Again in this case it is also possible to derive simple algorithms to numerically evaluate the different pmfs of $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ in case the usage time are independent of the failure counting process and the failure counting process is a nonhomogeneous Bernoulli counting process. Unfortunately it seems to to be difficult to evaluate the pmf of $\mathbf{V}(t)$.

In Section 5.5 we consider a purchase process consisting of independent purchases having a discrete gamma pmf. We introduce this class of pmfs and derive easy procedures to evaluate the pmfs of $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ using the same assumptions on the failure counting process and usage times as in Section 5.4. Finally, in Section 5.6, we list future research that involves implementing these recurrence relations and testing their efficiency and accuracy against the Fast Fourier transform technique. At the same time, we will give plots of the behaviour of these pmfs over time.

5.4.1 General properties of the pmfs under independent purchases

In this section we compute the probability generating function and hence the pmf of the installed base $\mathbf{B}(t)$ at time t, the total discarded items $\mathbf{D}(t)$ up to time t, the returned defective items $\mathbf{R}(t)$ at time t and the remaining total number of returned defective items $\mathbf{V}(t)$ from time t until the end of the life cycle. To start with this we first compute under general assumptions the probability generating function and Laplace-Stieltjes transform of the random variables $\mathbf{B}_k(t), \mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$, for every $k \leq t$ and $\mathbf{A}_k(t)$ for every $k \leq t-1$ and $\mathbf{A}(t)$. We denote by $\mathcal{P}_{\mathbf{X}}(z) = \mathbb{E}\left(z^{\mathbf{X}}\right)$ the probability generating function of the non-negative integer valued random variable $\mathbf{X}$ and by $\varphi_{\mathbf{X}}(s) = \mathbb{E}\left(e^{-s\mathbf{X}}\right)$ its Laplace-Stieltjes transform. It is obvious that $\mathcal{P}_{\mathbf{X}}(e^{-s}) = \varphi_{\mathbf{X}}(s)$ for every $s \geq 0$. By relations (5.22) and (5.195) it follows for every $t \in \mathbb{Z}_+$ and $k \leq t$, $k \in \mathbb{Z}_+$ that

$$\mathcal{P}_{\mathbf{B}_k(t)}(z) = \mathcal{P}_{\mathbf{P}_k}(a_{t-k}z + 1 - a_{t-k}). \tag{5.114}$$

and

$$\mathcal{P}_{\mathbf{D}_k(t)}(z) = \mathcal{P}_{\mathbf{P}_k}((1 - a_{t-k})z + a_{t-k}) \tag{5.115}$$

with the sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.23). Moreover, imposing the same conditions as done in Lemma 31 we obtain by relation (5.24) and (5.195) that

$$\mathcal{P}_{\mathbf{R}_k(t)}(z) = \mathcal{P}_{\mathbf{P}_k}(q_{t-k}z + 1 - q_{t-k}). \tag{5.116}$$

with the sequence $\mathbf{q} = (q_i)_{i \in \mathbb{Z}_+}$ defined in relation (5.25). To write down the pmf of the above random variables we assume for the stochastic purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ that the pmf of the random variable $\mathbf{P}_k$ is given by

$$p_{ik} := \mathbb{P}(\mathbf{P}_k = i), i \in \mathbb{Z}_+ \tag{5.117}$$

and compute the pmf of the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ for $k \leq t$, $t \in \mathbb{Z}_+$.

Lemma 41. *If the pmf of the random variable $\mathbf{P}_k$ is given by $p_{ik} = \mathbb{P}(\mathbf{P}_k = i)$, $i \in \mathbb{Z}_+$, then it follows for every $k \leq t$, $t \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$ that*

$$\mathbb{P}(\mathbf{B}_k(t) = i) = \sum_{m=i}^{\infty} \binom{m}{i} a_{t-k}^i (1 - a_{t-k})^{m-i} p_{mk} \tag{5.118}$$

and

$$\mathbb{P}(\mathbf{D}_k(t) = i) = \sum_{m=i}^{\infty} \binom{m}{i} (1 - a_{t-k})^i a_{t-k}^{m-i} p_{mk} \tag{5.119}$$

and

$$\mathbb{P}(\mathbf{R}_k(t) = i) = \sum_{m=i}^{\infty} \binom{m}{i} q_{t-k}^i (1 - q_{t-k})^{m-i} p_{mk} \tag{5.120}$$

Proof. Apply relation (5.26) and part 1 of Lemma 62. □

Although the above formulas for the pmfs have an easy probabilistic interpretation, it might be difficult to numerically evaluate these series unless the probability generating function of the random variable $\mathbf{P}(k)$ has an easy elementary expression or the distribution of $\mathbf{P}_k$ has a bounded support.

Using the previous results we can now verify the following result for the installed base random variable $\mathbf{B}(t)$, the discarded items random variable $\mathbf{D}(t)$ and the returned and defective items random variable $\mathbf{R}(t)$ at time t.

Lemma 42. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, \ldots \mathbf{T}_m^{(n)}, \ldots.)$, $n \in \mathbb{N}$ are independent and identically distributed satisfying $\mathbb{E}(\mathbf{N}_{\mathbf{U}}^2)$ is finite and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ satisfying $Var(\mathbf{S}(\infty))$ is finite then the next results hold*

10.1 *The probability generating function of the installed base $\mathbf{B}(t)$ at time t is given by*

$$\mathcal{P}_{\mathbf{B}(t)}(z) = \mathbb{E}(\Pi_{k=0}^t (a_{t-k} z + (1 - a_{t-k})^{\mathbf{P}_k}) \tag{5.121}$$

with the sequence $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ listed in relation (5.23).

10.2 *The probability generating function of the discarded items $\mathbf{D}(t)$ at time t is given by*

$$\mathcal{P}_{\mathbf{D}(t)}(z) = \mathbb{E}(\Pi_{k=0}^t ((1 - a_{t-k}) z + a_{t-k})^{\mathbf{P}_k}). \tag{5.122}$$

10.3 *The probability generating function of the returned and defective items $\mathbf{R}(t)$ at time t is given* *by*

$$\mathcal{P}_{\mathbf{R}(t)}(z) = \mathbb{E}(\Pi_{k=0}^{t}(q_{t-k}z + (1-q_{t-k})^{\mathbf{P}_k}). \tag{5.123}$$

with the sequence $\mathbf{q} = (q_i)_{i\in\mathbb{Z}_+}$ *listed in relation (5.71)*

Proof. Apply for the random variable $\mathbf{B}(t)$ relation (5.26) and part 4 of Theorem 19. For the other random variables a similar proof applies. □

If we additionally assume in Lemma 42 that the random purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are independent then the formulas in relations (5.121), (5.122) and (5.123) simplify and we obtain

$$\mathcal{P}_{\mathbf{B}(t)}(z) = \Pi_{k=0}^{t}\mathcal{P}_{\mathbf{P}_k}(a_{t-k}z + (1-a_{t-k})) \tag{5.124}$$

and

$$\mathcal{P}_{\mathbf{D}(t)}(z) = \Pi_{k=0}^{t}\mathcal{P}_{\mathbf{P}_k}((1-a_{t-k})z + a_{t-k}) \tag{5.125}$$

and

$$\mathcal{P}_{\mathbf{R}(t)}(z) = \Pi_{k=0}^{t}\mathcal{P}_{\mathbf{P}_k}(q_{t-k}z + 1 - q_{t-k}). \tag{5.126}$$

We will now identify for arbitrary random usage times and failure counting processes the class of pmfs of the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ for any t. To identify this class we first introduce the class of infinitely divisible pmfs on the non-negative integers (Steutel & Van Harn (2003)) and the compound Poissson pmfs.

Definition 12. *The non-negative discrete value random variable* $\mathbf{X}$ *is called infinitely divisible if for every* $n \in \mathbb{N}$

$$\mathbf{X} \stackrel{d}{=} \sum_{k=1}^{n} \mathbf{X}_{n,k}$$

with $\mathbf{X}_{n,k}, k = 1,...,n$ *a sequence of independent and identically distributed non-negative discrete valued random variables*

We next introduce the class of compound distributions.

Definition 13. *Let* $\mathbf{X},\mathbf{N},\mathbf{Z}$ *be integer valued non-negative random variables. The random variable* $\mathbf{X}$ *has a compound-*$(\mathbf{N},\mathbf{Z})$ *pmf if there exists some sequence of independent non-negative integer valued random variables* $\mathbf{Z}_n, n \in \mathbb{N}$ *satisfying* $\mathbf{Z}_n \stackrel{d}{=} \mathbf{Z}$ *for every* $n \in \mathbb{N}$ *and this sequence* $\mathbf{Z}_n$ *is independent of the random variable* $\mathbf{N}$ *having a positive atom at zero such that*

$$\mathbf{X} \stackrel{d}{=} \sum_{n=1}^{\mathbf{N}} \mathbf{Z}_n \tag{5.127}$$

The random variable $\mathbf{X}$ *has a compound Poisson pmf if additionally the random variable* $\mathbf{N}$ *has a Poisson pmf and has a compound geometric pmf if the random variable* $\mathbf{N}$ *has a geometric pmf.*

It is easy to verify using the independence between the non-negative integer valued random variable

$\mathbf{N}$ and the independent and identically distributed non-negative integer valued sequence $(\mathbf{Z}_k)_{k \in \mathbb{N}}$ that for any random variable $\mathbf{X}$ having a compound $(\mathbf{N}, \mathbf{Z})$ pmf that the probability generating function of the random variable $\mathbf{X}$ satisfies

$$\mathcal{P}_{\mathbf{X}}(z) = \mathcal{P}_{\mathbf{N}}(\mathcal{P}_{\mathbf{Z}}(z)) \tag{5.128}$$

The next result is surprising and well-known.

Theorem 14. *A non-negative discrete valued random variable $\mathbf{X}$ is infinitely indivisible if and only if the random variable $\mathbf{X}$ has a compound Poisson distribution*

Proof. see Theorem 3.2 of Steutel & Van Harn (2003) or Feller (1991) □

Using Theorem 14 one can now verify the following qualitative result for general failure counting processes and random usage times

Theorem 15. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the purchase $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ consists of independent random variables each having a compound Poisson pmf then for every $t \in \mathbb{Z}_+$ the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ have a compound Poisson pmf.*

Proof. Since the random variables $\mathbf{P}_k$, $k \in \mathbf{Z}_+$ are independent it follows by relation (5.22) and (5.24) that the random variables $\mathbf{B}_k(t)$, $0 \le k \le t$, $\mathbf{D}_k(t)$, $0 \le k \le t$ and $\mathbf{R}_k(t)$, $0 \le k \le t$ are independent. Applying Theorem 14 we obtain that the random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are infinitely divisible. This shows again applying relations (5.22) and (5.24) and the definition of infinitely divisible that for each k the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ are infinitely divisible. Since these random variables are independent and the sum of independent infinitely divisible random variables is infinitely divisible we obtain by relation (5.26) that the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ are infinitely divisible. Applying now Theorem 14 yields the desired result. □

The above result only characterizes under very general conditions the class of pmfs to which the pmfs of the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ belong. To compute these distributions we need to specify in more detail the pmfs of $\mathbf{P}_k$ and the used failure counting processes and pmfs of the usage times. We will now derive the probability generating function of the random variable $\mathbf{V}(t)$. To do so we introduce for any $t \in \mathbb{Z}_+$ and for every $0 \le k \le t-1$ the independent and identically distributed random variables $(\mathbf{Z}_n(k))_{n \in \mathbb{N}}$ given by

$$\mathbf{Z}_n(k) = \mathbf{N}_n(\mathbf{U}_n - 1) - \mathbf{N}_n((\mathbf{U}_n - 1)) \wedge (t - k - 1)). \tag{5.129}$$

Clearly $\mathbf{Z}_n(t-1) = \mathbf{N}_n(\mathbf{U}_n - 1)$. Since by relation (5.13) and (5.15) we know

$$\mathbf{A}(t) = \sum\nolimits_{n=\mathbf{S}(t-1)+1}^{\mathbf{S}(\infty)} \mathbf{Z}_n(t-1), \mathbf{A}_k(t) = \sum\nolimits_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} \mathbf{Z}_n(k), k \le t-1 \tag{5.130}$$

and by assumption for every $0 \leq k \leq t-1$ the independent and identically distributed random variables $\mathbf{Z}_n(k), n \in \mathbb{N}$ are independent of the sales process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ it follows that

$$\mathcal{P}_{\mathbf{A}(t)}(z) = \mathcal{P}_{\mathbf{S}(\infty)-\mathbf{S}(t-1)}(\mathcal{P}_{\mathbf{Z}(t-1)}(z)) \tag{5.131}$$

with

$$\mathbf{Z}(k) := \mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1) \wedge (t-k-1)) \stackrel{d}{=} \mathbf{Z}_n(k) \tag{5.132}$$

for every $k \leq t-1$. Also we obtain by the same arguments that

$$\mathcal{P}_{\mathbf{A}_k(t)}(z) = \mathcal{P}_{\mathbf{S}(k)-\mathbf{S}(k-1)}(\mathcal{P}_{\mathbf{Z}_k}(z)) = \mathcal{P}_{\mathbf{P}_k}(\mathcal{P}_{\mathbf{Z}_k}(z)). \tag{5.133}$$

For the random variable $\mathbf{V}(t)$ one can now show the following result.

