ON DYNAMICS OF ASYMPTOTICALLY MINIMAL POLYNOMIALS

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ABSTRACT

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In this thesis, we study dynamical properities of asymptotically extremal polynomials associated with a non-polar planar compact set E. In particular, we prove that if the zeros of such polynomials are uniformly bounded then their Brolin measures, w_n's, converge weakly to the equilibrium measure of E. For this, we observe that $\{w_n\}_n$ is sequentially pre-compact with respect to the weak*-topology and if ν is the weak* limit, then the support of this measure contained in the support of the equilibrium measure of E.

Approximating compact sets by fractals is a fruitful technique and used for different problems in complex analysis such as the universal dimension spectrum for harmonic measures. Another aspect of research in this context is approximating a given planar compact set by polynomial filled Julia sets (respectively Julia sets) with respect to the Hausdorff topology. In the second part of this thesis, we consider the problem of classifying all possible limit sets of a sequence of filled Julia sets of asymptotically minimal polynomials. First, we observe that the sequence of filled Julia sets (respectively Julia sets) of asymptotically minimal polynomials may not converge in the Hausdorff topology. On the other hand, we prove that if E is regular in the sense of Dirichlet problem and the zeros of such polynomials are sufficiently close to E then the filled Julia sets converge to the polynomial convex hull of E in the Klimek topology. Moreover, we prove that for any Hausdorff-limit set of filled Julia sets, the polynomial convex hull of this limit set coincide with the polynomial convex hull of E. Finally, we discuss possible generalizations of these results to multi-dimensional setting.

ÖZET

ASİMPTOTİK OLARAK MİNİMAL POLİNOMLARIN DİNAMİĞİ ÜZERİNE

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Bu tezde, polar olmayan düzlemsel bir kompakt küme olan E ile ilişkili asimptotik ekstremal polinomların dinamik özellikleri incelenmiştir. Özellikle, bu polinom ailesinin sıfırlarını içeren küme düzgün sınırlıysa, Brolin ölçüleri w_n'lerin, E'nin denge ölçüsüne zayıf bir şekilde yakınsadığı kanıtlanmıştır. Bunun için, $\{w_n\}_n$ dizisinin zayıf-yıldız topolojiye göre dizisel prekompakt olduğunu ve eğer ν zayıf-yıldız limit ise bu ölçünün dayanağının E'nin denge ölçüsünün dayanağı tarafından içerildiği gösterilmiştir.

Kompakt kümelere fraktallarla yaklaşmak verimli bir tekniktir ve harmonik ölçümler için evrensel boyut spektrumu gibi karmaşık analizdeki farklı problemler için kullanılır. Bu bağlamda araştırmanın başka bir yönü, belirli bir düzlemsel kompakt kümeye Hausdorff topolojisine göre doldurulmuş Julia kümeleri (sırasıyla Julia kümeleri) tarafından yaklaşmaktır. Bu tezin ikinci bölümünde, asimptotik olarak minimal polinomların doldurulmuş Julia kümeleri dizisinin, tüm olası limit kümelerini sınıflandırma problemi ele alınmıştır. İlk olarak, asimptotik olarak minimal polinomların doldurulmuş Julia kümeleri dizisinin (sırasıyla Julia kümeleri) Hausdorff topolojisinde yakınsamayabileceği gözlemlenmiştir. Öte yandan, eğer E, Dirichlet problemi anlamında düzenliyse ve bu polinomların sıfırları E'ye yeterince yakınsa, doldurulmuş Julia kümelerinin, Klimek topolojide E'nin polinom dışbükey örtüsüne yakınsadığı kanıtlanmıştır. Ayrıca, doldurulmuş Julia kümelerinin herhangi bir Hausdorff-limit kümesi için, bu kümenin polinom dışbükey örtüsü ile E'nin polinom dışbükey örtüsünün eşit olduğu ispatlanmıştır. Son olarak, bu sonuçların olası genelleştirmelerini çok boyutlu karmaşık uzay için tartışılmıştır.

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To my family

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1. INTRODUCTION

In this thesis, we study the relation between the potential theoretic equilibrium measure and the dynamically defined Brolin measures of asymptotically minimal polynomials on a compact non-polar subset of $\mathbb{C}$. We prove that if the zeros of such polynomials are uniformly bounded then their Brolin measures converge weakly to the equilibrium measure of a compact subset. In addition, we approximate regular compact subsets of $\mathbb{C}$ by using filled Julia sets of asymptotically minimal polynomials.

