HILBERT SERIES OF POLYOMINO IDEALS AND COHEN-MACAULAY POSETS

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ABSTRACT

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In this thesis, we study the algebraic and homological properties of polyomino ideals and characterize Cohen-Macaulay posets of dimension two.

In 2012, Qureshi introduced a class of binomial ideals called polyomino ideals, related to combinatorial objects called polyominoes. The polyomino ideals are defined by associating each polyomino $\mathcal{P}$ with the ideal generated by the inner 2-minors of $\mathcal{P}$ in the polynomial ring $S_{\mathcal{P}} = K[x_v : v \text{ is a vertex of } \mathcal{P}]$. A primary aim of this research is to investigate the algebraic and homological properties of $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$ depending on the shape of $\mathcal{P}$. We introduce a new class of non-simple polyominoes called frame polyominoes, and demonstrate that its Hilbert series can be described in terms of certain rook arrangements in polyominoes. We also define a new collection of cells called zig-zag collection and its Hilbert series is similarly characterized by certain rook arrangements. For a zig-zag collection of cells $\mathcal{P}$ we provide a necessary condition for the coordinate ring $K[\mathcal{P}]$ to be Gorenstein.

A key practical outcome of this research is the development of the `PolyominoIdeals` package for Macaulay2. This computational tool is tailored to assist in the study of polyomino ideals, enabling more effective exploration and analysis of these algebraic structures.

Additionally, we provide a characterization of Cohen-Macaulay posets of dimension two. We demonstrate that these posets are shellable and strongly connected.

ÖZET

POLİMİNO İDEALLERİNİN HİLBERT SERİLERİ VE COHEN-MACAULAY POSETLERİ

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Bu tezde, poliomino ideallerinin cebirsel ve homolojik özelliklerini inceleyip iki boyutlu Cohen-Macaulay posetlerini karakterize ediyoruz.

2012'de Qureshi, poliomino adı verilen kombinatorik nesnelerle ilişkili poliomino idealleri adında bir binom ideal sınıfı tanıttı. Poliomino idealler, her bir poliomino $\mathcal{P}$ ile $\mathcal{P}$'nin iç 2-minörleri tarafından üretilen idealin polinom halkası $S_{\mathcal{P}} = K[x_v :$ v bir $\mathcal{P}$ köşesi ise] ile ilişkilendirilerek tanımlanır. Bu araştırmanın birincil amacı, $\mathcal{P}$'nin şekline bağlı olarak $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$'nin cebirsel ve homolojik özelliklerini incelemektir. Basit olmayan poliomino sınıfı olan çerçeve poliomino adında yeni bir sınıf tanıtıyoruz ve bu sınıfın Hilbert serisinin, poliomino içindeki bazı piyon düzenlemeleri cinsinden tanımlandığını gösteriyoruz. Ayrıca zik-zak hücre koleksiyonu adında yeni bir hücre koleksiyonu tanımlıyoruz ve onun Hilbert serisi de benzer şekilde bazı piyon düzenlemeleri ile karakterize edilir. Zik-zak hücre koleksiyonu olan $\mathcal{P}$ için, koordinat halkası $K[\mathcal{P}]$'nin Gorenstein olması için gerekli bir koşul sağlıyoruz.

Bu araştırmanın ana pratik sonucu, Macaulay2 için `PolyominoIdeals` paketinin geliştirilmesidir. Bu hesaplama aracı, poliomino ideallerinin çalışılmasına yardımcı olmak için özel olarak tasarlanmıştır ve bu cebirsel yapıların daha etkili keşfi ve analizine olanak tanır.

Ek olarak, iki boyutlu Cohen-Macaulay posetlerinin karakterizasyonunu sağlıyoruz. Bu posetlerin kabuklanabilir ve güçlü bir şekilde bağlantılı olduğunu gösteriyoruz.

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To my family

Contents

List of Figures

Introduction

The study of determinantal ideals, a key area in algebraic geometry and commutative algebra, bridges various fields including invariant theory, representation theory, and combinatorics. Significant contributions to this area have been made by a range of authors, as indicated in references [5], [10], [15], [16], [35] and [51]. Expanding this focus, the study of ideals generated by arbitrary sets of t-minors of a $m \times n$ matrix of indeterminates has become a prominent subject in combinatorial commutative algebra since the 1990s. This specific aspect of determinantal ideals is further explored in works such as [11], [12], and [13]. Additionally, research on the ideals generated by adjacent 2-minors and those generated by any random set of 2-minors of a $2 \times n$ matrix has been conducted, as discussed in references [22], [23], [30], [39].

The focus of this research is twofold. The first objective is to examine polyomino ideals which are binomial ideals that emerge from certain combinatorial objects, and the second is to characterize Cohen-Macaulay posets of dimension two. The polyomino ideals class is an extension of various classes of ideals, including the join-meet ideals and determinantal ideals. Polyominoes are two-dimensional shapes composed of unit squares, known as cells, that are connected to one another along their edges. This concept initially emerged in the realm of recreational mathematics and combinatorics, with a particular focus on their application to plane tiling problems. Although certain problems, such as the enumeration of pentominoes, can be traced back to ancient times, Golomb introduced the formal definition of polyominoes in 1953 and later, in his monograph in 1996 [20].

In the paper of Qureshi [41], a link between polyominoes and commutative algebra is created by assigning a binomial ideal to a polyomino $\mathcal{P}$. The concept being referred to is called the polyomino ideal. It consists of all the inner 2-minors of $\mathcal{P}$ in the polynomial ring $S_{\mathcal{P}}$, where the variables are labeled according to the vertices of $\mathcal{P}$. The polyomino ideal is represented by $I_{\mathcal{P}}$, and the quotient ring $K[\mathcal{P}] = S_{\mathcal{P}}/I_{\mathcal{P}}$ is known as the coordinate ring of $\mathcal{P}$. Investigating the algebraic properties of $K[\mathcal{P}]$ by considering the geometric arrangement of $\mathcal{P}$ offers a promising avenue for research. To simplify the notation, we say that a polyomino has property Q if its associated polyomino ideal has the property Q. In recent times, there has been a substantial growth to the literature regarding polyomino ideals. Primality of $\mathcal{P}$ based on the shape of $\mathcal{P}$ has been discussed by several authors. It is shown in [25] and [43] that polyominoes without embedded holes, also known as simple polyominoes, are prime. In [6], [8], [26], [28] and [37] the primality of some classes of non-simple polyominoes has been shown. However, a complete characterization of non-simple prime polyominoes still remain open. A big breakthrough in this direction was the work of Mascia, Romeo and Rinaldo [36]. They showed that if a polyomino $\mathcal{P}$ has a zig-zag path which is a certain sequence of inner 2-minors, then $\mathcal{P}$ is not prime and conjectured that the converse is also true.

Recently the study of regularity and Hilbert series of $K[\mathcal{P}]$ in terms of the rook polynomial of the polyomino $\mathcal{P}$ is studied by many authors. The rook polynomial of a polyomino $\mathcal{P}$ is well-known in literature, see [34] and [46, Chapter 7]. It is a polynomial $r_{\mathcal{P}}(t) = \sum_{i=0}^{n} r_i t^i \in \mathbb{Z}[t]$, whose coefficient r_i is the number of distinct arrangements of i non-attacking rooks arranged on the cells of $\mathcal{P}$. The degree of $r_{\mathcal{P}}(t)$ is called the rook number of $\mathcal{P}$. In [18] the authors show that if $\mathcal{P}$ is an L-convex polyomino (see Definition **??**) then regularity of $K[\mathcal{P}]$ is equal to the rook number of $\mathcal{P}$. In [45] it is proved that for a simple thin polyomino $\mathcal{P}$, a simple polyomino that does not have a square tetromino in it, the Hilbert series of $K[\mathcal{P}]$ is $\frac{r_{\mathcal{P}}(t)}{(1-t)^d}$ where d is dimension of $K[\mathcal{P}]$. They also characterized Gorenstein simple thin polyominoes using the so-called S-property. Similar results are obtained in [9] for a particular class of non-simple thin polyominoes, called closed path. In [42] the authors associate $\mathcal{P}$ with another polynomial which is related to the certain rook arrangements in $\mathcal{P}$. This polynomial is denoted by $\tilde{r}_{\mathcal{P}}(t)$ and in this work we call it switching rook polynomial of $\mathcal{P}$, see Definition 1.29. They conjectured that the Hilbert series of $K[\mathcal{P}]$ for a simple polyomino $\mathcal{P}$ is $\frac{\tilde{r}_{\mathcal{P}}(t)}{(1-t)^d}$ ([42, Conjecture 3.2]) and provided a characterization of the parallelogram polyominoes whose coordinate ring is Gorenstein.

Inspired by the above-mentioned results and conjecture, we study the Hilbert series of $K[\mathcal{P}]$ for some classes of non-simple polyominoes. We provide all the needed definitions and notations in Chapter 1. In Chapter 2 we define a non-simple class of polyominoes called *frame polyomino*, see Definition 2.1. Roughly speaking, a frame polyomino $\mathcal{P}$ is a non-simple polyomino obtained by removing a parallelogram polyomino from a rectangle polyomino. Some algebraic properties of the K-algebras associated with a larger class of this kind of polyominoes are studied in [28] and [48], where the authors prove that the associated coordinate ring is a normal Cohen-Macaulay domain but without computing the dimension. In Section 2.1 we prove that the dimension of $K[\mathcal{P}]$ is equal to the difference between the number of the

vertices of $\mathcal{P}$ and the number of the cells of $\mathcal{P}$.
From Section 2.2 we start a deep study of the simplicial complex $\Delta(\mathcal{P})$ associated with the initial ideal of $I_{\mathcal{P}}$ with respect to a suitable monomial order on $S_{\mathcal{P}}$. We show that the facets of $\Delta(\mathcal{P})$ form a shelling order with respect to lexicographical order. We use McMullen-Walkup formula [4, Corollary 5.1.14] to interpret Hilbert series of $K[\mathcal{P}]$ based on this shelling order, see Theorem 2.12. Finally in Section 2.3 we prove the conjecture ([42, Conjecture 3.2]) for a non-simple class of polyominoes, see Theorem 2.20. Finally, we demonstrate in this section that the rook number of $\mathcal{P}$ is equal to the regularity of $K[\mathcal{P}]$ for a frame polyomino and conjecture that the Theorem 2.20 holds for every polyomino, see Conjecture 2.22.

Motivated by the work [36], in Chapter 3 we define another class of non-simple collection of cells and call it *zig-zag collection*, see Definition 3.1. We define a monomial order for the zig-zag collection that gives a squarefree Gröbner basis for the associated binomial ideal, see Lemma 3.4. Later in the Section 3.2 we show that the Conjecture 2.22 holds true for zig-zag collection. As a consequence we obtain that the regularity of a zig-zag collection $\mathcal{P}$ is the rook number $\mathcal{P}$. We conclude this section by providing a necessary condition for the coordinate ring of a zig-zag collection to be Gorenstein.

All the examples that served as inspiration for the findings related to polyomino ideals were examined using the computer algebra software `Macaulay2` [52]. In order to test multiple examples for our research, we have developed a package `PolyominoIdeals` [7] and implemented in `Macaulay2`. The objective of this software package is to offer a set of computational tools that can assist the study of polyomino ideals. The package offers three main functions and several related options for encoding a predetermined set of cells using a list of lists that contains the diagonal corners of each cell. One of the functions available is `polyoIdeal`, which defines the inner 2-minor ideal of a collection of cells $\mathcal{P}$. There are three options available for this function. The `RingChoice` option provides the ability to select between two rings with different monomial orders. The second option is the `Field`, which enables the modification of field K in the base polynomial ring of $I_{\mathcal{P}}$. The third option is `TermOrder`, which allows the substitution of the lexicographic order with other monomial orders. Next we have `polyoToric` function which facilitates the generation of the toric ideal as specified in reference [36]. This function proves to be valuable in investigating the primality of the polyomino ideal. Finally the third function is `polyoMatrix` which gives the matrix attached to $\mathcal{P}$. In the end we provide a useful method to obtain the input list for this package using `GeoGebra` [19].

For a poset P, the order complex $\Delta(P)$ of P is a simplicial complex on the underlying set of P whose faces are chains of P. We say that a poset has property $\mathcal{Q}$ if its order complex has the property $\mathcal{Q}$. In Section 5.1 we recall definitions of co-comparability graph, intersection graph and permutation graph. Using these concepts we observe that the dimension of a poset is at most two if and only if its co-comparability graph is a permutation graph. In Section 5.2 we characterize the Cohen-Macaulay posets (see Definition **??**) of dimension two, indeed we show that they are shellable and strongly connected, see Theorem 5.7.

Chapter 1

Basics on Combinatorial Commutative Algebra

This chapter includes a few definitions and associated results from combinatorics and commutative algebra. These concepts are necessary requirements for the following chapters. To gain a comprehensive understanding of these topics, one can consult the authoritative texts [1] and [4]. All the rings discussed in this work are commutative rings with an identity.

1.1 Krull dimension and Castelnuovo-Mumford regularity

Let R be a ring. A proper ideal $I \subset R$ is said to be a *prime ideal* if for any $a, b \in R$, the condition $ab \in I$ implies either $a \in I$ or $b \in I$. The radical of an ideal I, denoted by $\sqrt{I}$, is defined as $\sqrt{I} := \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$. An ideal $I \subset R$ is called a *radical ideal* if $I = \sqrt{I}$. The collection of all prime ideals in a ring R is referred to as the *spectrum* of R and is symbolized by $\mathrm{Spec}(R)$.
Let $P \subset R$ be a prime ideal. The *height* of P, denoted by $\mathrm{ht}(P)$, is defined as the maximum of lengths of all the chains of prime ideals in $\mathrm{Spec}(R)$ descending from P. For an ideal $I \subset R$, the height is defined as

$$\mathrm{ht}(I) = \min\{\mathrm{ht}(P) : P \in \mathrm{Spec}(R), I \subset P\}.$$

We now recall the definition of dimension of a ring.

Definition 1.1. Let R be a ring. The *Krull dimension* of R, referred to as dimension of R, is denoted by $\dim(R)$ and is defined as the supremum of lengths of all ascending chains of prime ideals in R. Formally,

$$\dim(R) = \sup\{\mathrm{ht}(P) : P \in \mathrm{Spec}(R)\}.$$

The dimension of an R-module M is defined as the dimension of the quotient ring R/I where $I = \mathrm{Ann}_R(M)$.
A K-algebra R is called *standard graded* if $R = \bigoplus_{i\in\mathbb{N}} R_i$ where $R_0 = K$ and for each $i, j \in \mathbb{N}$ we have $R_iR_j \subset R_{i+j}$. Each R_i is termed a *graded component* of degree i, and its elements are known as homogeneous elements of said degree. An R-module M is *graded* if $M = \bigoplus_{i\in\mathbb{N}} M_i$ and for each $i, j \in \mathbb{N}$ we have $R_iM_j \subset M_{i+j}$. An ideal I of R is called *homogeneous* or *graded ideal* if it is graded as an R-module. An R-module F is a *free module* if there exists a set B such that every element $m \in F$ can be expressed uniquely as

$$m = \sum_{b\in B} r_b\, b \quad \text{for some } r_b \in R.$$

Here the set B is termed a basis of the free module F. A *graded free module* over a graded ring R is an R-module that is both free and graded, with a basis B comprising homogeneous elements. Due to the uniqueness of decomposition of elements, a graded free module can be explicitly represented as a direct sum of copies of the graded ring R. For a comprehensive exploration of graded rings and modules, one can refer to the book by Peeva [40].
Let $S = K[x_1, x_2, \ldots, x_n]$ and M be a finitely generated graded S-module. A sequence $\mathcal{F}(M)$ comprising graded free S-modules F_j and S-module homomorphisms d_j as

$$\mathcal{F}(M) : \cdots \xrightarrow{d_{i+2}} F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

is called a *graded free resolution* of M if $\ker(d_i) = \mathrm{im}(d_{i+1})$ for all $i \neq 0$ and $M \cong F_0/\mathrm{im}(d_1)$. It is noteworthy that such resolutions are generally not unique.
The length of a graded free resolution $\mathcal{F}(M)$ is denoted as the supremum of indices j for which $F_j \neq 0$. A resolution is termed *minimal* if $d_{i+1}(F_{i+1}) \subset (x_1, x_2, \ldots, x_n)F_i$ for all $i \geq 0$.
Hilbert's syzygy theorem [29] asserts that a graded, finitely generated S-module has a minimal graded free resolution of length at most n. Furthermore, such resolutions are guaranteed to exist and are unique up to isomorphism.
In such a case $\mathcal{F}(M)$ is written as

$$0 \to \oplus_{j\in\mathbb{N}} S(-j)^{\beta_{k,j}} \xrightarrow{d_k} \ldots \to \oplus_{j\in\mathbb{N}} S(-j)^{\beta_{i,j}} \xrightarrow{d_i} \ldots \to \oplus_{j\in\mathbb{N}} S(-j)^{\beta_{0,j}} \xrightarrow{d_0} M \to 0.$$

In the above given representation, the coefficients $\beta_{i,j}$ are termed the graded *Betti numbers* of M. The *projective dimension* of M, denoted as $\mathrm{pd}(M)$, is the maximum index i for which the total Betti number $\beta_i \neq 0$, where $\beta_i = \sum_{j\in\mathbb{N}} \beta_{i,j}$.

Definition 1.2. The *Castelnuovo-Mumford regularity*, referred to simply as regularity, of M is denoted by $\mathrm{reg}(M)$ and is defined as $\max\{j : \beta_{i,i+j} \neq 0\}$ for the minimal free resolution of M.

Betti numbers are conventionally presented in tabular form for clarity and ease of reference as given below.

Example 1.3. Let $S = K[x, y]$ be a polynomial ring over a field K, and $I = (x^3, xy) \subset S$. The minimal free resolution of $M = S/I$ is given as

$$0 \longrightarrow S(-4) \xrightarrow{\begin{pmatrix} -y \\ x^2 \end{pmatrix}} S(-3) \oplus S(-2) \xrightarrow{\begin{pmatrix} x^3 & xy \end{pmatrix}} S \longrightarrow M \longrightarrow 0.$$

The graded Betti numbers for this resolution can be tabulated as

$\beta_{i,i+j}$	1	2
0	1	–
1	–	1
2	–	1

which gives us that $\text{pd}(M) = \text{reg}(M) = 2$.

1.2 Hilbert-Poincaré series

Let R be a standard graded ring and M a graded R-module. Each graded component M_i can be viewed as a vector space over K.

Definition 1.4. The *Hilbert function* of M is defined as the map $\text{H}(M, j) : \mathbb{N} \to \mathbb{N}$, where $j \mapsto \dim_K M_j$ and $\dim_K M_j$ represents the vector space dimension of jth graded component over K. The *Hilbert-Poincaré series* of M is given by the formal power series

$$\text{HS}_M(t) = \sum_{j \in \mathbb{N}} \text{H}(M, j) t^j.$$

By Hilbert-Serre theorem [47], the Hilbert series of M can be expressed as a rational function of the form

$$\text{HS}_M(t) = \frac{h(t)}{(1-t)^d}$$

where $h(t)$ is a polynomial with $h(1) \neq 0$, and d is the Krull dimension of M. This polynomial $h(t)$ is known as the h-polynomial of M. When expressed in this rational form, the Hilbert series is referred to as the *reduced Hilbert series* of M.

Example 1.5. Let $S = K[x, y]$ and $I = (x^2, xy) \subset S$. Then $M = S/I$ is a graded algebra, and we aim to compute its Hilbert series.
The Hilbert function is computed as $\text{H}(M, 0) = \dim_K M_0 = 1, \text{H}(M, 1) = \dim_K M_1 =$

$2, \mathrm{H}(M, 2) = \dim_K M_2 = 1$, and so on.
Thus, the Hilbert-Poincaré series for M is given by $\mathrm{HS}_M(t) = 1 + 2t + t^2 + \ldots$, which, when expressed in its rational form, is

$$\mathrm{HS}_M(t) = \frac{1 + t - t^2}{1 - t}.$$

The next proposition relates the Hilbert series of tensor products of standard graded K-algebras.

Proposition 1.6. [55, Lemma 5.1.11] "Let A and B be standard graded K-algebras over a field K. Then $\mathrm{HS}_{A \otimes_K B}(t) = \mathrm{HS}_A(t) \cdot \mathrm{HS}_B(t)$. In particular, $\dim(A \otimes_K B) = \dim(A) + \dim(B)$. "

1.3 Cohen-Macaulay and Gorenstein rings

Throughout this section M is a non-zero module.
Let M be a module over a ring R. An element $x \neq 0 \in R$ is called *M-regular* if $xm = 0$ for $m \in M$ implies $m = 0$. This leads us to the definition of a regular sequence, which is constructed using successive M-regular elements.

Definition 1.7. A sequence of elements $x_1, x_2, \ldots, x_n$ in R is called an *M-regular sequence* or simply *M-sequence* if, for each $i = 1, 2, \ldots, n$, the element x_i is regular with respect to the module $M/(x_1, x_2, \ldots, x_{i-1})M$ and $M/(x_1, x_2, \ldots, x_n)M \neq 0$.

For instance the sequence $x, yz, y + z$ in $S = K[x, y, z]$ is S-sequence.
Let M be a module over a ring R. An M-sequence $x_1, x_2, \ldots, x_n$ is called *maximal* if $x_1, \ldots, x_n, x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$.

Theorem 1.8. [4, Theorem 1.2.5] "Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length. "

The length of the longest M-sequences in I, as mentioned in the theorem above, is referred to as the *grade* of I on M. It is denoted as $\mathrm{grade}(I, M)$. The definition is enhanced by assigning a value of infinity to $\mathrm{grade}(I, M)$ when $IM = M$.
Consider a Noetherian local ring $(R, \mathfrak{m})$ and a finitely generated R-module M. The grade of the module M with respect to the maximal ideal $\mathfrak{m}$ is known as the *depth* of M and is denoted by $\mathrm{depth}(M)$. The following result establishes a relationship between the dimension and depth of a module M.

Proposition 1.9. [4, Proposition 1.2.12] "Let R be a Noetherian local ring and $M \neq 0$ a finitely generated R-module. Then $\mathrm{depth}(M) \leq \dim(M)$. "

We now define Cohen-Macaulay rings.

Definition 1.10. Let R be a Noetherian local ring. A finitely generated R-module $M \neq 0$ is called *Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(M)$. If R itself is a Cohen-Macaulay as an R-module, then it is called *Cohen-Macaulay ring*. If R is not local, then R is said to be Cohen-Macaulay if $R_{\mathfrak{m}}$ is a local Cohen-Macaulay ring for all maximal ideals $\mathfrak{m}$ of R.

Example 1.11. Let K be a field. The polynomial ring $K[x_1, x_2, \ldots, x_n]$ and the ring of formal power series $K[[x_1, x_2, \ldots, x_n]]$ are both Cohen-Macaulay rings.

Let I be an ideal of a ring R. In general it holds that $\dim(R/I) + \operatorname{ht}(I) \leq \dim(R)$. The following proposition asserts the equality in case of Cohen-Macaulay rings.

Proposition 1.12. [4] "Let R be a Cohen-Macaulay ring and $I \subset R$ be an ideal of R. Then $\dim(R/I) + \operatorname{ht}(I) = \dim(R)$. "

Let N', M' and N be R-modules. We call N' as *injective module* if for every R-module homomorphism $f : N \to N'$ and for any injective morphism $g : N \to M'$ there exist an R-module homomorphism $h : M' \to N'$ such that $f = h \circ g$.

$$\begin{array}{ccc} N & \xhookrightarrow{g} & M' \\ & \searrow^{f} & \downarrow{\scriptstyle h} \\ & & N' \end{array}$$

Let M be an R-module. A sequence $\mathcal{I}(M)$ comprising injective R-modules N_j and R-module homomorphisms f_j

$$\mathcal{I}(M): \; 0 \longrightarrow M \xrightarrow{f_0} N_0 \xrightarrow{f_1} N_1 \xrightarrow{f_2} N_2 \xrightarrow{f_3} \cdots$$

is called injective resolution of M if $\operatorname{im}(f_i) = \ker(f_{i+1})$ for all i. The *injective dimension* of M denoted as $\operatorname{injdim}(M)$, defined as the smallest integer n such that $I_j = 0$ for all $j > n$ in $\mathcal{I}(M)$. If such an integer does not exist, we say M has infinite injective dimension.
We now recall the definition of Gorenstein ring.

Definition 1.13. A Noetherian local ring R is called *Gorenstein* if the injective dimension of R as an R-module is finite. If R is not local, then R is said to be Gorenstein if R_m is local Gorenstein ring for all maximal ideals m of R.

Following proposition is the combination of Lemma 3.1.34, Lemma 3.1.35 and Proposition 11.5.8 from [55].

Proposition 1.14. [55] "Let A and B be standard graded K-algebras over a field K. Then $\operatorname{depth}(A \otimes_K B) = \operatorname{depth}(A) + \operatorname{depth}(B)$. In particular $A \otimes_K B$ is Cohen-Macaulay (Gorenstein) if and only if A and B are Cohen-Macaulay (Gorenstein). "

This section concludes with the following theorem that characterizes the regularity and Gorenstein property of Cohen-Macaulay rings.

Theorem 1.15. [49] "Let $S = K[x_1, \ldots, x_n]$ be a graded polynomial ring and I be a homogeneous ideal of S such that S/I is Cohen-Macaulay. Consider the reduced Hilbert series of S/I, that is

$$\mathrm{HS}_{S/I} = \frac{\sum_{j=0}^{s} h_j t^j}{(1-t)^d}.$$

Then the following hold:

1. $\mathrm{reg}(S/I) = s$;

2. R/I is Gorenstein if and only if $h_i = h_{s-i}$ for all $i = 0, \ldots, s$. "

1.4 Gröbner basis

Now we provide some definitions and results from the theory of Gröbner basis. A comprehensive reference on this topic is the book [17].

Let K be a field and $S = K[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n indeterminates. A monomial in S can be expressed in the form $\mathbf{x^a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. We use $\mathrm{Mon}(S)$ to denote the set of all monomials in S.

Definition 1.16. Let S be a polynomial ring. A monomial order on S is a total order $\leq$ on $\mathrm{Mon}(S)$ satisfying:

1. for every monomial $u \in \mathrm{Mon}(S)$, we have $1 \leq u$,

2. given any monomials u, v, and w in $\mathrm{Mon}(S)$, if $u \leq v$ then $uw \leq vw$.

Example 1.17. Let $\mathbf{x^a}, \mathbf{x^b} \in S = K[x_1, x_2, \ldots, x_n]$ be two monomials. Below are three examples of monomial orders induced by the ordering of variables as $x_1 > x_2 > \ldots > x_n$.

1. **The lexicographic order:** $\mathbf{x^a} < \mathbf{x^b}$, if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and the left-most component of $\mathbf{a} - \mathbf{b}$ is negative.

2. **The pure lexicographic order:** $\mathbf{x^a} < \mathbf{x^b}$, if the left-most component of $\mathbf{a} - \mathbf{b}$ is negative.

3. **The reverse lexicographic order:** $\mathbf{x^a} < \mathbf{x^b}$, if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and the right-most component of $\mathbf{a} - \mathbf{b}$ is positive.

A polynomial $f \in S$ is expressed in the form $f = \sum \alpha_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, where the coefficients $\alpha_{\mathbf{a}}$ are non-zero for only a finite number of terms. The support of f is denoted by $\mathrm{supp}(f)$ and is defined as

$$\mathrm{supp}(f) = \{\mathbf{x}^{\mathbf{a}} : \alpha_{\mathbf{a}} \neq 0\}.$$

Definition 1.18. Let S be a polynomial ring equipped with a monomial order $<$. For a non-zero polynomial $f \in S$, the *initial monomial* of f is denoted by $\mathrm{in}_<(f)$ and is defined as the largest monomial in $\mathrm{supp}(f)$ with respect to $<$. The coefficient c corresponding to $\mathrm{in}_<(f)$ in f is termed the *leading coefficient*. Consequently, the product $c\,\mathrm{in}_<(f)$ is referred as the *leading term* of f.

Conventionally we write $\mathrm{in}_<(0) = 0$ and $\mathrm{in}_<(0) < \mathrm{in}_<(f)$ for all non-zero polynomials $f \in S$.

Definition 1.19. Let I be an ideal of the polynomial ring S and let $<$ be a fixed monomial order on S. The *initial ideal* of I, denoted by $\mathrm{in}_<(I)$, is the monomial ideal generated by the initial monomials $\mathrm{in}_<(f)$ of all polynomials $f \in I$.

While it is natural to think that if $I = (f_1, \dots, f_k)$ then $\mathrm{in}_<(I) = (\mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_k))$, but this is not generally the case. Below is the definition of a special generating set of I which satisfy this property.

Definition 1.20. Let S be a polynomial ring with a monomial order $<$. Let $I \subset S$ be an ideal. A generating set $\mathcal{G} = \{g_1, g_2, \dots, g_m\}$ of I is called *Gröbner basis* if $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), \mathrm{in}_<(g_2), \dots, \mathrm{in}_<(g_m))$.

It is worth noting that while every ideal I has a Gröbner basis, such a basis is not necessarily unique. For instance, if $\{g_1, g_2, \dots, g_m\}$ forms a Gröbner basis, then the set $\{g_1, g_2, \dots, g_m, g_1 + g_m\}$ is also a Gröbner basis.

Definition 1.21. Let S be a polynomial ring equipped with a monomial order $<$. Let I be an ideal of S and $\mathcal{G} = \{g_1, \dots, g_m\}$ a Gröbner basis of I with respect to $<$. We say $\mathcal{G}$ is *reduced Gröbner basis* if:

1. the leading coefficient of each g_i is 1, for all $i \in \{1, \dots, m\}$,
2. no monomial in $\mathrm{supp}(g_i)$ is divisible by the leading monomial of g_j for any $i \neq j$.

Example 1.22. Let $S = K[x, y]$ be a polynomial ring with the lexicographic order. Let $I = (x^2 - y, y - x^2) \subset S$. The generating set $\{g_1 = x^2 - y, g_2 = y^2 - x\}$ forms a Gröbner basis. This Gröbner basis is reduced since the leading coefficient of both g_1 and g_2 is 1, and no monomial in g_1 is divisible by the leading monomial of g_2 and vice versa.