Lemma 43. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ *are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process* $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ *then it follows that*

$$\mathcal{P}_{\mathbf{V}(t)}(z) = \mathbb{E}\left(\Pi_{k=0}^{t-1} \mathcal{P}_{\mathbf{Z}(k)}(z)^{\mathbf{P}_k} \mathcal{P}_{\mathbf{Z}(t-1)}(z)^{\mathbf{S}(\infty)-\mathbf{S}(t-1)}\right). \tag{5.134}$$

Proof. If $(\mathcal{F}_t)_{t=0}^{\infty}$ denotes the increasing filtration of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+$ and $\mathcal{F}_\infty = \cup_{t=0}^{\infty} \mathcal{F}_t$ then by the power property of conditional expectation (Williams (1991)) we obtain

$$\mathcal{P}_{\mathbf{V}(t)}(z) = \mathbb{E}(z^{\mathbf{V}(t)}) = \mathbb{E}\left(\Pi_{k=0}^{t-1} z^{\mathbf{A}_k(t)} z^{\mathbf{A}(t)}\right) = \mathbb{E}\left(\mathbb{E}\left(\Pi_{k=0}^{t-1} z^{\mathbf{A}_k(t)} z^{\mathbf{A}(t)} \mid \mathcal{F}_\infty\right)\right) \tag{5.135}$$

Since the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ this shows that the conditional expectation $\mathbb{E}\left(\Pi_{k=0}^{t-1} z^{\mathbf{A}_k(t)} z^{\mathbf{A}(t)} \mid \mathcal{F}_\infty\right)$ equals

$$\mathbb{E}\left(\Pi_{k=0}^{t-1} z^{\mathbf{A}_k(t)} z^{\mathbf{A}(t)} \mid \mathcal{F}_\infty\right) = \Pi_{k=0}^{t-1} \mathcal{P}_{\mathbf{Z}(k)}(z)^{\mathbf{P}_k} \mathcal{P}_{\mathbf{Z}(t-1)}^{\mathbf{S}(\infty)-\mathbf{S}(t-1)}(z)$$

and this shows the result. $\square$

Since $\mathbf{S}(k) \uparrow \mathbf{S}(\infty)$ a.s as $k \uparrow \infty$ we obtain by the monotone convergence theorem that for independent purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$

$$\mathcal{P}_{\mathbf{S}(\infty)-\mathbf{S}(t-1)}(z) = \lim_{k \uparrow \infty} \mathcal{P}_{\mathbf{S}(k)-\mathbf{S}(t-1)}(z) = \Pi_{k=t}^{\infty} \mathcal{P}_{\mathbf{P}_k}(z). \tag{5.136}$$

Under the same additional condition the formula in relation (5.134) simplifies and we obtain

$$
\begin{aligned}
\mathcal{P}_{\mathbf{V}(t)}(z) &= \Pi_{k=0}^{t-1}\mathbb{E}\left(\mathcal{P}_{\mathbf{Z}(k)}(z)^{\mathbf{P}_k}\right)\mathbb{E}\left(\mathcal{P}_{\mathbf{Z}(t-1)}(z)^{\mathbf{S}(\infty)-\mathbf{S}(t-1)}\right) \\
&= \Pi_{k=0}^{t-1}\mathcal{P}_{\mathbf{P}_k}\left(\mathcal{P}_{\mathbf{Z}(k)}(z)\right)\mathcal{P}_{\mathbf{S}(\infty)-\mathbf{S}(t-1)}\left(\mathcal{P}_{\mathbf{Z}(t-1)}(z)\right) \\
&= \Pi_{k=0}^{t-1}\mathcal{P}_{\mathbf{P}_k}\left(\mathcal{P}_{\mathbf{Z}(k)}(z)\right)\Pi_{k=t}^{\infty}\mathcal{P}_{\mathbf{P}_k}\left(\mathcal{P}_{\mathbf{Z}(t-1)}(z)\right)
\end{aligned}
\tag{5.137}
$$

Hence for $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ having an elementary expression for its probability generating function we can get an elementary expression for the probability generating function $\mathcal{P}_{\mathbf{B}(t)}$ of the installed base $\mathbf{B}(t)$ at time t, the probability generating function $\mathcal{P}_{\mathbf{D}(t)}$ of the discarded items $\mathbf{D}(t)$ up to time t and the probability generating function $\mathcal{P}_{\mathbf{R}(t)}$ of the returned and defective items $\mathbf{R}(t)$ at time t. This means we can use the discrete Fast Fourier Transform technique to calculate these probabilities (Abate & Whitt (1992b)). Again for this random variable, we can characterize under very general conditions to which class of pmfs the pmf of $\mathbf{V}(t)$ belongs.

Theorem 16. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the purchase $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ consists of independent random variables each having a compound Poisson pmf then for every $t \in \mathbb{Z}_+$ the random variables $\mathbf{V}(t)$ has a compound Poisson pmf*

Proof. Again by the independence of the random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ we obtain by relation (5.130) that the random variables $\mathbf{A}_k(t)$, $0 \le k \le t-1$ and $\mathbf{A}(t)$ are independent. Also by the same arguments as used in Theorem 15 we obtain that the random variables $\mathbf{A}_k(t)$, $0 \le k \le t-1$ and $\mathbf{A}(t)$ are infinitely divisible and this shows since $\mathbf{V}(t)$ is the convolution of infinitely divisible independent random variables the desired result. □

In the next subsections, we will analyse in more detail the probability generating functions for independent random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ having a pmf belonging to a certain class of pmfs. We start with a deterministic purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$.

5.4.2 On a deterministic purchase process.

In Minner (2011) it is assumed that the sales process is a deterministic process given by the constants p_k^*, $k \in \mathbb{Z}_+$. In this particular case, it is obvious that the degenerate random variable $\mathbf{P}_k$ is not infinitely divisible since the pmf of any infinitely divisible non-negative integer valued random variable must have unbounded support. For deterministic purchase processes, the following result holds (Minner (2011)).

Lemma 44. *If the purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ is deterministic and given by the constants p_k^*, $k \in \mathbb{Z}_+$ then for every $k \le t$, $t \in \mathbb{Z}_+$ the random variables $\mathbf{B}_k(t)$ and $\mathbf{D}_k(t)$ have a binomial* *distribution with number of trials p_k^* and success probability a_{t-k}, respectively $1-a_{t-k}$. Also the random variable $\mathbf{R}_k(t)$ has a binomial distribution with number of trials p_k^* and success probability q_{t-k}.*

Proof. Since $\mathcal{P}_{\mathbf{P_k}}(z) = z^{p_k^*}$ the result follows applying relations (5.114), (5.115) and (5.116). $\square$

Since computing binomial coefficients can be numerically unstable and the probability generating function of a binomial distributed random variable with number of trials n and success probability $0 \leq p \leq 1$ is given by $\mathbb{E}(z^{\mathbf{X}}) = (1-p+pz)^n$ we can either use the discrete fast Fourier Transform Technique to compute the pmf of the random variable $\mathbf{X}$ (see Appendix D of Tijms (2003)) or apply the next recurrence relation. To evaluate the pmf of the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ or $\mathbf{R}_k(t)$ one could use the next lemma with the proper success probabilities given in Lemma 44. In Minner (2011) an alternative way is presented by building up these convolutions.

Lemma 45. *If the non-negative integer valued random variable $\mathbf{X}$ has a binomial pmf with number of independent trials $n \in \mathbb{N}$ and success probability $0 \leq p < 1$ then it follows that*

$$\mathbb{P}(\mathbf{X} = 0) = (1-p)^n \tag{5.138}$$

and for every $m = 1, ..., n$

$$m\mathbb{P}(\mathbf{X} = m) = n\sum_{j=0}^{m-1}(-1)^{m-j-1}\mathbb{P}(\mathbf{X} = j)\left(\frac{p}{1-p}\right)^{m-j} \tag{5.139}$$

Proof. It follows for a binomial distribution having n independent trials and success probability p that

$$\mathcal{C}_{\mathbf{X}}(z) := \ln(\mathcal{P}_{\mathbf{X}}(z)) = n\ln(1-p+pz) = n\left(\ln(1-p) + \ln\left(1+\frac{p}{1-p}z\right)\right). \tag{5.140}$$

This shows for every $j \in \mathbb{N}$ that

$$\mathcal{C}_{\mathbf{X}}^{(j)}(z) = \frac{n(-1)^{j-1}\left(\frac{p}{1-p}\right)^j (j-1)!}{\left(1+\frac{p}{1-p}z\right)^j} \tag{5.141}$$

and so we obtain

$$\rho_j(\mathbf{X}) := \mathcal{C}_{\mathbf{X}}^{(j)}(0) = n(-1)^{j-1}\left(\frac{p}{1-p}\right)^j (j-1)!. \tag{5.142}$$

Applying Lemma 63 and in particular relation (5.210) yields the desired result. $\square$

We next recover the result for the random variables $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ also discussed in Minner (2011).

Lemma 46. *The next results holds for a deterministic purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ given by $\mathbf{P}_k = p_k^*$, $k \in \mathbb{Z}_+$.*

11.1 For every $t \in \mathbb{Z}_+$ it follows for the installed base at time t that

$$\mathbf{B}(t) = \sum_{k=0}^{t} \mathbf{B}_k(t) \tag{5.143}$$

with the random variables $\mathbf{B}_k(t)$, $k = 0,...,t$ independent and $\mathbf{B}_k(t)$ having a binomial pmf with number of independent trials given by p_k^* and probability of success given by a_{t-k}.

11.2 For every $t \in \mathbb{Z}_+$ it follows for the discarded items at time t that

$$\mathbf{D}_t = \sum_{k=0}^{t} \mathbf{D}_k(t) \tag{5.144}$$

with the random variables $\mathbf{D}_k(t)$, $k = 0,...,t$ independent and $\mathbf{D}_k(t)$ having a binomial pmf with number of independent trials given by p_k^* and probability of success given by $1 - a_{t-k}$.

11.3 If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent then for the returned and defective items process it follows that

$$\mathbf{R}(t) = \sum_{k=0}^{t} \mathbf{R}_k(t) \tag{5.145}$$

with the random variables $\mathbf{R}_k(t)$, $k = 0,...,t$ independent and $\mathbf{R}_k(t)$ having a binomial pmf with number of independent trials given by p_k^* and probability of success given by q_{t-k}.

Proof. Apply Lemma 44 and relations (5.6), (5.7) and (5.11). □

To compute the pmf of the random variable $\mathbf{B}(t)$, $\mathbf{D}(t)$ and $\mathbf{R}(t)$ for any $t \in \mathbb{Z}_+$ we need to evaluate a convolution of a finite number of independent binomial distributed random variables with an unequal number of independent trials and different probabilities of success. Such a general distribution is called in statistics and probability theory a Poisson binomial distribution. In Hong (2013) several efficient algorithms to compute the pmf of a Poisson binomial distributed random variable based on the discrete Fourier Transform technique are discussed. There exists an *R*-package package called poibin and a package within Python based on the algorithms discussed in Hong (2013) to compute this pmf (See Wikipedia Poisson binomial distribution). We will now give an alternative way, not discussed in Hong (2013), to calculate these probabilities using recurrence relations. In Minner (2011) a more standard way to calculate these probabilities is proposed.