Let $E \subset \mathbb{C}$ be a compact set with positive logarithmic capacity $\text{cap}(E)$. We denote by Ω the unbounded component of $\widehat{\mathbb{C}} \setminus E$ so that $\text{Pc}(E) := \mathbb{C} \setminus \Omega$ is the polynomial convex hull of E. We also let ω_E be the equilibrium measure of E that is the unique maximizer of the logarithmic energy among all Borel probability measures supported on E.

A Borel measure belongs to the class **Reg** if the n^{th} root of the leading coefficient of the n^{th} orthonormal polynomial is asymptotic to logarithmic capacity of the support of the measure as the degree n grows to infinity (see §2.1 for precise definition). For example, the equilibrium measure of a regular compact set is of class **Reg**. In the first part of this thesis, we focus on dynamical properties of asymptotically minimal polynomials associated with planar compact sets and regular measures. In this context, Christiansen et. al. studied the limit behavior of the filled Julia sets K_{p_n} of orthogonal polynomials with respect to the Hausdorff topology [Christiansen, Henriksen, Pedersen & Petersen (2019)]. Later, in [Petersen & Uhre (2021)], Petersen and Uhre showed that Brolin measures of orthogonal polynomials associated with a **Reg** class measure converge in weak* topology to the equilibrium measure of the set. More recently, this result was generalized to the case of normalized Chebyshev polynomials in [Christiansen, Henriksen, Pedersen & Petersen (2021)]. By adapting the techniques of [Christiansen et al. (2021); Petersen & Uhre (2021); Saff & Totik (1997)] to our setting, we obtain another generalization for asymptotically minimal polynomials:

Theorem 1.0.1. *Let $E \subset \mathbb{C}$ be a compact set with positive logarithmic capacity and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials on E. Assume that the zeros of $p_n(z)$ are uniformly bounded for $n \in \mathbb{N}$. Then, the Brolin measures $\omega_{J_{p_n}} \to \omega_E$ in the weak* topology.*

In the second part of this thesis, we consider the problem that is classifying all possible geometric limit sets of a sequence of filled Julia sets of asymptotically minimal polynomials. Julia sets of polynomials have been studied in many aspects. One line of research is approximating compact sets by fractals which is a fruitful technique in the study of important problems in complex analysis, such as the universal dimension spectrum for harmonic measure (cf. [Binder, Makarov & Smirnov (2003)], [Carleson & Jones (1992)]). Another aspect of research in this context is approximating a given planar compact set by polynomial filled Julia sets (respectively Julia sets) with respect to the Hausdorff topology (cf. [Bialas-Ciez, Kosek & Stawiska (2018); Bishop & Pilgrim (2015); Lindsey (2015); Lindsey & Younsi (2019)]). By using a similar method given in [Lindsey & Younsi (2019)], [Bialas-Ciez et al. (2018)], we can approximate a polynomially convex compact non-polar subset of $\mathbb{C}$ by using filled Julia sets of polynomials which are defined by using asymptotically minimal polynomials with respect to the Hausdorff topology. However, we observe that the sequence of filled Julia sets (respectively Julia sets) of asymptotically minimal polynomials may not converge in the Hausdorff topology (see Example 4.2.1). On the other hand, under the stronger assumption on zeros of asymptotically minimal polynomials associated with a regular compact set we show that the filled Julia sets of the extremal polynomials converge with respect to a natural metric Γ, called the Klimek metric [Klimek (1995)], that is defined in terms of Green's functions of the corresponding regular polynomially convex compact sets in $\mathbb{C}$ (see (2.46) for the definition). More precisely, we prove the following result:

Theorem 1.0.2. *Let E be a regular compact set and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials on E. Assume that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the zeros of p_n are contained in ε-dilation of $\mathrm{Pc}(E)$. Then $\Gamma(K_{p_n}, \mathrm{Pc}(E)) \to 0$ as $n \to \infty$.*

In the next proposition, for any Hausdorff limit K_∞ of the filled Julia sets $\{K_n\}_n$, we also prove:

Proposition 1.0.3. *Let E be a regular compact set in $\mathbb{C}$ and $\{p_n\}_n$ be an asymptotically minimal sequence. Assume that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that the zeros of p_n's are contained in $\mathrm{Pc}(E)^\varepsilon$ for all $n \geq N$. Let $K_\infty \subset \mathbb{C}$ be a compact set that is a limit point of $\{K_n\}_n$ with respect to Hausdorff metric then $\mathrm{Pc}(K_\infty) = \mathrm{Pc}(E)$.*

Lastly, we focus on the Hausdorff limit of the Julia sets of asymptotically minimal polynomials. We denote the outer boundary of E by $J_E := \partial\Omega$. We also denote the exceptional set (see [Ransford (1995)] for definition) for the Green's function g_Ω by F_E. This means that $F_E = \{z \in E : g_\Omega(z) > 0\}$. We prove that the limit of Julia sets of asymptotically minimal polynomials contain the regular points of the outer boundary:

Theorem 1.0.4. *Let E be a compact non-polar subset of $\mathbb{C}$ and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials whose zeros are contained in* $\mathrm{Pc}(E)$. *Then,*

$$\overline{J \setminus F} \subseteq \liminf_{n\to\infty} J_n. \tag{1.1}$$

In particular, if J is regular, then

$$J \subseteq \liminf_{n\to\infty} J_n. \tag{1.2}$$

2. Preliminaries

In this chapter, we will give the basic concepts of potential theory and holomorphic dynamics. Also, we will introduce asymptotically minimal polynomials and some important properties of these polynomials. In addition to these, we will give the definitions of Hausdorff and Klimek metrics and some relations between them.

2.1 Potential Theory

Throughout this section, we will follow [Ransford (1995)] for the basic concepts of potential theory in the complex plane. In the first part of this thesis, we will work on the equilibrium measure of compact sets. This measure is the unique one that has maximal energy among all Borel probability measures supported on a non-polar compact set. In order to define it, we need to define logarithmic potential of a measure. Let E be a compact subset of $\mathbb{C}$ and μ be a finite Borel measure with support E. The *logarithmic potential* of μ is a function $U_\mu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$U_\mu(z) = \int \log|z-w| d\mu(w) \quad (z \in \mathbb{C}). \tag{2.1}$$

Recall that a function $f : \Omega \to [-\infty, \infty)$ is *subharmonic* on a domain $\Omega \in \mathbb{C}$ if it is upper semicontinuous and for all $z \in \Omega$ and for each $\epsilon > 0$ such that the closed disk $\overline{D_\epsilon(z)} := \{w : |z-w| \leq \epsilon\} \subset \Omega$, satisfies

$$f(z) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z + \epsilon e^{i\theta}) d\theta. \tag{2.2}$$

It is known that the logarithmic potential of finite Borel measure is subharmonic:

Theorem 2.1.1 (Ransford (1995))**.** *U_μ is subharmonic on $\mathbb{C}$, and harmonic on*

$\mathbb{C}\setminus(\text{supp }\mu)$. *Moreover,*

$$\limsup_{z\to\infty} |z(U_\mu(z) - \mu(\mathbb{C})\log|z|)| < \infty. \tag{2.3}$$

The logarithmic energy of μ is defined by

$$I(\mu) := \int\int \log|z-w| d\mu(w) d\mu(z) = \int U_\mu(z) d\mu(z). \tag{2.4}$$

It is possible that $I(\mu) = -\infty$. If E is singleton, then $I(\mu) = -\infty$ for all finite Borel measure $\mu \neq 0$. A subset E of $\mathbb{C}$ is called *polar* if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which $\text{supp}(\mu)$ is a compact subset of E. We say that a property holds *nearly everywhere* on a subset E of $\mathbb{C}$ if it holds everywhere on $E\setminus F$ for some Borel polar set F. Denote by $\mathcal{M}(E)$ the collection of all Borel probability measures on E. If there exists $\omega \in \mathcal{M}(E)$ such that

$$I(\omega) = \sup_{\mu\in\mathcal{M}(E)} I(\mu), \tag{2.5}$$

then ω is called *an equilibrium measure* for E.

Remark 2.1.1. *In this thesis, we define the equilibrium measure of E as a maximizer of the logarithmic energy among all Borel probability measures on E. In some books, it is defined as the minimizer of the logarithmic energy among all Borel probability measures on E by taking $-U_\mu$ for the potential of μ instead of U_μ (see [Tsuji (1959)], [Saff & Totik (1997)]).*

Example 2.1.1. *Let E be the closed disk $\overline{D_\epsilon(a)}$ or the circle $S_\epsilon(a) := \{w : |a-w| = \epsilon\}$. Then, we have $dw_E = d\theta/2\pi r$ where $d\theta$ denotes arc measure on $S_\epsilon(a)$ and*

$$U_{w_E} = \begin{cases} \log\epsilon & \text{if } |z-a| \le \epsilon \\ \log|z-a| & \text{if } |z-a| > \epsilon \end{cases} \tag{2.6}$$

If E is a compact subset of $\mathbb{C}$, then an equilibrium measure for E always exists:

Theorem 2.1.2 (Ransford (1995))**.** *Every compact set E has an equilibrium measure. Moreover, if E is a non-polar compact set, then this equilibrium measure is unique and* $\text{supp}(\omega_E)$ *is subset of the boundary of the unbounded component of $\widehat{\mathbb{C}}\setminus E$.*