Now we present some results relating the ideal I and the initial ideal $\text{in}_<(I)$ with respect to a monomial order $<$.

Proposition 1.23. [55] "Given a homogeneous ideal I in a polynomial ring S, the following hold:

1. The Hilbert functions of S/I and $S/\text{in}_<(I)$ coincide for any monomial order $<$ [55, Corollary 3.3.15].

2. If $S/\text{in}_<(I)$ is Cohen–Macaulay (or Gorenstein), then so is S/I [55, Proposition 9.6.17]. "

From [14], it can be inferred that the converse of the Cohen-Macaulay property in item (2) of Proposition 1.23 is valid for a homogeneous ideal whose initial ideal is squarefree.

Proposition 1.24. [14] "Let I be a homogeneous ideal in a polynomial ring S. If $\text{in}_<(I)$ is radical with respect to a given monomial order $<$, then:

a) $\text{reg}(S/I) = \text{reg}(S/\text{in}_<(I))$,

b) $\text{depth}(S/I) = \text{depth}(S/\text{in}_<(I))$,

c) S/I is Cohen-Macaulay if and only if $S/\text{in}_<(I)$ is Cohen-Macaulay. "

1.5 Polyominoes and polyomino ideal

Polyominoes, configurations formed by joining unit squares edge-to-edge, have long held a significant place in the field of combinatorics. For a comprehensive study on We consider the natural partial order on $\mathbb{Z}^2$. Let $(i, j), (k, l) \in \mathbb{Z}$ then $(i, j) \leq (k, l)$ if $i \leq k$ and $j \leq l$. For any two points $a = (i, j)$ and $b = (k, l)$ in $\mathbb{Z}^2$ with $a \leq b$, the set

$$[a, b] = \{(m, n) \in \mathbb{Z}^2 : i \leq m \leq k,\ j \leq n \leq l\}$$

is termed an *interval* of $\mathbb{Z}^2$. If $i < k$ and $j < l$, then $[a, b]$ is referred to as a *proper interval*. The points a and b are called the *diagonal corners* of $[a, b]$, while the points $c = (i, l)$ and $d = (k, j)$ are termed the *anti-diagonal corners* of $[a, b]$. If $j = l$ (or $i = k$), then the points a and b are said to be in a horizontal (or vertical) position. A *cell* in $\mathbb{Z}^2$ is a proper interval $C = [a, b]$ where $b = a + (1, 1)$. The vertices a, b, c, and d are respectively *lower left*, *upper right*, *upper left*, and *lower right* corners of C. The sets $\{a, c\}$, $\{c, b\}$, $\{b, d\}$, and $\{a, d\}$ are the *edges* of C. We define $V(C) = \{a, b, c, d\}$ and $E(C) = \{\{a, c\}, \{c, b\}, \{b, d\}, \{a, d\}\}$.

Let $\mathcal{S}$ be a non empty collection of cells in $\mathbb{Z}^2$. The vertices and edges of $\mathcal{S}$ are defined by $V(\mathcal{S}) = \bigcup_{C\in\mathcal{S}} V(C)$ and $E(\mathcal{S}) = \bigcup_{C\in\mathcal{S}} E(C)$, respectively. The *rank* of $\mathcal{S}$, represented as $\operatorname{rank}\mathcal{S}$, indicates the number of cells that belong to $\mathcal{S}$. For two distinct cells, C and D in $\mathcal{S}$, a *path* from C to D in $\mathcal{S}$ is described as a sequence $\mathcal{C} : C = C_1, \dots, C_m = D$ of cells in $\mathbb{Z}^2$ such that $C_i \cap C_{i+1}$ is an edge of both C_i and C_{i+1} for all $i = 1, \dots, m-1$, and $C_i \neq C_j$ when $i \neq j$. Furthermore, two cells, C and D, are said to be *connected* in $\mathcal{S}$ if a path exists between them within $\mathcal{S}$.

Definition 1.25. A *polyomino* $\mathcal{P}$ is a non-empty, finite collection of cells in $\mathbb{Z}^2$ such that any two cells of $\mathcal{P}$ are connected in $\mathcal{P}$.

An example of a polyomino is illustrated in Figure 1.1.

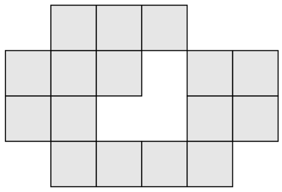


Figure 1.1: An example of a polyomino.

Let A and B be two cells of $\mathbb{Z}^2$ with $a = (i, j)$ and $b = (k, l)$ as the lower left corners of A and B with $a \leq b$. A *cell interval* $[A, B]$ is the set of the cells of $\mathbb{Z}^2$ with lower left corner (r, s) such that $i \leqslant r \leqslant k$ and $j \leqslant s \leqslant l$. If (i, j) and (k, l) are in horizontal (or vertical) position, we say that the cells A and B are in a *horizontal* (or *vertical*) *position*.
Let $\mathcal{P}$ be a polyomino with two cells A and B of $\mathcal{P}$ in a vertical or horizontal position. The cell interval $[A, B]$, containing $n > 1$ cells, is called a *block of* $\mathcal{P}$ *of rank* n if all cells of $[A, B]$ belong to $\mathcal{P}$. The cells A and B are called *extremal* cells of $[A, B]$. Moreover, a block $\mathcal{B}$ of $\mathcal{P}$ is *maximal* if there does not exist any block of $\mathcal{P}$ which contains properly $\mathcal{B}$. It is clear that an interval of $\mathbb{Z}^2$ identifies a cell interval of $\mathbb{Z}^2$ and vice versa, so we associate to an interval I of $\mathbb{Z}^2$ the corresponding cell interval, which we denote by $\mathcal{P}_I$. Note that $\mathcal{P}_I$ is a polyomino itself. If $I = [a, b]$ and c, d are the anti-diagonal corner of I, then the two cells containing a or b (c or d) are called *diagonal* (*anti-diagonal*) cells of $\mathcal{P}_I$. A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $\mathcal{P}_{[a,b]}$ belong to $\mathcal{P}$. An interval $[a, b]$ with $a = (i, j)$, $b = (k, j)$ and $i < k$ is called a *horizontal edge interval* of $\mathcal{P}$ if the sets $\{(\ell, j), (\ell + 1, j)\}$ are edges of cells of $\mathcal{P}$ for all $\ell = i, \dots, k-1$. In addition, if $\{(i-1, j), (i, j)\}$ and $\{(k, j), (k+1, j)\}$ do not belong to $E(\mathcal{P})$, then $[a, b]$ is called a *maximal* horizontal edge interval of $\mathcal{P}$. We define similarly a *vertical edge interval* and a *maximal* vertical edge interval.

Definition 1.26. Let $\mathcal{P}$ be a non-empty finite collection of cells in $\mathbb{Z}^2$. Let K be a field and $S = K[x_v : v \in V(\mathcal{P})]$. For a proper interval $[a, b]$ of $\mathbb{Z}^2$, with a, b being diagonal corners and c, d being anti-diagonal ones, we associate the binomial $x_a x_b - x_c x_d$ to $[a, b]$. If $[a, b]$ is an inner interval, then the binomial $x_a x_b - x_c x_d$ is termed an *inner 2-minor* of $\mathcal{P}$. We denote by $I_{\mathcal{P}}$ the ideal in S generated by all the inner 2-minors of $\mathcal{P}$. The ring $K[\mathcal{P}] = S/I_{\mathcal{P}}$ represents the coordinate ring of $\mathcal{P}$. If $\mathcal{P}$ is a polyomino, the ideal $I_{\mathcal{P}}$ is termed the *polyomino ideal* of $\mathcal{P}$.

The common approach to calculate the Hilbert series of a binomial ideal is to look for the Hilbert series of its initial ideal as they are the same.
The conditions for the set of generators of $I_{\mathcal{P}}$ to form the reduced Gröbner basis with respect to a given order $<$ are provided by Qureshi [41] as the following theorem.

Theorem 1.27. [41] "Let $\mathcal{P}$ be a collection of cells. The set of inner 2-minors of $\mathcal{P}$ forms the reduced Gröbner basis with respect to $<$ if and only if for any two inner intervals $[b, a]$ and $[d, c]$ of $\mathcal{P}$ with anti-diagonal corners e, f and f, g respectively, either b, g or e, c are anti-diagonal corners of an inner interval of $\mathcal{P}$. "

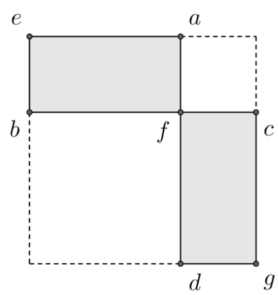
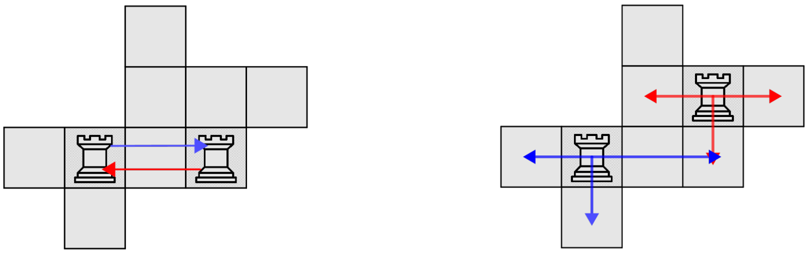


Figure 1.2: Conditions for the Gröbner basis with respect to $<$.

A polyomino $\mathcal{P}$ is termed *simple* if for any two cells C and D not in $\mathcal{P}$, there exists a path of cells not in $\mathcal{P}$ from C to D. A finite collection of cells $\mathcal{H}$ not in $\mathcal{P}$ is a *hole* of $\mathcal{P}$ if any two cells of $\mathcal{H}$ are connected in $\mathcal{H}$ and $\mathcal{H}$ is maximal with respect to set inclusion.
A polyomino $\mathcal{P}$ is called *convex* if for any two cells $A, B \in \mathcal{P}$ in horizontal or vertical position the cell interval $[A, B]$ is in $\mathcal{P}$.

Proposition 1.28. [28, Corollary 1.2] "Let $I \subset \mathbb{N}^2$ be an interval of $\mathbb{N}^2$ and $\mathcal{P}$ a convex polyomino which is a subpolyomino of $\mathcal{P}_I$. Let $\mathcal{P}^c = \mathcal{P}_I \setminus \mathcal{P}$. Then the set of inner 2-minors of $\mathcal{P}^c$ forms a reduced Gröbner basis of $I_{\mathcal{P}^c}$ with respect to $<_{\text{lex}}$. "

Let $\mathcal{P}$ be a collection of cells. A polyomino or a collection of cells can be considered as pruned chessboard. Two rooks R_1 and R_2 arranged on the cells of $\mathcal{P}$ are in *attacking position* in $\mathcal{P}$ if they are in a same row or column of cells. In such a case we say that R_1 and R_2 are two *attacking rooks*. Moreover, two rooks are in *non-attacking position*



(a) Attacking rooks (b) Non-attacking rooks

Figure 1.3: Positions of two rooks in a polyomino.

in $\mathcal{P}$ (or they are two *non-attacking rooks*) if they are not in attacking position in $\mathcal{P}$. For instance see Figure 1.3.

A j*-rook configuration* in $\mathcal{P}$ is a configuration of j rooks which are arranged in $\mathcal{P}$ in non-attacking positions. Figure 1.4 shows a 6-rook configuration.

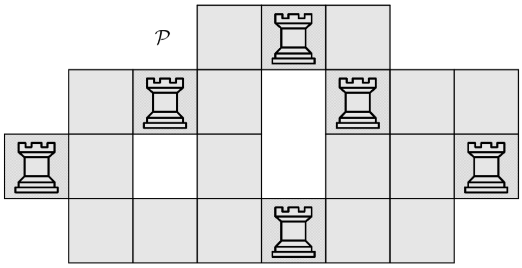


Figure 1.4: An example of a 6-rook configuration in $\mathcal{P}$.

The *rook number* $r(\mathcal{P})$ is the maximum number of rooks that can be placed in $\mathcal{P}$ in non-attacking position. We denote by $\mathcal{R}_j$ the set of all j-rook configurations in $\mathcal{P}$ for all $j \in \{0, 1, 2, \dots, r(\mathcal{P})\}$ and we set conventionally $R_0 = \emptyset$. Two non-attacking rooks in $\mathcal{P}$ are said to be in *switching position* or they are called *switching rooks* if they are placed in the diagonal (resp. anti-diagonal) cells of $\mathcal{P}_I$, where I is an inner interval of $\mathcal{P}$. In such a case we say that the rooks are in a diagonal (resp. anti-diagonal) position.

Fix $j \in \{0, \dots, r(\mathcal{P})\}$. Let $F \in \mathcal{R}_j$ and R_1 and R_2 be two switching rooks of F in diagonal (resp. anti-diagonal) position in $\mathcal{P}_I$, where I is an inner interval of $\mathcal{P}$. Let R_1' and R_2' be the rooks in anti-diagonal (resp. diagonal) cells of $\mathcal{P}_I$. Then the set $(F \backslash \{R_1, R_2\}) \cup \{R_1', R_2'\}$ belongs to $\mathcal{R}_j$. The operation of replacing R_1 and R_2 by R_1' and R_2' is called *switch of* R_1 *and* R_2. This induces the following equivalence relation $\sim$ on $\mathcal{R}_j$: let $F_1, F_2 \in \mathcal{R}_j$, so $F_1 \sim F_2$ if F_2 can be obtained from F_1 after some switches. Figure 1.5 illustrates an example of four 3-rook configurations which are equivalent with respect to $\sim$.

Definition 1.29. Let $\tilde{\mathcal{R}}_j = \mathcal{R}_j / \sim$ be the quotient set. We set $\tilde{r}_j = |\tilde{\mathcal{R}}_j|$ for all $j \in \{0, \dots, r(\mathcal{P})\}$; conventionally $\tilde{r}_0 = 1$. The *switching rook-polynomial* of $\mathcal{P}$ is the

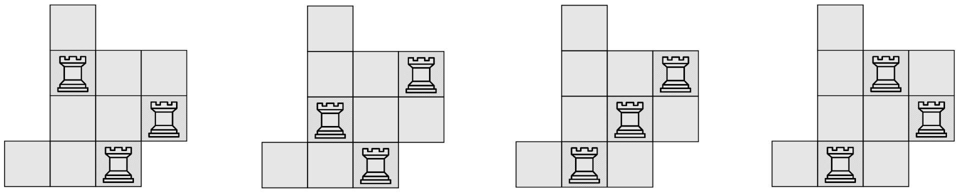


Figure 1.5: Equivalent 3-rook configurations.

polynomial in $\mathbb{Z}[t]$ defined as $\tilde{r}_{\mathcal{P}}(t) = \sum_{j=0}^{r(\mathcal{P})} \tilde{r}_j t^j$.

We now recall some classes of polyominoes and their relation with rook polynomial. Let $\mathcal{C} : C_1, C_2, \dots, C_m$ be a path of cells and (i_k, j_k) be the lower left corner of C_k for $1 \leq k \leq m$. Then $\mathcal{C}$ has a change of direction at C_k for some $2 \leq k \leq m-1$ if $i_{k-1} \neq i_{k+1}$ and $j_{k-1} \neq j_{k+1}$.

Definition 1.30. A convex polyomino $\mathcal{P}$ is called k-convex if any two cells in $\mathcal{P}$ can be connected by a path of cells in $\mathcal{P}$ with at most k change of directions. The 1-convex polyominoes are simply called *L-convex polyomino*.

Theorem 1.31. [18, Theorem 3.3] "Let $\mathcal{P}$ be an L-convex polyomino. Then $\operatorname{reg}(K[\mathcal{P}]) = r(\mathcal{P})$. "

A polyomino $\mathcal{P}$ is considered *thin* if it does not include the square tetromino as a subpolyomino.

Theorem 1.32. [45, Theorem 1.1] "Let $\mathcal{P}$ be a simple thin polyomino such that the reduced Hilbert series of $K[\mathcal{P}]$ is

$$\mathrm{HS}_{K[\mathcal{P}]}(t) = \frac{h(t)}{(1-t)^d}.$$

Then $h(t)$ is the rook polynomial of $\mathcal{P}$. "

For the general simple class of polyomino we have following conjecture.

Conjecture 1.33. [42, Conjecture 3.2] "Let $\mathcal{P}$ be a simple polyomino. The h-polynomial in the Hilbert series of $K[\mathcal{P}]$ is the same as switching rook polynomial of $\mathcal{P}$. "

1.6 Shellable simplicial complexes and posets

Now we present some basics about simplicial complexes. While this section outlines the essentials, for a comprehensive understanding, one may refer to [50].

Definition 1.34. A *finite simplicial complex* Δ on a set $[n] := \{1, \ldots, n\}$ is a collection of its subsets that adhere to:

1. For any $F' \in \Delta$ and $F \subseteq F'$, $F \in \Delta$.

2. Every single element set $\{i\}$ is in Δ for all $i \in [n]$.

The members of Δ are termed *faces*. The dimension of a face is one less than it's cardinality. Faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. The largest faces in terms of set inclusion are *facets*. The dimension of Δ is the maximum dimension of its faces. A *pure* simplicial complex has facets of uniform dimension.
Given a collection $\mathscr{F} = \{F_1, \ldots, F_m\}$ of subsets of $[n]$, the simplicial complex formed by all subsets of $[n]$ contained in some F_i is denoted by $\langle \mathscr{F} \rangle$. If $\mathcal{F}$ represents the facets of a simplicial complex Δ, then Δ is generated by $\mathcal{F}$.

Definition 1.35. [4, Definition 5.1.11] "A pure simplicial complex Δ is called *shellable* if the facets of Δ can be ordered as $F_1, \ldots, F_m$ in such a way that $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i \rangle$, for all $i \in \{2, \ldots, m\}$. In such a case $F_1, \ldots, F_m$ is called a *shelling* of Δ. "

Let Δ be a simplicial complex on $[n]$ and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K. To every collection $F = \{i_1, \ldots, i_r\}$ of r distinct vertices of Δ, there is an associated monomial x_F in S where $x_F = x_{i_1} \cdots x_{i_r}$. The monomial ideal generated by all monomials x_F such that F is not a face of Δ is called *Stanley-Reisner ideal* and it is denoted by I_Δ. The *face ring* of Δ, denoted by $K[\Delta]$, is defined to be the quotient ring S/I_Δ. From [55, Corollary 6.3.5], if Δ is a simplicial complex on $[n]$ of dimension d, then $\dim K[\Delta] = d + 1 = \max\{s : x_{i_1} \ldots x_{i_s} \notin I_\Delta, i_1 < \cdots < i_s\}$. We recall a nice combinatorial interpretation of the h-vector of a shellable simplicial complex.

Proposition 1.36. [4, Corollary 5.1.14] "Let Δ be a shellable simplicial complex of dimension d with shelling $F_1, \ldots, F_m$. For $j \in \{2, \ldots, m\}$ we denote by r_j the number of facets of $\langle F_1, \ldots, F_{j-1} \rangle \cap \langle F_j \rangle$ and we set $r_1 = 0$. Let $(h_0, \ldots, h_{d+1})$ be the h-vector of $K[\Delta]$. Then $h_i = |\{j : r_j = i\}|$ for all $i \in [d+1]$. In particular, up to their order, the numbers r_j do not depend on the particular shelling. "

We now introduce an association of simplicial complex and polyomino. Let $\mathcal{P}$ be a polyomino satisfying the conditions of Theorem 1.27. Then $G(\mathcal{P})$, the set of generators, forms the reduced Gröbner basis of $I_\mathcal{P}$ with respect to $<$, in particular $\text{in}_<(I_\mathcal{P})$ is squarefree and it is generated in degree two. We denote by $\Delta(\mathcal{P})$ the simplicial complex on $V(\mathcal{P})$ with $\text{in}_<(I_\mathcal{P})$ as Stanley-Reisner ideal and we call it the *simplicial complex attached to* $\mathcal{P}$.

Definition 1.37. Let K be field and let Δ be a finite simplicial complex. For a face σ of Δ, define the *link* of σ in Δ, denoted by $\text{link}(\Delta, \sigma)$ to be the subcomplex $\{\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$. The simplicial complex Δ is said to be *Cohen-Macaulay* over K if $\widetilde{H}_i(\text{link}(\Delta, \sigma), K) = 0$ for all $i < \dim(\text{link}(\Delta, \sigma))$ for every face σ of Δ. Here, $\widetilde{H}_i(\_, K)$ is the i-th reduced homology group with coefficients in K.
The following result is due to Reisner.

Theorem 1.38. [44, Theorem 1] "A simplicial complex Δ is Cohen-Macaulay over a field K if and only if the Stanley-Reisner ring associated to Δ (over K) is Cohen-Macaulay. "

Theorem 1.39. [4, Theorem 5.1.13] "A shellable simplicial complex Δ is Cohen-Macaulay over any field. "

We now recall definitions and related results from poset theory. For an in-depth study of this theory one can consult to the book [54].

Definition 1.40. Let P be a set and $\leq$ a binary relation on P. The relation $\leq$ is defined as a partial order if it satisfies the following axioms:

1. Reflexivity: For every $x \in P$, $x \leq x$.

2. Antisymmetry: For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.

3. Transitivity: For any $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

A set P equipped with a partial order $\leq$ is termed a *partially ordered set* or *poset*, denoted as $(P, \leq)$.

Let P be a poset. For $x, y \in P$, we say that y *covers* x, denoted by $x \lessdot y$, if $x < y$ and there is no $z \in P$ with $x < z < y$. A *chain* C of P is a totally ordered subset of P. The *length* of a chain C of P is $|C| - 1$. The *rank* of P, denoted by $\text{rank}(P)$, is the maximum of the lengths of chains in P. A poset is called *pure* if all maximal chains of P have the same length. An *induced subposet* Q of P is a poset on a subset of the underlying set P such that for every $x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P.
The *order complex* $\Delta(P)$ of P is a simplicial complex on the underlying set of P whose faces are chains of P. To simplify the notation, we say that a poset has property $\mathcal{Q}$ if its order complex has the property $\mathcal{Q}$. Let P be a pure poset and $\Delta(P)$ be its order complex. We say that $\Delta(P)$ is *strongly connected* if for any two maximal chains γ and γ' of P, there is a sequence $\sigma_0, \sigma_1, \ldots, \sigma_k$ of maximal chains of P such that $\sigma_0 = \gamma$, $\sigma_k = \gamma'$, and $\sigma_i \cap \sigma_{i+1}$ is a chain of length $\text{rank}(P) - 1$.

Proposition 1.41. [2, Proposition 11.7] "Every Cohen-Macaulay complex is pure and strongly connected. "

Let P and Q be two posets on disjoint sets. The *disjoint union* of posets P and Q is the poset $P + Q$ on the set $P \cup Q$ with the following order: if $x, y \in P + Q$, then $x \leq y$ if either $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q. A poset which can be written as disjoint union of two posets is called *disconnected*; otherwise the poset is called *connected*.

Chapter 2

Hilbert Series of Frame Polyominoes

This chapter presents frame polyominoes, which are a novel category of non-simple polyominoes. The findings given in this chapter are documented in [31]. A frame polyomino can be defined as a polyomino that is derived from a rectangular polyomino by deleting a parallelogram-shaped polyomino.

We start by recalling the definition of parallelogram polyomino given in [42]. "Let $(a, b) \in \mathbb{Z}^2$. The sets $\{(a, b), (a + 1, b)\}$ and $\{(a, b), (a, b + 1)\}$ are called respectively *east step* and *north step* in $\mathbb{Z}^2$. A sequence of vertices $(a_0, b_0), (a_1, b_1), \dots, (a_k, b_k)$ in $\mathbb{Z}^2$ is called a *north-east path* if $\{(a_i, b_i), (a_{i+1}, b_{i+1})\}$ is either an east or a north step. The vertices (a_0, b_0) and (a_k, b_k) are called the *endpoints* of S. Let $S_1 : (a_0, b_0), (a_1, b_1), \dots, (a_k, b_k)$ and $S_2 : (c_0, d_0), (c_1, d_1), \dots, (c_k, d_k)$ be two north-east paths such that $(a_0, b_0) = (c_0, d_0)$ and $(a_k, b_k) = (c_k, d_k)$. If for all $1 \leq i$ and $j \leq k-1$, we have $b_i > d_j$ when $a_i = c_j$, then S_1 is said to lie above S_2.

If S_1 lies above S_2, we define *parallelogram* polyomino, determined by S_1 and S_2, the set of cells in the region of $\mathbb{Z}^2$ bounded above by S_1 and below by S_2. "

Definition 2.1. Let $I = [(1, 1), (m, n)]$ be an interval of $\mathbb{Z}^2$ and $\mathcal{S}$ be a parallelogram polyomino determined by $S_1 : (a_0, b_0), \dots, (a_k, b_k)$ and $S_2 : (c_0, d_0), \dots, (c_k, d_k)$, where $1 < a_0 < a_k < m$ and $1 < b_0 < b_k < n$. We call *frame polyomino* the non-simple polyomino obtained by removing the cells of $\mathcal{S}$ from $\mathcal{P}_I$.

Figure 2.1 shows three examples of frame polyominoes.

2.1 Krull dimension of frame polyominoes

For a frame polyomino $\mathcal{P}$ we provide an elementary decomposition, which we use along the work. It consists of two suitable parallelogram sub-polyominoes, denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. Referring to Figure 2.2, $\mathcal{P}_1$ is the sub-polyomino of $\mathcal{P}$ highlighted with

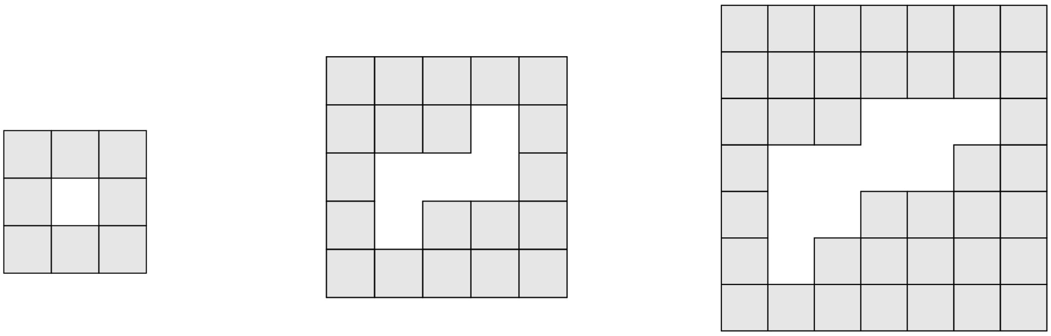


Figure 2.1: Examples of frame polyominoes.

a red color, and $\mathcal{P}_2$ is the other one with a hatching filling. Observe that $\mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{Q}$, with $\mathcal{Q} = \mathcal{P}_{[(1,1),(a_0,b_0)]} \cup \mathcal{P}_{[(a_k,b_k),(m,n)]}$, where $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and $\mathcal{P}_{[(a_k,b_k),(m,n)]}$ are the cell intervals attached respectively to $[(1,1),(a_0,b_0)]$ and $[(a_k,b_k),(m,n)]$.

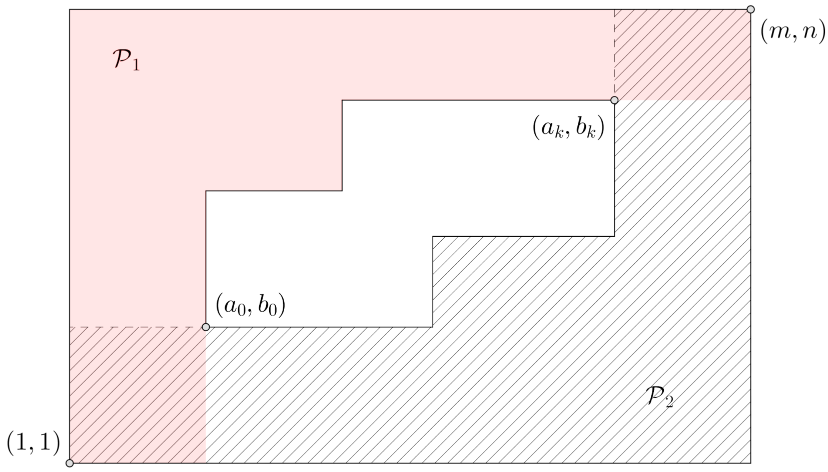


Figure 2.2: Elementary decomposition of frame polyomino $\mathcal{P}$.

In the following proposition we show some basic algebraic properties of the polyomino ideal of a frame polyomino. In particular, we determine the Krull dimension of the related coordinate ring using the simplicial complex theory.

Proposition 2.2. Let $\mathcal{P}$ be a frame polyomino defined by $I = [(1,1),(m,n)]$ and by a parallelogram polyomino $\mathcal{S}$ determined by S_1 and S_2 with endpoints (a_0,b_0) and (a_k,b_k). Then:

1. the set of generators $G(\mathcal{P})$ forms the reduced Gröbner basis of $I_\mathcal{P}$ with respect to a monomial order $<$;

2. the initial ideal $\mathrm{in}_<(I_\mathcal{P})$ is generated by the monomials $x_c x_d$ where c,d are the anti-diagonal corners of an inner interval $[a,b]$ of $\mathcal{P}$;

3. $K[\mathcal{P}]$ is a normal Cohen-Macaulay domain of Krull dimension $|V(\mathcal{P})| - \mathrm{rank}(\mathcal{P})$.