Lemma 47. *Let $\mathbf{X}_k, k = 0,...,t$ with $t \in \mathbb{N}$ be a sequence of independent non-negative integer valued random variables and $\mathbf{X}_k$ has a binomial pmf with number of trials given by n_k and success probability $0 < p_k < 1$. If $\mathbf{X} = \sum_{k=0}^{t} \mathbf{X}_k$ then*

$$\mathbb{P}(\mathbf{X} = 0) = \Pi_{k=0}^{t} (1 - p_k)^{n_k} \tag{5.146}$$

and for every $m = 1, ..., \sum_{k=0}^{t} n_k$ and $v_j := \sum_{k=0}^{t} n_k \left(\frac{p_k}{1-p_k}\right)^j$

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X} = j)(-1)^{m-j-1} v_{m-j} \tag{5.147}$$

Proof. It follows using the independence of the random variables $\mathbf{X}_k$, $k = 1, ..., t$ that

$$\mathcal{P}_{\mathbf{X}}(z) = \Pi_{k=0}^{t} \mathcal{P}_{\mathbf{X}_k}(z) = \Pi_{k=0}^{t}(p_k z + 1 - p_k)^{n_k} \tag{5.148}$$

This shows $\mathbb{P}(\mathbf{X} = 0) = \mathcal{P}_{\mathbf{X}}(0) = \Pi_{k=0}^{t}(1 - p_k)^{n_k}$. Since $0 < p_k < 1$ for every $0 \leq k \leq t$ and hence $\mathcal{P}_{\mathbf{X}}(z)$ is positive in a neighborhood $\mathcal{N}$ of zero we also obtain for any $z \in \mathcal{N}$

$$\mathcal{C}_{\mathbf{X}}(z) = \ln(\mathcal{P}_{\mathbf{X}})(z) = \sum_{k=0}^{t} \ln(\mathcal{P}_{\mathbf{X}_k})(z) = \sum_{k=0}^{t} n_k \ln(p_k z + 1 - p_k) \tag{5.149}$$

Applying relation (5.142) this implies

$$\rho_j(\mathbf{X}) = \mathcal{C}_{\mathbf{X}}^{(j)}(0) = \sum_{k=0}^{t} \mathcal{C}_{\mathbf{X}_k}^{(j)}(0) = (-1)^{j-1} \sum_{k=0}^{t} n_k \left(\frac{p_k}{1 - p_k}\right)^j$$

and by relation (5.210) we obtain the desired result. □

Using Lemma 46 we can now evaluate for a deterministic purchase process $\mathbf{P} = \{\mathbf{P}_k : k \in \mathbb{Z}_+\}$ the pmf of the installed base, the discarded items and the returned and defective items process at any time t by substituting the proper parameters for n_k and p_k in Lemma 53. An important subclass of the Poison binomial distributed random variables is given by the convolution of a finite number of independent Bernoulli distributed random variables each having a different probability of success. This is useful in computing the pmf of the random variable $\mathbf{N}(t)$ with $\mathbf{N} = \{\mathbf{N}(t) : t \in \mathbb{N}\}$ denoting a nonhomogeneous Bernoulli counting process to be introduced later in this section. In this particular case we obtain from relation (5.163) and (5.147) for $n_k = 1$, $0 \leq k \leq t$ that

$$\mathbb{P}(\mathbf{X} = 0) = \Pi_{k=0}^{t}(1 - p_k) \tag{5.150}$$

and for $1 \leq m \leq t$

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X} = j)(-1)^{m-j-1} \sum_{k=0}^{t} \left(\frac{p_k}{1 - p_k}\right)^{m-j}. \tag{5.151}$$

In the next subsection, we consider the case that the random purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are independent and Poisson distributed.

5.4.3 On a Poisson Purchase Stochastic Process

In this subsection, we consider a purchase process consisting of independent and Poisson distributed random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ having parameter $0 < \beta_k < \infty$. Observe this class of pmfs is the most well-known representative of the class of pmfs of infinitely divisible non-negative integer valued random variables. Since for these random variables $\mathbb{E}(\mathbf{P}_k) = \beta_k$ and $S(\infty)$ is finite this means that $S(\infty) = \sum_{k=0}^{\infty} \beta_k$ is finite. One can now show the following result.

Lemma 48. *If the integer valued non-negative random purchase* $\mathbf{P}_k$*,* $k \in \mathbb{Z}_+$ *at time* k *has a Poisson pmf with parameter* $0 < \beta_k < \infty$*, then the random variables* $\mathbf{B}_k(t)$*,* $\mathbf{D}_k(t)$ *and* $\mathbf{R}_k(t)$ *are Poisson distributed with parameters* $\beta_k a_{t-k}, \beta_k(1-a_{t-k})$ *and* $\beta_k q_{t-k}$.

Proof. If the random demand $\mathbf{P}_k$ at time k has a Poisson pmf with parameter $0 < \beta_k < \infty$ then it follows for every $z \in \mathbb{R}$ that $\mathcal{P}_{\mathbf{P}_k}(z) = e^{-\beta(1-z)}$. This implies by relation (5.114) that

$$\mathcal{P}_{\mathbf{B}_k(t)}(z) = e^{-\beta_k a_{t-k}(1-z)}. \tag{5.152}$$

Also by relations (5.115) and relation (5.116) we obtain

$$\mathcal{P}_{\mathbf{D}_k(t)}(z) = e^{-\beta_k(1-a_{t-k})(1-z)}, \mathcal{P}_{\mathbf{R}_k(t)}(z) = e^{-\beta_k q_{t-k}(1-z)} \tag{5.153}$$

Since these probability generating functions are the probability generating functions of a Poisson pmf with the properly selected parameters we have verified the desired result. □

For the pmf of the installed base $\mathbf{B}(t)$ at time t, discarded items $\mathbf{D}(t)$ up to time t and returned and defective items $\mathbf{R}(t)$ at time t one can show the following result.

Theorem 17. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$*,* $n \in \mathbb{N}$ *are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process* $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ *and the purchase time random variables* $\mathbf{P}_k$*,* $k \in \mathbb{Z}_+$ *are independent and Poisson distributed with parameter* $0 < \beta_k < \infty$ *then for every* $t \in \mathbb{Z}_+$ *the next results hold.*

12.1 The installed base $\mathbf{B}(t)$ *at time* t *has a Poisson distribution with parameter* $\sum_{k=0}^{t} \beta_k a_{t-k}$ *with the sequence* $\mathbf{a} = (a_i)_{i \in \mathbb{Z}_+}$ *defined in relation (5.23).*

12.2 The total number of discarded items $\mathbf{D}(t)$ *up to time* t *has a Poisson distribution with parameter* $\sum_{k=0}^{t} \beta_k(1-a_{t-k})$.

12.3 The total number of returned defective items $\mathbf{R}(t)$ *at time* t *has a Poisson distribution with parameter* $\sum_{k=0}^{t} \beta_k q_{t-k}$*. with the sequence* $\mathbf{q} = (q_i)_{i \in \mathbb{Z}+}$ *defined in relation (5.25)*

Proof. Apply Lemma 48 and relations (5.124), (5.125) and (5.126). □

The above results are not surprising due to the thinning property of a Poisson pmf and the convolution of a finite number of independent Poisson distributed random variables is again Poisson distributed. If the purchase in period k is Poisson distributed with parameter β_k, then $\mathbb{E}(\mathbf{B}(t)) = \mathrm{Var}(\mathbf{B}(t)) = \beta_t$ and so this distribution is not very flexible with respect to its variance mean ratio (Boylan & Syntetos (2021)). Hence it might not fit well to statistical purchase data and so we need to find more flexible distributions. The most flexible class of pmfs will be considered in the last subsection. We will now show a more detailed result as presented in Theorem 16 for the random variable $\mathbf{A}(t)$ introduced in relation (5.13) and the random variables $\mathbf{A}_k(t), 0 \le t \le t-1$ introduced in relation (5.15).

Lemma 49. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$*,* $n \in \mathbb{N}$ *are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process*

$\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ *and the integer valued non-negative random purchase* $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ *at time* k *has a Poisson pmf with parameter* $0 < \beta_k < \infty$ *and these random purchases are independent then*

13.1 The random variable $\mathbf{A}(t)$ *has a compound* $(\mathbf{N}, \mathbf{Z}(t-1))$ *distribution with* $\mathbf{N}$ *Poisson distributed with parameter* $\sum_{k=t}^{\infty} \beta_k$ *and* $\mathbf{Z}(t-1) = \mathbf{N}(\mathbf{U}-1)$

13.2 The random variable $\mathbf{A}_k(t)$ *has a compound* $(\mathbf{N}, \mathbf{Z}(k))$ *distribution with* $\mathbf{N}$ *Poisson distributed with parameter* β_k *and* $\mathbf{Z}(k) = \mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1) \wedge (t-k-1))$

Proof. Clearly by relation (5.136) the random variable $\mathbf{S}(\infty) - \mathbf{S}(t-1)$ has a Poisson pmf with parameter $\sum_{k=t}^{\infty} \beta_k$. By relation (5.130) and the definition of a compound Poisson pmf the first part follows. The second part can be shown similarly using relation (5.130). □

In the next lemma we give a recurrence relation to evaluate the pmf of a non-negative integer valued random variable having a compound Poisson pmf. This recurrence relation is also known as Adelsons recursion scheme (Tijms (2003)).

Lemma 50. *If the non-negative integer valued random variable* $\mathbf{X}$ *has a compound* $(\mathbf{N}, \mathbf{Z})$ *Poisson pmf with* $\mathbf{N}$ *Poisson distributed with parameter* $\beta > 0$ *and* $z_n = \mathbb{P}(\mathbf{Z} = n)$, $n \in \mathbb{Z}_+$ *then*

(5.154) $$\mathbb{P}(\mathbf{X} = 0) = e^{-\beta(1-z_0)}$$

and

(5.155) $$m\mathbb{P}(\mathbf{X} = m) = \beta \sum_{j=1}^{m} j\mathbb{P}(\mathbf{X} = j) z_{m-j}$$

Proof. Applying relation (5.128) and using $\mathbf{N}$ has a Poisson pmf with parameter $0 \leq \beta \leq \infty$ we obtain

(5.156) $$\mathcal{P}_{\mathbf{X}}(z) = e^{-\beta(1-\mathcal{P}_{\mathbf{Z}}(z)}$$

This shows by relation (5.156) that $\mathbb{P}(\mathbf{X} = 0) = \mathcal{P}_{\mathbf{X}}(0) = e^{-\beta(1-z_0)}$. Also we obtain by relation (5.156) that

(5.157) $$\mathcal{C}_{\mathbf{X}}(z) = \ln(\mathcal{P}_{\mathbf{X}}(z)) = -\beta + \beta \mathcal{P}_{\mathbf{Z}}(z).$$

This shows for every $n \in \mathbb{N}$ that $\rho_n(\mathbf{X}) = \mathcal{C}_{\mathbf{X}}^{(n)}(0) = \beta n! z_n$ and by Lemma 63 we obtain the desired result. □

To compute the pmf of the random variable $\mathbf{A}_k(t)$,$0 \leq k \leq t-1$ and $\mathbf{A}(t)$ we know by relation (5.130) that

(5.158) $$\mathbf{A}(t) = \sum_{n=\mathbf{S}(t-1)+1}^{\mathbf{S}(\infty)} \mathbf{Z}_n(t-1), \mathbf{A}_k(t) = \sum_{n=\mathbf{S}(k-1)+1}^{\mathbf{S}(k)} \mathbf{Z}_n(k), k \leq t-1$$

with for every $0 \leq k \leq t-1$ the independent and identically distributed random variables $\mathbf{Z}_{k,n}$,$n \in \mathbb{N}$

given by

$$\mathbf{Z}_n(k) := \mathbf{N}_n(\mathbf{U}_n - 1) - \mathbf{N}_n((\mathbf{U}_n - 1) \wedge (t-k-1)) \stackrel{d}{=} \mathbf{Z}(k) \tag{5.159}$$

To use the above recurrence relation we need to compute the pmf of the independent and identically distributed random variables $\mathbf{Z}_n(k)$ and this means we need to compute the pmf of the random variable $\mathbf{Z}(k)$ defined in relation (5.132).