If E is non-polar compact set, then the equilibrium measure is denoted by w_E. In the next chapter, we will show the convergence of the equilibrium measures in the weak* topology. A sequence $\{\mu_n\}_{n\geq 1}$ in $\mathcal{M}(E)$ is *weak*-convergent* to $\mu \in \mathcal{M}(E)$, and we write $\mu_n \xrightarrow{w^*} \mu$, if

$$\int_E \phi d\mu_n \to \int_E \phi d\mu \tag{2.7}$$

for each continuous function ϕ on E. We shall use the following theorem in the next chapters:

Theorem 2.1.3 (Helly's Selection Theorem)**.** *If $\{\mu_n\}_n$ is a sequence of complex valued measures on a compact set E with bounded total mass, then $\{\mu_n\}_n$ has a weak*-convergent subsequence.*

If the sequence $\{\mu_n\}_n$ converges the measure μ in weak* topology, then limit of the logarithmic energies of μ_n's are bounded by the logarithmic energy of μ:

Lemma 2.1.4 (Ransford (1995))**.** *If $\mu_n \xrightarrow{w^*} \mu$ in $\mathcal{M}(E)$, then*

$$\limsup_{n\to\infty} I(\mu_n) \leq I(\mu).$$

Now, we will define a very important and useful concept of potential theory: Green's function. Let Ω be the unbounded component of $\widehat{\mathbb{C}} \setminus E$. The *Green's function* g_Ω for Ω with a pole at ∞ is the non-negative subharmonic function with the properties

(i) g_Ω is harmonic on $\Omega \setminus \{\infty\}$ and bounded outside each neighborhood of ∞;

(ii) $g_\Omega(z) = \log|z| + O(1)$ as $z \to \infty$;

(iii) $g_\Omega(z) \to 0$ as $z \to \xi$, for nearly everywhere $\xi \in \partial\Omega$.

One can see that the Green's function g_Ω satisfies

$$g_\Omega(z) = U_{\omega_E}(z) - I(\omega_E) = U_{\omega_E}(z) - \log \text{cap}(E) \tag{2.8}$$

where $\text{cap}(E) = e^{I(\omega_E)}$ is the *logarithmic capacity* of E. If $\partial\Omega$ is non-polar, then there exists a unique Green's function g_Ω for Ω [Ransford (1995)].

Assume that $\text{cap}(E) > 0$. By (2.8), we see that the continuity points of g_Ω and U_{w_E} coincide. A point z on $\partial\Omega$ is called a *regular (boundary) point* of Ω if g_Ω is continuous at z; otherwise it is called *irregular*. We can observe that $z \in \partial\Omega$ is regular if and only if $g_\Omega(z) = 0$ which is equivalent to

$$U_{w_E} = \log \text{cap}(E).$$

In particular, the set of irregular points has zero capacity [Saff & Totik (1997)]. If every $z \in \partial\Omega$ is regular, then E is called a *regular* domain.

2.2 Asymptotically Minimal Polynomials

Asymptotically minimal polynomials are defined via regular measures. Given a Borel measure μ with support $S_\mu := \text{supp}(\mu) \subset \mathbb{C}$ one can define the inner product

$$\langle f, g \rangle := \int_{\mathbb{C}} f\overline{g} d\mu$$

on the space of polynomials $\mathcal{P}_n$. Then one can find uniquely defined orthonormal polynomials

$$P_n^\mu(z) = \gamma_n(\mu) z^n + \cdots, \text{ where } \gamma_n(\mu) > 0 \text{ and } n \in \mathbb{N}.$$

We say that μ is *regular*, denoted by $\mu \in \textbf{Reg}$, if

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\text{cap}(S_\mu)}. \tag{2.9}$$

Let $E \in \mathbb{C}$ be a non-polar compact set and

$$p_n(z) = z^n + \text{ lower order terms.}$$

Then we have that $\text{cap}(E)^n \leq \|p_n\|_E$, see [Saff & Totik (1997), Theorem I.3.6]. So, a sequence $\{p_n\}_n$ of monic polynomials of degree n is called asymptotically minimal if

$$\limsup_{n\to\infty} \|p_n\|_E^{1/n} \leq \text{cap}(E).$$

Now, following [Dauvergne (2021)], we give the following definition:

Definition 2.2.1. *We say that a sequence of polynomials* $p_n(z) = \sum_{j=0}^n a_{n,j} z^j$ *with* $a_{n,n} \neq 0$ *is asymptotically minimal on a compact set* $E \subset \mathbb{C}$ *if there exists a regular measure* $\tau \in$ ***Reg*** *with* $\text{supp}(\tau) = E$ *and a constant* $p \in (0, \infty]$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \log |a_{n,n}| = -\log \text{cap}(E) \tag{2.10}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \| p_n \|_{L^p(\tau)} = 0. \tag{2.11}$$