Proof. (1) follows directly by Proposition 1.28 and (2) is an immediate consequence of (1). For the proof of (3) we follow the below arguments.
The fact that $I_{\mathcal{P}}$ is a toric ideal can be inferred from [28, Theorem 2.1]. Applying Corollary 4.26 from the reference [24] and Proposition1.28, we can conclude that the ring $K[\mathcal{P}]$ is normal. Furthermore, by utilizing Theorem 6.3.5 from the reference [4], we can deduce that $K[\mathcal{P}]$ is Cohen-Macaulay. Next, we calculate the Krull dimension of the ring $K[\mathcal{P}]$. We examine the simplicial complex $\Delta(\mathcal{P})$ associated with the frame polyomino $\mathcal{P}$. Observe that $\Delta(\mathcal{P})$ is a shellable simplicial complex by [55, Theorem 9.6.1]. Moreover, we have that $|V(\mathcal{P})| = |V(\mathcal{P}_I)| - |V(\mathcal{S})| + |S_1| + |S_2| - 2$ and $\mathrm{rank}(\mathcal{P}) = \mathrm{rank}(\mathcal{P}_I) - \mathrm{rank}(\mathcal{S})$. Hence:

$$|V(\mathcal{P})| - \mathrm{rank}(\mathcal{P}) = |V(\mathcal{P}_I)| - \mathrm{rank}(\mathcal{P}_I) - (|V(\mathcal{S})| - \mathrm{rank}(\mathcal{S})) + |S_1| + |S_2| - 2. \quad (2.1)$$

Observe that $\mathcal{P}_I$ and $\mathcal{S}$ satisfy the conditions in Theorem 1.27, so we denote by $\Delta(\mathcal{P}_I)$ and $\Delta(\mathcal{S})$ the simplicial complexes attached to $\mathcal{P}_I$ and $\mathcal{S}$ respectively. In particular we have that $\dim K[\mathcal{P}_I] = \dim K[\Delta(\mathcal{P}_I)]$ and $\dim K[\mathcal{S}] = \dim K[\Delta(\mathcal{S})]$. Since $\mathcal{P}_I$ and $\mathcal{S}$ are simple polyominoes, from [25, Theorem 2.1] and [26, Corollary 3.3] we know that $K[\mathcal{P}_I]$ and $K[\mathcal{S}]$ are normal Cohen-Macaulay domain with respectively $\dim K[\mathcal{P}_I] = |V(\mathcal{P}_I)| - \mathrm{rank}(\mathcal{P}_I)$ and $\dim K[\mathcal{S}] = |V(\mathcal{S})| - \mathrm{rank}(\mathcal{S})$. As a consequence $\Delta(\mathcal{P}_I)$ and $\Delta(\mathcal{S})$ are pure, so $\dim(\Delta(\mathcal{P}_I)) = |F_I|$ and $\dim(\Delta(\mathcal{S})) = |S_1| = |S_2|$, where $F_I = [(1,1),(1,n)] \cup [(1,n),(m,n)]$. Set $S^* = S_2 \setminus \{(a_0,b_0),(a_k,b_k)\}$. Therefore, from (1) and from the previous arguments, we have that

$$|V(\mathcal{P})| - \mathrm{rank}(\mathcal{P}) = |F_I| - |S_1| + |S_1| + |S_2| - 2 = |F_I| + |S^*| = |F_I \sqcup S^*|.$$

We prove that $F_I \sqcup S^*$ is a facet of $\Delta(\mathcal{P})$. Firstly observe that $F_I \sqcup S^*$ is a face of $\Delta(\mathcal{P})$ because there does not exist any inner interval of $\mathcal{P}$ whose anti-diagonal corners are in $F_I \sqcup S^*$. Due to the maximality, suppose for the sake of contradiction, that there is a face K of $\Delta(\mathcal{P})$ such that $F_I \sqcup S^* \subset K$. Let $w \in K \setminus (F_I \sqcup S^*)$. If $w \in V(\mathcal{P}_1) \setminus F_I$ then the interval with $(1,n)$ and w as anti-diagonal vertices is an inner interval of $\mathcal{P}$, which is a contradiction with (2). If $w \in V(\mathcal{P}_2) \setminus (V(\mathcal{P}_1) \cup S^*)$, then it is easy to see that there is an inner interval of $\mathcal{P}$ whose anti-diagonal vertices are w and a vertex in $\{(1,b_0),(a_k,n)\} \sqcup S^*$, that is a contradiction with (2). Hence $F_I \sqcup S^*$ is a facet of $\Delta(\mathcal{P})$, so we get the desired conclusion. □

2.2 Shellability of attached simplicial complex

As seen in (3) of Proposition 2.2, the simplicial complex $\Delta(\mathcal{P})$ attached to a frame polyomino $\mathcal{P}$ is shellable. In order to define a suitable shelling order of $\Delta(\mathcal{P})$, we introduce the notion of a *step* of a face of $\Delta(\mathcal{P})$.

Definition 2.3. Let $\mathcal{P}$ be a polyomino satisfying Theorem 1.27 and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. Let F be a face of $\Delta(\mathcal{P})$ with $|F| \geq 3$ and $F' = \{(a,b),(c,b),(c,d)\} \subseteq F$. We say that F' forms a *step* in F or that F has a *step* F' if:

1. $a < c$ and $b < d$;
2. for every integer $i \in \{a+1,\dots,c-1\}$ there does not exist (i,b) in F;
3. for every integer $j \in \{b+1,\dots,d-1\}$ there does not exist (c,j) in F;
4. (c,b) is the lower right corner of a cell of $\mathcal{P}$.

In such a case the vertex (c,b) is said to be the *lower right corner* of F'.

Example 2.4. Let $\mathcal{P}$ be the polyomino in Figure 2.3. It is trivial to see that $\mathcal{P}$ satisfies Theorem 1.27, so we consider the simplicial complex $\Delta(\mathcal{P})$ attached to $\mathcal{P}$. The blue vertices represent a facet F of $\Delta(\mathcal{P})$. F has five steps, which are $\{a_1,a_2,a_7\}$, $\{a_2,a_3,a_4\}$, $\{a_7,a_8,a_{11}\}$, $\{a_{12},a_{13},a_{15}\}$ and $\{a_9,a_{10},a_{14}\}$. Note that $\{a_5,a_6,a_9\}$ is not a step of such a facet because a_6 is not the lower right corner of a cell of $\mathcal{P}$.

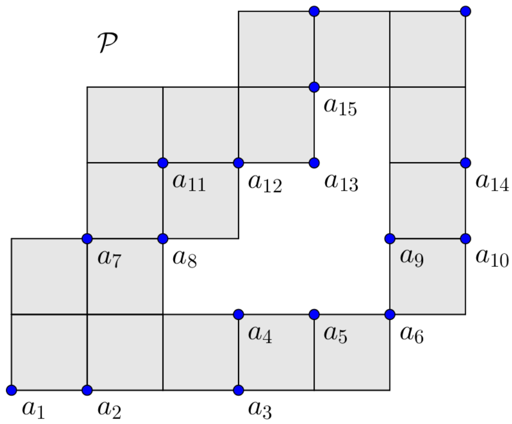


Figure 2.3: Example of steps in a facet of $\Delta(\mathcal{P})$.

We show a useful property of a step of a facet of the simplicial complex attached to a frame polyomino.

Lemma 2.5. Let $\mathcal{P}$ be a frame polyomino. Let $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$ and $F' = \{(a,j),(i,j),(i,b)\}$ be a step of a facet of $\Delta(\mathcal{P})$. Then $[(a,j),(i,b)]$ is an inner interval of $\mathcal{P}$.

Proof. If $a = i-1$ and $b = j+1$ then we have the conclusion immediately from (4) of Definition 2.3. Assume that $a \neq i-1$ or $b \neq j+1$. If $(i,j) \in V(\mathcal{P}_1)$ then

$[(a,j),(i,b)]$ is an inner interval of $\mathcal{P}$, by the structure of $\mathcal{P}$ and by Definition 2.3. Assume that $(i,j) \notin V(\mathcal{P}_1)$, so $(i,j) \in V(\mathcal{P}_2) \setminus (\mathcal{P}_{[(1,1),(a_0,b_0)]} \cup \mathcal{P}_{[(a_k,b_k),(m,n)]})$. Suppose by contradiction that $[(a,j),(i,b)]$ is not an inner interval of $\mathcal{P}$. Then there exists a cell C not belonging to $\mathcal{P}$ with lower right corner (h,l) such that $(h,l) \neq (i,j)$ and $a < h \leq i$, $j \leq l < b$. Observe that all vertices of $\mathcal{P}$ in $[(a+1,1),(m,j)] \setminus \{(i,j)\}$ and $[(i,1),(m,b-1)] \setminus \{(i,j)\}$ do not belong to F. Suppose $h = i$. Then there does not exist any inner interval of $\mathcal{P}$ having (h,l) and another vertex in F as anti-diagonal corners, so $(h,l) \in F$ due to the maximality of F, but this is a contradiction with (3) of Definition 2.3. A similar contradiction arises if $l = j$. Therefore $h \neq i$ and $l \neq j$. It is not restrictive to assume that l is the minimum integer such that the cell C with lower right corner (h,l) belongs to $[(a,j),(i,b)]$ but not to $\mathcal{P}$. Let J be the maximal inner interval of $\mathcal{P}$ having (i,j) as the lower right corner and containing (a,j); moreover, we denote by H the maximal edge interval of $\mathcal{P}$ containing (a,j) and (i,j). Note that no vertex of $J \setminus H$ belongs to F. Therefore, as explained before, $(h,j) \in F$, so we get again a contradiction with (2) of Definition 2.3. In conclusion $[(a,j),(i,b)]$ is an inner interval of $\mathcal{P}$. □

Remark 2.6. The assumption that $\mathcal{P}$ is a frame polyomino is important for the claim of Lemma 2.5. In fact, if $\mathcal{P}$ is the polyomino in Figure 2.4 then $\mathcal{P}$ satisfies the condition of Theorem 1.27 and the set of the orange vertices determines a facet F of the simplicial complex $\Delta(\mathcal{P})$ attached to $\mathcal{P}$, where $\{v_1, v_2, v_3\}$ is a step of F but $[v_1, v_3]$ is not an inner interval of $\mathcal{P}$.

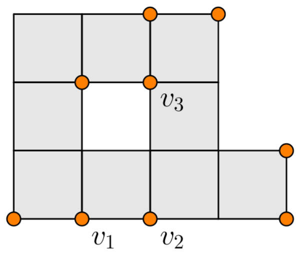


Figure 2.4: A facet in a non-frame polyomino.

Along the work, as done in [42], if a polyomino $\mathcal{P}$ has a structure of a distributive lattice on $V(\mathcal{P})$, then with abuse of notation we refer to $\mathcal{P}$ as a distributive lattice. From [42, Proposition 2.3] we know that a finite collection of cells $\mathcal{P}$ is a parallelogram polyomino if and only if $\mathcal{P}$ is a simple planar distributive lattice.

Let $\mathcal{P}$ be a parallelogram polyomino. Let $\operatorname{rank}(\mathcal{P}) = d+1$ as distributive lattice, $\mathfrak{m} : \min L = x_0 \lessdot x_1 \lessdot x_2 \lessdot \dots x_{d+1} = \max L$ be a maximal chain of $\mathcal{P}$ and $C = [(i,j),(i+1,j+1)]$ be a cell of $\mathcal{P}$. We say that $\mathfrak{m}$ has a *descent* at C if $\mathfrak{m}$ passes through the edges $(i,j) \lessdot (i+1,j)$ and $(i+1,j) \lessdot (i+1,j+1)$.

Remark 2.7. Consider a polyomino $\mathcal{P}$ that is in the shape of a parallelogram. Note that $\mathcal{P}$ fulfills the requirements stated in Theorem 1.27, so the collection of generators of $I_{\mathcal{P}}$ constitutes the (quadratic) reduced Gröbner basis of $I_{\mathcal{P}}$ in relation to $<$. Let $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. For all $j \geq 0$, it is easy to see that every maximal chain of $\mathcal{P}$ with j descents as a distributive lattice is a facet of $\Delta(\mathcal{P})$ with j steps, and vice versa.

Lemma 2.8. Let $\mathcal{P}$ be a frame polyomino defined by $I = [(1,1),(m,n)]$ and by a parallelogram polyomino $\mathcal{S}$ determined by $S_1 : (a_0,b_0),(a_1,b_1),\dots,(a_k,b_k)$ and $S_2 : (c_0,d_0),(c_1,d_1),\dots,(c_k,d_k)$ with $(a_0,b_0) = (c_0,d_0)$ and $(a_k,b_k) = (c_k,d_k)$. Consider the simplicial complex $\Delta(\mathcal{P})$ associated with $\mathcal{P}$ and F be a facet of $\Delta(\mathcal{P})$. Then for all maximal edge intervals $\mathcal{L}$ of $\mathcal{P}$ there exists $v \in \mathcal{L}$ belonging to F.

Proof. Let V be a maximal vertical edge interval of $\mathcal{P}$. We prove that there exists v in V belonging to F. We distinguish the following four cases.

Case 1. Assume that $V = [(a,1),(a,n)]$ with $a \in \{1,\dots,a_0\}$. Suppose that $a = 1$. Then $(1,1) \in F$, since $(1,1)$ is not the anti-diagonal corner of an inner interval of $\mathcal{P}$, and trivially $(1,1) \in V$. Therefore we get the claim for $a = 1$. Suppose now that $1 < a \leq a_0$. Consider $G = \{(i,j) \in F : 1 \leq i < a, 1 \leq j \leq n\}$. Observe that $G \neq \emptyset$ since $(1,1) \in G$. Let $(i_1,k_1) \in G$ with $k_1 = \max\{j : (i,j) \in G\}$. We want to show that $(a,k_1) \in F$. First of all, we observe that for every inner interval $\mathcal{I}$ of $\mathcal{P}$ having (i_1,k_1) as anti-diagonal corner the other anti-diagonal corner of $\mathcal{I}$ does not belong to F, otherwise there exists an inner interval of $\mathcal{P}$ whose two anti-diagonal corners are in F, which is a contradiction with (2) of Proposition 2.2. Moreover, the vertices in $V_1 = \{(i,j) \in V(\mathcal{P}) : 1 \leq i < a, k_1 < j \leq n\}$ are not in F due to the maximality of k_1. In order to prove that $(a,k_1) \in F$, it is sufficient to prove that, for every inner interval of $\mathcal{P}$ with anti-diagonal corner (a,k_1), the other anti-diagonal corner is not in F; in fact, due the maximality of F it follows necessarily that $(a,k_1) \in F$. Let $\mathcal{K}$ be an inner interval of $\mathcal{P}$ having (a,k_1) as anti-diagonal corner and $v = (r,s)$ be the other anti-diagonal corner of $\mathcal{K}$. We have just two cases to examine. If $r < a$ and $s > k_1$ then $v \in V_1$, so $v \notin F$. If $r > a$ and $s < k_1$, then we show that $v \notin F$. In fact, suppose by contradiction that $v \in F$. Let $\tilde{\mathcal{K}}$ be the interval of $\mathbb{Z}^2$ with anti-diagonal corners (i_1,k_1) and v. Denote by $\mathcal{C}$ the interval of $\mathbb{Z}^2$ having (i_1,k_1) and (a,s) as anti-diagonal corners. Note that $\mathcal{C}$ is an inner interval of $\mathcal{P}$ due to the structure of $\mathcal{P}$ as $a \leq a_0$, and that $\tilde{\mathcal{K}} = \mathcal{K} \cup \mathcal{C}$. Since $\mathcal{K}$ and $\mathcal{C}$ are inner intervals of $\mathcal{P}$, then $\tilde{\mathcal{K}}$ is an inner interval of $\mathcal{P}$. This is a contradiction because (i_1,k_1) and v are anti-diagonal corners of $\tilde{\mathcal{K}}$ and they belong to F at the same time. Hence v cannot be in F. In conclusion, we get the desired claim.

Case 2. Assume that $V = [(a,1),(a,d_h)]$, where $a_0 < a < a_k$ and $(a,d_h) \in S_2$ for a suitable $h \in [k-1]$. Suppose that $a = a_0+1$. Let $k_2 = \max\{j : (i,j) \in F \cap [(1,1),(a_0,b_0)]\}$. Then it is easy to see for every inner interval of $\mathcal{P}$ with anti-diagonal corner (a_0+1,k_2), the other anti-diagonal corner is not in F, so $(a_0+1,k_2) \in F$. Suppose that $a_0+1 < a \leq a_k$. Let $S_2' \subset S_2$ be the north-east path $(a_0+1,d_1),\dots,(a,d_h)$. We denote by $\mathcal{R}$ the parallelogram polyomino determined by the two north-east paths $[(a_0+1,1),(a_0+1,d_1)] \cup S_2'$ and $[(a_0+1,1),(a,1)] \cup [(a,1),(a,d_h)]$. Let $G' = V(\mathcal{R}) \cap F$. $G' \neq \emptyset$ because $(a_0+1,k_2) \in G'$. Set $k_3 = \max\{j : (i,j) \in G'\}$. Using similar arguments as done in Case 1, we prove that $(a,k_3) \in F$, so the claim follows.

Case 3. The case when $V = [(a,b_j),(a,n)]$, where $a_0 < a < a_k$ and $(a,b_j) \in S_1$ for an opportune $h \in [k-1]$, can be proved similarly as done in Case 2.

Case 4. Assume that $V = [(a,1),(a,n)]$, where $a_k \leq a \leq m$. If $a = a_k$, then we set $k_4 = \max\{j \in [b_k] : (a_k-1,j) \in F\}$ and, using the arguments explained in Case 2, it is easy to show that $(a_k,k_4) \in F$. If $a_k < a \leq m$, then we get the claim arguing as done in Cases 1 and 2.

For a maximal horizontal edge interval of $\mathcal{P}$, the claim follows by the same arguments. Therefore the Lemma is completely proven. □

Discussion 2.9. Let $\mathcal{P}$ be a frame polyomino and F be a facet of the simplicial complex $\Delta(\mathcal{P})$ attached to $\mathcal{P}$. From Lemma 2.8 we know that in every maximal edge interval of $\mathcal{P}$ we can find an element of F. Now, we want to describe how the elements of F are arranged in $\mathcal{P}$.

Let $v = (i,j)$ be the maximal vertex of F in $V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ with respect to $<$ and $w = (t,l)$ be the minimal vertex of F in $V(\mathcal{P}_{[(a_k,b_k),(m,n)]})$ with respect to $<$. Observe that v and w are unique. In fact, if $v = (i',j')$ is another maximal vertex of F in $V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ with respect to $<$, then either $i' < i$ and $j' > j$ or $i' > i$ and $j' < j$, so there exists an inner interval of $\mathcal{P}$ with v, v' as anti-diagonal corners. But this is a contradiction with (2) of Proposition 2.2, since $v, v' \in F$. The same argument follows for w. Moreover, we point out that $(1,1),(m,n) \in F$, because they cannot be the anti-diagonal vertices of an inner interval of $\mathcal{P}$.

Consider $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and $v = (i,j)$. We examine the following four cases.

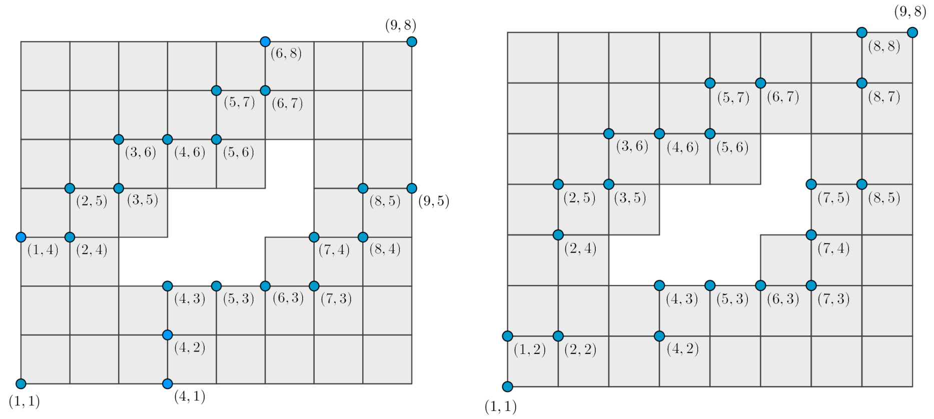
1. If $i = 1$ and $j = 1$ then no vertex from $[(1,1),(a_0,b_0)]$ belongs to F.

2. If $i > 1$ and $j = 1$, then $[(1,1),(i,1)] \subset F$, because no vertex from $\{(p,q) \in V(\mathcal{P}) : p < i, q > 1\}$ can be in F and due to the maximality of F.

3. If $i = 1$ and $j > 1$, then $[(1,1),(1,j)] \subset F$, arguing as in (1).

4. Assume that $i > 1$ and $j > 1$. Consider $(1,2)$ and $(2,1)$ and we show that either $(1,2)$ or $(2,1)$ belongs to F. First of all, both of them cannot be in F, otherwise, we have a contradiction with (2) of Proposition 2.2. Moreover, since the vertices in $\{(p,q) \in V(\mathcal{P}) : p < i, q > j\} \cup \{(p,q) \in V(\mathcal{P}) : p > i, q < j\}$ are not in F and due to the maximality of F, at least one of $(1,2)$ and $(2,1)$ belongs to F. We may assume that $(2,1) \in F$, because the arguments for $(1,2)$ are the same. We have two sub-cases. If $i = 2$, that is $v = (2,j)$, then the only vertices of F in $V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ are given by $\{(1,1),(2,1)\} \cup [(2,1),(2,j)]$. If $i > 2$, then the cell having $(2,1)$ as lower left corner is in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$, so $(2,2),(3,1) \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. Arguing as before, we have that either $(2,2)$ or $(3,1)$ belongs to F, and finally we iterate that procedure until v.

Hence the elements of F form a chain $\mathfrak{c}_1 : (1,1) \lessdot \ldots \lessdot v = (i,j)$ in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$.
Now, we focus on $\mathcal{P}_1 \setminus \mathcal{Q}$. Observe that no vertex in $\{(p,q) \in V(\mathcal{P}) : p < i, q > j\} \cup \{(p,q) \in V(\mathcal{P}) : p > i, q < j\}$ belongs to F because $v = (i,j) \in F$. Moreover, due to the maximality of v in $V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ with respect to $<$, we have that the vertices in $\{(p,q) \in V(\mathcal{P}) : p > i, j \leq q < b_0 + 1\}$ do not belong to F. Therefore, for each inner interval of $\mathcal{P}$ with an anti-diagonal corner at (i, b_0+1), the opposite anti-diagonal corner is not in F. Consequently, (i, b_0+1) is included in F due to the maximality of F. Starting from (i, b_0+1), we argue similarly as done before in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and we continue that procedure until $(a_k - 1, l)$. Therefore, the elements of F form a chain $\mathfrak{c}_2 : (i, b_0+1) \lessdot \ldots \lessdot (a_k - 1, l)$ in $\mathcal{P}_1 \setminus \mathcal{Q}$.
By similar arguments, the elements of F provide a chain $\mathfrak{c}_3 : (a_0+1, j) \lessdot \ldots \lessdot (t, b_k - 1)$ in $\mathcal{P}_2 \setminus \mathcal{Q}$ and another one $\mathfrak{c}_4 : w = (t,l) \lessdot \ldots \lessdot (m,n)$ in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$.
Therefore F is described by the chains $\mathfrak{c}_1$, $\mathfrak{c}_2$, $\mathfrak{c}_3$ and $\mathfrak{c}_4$. In Figure 2.5 we show two facets of the simplicial complex attached to a frame polyomino.



(a) (b)

Figure 2.5: Examples of facets in a frame polyomino.

Furthermore, let us represent the number of descents in $\mathfrak{c}_i$ as $n(\mathfrak{c}_i)$ for all $i \in [4]$, and let $n(F)$ denote the number of steps in F, we invite the reader to observe that $\sum_{i=1}^{4} n(\mathfrak{c}_i) \leq n(F) \leq \sum_{i=1}^{4} n(\mathfrak{c}_i)+4$. In fact, every descent of $\mathfrak{c}_i$ corresponds to a step of F, so $\sum_{i=1}^{4} n(\mathfrak{c}_i) \leq n(F)$, and there are at most four steps in F that are not descents of a chain $\mathfrak{c}_i$, as shown in Figure 2.5 (B) by $\{(1,2),(2,2),(2,4)\}$, $\{(2,2),(4,2),(4,3)\}$, $\{(6,7),(8,7),(8,8)\}$ and $\{(7,5),(8,5),(8,7)\}$, so $n(F) \leq \sum_{i=1}^{4} n(\mathfrak{c}_i) + 4$.

In the following definition we introduce a way to compare two facets of the simplicial complex attached to a polyomino having the property described in Theorem 1.27.

Definition 2.10. Consider a polyomino $\mathcal{P}$ that satisfies the requirement stated in Theorem 1.27. Let $\Delta(\mathcal{P})$ denote the simplicial complex associated with $\mathcal{P}$. The set of facets of $\Delta(\mathcal{P})$ is denoted by $\mathcal{F}_{\mathcal{P}}$, and we establish a well-defined order on $\mathcal{F}(\mathcal{P})$. To begin with, we establish the subsequent total order on the set of vertices in $\mathcal{P}$. Consider two elements a and b belonging to the set $V(\mathcal{P})$. Let a be represented as (i,j) and b as (k,l). We define b to be less than a ($b <' a$) if j is greater than l, or if j is equal to l and i is greater than k. Now, consider two distinct facets $F = \{a_1,\dots,a_d\}$ and $G = \{b_1,\dots,b_d\}$ of $\Delta(\mathcal{P})$, where $a_{i+1} <' a_i$ and $b_{i+1} <' b_i$ for all $i = 1,\dots,d-1$. Let j be the smallest integer in $[d]$ such that $b_j \neq a_j$. Then we define $F <_{\text{lex}} G$ if $a_j <' b_j$. Moreover, if $\mathcal{F}_{\mathcal{P}} = \{F_0, F_1, \dots, F_r\}$, then we say that $\mathcal{F}_{\mathcal{P}}$ is *lexicographically order in descending* if $F_{i+1} <_{\text{lex}} F_i$ for all $i = 0,\dots,r-1$.

Example 2.11. Let $\mathcal{P}$ be the polyomino in Figure 2.5 and F and G respectively the facets of $\Delta(\mathcal{P})$ shown in (a) and (b), that are

$$\begin{aligned} F = \ &\{(9,8),(6,8),(6,7),(5,7),(5,6),(4,6),(3,6),(9,5),(8,5),(3,5),(2,5)(8,4), \\ &(7,4),(2,4),(1,4),(7,3),(6,3),(5,3),(4,3),(4,2),(4,1),(1,1)\}, \end{aligned}$$

$$\begin{aligned} G = \ &\{(9,8),(8,8),(8,7),(6,7),(5,7),(5,6),(4,6),(3,6),(8,5),(7,5),(3,5)(2,5), \\ &(7,4),(2,4),(7,3),(6,3),(5,3),(4,3),(4,2),(2,2),(1,2),(1,1)\}. \end{aligned}$$

Observe that in F and G the first different vertices from the left to right are in the second position and $(6,8) <' (8,8)$, so $F <_{\text{lex}} G$.

Consider a frame polyomino $\mathcal{P}$ and its associated simplicial complex $\Delta(\mathcal{P})$. We set $F_0 = [(1,1),(1,n)] \cup [(1,n),(m,n)] \cup \big(S_2 \backslash \{(a_0,b_0),(a_k,b_k)\}\big)$, which is a facet of $\Delta(\mathcal{P})$ as proved in (3) of Proposition 2.2. Moreover, observe from Discussion 2.9 that F_0 is the unique facet in $\Delta(\mathcal{P})$ with no step and $F <_{\text{lex}} F_0$ for all $F \in \mathcal{F}_{\mathcal{P}}$. Now, we are prepared to demonstrate the principal outcome of this section.

Theorem 2.12. Let $\mathcal{P}$ be a frame polyomino and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. Suppose that $\mathcal{F}_{\mathcal{P}}$ is lexicographically ordered in descending and consider a facet $F \neq F_0$ of $\Delta(\mathcal{P})$. Set $\mathcal{S}(F) = \{G \in \mathcal{F}_{\mathcal{P}} : F <_{\text{lex}} G\}$ and $\mathcal{K}_F = \{F \backslash \{v\} : v \text{ is the lower right corner of a step of } F\}$. Then:

1. $\langle \mathcal{S}(F)\rangle \cap \langle F\rangle = \langle \mathcal{K}_F\rangle$ and, in particular, $\mathcal{F}_\mathcal{P}$ forms a shelling order of $\Delta(\mathcal{P})$;

2. the i-th coefficient of the h-polynomial of $K[\mathcal{P}]$ is the number of the facets of $\Delta(\mathcal{P})$ having i steps.

Proof. (1) Firstly, we show that $\langle \mathcal{K}_F\rangle \subseteq \langle \mathcal{S}(F)\rangle \cap \langle F\rangle$. Let $F\backslash\{(i,j)\}$, where (i,j) is the lower right corner of a step $F' = \{(i,b),(i,j),(a,j)\}$ of F. Trivially $F\backslash\{(i,j)\} \subset F$. We may assume that $(i,j),(i,b) \in V(\mathcal{P}_2) \setminus (V(\mathcal{P}_{[(1,1),(a_0,b_0)]}) \cup V(\mathcal{P}_{[(a_k,b_k),(m,n)]}))$ and $(a,j) \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$, as in Figure Figure 2.6, since all other cases can be proved by similar arguments. From Lemma 2.5, $[(a,j),(i,b)]$ is an inner interval of $\mathcal{P}$. Since $(i,b),(a,j),(i,j) \in F$, observe that no vertex in

$$\begin{aligned}\mathcal{N} =&\Big([(i,1),(m,b-1)] \setminus \{(i,j)\}\Big) \cup \Big([(a+1,1),(i,j)] \setminus \{(i,j)\}\Big)\cup \\ &\cup \Big([(1,j+1),(a-1,n)]\Big) \cup \Big([(a,j),(i,b)] \setminus \{(a,j),(i,j),(i,b)\}\Big)\end{aligned}$$

belongs to F. For instance, $\mathcal{N}$ consists of the white vertices in the blue, red, and grey parts in Figure 2.6. We consider the set $H = (F\backslash\{(i,j)\}) \cup \{(a,b)\}$ and we prove that H is a facet of $\Delta(\mathcal{P})$. In order to show that H is a face of $\Delta(\mathcal{P})$, it is sufficient to note that there does not exist an inner interval of $\mathcal{P}$ having (a,b) and another vertex $w \in F \setminus \{(i,j)\}$ as anti-diagonal corners; in fact, if such an inner interval of $\mathcal{P}$ exists, then $w \in \mathcal{N}$, which is a contradiction. To prove the maximality of H, we observe that if $H \subset K$ for some face K of $\Delta(\mathcal{P})$ then there exists a facet $K_{\max}$ of $\Delta(\mathcal{P})$ such that $H \subset K_{\max}$, so $|K_{\max}| > |H| = |F|$, which is a contradiction with the pureness of $\Delta(\mathcal{P})$. Therefore H is a facet of $\Delta(\mathcal{P})$. Observe that $F <_{\text{lex}} H$ since we replace (i,j) in F with (a,b), so $H \in \mathcal{S}(F)$. Hence $F\backslash\{(i,j)\} \in \langle \mathcal{S}(F)\rangle$ and $\langle \mathcal{K}_F\rangle \subseteq \langle \mathcal{S}(F)\rangle \cap \langle F\rangle$.
Now, we prove $\langle \mathcal{S}(F)\rangle \cap \langle F\rangle \subseteq \langle \mathcal{K}_F\rangle$. Let G be in $\langle \mathcal{S}(F)\rangle \cap \langle F\rangle$. Since $G \in \langle F\rangle$, then $G \subseteq F$. Moreover, $G \neq F$ because $G \in \mathcal{S}(F)$ and $F \notin \mathcal{S}(F)$, so $G \subset F$. Therefore, $G = F\backslash\{v_h : h \in [t]\}$, where $t \in [|F|]$ and $v_h \in F$ for all $h \in [t]$. We discuss two cases.