Lemma 51. *If the failure process is a nonhomogeneous Bernoulli counting process with parameters β_j, $j \in \mathbb{N}$ and the usage time $\mathbf{U}$ is independent of the failure counting process then for every $0 \leq k \leq t-1$ and $\mathbf{Z}(k)$ defined in relation (5.159) we obtain*

$$\mathbb{P}(\mathbf{Z}(k) = 0) = \mathbb{P}(\mathbf{U} \leq t-k) + \mathbb{E}(\Pi_{j=t-k}^{\mathbf{U}-1}(1-\beta_j)1_{\{\mathbf{U} \geq t-k+1\}}). \tag{5.160}$$

and for every $m \in \mathbb{N}$

$$\mathbb{P}(\mathbf{Z}(k) = m) = \sum_{i=t-k+1}^{\infty} \mathbb{P}\left(\sum_{j=t-k}^{i} 1_{\{\mathbf{B}_j = 1\}} = m\right) \mathbb{P}(\mathbf{U} = i) \tag{5.161}$$

with $\sum_{j=t-k}^{i} 1_{\{\mathbf{B}_j=1\}}$ a Poisson binomial distributed random variable having $i+k-t$ independent trials and success probabilities $\beta_{t-j}, ..., \beta_i$

Proof. It follows for every $0 \leq k \leq t = 1$ that by the conditional expectation formula

$$\begin{aligned}
\mathbb{P}(\mathbf{Z}(k) = 0) &= \mathbb{P}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1) \wedge (t-k-1) = 0) \\
&= \mathbb{P}(\mathbf{U} \leq t-k) + \mathbb{E}(\mathbf{1}_{\{\mathbf{N}(\mathbf{U}-1) - \mathbf{N}(t-k-1) = 0\}} \mathbf{1}_{\{\mathbf{U} \geq t-k+1\}}) \\
&= \mathbb{P}(\mathbf{U} \leq t-k) + \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{\mathbf{N}(\mathbf{U}-1) - \mathbf{N}(t-k-1) = 0\}} \mid \mathbf{U}) \mathbf{1}_{\{\mathbf{U} \geq t-k+1\}})
\end{aligned}$$

Since the random variable $\mathbf{U}$ is independent of the failure process $\mathbf{N} = \{\mathbf{N}(i) : i \in \mathbb{Z}_+\}$ we obtain

$$\mathbb{E}(\mathbf{1}_{\{\mathbf{N}(\mathbf{U}-1) - \mathbf{N}(t-k-1) = 0\}} \mid \mathbf{U}) = \mathbf{\Pi}_{j=t-k}^{\mathbf{U}-1}(1-\beta_j)$$

and this shows the desired result. To show the other formula we observe for every $m \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{P}(\mathbf{Z}(k) = m) &= \mathbb{P}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1) \wedge (t-k-1) = m) \\
&= \mathbb{P}(\mathbf{N}(\mathbf{U}-1) - \mathbf{N}(t-k-1) = m, \mathbf{U} \geq t-k+1) \\
&= \mathbb{P}\left(\sum_{j=t-k}^{\mathbf{U}} 1_{\{\mathbf{B}_j=1\}} = m, \mathbf{U} \geq t-k+1\right)
\end{aligned}$$

Since the random variable $\mathbf{U}$ is independent of the failure process $\mathbf{N} = \{\mathbf{N}(i) : i \in \mathbb{Z}_+\}$ we obtain conditioning on $\mathbf{U}$ that

$$\mathbb{P}\left(\sum_{j=t-k}^{\mathbf{U}} 1_{\{\mathbf{B}_j=1\}} = m, \mathbf{U} \geq t-k+1\right) = \sum_{i=t-k+1}^{\infty} \mathbb{P}(\sum_{j=t-k}^{i} 1_{\{\mathbf{B}_j=1\}} = m\} \mathbb{P}(\mathbf{U} = i)$$

and $\sum_{j=t-k}^{i} 1_{\{\mathbf{B}_j=1\}}$ is a Poisson binomial distributed random variable having $i+k-t$ trials and success probabilities $\beta_{t-k}, \ldots, \beta_i$. $\square$

We can now determine the pmf of the random variable $\mathbf{V}(t)$ of the total number of returned defective items from time t until the end of the life cycle. This result is more detailed than the one presented in Theorem 16.

Lemma 52. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, \ldots, \mathbf{T}_m^{(n)}, \ldots)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the random purchases $\mathbf{P}_k, k \in \mathbb{Z}_+$ are independent and Poisson distributed with parameter $0 < \beta_k < \infty$ then for every $t \in \mathbb{Z}_+$ the pmf of the random variable $\mathbf{V}(t)$ is the convolution of a finite number of independent compound Poisson distributed random variables. In particular, it follows that*

$$\mathbf{V}(t) = \sum_{k=0}^{t-1} \mathbf{A}_k(t) + \mathbf{A}(t) \tag{5.162}$$

with $\mathbf{A}_k(t)$, $0 \leq k \leq t-1$ and $\mathbf{A}(t)$ independent random variables and $\mathbf{A}(t)$ having a compound $(\mathbf{N}, \mathbf{Z}(t-1))$ distribution with $\mathbf{N}$ Poisson distributed with parameter $\sum_{k=t}^{\infty} \beta_k$ and $\mathbf{Z}(t-1) = \mathbf{N}(\mathbf{U}-1)$ and $\mathbf{A}_k(t)$ having a compound $(\mathbf{N}, \mathbf{Z}(k))$ distribution with $\mathbf{N}$ Poisson distributed with parameter β_k and $\mathbf{Z}(k) = \mathbf{N}(\mathbf{U}-1) - \mathbf{N}((\mathbf{U}-1) \wedge (t-k-1))$.

Proof. Since the random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are independent it follows by relations (5.13) and (5.15) that the random variables $\mathbf{A}_k(t)$, $0 \leq k \leq t-1$ and $\mathbf{A}(t)$ are independent. Applying now Lemma 49 and relation (5.17) yields the desired result. $\square$

By the above lemma we need to compute the convolution of a finite number of independent compound Poisson distributed random variables. An easy recurrence relation to compute such a convolution is listed in the next lemma.

Lemma 53. *Let $\mathbf{X}_k, k = 0, \ldots, t$ with $t \in \mathbb{N}$ be a sequence of independent non-negative integer valued random variables with $\mathbf{X}_k$ having a compound $(\mathbf{N}_k, \mathbf{Z}_k)$ distribution with $\mathbf{N}_k$ Poisson distributed with parameter β_k and $z_{nk} = \mathbb{P}(\mathbf{Z}_k = n)$, $n \in \mathbf{Z}_+$ then*

$$\mathbb{P}(\mathbf{X} = 0) = e^{-\sum_{k=0}^{t} \beta_k (1 - z_{0k})} \tag{5.163}$$

and for every $m \in \mathbb{N}$ and $v_j := \sum_{k=0}^{t} \beta_k z_{jk}$

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} (m-j)\mathbb{P}(\mathbf{X} = j) v_{m-j} \tag{5.164}$$

Proof. It follows using the independence of the random variables $\mathbf{X}_k$, $k = 1, \ldots, t$ that

$$\mathcal{P}_{\mathbf{X}}(z) = \Pi_{k=0}^{t} \mathcal{P}_{\mathbf{X}_k}(z) = e^{-\sum_{k=0}^{t} \beta_k (1 - \mathcal{P}_{\mathbf{Z}_k}(z))} \tag{5.165}$$

This shows

$$\mathbb{P}(\mathbf{X} = 0) = \mathcal{P}_{\mathbf{X}}(0) = e^{-\sum_{k=0}^{t} \beta_k (1 - z_{0k})}.$$

Since $\mathcal{P}_{\mathbf{X}}(z)$ is positive we also obtain for any $z \in \mathbb{R}$

$$(5.166) \qquad \mathcal{C}_{\mathbf{X}}(z) = \ln(\mathcal{P}_{\mathbf{X}})(z) = -\sum_{k=0}^{t} \beta_k (1 - \mathcal{P}_{\mathbf{Z}_k}(z))$$

Applying relation (5.142) this implies

$$(5.167) \qquad \rho_j(\mathbf{X}) = \mathcal{C}_{\mathbf{X}}^{(j)}(0) = \sum_{k=0}^{t} \mathcal{C}_{\mathbf{X}_k}^{(j)}(0) = j! \sum_{k=0}^{t} \beta_k z_{jk}$$

and by relation (5.210) we obtain the desired result. □

Using the above recurrence relation we can easily evaluate the pmf of the random variable $\mathbf{V}(t)$ once we have computed in a separate code the pmf of the random variable $\mathbf{Z}_k$, $0 \leq k \leq t-1$ listed in relation (5.159) and computed in lemma 51.

5.4.4 On a Geometric Purchase Stochastic Process

In this subsection we consider a purchase process consisting of independent and geometric distributed random variables $\mathbf{P}_k$, $k \in \mathbb{Z}_+$. This means (Steutel & Van Harn (2003)) that for some $0 < p_k < 1$ it follows for every $m \in \mathbb{Z}_+$ that

$$(5.168) \qquad \mathbb{P}(\mathbf{P}_k = m) = (1 - p_k) p_k^m$$

It is well know that $\mathbb{E}(\mathbf{P}_k) = p_k(1-p_k)^{-1}$ and $\mathrm{Var}(\mathbf{P}_k) = p_k(1-p_k)^{-2}$ and its probability generating function is given by

$$(5.169) \qquad \mathcal{P}_{\mathbf{P}_k}(z) = (1 - p_k)(1 - p_k z)^{-1}.$$

This shows $\mathrm{Var}(\mathbf{P}_k) \geq \mathbb{E}(\mathbf{P}_k)$ and since $\mathbf{S}(\infty)$ is finite we obtain that the parameters p_k, $k \in \mathbb{Z}_+$ satisfy $\sum_{k=0}^{\infty} p_k(1-p_k)^{-1}$ is finite. It is well known that the geometric pmf represents the pmf of the number of failures before encountering the first success in independent trials with failure probability p_k and that this pmf is infinitely divisible. The next result is easy to show. Before discussing this result we introduce for different parameters $0 < p_k < 1$, $k \in \mathbb{Z}_+$ the functions $h_k : (0,1) \to [0,1]$ defined by

$$(5.170) \qquad h_k(x) := \frac{p_k x}{1 - p_k(1-x)}.$$

Lemma 54. *If the random vectors* $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, \ldots, \mathbf{T}_m^{(n)}, \ldots)$, $n \in \mathbb{N}$ *are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process* $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ *and the random purchases* $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ *are geometric distributed with parameter* $0 < p_k < 1$ *then for every* t *and* $k \leq t$ *the next result holds.*

14.1 The random variable $\mathbf{B}_k(t)$ is geometric distributed with parameter $h_k(a_{t-k})$.

14.2 The random variable $\mathbf{D}_k(t)$ is geometric distributed with parameter $h_k(1-a_{t-k})$.

14.3 The random variable $\mathbf{R}_k(t)$ is geometric distributed with parameter $h_k(q_{t-k})$.

Proof. It follows by relations (5.114) and (5.169) that

$$\mathcal{P}_{\mathbf{B}_k(t)}(z) = \frac{1-p_k}{1-p_k(a_{t-k}z+1-a_{t-k})} = \frac{1-h_k(a_{t-k})}{1-h_k(a_{t-k})z} \tag{5.171}$$

and this shows the desired result. The proof for the other random variables is similar using relations (5.115) and (5.116) and so it is submitted. □

As for the Poisson case, one can now show the following result.

Lemma 55. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the random purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are independent and geometric distributed with parameter $0 < p_k < 1$ then the next results hold.*

15.1 For every $t \in \mathbb{Z}_+$ the installed base process at time t is given by

$$\mathbf{B}(t) = \sum_{k=0}^{t} \mathbf{B}_k(t) \tag{5.172}$$

with the random variables $\mathbf{B}_k(t)$, $k = 0, ..., t$ independent and the random variable $\mathbf{B}_k(t)$ has a discrete geometric distribution with parameters $h_k(a_{t-k})$ listed in relation (5.170).

15.2 For every $t \in \mathbb{Z}_+$ the total number of discarded items up to time t is given by

$$\mathbf{D}(t) = \sum_{k=0}^{t} \mathbf{D}_k(t)$$

with the random variables $\mathbf{D}_k(t)$, $k = 0, ..., t$ independent and the random variable $\mathbf{D}_k(t)$ has a geometric distribution with parameters $h_k(1-a_{t-k})$.

15.3 For every $t \in \mathbb{Z}_+$ the total number of returned defective items at time t is given by

$$\mathbf{R}(t) = \sum_{k=0}^{t} \mathbf{R}_k(t)$$

with the random variables $\mathbf{R}_k(t)$, $k = 0, ..., t$ independent and the random variable $\mathbf{R}_k(t)$ has a geometric distribution with parameters $h_k(q_{t-k})$

Proof. Apply relations (5.6), (5.7),(5.11) and Lemma 42 and 54. □

To compute the pmf of $\mathbf{B}(t)$, $\mathbf{D}(t)$ or $\mathbf{R}(t)$ we need to evaluate by the above lemma the convolution of $t+1$ independent geometrically distributed random variables with the proper selected parameters. Since the random variable $\mathbf{X} = \sum_{i=0}^{t} \mathbf{X}_i$ with $\mathbf{X}_i$, $0 \le i \le t$ independent and $\mathbf{X}_i$ having a geometric distribution with parameter $0 < \gamma_i < 1$ has probability generating function $\mathcal{P}_{\mathbf{X}}(z) = \Pi_{i=0}^{t}(1-\gamma_i)(1-$

$\gamma_i z)^{-1}$ we can apply the inverse Fourier transfrom technique (Abate & Whitt (1992a) or use the following recurrence relation.