Note that for the case $p=\infty$ we do not need the reference measure τ in the definition. The following result is well known (see [Dauvergne (2021)] and references therein):

Proposition 2.2.1. *Let E be a compact non-polar set and $\tau \in \boldsymbol{Reg}$ be a finite measure supported on E. A sequence $p_n(z)=a_{n,n}z^n+\cdots$ is asymptotically minimal on E for $p=\infty$ if and only if it is asymptotically minimal with respect to τ and for some (equivalently for all) $p\in(0,\infty)$.*

Proof. It is known that every measure $\tau \in \mathbf{Reg}$ on E satisfies Nikolskiĭ type inequality (cf. [Stahl & Totik (1992), Chapter 3]) for some (equivalently for all) $p\in(0,\infty)$. Namely, there exist constants $M_n>0$ such that

$$\limsup_{n\to\infty} M_n^{1/n}=1 \tag{2.12}$$

and for all polynomials p_n whose degree is at most n we have

$$\|p_n\|_E \le M_n\|p_n\|_{L^p(\tau)}. \tag{2.13}$$

Recall that by minimality we always have

$$\mathrm{cap}(E)^n \le \|\frac{p_n}{a_{n,n}}\|_E. \tag{2.14}$$

Assume $\{p_n\}_n$ is asymptotically minimal on E for some $p\in(0,\infty)$. Then, by (2.13) we have

$$\mathrm{cap}(E)\le \limsup_{n\to\infty}\frac{\|p_n\|_E^{1/n}}{|a_{n,n}|^{1/n}} \le \lim_{n\to\infty}\frac{(M_n\|p_n\|_{L^p(\tau)})^{1/n}}{|a_{n,n}|^{1/n}}=\mathrm{cap}(E). \tag{2.15}$$

This gives us

$$\lim_{n\to\infty}\frac{1}{n}\log\|p_n\|_E=0 \tag{2.16}$$

and hence $\{p_n\}_n$ be asymptotically minimal on E for $p=\infty$.

In order to prove reverse direction observe that

$$\|p_n\|_{L^p(\tau)}\le \|p_n\|_E\sqrt{\tau(E)}. \tag{2.17}$$

If $\{p_n\}_n$ is asymptotically minimal with $p=\infty$ then

$$\limsup_{n\to\infty}\frac{\|p_n\|_{L^p(\tau)}^{1/n}}{|a_{n,n}|^{1/n}}\le \mathrm{cap}(E). \tag{2.18}$$

Moreover, by (2.13) we always have

$$\operatorname{cap}(E) \le \liminf_{n\to\infty} \frac{\|p_n\|_{L^p(\tau)}^{1/n}}{|a_{n,n}|^{1/n}}.$$

Hence, $\{p_n\}_n$ is asymptotically minimal on E with respect to τ and $p \in (0,\infty)$. $\square$

Remark 2.2.1. *Our definition of asymptotic minimality is not standart (cf. [Saff & Totik (1997)]). More precisely, if a sequence of polynomials $p_n(z)$ is asymptotically minimal in the sense of Definition 2.2.1 then $\frac{1}{n}\log(\|p_n\|_{L^p(\tau)}/|a_{n,n}|)$ converges to the minimal value $\log\operatorname{cap}(E)$. However, solely this last condition is not sufficient for our purposes (cf. Lemma 3.1.5). For example, let $p_n(z) = 2^{n^2}z^n$. Then $p_n(z)/a_{n,n} = z^n$ is minimal on the unit circle but the filled Julia sets decrease to the point at the origin. Thus, we require both the convergence of n^{th} roots of the leading coefficients and the corresponding norms.*

Now, we review natural classes of polynomials that fit into the framework of Theorem 1.0.1.

Example 2.2.1 (L^p-minimal polynomials)**.** *Let $E \subset \mathbb{C}$ be a non-polar compact set and $\tau \in$ **Reg**. For $p \in (1,\infty]$ there exist unique monic polynomials p_n with minimal $L^p(\tau)$-norm satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log \|p_n\|_{L^p(\tau)} = \log cap(E). \tag{2.19}$$

Then the sequence $\{\frac{p_n}{\|p_n\|_{L^p(\tau)}}\}_n$ is asymptotically minimal on E [Stahl & Totik (1992)]. We remark that the case $p = \infty$ corresponds to normalized Chebyshev polynomials.