Case 1. Assume that $t = 1$, so $G = F \setminus \{v_1\}$. We set $v_1 = (i,j)$. Since $G = F\backslash\{v_1\} \in \langle \mathcal{S}(F)\rangle$, then there exists a facet $H \in \mathcal{S}(F)$ such that $F\backslash\{v_1\} \subset H$. Moreover, we recall that $\Delta(\mathcal{P})$ is pure and F and H are two facets of $\Delta(\mathcal{P})$ with $F <_{\text{lex}} H$, so we can obtain H from G adding a vertex $w = (k,l)$, where $l > j$ or $j = l$ and $k > i$. We want to show that v_1 is the lower right corner of a step in F. Suppose by contradiction that v_1 is not the lower right corner of a step in F. With reference to Discussion 2.9, if we add to $G = F \setminus \{v_1\}$ a vertex $w = (k,l)$ (with $l > j$ or $j = l$ and $k > i$) then we find an inner interval of $\mathcal{P}$ having w and a vertex in $F \setminus \{v_1\}$ as anti-diagonal corners, so we have a contradiction with (2) of Proposition 2.2. The operation to get H from G adding a vertex

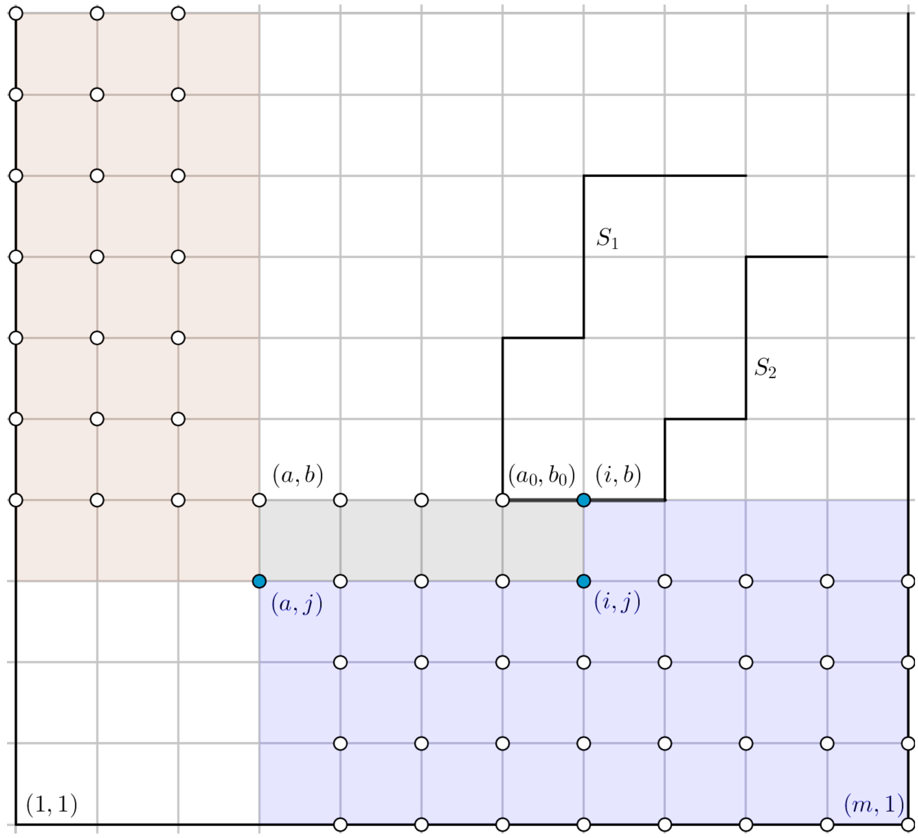


Figure 2.6: An arrangement of intervals.

$w = (k, l)$ (with $l > j$ or $j = l$ and $k > i$) can be done just when v_1 is the lower right corner of a step F' of F. In fact, in such a case, we can replace v_1 in F with the anti-diagonal corner of the inner interval given by the step F' to get H. Hence v_1 is necessarily the lower right corner of a step of F.

Case 2. Assume that $t > 1$, so $G = F \setminus \{v_1, v_2, \dots, v_t\}$. Arguing as in case (a), it is easy to show that there exists $q \in [t]$ such that v_q is the lower right corner of a step of F.

Hence we have that $G \in \langle \mathcal{K}_F \rangle$ and $\langle \mathcal{S}(F) \rangle \cap \langle F \rangle \subseteq \langle \mathcal{K}_F \rangle$. In conclusion $\langle \mathcal{S}(F) \rangle \cap \langle F \rangle = \langle \mathcal{K}_F \rangle$.

(2) It follows directly from (1) and Proposition 1.36. □

Remark 2.13. The above theorem does not hold in general. Consider the polyomino in Figure 2.7 and assume that $V(\mathcal{P}) = [(1, 1), (6, 4)]$. Note that $\mathcal{P}$ is a prime polyomino (see to [36]) and $G(\mathcal{P})$ is the reduced Gröbner basis of $I_{\mathcal{P}}$ with regard to $<$. As a result, $\Delta(\mathcal{P})$ is a shellable simplicial complex. The first three facets of $\Delta(\mathcal{P})$ lexicographically ordered in descending are:

$$F_0 = \{(6, 4), (5, 4), (4, 4), (3, 4), (2, 4), (1, 4), (1, 3), (5, 2), (3, 2), (1, 2), (1, 1)\};$$

$$F_1 = \{(6, 4), (5, 4), (4, 4), (3, 4), (2, 4), (1, 4), (1, 3), (5, 2), (3, 2), (3, 1), (1, 1)\};$$

$F_2 = \{(6,4),(5,4),(4,4),(3,4),(2,4),(1,4),(1,3),(5,2),(5,1),(3,1),(1,1)\}.$

Note that F_2 contains the two steps $\{(1,1),(3,1),(3,4)\}$ and $\{(3,1),(5,1),(5,2)\}$ but $\langle F_0, F_1\rangle \cap \langle F_2\rangle$ is generated just by $\{(6,4),(5,4),(4,4),(3,4),(2,4),(1,4),(1,3),(5,2),(3,1),(1,1)\} = F_2 \setminus \{(5,1)\}$ and not by $F_2 \setminus \{(3,1)\}$ and $F_2 \setminus \{(5,1)\}$.

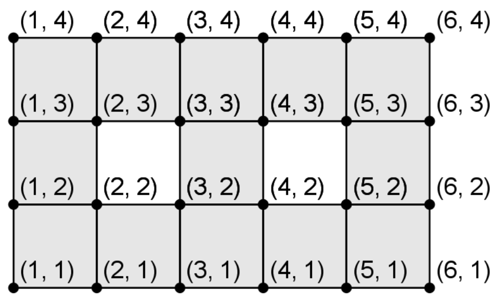


Figure 2.7: A polyomino.

2.3 Hilbert series and rook polynomial of frame polyominoes

This section focuses on analyzing the Hilbert-Poincaré series of the coordinate ring associated with a frame polyomino. We start with the following proposition, where we show a natural representative of an equivalence class of $\tilde{\mathcal{R}}_j$ associated with a frame polyomino.

Proposition 2.14. Let $\mathcal{P}$ be a frame polyomino and $\mathcal{R}$ be a j-rook configuration in $\mathcal{P}$, with $j \in [r(\mathcal{P})]$. Then there exists a j-rook configuration $\mathcal{C}$ in $\mathcal{P}$ such that $\mathcal{C} \sim \mathcal{R}$ and any two switching rooks of $\mathcal{C}$ are placed in diagonal position.

Proof. If $j = 1$, then the claim follows trivially. Suppose that $2 \leq j \leq r(\mathcal{P})$. We distinguish the following cases with reference to Figure 2.2.

Case 1. Assume that no rook of $\mathcal{R}$ is in $\mathcal{Q}$. Then, every rook of $\mathcal{R}$ is placed in $\mathcal{P}_1 \setminus \mathcal{Q}$ or in $\mathcal{P}_2 \setminus \mathcal{Q}$. Moreover, the rooks in $\mathcal{P}_1 \setminus \mathcal{Q}$ are never in attacking and switching position with the rooks in $\mathcal{P}_2 \setminus \mathcal{Q}$. Hence, the claim follows immediately from [42, Lemma 3.12] applying to $\mathcal{P}_1 \setminus \mathcal{Q}$ and to $\mathcal{P}_2 \setminus \mathcal{Q}$.

Case 2. Suppose that the rooks of $\mathcal{R}$ belong just to $\mathcal{Q}$. Then the claim follows trivially from [42, Lemma 3.12], since the cell intervals $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and $\mathcal{P}_{[(a_k,b_k),(m,n)]}$ associated respectively to $[(1,1),(a_0,b_0)]$ and $[(a_k,b_k),(m,n)]$ are parallelogram polyominoes.

Case 3. The same conclusion holds if all rooks of $\mathcal{R}$ are either in $\mathcal{P}_1$ or in $\mathcal{P}_2$.

Case 4. Suppose that there exists a rook of $\mathcal{R}$ in $\mathcal{P}_1 \setminus \mathcal{Q}$, another one in $\mathcal{P}_2 \setminus \mathcal{Q}$ and another in $\mathcal{Q}$. We consider non-empty sets of rooks $\mathcal{R}_1, \mathcal{R}_2, \tilde{\mathcal{R}} \subset \mathcal{R}$ such that the rooks in $\mathcal{R}_1$, $\mathcal{R}_2$ and $\tilde{\mathcal{R}}$ are placed in $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{Q}$ respectively, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}} = \mathcal{R}_1 \cap \mathcal{R}_2$. We proceed in two steps.

Step 1. First of all, we focus on $\mathcal{P}_1$ and $\mathcal{R}_1$. Since $\mathcal{P}_1$ is a parallelogram polyomino, then from [42, Lemma 3.12] there exists a $|\mathcal{R}_1|$-rook configuration $\mathcal{C}_1$ in $\mathcal{P}_1$ such that $\mathcal{C}_1 \sim \mathcal{R}_1$ and any two switching rooks in $\mathcal{C}_1$ are placed in diagonal position. Denote by $\mathcal{W}$ the set of the rooks of $\mathcal{C}_1$ placed in $\mathcal{Q}$. We show that $|\mathcal{W}| = |\tilde{\mathcal{R}}|$. With reference to Figure 2.8, let R_1 and R_2 be two switching rooks of $\mathcal{R}_1$ in anti-diagonal position and R_1' and R_2' the related switching rooks in a diagonal position.

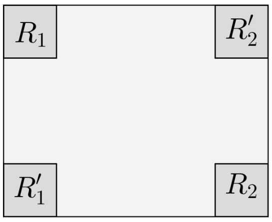


Figure 2.8: Placement of switching rooks of $\mathcal{R}_1$.

From the structure of $\mathcal{P}_1$ the following remarks follow:

if R_1 and R_2 are placed in $\mathcal{P}_1 \setminus \mathcal{Q}$, then R_1' and R_2' are also in $\mathcal{P}_1 \setminus \mathcal{Q}$;

if R_1 and R_2 are in $\mathcal{Q}$, then R_1' and R_2' are also in $\mathcal{Q}$;

if R_1 and R_2 are placed respectively in $\mathcal{P}_1 \setminus \mathcal{P}_{[(1,1),(a_0,b_0)]}$ and $\mathcal{P}_{[(1,1),(a_0,b_0)]}$, then R_1' and R_2' are respectively in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and $\mathcal{P}_1 \setminus \mathcal{P}_{[(1,1),(a_0,b_0)]}$;

if R_1 and R_2 are respectively in $\mathcal{P}_1 \setminus \mathcal{P}_{[(a_k,b_k),(m,n)]}$ and $\mathcal{P}_{[(a_k,b_k),(m,n)]}$, then R_1' and R_2' are respectively in $\mathcal{P}_1 \setminus \mathcal{P}_{[(a_k,b_k),(m,n)]}$ and $\mathcal{P}_{[(a_k,b_k),(m,n)]}$.

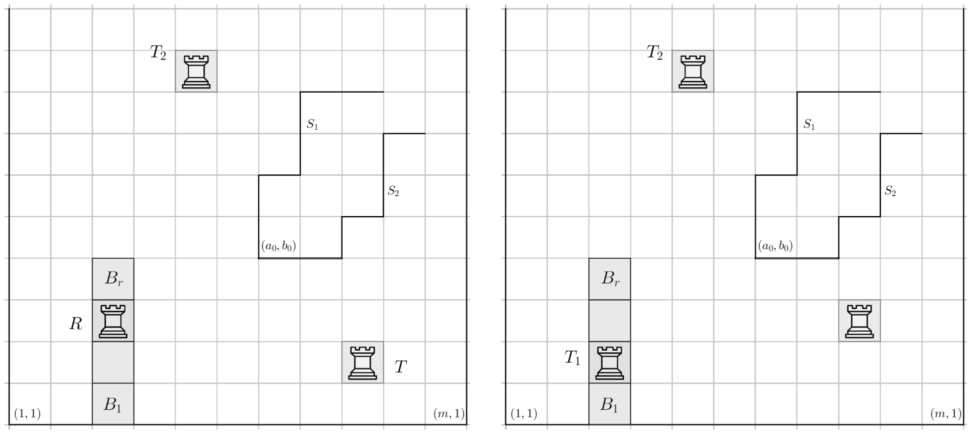
Therefore it easy to see that $|\mathcal{W}| = |\tilde{\mathcal{R}}|$.

Step 2 Now, we consider $\mathcal{P}_2$ and the $|\mathcal{R}_2|$-rook configuration $(\mathcal{R}_2 \setminus \tilde{\mathcal{R}}) \cup \mathcal{W}$ in $\mathcal{P}_2$. As before, since $\mathcal{P}_2$ is a parallelogram polyomino, there exists a $|\mathcal{R}_2|$-rook configuration $\mathcal{C}_2$ in $\mathcal{P}_2$ such that $\mathcal{C}_2 \sim (\mathcal{R}_2 \setminus \tilde{\mathcal{R}}) \cup \mathcal{W}$ and any two switching rooks in $\mathcal{C}_2$ are placed in diagonal position. Moreover, if $\mathcal{Z}$ is the subset of $\mathcal{C}_2$ of the rooks placed in $\mathcal{Q}$, then $|\mathcal{Z}| = |\mathcal{W}| = |\tilde{\mathcal{R}}|$.

Set $\mathcal{C} = (\mathcal{C}_1 \setminus \mathcal{W}) \cup \mathcal{C}_2$ and we show that $\mathcal{C}$ is the desired j-rook configuration. Observe trivially that $\mathcal{C} \sim \mathcal{R}$ because we get $\mathcal{C}$ from $\mathcal{R}$ with suitable switches. We need to prove that any two switching rooks in $\mathcal{C}$ are in a diagonal position. Let T_1 and T_2 be two switching rooks in $\mathcal{C}$. We analyze the following cases.
(1) If T_1 and T_2 are placed in $\mathcal{P}_1 \setminus \mathcal{Q}$ then they are in diagonal position by Step

1, as well as if T_1 and T_2 are in $\mathcal{P}_2$ by Step 2.

(2) Suppose that T_1 is in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and T_2 in $\mathcal{P}_1 \setminus \mathcal{Q}$. We use the notation $[B_1, B_r]$ to represent the largest vertical block of $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ where the rook T_1 is placed. After Step 1, a rook R is placed in $[B_1, B_r]$. In fact, if no rook is in $[B_1, B_r]$ after Step 1, then there is no rook in $[B_1, B_r]$ after Step 2; but this is not possible since we are assuming that the rook T_1 is in $[B_1, B_r]$ after Step 2. Moreover, by Step 1 we have that R and T_2 are in diagonal position (see Figure 2.9 (A)). Now, observe that applying Step 2 we remove R and we put T_1 in $[B_1, B_r]$. Since R and T_2 were in diagonal position, then also T_1 and T_2 are in diagonal position (see Figure 2.9 (B)).



(a) After Step 1 (b) After Step 2

Figure 2.9: T_1 in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and T_2 in $\mathcal{P}_1 \setminus \mathcal{Q}$.

(3) If T_1 is in $\mathcal{P}_1 \setminus \mathcal{Q}$ and T_2 in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$, then we get the claim arguing as done in (2).

Therefore, $\mathcal{C}$ is a j-rook configuration such that $\mathcal{C} \sim \mathcal{R}$ and any two switching rooks of $\mathcal{C}$ are placed in a diagonal position, as claimed.

□

Definition 2.15. Consider a frame polyomino $\mathcal{P}$ and a j-rook configuration $\mathcal{R}$ within $\mathcal{P}$, where j is an element of the set of integers from 1 to $r(\mathcal{P})$. We say that the j-rook configuration $\mathcal{C}$ defined in Proposition 2.14 is a *canonical configuration* of $\mathcal{R}$.

Now we provide some lemmas and a remark which is useful to prove Theorem 2.19.

Lemma 2.16. Let $\mathcal{P}$ be a frame polyomino and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. Let F be a facet of $\Delta(\mathcal{P})$ with j steps, where $j \geq 2$. If v and w are the lower right corners of two distinct steps of F belonging on the same maximal horizontal edge interval of $\mathcal{P}$ with $v < w$ then $v \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $w \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$.

Proof. Let $G = \{g_1, v, g_2\}$ and $U = \{u_1, w, u_2\}$ be the two steps of F having respectively v and w as lower right corners. Since v and w are on the same maximal horizontal edge interval H of $\mathcal{P}$ and $v < w$, then $g_1, u_1 \in H$ and $g_1 < v \leq u_1 < w$. By contradiction suppose that $v \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ or $w \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. We examine the three cases. If $v \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $w \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$, then necessarily $w < v$, which is a contradiction with the assumption that $v < w$. Assume that $v \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $w \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. In such a case, for the structure of $\mathcal{P}$, g_2 and w are the anti-diagonal corner of an inner interval of $\mathcal{P}$ but $g_2, w \in F$, so we get a contradiction. The same comes when $v \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $w \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. In every case, a contradiction arises. Hence the conclusion follows □

Lemma 2.17. Let $\mathcal{P}$ be a frame polyomino and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. Let F be a facet of $\Delta(\mathcal{P})$ with j steps, with $j \geq 2$. If v and w are the lower right corners of two distinct steps of F belonging on the same maximal vertical edge interval of $\mathcal{P}$ with $w < v$ then $v \in V(\mathcal{P}_{[(a_k,b_k),(m,n)]})$ and $w \notin V(\mathcal{P}_{[(a_k,b_k),(m,n)]})$.

Proof. The claim follows arguing similarly as done in the proof of Lemma 2.16. □

Remark 2.18. Let $\mathcal{P}$ be a parallelogram polyomino and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. Recall that, for all $j \geq 0$, every maximal chain of $\mathcal{P}$ with j descents as distributive lattice is a facet of $\Delta(\mathcal{P})$ with j steps, and vice versa. As a consequence of [42, Proposition 3.11, Lemma 3.12], we observe that there is a one-to-one correspondence between the canonical configurations in $\mathcal{P}$ of j rooks and the facets of $\Delta(\mathcal{P})$ with j steps.

Now we are ready to prove one of the crucial results of this chapter, which provides a bijection between the facets with j steps of the simplicial complex attached to a frame polyomino $\mathcal{P}$ and the canonical configurations of j rooks in $\mathcal{P}$.

Theorem 2.19. Let $\mathcal{P}$ be a frame polyomino and $\Delta(\mathcal{P})$ be the simplicial complex attached to $\mathcal{P}$. For all $j \geq 0$, there exists a bijection between the facets with j steps of $\Delta(\mathcal{P})$ and the canonical configurations in $\mathcal{P}$ of j rooks.

Proof. The first part of the proof is devoted to defining a desired bijective function that uniquely assigns to a facet of $\Delta(\mathcal{P})$ a canonical configuration in $\mathcal{P}$. Let F be a facet of $\Delta(\mathcal{P})$ with j steps.

- If $j = 0$, then F has no step. Observe from Discussion 2.9 that $F = F_I \cup S^*$, where $F_I \cup S^*$ is the facet defined in the proof of Proposition 2.2. We associate to F the 0-rook configuration, which is the empty set.

- If $j = 1$, then the facet F has just one step, whose lower right corner is denoted by v_F. In such a case, we attach to F the 1-rook configuration defined by placing a rook in the cell of $\mathcal{P}$ having v_F as the lower right corner.

- Assume that $j \geq 2$. Consider a step F' of F having w as the lower right corner and we denote by H_w and V_w respectively the maximal horizontal and vertical edge intervals of $\mathcal{P}$ containing w. We look at the following two possibilities.

  1. There does not exist any step with a lower right corner belonging to H_w and to V_w. In such a case, we place a rook in the cell of $\mathcal{P}$ having w as the lower right corner.
  2. There exists a step with a lower right corner $v = (i, h)$ belonging to H_w or to V_w.

     a) Assume that $v \in H_w$. It is not restrictive to suppose that $v < w$ because the other case will follow similarly. From Lemma 2.16, we have that $v \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $w \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. Therefore we assign a rook to the cell of $\mathcal{P}$ having w as the lower right corner and another one to the cell having (i, b_0) as a lower right corner.

     b) Assume that $v \in V_w$. We may take $w < v$ because the other case is similar. From Lemma 2.17, we get $v \in V(\mathcal{P}_{[(a_k,b_k),(m,n)]})$ and $w \notin V(\mathcal{P}_{[(a_k,b_k),(m,n)]})$. Therefore we attach a rook to the cell of $\mathcal{P}$ having w as the lower right corner and another one to the cell having (a_k, h) as a lower right corner.

In this way, we define a configuration $\mathcal{R}$ of j rooks in $\mathcal{P}$, related to the facet F of $\Delta(\mathcal{P})$ with j steps. We need to show that $\mathcal{R}$ is a canonical configuration in $\mathcal{P}$. Firstly, we prove that every pair of rooks of $\mathcal{R}$ is in non-attacking position. Let $R_1, R_2 \in \mathcal{R}$ and $F_1 = \{a, v_1, b\}, F_2 = \{c, v_2, d\}$ be the two steps related respectively to R_1 and R_2 having $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$ as lower right corners, with $v_1 < v_2$. The only case that we need to examine is when v_1 and v_2 are on the same maximal edge interval of $\mathcal{P}$. We may assume that $j_1 = j_2$ because the case $i_1 = i_2$ can be shown similarly. From Lemma 2.16, we have that $v_1 \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$ and $v_2 \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. Hence, R_1 is in the cell of $\mathcal{P}$ having (i_1, b_0) as lower right corner and R_2 is in the cell of $\mathcal{P}$ with v_2 as lower right corner, so R_1 and R_2 are not in attacking position. Moreover, we cannot find any rook in the vertical and horizontal blocks of $\mathcal{P}$ containing R_1 or R_2. We show just that there is no rook in the vertical and horizontal blocks of $\mathcal{P}$ containing R_1, because the other case for R_2 can be shown similarly. By contradiction, if there is a rook different from R_1 in the vertical block of $\mathcal{P}$ containing R_1, then there exists a step $G = \{p, q, r\}$, where q is its lower right corner with $q = (i_1, h_1)$ and $1 \leq h_1 \leq n$. But in such a case, we have that either p, v_1 or a, q are the anti-diagonal corners of an inner interval of $\mathcal{P}$, a contradiction. In a similar way, we show that there is no rook in the horizontal block of $\mathcal{P}$ containing R_1. Hence all pairs of rooks in $\mathcal{R}$ are in non-attacking position.

Moreover, we observe that two switching rooks of $\mathcal{R}$ cannot be in anti-diagonal position; in fact, by contradiction, if two switching rooks of $\mathcal{R}$ are in anti-diagonal position then the lower right corners of the related steps are the anti-diagonal corners of an inner interval of $\mathcal{P}$, which is a contradiction with (2) of Proposition 2.2. We conclude that $\mathcal{R}$ is a canonical configuration.
Set $\mathcal{R} = \mathcal{C}_F$ and denote by $\mathcal{F}_{\mathcal{P},j}$ the set of the facets of $\Delta(\mathcal{P})$ with j steps and by $\mathfrak{C}_{\mathcal{P},j}$ the set of all canonical configurations in $\mathcal{P}$ of j rooks. We introduce the map $\psi : \mathcal{F}_{\mathcal{P},j} \to \mathfrak{C}_{\mathcal{P},j}$ where $\psi(F)$ is the canonical configuration $\mathcal{C}_F$ defined before, for all $F \in \mathcal{F}_{\mathcal{P},j}$. We prove that ψ is bijective.
Firstly, we show that ψ is injective. Let $F_1, F_2 \in \mathcal{F}_{\mathcal{P},j}$ such that $F_1 \neq F_2$. We prove that $\mathcal{C}_{F_1} \neq \mathcal{C}_{F_2}$. Since $F_1 \neq F_2$, there exists $a \in V(\mathcal{P})$ such that $a \in F_1$ and $a \notin F_2$. Set $a = (a_x, a_y)$. Firstly, assume that $a \in V(\mathcal{P}_{[(1,1),(a_0,b_0)]}) \setminus [(1, b_0), (a_0, b_0)]$. We distinguish two cases.

Case 1. Suppose that a is the lower right corner of a step $\{p, a, q\}$ of F_1. Set $p = (p_x, a_y)$ and $q = (a_x, q_y)$. It follows from Discussion 2.9 that $p_x = a_x - 1$ and q_y is either $a_y + 1$ or $b_0 + 1$. We examine the following two sub-cases.

(1) If $q_y = a_y + 1$, then a rook R_1 of $\mathcal{C}_{F_1}$ is in the cell C of $\mathcal{P}$ with a as lower right corner. Observe from the definition of ψ that the only possibility in order to $R_1 \in \mathcal{C}_{F_2}$ is that $\{p, a, q\}$ is a step of F_2. Since $a \notin F_2$, $\{p, a, q\}$ is not a step of F_2, so $R_1 \notin \mathcal{C}_{F_2}$, that is $\mathcal{C}_{F_1} \neq \mathcal{C}_{F_2}$.

(2) If $q_y = b_0 + 1$, then from Discussion 2.9 there exists a step of F_1 with lower right corner a' such that a and a' are on the same maximal horizontal edge interval of $\mathcal{P}$ and $a' \notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})$. Hence a rook R_1 of $\mathcal{C}_{F_1}$ is in the cell of $\mathcal{P}$ having (a_x, b_0) as lower right corner and another rook R_2 of $\mathcal{C}_{F_1}$ is in the cell of $\mathcal{P}$ having a' as lower right corner. We show that R_1 or R_2 do not belong to $\mathcal{C}_{F_2}$. If $R_1 \notin \mathcal{C}_{F_2}$, then we have finished. Suppose that $R_1 \in \mathcal{C}_{F_2}$ and we prove that $R_2 \notin \mathcal{C}_{F_2}$. Since $a \notin F_2$ and $R_1 \in \mathcal{C}_{F_2}$, then from Discussion 2.9 the only possibility is that $\{(a_x - 1, b_0), (a_x, b_0), (a_x, b_0 + 1)\}$ is a step of F_2. As a consequence, the vertices in $\{(i, j) \in V(\mathcal{P}) : i \geq a_x, j < b_0\}$ do not belong to F_2, so no rook of $\mathcal{C}_{F_2}$ is placed in a cell of the cell interval $\mathcal{P}_{[(a_0,1),(m,b_0)]}$ and $R_2 \notin \mathcal{C}_{F_2}$. Therefore, $\mathcal{C}_{F_1} \neq \mathcal{C}_{F_2}$.

Case 2. Suppose that a is not the lower right corner of a step of F_1. Recall that $a \neq (1, 1)$ since $a \notin F_2$. We examine the following sub-cases.