Lemma 56. *Let* $\mathbf{X}_i,\ 0 \leq i \leq t$ *be a sequence of independent random variables with* $\mathbf{X}_i$ *having a geometric pmf with parameter* $0 < \gamma_k < 1$ *and* $\mathbf{X} = \sum_{i=0}^t \mathbf{X}_i$ *then*

$$\mathbb{P}(\mathbf{X}=0) = \Pi_{i=0}^t (1-\gamma_i) \tag{5.173}$$

and for every $m \in \mathbb{N}$ *and* $v_j = \sum_{i=0}^t \gamma_i^j$

$$m\mathbb{P}(\mathbf{X}=m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X}=j) v_{m-j} \tag{5.174}$$

Proof. Clearly we obtain

$$\mathbb{P}(\mathbf{X}=0) = \mathcal{P}_{\mathbf{X}}(0) = \Pi_{i=0}^t (1-\gamma_i). \tag{5.175}$$

Since for every $0 \leq i \leq t$ it follows by relation (5.169) that $\mathcal{P}_{\mathbf{X}_i}(z) = (1-\gamma_i)(1-\gamma_i z)^{-1}$ we obtain $\mathcal{C}_{\mathbf{X}_i}(z) := \ln(\mathcal{P}_{\mathbf{X}_i}(z)) = \ln((1-\gamma_i) - \ln(1-\gamma_i z)$. This shows

$$\mathcal{C}_{\mathbf{X}_i}^{(1)}(z) = \frac{\gamma_i}{1-\gamma_i z} = \frac{\gamma_i}{1-\gamma_i}\mathcal{P}_{\mathbf{X}_i}(z).$$

Hence we obtain for every $j \in \mathbb{N}$ that $\mathcal{C}_{\mathbf{X}_i}^{(j)}(z) = \frac{\gamma_i}{1-\gamma_i}\mathcal{P}_{\mathbf{X}_i}^{(j-1)}(z)$ and so

$$\frac{\rho_j(\mathbf{X}_i)}{(j-1)!} = \frac{\mathcal{C}_{\mathbf{X}_i}^{(j)}(0)}{(j-1)!} = \frac{\gamma_i}{1-\gamma_i}\frac{\mathcal{P}_{\mathbf{X}_i}^{(j-1)}(0)}{(j-1)!} = \frac{\gamma_i}{1-\gamma_i}\mathbb{P}(\mathbf{X}_i = j-1) = \gamma_i^j \tag{5.176}$$

Applying now part 2 of Lemma 63 the recurrence formula in relation 5.174 follows. □

Since it seems difficult to give a closed-form expression for the probability generating function of the random variable $\mathbf{S}(\infty) - \mathbf{S}(t)$ in case the independent purchases are geometrically distributed with parameter γ_i satisfying $\sum_{k=0}^{\infty} \gamma_k (1-\gamma_k)^{-1}$ is finite we cannot derive an easy expression for the pmf of the random variable $\mathbf{A}_t$ given by relation (5.13) and so we cannot calculate the pmf of $\mathbf{V}_t$ listed in relation (5.17). However, it is possible to give under certain conditions a simple expression for the variance and expectation of this random variable as done in Section 3.4.

5.4.5 On a Discrete Gamma Purchase Stochastic Process

Before discussing another class of discrete pmfs for which we can easily identify the pmf of the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ we first introduce this class. In the computational section of Minner (2011) it is assumed that the sales in a given period are gamma distributed. Notice this is a continuous random variable while sales volumes are always integer valued. Therefore we introduce in the next definition the discrete equivalence of a continuous gamma distribution.

Definition 18. *An integer valued non-negative random variable* $\mathbf{X}$ *has a discrete gamma pmf with parameter* $\alpha > 0$ *and* $0 < p < 1$ *if its probability generating function is given by*

$$\mathbb{E}(z^{\mathbf{X}}) = \left(\frac{1-p}{1-pz}\right)^{\alpha}. \tag{5.177}$$

It is easy to verify for $\mathbf{X}$ having a discrete gamma distribution with unknown parameters $0 < p < 1$ and $\alpha > 0$ that

$$\mathbb{E}(\mathbf{X}) = \frac{\alpha p}{1-p}, \mathrm{Var}(\mathbf{X}) = \frac{\alpha p}{(1-p)^2}$$

and so we obtain $\mathrm{Var}(\mathbf{X}) > \mathbb{E}(\mathbf{X})$. This implies if, for a given sample of sales data, the sample variance is bigger than the sample mean we can construct for this data a unique two-moment discrete gamma pmf fit. Hence the above class of pmfs is more flexible than the class of Poisson pmfs. It is easy to verify for $\mathbf{X}_{n,\beta n^{-1}}$ having a discrete gamma distribution with parameters $\alpha = n$ and $p = \beta n^{-1}$ for any given $\beta > 0$ that for any z we obtain

$$\lim_{n\uparrow\infty} \ln \mathbb{E}(z^{\mathbf{X}_{n,\beta n^{-1}}}) = \lim_{n\uparrow\infty} n \ln\left(\frac{1-\beta n^{-1}}{1-\beta n^{-1} z}\right) = -\beta(1-z).$$

This shows by the continuity theorem for probability generating functions (see Theorem 4.1 on page 490 of Steutel & Van Harn (2003)) that $\mathbf{X}_{n,\beta n^{-1}}$ converges in distribution to a discrete non-negative integer-valued random variable $\mathbf{X}$ having a Poisson pmf with parameter $\beta > 0$. Hence the Poisson pmf is a limiting case of the class of discrete gamma pmfs. Also a discrete gamma distribution belongs to the class of compound Poisson distributions as shown by the following argument. If $\mathbf{X}$ has a discrete gamma distribution with parameters $\alpha > 0$ and $0 < p < 1$ then

$$\mathbb{E}(z^{\mathbf{X}}) = \left(\frac{1-p}{1-pz}\right)^{\alpha} = e^{\alpha g(z)} \tag{5.178}$$

with $g(z) = \ln\left(\frac{1-p}{1-pz}\right)$. Introduce now the function $\mathcal{P} : (-1,1] \to \mathbb{R}$ given by

$$\mathcal{P}(z) = 1 + g(z) \tag{5.179}$$

It follows using $g(1) = 0$ that $\mathcal{P}(1) = 1$. Also, since

$$g^{(1)}(z) = \frac{p}{1-pz} = p \sum_{n=0}^{\infty} (1-p)^n z^n$$

we obtain $g^{(n)}(z) \geq 0$ for every $n \in \mathbb{N}$. This shows by relation (5.179) that $\mathcal{P}^{(n)}(z) = g^{(n)}(z) \geq 0$ for every $n \in \mathbb{N}$ and so applying Theorem 4.3 of Appendix A in Steutel & van Harn (1979) and $\mathcal{P}(1) = 1$ the function $\mathcal{P}$ is a probability generating function of a non-negative integer valued random variable $\mathbf{X}$. In the remainder we denote this by $\mathcal{P}_{\mathbf{X}}$. Hence by relations (5.178) and (5.179) we obtain

$$\left(\frac{1-p}{1-pz}\right)^{\alpha} = e^{-\alpha(1-\mathcal{P}_{\mathbf{X}}(z))}$$

and so a discrete gamma pmf is a compound Poisson pmf.

To explain the above definition of a discrete gamma pmf and relate its definition to a continuous gamma cdf we observe for $\mathbf{X}$ having an exponential distribution with parameter $\beta > 0$ that the random variable $\lfloor \mathbf{X} \rfloor$ with $\lfloor x \rfloor$ denoting the biggest integer smaller than or equal to x (the so-called lower entier function) has a geometric distribution with parameter $p = e^{-\beta}$. If we consider a random variable $\mathbf{Z}$ having a gamma distribution with scale parameter $\beta > 0$ and integer valued shape parameter $\alpha > 0$ and introducing $\mathbf{Z}_1 \stackrel{d}{=} \mathbf{Z}_2$ meaning that the random variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have the same cdf it is well know that

$$\mathbf{Z} \stackrel{d}{=} \sum_{i=1}^{\alpha} \mathbf{Z}_i$$

with $\mathbf{Z}_i, i = 1, ..., \alpha$ a sequence of independent and exponentially distributed random variables having parameter $\beta > 0$. According to our definition the discrete random variable $\mathbf{X}$ has a discrete Gamma pmf with parameter $0 < p < 1$ and α a non-negative integer if

$$\mathbf{X} \stackrel{d}{=} \sum_{i=1}^{\alpha} \mathbf{X}_i$$

with $\mathbf{X}_i$ a sequence of independent and geometric distributed random variables having the same parameter $0 < p < 1$. The pmf of this random variable $\mathbf{X}$ is known as the negative binomial pmf having parameters p and α and it is given by

$$\mathbb{P}(\mathbf{X} = k) = \binom{\alpha + k - 1}{k} p^k (1-p)^{\alpha}, k \in \mathbb{Z}_+$$

One can easily show the following result for the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$.

Lemma 57. *If the random vectors $(\mathbf{U}_n, \mathbf{T}_1^{(n)}, ..., \mathbf{T}_m^{(n)}, ...)$, $n \in \mathbb{N}$ are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process $\mathbf{S} = \{\mathbf{S}(t) : t \in \mathbb{Z}_+\}$ and the random purchases $\mathbf{P}_k$, $k \in \mathbb{Z}_+$ are discrete gamma distributed with parameter $0 < p_k < 1$ and $\alpha_k > 0$ then for every t and $k \leq t$ the next result holds.*

16.1 The random variable $\mathbf{B}_k(t)$ is discrete gamma distributed with parameter $p = h_k(a_{t-k})$ and $\alpha_k > 0$ and h_k defined in relation (5.170)

16.2 The random variable $\mathbf{D}_k(t)$ is discrete gamma distributed with parameter $p = h_k(1 - a_{t-k})$ and $\alpha_k > 0$ and h_k defined in relation (5.170) .

16.3 The random variable $\mathbf{R}_k(t)$ is discrete gamma distributed with parameter $p = h_k(q_{t-k})$ and $\alpha_k > 0$ and h_k defined in relation (5.170).

Proof. Apply relations (5.22) and (5.24) and part 1 of Lemma 62. □

Observe for α_k being a positive integer it follows that the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ have a negative binomial distribution and this pmf can be easily evaluated. For α_k not an integer we may also use the discrete Fast Fourier transform method (Abate & Whitt (1992b)) for probability sequences having unbounded support to approximate the pmf. Notice in this case the probability generating function is given by $(1-p)(1-pz)^{-\alpha_k}$ with the parameter p listed in Lemma 57 for the different random variables and so this is given by an elementary expression. It is also possible to apply the following recurrence relation to compute the pmf of a discrete gamma distribution with parameters $0<p<1$ and $\alpha>0$. This recurrence relation can be used to compute the pmf of the random variables $\mathbf{B}_k(t)$, $\mathbf{D}_k(t)$ and $\mathbf{R}_k(t)$ by plugging in the proper expression for p and α listed in Lemma 57.

Lemma 58. *If the integer valued non-negative random variable* $\mathbf{X}$ *has a discrete gamma pmf with parameters* $0<p<1$ *and* $\alpha>0$ *then*

$$\mathbb{P}(\mathbf{X}=0)=(1-p)^{\alpha} \tag{5.180}$$

and for every $m\in\mathbb{N}$

$$m\mathbb{P}(\mathbf{X}=m)=\alpha\sum_{j=0}^{m-1}\mathbb{P}(\mathbf{X}=j)p^{m-j} \tag{5.181}$$

Proof. It follows using the definition of the probability generating function of a discrete gamma pmf that $\mathbb{P}(\mathbf{X}=0)=\mathcal{P}_{\mathbf{X}}(0)=(1-p)^{\alpha}$. Also we obtain

$$\mathcal{C}_{\mathbf{X}}(z):=\ln(\mathcal{P}_{\mathbf{X}}(z))=\alpha(\ln(1-p)-\ln(1-pz)). \tag{5.182}$$

This shows for every $n\in\mathbb{N}$ that $\mathcal{C}_{\mathbf{X}}^{(n)}(z)=\frac{\alpha p^n(n-1)!}{(1-pz)^n}$ and so

$$\rho_n(\mathbf{X}):=\mathcal{C}_{\mathbf{X}}^{(n)}(0)=\alpha p^n(n-1)!. \tag{5.183}$$

Applying Lemma 63 and in particular relation (5.210) yields the desired result. □

We will now discuss the case that the random purchases $\mathbf{P}_k$ are independent and in period k have a discrete gamma pmf with parameter $p_k>0$ and $\alpha_k>0$.

Lemma 59. *If the random vectors* $(\mathbf{U}_n,\mathbf{T}_1^{(n)},\ldots,\mathbf{T}_m^{(n)},\ldots)$, $n\in\mathbb{N}$ *are identically distributed and independent and these random vectors are independent of the cumulative sales stochastic process* $\mathbf{S}=\{\mathbf{S}(t):t\in\mathbb{Z}_+\}$ *and the random purchases* $\mathbf{P}_k$, $k\in\mathbb{Z}_+$ *are independent and discrete gamma distributed with parameter* $0<p_k<1$ *and* $\alpha_k>0$ *then the next results hold.*

17.1 For every $t\in\mathbb{Z}_+$ *the installed base process at time* t *is given by*

$$\mathbf{B}(t)=\sum_{k=0}^{t}\mathbf{B}_k(t) \tag{5.184}$$

with the random variables $\mathbf{B}_k(t)$, $k=0,\ldots,t$ *independent and the random variable* $\mathbf{B}_k(t)$ *has a discrete gamma distribution with parameters* $p=h_k(a_{t-k})$ *and* $\alpha_k>0$ *with* h_k *listed in relation (5.170).*

17.2 For every $t\in\mathbb{Z}_+$ *the total number of discarded items up to time* t *is given by*

$$\mathbf{D}(t)=\sum_{k=0}^{t}\mathbf{D}_k(t) \tag{5.185}$$

with the random variables $\mathbf{D}_k(t)$, $k=0,\ldots,t$ *independent and the random variable* $\mathbf{D}_k(t)$ *has* *a discrete gamma distribution with parameters $p = h_k(1-a_{t-k})$ and $\alpha_k > 0$ with h_k listed in relation (5.170).*

17.3 *For every $t \in \mathbb{Z}_+$ the total number of returned defective items at time t is given by*

$$\mathbf{R}(t) = \sum_{k=0}^{t} \mathbf{R}_k(t) \tag{5.186}$$

with the random variables $\mathbf{R}_k(t)$, $k = 0,...,t$ independent and the random variable $\mathbf{R}_k(t)$ has discrete gamma distribution with parameters $p = h_k(q_{t-k})$ and $\alpha_k > 0$ with h_k listed in relation (5.170).