Example 2.2.2 (Fekete Polynomials)**.** *Let E be a non-polar compact set. For $n \geq 2$, we denote n-tuple of points $w_1, \ldots, w_n \in E$ such that the supremum $\sup\{\prod_{j<k}|z_j - z_k|^{\frac{2}{n(n-1)}}\}$ is attained at these points. The points $F_n := (w_1, \cdots, w_n)$ are called Fekete points of order n. We remark that Fekete points are not unique in general but do exist by compactness of E. The Fekete polynomial associated with F_n is given by*

$$q_n(z) = \prod_{j=1}^{n} (z - w_j). \tag{2.20}$$

By [Ransford (1995)]

$$\lim_{n\to\infty} \|q_n\|_E^{\frac{1}{n}} = \operatorname{cap}(E). \tag{2.21}$$

Thus, the normalized sequence $\{p_n = \dfrac{q_n}{\|q_n\|_E}\}_n$ *is asymptotically minimal on* E*. Note that by construction all the zeros of* p_n *lie in* E*.*

Example 2.2.3 (Faber Polynomials)**.** *Let* E *be a non-polar compact set such that the unbounded component* Ω *of* $\widehat{\mathbb{C}} \setminus E$ *is simply connected. Let* $\phi : \Omega \to \widehat{\mathbb{C}} \setminus \{z : \| z \| \leq 1\}$ *be the (unique) conformal map such that* $\phi(\infty) = \infty$ *and* $\phi'(\infty) > 0$*. It is well known that*

$$\phi(z) = \frac{z}{\operatorname{cap}(E)} + a_0 + \frac{a_1}{z} + \cdots . \tag{2.22}$$

The Faber polynomial F_n *of degree* n *is defined by the equation*

$$F_n(z) = \phi(z)^n + \mathcal{O}(\frac{1}{z}) \quad z \to \infty. \tag{2.23}$$

These polynomials satify equation (2.21) (see [Levenberg & Wielonsky (2020)]). Thus, the normalized sequence $\{p_n = \dfrac{F_n}{\|F_n\|_E}\}_n$ *is asymptotically minimal on* E*. Furthermore, if* E *is convex then all zeros of* p_n *lie in the interior of* E *(see [Kövari & Pommerenke (1967), Theorem 2]).*

Example 2.2.4 (Polynomials with bounded coefficients)**.** *Let*

$$p_n(z) = z^n + a_{n-1}^n z^{n-1} + \cdots + a_0^n$$

where $|a_j^n| < M$ *for some fixed* $M > 0$*. Then it is easy to see that* $\{p_n\}_n$ *is asymptotically minimal on the unit circle* $S^1 := \{z \in \mathbb{C} : |z| = 1\}$*. Moreover, Cauchy bounds imply that if* $p_n(z) = 0$ *then* $|z| \leq 1 + \max_{0 \leq j \leq n-1} |a_j^n|$*. In particular, zeros of* p_n *are uniformly bounded and contained in the disc* $D(0, M+1)$*.*

In all of these examples, we consider the polynomials with bounded zeros. Howeover, there exist asymptotically minimal polynomials with unbounded zeros as in the following example:

Example 2.2.5. *Let* $c_n \in \mathbb{C}$ *be a sequence such that* $|c_n| \to \infty$ *and* $\frac{1}{n} \log |c_n| \to 0$ *as* $n \to \infty$*. Let also*

$$p_n(z) = z^{n-1}(z - c_n) \quad \textit{for} \quad n \geq 1. \tag{2.24}$$

Then it is easy to see that $\{p_n\}_{n \geq 1}$ *is asymptotically minimal on the unit circle* S^1 *with respect to the equilibrium measure* $\frac{d\theta}{2\pi}$*.*

Let E be a compact subset of $\mathbb{C}$ and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials on E. We denote the *counting measures* of zeros of p_n by

$$\mu_n := \frac{1}{n} \sum_{p_n(z)=0} \delta_z \tag{2.25}$$

where δ_z denotes the unit mass at z. There exist some relations between weak* limit of these measures and the equilibrium measure for E. Moreover, for some special compact sets, these two limits coincide:

Theorem 2.2.2 (Saff & Totik (1997)). *Let E be a compact non-polar subset of $\mathbb{C}$, $\{q_n\}_n$ be a sequence of monic polynomials such that*

$$\lim_{n\to\infty} \|q_n\|_E^{\frac{1}{n}} = \operatorname{cap}(E) \tag{2.26}$$

and μ_n be the counting measure of zeros of q_n. Then $\mu_n \to w_E$ as $n \to \infty$ in the weak-topology if and only if for every bounded component R of $\mathbb{C} \setminus \operatorname{supp}(w_E)$ and every subsequences $\mathcal{N}$ of the natural numbers there is a $z_0 \in R$ and a subsequence $\mathcal{N}_1 \subset \mathcal{N}$ such that*

$$\lim_{n\to\infty, n\in\mathcal{N}_1} |q_n(z_0)|^{\frac{1}{n}} = \exp(U_{w_E}(z_0)). \tag{2.27}$$