(1) If $1 < a_x \leq a_0$ and $a_y = 1$, then $[(1, 1), (a_x, 1)] \subset F_1$ as explained in Discussion 2.9. We show firstly that $(a_x + 1, 1)$ belongs to F_1. Suppose

by contradiction that $(a_x+1,1)\notin F_1$. From Discussion 2.9 it follows that $(a_x,2)\in F_1$ or $(a_x,b_0+1)\in F_1$ and no vertex in $[(a_x,2),(a_x,b_0)]$ is in F_1. In both cases $(a_x,1)$ is a lower right corner of a step of F_1, which is a contradiction. Hence $(a_x+1,1)\in F_1$. Let k_1 be the maximum integer in $\{a_x+1,\dots,a_0\}$ such that $(k_1,1)\in F_1$. So, we have only two possibilities: either $(k_1,2)\in F_1$ or $(k_1,2)\notin F_1$. If $(k_1,2)\in F_1$ then $\{(k_1-1,1),(k_1,1),(k_1,2)\}$ is a step of F_1 so a rook Q_1 of $\mathcal{C}_{F_1}$ is in the cell of $\mathcal{P}$ having $(k_1,1)$ as lower right corner. If $(k_1,2)\notin F_1$, then it easily follows from Discussion 2.9 that $\{(k_1-1,1),(k_1,1),(k_1,b_0+1)\}$ is a step of F_1 and there exists another step of F_1 whose lower right corner v belongs to $[(a_0+1,1),(m,1)]$. Hence, from the definition of ψ, there is a rook T_1 of $\mathcal{C}_{F_1}$ in the cell of $\mathcal{P}$ with (k_1,b_0) as lower right corner and another rook T_2 of $\mathcal{C}_{F_1}$ in the cell of $\mathcal{P}$ with v as lower right corner.
Now, we consider F_2. Let $N=\{(i,j)\in V(\mathcal{P}): i<a_x, j>1\}$. We prove that there exists a vertex w in N belonging to F_2. In fact, by contradiction, if there does not exist any vertex in N belonging to F_2, then there is no inner interval of $\mathcal{P}$ having as anti-diagonal corners a and a vertex in N, hence a belongs to F_2 due to the maximality of F_2. But this is a contradiction because $a\notin F_2$. Therefore, let $w=(w_x,w_y)\in N\cap F_2$. If $w\in[(1,2),(a_x-1,b_0)]$, then no vertex in $[(w_x+1,1),(m,w_y-1)]$ is in F_2 so no rook of $\mathcal{C}_{F_2}$ is in a cell of $\mathcal{P}_{[(w_x,1),(m,w_y)]}$. Hence neither Q_1 or T_2 belong to $\mathcal{C}_{F_2}$, that is $\mathcal{C}_{F_1}\neq\mathcal{C}_{F_2}$. If $w\in N\setminus[(1,2),(a_x-1,b_0)]$, then any vertex in $[(w_x+1,1),(a_0,b_0)]$ is in F_2 so no rook of $\mathcal{C}_{F_2}$ is in a cell of $\mathcal{P}_{[(w_x+1,1),(a_0,b_0+1)}$ and neither Q_1 or T_1 belong to $\mathcal{C}_{F_2}$, so $\mathcal{C}_{F_1}\neq\mathcal{C}_{F_2}$.

(2) If $a_x=1$ and $1<a_y<b_0$, then we use similar arguments as done in the previous sub-case (1) to prove that no rook of $\mathcal{C}_{F_1}$ is in a cell of $\mathcal{P}_{[(1,1),(m,a_y)]}$ and at least one rook of $\mathcal{C}_{F_2}$ is in a cell of a sub-polyomino of $\mathcal{P}_{[(1,1),(m,a_y)]}$.

(3) If $1<a_x\leq a_0$ and $1<a_y<b_0$, then we can argue as done in (1) for F_1. For the discussion about F_2, we may consider $N_1=\{(i,j)\in V(\mathcal{P}): i<a_x, j>a_y\}$ and $N_2=\{(i,j)\in V(\mathcal{P}): i>a_x, j<a_y\}$. In particular, if $N_1\cap F_2\neq\emptyset$, so we get the claim arguing similarly as done in the sub-case (1). If $N_1\cap F_2=\emptyset$ and $N_2\cap F_2\neq\emptyset$, then we can argue as in sub-case (2). In both we get the desired claim.

The two examined cases lead to $\mathcal{C}_{F_1}\neq\mathcal{C}_{F_2}$. Moreover, all the other situations when $a\notin V(\mathcal{P}_{[(1,1),(a_0,b_0)]})\setminus[(1,b_0),(a_0,b_0)]$ give us $\mathcal{C}_{F_1}\neq\mathcal{C}_{F_2}$, using an approach similar to the previous one. Hence we conclude that $\mathcal{C}_{F_1}\neq\mathcal{C}_{F_2}$, so ψ is injective.
We prove that ψ is surjective. Let $\mathcal{T}$ be a canonical configuration of j rooks and we prove that there exists a facet F of $\Delta(\mathcal{P})$ of j steps such that $\psi(F)=\mathcal{T}$. If

$j = 0$, then we set $F = F_I \cup S^*$, where $F_I \cup S^*$ is the facet defined in the proof of Proposition 2.2, so F has no step and $\psi(F) = \mathcal{T}$.
Suppose that $j \geq 1$. Recall that the parallelogram polyomino $\mathcal{S}$, which is the hole of $\mathcal{P}$, is determined by the north-east paths S_1 and S_2 with endpoints (a_0, b_0) and (a_k, b_k). Referring to Figure 2.2, we consider several cases.

Case 1. Assume that all rooks of $\mathcal{T}$ are in $\mathcal{P}_1$. From Remark 2.18, there exists a facet F_1 of $\Delta(\mathcal{P}_1)$ of j steps corresponding to $\mathcal{T}$. Set $F = F_1 \cup (S_2 \setminus \{(a_0, b_0), (a_k, b_k)\})$. It is easy to see that F is a facet with j steps of $\Delta(\mathcal{P})$ and, in particular, $\psi(F) = \mathcal{T}$, that is the claim.

Case 2. Suppose that all rooks of $\mathcal{T}$ are in $\mathcal{P}_2$. Applying Remark 2.18, there exists a facet F_2 of $\Delta(\mathcal{P}_2)$ of j steps corresponding to $\mathcal{T}$. Now we examine four sub-cases depending on the placement of the rooks in $\mathcal{Q}$.

1. Assume that there is no rook in $\mathcal{Q}$. Since $j \geq 1$, then at least one rook is in $\mathcal{P}_2 \setminus \mathcal{Q}$. We denote by (t_x, t_y) and (r_x, r_y) respectively the lower right corner of the cells in $\mathcal{P}_2$ where the most left and the most right rooks of $\mathcal{T}$ are placed. We set $L_1 = [(1, b_0 + 1), (1, n)] \cup [(1, n), (a_k - 1, n)]$ and we distinguish the following sub-cases.

   (a) If $t_y \leq b_0$ and $r_x > a_k$, then $F = \big(F_2 \setminus ([(2, t_y), (a_0, t_y)] \cup [(r_x, b_k), (r_x, n-1)])\big) \cup L_1$ (see Figure 2.10);

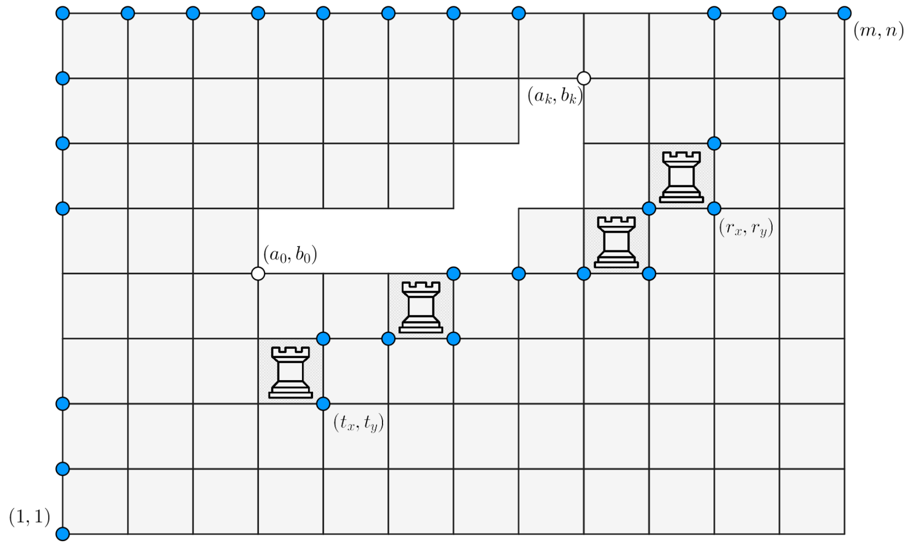


Figure 2.10: A canonical configuration and the related facet.

   (b) If $t_y \leq b_0$ and $r_x \leq a_k$, then $F = \big(F_2 \setminus ([(2, t_y), (a_0, t_y)] \cup [(a_k, b_k), (a_k, n-1)])\big) \cup L_1$;

   (c) If $t_y > b_0$ and $r_x \leq a_k$, then $F = \big(F_2 \setminus ([(2, b_0), (a_0, b_0)] \cup [(a_k, b_k), (a_k, n-1)])\big) \cup L_1$;

(d) If $t_y > b_0$ and $r_x > a_k$, then $F = \big(F_2 \setminus ([(2,b_0),(a_0,b_0)] \cup [(r_x,b_k),(r_x,n-1)])\big) \cup L_1$.

It is easy to see in every previous sub-case that F is a facet of j steps of $\Delta(\mathcal{P})$ and $\psi(F) = \mathcal{T}$, which is the desired claim.

2. Suppose that at least one rook is in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and no one in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$. Let (a',b') represent the coordinates of the lower right corner of the cell in $\mathcal{P}$ where the rightmost rook of $\mathcal{T}$ is located in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$. Set $L_2 = [(a',b_0+1),(a',n)] \cup [(a',n),(a_k-1,n)]$.

2.1 If no rook is in $\mathcal{P}_2 \setminus \mathcal{Q}$, then we define $F = \big(F_2 \setminus ([(a'+1,b_0),(a_0,b_0)] \cup [(a_k,b_k),(a_k,n-1)])\big) \cup L_2$.

2.2 If a rook is in $\mathcal{P}_2 \setminus \mathcal{Q}$, we indicate by (t_x,t_y) and (r_x,r_y) respectively the lower right corner of the cells in $\mathcal{P}_2$ where the most left and the most right rooks of $\mathcal{T}$ are placed. We have the following four sub-cases.

(a) If $t_y \leq b_0$ and $r_x > a_k$, then $F = (F_2 \setminus [(a'+1,t_y),(a_0,t_y)] \cup [(r_x,b_k),(r_x,n-1)]) \cup L_2$ (see Figure 2.11).

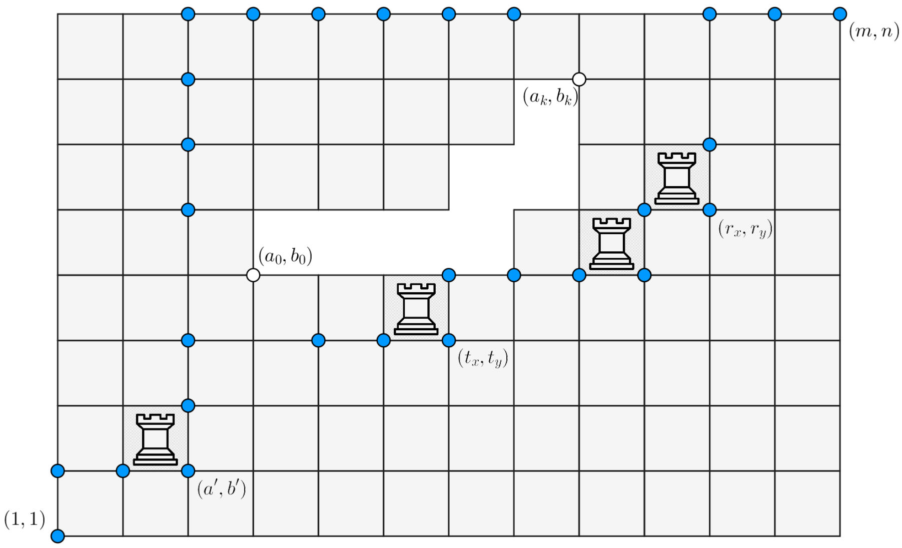


Figure 2.11: A canonical configuration and the related facet.

(b) If $t_y \leq b_0$ and $r_x \leq a_k$, then $F = (F_2 \setminus [(a'+1,t_y),(a_0,t_y)] \cup [(a_k,b_k),(a_k,n-1)]) \cup L_2$;

(c) If $t_y > b_0$ and $r_x \leq a_k$, then $F = (F_2 \setminus [(a'+1,b_0),(a_0,b_0)] \cup [(a_k,b_k),(a_k,n-1)]) \cup L_2$;

(d) If $t_y > b_0$ and $r_x > a_k$, then $F = (F_2 \setminus [(a'+1,b_0),(a_0,b_0)] \cup [(r_x,b_k),(r_x,n-1)]) \cup L_2$.

In every case F is a facet of j steps of $\Delta(\mathcal{P})$ and $\psi(F) = \mathcal{T}$.

3. The case when no rook is placed in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and at least one in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$ can be proved similarly, as well as when a rook is both in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$.

Case 3 It may be inferred that one rook from the set $\mathcal{T}$ is located in $\mathcal{P}_1$ and another rook is located in $\mathcal{P}_2$. Take into account the following two cell intervals: E_{h} represents the collection of cells in $\mathcal{P}$ that have their lower right corner. For any i ranging from 2 to a_0, (i, b_0) is true. Additionally, E_{v} refers to one of the cells in $\mathcal{P}$ with a lower right corner at (a_k, l), where l ranges from b_k to $n-1$. Here, we must differentiate between several scenarios based on the positioning of the rooks in $\mathcal{Q}$, E_{h}, or E_{v}.

1. Suppose firstly that no rook of $\mathcal{T}$ is in $\mathcal{Q} \cup E_{\mathrm{h}} \cup E_{\mathrm{v}}$. Consider the parallelogram sub-polyomino $\mathcal{K}_1 = \mathcal{P}_1 \setminus (\mathcal{Q} \cup E_{\mathrm{h}} \cup E_{\mathrm{v}})$ of $\mathcal{P}_1$. Note that all rooks of $\mathcal{T}$ are placed in $\mathcal{K}_1$ and $\mathcal{P}_2 \setminus \mathcal{Q}$. From Remark 2.18 it follows that there exist two facets K_1 and K_2 respectively of $\Delta(\mathcal{K}_1)$ and $\Delta(\mathcal{P}_2)$ corresponding to the canonical configurations in $\mathcal{K}_1$ and $\mathcal{P}_2$. Denote by (t_x, t_y) and (r_x, r_y) respectively the lower right corner of the cells in $\mathcal{P}_2$ where the most left and the most right rooks of $\mathcal{T}$ are placed. We define F as follows.
   (a) If $t_y \leq b_0$ and $r_x > a_k$, then $F = K_1 \cup (K_2 \setminus ([(2,t_y),(a_0,t_y)] \cup [(r_x,b_k),(r_x,n-1)])$ (see Figure 2.12);

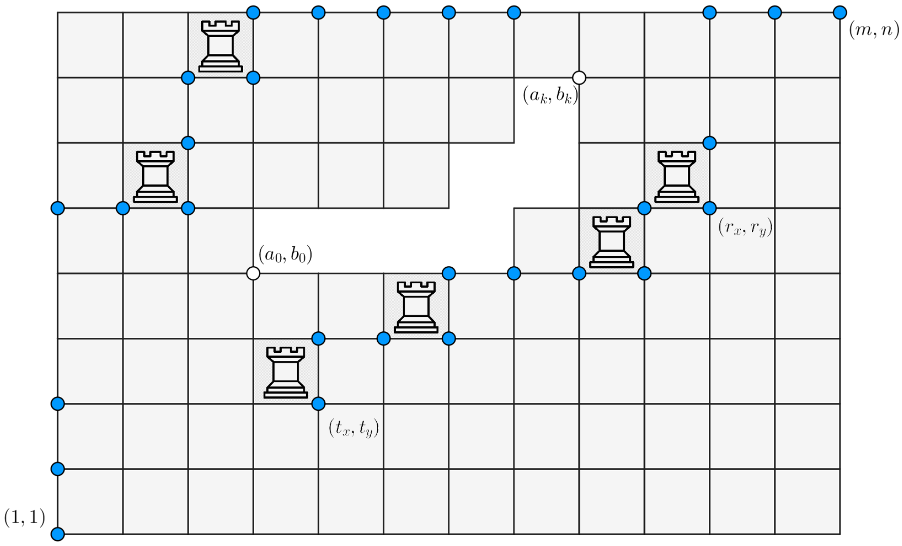


Figure 2.12: A canonical configuration and the related facet.

   (b) If $t_y \leq b_0$ and $r_x \leq a_k$, then $F = K_1 \cup (K_2 \setminus ([(2,t_y),(a_0,t_y)] \cup [(a_k,b_k),(a_k,n-1)])$;
   (c) If $t_y > b_0$ and $r_x \leq a_k$, then $F = K_1 \cup (K_2 \setminus ([(2,b_0),(a_0,b_0)] \cup [(a_k,b_k),(a_k,n-1)])$;

(d) If $t_y > b_0$ and $r_x > a_k$, then $F = K_1 \cup (K_2 \setminus ([(2,b_0),(a_0,b_0)] \cup [(r_x,b_k),(r_x,n-1)])$.

It is easy to see that, in every case, F is a facet of j steps of $\Delta(\mathcal{P})$ and, moreover, that $\psi(F) = \mathcal{T}$, which is the claim.

2. Suppose that none of the rooks in $\mathcal{T}$ are put in $E_{\mathrm{h}} \cup E_{\mathrm{v}}$, but there is at least one rook in $\mathcal{Q}$. Moreover, we may suppose that a rook is in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and another one in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$, because the other cases can be proved similarly. Observe that if there are no rooks in $\mathcal{P}_1 \setminus \mathcal{Q}$ (resp. $\mathcal{P}_2 \setminus \mathcal{Q}$) then we are in Case 2 (resp. Case 1) so the proof is done. Assume that at least a rook is both in $\mathcal{P}_1 \setminus \mathcal{Q}$ and in $\mathcal{P}_2 \setminus \mathcal{Q}$. Let (t_x,t_y) and (r_x,r_y) be respectively the lower right corner of the cells in $\mathcal{P}_2$ where the most left and the most right rooks of $\mathcal{T}$ are placed. Denote by (c_x,c_y) the lower right corner of the cell in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ where the most right rook of $\mathcal{T}$ is placed, and by (d_x,d_y) the lower right corner of the cell in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$ where the most left rook of $\mathcal{T}$ is placed. Observe that $c_y < t_y$ and $r_x < d_x$. In fact, $c_y \leq t_y$ and $r_x \leq d_x$ from Discussion 2.9 and, moreover, $c_y \neq t_y$ and $r_x \neq d_x$ since no rook of $\mathcal{T}$ is placed in $E_{\mathrm{h}} \cup E_{\mathrm{v}}$. Now, we examine the following sub-cases.

(a) If $t_y < b_0$ and $r_x > a_k$, then consider $\mathcal{P}_1$ and the parallelogram polyomino $\mathcal{K}_2$ given by the north-east paths $[(a_0,t_y),(r_x,t_y)] \cup [(r_x,t_y),(r_x,b_k)]$ and $[(a_0,t_y),(a_0,b_0)] \cup (S_2 \setminus \{(a_0,b_0),(a_k,b_k)\} \cup [(a_k,b_k),(r_x,b_k)]$. From Remark 2.18 there exist two facets K_1 and K_2 respectively of $\Delta(\mathcal{P}_1)$ and $\Delta(\mathcal{K}_2)$ corresponding to the canonical configurations in $\mathcal{P}_1$ and $\mathcal{K}_2$. Then we set $F = \big(K_1 \setminus ([(c_x,t_y+1),(c_x,b_0)] \cup [(a_k,d_y),(r_x-1,d_y)])\big) \cup \big(K_2 \setminus \{(a_0,t_y),(r_x,b_k)\}\big)$ (see Figure 2.13). By construction, it is easy to see that F is a facet of j steps of $\Delta(\mathcal{P})$ and, moreover, that $\psi(F) = \mathcal{T}$.

(b) If $t_y \geq b_0$ and $r_x \leq a_k$, then we need to consider $\mathcal{P}_1$ and the parallelogram polyomino $\mathcal{K}_2$ given by the north-east paths $[(a_0+1,b_0-1),(a_k+1,b_0-1)] \cup [(a_k+1,b_0-1),(a_k+1,b_k-1)]$ and $[(a_0+1,b_0-1),(a_0+1,b_0)] \cup (S_2 \setminus \{(a_0,b_0),(a_k,b_k)\} \cup [(a_k,b_k-1),(a_k+1,b_k-1)]$. As before, there exist two facets K_1 and K_2 respectively of $\Delta(\mathcal{P}_1)$ and $\Delta(\mathcal{K}_2)$ corresponding to the canonical configurations in $\mathcal{P}_1$ and $\mathcal{K}_2$, and in such a case the desired facet is $F = K_1 \cup K_2 \setminus (\{(a_0+1,b_0-1),(a_k+1,b_k-1)\})$.

(c) If $t_y < b_0$ and $r_x \leq a_k$, then it is sufficient to consider $\mathcal{P}_1$ and the parallelogram polyomino $\mathcal{K}_2$ given by the north-east paths $[(a_0,t_y),(a_k+1,t_y)] \cup [(a_k+1,t_y),(a_k+1,b_k-1)]$ and $[(a_0,t_y),(a_0,b_0)] \cup (S_2 \setminus$

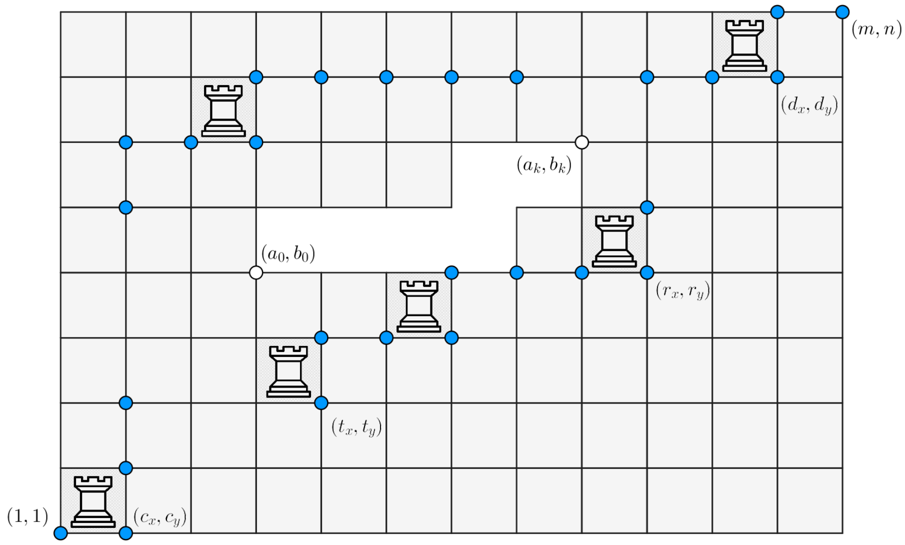


Figure 2.13: A canonical configuration and the related facet.

$\{(a_0,b_0),(a_k,b_k)\} \cup [(a_k,b_k-1),(a_k+1,b_k-1)]$. As before, K_1 and K_2 are the two facets respectively of $\Delta(\mathcal{P}_1)$ and $\Delta(\mathcal{K}_2)$ corresponding to the canonical configurations in $\mathcal{P}_1$ and $\mathcal{K}_2$ and $F = \big(K_1 \setminus [(c_x,t_y+1),(c_x,b_0)]\big) \cup \big(K_2 \setminus (\{(a_0,t_y),(a_k+1,b_k-1)\})\big)$.

(d) If $t_y \geq b_0$ and $r_x > a_k$, it is clear how the facet F can be defined similarly.

3. Suppose now that there exists a rook in $E_{\mathrm{h}} \cup E_{\mathrm{v}}$. It is not restrictive to assume that there is a rook both in E_{h} and in E_{v}, one in $\mathcal{P}_{[(1,1),(a_0,b_0)]}$ and another one in $\mathcal{P}_{[(a_k,b_k),(m,n)]}$. Moreover, if there is not any rook in $\mathcal{P}_2 \setminus \mathcal{Q}$ then we are in Case 1, so we can assume that a rook is also in $\mathcal{P}_2 \setminus \mathcal{Q}$. We denote by (f_x,b_0) and (a_k,g_y) respectively the lower right corner of the rook in E_{h} and E_{v}, and we set (t_x,t_y), (r_x,r_y), (c_x,c_y) and (d_x,d_y) as before. Here we examine just the case when $t_x < b_0$ and $r_x > a_k$ because the other ones can be proved using the same approach and some considerations as done in (2) of Case 3. Consider now the following four parallelogram polyominoes: $\mathcal{B}_1$ and $\mathcal{B}_2$ are the rectangular polyominoes given respectively by $[(1,1),(f_x,t_y)]$ and $[(r_x,d_y),(m,n)]$, $\mathcal{G}_1$ is the parallelogram sub-polyomino of $\mathcal{P}_1$ determined by the north-east paths $[(f_x-1,b_0),(f_x-1,g_y+1)] \cup [(f_x-1,g_y+1),(a_k,g_y+1)]$ and $[(f_x-1,b_0),(a_0,b_0)] \cup S_1 \cup [(a_k,b_k),(a_k,g_y+1)]$ and $\mathcal{G}_2$ is the parallelogram sub-polyomino of $\mathcal{P}_2$ determined by $[(a_0,t_y),(a_0,b_0)] \cup S_2 \cup [(a_k,b_k),(r_x,b_k)]$ and $[(a_0,t_y),(r_x,t_y)] \cup [(r_x,t_y),(r_x,b_k)]$. From Remark 2.18, there exist four facets F_1, F_2, F_3 and F_4 of respectively $\Delta(\mathcal{B}_1)$, $\Delta(\mathcal{B}_2)$, $\Delta(\mathcal{G}_1)$ and $\Delta(\mathcal{G}_2)$ corresponding to the canonical configurations in that four parallelogram polyominoes. Consider $F = F_1 \cup F_2 \cup (F_3 \setminus$

$\{(f_x-1,b_0),(f_x,b_0),(a_k,g_y),(a_k,g_y+1)\}) \cup (F_4 \setminus \{(a_0,t_y),(a_k,b_k)\})$. Look at Figure 2.14 for an example of such a case. We want to point out that when we remove $(f_x-1,b_0),(f_x,b_0)$ from F_3, we are removing the step in F_3 corresponding to the rook in E_{h}, but it is substituted by $\{(f_x-1,t_y),(f_x,t_y),(f_x,b_0+1)\}$ as step of F. The same holds for the vertices $(a_k,g_y),(a_k,g_y+1)$. In light of this, it is easy to see that F is a facet of $\Delta(\mathcal{P})$ with j steps and $\psi(F)=\mathcal{T}$.

Hence ψ is surjective. In conclusion, ψ is bijective. □

In Figure 2.14 we figure out a canonical configuration $\mathcal{C}$ in $\mathcal{P}$ of seven rooks and the facet of $\Delta(\mathcal{P})$ attached to $\mathcal{C}$.

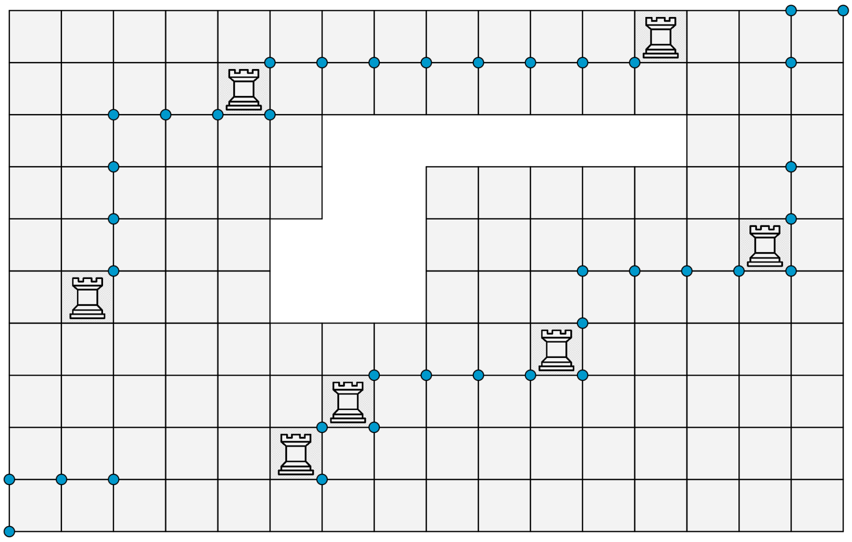


Figure 2.14: A canonical configuration of seven rooks and the related facet.

At last, we are prepared to demonstrate the principal outcome of this chapter.

Theorem 2.20. Let $\mathcal{P}$ be a frame polyomino. The h-polynomial of $K[\mathcal{P}]$ is the switching rook polynomial of $\mathcal{P}$.

Proof. From (2) of Theorem 2.12 we know that the j-th coefficient of the h-polynomial of $K[\mathcal{P}]$ is the number of the facets of $\Delta(\mathcal{P})$ having j steps. From Theorem 2.19, the latter is equal to the number of the canonical configurations in $\mathcal{P}$ of j rooks, which is the j-th coefficient of the switching rook polynomial of $\mathcal{P}$. □

Corollary 2.21. Let $\mathcal{P}$ be a frame polyomino. Then the Castelnuovo-Mumford regularity of $K[\mathcal{P}]$ is the rook number of $\mathcal{P}$.

Proof. Let $\mathcal{P}$ is a frame polyomino, then by Proposition 2.2 $K[\mathcal{P}]$ is a Cohen-Macaulay domain. From Theorem 1.15 iit implies that regularity of $K[\mathcal{P}]$ is the degree of the h-polynomial of $K[\mathcal{P}]$, which is the rook number of $\mathcal{P}$ by Theorem 2.20. □

We conclude this chapter observing that Conjecture 1.33 is given just for simple polyominoes. Actually, a frame polyomino is a non-simple polyomino so it is natural to think that this conjecture could be extended to every polyomino.

Conjecture 2.22. Let $\mathcal{P}$ be a polyomino. The h-polynomial of $K[\mathcal{P}]$ is the switching rook polynomial of $\mathcal{P}$.