Proof. Apply relations (5.124), (5.125) and (5.126) and lemma 57. □

To compute the above probabilities one can apply the discrete Fast Fourier Transform method (see again Abate & Whitt (1992b)) since the probability generating function has an elementary expression. One can also use the following recurrence relation for convolutions of discrete gamma pmfs having different parameters.

Lemma 60. *Let $\mathbf{X}_k, k = 0,...,t$ with $t \in \mathbb{N}$ be a sequence of independent non-negative integer valued random variables and $\mathbf{X}_k$ has a discrete gamma pmf with parameters $0 < p_k < 1$ and $\alpha_k > 0$. If $\mathbf{X} = \sum_{k=0}^{t} \mathbf{X}_k$ then*

$$\mathbb{P}(\mathbf{X} = 0) = \Pi_{k=0}^{t}(1-p_k)^{\alpha_k} \tag{5.187}$$

and for every $m \in \mathbb{N}$

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X} = j)v_{m-j}, v_j = \sum_{k=0}^{t} \alpha_k p_k^j \tag{5.188}$$

Proof. Apply the same proof as in Lemma 58. □

Using Theorem 59 we can now compute for a discrete gamma sales process the pmf of the installed base, the discarded items and the returned and defective items process at time t by substituting the proper parameters for α and p_k in Lemma 60. To compute the breakdown probabilities $(u_t)_{t\in\mathbb{N}}$ listed in relation (5.72) we need to model the type of repair. First, we assume that any type of repair does take negligible time. In case the repair turns a defective product into a functioning one but does not change its age (this is called minimal repair!) it follows for every $t \in \mathbb{N}$ that

$$u_t = r(t) = \frac{f_t}{1-F(t)} \tag{5.189}$$

with $(f_t)_{n=0}^{\infty}$, $f_0 = 0$ denoting the pmf of the random time $\mathbf{T}_1$ of the first breakdown and

$$F(t) = \mathbb{P}(\mathbf{T}_1 \leq t) = \sum_{k=0}^{t} f_k. \tag{5.190}$$

Observe the value $r(t)$ represents the discrete hazard rate function at age t. This type of breakdown process was given in Definition 10. If the repair changes the product in as good as new the failure counting process is given by a renewal process. It follows by the renewal argument (Ross (2023)) that

$$u_0 = 0, u_t = f_t + \sum_{k=1}^{t} f_{t-k} u_k$$

with u_t defined in relation (5.72) and so the sequence $(u_t)_{t=0}^{\infty}$ satisfies the discrete renewal type equation. (see Frenk (1987) or Omey & Van Gulck (2015)). Hence knowing $f_t, t \in \mathbb{N}$ we can easily compute $u_t, t \in \mathbb{N}$ applying this recurrence relation. If the pmf $f_t, t \in \mathbb{N}$ is nonlattice it follows that

$$\lim_{t \uparrow \infty} u_t = \frac{1}{\mathbb{E}(\mathbf{T}_1)}$$

(strong renewal theorem discussed in Feller (1991)). This shows we only need to compute u_t for t large enough. If we model the cdf of the usage time and assume this is a discrete Weibull cdf we should consult the overview paper Almalki & Nadarajah (2014) discussing all known modifications of discrete Weibull distributions.

5.5 Directions of Future Research

In future research the above recurrence relations will be implemented on a computer to test the quality of the proposed procedures against the efficiency of the Fast Fourier transform technique and show the behaviour of the pmfs of these different random variables over time.

5.6 Appendix.

In the next lemma we derive results for random variables having a compound pmf.

Lemma 61. *The next results hold.*

18.1 *If the non-negative integer valued random variable* $\mathbf{X}$ *has a compound* $(\mathbf{N}, \mathbf{Z})$ *pmf with the random variables* $\mathbf{N}$ *and* $\mathbf{Z}$ *having a finite second moment then the expectation and variance of the random variable* $\mathbf{X}$ *is finite and given by*

(5.191) $$\mathbb{E}(\mathbf{X}) = \mathbb{E}\left(\sum_{n=1}^{\mathbf{N}} \mathbf{Z}_n\right) = \mathbb{E}(\mathbf{N})\mathbb{E}(\mathbf{Z}_1).$$

and

(5.192) $$Var(\mathbf{X}) = \mathbb{E}(\mathbf{N})\, Var(\mathbf{Z_1}) + Var(\mathbf{N})\mathbb{E}(\mathbf{Z}_1)^2.$$

18.2 If the non-negative integer valued random variable $\mathbf{X}_1$ has a compound $(\mathbf{N}_1, \mathbf{Y})$ pmf and the non-negative integer valued random variable $\mathbf{X}_2$ has a compound $(\mathbf{N}_2, \mathbf{Z})$ pmf and the sequence $\mathbf{Y}_n, n \in \mathbb{N}$ of non-negative integer valued random variables is independent of the sequence $\mathbf{Z}_n$, $n \in \mathbb{N}$ of non-negative integer valued random variables and the random variables $\mathbf{N}_i, i = 1,2$ and $\mathbf{Z}, \mathbf{Y}$ have a finite second moment then the covariance of the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ is finite and given by

$$\begin{aligned} Cov(\mathbf{X}_1, \mathbf{X}_2) &= Cov\left(\sum_{n=1}^{\mathbf{N}_1} \mathbf{Y}_n, \sum_{n=1}^{\mathbf{N}_2} \mathbf{Z}_n\right) \\ &= \mathbb{E}(\mathbf{Z}_1)\mathbb{E}(\mathbf{Y}_1) Cov(\mathbf{N}_1, \mathbf{N}_2). \end{aligned} \tag{5.193}$$

Proof. The first part is shown in Appendix A of Tijms (2003) and so we only will verify the second part. By our assumption we know that $\mathrm{Var}(\mathbf{N}_i)$, $i = 1,2$ is finite and so we obtain by the Cauchy-Schwarz inequality that $\mathrm{Cov}(\mathbf{N}_1, \mathbf{N}_2)$ is finite. It follows by the first part

$$\begin{aligned} \mathrm{Cov}(\mathbf{X}_1, \mathbf{X}_2) &= \mathbb{E}(\mathbf{X}_1\mathbf{X}_2) - \mathbb{E}(\mathbf{X}_1)\mathbb{E}(\mathbf{X}_2) \\ &= \mathbb{E}(\mathbf{X}_1\mathbf{X}_2) - \mathbb{E}(\mathbf{N}_1)\mathbb{E}(\mathbf{N}_2)\mathbb{E}(\mathbf{Y}_1)\mathbb{E}(\mathbf{Z}_1) \end{aligned} \tag{5.194}$$

Also we obtain by conditioning on the random vector $(\mathbf{N}_1, \mathbf{N}_2)$ and using that this random vector is independent of the independent i.i.d sequences $\mathbf{Y}_n, n \in \mathbb{N}$ and $\mathbf{Z}_n, n \in \mathbb{N}$ that

$$\mathbb{E}(\mathbf{X}_1\mathbf{X}_2) = \mathbb{E}\left(\sum_{n=1}^{\mathbf{N}_1} \sum_{m=1}^{\mathbf{N}_2} \mathbf{X}_n \mathbf{Y}_n\right) = \mathbb{E}(\mathbf{X}_1)\mathbb{E}(\mathbf{Y}_1)\mathbb{E}(\mathbf{N}_1\mathbf{N}_2)$$

This shows by relation (5.194) the desired result □

We will now derive some useful properties of the Steutel van Harn thinning operator $\circ$ given by

$$\alpha \circ \mathbf{X} = \sum_{j=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha\}}$$

with $\mathbf{Z}_j, j \in \mathbb{Z}_+$ a sequence of independent and standard uniform distributed random variables independent of the integer valued non-negative random variable $\mathbf{X}$ and $0 \leq \alpha \leq 1$. For this thinning operator, one can show the following properties.

Theorem 19. *The next results hold.*

19.1 If $0 \leq \alpha \leq 1$ and $\mathbf{X}$ is a non-negative integer valued random variable then

$$\mathbb{E}(z^{\alpha \circ \mathbf{X}}) = \mathcal{P}_{\mathbf{X}}(\alpha z + 1 - \alpha) \tag{5.195}$$

with $\mathcal{P}_{\mathbf{X}}$ the probability generating function of the random variable $\mathbf{X}$.

19.2 If $0 \leq \alpha \leq 1$ and $\mathbf{X}$ is a non-negative integer valued random variable having a finite second moment then

$$\mathbb{E}(\alpha \circ \mathbf{X}) = \alpha \mathbb{E}(\mathbf{X}),\ Var(\alpha \circ \mathbf{X}) = \alpha(1-\alpha)\mathbb{E}(\mathbf{X}) + \alpha^2 Var(\mathbf{X}). \tag{5.196}$$

19.3 If $0 \le \alpha_1 \le 1, 0 \le \alpha_2 \le 1$ *and* $\mathbf{X}$ *is a non-negative integer valued random variable having a finite second moment then*

$$Cov(\alpha_1 \circ \mathbf{X}, \alpha_2 \circ \mathbf{X}) = \alpha_1 \alpha_2 \, Var(\mathbf{X}) + ((\alpha_1 \wedge \alpha_2) - \alpha_1 \alpha_2) \mathbb{E}(\mathbf{X}). \tag{5.197}$$

19.4 If $0 \le \alpha_1 \le 1, 0 \le \alpha_2 \le 1$ *and for any two given nonnegative integer valued random variables* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *having a finite second moment the Steutel van Harn thinning operator* $\circ$ *given by*

$$\alpha_i \circ \mathbf{X}_i = \sum_{j=1}^{\mathbf{X}_i} \mathbf{1}_{\{\mathbf{Z}_{ij} \le \alpha_i\}} \tag{5.198}$$

satisfies the additional condition that the standard uniform distributed sequence $(\mathbf{Z}_{1j})_{j \in \mathbb{N}}$ *is independent of the standard uniform distributed sequence* $(\mathbf{Z}_{2j})_{j \in \mathbb{N}}$ *then*

$$Cov(\alpha_1 \circ \mathbf{X}_1, \alpha_2 \circ \mathbf{X}_2) = \alpha_1 \alpha_2 \, Cov(\mathbf{X}_1, \mathbf{X}_2). \tag{5.199}$$

19.5 If $0 \le \alpha_i \le 1, i = 1, ..., q$ *and* $\mathbf{X}_i, i = 1, ..., q$ *are non-negative integer valued random variables then*

$$\mathbb{E}\left(z^{\sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i}\right) = \mathbb{E}(\Pi_{i=1}^{q} (\alpha_i z + 1 - \alpha_i)^{\mathbf{X}_i}). \tag{5.200}$$

19.6 If $0 \le \alpha_i \le 1, i = 1, ...q$ *and* $\mathbf{X}_i, i = 1, ..., q$ *are non-negative integer valued random variables and* $0 \le \alpha \le 1$ *then*

$$\alpha \circ \left(\sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i\right) \stackrel{d}{=} \sum_{i=1}^{q} \alpha \alpha_i \circ \mathbf{X}_i. \tag{5.201}$$

19.7 If $\mathbf{N}$ *is an integer valued non-negative random variable independent of the sequence of integer valued non-negative random variable* $\mathbf{X}_i, i \in \mathbb{Z}_+$ *then for every sequence* $0 \le \alpha_i \le 1$ *and* $0 \le \alpha \le 1$

$$\alpha \circ \left(\sum_{i=1}^{\mathbf{N}} \alpha_i \circ \mathbf{X}_i\right) \stackrel{d}{=} \sum_{i=1}^{\mathbf{N}} \alpha \alpha_i \circ \mathbf{X}_i. \tag{5.202}$$