By using Theorem 2.2.2 and the definition of asymptotically minimal polynomials, we observe the following:

Corollary 2.2.3. *Let E be a compact subset of $\mathbb{C}$ and $\{p_n\}_n$ be asymptotically minimal on E and μ_n be the counting measure of zeros of p_n. Then $\mu_n \to w_E$ as $n \to \infty$ in the weak*-topology if and only if for every bounded component R of $\mathbb{C} \setminus \operatorname{supp}(w_E)$ and every subsequences $\mathcal{N}$ of the natural numbers there is a $z_0 \in R$ and a subsequence $\mathcal{N}_1 \subset \mathcal{N}$ such that*

$$\lim_{n\to\infty, n\in\mathcal{N}_1} |q_n(z_0)|^{\frac{1}{n}} = e^{g_\Omega(z_0)}. \tag{2.28}$$

Corollary 2.2.3 states that we need condition (ii) to have the equilibrium measure as a limit point of counting measures of zeros of asymptotically minimal polynomials. In general case, we have a relation between these two measures although we do not have convergence. For this, we need to recall the balayage measure.

Let E be a bounded domain and μ be a measure with compact support in E. Then we can find a unique measure $\hat{\mu}$ supported on ∂E such that $\|\hat{\mu}\| = \|\mu\|$, the potential

$U_{\hat{\mu}}$ is bounded on ∂E, and

$$U_{\hat{\mu}}(z) = U_{\mu}(z) \tag{2.29}$$

for quasi-every $z \in \partial E$ [Saff & Totik (1997)]. This measure is called the *balayage measure* associated with μ.

Theorem 2.2.4 (Saff & Totik (1997)). *Let E be a compact non-polar subset of $\mathbb{C}$, Ω be the unbounded component of $\widehat{\mathbb{C}} \setminus \mathrm{supp}(w_E)$ and $\{q_n\}_n$ be a sequence of monic polynomials satisfying (2.26). If μ is a weak* limit of the counting measures $\{\mu_n\}_n$ of zeros of the q_n's, then*

(i) The logarithmic potential of μ coincide with the logaritmic potential of w_E in Ω. Moreover, $\mathrm{supp}(\mu) \subset \widehat{\mathbb{C}} \setminus \Omega$*;*

(ii) $\hat{\mu_n} \to \hat{w_E}$ in the weak sense; furthermore, $\hat{\mu} = \hat{w_E}$.*

This theorem yields us the following corollary:

Corollary 2.2.5. *Let E be a compact non-polar subset of $\mathbb{C}$, Ω be the unbounded component of $\widehat{\mathbb{C}} \setminus E$ and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials on E. If μ is a weak* limit of the counting measures $\{\mu_n\}_n$ of zeros of the p_n's, then*

(i) The logarithmic potential of μ coincide with the logarithmic potential of w_E in Ω. Moreover, $\mathrm{supp}(\mu) \subset \widehat{\mathbb{C}} \setminus \Omega$*;*

(ii) $\hat{\mu_n} \to \hat{w_E}$ in the weak sense; furthermore, $\hat{\mu} = \hat{w_E}$.*

2.3 Polynomial Dynamics

In this section, we will give the basic definitions and some important theorems in polynomial dynamics which we will use throughout this thesis. For details, we refer the reader to the manuscripts [Ransford (1995), Carleson & Gamelin (1993)]. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \geq 2$. The *attracting basin* of ∞ for p is the set of points with unbounded forward orbit under p that is

$$\Omega_p := \{z \in \widehat{\mathbb{C}} : p^k(z) \to \infty \quad \text{as} \quad k \to \infty\} \tag{2.30}$$

where $p^k = \overbrace{p \circ \cdots \circ p}^{k \text{ many}}$. Let us denote the escape radius for p by

$$R_p := \frac{1 + |a_n| + \cdots + |a_0|}{|a_n|}.$$

It follows that

$$\Omega_p = \bigcup_{k \geq 0} p^{-k}(\mathbb{C} \setminus \overline{D(0, R_p)}). \tag{2.31}$$