Chapter 3

Weakly Connected Collection of Cells

This chapter introduces a novel class of cell collections known as zig-zag collections. We offer Hilbert series and associated findings for this assemblage of cells.
"A collection of cells $\mathcal{S}$ is called *weakly connected* if for any two cells C and D in $\mathcal{S}$ there exists a sequence of cells $\mathcal{C} : C = C_1, \dots, C_m = D$ of $\mathcal{S}$ such that $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for all $i = 1, \dots, m-1$. "

3.1 Zig-zag collection

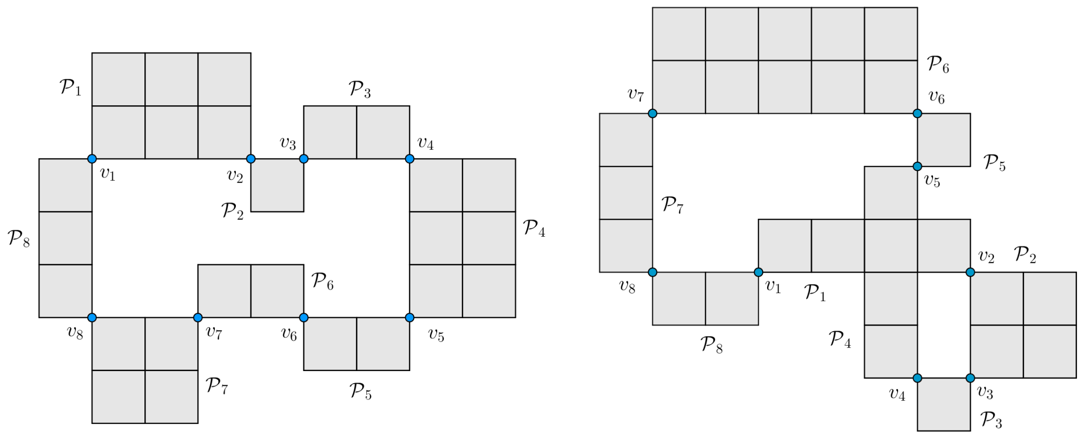
Definition 3.1. Let $I_1, \dots, I_\ell$ be a sequence of distinct intervals in $\mathbb{Z}^2$. For all $i \in \{1, \dots, \ell\}$ denote by $\mathcal{P}_i$ the collection of cells related to I_i and $\mathcal{P} = \bigcup_{i=1}^{\ell} \mathcal{P}_i$. $\mathcal{P}$ is a *zig-zag collection* if satisfies:

1. $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$, for all $i = 1, \dots, \ell - 1$;
2. v_i and v_{i+1} are on the same edge interval of $\mathcal{P}$, for all $i = 1, \dots, \ell$.
3. $I_i \cap I_j = \emptyset$ for all $\{i, j\} \subseteq [\ell]$ with $i < j$, such that $j \neq i + 1$ and $(i, j) \neq (1, \ell)$.

In such a case, we say that $\mathcal{P}$ is supported by $I_1, \dots, I_\ell$.

Figure 3.1a represent a zig-zag collection while in Figure 3.1b we present an example of a non zig-zag collection, in fact $|I_1 \cap I_4| = 4$.

Remark 3.2. Let $\mathcal{P}$ be a zig-zag collection. From Definition 3.1 it follows that $\mathcal{P}$ is a non-simple weakly connected collection of cells having just one hole and ℓ is an even number.
Moreover, by extending the arguments from [36, Section 3] in terms of weekly connected collection of cells, it follows that $I_{\mathcal{P}}$ is not prime because $I_1, \dots, I_\ell$ is a zig-zag walk of $\mathcal{P}$.



(a) A zig-zag collection.

(b) A non-zig-zag collection.

Figure 3.1: Example of collections of cells.

Our first step is to study the Gröbner basis of the inner 2-minors ideal of a zig-zag collection. In order to prove the next lemma, we introduce four total orders on $\mathbb{Z}^2$. Let $(i,j),(k,l) \in \mathbb{Z}^2$, then we say that:

- $(k,l) <^{(1)} (i,j)$, if $l < j$, or $l = j$ and $i < k$;
- $(k,l) <^{(2)} (i,j)$, if $l < j$, or $l = j$ and $k < i$;
- $(k,l) <^{(3)} (i,j)$, if $l > j$, or $l = j$ and $k < i$;
- $(k,l) <^{(4)} (i,j)$, if $l > j$, or $l = j$ and $i < k$.

Remark 3.3. Let $\mathcal{P}$ be a collection of cells. For $i \in \{1,2,3,4\}$, denote by $<_{\text{lex}}^{(i)}$ the lexicographic order in $K[x_v \mid v \in V(\mathcal{P})]$, induced by the total order on the variables: $x_a <_{\text{lex}}^{(i)} x_b$ if and only if $a <^{(i)} b$ for $a, b \in V(\mathcal{P})$. If f is an inner 2-minor in $I_{\mathcal{P}}$, observe that for $i \in \{1,3\}$ the initial monomial of f with respect to $<_{\text{lex}}^{(i)}$ is related to the anti-diagonal corners of the inner interval associated. While, for $i \in \{2,4\}$ the initial monomial of f is related to the diagonal corners. Moreover, [41, Theorem 4.1] and the arguments of its proof hold also considering the monomial orders $<_{\text{lex}}^{(i)}$ for $i \in \{2,4\}$. The same can be considered for [41, Remark 4.2] with respect to the monomial orders $<_{\text{lex}}^{(i)}$ for $i \in \{1,3\}$.

Lemma 3.4. Let $\mathcal{P}$ be a zig-zag collection supported by $I_1, \dots, I_\ell$. Then there exists a monomial order $\prec$ on $S_{\mathcal{P}}$ such that:

1. the set of generators $G(\mathcal{P})$ of $I_{\mathcal{P}}$ forms the reduced Gröbner basis of $I_{\mathcal{P}}$ with respect $\prec$;
2. the order $\prec$ induces a monomial order $<_i$ on $S_{\mathcal{P}_i}$ for all $i \in [\ell]$, such that for all $k \in [\ell/2]$ we have:

- $\text{in}_{<_{2k-1}}(I_{\mathcal{P}_{2k-1}})$
  $= (\{x_a x_b \mid a, b \text{ are anti-diagonal corners of an inner interval of } I_{2k-1}\})$;
- $\text{in}_{<_{2k}}(I_{\mathcal{P}_{2k}}) = (\{x_c x_d \mid c, d \text{ are diagonal corners of an inner interval of } I_{2k}\})$;
- $\text{in}_{\prec}(I_{\mathcal{P}}) = \sum_{i=1}^{\ell} \text{in}_{<_i}(I_{\mathcal{P}_i})$.

In particular, $I_{\mathcal{P}}$ is radical.

Proof. We start by defining inductively a suitable total order $<_{V(\mathcal{P})}$ on $V(\mathcal{P})$. It is not restrictive to suppose that I_1, I_2 and I_ℓ are arranged as in Figure 3.2 (a), otherwise it is sufficient to do some suitable reflections of $\mathcal{P}$ or to relabel the intervals $\{I_i\}_{i\in[\ell]}$.

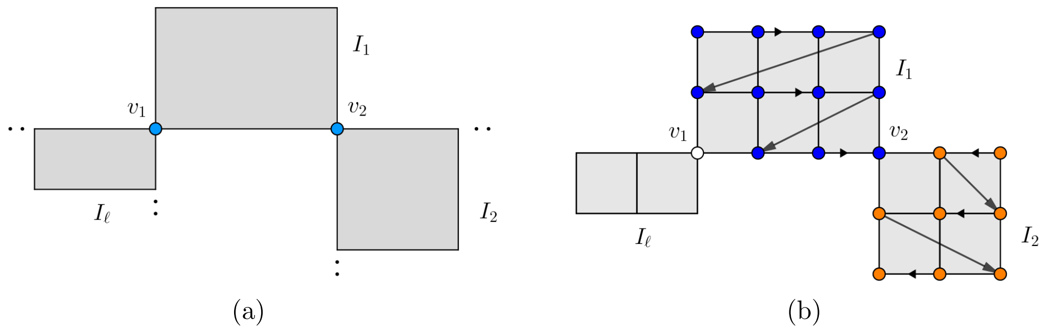


Figure 3.2: Arrangement of I_1, I_2 and I_ℓ and the related orders.

In the first step we give a total order on $(I_1 \setminus \{v_1\}) \cup I_2$. We use $<^{(1)}$ and $<^{(2)}$ respectively to order the vertices in $I_1 \setminus \{v_1\}$ and $I_2 \setminus \{v_2\}$; in Figure 3.2b, where the arrows denote the sense of the orders of the vertices in $I_1 \setminus \{v_1\}$ and $I_2 \setminus \{v_2\}$ respectively. Moreover, we set that every vertex of $\mathcal{P}$ in $I_1 \setminus \{v_1\}$ is smaller than all vertices in $I_2 \setminus \{v_2\}$. Let $k \in [\ell/2]$ with $k \geq 2$ and consider the intervals I_{2k-2}, I_{2k-1} and I_{2k}. Assume that a total order in $I_{2k-2} \setminus \{v_{2k-2}\}$ is already defined; we want to define a total order on $(I_{2k-1} \setminus \{v_{2k-1}\}) \cup I_{2k}$. First of all, we introduce two total orders on $I_{2k-1} \setminus \{v_{2k-1}\}$ and on $I_{2k} \setminus \{v_{2k}\}$ in the following way.

1. If I_{2k-2}, I_{2k-1} and I_{2k} are arranged as in Figures 3.3a and 3.3d or as in Figures 3.4a, 3.4b and 3.4d, then we use $<^{(1)}$ and $<^{(2)}$ to order the vertices in $I_1 \setminus \{v_1\}, I_2 \setminus \{v_2\}$ and $I_{2k-1} \setminus \{v_{2k-1}\}, I_{2k} \setminus \{v_{2k}\}$ respectively.

2. In the case that I_{2k-2}, I_{2k-1} and I_{2k} are as in Figure 3.3b, then we define an order of the vertices in $I_1 \setminus \{v_1\}, I_2 \setminus \{v_2\}$ and $I_{2k-1} \setminus \{v_{2k-1}\}, I_{2k} \setminus \{v_{2k}\}$ using $<^{(3)}$ and $<^{(2)}$ respectively.

3. When I_{2k-2}, I_{2k-1} and I_{2k} are in the position of Figure 3.3c, then use respectively $<^{(1)}$ and $<^{(4)}$ to make order in $I_1 \setminus \{v_1\}, I_2 \setminus \{v_2\}$ and $I_{2k-1} \setminus \{v_{2k-1}\}, I_{2k} \setminus \{v_{2k}\}$.

4. If I_{2k-2}, I_{2k-1} and I_{2k} are in the position of Figure 3.3c, Figure 3.4c then apply respectively $<^{(3)}$ and $<^{(4)}$ to define an order in $I_1 \setminus \{v_1\}, I_2 \setminus \{v_2\}$ and $I_{2k-1} \setminus \{v_{2k-1}\}, I_{2k} \setminus \{v_{2k}\}$.

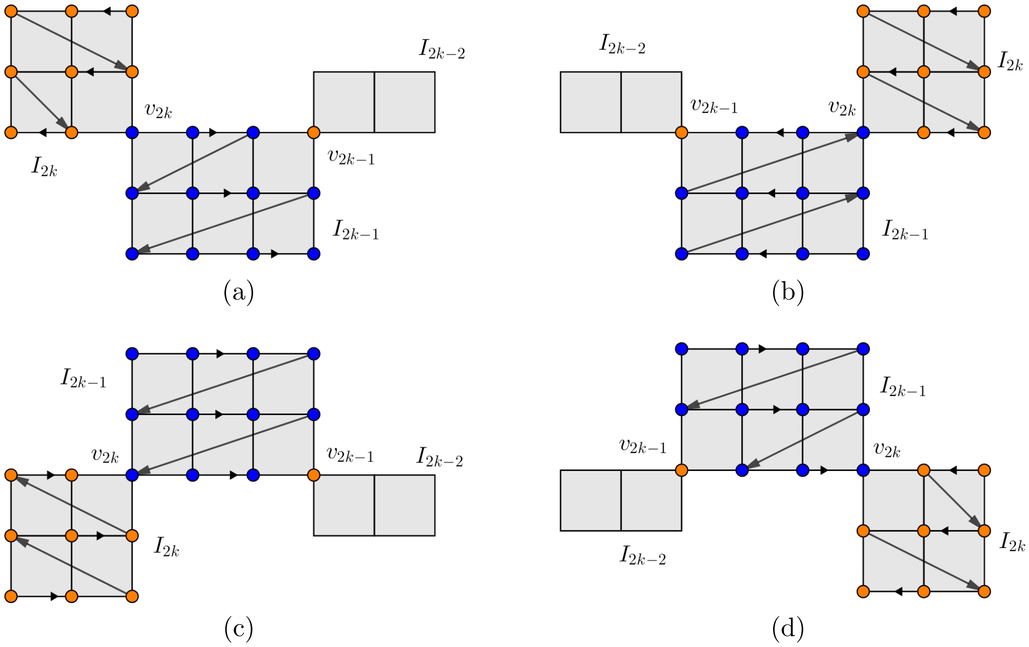


Figure 3.3: Horizontal arrangements of I_{2k-2}, I_{2k-1} and I_{2k} and the related orders.

Therefore, depending on the described situations, we have two total orders on $I_{2k-1} \setminus \{v_{2k-1}\}$ and on $I_{2k} \setminus \{v_{2k}\}$; moreover, putting that every vertex of $\mathcal{P}$ in $I_{2k-1} \setminus \{v_{2k-1}\}$ is smaller than every vertex in $I_{2k} \setminus \{v_{2k}\}$, we get a total order on $(I_{2k-1} \setminus \{v_{2k-1}\}) \cup I_{2k}$. Applying inductively the procedure until the interval I_ℓ, we can get a total order $<_{V(\mathcal{P})}$ for the set of the vertices of $\mathcal{P}$. Now, define the lexicographic order $<_{\text{lex}}$ on $S_\mathcal{P}$ induced by the order on the variables: $x_b <_{\text{lex}} x_a$, where $a, b \in V(\mathcal{P})$, if $b <_{V(\mathcal{P})} a$. We prove that $<_{\text{lex}}$ provides the desired claim.
We start proving (1). Let $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_r x_s$ be two generators of $I_\mathcal{P}$ related to the inner intervals $[a, b]$ and $[p, q]$ of $\mathcal{P}$, respectively. We want to show that the S-polynomial $S(f, g)$ of f and g reduces to 0 with respect to $G(\mathcal{P})$. Let $k \in [\ell/2]$. We may consider just the case when I_{2k-2}, I_{2k-1} and I_{2k} are arranged as in Figures 3.3a (with the convention that $I_0 = I_\ell$); indeed, all the other situations described in Figures 3.3 and 3.4 can be proved similarly. If $[a, b], [p, q] \subseteq I_{2k-1}$ or $[a, b], [p, q] \subseteq I_{2k}$ then we consider Remark 3.3. In particular, suppose that $[a, b], [p, q] \subseteq I_{2k}$, so $\text{in}_<(f) = x_a x_b$ and $\text{in}_<(g) = x_p x_q$. Except the trivial case $\gcd(\text{in}_<(f), \text{in}_<(g)) = 1$, then $S(f, g)$ reduces to 0 with respect to $G(\mathcal{P})$ by the arguments done in [41, Theorem 4.1]. Suppose that $[a, b], [p, q] \subseteq I_{2k-1}$, so $\text{in}_<(f) = x_c x_d$ and $\text{in}_<(g) = x_r x_s$ and $S(f, g)$ reduces to 0 with respect to $G(\mathcal{P})$ by using [41, Remark 4.2]. Assume that $[a, b] \subseteq I_{2k-1}$ and $[p, q] \subseteq I_{2k}$. In such a

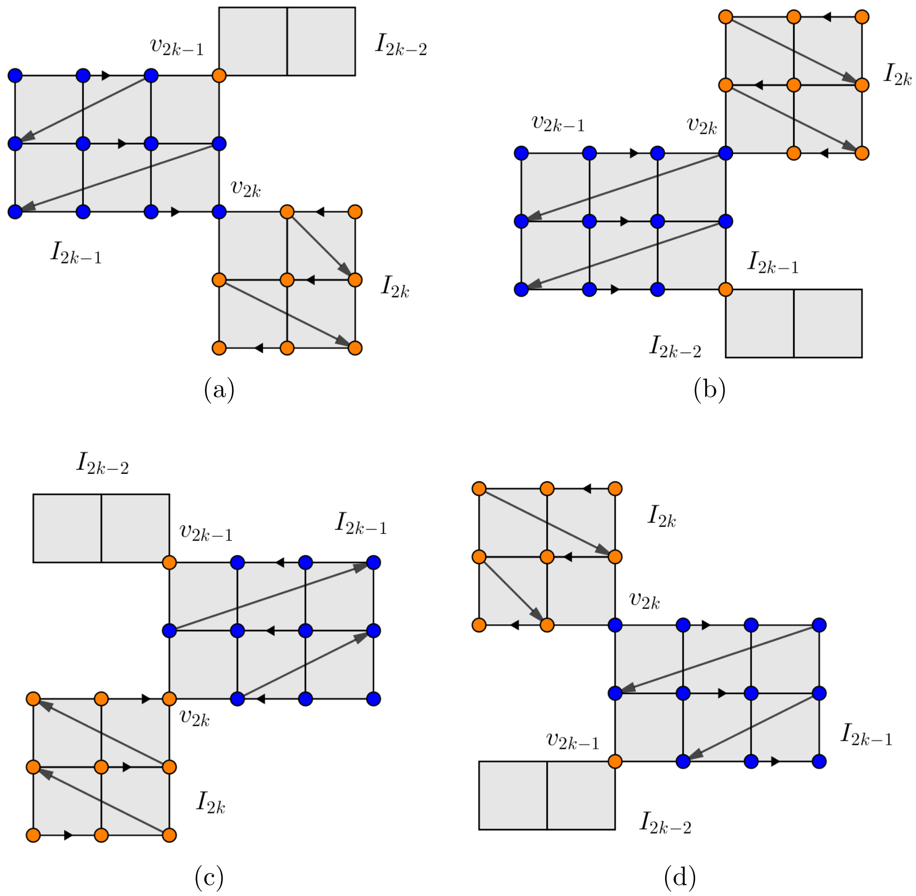


Figure 3.4: Vertical arrangements of I_{2k-2}, I_{2k-1} and I_{2k} and the related orders.

case, observe that $[a,b]\cap[p,q]=\emptyset$ or $[a,b]\cap[p,q]=\{a\}$ with $a=q$. Since in this case we have also $\mathrm{in}_<(f)=x_cx_d$ and $\mathrm{in}_<(g)=x_px_q$, then in both cases we obtain $\gcd(\mathrm{in}_<(f),\mathrm{in}_<(g))=1$. Hence $S(f,g)$ reduces to 0 with respect to $G(\mathcal{P})$. In such a case, we have $d <_{V(\mathcal{P})} a <_{V(\mathcal{P})} b <_{V(\mathcal{P})} c <_{V(\mathcal{P})} p <_{V(\mathcal{P})} s <_{V(\mathcal{P})} r <_{V(\mathcal{P})} q$ so $\mathrm{in}_<(f)=x_cx_d$ and $\mathrm{in}_<(g)=x_px_q$, that is $\gcd(\mathrm{in}_<(f),\mathrm{in}_<(g))=1$, hence $S(f,g)$ reduces to 0 with respect to $G(\mathcal{P})$. It is trivial to see that $\gcd(\mathrm{in}_<(f),\mathrm{in}_<(g))=1$ occurs when $[a,b]\subseteq I_i$ and $[p,q]\subseteq I_j$ with $i,j\in[\ell]$ and $j>i+1$. In conclusion, the claim (1) is completely proved.

Now, observe that $<_{\mathrm{lex}}$ induces in a natural way a monomial order $<_i$ on $S_{\mathcal{P}_i}$, for $i\in[\ell]$, which is the restriction of $<_{\mathrm{lex}}$ on $S_{\mathcal{P}_i}$. In particular, the claim (2) follows from the considerations done for claim (1) and Remark 3.3.

Finally, it is known that if the initial ideal of an homogeneous ideal I of $K[x_1,\dots,x_n]$, with respect to a monomial order, is squarefree then I is radical (see [23, Corollary 2.2]). Since $\mathrm{in}_{<_{\mathrm{lex}}}(I_{\mathcal{P}})$ is squarefree, then $I_{\mathcal{P}}$ is radical. □

3.2 Hilbert series of zig-zag collection

For shortness we will write $\mathrm{in}(I_{\mathcal{P}})$ and $\mathrm{in}(I_{\mathcal{P}_i})$ for respectively $\mathrm{in}_<(I_{\mathcal{P}})$ and $\mathrm{in}_{<_i}(I_{\mathcal{P}_i})$. Moreover, without loss of generality we can always assume that v_1 is a diagonal corner in I_1, as in Figure 3.2. For all $i \in \{1 \dots, \ell\}$ denote by $S'_i = K[x_a \mid a \in V(I_i) \setminus \{v_{i+1}\}]$. Set also $\mathrm{HS}_{K[\mathcal{P}_i]}(t) = \frac{h_i(t)}{(1-t)^{n_i}}$, in particular we know that $n_i = |V(\mathcal{P}_i)| - |\mathcal{P}_i|$ since each $\mathcal{P}_i$ is a simple polyomino.

Theorem 3.5. Let $\mathcal{P}$ be a zig-zag collection supported by $I_1, \dots, I_\ell$. Then:

1. $$\mathrm{HS}_{K[\mathcal{P}]}(t) = (1-t)^\ell \prod_{i=1}^{\ell} \mathrm{HS}_{K[\mathcal{P}_i]}(t) = \frac{h_1(t) \cdot h_2(t) \cdots h_\ell(t)}{(1-t)^{|V(\mathcal{P})|-|\mathcal{P}|}},$$

   where $h_i(t)$ is the h-polynomial of $K[\mathcal{P}_i]$.

2. $K[\mathcal{P}]$ is Cohen-Macaulay with Krull dimension $|V(\mathcal{P})| - |\mathcal{P}|$.

Proof. (1) By Lemma 3.4 we obtain that $S_{\mathcal{P}}/\mathrm{in}(I_{\mathcal{P}}) = \otimes_{i=1}^{\ell} S'_{\mathcal{P}_i}/\mathrm{in}(I_{\mathcal{P}_i})$. Observe that for all $i \in [\ell]$ then $S_{\mathcal{P}_i}/\mathrm{in}(I_{\mathcal{P}_i}) = S'_{\mathcal{P}_i}/\mathrm{in}(I_{\mathcal{P}_i}) \otimes_K K[x_{v_{i+1}}]$. By Proposition 1.6 we obtain that $\mathrm{HS}_{S'_i/I_{\mathcal{P}_i}}(t) = (1-t)\mathrm{HS}_{K[\mathcal{P}_i]}(t)$ and continuing in this way we obtain our claim on $\mathrm{HS}_{K[\mathcal{P}]}(t)$.
(2) Furthermore, by [41, Theorem 2.2] we know that $K[\mathcal{P}_i]$ is a Cohen-Macaulay domain of dimension $|V(\mathcal{P}_i)| - |\mathcal{P}_i|$, so by Proposition 1.24 also $S_{\mathcal{P}_i}/\mathrm{in}(I_{\mathcal{P}_i})$ is Cohen-Macaulay of dimension $|V(\mathcal{P}_i)| - |\mathcal{P}_i|$. Therefore, by Proposition 1.14 we obtain that $S'_{\mathcal{P}_i}/\mathrm{in}(I_{\mathcal{P}_i})$ is Cohen-Macaulay of dimension $|V(\mathcal{P}_i)| - |\mathcal{P}_i| - 1$ and as consequence $S_{\mathcal{P}}/\mathrm{in}(I_{\mathcal{P}})$ is Cohen-Macaulay of dimension $\sum_{i=1}^{\ell}(|V(\mathcal{P}_i)| - |\mathcal{P}_i| - 1) = V(\mathcal{P})| - |\mathcal{P}|$. By Proposition 1.24 the same property holds for $K[\mathcal{P}]$.

□

Example 3.6. We provide an example of non zig-zag collection of cells, whose coordinate ring is Cohen-Macaulay. Let $\mathcal{P}$ be the collection of cells given in Figure 3.5.
Denote by $<$ the lexicographic order on $S_{\mathcal{P}}$ induced by the following total order of the variables:

$$x_{34} > x_{42} > x_{45} > x_{44} > x_{54} > x_{73} > x_{61} > x_{65} > x_{57} > x_{47} > x_{37} > x_{16} > x_{15} > x_{14} >$$
$$x_{22} > x_{72} > x_{66} > x_{64} > x_{63} > x_{62} > x_{56} > x_{55} > x_{53} > x_{52} > x_{51} > x_{46} > x_{43} > x_{36} >$$
$$x_{33} > x_{32} > x_{27} > x_{26} > x_{25} > x_{24} > x_{23} > x_{13}$$

By some computations we get that the set of the generators of $I_{\mathcal{P}}$ forms the reduced

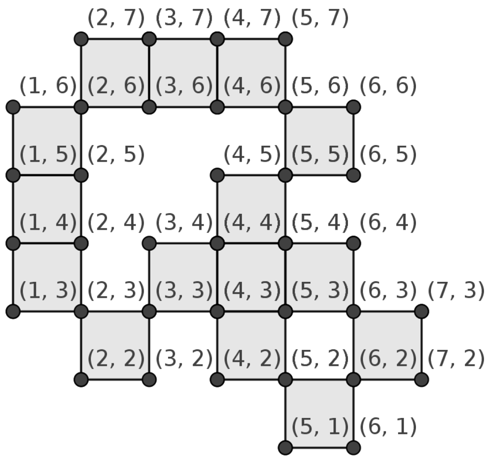


Figure 3.5: A non zig-zag collection of cells.

Gröbner basis of $I_{\mathcal{P}}$ with respect to $<$ and the initial ideal of $I_{\mathcal{P}}$ is generated by

$$x_{42}x_{54}, x_{45}x_{54}, x_{34}x_{63}, x_{44}x_{63}, x_{54}x_{63}, x_{73}x_{62}, x_{65}x_{56}, x_{42}x_{55}, x_{34}x_{53}, x_{42}x_{53}, x_{45}x_{53}, x_{44}x_{53},$$
$$x_{61}x_{52}, x_{57}x_{46}, x_{34}x_{43}, x_{57}x_{36}, x_{47}x_{36}, x_{22}x_{33}, x_{57}x_{26}, x_{47}x_{26}, x_{37}x_{26}, x_{16}x_{25}, x_{16}x_{24}, x_{15}x_{24},$$
$$x_{16}x_{23}, x_{15}x_{23}, x_{14}x_{23}.$$

Hence $I_{\mathcal{P}}$ is radical but it is not prime since $[(2,2),(3,3)], [(3,3),(6,4)], [(6,2),(7,3)], [(5,1),(6,2)], [(4,2),(5,5)], [(5,5),(6,6)], [(2,6),(5,7)], [(1,3),(2,6)]$ is a zig-zag walk of $\mathcal{P}$. Using `Macaulay2` ([52]) we obtain that $S_{\mathcal{P}}/\mathrm{in}_<(I_{\mathcal{P}})$ is a Cohen-Macaulay ring. Hence, from Proposition 1.24, we have that $K[\mathcal{P}]$ is Cohen-Macaulay.
We point out that we needed to check the Cohen-Macaulayness of $S_{\mathcal{P}}/\mathrm{in}_<(I_{\mathcal{P}})$ and not of $S_{\mathcal{P}}/I_{\mathcal{P}}$ using `Macaulay2`, since the process for $S_{\mathcal{P}}/I_{\mathcal{P}}$ is too hard for `Macaulay2`.

Lemma 3.7. Let $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be three collections of cells. Assume that:

1. $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}$;
2. $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$;
3. either $|V(\mathcal{P}_1) \cap V(\mathcal{P}_2)| = 1$ or there exist two distinct and non-adjacent cells E, F belonging to $\mathcal{P}_1$ (or $\mathcal{P}_2$) such that $V(\mathcal{P}_1) \cap V(\mathcal{P}_2) = \{e, f\}$, where $e \in E$ and $f \in F$ (see Figure 3.6).

Denote by $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ respectively the switching rook polynomial of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then $\tilde{r}_1(t)\tilde{r}_2(t)$ is the switching rook polynomial of $\mathcal{P}$.

Proof. We denote by $\tilde{r}(t)$ the switching rook polynomial of $\mathcal{P}$ and we set

$$\tilde{r}_1(t) = \sum_{i=0}^{r(\mathcal{P}_1)} r_i^{(1)} t^i \quad \text{and} \quad \tilde{r}_2(t) = \sum_{j=0}^{r(\mathcal{P}_2)} r_j^{(2)} t^j.$$

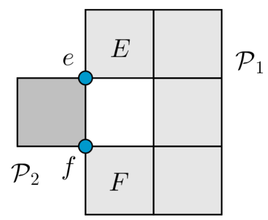


Figure 3.6: $V(\mathcal{P}_1) \cap V(\mathcal{P}_2) = \{e, f\}$.

Hence

$$\tilde{r}_1(t)\tilde{r}_2(t) = \left(\sum_{i=0}^{r(\mathcal{P}_1)} r_i^{(1)} t^j\right)\left(\sum_{j=0}^{r(\mathcal{P}_2)} r_j^{(2)} t^j\right) = \sum_{k=0}^{r(\mathcal{P}_1)+r(\mathcal{P}_2)} c_k t^k.$$

where $c_k = r_0^{(1)} r_k^{(2)} + r_1^{(1)} r_{k-1}^{(2)} + r_2^{(1)} r_{k-2}^{(2)} + \cdots + r_k^{(1)} r_0^{(2)}$. From the structure of $\mathcal{P}$ coming from the assumptions (1), (2) and (3), it follows that c_k represents the number of the canonical configurations of k-rooks in $\mathcal{P}$, which is the k-th coefficient of $\tilde{r}(t)$. Therefore $\tilde{r}_1(t)\tilde{r}_2(t)$ is the switching rook polynomial of $\mathcal{P}$. □

Corollary 3.8. Let $\mathcal{P}$ be a zig-zag collection supported by $I_1, \ldots, I_\ell$. Then

$$\mathrm{HS}_{K[\mathcal{P}]}(t) = \frac{h(t)}{(1-t)^{|V(\mathcal{P})|-|\mathcal{P}|}},$$

where $h(t)$ is the switching rook polynomial of $\mathcal{P}$. Moreover $\mathrm{reg}(K[\mathcal{P}]) = r(\mathcal{P})$.