Proof. Since the random variable $\mathbf{X}$ is independent of the sequence of of independent and standard uniform distributed random variables $\mathbf{Z}_j, j \in \mathbb{Z}_+$ relation (5.195) is easy to verify. To show relation (5.196) we observe by part 1 of Lemma 61 that $\mathbb{E}(\alpha \circ \mathbf{X}) = \alpha \mathbb{E}(\mathbf{X})$. Again by part 1 of Lemma 61 we obtain using the independence between the independent and standard uniformly distributed random variables $\mathbf{Z}_j, j \in \mathbb{Z}_+$ and the non-negative integer valued random variable $\mathbf{X}$ that

$$\begin{aligned} \mathrm{Var}(\alpha \circ \mathbf{X}) &= \mathrm{Var}\left(\sum_{j=0}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{Z}_j \le \alpha\}}\right) \\ &= \mathbb{E}(\mathbf{X}) \mathrm{Var}(\mathbf{1}_{\{\mathbf{Z}_1 \le \alpha\}}) + \mathrm{Var}(\mathbf{X}) \mathbb{E}^2(\mathbf{1}_{\{\mathbf{Z}_j \le \alpha\}}) \\ &= \alpha(1-\alpha)\mathbb{E}(\mathbf{X}) + \alpha^2 \mathrm{Var}(\mathbf{X}) \end{aligned}$$

and we have verified relation (5.196). To show relation (5.197) we first observe that the covariance using part 1 can be written as

$$\operatorname{Cov}(\alpha_1 \circ \mathbf{X}, \alpha_2 \circ \mathbf{X}) = \mathbb{E}((\alpha_1 \circ \mathbf{X})(\alpha_2 \circ \mathbf{X})) - \alpha_1\alpha_2\mathbb{E}(\mathbf{X})^2 \tag{5.203}$$

Clearly by the definition of the Steutel van Harn thinning operator we obtain

$$\mathbb{E}((\alpha_1 \circ \mathbf{X})(\alpha_2 \circ \mathbf{X})) = \mathbb{E}\left(\sum_{j=1}^{\mathbf{X}} \sum_{i=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1\}} \mathbf{1}_{\{\mathbf{Z}_i \leq \alpha_2\}}\right)$$

Using the independence between $\mathbf{X}$ and the sequence of independent standard uniform distributed random variables $\mathbf{Z}_j, j \in \mathbb{N}$ it follows that

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^{\mathbf{X}} \sum_{i=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1\}} \mathbf{1}_{\{\mathbf{Z}_i \leq \alpha_2\}}) \mid \mathbf{X} = k\right) &= \mathbb{E}\left(\sum_{j=1}^{k} \sum_{i=1}^{k} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1\}} \mathbf{1}_{\{\mathbf{Z}_i \leq \alpha_2\}})\right) \\
&= \sum_{j=1}^{k} \sum_{i=1}^{k} \mathbb{E}(\mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1\}} \mathbf{1}_{\{\mathbf{Z}_i \leq \alpha_2\}}) \\
&= k(k-1)\alpha_1\alpha_2 + \sum_{j=1}^{k} \mathbb{E}(\mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1\}} \mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_2\}}) \\
&= k(k-1)\alpha_1\alpha_2 + \sum_{j=1}^{k} \mathbb{E}(\mathbf{1}_{\{\mathbf{Z}_j \leq \alpha_1 \wedge \alpha_2\}}) \\
&= k(k-1)\alpha_1\alpha_2 + k(\alpha_1 \wedge \alpha_2)
\end{aligned}$$

This implies applying the conditional expectation formula

$$\begin{aligned}
\mathbb{E}((\alpha_1 \circ \mathbf{X})(\alpha_2 \circ \mathbf{X})) &= \alpha_1\alpha_2\mathbb{E}(\mathbf{X}(\mathbf{X}-1)) + (\alpha_1 \wedge \alpha_2)\mathbb{E}(\mathbf{X}) \\
&= \alpha_1\alpha_2\mathbb{E}(\mathbf{X}^2) + ((\alpha_1 \wedge \alpha_2) - \alpha_1\alpha_2)\mathbb{E}(\mathbf{X})
\end{aligned} \tag{5.204}$$

Substituting this into relation (5.203) yields the formula in relation (5.197). The formula in relation (5.199) is a special case of part 2 of Lemma 61. To show relation (5.200) we observe by the conditional expectation formula that

$$\mathbb{E}\left(z^{\sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i}\right) = \mathbb{E}\left(\mathbb{E}\left(z^{\sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i} \mid \mathbf{X}_0, ..., \mathbf{X}_q)\right)\right)$$

It follows using the independence between the independent and standard uniform distributed random variables $\mathbf{Z}_{ij}, i = 0, .., q$ and $j \in \mathbb{Z}_+$ and $\mathbf{X}_0, ..., \mathbf{X}_q$ that

$$\begin{aligned}
\mathbb{E}\left(z^{\sum_{i=0}^{q} \alpha_i \circ \mathbf{X}_i} \mid \mathbf{X}_0, ..., \mathbf{X}_q)\right) &= \mathbb{E}\left(\left(z^{\sum_{i=0}^{q} \sum_{j=0}^{\mathbf{X}_i} \mathbf{1}_{\{\mathbf{Z}_{ij} \leq \alpha_i\}}} \mid \mathbf{X}_0, ..., \mathbf{X}_q\right)\right) \\
&= \Pi_{i=0}^{q} \Pi_{j=0}^{\mathbf{X}_i} \mathbb{E}(z^{\mathbf{1}_{\{\mathbf{Z}_{ij} \leq \alpha_i\}}} \mid \mathbf{X}_0, ..., \mathbf{X}_q) \\
&= \Pi_{i=0}^{q} \Pi_{j=0}^{\mathbf{X}_i} \mathbb{E}(z^{\mathbf{1}_{\{\mathbf{Z}_{ij} \leq \alpha_i\}}}) \\
&= \Pi_{i=0}^{q} (\alpha_i z + 1 - \alpha_i)^{\mathbf{X}_i}
\end{aligned}$$

and this shows the second part. To verify the third part we observe for $\mathbf{Y} = \sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i$

$$\mathbb{E}\left(z^{\alpha \circ (\sum_{i=1}^{q} \alpha_i \circ \mathbf{X}_i)}\right) = \mathcal{P}_{\mathbf{Y}}(\alpha z + 1 - \alpha))$$

Applying now the second part we obtain

$$\begin{aligned}\mathcal{P}_{\mathbf{Y}}(\alpha z+1-\alpha) &= \mathbb{E}(\Pi_{i=0}^{q}((\alpha z+1-\alpha)\alpha_i+1-\alpha_i)^{\mathbf{X}_i}) \\ &= \mathbb{E}(\Pi_{i=0}^{q}(\alpha\alpha_i z+1-\alpha\alpha_i)^{\mathbf{X}_i})\end{aligned}$$

and applying relation (5.200) replacing α_i by $\alpha\alpha_i$ we obtain the desired result. To show relation (5.202) we observe by relation (5.195)

$$\mathbb{E}\left(z^{\alpha\circ\sum_{i=1}^{\mathbf{N}}\alpha_i\circ\mathbf{X}_i}\right)=\mathbb{E}((\alpha z+1-\alpha)^{\mathbf{Z}})$$

with $\mathbf{Z}=\sum_{i=1}^{\mathbf{N}}\alpha_i\circ\mathbf{X}_i$. It follows by the independence between the random variable $\mathbf{N}$ and the sequence $\mathbf{X}_i, i\in\mathbb{Z}_+$ that

$$\mathbb{E}((\alpha z+1-\alpha)^{\mathbf{Z}})=\sum_{k=0}^{\infty}\mathbb{P}(\mathbf{N}=k)\mathbb{E}\left((\alpha z+1-\alpha)^{\sum_{i=1}^{k}\alpha_i\circ\mathbf{X}_i}\right)$$

By relation (5.201) we obtain

$$\mathbb{E}\left((\alpha z+1-\alpha)^{\sum_{i=1}^{k}\alpha_i\circ\mathbf{X}_i}\right)=\mathbb{E}\left(z^{\alpha\circ\sum_{i=1}^{k}\alpha_i\circ\mathbf{X}_i}\right)=\mathbb{E}\left(z^{\sum_{i=1}^{k}\alpha\alpha_i\circ\mathbf{X}_i}\right)$$

This implies

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbb{P}(\mathbf{N}=k)\mathbb{E}\left((\alpha z+1-\alpha)^{\sum_{i=1}^{k}\alpha_i\circ\mathbf{X}_i}\right) &= \sum_{k=0}^{\infty}\mathbb{P}(\mathbf{N}=k)\mathbb{E}\left(z^{\sum_{i=1}^{k}\alpha\alpha_i\circ\mathbf{X}_i}\right) \\ &= \mathbb{E}\left(z^{\sum_{i=1}^{\mathbf{N}}\alpha\alpha_i\circ\mathbf{X}_i}\right)\end{aligned}$$

and we have shown the result. $\square$

We next verify the following results.

Lemma 62. *The next results hold*

20.1 If $0\leq\beta\leq 1$ and $\mathbf{X}$ is a non-negative integer valued random variable having pmf

$$q_n=\mathbb{P}(\mathbf{X}=n), n\in\mathbb{Z}_+$$

then for every $n\in\mathbb{Z}_+$ it follows

$$\mathbb{P}(\beta\circ\mathbf{X}=n)=\sum_{k=n}^{\infty}\binom{k}{n}\beta^n(1-\beta)^{k-n}q_k. \tag{5.205}$$

20.2 If the integer non-negative valued random variable $\mathbf{X}$ has a discrete gamma distribution with parameter $\alpha>0$ and $0<p<1$ then the random variable $\beta\circ\mathbf{X}, 0\leq\beta\leq 1$ has a discrete gamma distribution with parameters $\alpha>0$ and $\frac{p\beta}{1-p(1-\beta)}$.

20.3 If the integer non-negative valued random variable $\mathbf{X}$ *has a compound* $\mathbf{N}$ *distribution given by*

$$\mathbf{X} = \sum_{n=0}^{\mathbf{N}} \mathbf{Z}_n$$

with $\mathbf{N}$ *a non-negative integer valued random variable independent of the sequence of independent and identically distributed integer valued random variables* $\mathbf{Z}_n$ *then the random variable* $\beta \circ \mathbf{X}, 0 \le \beta \le 1$ *has a compound* $\mathbf{N}$ *distribution. In particular, it holds that*

$$\beta \circ \mathbf{X} \stackrel{d}{=} \sum_{n=0}^{\mathbf{N}} \beta \circ \mathbf{Z}_n. \tag{5.206}$$

Proof. To verify relation (5.205) we observe by relation (5.195) and Newtons binomial formula that

$$\begin{aligned}
\mathbb{E}(z^{\beta \circ \mathbf{X}}) &= \mathcal{P}_{\mathbf{X}}(\beta z + (1-\beta)) \\
&= \sum_{k=0}^{\infty} q_k (\beta z + (1-\beta))^k \\
&= \sum_{k=0}^{\infty} q_k \sum_{n=0}^{k} \binom{k}{n} \beta^n (1-\beta)^{k-n} z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \binom{k}{n} \beta^n (1-\beta)^{k-n} q_k z^n
\end{aligned}$$

and this shows relation (5.205). To show part 2 we observe by relation (5.195) and the definition of a discrete gamma pmf that

$$\mathbb{E}(z^{\beta \circ \mathbf{X}}) = \mathcal{P}_{\mathbf{X}}(\beta z + 1 - \beta)) = \left(\frac{1-p}{1-p(\beta z + 1 - \beta)}\right)^{\alpha} = \left(\frac{1-p}{1-p(1-\beta) - p\beta z}\right)^{\alpha}.$$

Since

$$\frac{1-p}{1-p(1-\beta)} = \frac{1-p}{1-p+p\beta} = 1 - \frac{p\beta}{1-p(1-\beta)}$$

this shows part 1. To prove the last part we observe by relation (5.202) that

$$\beta \circ \mathbf{X} = \beta \circ \sum_{n=0}^{\mathbf{N}} \mathbf{Z}_n \stackrel{d}{=} \sum_{n=0}^{\mathbf{N}} \beta \circ \mathbf{Z}_n$$

and this shows relation (5.206). $\square$

If the integer non-negative random variable $\mathbf{X}$ satisfying $\mathbb{P}(\mathbf{X} = 0) > 0$ has the probability generating function $\mathbb{P}_{\mathbf{X}}$ introduce on $D = \{z \in \mathbb{R} : \mathbb{P}_{\mathbf{X}}(z) > 0\}$ the function $C : D \to \mathbb{R}$ given by

$$\mathcal{C}_{\mathbf{X}}(z) = \ln(\mathcal{P}_{\mathbf{X}}(z)) \tag{5.207}$$

Since the open set D contains 0 and the function $\mathcal{P}_{\mathbf{X}}$ is infinitely differentiable in a neighborhood of zero the function $\mathcal{C}_{\mathbf{X}}$ has a Taylor series expansion in a neighborhood of zero and so

$$\mathcal{C}_{\mathbf{X}}(z) = \ln(\mathcal{P}_{\mathbf{X}}(z)) = \sum_{n=0}^{\infty} \rho_n(\mathbf{X}) \frac{z^n}{n!} \tag{5.208}$$

with $\rho_n(\mathbf{X}) = \mathcal{C}_{\mathbf{X}}^{(n)}(0)$ with $\mathcal{C}_{\mathbf{X}}^{(n)}$ denoting the nth derivative of the function $\mathcal{C}_{\mathbf{X}}$. Observe, if $q \in \mathbb{N}$

and $\mathbf{X} \stackrel{d}{=} \sum_{j=1}^{q} \mathbf{X}_j$ with $\mathbf{X}_j, j = 1, ..., q$ a sequence of independent integer valued non-negative random variables then it is easy to check for every $n \in \mathbb{Z}_+$ that

$$\rho_n(\mathbf{X}) = \sum_{j=1}^{q} \rho_n(\mathbf{X}_j) \tag{5.209}$$

One can now can show the following recurrence relation for the pmf of the random variable $\mathbf{X}$. Observe a special case of the next result is Adelsons recurrence relations for a compound Poisson pmf (Tijms (2003)) and this is shown using a similar type of proof.

Lemma 63. *The next results hold*

21.1 If $\mathbf{X}$ is an integer valued non-negative random variable satisfying $\mathbb{P}(\mathbf{X} = 0) > 0$ it follows that $\mathbb{P}(\mathbf{X} = 0) = \mathcal{P}_{\mathbf{X}}(0) > 0$ and for every $m \in \mathbb{N}$ that

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X} = j) \frac{\rho_{m-j}(\mathbf{X})}{(m-1-j)!} \tag{5.210}$$

21.2 If the random variables $\mathbf{X}_i$, $i = 1, ..., n$ are a finite sequence of independent integer valued non-negative random variables satisfying $\mathbb{P}(\mathbf{X}_i = 0) > 0$ for every $i = 1, ..., n$ having probability generating functions $\mathcal{P}_{\mathbf{X}_i}$ and

$$\mathbf{X} = \sum_{i=1}^{n} \mathbf{X}_i$$

then $\mathbb{P}(\mathbf{X} = 0) = \Pi_{i=1}^{n} \mathcal{P}_{\mathbf{X}_i}(0) > 0$ and for every $m \in \mathbb{N}$

$$m\mathbb{P}(\mathbf{X} = m) = \sum_{j=0}^{m-1} \mathbb{P}(\mathbf{X} = j) \frac{\rho_{m-j}(\mathbf{X})}{(m-1-j)!} \tag{5.211}$$

with $\rho_j(\mathbf{X}) = \sum_{i=1}^{n} \rho_j(\mathbf{X}_i)$.

Proof. To show part 1 it is obvious that $\mathbb{P}(\mathbf{X} = 0) = \mathcal{P}_{\mathbf{X}}(0) > 0$ and so we consider some $m \in \mathbb{N}$. It follows since $\mathcal{P}_{\mathbf{X}}(0) > 0$ that there exist some $\epsilon > 0$ satisfying $\mathcal{P}_{\mathbf{X}}(z) > 0$ for every $-\epsilon \leq z \leq \epsilon$. This shows for every z in this neighborhood of 0 that

$$\mathcal{P}_{\mathbf{X}}^{(1)}(z) = \frac{\mathcal{P}_{\mathbf{X}}^{(1)}(z)}{\mathcal{P}_{\mathbf{X}}(z)} \mathcal{P}_{\mathbf{X}}(z) = \mathcal{C}_{\mathbf{X}}^{(1)}(z) \mathcal{P}_{\mathbf{X}}(z)$$

Applying now Leibniz's formula we obtain

$$\mathcal{P}_{\mathbf{X}}^{(m)}(z) = \sum_{j=0}^{m-1} \binom{m-1}{j} \mathcal{P}_{\mathbf{X}}^{(j)}(z) \mathcal{C}_{\mathbf{X}}^{(m-j)}(z) \tag{5.212}$$

Since for every $j \in \mathbb{Z}_+$ it follows that $\mathbb{P}(\mathbf{X} = j) = \frac{\mathcal{P}_{\mathbf{X}}^{(j)}(0)}{j!}$ this shows applying relation (5.212) that

$$
\begin{aligned}
\mathbb{P}(\mathbf{X} = m) &= \frac{\mathcal{P}_{\mathbf{X}}^{(m)}(0)}{m!} \\
&= \sum_{j=0}^{m-1} \frac{(m-1)!}{m!(m-1-j)!} \frac{\mathcal{P}_{\mathbf{X}}^{(j)}(0)}{j!} \mathcal{C}_{\mathbf{X}}^{(m-j)}(0) \\
&= \sum_{j=0}^{m-1} \frac{1}{m(m-1-j)!} \mathbb{P}(\mathbf{X} = j) \rho_{m-j}(\mathbf{X})
\end{aligned}
$$

and we have verified relation (5.210). To show relation (5.211) we apply the first part and relation (5.209). □

We also show the following intuitive clear result needed in our analysis.

Lemma 64. *If $\mathbf{P}$ is a purchase process with $\mathbf{P}_k, k \in \mathbb{Z}_+$ having a finite variance and the nonnegative series $\sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k)$ is finite then for every $t \in \mathbb{Z}_+$*

$$
Var(\mathbf{S}(\infty) - \mathbf{S}(t)) = \lim_{k \uparrow \infty} Var(\mathbf{S}(k) - \mathbf{S}(t)). \tag{5.213}
$$

Proof. Since $\sum_{k=0}^{\infty} \mathbb{E}(\mathbf{P}_k)$ is finite it follows by by Markovs inequality $\mathbf{S}(\infty) < \infty$ with probability 1. Also by the monotone convergence theorem we obtain

$$
\mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1)) = \sum_{k=t}^{\infty} \mathbb{E}(\mathbf{P}_k) < \infty
$$

Since $\mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))$ is finite we know

$$
\text{Var}(\mathbf{S}(\infty) - \mathbf{S}(t-1)) = \mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))^2) - \mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))^2 \leq \infty
$$

Using now that $\mathbf{P}_k, k \in \mathbb{Z}_+$ has a finite variance it follows by the Cauchy-Schwarz inequality that $\text{Cov}(\mathbf{P}_k, \mathbf{P}_j)$ is also finite for every k, j and this shows $\text{Var}(\mathbf{S}(k) - \mathbf{S}(t-1))$ is finite for every k. Again applying the monotone convergence theorem and using $\mathbf{S}(k) \uparrow \mathbf{S}(\infty)$ we obtain

$$
\begin{aligned}
\mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))^2) &= \mathbb{E}(\lim_{k \uparrow \infty} (\mathbf{S}(k) - \mathbf{S}(t-1))^2) \\
&= \lim_{k \uparrow \infty} \mathbb{E}(\mathbf{S}(k) - \mathbf{S}(t))^2) \\
&= \lim_{k \uparrow \infty} \text{Var}(\mathbf{S}(k) - \mathbf{S}(t-1)) + \mathbb{E}(\mathbf{S}(k) - \mathbf{S}(t-1))^2 \\
&= \lim_{k \uparrow \infty} \text{Var}(\mathbf{S}(k) - \mathbf{S}(t-1)) + \lim_{k \uparrow \infty} \mathbb{E}(\mathbf{S}(k) - \mathbf{S}(t-1))^2 \\
&= \lim_{k \uparrow \infty} \text{Var}(\mathbf{S}(k) - \mathbf{S}(t-1)) + \mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))^2
\end{aligned}
$$

This implies by the finiteness of $\mathbb{E}(\mathbf{S}(\infty) - \mathbf{S}(t-1))$ that

$$
Var(\mathbf{S}(\infty) - \mathbf{S}(t)) = \lim_{k \uparrow \infty} Var(\mathbf{S}(k) - \mathbf{S}(t))
$$

and we have shown the desired result. □

6. Conclusion

Spare parts can significantly impact a manufacturer's profitability. While maintaining an inventory of spare parts more than necessary can tie up capital and warehouse space, keeping low volumes of spare parts may lead to a shortage of parts which in turn leads to the dissatisfaction of the customers or replacement of the product with a new one. Both cases are considered a loss for the manufacturer.

In this study, we develop a stochastic modeling framework for the life cycle features (installed base size and total spare parts demand) of products based on three characteristic features: the sales rate, usage time (consumption time), and time until the first failure. The relation between these life cycle features is depicted in Figure 3.2. The sales rate is intricately tied to the life cycle pattern of the product, which typically encompasses three primary phases: initial, maturity, and decline. Each phase influences the demand for the product, thereby shaping the overall sales trajectory. Concurrently, the usage behavior of customers directly contributes to the formation of the installed base of the sold products. This installed base size pattern essentially represents a projection of the sales pattern, as the accumulation of sold items over time leads to the establishment of the installed base. As the product progresses through its life cycle, the installed items eventually experience failure during their consumption period. This occurrence fundamentally influences the stochastic counting process associated with the total number of failures, thereby playing a pivotal role in understanding and modeling the demand for spare parts and maintenance services. This interplay between sales patterns, installation base dynamics, and failure occurrences is crucial for developing accurate and effective models for spare parts demand estimation and inventory management.

Our research operates under the assumption that every product failure results in a corresponding demand for spare parts, a premise we acknowledge could be subject to refinement. We have chosen to use the total number of returned defectives and spare parts demand interchangeably, a simplification that we recognize may warrant further investigation. Furthermore, we have assumed that the usage time of the product remains independent of its failure rate. This assumption, while convenient for modeling purposes, may not fully align with real-world scenarios. To address this, a more realistic approach would require insights into customer expectations and usage behaviors, such as the probability of discarding an item after experiencing a certain number of failures. In the interest of maintaining the model's tractability, we have opted not to relax this assumption in the present study. However, future research endeavors could potentially explore these aspects to enhance the model's fidelity to real-world dynamics, thereby broadening its applicability and relevance in practical settings.

We have structured our framework in two ways: continuous time and discrete time indexed. This classification is directly tied to the sales process. While sales can occur continuously, such as 24/7, in practical terms, we typically record sales not at the exact moment of transaction (e.g., 10 pm), but rather in daily, weekly, or monthly intervals. Consequently, we have introduced the discrete time model to accommodate this reality.

In our continuous model, we make the assumption that the sales process adheres to a nonhomogeneous Poisson process, which can be represented by a continuous function such as the Brockoff or Bass diffusion functions. Through theoretical derivation, we establish the necessary conditions for calculating saturation points within the life cycle elements. Subsequently, we demonstrate that the Weibull distribution family, when applied to a nonhomogeneous Poisson process using Brockoff's sales model, meets these conditions. Leveraging these theoretical findings, we conduct a numerical exploration of the impacts of usage time and failure rate on the life cycle features. Our investigation reveals a direct correlation: as customer consumption time lengthens or the failure rate increases, there is a corresponding rise in spare parts demand. This observation intuitively aligns with expectations and underscores the practical significance of our theoretical framework in understanding spare parts demand dynamics within the context of product life cycles.

We expanded our framework to calculate spare parts demand within a defined time interval or from a specific point in time, t, until the end of the product's life cycle. This extension enables us to compute the probabilities associated with spare parts demand, facilitating the determination of a 95 percent demand covering safety stock. Our research underscores the importance for Original Equipment Manufacturers (OEMs) to invest in estimating future sales intensity and failure distribution of their products. Such insights are crucial for informing strategic decisions related to the design and optimization of their service networks, ultimately enhancing operational efficiency and customer satisfaction.

In the discrete model, the number of purchases at time t follows a discrete distribution such as Poisson distribution, Geometric distribution, or Discrete Gamma distribution. As in the continuous model, we formulated the relations to compute the pmf of the number of returned defective items (spare parts demand) by time t.

Based on the findings of this study, it can be generally concluded that employing stochastic processes to model spare parts demand effectively estimates the moments and distribution of such demand. However, this study also reveals potential avenues for extending this research, which could significantly enhance its comprehensiveness. These potential extensions might involve exploring the impact of specific variables on the stochastic process, investigating alternative modeling techniques to capture demand variability, and examining the applicability of the approach to different industry contexts. By addressing these areas, the research can offer a more holistic understanding of spare parts demand and its modeling, thus contributing to the broader body of knowledge in this field.

There are several potential extensions to our research. Firstly, the evaluation of the proposed models using real-life data is yet to be conducted. Successful validation with real-life data could establish our model as a benchmark for future researchers. Secondly, delving into the main assumptions of the models is vital. Investigating cases where our assumptions, such as the independence of usage time on the failure rate, may not hold true is crucial for refining our model. Additionally, a third extension involves considering special cases of a product's spare parts demand, including seasonal variations, geographic variances, and technological changes. These cases require meticulous attention and proactive management to ensure the availability of the right spare parts when needed, while also minimizing excess inventory. Addressing these extensions will not only fortify the robustness of our research but also provide valuable insights for the practical application of spare parts demand modeling.

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