Proof. For $i \in [\ell]$ we denote by $h_i(t)$ the h-polynomial of $K[\mathcal{P}_i]$, so from [42, Theorem 3.5] we have that $h_i(t)$ is the switching rook-polynomial of $\mathcal{P}_i$ for all $i \in [\ell]$. Recall that Theorem 3.5 states in particular that $h(t) = \prod_{i=1}^{\ell} h_i(t)$. Consider $\mathcal{P}_1 \cup \mathcal{P}_2$, so from Lemma 3.7 we have that $h_1(t)h_2(t)$ is the switching rook polynomial of $\mathcal{P}_1 \cup \mathcal{P}_2$. Now, we consider $(\mathcal{P}_1 \cup \mathcal{P}_2) \cup \mathcal{P}_3$. Applying Lemma 3.7, we have that $(h_1(t)h_2(t))h_3(t)$ is the switching rook polynomial of $(\mathcal{P}_1 \cup \mathcal{P}_2) \cup \mathcal{P}_3$. We continue these arguments until $\mathcal{P}_\ell$, getting that $\prod_{i=1}^{\ell} h_i(t)$ is the switching rook polynomial of $\cup_{i=1}^{\ell} \mathcal{P}_i$, that is $h(t)$ is the switching rook polynomial of $\mathcal{P}$. Moreover, since $K[\mathcal{P}]$ is Cohen-Macaulay from Theorem 3.5, then it follows by Theorem 1.15 that $\mathrm{reg}(K[\mathcal{P}]) = \deg(h(t)) = r(\mathcal{P})$. □

Let I be an interval of $\mathbb{Z}^2$ and $\mathcal{P}_I$ is the polyomino obtained from I. We say that $\mathcal{P}_I$ is a *square* if $I = [a, a + n(1, 1)]$ for $a \in \mathbb{Z}^2$ and for $n > 0$.

Corollary 3.9. Let $\mathcal{P}$ be a zig-zag collection supported by $I_1, \ldots, I_\ell$. If $K[\mathcal{P}]$ is Gorenstein then $\mathcal{P}_i$ is a square for all $i \in [\ell]$.

Proof. Let $h(t) = \sum_{k=1}^{s} h_k t^k$ be the h-polynomial of $K[\mathcal{P}]$. From Corollary 3.8 we have that $h(t)$ is the switching rook polynomial of $\mathcal{P}$ and $s = r(\mathcal{P})$. Since $K[\mathcal{P}]$ is Gorenstein, it follows from [55, Corollary 5.3.10] that $h_k = h_{s-k}$ for all $k \in [s]$. In particular we have that $h_s = 1$. Suppose by contradiction that there exists $h \in [\ell]$ such that $\mathcal{P}_h$ is not a square. Up to reflections, rotations or translations of $\mathcal{P}$, we may assume that $\mathcal{P}_h = \mathcal{P}_{[(1,1),(m.n)]}$, where $m > n$. Observe that $r(\mathcal{P}) = n - 1$. We define a canonical configuration $\mathcal{T}_1$ of $n-1$ rooks in $\mathcal{P}_h$, placing a rook in the cell of $\mathcal{P}_h$ having lower left corner (j, j) for all $j \in [n-1]$. Moreover, since $m > n$, we can define another canonical configuration $\mathcal{T}_2$ of $n-1$ rooks in $\mathcal{P}_h$, placing a rook in the cell of $\mathcal{P}_h$ having lower left corner (j, j) for all $j \in [n-2]$ and a rook in that one with $(n, n-1)$ as lower left corner. Denote by $\mathcal{C}_i$ a canonical configuration of $r(\mathcal{P}_i)$ rooks in $\mathcal{P}_i$, for all $i \in [s] \setminus \{h\}$, and set $\mathcal{C} = \cup_{i \in [s] \setminus \{h\}} \mathcal{C}_i$. It is easy to see that $\mathcal{C} \cup \mathcal{T}_1$ and $\mathcal{C} \cup \mathcal{T}_2$ are two canonical configuration of $r(\mathcal{P})$ rooks in $\mathcal{P}$, so $h_s > 1$. This is a contradiction because $h_s = 1$. In conclusion $\mathcal{P}_i$ is a square for all $i \in [\ell]$. $\square$

Chapter 4

PolyominoIdeals: A package for Macaulay2

In this chapter, we describe the package `PolyominoIdeals` [7] for the computer algebra software `Macaulay2` [52]. This package enables the users to define and manipulate the binomial ideal associated to the collection of cells. This guide is also available at [38].

4.1 Functions and Options

Consider that a collection of cells $\mathcal{P}$ is encoded, for the package, with a list of lists. Each list represents a cell of the collection and contains two lists representing the diagonal corners of a cell, the first for the lower left corner, the second for the upper right corner. For instance fixing the lower left corner of cell A as $(1, 1)$, the collection of cells in Figure 4.1 is encoded with the list `Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}}`.

4.1.1 `polyoIdeal` function

Consider a polyomino $\mathcal{P}$ and let $I_{\mathcal{P}}$ denote the polyomino ideal of $\mathcal{P}$. The `polyoIdeal` function gives the polyomino ideal $I_{\mathcal{P}}$. The polynomial ring defined as $S_{\mathcal{P}} = K[x_v : v \in V(\mathcal{P})]$ is auto-declared in the `polyoIdeal` function and can be accessed with the command `ring polyoIdeal Q`, where Q is the input list which comprises of the diagonal corners of each cell in $\mathcal{P}$. The function works to generate the binomial ideal $I_{\mathcal{S}}$ associated to a weekly connected collection of cells $\mathcal{S}$ or any collection of cells.

Example 4.1. Consider the polyomino on 6 cells shown in Figure 4.1. By fixing the lower left corner A as $(1, 1)$ we embed the polyomino with the list `Q = {{{1, 1}, {2,` 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}}. Using the `polyoIdeal Q` function we obtain the polyomino ideal and with `gens` function we can view the generators.

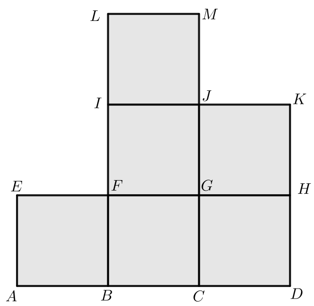


Figure 4.1: A polyomino of rank six.

“

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2},
     {3, 3}}, {{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}};
i3 : I = polyoIdeal Q;
i4 : g = gens I
o4 : | x_(4,3)x_(3,2)-x_(4,2)x_(3,3) x_(2,2)x_(1,1)-x_(2,1)x_(1,2)
---------------------------------------------------------------------------
x_(4,3)x_(2,1)-x_(4,1)x_(2,3) x_(3,2)x_(2,1)-x_(3,1)x_(2,2)
---------------------------------------------------------------------------
x_(4,3)x_(2,2)-x_(4,2)x_(2,3) x_(3,3)x_(2,1)-x_(3,1)x_(2,3)
---------------------------------------------------------------------------
x_(4,2)x_(1,1)-x_(4,1)x_(1,2) x_(3,4)x_(2,1)-x_(3,1)x_(2,4)
---------------------------------------------------------------------------
x_(3,3)x_(2,2)-x_(3,2)x_(2,3) x_(4,2)x_(3,1)-x_(4,1)x_(3,2)
---------------------------------------------------------------------------
x_(3,4)x_(2,2)-x_(3,2)x_(2,4) x_(3,2)x_(1,1)-x_(3,1)x_(1,2)
---------------------------------------------------------------------------
x_(4,3)x_(3,1)-x_(4,1)x_(3,3) x_(3,4)x_(2,3)-x_(3,3)x_(2,4)
---------------------------------------------------------------------------
x_(4,2)x_(2,1)-x_(4,1)x_(2,2) |
```

”

4.1.2 `polyoMatrix` function

Let $\mathcal{P}$ be a collection of cells and $[(p,q),(r,s)]$ be the smallest interval of $\mathbb{Z}^2$ containing $\mathcal{P}$. The matrix $M(\mathcal{P})$ has $s-q+1$ rows and $r-p+1$ columns with $M(\mathcal{P})_{i,j} = x_{(i,j)}$ if (i,j) is a vertex of $\mathcal{P}$, otherwise it is zero.

Example 4.2. Consider the same polyomino given in Figure 4.1 encoded by `Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}}, {{3, 2}, {4, 3}}, {{2,3}, {3, 4}}}`. The associated matrix is obtained using `polyoMatrix Q` command as described below. "

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{1, 1},{2, 2}}, {{2, 1},{3, 2}}, {{3, 1},{4, 2}}, {{2, 2},{3, 3}},
{{3, 2},{4, 3}}, {{2, 3}, {3, 4}}};
i3 : M = polyoMatrix Q
o3 : | 0       x_(2,4) x_(3,4) 0       |
     | 0       x_(2,3) x_(3,3) x_(4,3) |
     | x_(1,2) x_(2,2) x_(3,2) x_(4,2) |
     | x_(1,1) x_(2,1) x_(3,1) x_(4,1) |
```

"

The associated matrix for a collection of cells can help to order the variables to define a polynomial ring with another monomial order. In particular, this function is fundamental for coding the option when `RingChoice` has a different value by 1 (see Subsection 4.1.5).

4.1.3 `polyoToric` function

Let $\mathcal{P}$ be a weakly connected collection of cells. We introduce a suitable toric ideal attached to $\mathcal{P}$ based on that one given in [36] for polyominoes. Consider the following total order on $V(\mathcal{P})$: $a = (i,j) > b = (k,l)$, if $i > k$, or $i = k$ and $j > l$. If $\mathcal{H}$ is a hole of $\mathcal{P}$, then we call the lower left corner e of $\mathcal{H}$ the minimum, with respect to $<$, of the vertices of $\mathcal{H}$. Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be the holes of $\mathcal{P}$ and $e_k = (i_k, j_k)$ be the lower left corner of $\mathcal{H}_k$. For $k \in K = [r]$, we define the following subset $F_k = \{(i,j) \in V(\mathcal{P}) : i \leq i_k, j \leq j_k\}$. Denote by $\{V_i\}_{i\in I}$ the set of all the maximal vertical edge intervals of $\mathcal{P}$, and by $\{H_j\}_{j\in J}$ the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}$, $\{h_j\}_{j\in J}$, and $\{w_k\}_{k\in K}$ be three sets of variables. We consider the map

$$\alpha : V(\mathcal{P}) \to K[h_i, v_j, w_k : i \in I, j \in J, k \in K]$$

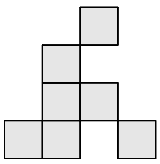
$$a \to \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in F_k} w_k$$

The toric ring $T_{\mathcal{P}}$ associated to $\mathcal{P}$ is defined as $T_{\mathcal{P}} = K[\alpha(a) : a \in V(\mathcal{P})]$. The homomorphism $\psi : S \to T_{\mathcal{P}}$ with $x_a \to \alpha(a)$ is surjective and the toric ideal $J_{\mathcal{P}}$ is the kernel of ψ. Observe that the latter generalizes in a natural way that is given in [8] and [43].

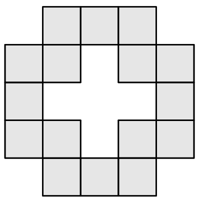
Theorem 4.3. [8, Theorem 3.3] "Let $\mathcal{P}$ be a simple and weakly-connected collection of cells. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$. "

The same holds for the class of grid polyominoes, defined in Section 4 of [36]. Moreover, with some suitable changes in the choice of the vertices of $\mathcal{P}$ to assign the variable w, similar statements can be proved for the classes of closed paths and weakly closed paths (see [6, Section 4-5] and [8, Section 4]).
The function `PolyoToric(Q,H)` provides the toric ideal $J_{\mathcal{P}}$ defined before, where `Q` is the list encoding the collection of cells and `H` is the list of the lower left corners of the holes. It provides a nice tool to study the primality of the inner 2-minor ideals of weakly connected collections of cells. Here we illustrate some examples.

Example 4.4. Consider the simple and weakly-connected collection $\mathcal{P}$ of cells in Figure 4.2a, encoded by the list `Q={{{1, 1}, {2, 2}}, {{2, 2}, {3, 3}}, {{2, 1}, {3, 2}},{{3, 2}, {4, 3}},{{2, 3}, {3, 4}}, {{4, 1}, {5, 2}}, {{3, 4}, {4, 5}}}`.
We can compute the ideal $I_{\mathcal{P}}$ using the function `polyoIdeal(Q)`, the toric ideal $J_{\mathcal{P}}$ with `polyoToric(Q,{})` and finally we do a comparison between the two ideals. We underline that, to verify the equality, we need to bring the ideal `J=polyoToric(Q,{})` in the ring `R` of `polyoIdeal(Q)`, using the command `substitute(J,R)`. In according to Theorem 4.3, we find that $I_{\mathcal{P}} = J_{\mathcal{P}}$.



(a)



(b)

Figure 4.2: Polyminoes for `polyoToric` function.

"

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
```

```
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{1, 1}, {2, 2}}, {{2, 2}, {3, 3}}, {{2, 1}, {3, 2}},{{3, 2}, {4, 3}},
{{2, 3}, {3, 4}}, {{4, 1}, {5, 2}}, {{3, 4}, {4, 5}}};
i3 : I = polyoIdeal Q;
i4 : J = polyoToric (Q,{});
i5 : R = ring I;
i6 : J = substitute (J,R);
o6 : Ideal of R
i7 : J == I
o7 = true
```

"

Now consider the polyomino $\mathcal{P}$ in Figure 4.2b. The polyomino ideal is not prime (see [6]), so $I_\mathcal{P} \subset J_\mathcal{P}$ since $I_\mathcal{P} = (J_\mathcal{P})_2$ ([36, Lemma 3.1]). We can also compute the set of the binomials generating $J_\mathcal{P}$ but not $I_\mathcal{P}$.

"

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{2, 1}, {3, 2}}, {{2, 2}, {3, 3}}, {{1, 2}, {2, 3}}, {{1, 3}, {2, 4}},
{{1, 4}, {2, 5}}, {{2, 4}, {3, 5}}, {{2, 5}, {3, 6}}, {{3, 5}, {4, 6}},
{{4, 5}, {5, 6}}, {{4, 4}, {5, 5}}, {{5, 4}, {6, 5}}, {{5, 3}, {6, 4}},
{{5, 2}, {6, 3}}, {{4, 2}, {5, 3}}, {{4, 1}, {5, 2}}, {{3, 1}, {4, 2}}};
i3 : I = polyoIdeal Q;
i4 : J = polyoToric (Q,{{2,3}});
i5 : R = ring I;
i6 : J = substitute(J,R);
i7 : J == I
o7 = false
i8 : select(first entries mingens J,f->first degree f>=3)
o8 = {x    x    x    x    - x    x    x    x   }
       6,5  5,1  2,6  1,2    6,2  5,6  2,1  1,5
```

"

4.1.4 The options Field and TermOrder

Let $\mathcal{P}$ be a collection of cells. The option `Field` for the function `polyIdeal` allows changing the base ring of the polynomial ring embedded in $I_\mathcal{P}$. One can choose every base ring that `Macaulay2` provides. The option `TermOrder` allows changing the monomial order of the ambient ring of $I_\mathcal{P}$ as given by the function `polyoIdeal`. In particular, by default, it provides the lexicographic order but one can replace it with other monomial orders defined in Macaulay2. See, for instance, the following example. “

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{1,1},{2,2}},{{2,2},{3,3}},{{3,3},{4,4}}};
i3 : I = polyoIdeal(Q,RingChoice=>1,TermOrder=> GRevLex);
o3 : Ideal of QQ[x   , x   , x   , x   , x   , x   , x   , x   , x   , x   ]
                  4,4   4,3   3,4   3,3   3,2   2,3   2,2   2,1   1,2   1,1
i4 : R = ring I;
i5 : describe R
o5 = QQ[x   , x   , x   , x   , x   , x   , x   , x   , x   , x   ,Degrees=>{10:1},Heft=>{1}]
         4,4   4,3   3,4   3,3   3,2   2,3   2,2   2,1   1,2   1,1
```

”

4.1.5 RingChoice: an option for the function polyoIdeal

Let $\mathcal{P}$ be a collection of cells. Recall that the definition of a ring in Macaulay2 needs to provide, together with a base ring and a set of variables, also a monomial order. RingChoice is an option that allows choosing between two available rings that one can define into $I_{\mathcal{P}}$.

If RingChoice is equal to 1, or by default, the function polyoIdeal gives the ideal $I_{\mathcal{P}}$ in the polynomial ring $S_{\mathcal{P}} = K[x_a : a \in V(\mathcal{P})]$, where K is a field and the monomial order is defined by TermOrder induced by the following order of the variables: $x_a > x_b$ with $a = (i, j)$ and $b = (k, l)$, if $i > k$, or $i = k$ and $j > l$.

Now we describe what is the ambient ring in the case RingChoice has a value different from 1. Consider the edge ring $R = K[s_it_j : (i, j) \in V(\mathcal{P})]$ associated to the bipartite graph G with vertex set $\{s_1, \dots, s_m\} \cup \{t_1, \dots, t_n\}$ such that each vertex $(i, j) \in V(\mathcal{P})$ determines the edge $\{s_i, t_j\}$ in G. Let $S = K[x_a : a \in V(\mathcal{P})$ and $\phi : S \to R$ be the K-algebra homomorphism defined by $\phi(x_{ij}) = s_it_j$, for all $(i, j) \in V(\mathcal{P})$ and set $J_{\mathcal{P}} = \ker(\phi)$. From [41, Theorem 2.1], we know that $I_{\mathcal{P}} = J_{\mathcal{P}}$, if $\mathcal{P}$ is a weakly connected and convex collection of cells. According to the findings in [27], it can be concluded that the generators of $I_{\mathcal{P}}$ constitute the reduced Gröbner basis when considering an appropriate order $<$. Furthermore, it is worth noting that the initial ideal $\text{in}_<(I_{\mathcal{P}})$ is both squarefree and generated in degree two. Following the proof in [27], the implemented routine provides the polynomial ring $S_{\mathcal{P}}$ with monomial order $<$.

Example 4.5. The polyomino $\mathcal{P}$ in Figure 4.1 is convex. Using the options `RingChoice => 2` to define $I_{\mathcal{P}}$, the ambient ring of $I_{\mathcal{P}}$ is given by `PolyoRingConvex`. Hence the initial ideal is squarefree in degree two. "

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : loadPackage "PolyominoIdeals";
i2 : Q = {{{1, 1}, {2, 2}}, {{2, 1}, {3, 2}}, {{3, 1}, {4, 2}}, {{2, 2}, {3, 3}},
{{3, 2}, {4, 3}}, {{2, 3}, {3, 4}}};
i3 : I = polyoIdeal Q;
i4 : In = monomialIdeal leadTerm I

o4 = monomialIdeal (x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
                     2,4 3,3   3,3 4,2   2,3 4,2   2,4 3,2   2,3 3,2   4,2 1,1
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
3,2 1,1   2,2 1,1   3,3 4,1   2,3 4,1   3,2 4,1   2,2 4,1   2,4 3,1
------------------------------------------------------------------------------
x    x   , x    x   )
2,3 3,1   2,2 3,1

i5 : Q = {{{1, 3}, {2, 4}}, {{2, 2}, {3, 3}}, {{2, 3}, {3, 4}}, {{2, 4}, {3, 5}},
{{3, 4}, {4, 5}}, {{3, 3}, {4, 4}}, {{3, 2}, {4, 3}}, {{3, 1}, {4, 2}},
{{3, 5}, {4, 6}}, {{4, 4}, {5, 5}}, {{4, 3}, {5, 4}}, {{5, 4}, {6, 5}}};

i6 : I = polyoIdeal(Q,RingChoice=>2);

i7 : In = monomialIdeal leadTerm I

o7 = monomialIdeal (x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
                     4,6 3,1   4,6 3,2   4,1 3,2   3,2 2,3   4,2 2,3   4,6 3,3
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
4,1 3,3   4,2 3,3   2,3 5,5   3,3 5,5   4,3 5,5   3,2 2,5   4,2 2,5
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
3,3 2,5   4,3 2,5   4,6 3,5   4,1 3,5   4,2 3,5   4,3 3,5   5,3 1,4
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
2,3 1,4   3,3 1,4   4,3 1,4   5,5 6,4   2,5 6,4   3,5 6,4   4,5 6,4
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
2,3 5,4   3,3 5,4   4,3 5,4   2,5 5,4   3,5 5,4   4,5 5,4   3,2 2,4
------------------------------------------------------------------------------
x    x   , x    x   , x    x   , x    x   , x    x   , x    x   , x    x   ,
```

```
4,2 2,4   3,3 2,4   4,3 2,4   3,5 2,4   4,5 2,4   4,6 3,4   4,1 3,4
---------------------------------------------------------------------------
x   x   , x   x   , x   x   )
4,2 3,4   4,3 3,4   4,5 3,4
```

"

4.2 A method to obtain the input list `Q` from GeoGebra

In order to make tests for some big polyominoes or collections of cells, it could be not so easy to input to `Macaulay2` the list of the diagonal corners of each cell that encodes our object. In this section, we give a method to obtain the encoding of a desired collection of cells, after drawing it in `GeoGebra`.

First of all, we explain how a collection of cells must be drawn in `GeoGebra`. In the `Graphics View`, by default, the coordinate axes and the grid appear. In the `Graphics View Toolbar` select the tool `Regular polygon`. This tool works in the following way: if one selects two points A and B in the plane and specifies the number n of vertices in the input field of the appearing dialog window, then a regular polygon with n vertices including points A and B is drawn. We use this tool to draw each cell of the collection we want to study, as a regular polygon of 4 vertices. For our purpose, each cell must be drawn by selecting first the lower left corner, say A, and then the lower right corner $A + (1, 0)$, using only integer points (see Figure 4.3).

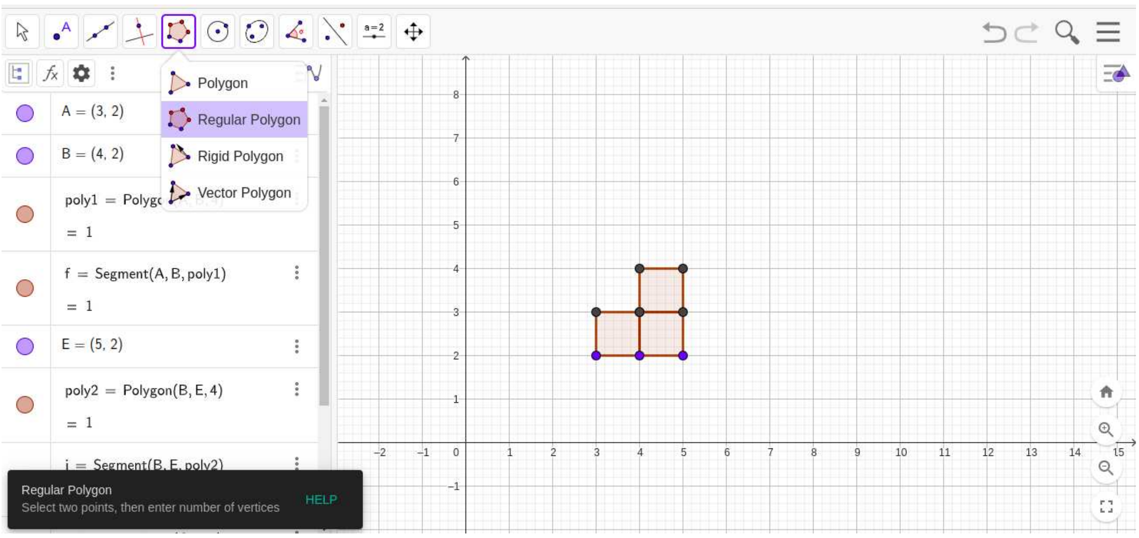


Figure 4.3: Regular polygon in GeoGebra.

In order to continue, we provide here the code of the function "GeoPoly" that we need to import in `Macaualy2`. If one uses a different language in `GeoGebra`, the code of the function must be changed accordingly.

```
GeoPoly=(n)->(
out={};
for i from 1 to n do (
out=join(out,{"{Vertex(poly"|i|",1),Vertex(poly"|i|",3)}"});
);
"vertices" << out << endl << close;
return out;
);
```

Now suppose the collection of cells drawn in `GeoGebra` has n cells. In the command line of `Macaulay2`, type the command "GeoPoly(n)". This function creates the text file "vertices.txt" in the local folder where `Macaulay2` is launched. So, open the file "vertices.txt" and copy its content in the `Input Bar` of the `GeoGebra` window (to activate it, go to the options up-right) where the collection of cells is drawn, and type Enter (see Figure 4.4).

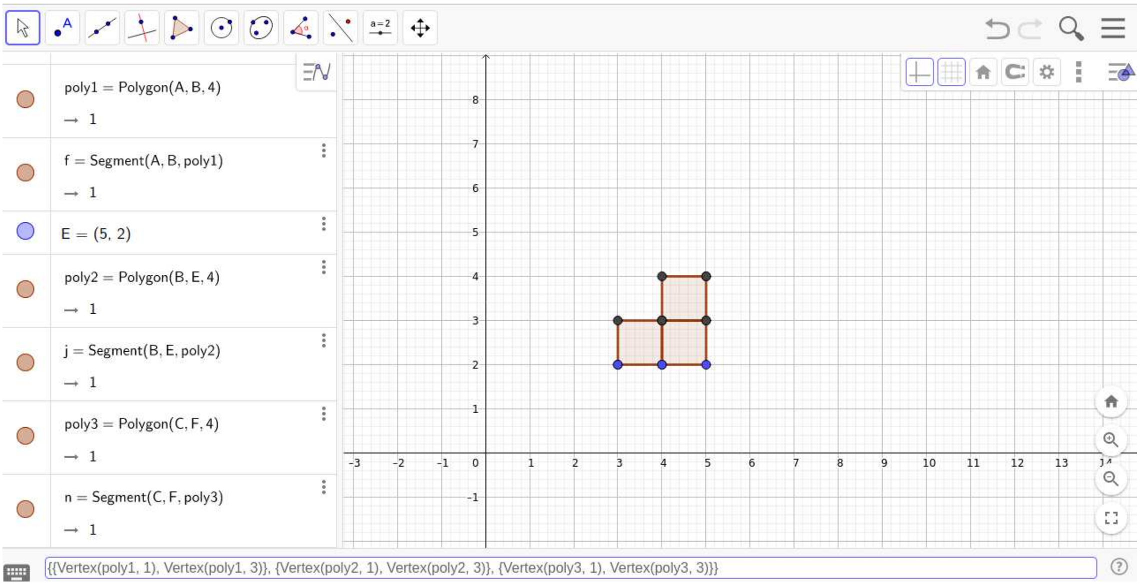


Figure 4.4: Input bar in GeoGebra.

In this way a list is generated, where each element is the list of the diagonal corners of each cell. Let $l1$ be the name of this list and, in the `Input bar`, type the command "Text(l1)" (see Figure 4.5).
Observe that, at this point, we can obtain the input data in `Macaulay2` of the drawn collection of cells by the text appearing in the `Graphic view`. We only need to change parentheses with braces.
A possible way to obtain the character string of the text in the `Graphic view`, is to export it as PGF/TikZ (see Figure 4.6).
In the exported file (that is a .txt file), one can find the desired character string in a particular line (see Figure 4.7 for the example explained in the previous pictures).

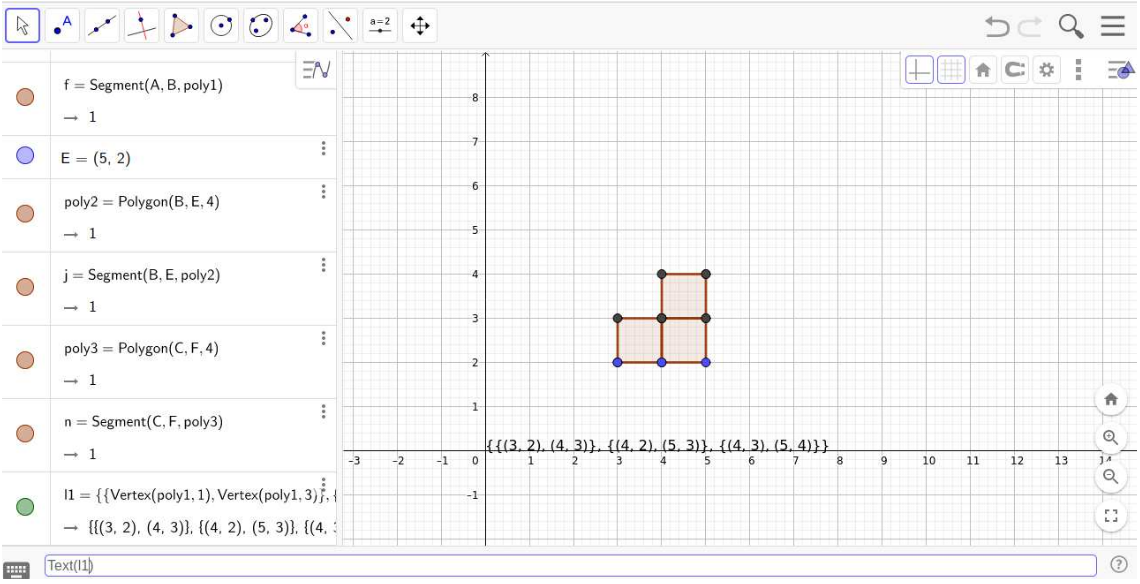


Figure 4.5: Adding text through input bar in GeoGebra.

Finally, copy the character string, open `Macaulay2` and paste it into the command line creating a variable `P`. That is, in the command line of `Macaulay2` you have something like `P={{(3, 2), (4, 3)}, {(4, 2), (5, 3)}, {(4, 3), (5, 4)}}` (considering the polyomino in the previous pictures). In order to replace parentheses in braces, and so obtain the right encoding of the desired collection of cells in a variable `Q`, it suffices to type the following commands: “

```
Macaulay2, version 1.21
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, Isomorphism, LLLBases, MinimalPrimes, OnlineLookup,
PrimaryDecomposition, ReesAlgebra, Saturation, TangentCone

i1 : P={{(3, 2), (4, 3)}, {(4, 2), (5, 3)}, {(4, 3), (5, 4)}};
i2 : Q={};
for i from 0 to #P-1 do(
Q=join(Q,{{toList(P#i#0),toList(P#i#1)}});
);
i4 : Q
o4 = {{{3, 2}, {4, 3}}, {{4, 2}, {5, 3}}, {{4, 3}, {5, 4}}}
o4 : List
```

”

The list `Q` encodes the desired collection of cells drawn in the `Graphic view` of `GeoGebra`, and from it, we are able to use the functions of the package `PolyominoIdeals`.

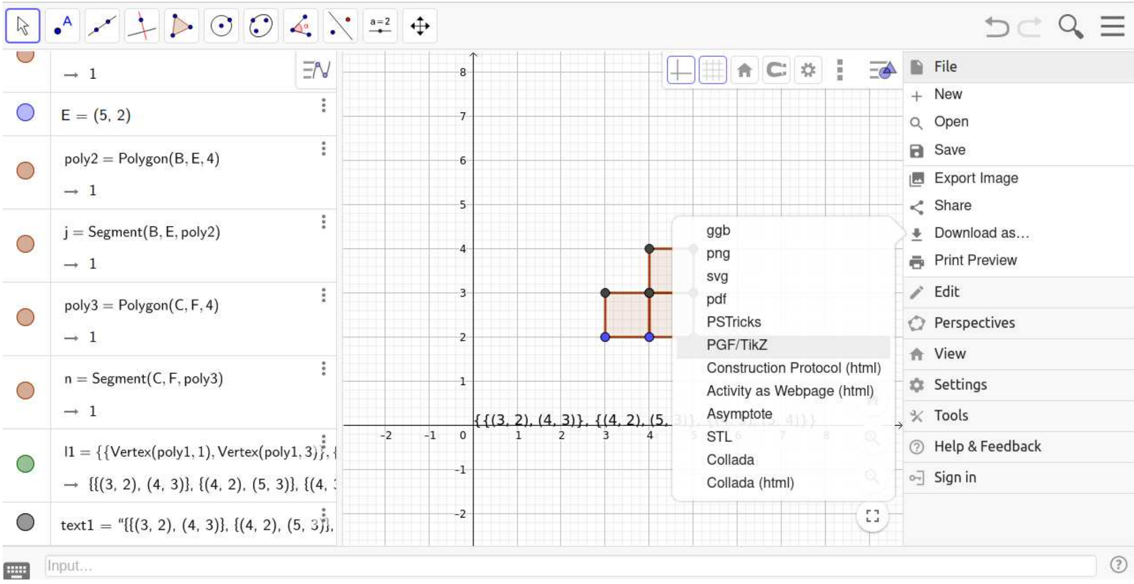
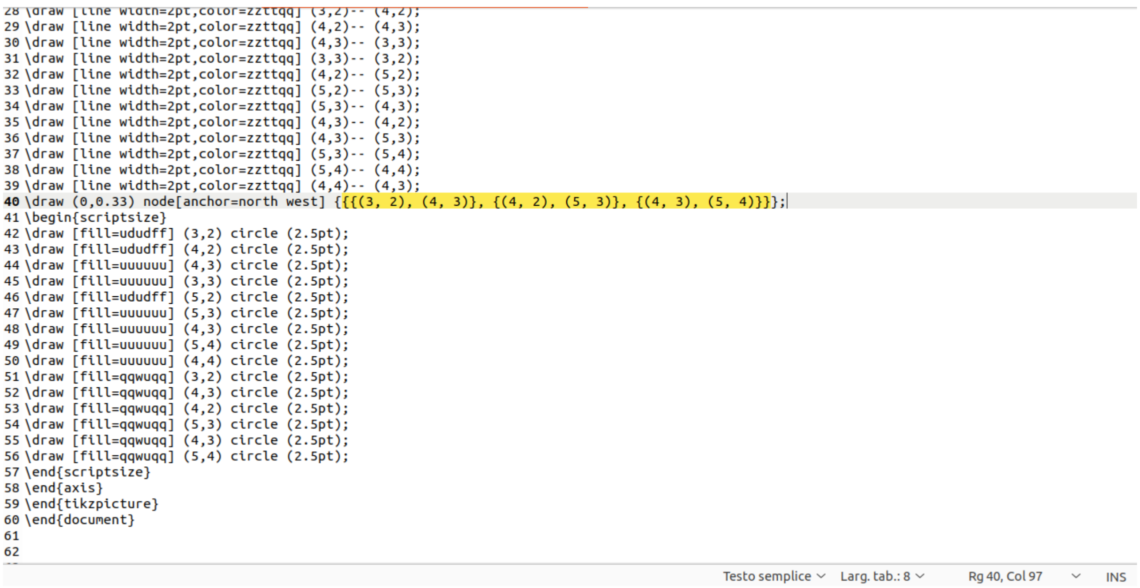


Figure 4.6: Export as PGF/TikZ option in GeoGebra.



```
\draw [line width=2pt,color=zzttqq] (3,2)-- (4,2);
\draw [line width=2pt,color=zzttqq] (4,2)-- (4,3);
\draw [line width=2pt,color=zzttqq] (4,3)-- (3,3);
\draw [line width=2pt,color=zzttqq] (3,3)-- (3,2);
\draw [line width=2pt,color=zzttqq] (4,2)-- (5,2);
\draw [line width=2pt,color=zzttqq] (5,2)-- (5,3);
\draw [line width=2pt,color=zzttqq] (5,3)-- (4,3);
\draw [line width=2pt,color=zzttqq] (4,3)-- (4,2);
\draw [line width=2pt,color=zzttqq] (4,3)-- (5,3);
\draw [line width=2pt,color=zzttqq] (5,3)-- (5,4);
\draw [line width=2pt,color=zzttqq] (5,4)-- (4,4);
\draw [line width=2pt,color=zzttqq] (4,4)-- (4,3);
\draw (0,0.33) node[anchor=north west] {{{(3, 2), (4, 3)}, {(4, 2), (5, 3)}, {(4, 3), (5, 4)}}};
\begin{scriptsize}
\draw [fill=ududff] (3,2) circle (2.5pt);
\draw [fill=ududff] (4,2) circle (2.5pt);
\draw [fill=uuuuuu] (4,3) circle (2.5pt);
\draw [fill=uuuuuu] (3,3) circle (2.5pt);
\draw [fill=ududff] (5,2) circle (2.5pt);
\draw [fill=uuuuuu] (5,3) circle (2.5pt);
\draw [fill=uuuuuu] (4,3) circle (2.5pt);
\draw [fill=uuuuuu] (5,4) circle (2.5pt);
\draw [fill=uuuuuu] (4,4) circle (2.5pt);
\draw [fill=qqwuqq] (3,2) circle (2.5pt);
\draw [fill=qqwuqq] (4,3) circle (2.5pt);
\draw [fill=qqwuqq] (4,2) circle (2.5pt);
\draw [fill=qqwuqq] (5,3) circle (2.5pt);
\draw [fill=qqwuqq] (4,3) circle (2.5pt);
\draw [fill=qqwuqq] (5,4) circle (2.5pt);
\end{scriptsize}
\end{axis}
\end{tikzpicture}
\end{document}
```


Figure 4.7: Character string in the export window.

Chapter 5

Cohen-Macaulay Posets

In this chapter we provide a characterization of Cohen-Macaulay posets of dimension two. The results that are presented in this chapter are contained in [32]. Along the chapter, all the posets are finite and all graphs are simple and finite.

5.1 Dimension two posets and permutation graphs

Let P be a poset. The *co-comparability graph* G of P is a graph on the underlying set of P such that $\{x, y\}$ is an edge of G if and only if x and y are incomparable in P. We say that a graph is a co-comparability graph if it is a co-comparability graph of some poset.

Let $l_0, \ldots, l_k$ be horizontal lines each labeled from left to right by permutations of $[n] = \{1, \ldots, n\}$. For each $i \in [n]$, the curve f_i consists of k straight line segments which join i on l_r to i in l_{r+1}, for $0 \leq r \leq k-1$. When $k = 1$, such a diagram is called a *permutation diagram*. When $k \geq 2$, it is called concatenation of k permutation diagrams. Figure 5.1 gives an example of a concatenation of 2 permutation diagrams for $n = 4$.

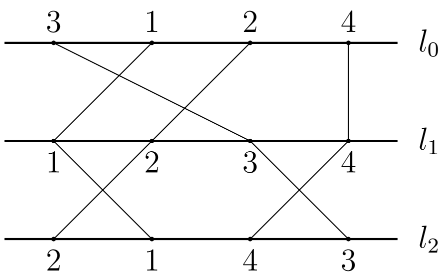


Figure 5.1: A concatenation of 2 permutation diagrams.

The *intersection graph* G of the concatenation of k permutation diagrams is a graph on $[n]$ such that $\{i, j\}$ is an edge of G if and only if f_i intersects with f_j. It was shown by Golumbic et al. [21, Theorem 1] that a graph is a co-comparability graphs if and only if it is an intersection graph of concatenation of k permutation diagrams. A graph G is called a *permutation graph* if it is the intersection graph of a permutation diagram (i.e., $k = 1$). Observe that the dimension of a poset is at most two if and only if its co-comparability graph is a permutation graph. Also, it follows from the definition that the dimension of a poset is one if and only if it is a linear order.

Let P be a poset and $\Delta(P)$ be its order complex. We interpret the definition of shelling for $\Delta(P)$ in terms of its defining chains. The order complex $\Delta(P)$ is called *shellable* if the maximal chains of P admit a linear order $\gamma_0, \ldots, \gamma_m$ such that for all $1 \leq j < i \leq m$, there exists a $v \in \gamma_i \setminus \gamma_j$ and some $k \in [i-1]$ with $\gamma_i \setminus \gamma_k = \{v\}$. A linear order satisfying the definition is called a *shelling order* on P.

5.2 Characterization of Cohen-Macaulay posets of dimension two

Let P be a poset. We say that P is an *antichain* if any two distinct elements of P are incomparable. For $p \in P$, *height* of p is the rank of the induced subposet of P which consists of all $q \in P$ with $q \leq p$.

Lemma 5.1. Let P be a strongly connected poset. Then P is an antichain or the induced subposet of P consisting of height i and height $i+1$ elements is connected for all $0 \leq i \leq \operatorname{rank}(P) - 1$.

Proof. Clearly, antichains are strongly connected. So we may assume that $\operatorname{rank}(P) \geq 1$. We proceed by contradiction. Fix an i with $0 \leq i \leq \operatorname{rank}(P) - 1$ such that the induced subposet Q of P consisting of height i and height $i+1$ elements is disconnected. Assume that Q is the disjoint union of two subposets Q_1 and Q_2. Since P is pure, every maximal chain of P contains an element of height i and an element of height $i+1$. Thus, Q_j contains at least one element of height i and at least one element of height $i+1$ for all $j = 1, 2$.

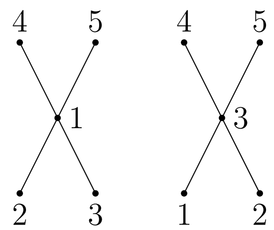
Let m_1 and m_2 be two maximal chains of P such that $m_1 \cap Q_2 = \emptyset$ and $m_2 \cap Q_1 = \emptyset$ (here, $\cap$ denotes the set theoretic intersection). Since P is strongly connected, there exists a sequence $\sigma_0, \sigma_1, \ldots, \sigma_k$ of maximal chains such that $m_1 = \sigma_0$, $m_2 = \sigma_k$, and $\sigma_j \cap \sigma_{j+1}$ is a chain of length $\operatorname{rank}(P) - 1$ for all $0 \leq j \leq k-1$. Let l be the smallest integer such that $\sigma_l \cap Q_2 \neq \emptyset$. Since $\sigma_{l-1} \cap Q_2 = \emptyset$ by the choice of l and $\sigma_{l-1} \cap \sigma_l$ is a chain of length $\operatorname{rank}(P) - 1$, we get that $\sigma_l \cap Q_2$ is a singleton, say $\{a\}$. First, assume that the height of a is $i+1$ in P. Let $b \in \sigma_l$ be such that $b \lessdot a$. Since P is pure, height of b is i in P. Also, $b \notin Q_2$ because $\sigma_l \cap Q_2 = \{a\}$; thus $b \in Q_1$ which is a contradiction. Similar argument follows when height of a is i in P. This completes the proof. □

Let τ be a permutation on $[n]$. Then τ gives a linear order on $[n]$ as follows: for $i, j \in [n]$, $i < j$ in τ if there exist $a, b \in [n]$ with $a < b$ in $\mathbb{N}$ such that $\tau(a) = i$ and $\tau(b) = j$. By abuse of terminology, we say that the permutation τ is a linear order. We write τ as $[\tau_1, \ldots, \tau_n]$ where $\tau_a := \tau(a)$ for all $a \in [n]$. Note that $\tau_a < \tau_b$ in τ if and only if τ_b is on the right side of τ_a in τ for any $a, b \in [n]$. For two permutations σ and τ, $P_{\sigma,\tau}$ denotes the poset that is the intersection of σ and τ. We start with an observation that a dimension two poset is isomorphic to a poset that is an intersection of the identity permutation and an another permutation.

Proposition 5.2. Let σ and τ be two permutations on $[n]$. Then, there exists a permutation π such that $P_{\sigma,\tau} \simeq P_{\mathrm{id},\pi}$, where id is the identity permutation.

Proof. Let $\pi = \sigma^{-1}\tau$. Define a map $\varphi : P_{\mathrm{id},\pi} \to P_{\sigma,\tau}$ by $\varphi(j) = \sigma(j)$. Clearly, φ is well-defined and it is a bijection. It suffices to show that $i < j$ in $P_{\mathrm{id},\pi}$ if and only if $\sigma(i) < \sigma(j)$ in $P_{\sigma,\tau}$.
Suppose that $i < j$ in $P_{\mathrm{id},\pi}$, i.e., $i < j$ in $\mathbb{N}$ and there exist $a, b \in [n]$ with $a < b$ such that $\pi(a) = i$ and $\pi(b) = j$. Clearly, $\sigma(i) < \sigma(j)$ in σ. Also, we have $\pi(a) = \sigma^{-1}\tau(a) = i$; thus $\tau(a) = \sigma(i)$. Similarly, $\tau(b) = \sigma(j)$. So, $\tau(a) = \sigma(i) < \sigma(j) = \tau(b)$ in τ. Hence $\sigma(i) < \sigma(j)$ in $P_{\sigma,\tau}$.
Conversely, suppose that $\sigma(i) < \sigma(j)$ in $P_{\sigma,\tau}$, i.e., $i < j$ in $\mathbb{N}$ and there exist $a, b \in [n]$ with $a < b$ such that $\tau(a) = \sigma(i)$ and $\tau(b) = \sigma(j)$. Therefore, $\sigma^{-1}\tau(a) = i$ and $\sigma^{-1}\tau(b) = j$. So $i < j$ in π. Hence, $i < j$ in $P_{\mathrm{id},\pi}$. □

Example 5.3. We illustrate the above proposition now. Let $\sigma = [2, 3, 1, 4, 5]$ and $\tau = [3, 2, 1, 5, 4]$ be two permutations. Then, $P_{\sigma,\tau}$ is as shown in Figure 5.2a. Let $\pi = \sigma^{-1}\tau$. Note that $\pi = [2, 1, 3, 5, 4]$ and $P_{\mathrm{id},\pi}$ is as shown in Figure 5.2b. Also, it is immediate that $j \mapsto \sigma(j)$ is an isomorphism from $P_{\mathrm{id},\pi}$ to $P_{\sigma,\tau}$.



(a) $P_{\sigma,\tau}$ (b) $P_{\mathrm{id},\pi}$

Figure 5.2: Isomorphism of two posets.

Definition 5.4. Let P be a poset of dimension two. By relabeling, we may assume that P is a poset on the set $[n]$. So P can be written as an intersection of two permutations, say σ and τ. By Proposition 5.2, there exists a permutation π such that $P \simeq P_{\mathrm{id},\pi}$, where $P_{\mathrm{id},\pi}$ is the poset that is the intersection of the identity permutation and π. We denote $P_{\mathrm{id},\pi}$ by P_π.

The following idea is motivated by [33]. Let P_π be a poset as defined in Definition 5.4. For $0 \leq i \leq \mathrm{rank}(P_\pi)$, let P_i be the set of all height i elements of P_π. For all i, define a linear order $<_i$ on P_i as following:

$$x <_i y \text{ if and only if } x > y \text{ in } \mathbb{N}.$$

For $x \in P_\pi$, let $U(x)$ be the set of all elements of P_π that covers x. If $x \in P_i$, then $y \in P_{i+1}$ for all $y \in U(x)$ if P_π is pure. For $x \in P_\pi$, define $x_{\min} := \min(U(x))$ and $x_{\max} := \max(U(x))$.
We make few observations which directly follows from the definition of P_π and of the linear order $<_i$.

Observation 5.5. Let P_π be a pure poset. We have
1. If $x <_i y$ in P_i, then π has the form $[\ldots, x, \ldots, y, \ldots]$ because x and y are incomparable in P_π and $x > y$ in $\mathbb{N}$.
2. If $y \in U(x)$ for some $x, y \in P_\pi$, then $x < y$ in $\mathbb{N}$. Also, y is on the right side of x in π.
3. If $x_{\min} <_i y <_i x_{\max}$ for some $x \in P_{i-1}$ and $y \in P_i$, then $y \in U(x)$. In fact, by (1) and (2), π has the form $[\ldots, x, \ldots, x_{\min}, \ldots, y, \ldots, x_{\max}, \ldots]$. Also, note that $x < x_{\max}$ in $\mathbb{N}$ because $x_{\max} \in U(x)$; thus $x < y$ in $\mathbb{N}$. Hence, $y \in U(x)$.

□

Lemma 5.6. Let P_π be a poset that satisfies (4) of the Theorem 5.7. Let $[x, y]$ be an interval in P_π such that $\mathrm{ht}(x) = i$, $\mathrm{ht}(y) = j$ and $j - i \geq 2$. Let $x = y_i \lessdot y_{i+1} \lessdot \cdots \lessdot y_j = y$ be a chain in $[x, y]$ such that for all k with $i < k < j$ there exists an $x_k \in [x, y]$, $x_k \lessdot_k y_k$. Then, there exists some integer k', $i < k' < j$ such that $y_{k'-1} \lessdot x_{k'} \lessdot y_{k'+1}$.

Proof. First, we claim the following: if $x \lessdot_l y$ in P_l for some $x, y \in P_l$ and $0 \leq l \leq \mathrm{rank}(P_\pi) - 1$, then $y_{\min} \leq_{l+1} x_{\max}$ in P_{l+1}.
Assume that the claim holds. If $j - i = 2$, then we can take $k' = i + 1$. Now assume that $j - i > 2$. Consider x_{i+1}. If $x_{i+1} \lessdot y_{i+2}$, then we can take $k' = i + 1$. Otherwise, if $x_{i+1} \not\lessdot y_{i+2}$, then $y_{i+1} \lessdot x_{i+2}$ by the claim. Now, consider x_{i+2}. If $x_{i+2} \lessdot y_{i+3}$, then we can take $k' = i + 2$. Otherwise if $x_{i+2} \not\lessdot y_{i+3}$, then $y_{i+2} \lessdot x_{i+3}$ by the claim. Proceeding in this way and using the case $j - i = 2$, we find the desired $x_{k'}$, which completes the proof.

We now prove the claim we made. On the contrary, suppose that there exists a $l \in \{0, 1, \ldots, \text{rank}(P_\pi) - 1\}$ and $x, y \in P_l$ such that $x \lessdot_l y$ in P_l and $x_{\max} <_{l+1} y_{\min}$ in P_{l+1}. We show that the induced subposet Q of P_π consisting P_l and P_{l+1} is disconnected.
Define

$$Q_1 = \{p \in P_\pi : \text{either } p \in P_l \text{ and } p \leq_l x \text{ or } p \in P_{l+1} \text{ and } p <_{l+1} y_{\min}\}$$

and

$$Q_2 = \{p \in P_\pi : \text{either } p \in P_l \text{ and } y \leq_l p \text{ or } p \in P_{l+1} \text{ and } y_{\min} \leq_{l+1} p\}.$$

We show that Q is the disjoint union of the subposets Q_1 and Q_2. Suppose that there exists an edge between Q_1 and Q_2. So, either there exists an $x' <_l x$ with $y_{\min} \leq_{l+1} x'_{\max}$ or there exists an $y' \in P_l$ with $y <_l y'$ and $y'_{\min} <_{l+1} y_{\min}$. We consider both cases separately:
(i) There exists an $x' <_l x$ with $y_{\min} \leq_{l+1} x'_{\max}$. Then, $x < x'$, $x'_{\max} \leq y_{\min}$ in $\mathbb{N}$. Using Observation 5.5, we get that π has the form

$$[\ldots, x', \ldots, x, \ldots, x_{\max}, \ldots, y_{\min}, \ldots, x'_{\max}, \ldots],$$

in fact x is on the right side of x' in π by (1) of the Observation 5.5, $x_{\max}$ is on the right side of x in π by (2) of the Observation 5.5, and the position of $x_{\max}, y_{\min}, x'_{\max}$ is by (1) of the Observation 5.5. By (2) of the Observation 5.5, $x' < x'_{\max}$ in $\mathbb{N}$. Since $x < x' < x'_{\max}$ in $\mathbb{N}$ and $x'_{\max}$ is on the right side of x in π, we get that $x'_{\max} \in U(x)$. Which is a contradiction.
(ii) There exists an $y' \in P_l$ with $y <_l y'$ and $y'_{\min} <_{l+1} y_{\min}$. Then, $y' < y$, $y_{\min} < y'_{\min}$ in $\mathbb{N}$. Using Observation 5.5, π has the form $[\ldots, y, \ldots, y', \ldots, y'_{\min}, \ldots, y_{\min}, \ldots]$. Note that $y'_{\min}$ is on the right side of y in π. Also, note that $y < y'_{\min}$ in $\mathbb{N}$ because $y \leq y_{\min} \leq y'_{\min}$ in $\mathbb{N}$. Therefore, it follows that $y'_{\min}$ belongs to the set $U(y)$, which contradicts the previous statement. The proof of the claim is now finished. □

Here, we present the primary outcome of this chapter.

Theorem 5.7. Let P be a finite poset of dimension two. Then the following are equivalent:

1. P is shellable.

2. P is Cohen-Macaulay.

3. P is strongly connected.

4. P is an antichain or P is pure and the induced subposet of P consisting of height i and height $i+1$ elements is connected for all $0 \leq i \leq rank(P) - 1$.

Proof. (1) $\implies$ (2) follows by Theorem 1.39, and (2) $\implies$ (3) follows from Proposition 1.41. In Lemma 5.1 we prove a more general result then (3) $\implies$ (4). We now prove (4) $\implies$ (1) as below.

According to Definition 5.4, it is enough to demonstrate the outcome for the posets P_π, which are defined in Definition 5.4. Assume that P_π is a poset of rank r that satisfies (4). If $\pi = [n, n-1, \ldots, 1]$, then P_π is an antichain. So antichains have dimension two. It follows from the definition of the shellability that antichains are shellable. So we may assume that P_π is not an antichain.

Consider the permutation $\pi' = [0, \pi, n+1]$ on $n+2$ elements. Then, $P_{\pi'} = P_\pi \cup \{0, n+1\}$, where 0 and $n+1$ are the minimal and the maximal elements of $P_{\pi'}$ respectively. Note that $P_{\pi'}$ is a pure poset of rank $r+2$ and it satisfies the hypothesis of (4). Since every interval of a shellable poset is also shellable [3, Proposition 8.2], we may replace π by π'.

Let $C : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{r+2}$ and $C' : y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{r+2}$ be two maximal chains of P_π. Note that $x_0 = y_0 = 0$ and $x_{r+2} = y_{r+2} = n+1$. Let $j' = \max\{i \in [r+2] : x_i \neq y_i\}$. Define a linear order $<_E$ on the maximal chains of P_π as follows:

$$C <_E C' \text{ if and only if } x_{j'} <_{j'} y_{j'}.$$

Under the above notations, assume that $C <_E C'$. Let $i' = \max\{i < j' : x_i = y_i\}$. Then, $x_{i'} = y_{i'} \lessdot y_{i'+1} \lessdot \cdots \lessdot y_{j'+1} = x_{j'+1}$ is a maximal chain in $[x_{i'}, y_{j'+1}]$. We proceed in the following cases:

1. If $x_k <_k y_k$ for all $i' < k < j'+1$, then the maximal chain $x_{i'} = y_{i'} \lessdot y_{i'+1} \lessdot \cdots \lessdot y_{j'+1} = x_{j'+1}$ in the interval $[x_{i'}, y_{j'+1}]$ satisfies the hypothesis of Lemma 5.6, i.e., for all k, $i' < k < j'+1$ there exists a $z_k \in [x_{i'}, y_{j'+1}]$ such that $z_k \lessdot_k y_k$ because $<_k$ is linear order and $x_k <_k y_k$ for all k. So there exists a k', $i' < k' < j'+1$ such that $y_{k'-1} \lessdot z_{k'} \lessdot y_{k'+1}$ by Lemma 5.6. If we let

$$C'' : y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{k'-1} \lessdot z_{k'} \lessdot y_{k'+1} \lessdot \cdots \lessdot y_{r+2},$$

then $C'' <_E C'$, $y_{k'} \in C' \setminus C$ and $C' \setminus C'' = \{y_{k'}\}$.

2. There exists a k with $i' < k < j'+1$ such that $y_k <_k x_k$. Let $l = \max\{k : i' < k < j'+1 \text{ and } y_k <_k x_k\}$. First, we show that $y_l < x_{l+1}$ in P_π. By the choice of l, we have $x_{l+1} <_{l+1} y_{l+1}$; so $y_{l+1} < x_{l+1}$ in $\mathbb{N}$. Also, $y_l < y_{l+1}$ in P_π. Thus $y_l < y_{l+1} < x_{l+1}$ in $\mathbb{N}$. Under the given conditions, π has the form $[\ldots, y_l, \ldots, x_l, \ldots, x_{l+1}, \ldots, y_{l+1}, \ldots]$. Observe that x_{l+1} is on the right side of y_l in π. Therefore, $y_l < x_{l+1}$ in P_π.

Now consider the interval $[y_l, y_{j'+1}]$. Note that $j' + 1 - l \geq 2$ because $x_{j'} <_{j'} y_{j'}$.

Also, observe that for all $l < k < j' + 1$, $x_k \in [y_l, y_{j'+1}]$ because $y_l < x_{l+1}$ in P_π, and $x_k <_k y_k$ by the choice of l. Thus for all k, $l < k < j' + 1$ there exists a $z_k \in [y_l, y_{j'+1}]$ such that $z_k \lessdot_k y_k$. Thus, the maximal chain $y_l \lessdot y_{l+1} \lessdot \cdots \lessdot y_{j'+1} = x_{j'+1}$ in the interval $[y_l, y_{j'+1}]$ satisfies the hypothesis of Lemma 5.6. So there exists a k', $l < k' < j' + 1$ such that $y_{k'-1} \lessdot z_{k'} \lessdot y_{k'+1}$. Therefore, we can repeat the argument of (1) to complete the proof.
Therefore, $<_E$ is a shelling order on the maximal chains of P_π. This completes the proof. □

We see that the Theorem 5.7 helps us to characterize the Cohen-Macaulay permutation graphs. Let G be a graph on $\{1, 2, \ldots, n\}$. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. Let $I_G = (x_i x_j : \{i, j\}$ is an edge of $G)$ be the *edge ideal* of G. We say that G is *Cohen-Macaulay* if S/I_G is Cohen-Macaulay.
Now assume that G is a permutation graph. Then G is the intersection graph of the permutation diagram consisting of horizontal lines l_0 and l_1. Assume that l_0 and l_1 are labeled by the permutations π_0 and π_1 respectively. Let P be the poset that is the intersection of π_0 and π_1. Then G is the co-comparability graph of P by [21, Theorem 1], and the dimension of P is at most two. The Stanley-Reisner ideal of the order complex of P coincide with the edge ideal of G. Thus, by [44, Theorem 1], we get that P is Cohen-Macaulay if and only if G is Cohen-Macaulay. If the dimension of P is one, then P is a linear order; thus, I_G is the trivial ideal. Hence G is Cohen-Macaulay. When the dimension of P is two, we can use Theorem 5.7 to check whether G is Cohen-Macaulay. Consequently, G is Cohen-Macaulay over any field.

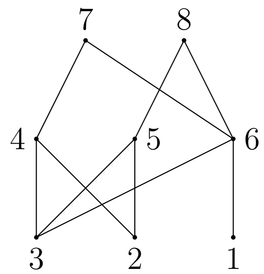


Figure 5.3: A dimension three poset.

It follows from the [2, Proposition 11.7] and Lemma 5.1 that the (4) of Theorem 5.7 is a necessary condition for a poset to be Cohen-Macaulay. In the following example, we show that it is not a sufficient condition. More precisely, we show that (4) $\implies$ (2) and (4) $\implies$ (1) of the Theorem 5.7 may not be true when the dimension of the poset is ≥ 3.

Example 5.8. Consider the poset P as shown in Figure 5.3. A SageMath [53] computation shows that the dimension of P is three. In fact, P is the intersection of the following permutations: $\tau_1 = [1, 3, 6, 2, 4, 7, 5, 8]$, $\tau_2 = [2, 3, 4, 5, 1, 6, 7, 8]$ and $\tau_3 = [3, 1, 2, 6, 5, 8, 4, 7]$.
It is clear that P satisfies the hypothesis of (4) of the Theorem 5.7. Note that $\text{link}(\Delta(P), \{2\}) = \{\{4, 7\}, \{5, 8\}\}$ which is disconnected. Since the dimension, $\dim(\text{link}(\Delta(P), \{2\})) = 1$ and $\widetilde{H}_0(\text{link}(\Delta(P), \{2\}), K) = K$, P is not Cohen-Macaulay. Hence, P is not shellable.

All examples we have computed suggest that shellability and Cohen-Macaulayness coincide for dimension three posets. This, in conjunction with Theorem 5.7, prompts us to pose the following inquiry: What is the smallest value of $d \in \mathbb{N}$ for which there exists a Cohen-Macaulay poset of dimension d that is not shellable?

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