The set Ω_p is a connected, open and completely invariant, i.e., $p^{-1}(\Omega_p) = \Omega_p$. The complement $K_p := \mathbb{C} \setminus \Omega_p$ is called the *filled Julia set*. The *Julia set* of p is defined by $J_p := \partial \Omega_p = \partial K_p$. It is well-known that K_p and J_p are compact and completely invariant. One can observe that

$$K_p = \bigcap_{k \geq 0} p^{-k}(\overline{D(0, R_p)}). \tag{2.32}$$

Example 2.3.1. *Let $p_n(z) = z^n$ for $n > 1$. Then $\Omega_{p_n} = \{z : \|z\| > 1\}$ and $J_{p_n} = \{z : \|z\| = 1\}$. One can observe that $p_n^{-1}(\Omega_{p_n}) = \Omega_{p_n}$ and*

$$\Omega_{p_n} = \bigcup_{k \geq 0} p_n^{-k}(\mathbb{C} \setminus \overline{D(0,1)}), \quad K_{p_n} = \bigcap_{k \geq 0} p_n^{-k}(\overline{D(0,1)}). \tag{2.33}$$

Example 2.3.2. *Let $p(z) = z^2 - 2$. By induction, we can observe that $p^n(z + \frac{1}{z}) = z^{2^n} + \frac{1}{z^{2^n}}$. This gives us $\Omega_p = \widehat{\mathbb{C}} \setminus [-2, 2]$ and $J_p = [-2, 2]$.*

The Julia sets of polynomials are always non-polar:

Theorem 2.3.1 (Ransford (1995))**.** *Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomials with $n > 1$, then*

$$\operatorname{cap}(K_p) = \operatorname{cap}(J_p) = 1/|a_n|^{1/(n-1)}.$$

We denote the Dynamical Green's function of p by $g_p(z) : \mathbb{C} \to [0, \infty)$ where

$$g_p(z) = \lim_{k \to \infty} \frac{1}{n^k} \log^+ |p^k(z)| \tag{2.34}$$

where $\log^+ = \max\{\log, 0\}$. By a theorem of Brolin [Brolin (1965)] the function g_p coincides with the Green's function of Ω_p with the pole at infinity. Note that g_p vanishes precisely on K_p and has the invariant property

$$g_p(p(z)) = n \cdot g_p(z). \tag{2.35}$$

It follows from [Brolin (1965)] that the measure $\omega_p = \frac{1}{2\pi}\Delta g_p$ is p-invariant and coincides with the equilibrium measure of J_p. The measure $\omega := \omega_p$ is balanced i.e. for every set $X \subset \widehat{\mathbb{C}}$ on which p is injective we have $\omega(p(X)) = n \cdot \omega(X)$. Moreover, Lyubich [Lyubich (1982)] proved that ω is the unique measure of maximal entropy for p.

Let $\{p_n = \sum_{j=0}^{n} a_{n,j} z^j\}$ be a sequence of polynomials and E be a compact non-polar subset of $\mathbb{C}$. Then, by Theorem 2.3.1, we have the following:

Lemma 2.3.2 (Petersen & Uhre (2021))**.** *The following are equivalent;*

(i)

$$\lim_{n\to\infty} (a_{n,n})^{1/n} = \frac{1}{\text{cap}(E)} \tag{2.36}$$

(ii)

$$\text{cap}(K_{p_n}) \xrightarrow{n\to\infty} \text{cap}(E) \tag{2.37}$$

(iii)

$$I(w_{p_n}) \xrightarrow{n\to\infty} I(w_E) \tag{2.38}$$

Proof. We can observe that

$$\lim_{n\to\infty} (a_{n,n})^{1/n} = \frac{1}{\text{cap}(E)} \quad \Leftrightarrow \quad \lim_{n\to\infty} (a_{n,n})^{1/(n-1)} = \frac{1}{\text{cap}(E)}. \tag{2.39}$$

On the other hand, by Theorem 2.3.1, we have that

$$e^{I(w_{p_n})} = \text{cap}(K_{p_n}) = 1/|a_{n,n}|^{1/(n-1)}. \tag{2.40}$$

Hence, the assertion follows by combining (2.39) and (2.40). □

2.4 Topology of Compact Sets

We denote the collection of all non-empty compact subsets of $\mathbb{C}$ by $\mathscr{K}$. The classical *Hausdorff metric* χ on $\mathscr{K}$ is defined by

$$\chi(A,B) = \max(h(A,B), h(B,A)) = \inf\{r > 0 : B \subset D_r(A), A \subset D_r(B)\}$$

where

$$h(A,B) = \sup_{a\in A} \inf_{b\in B} \parallel a - b \parallel$$

and $D_r(A) = \{z \in \mathbb{C} : d(z,A) < r\}$. The pair $(\mathscr{K}, \chi)$ forms a complete metric space. Let $\{A_n\} \subset \mathscr{K}$ be a sequence of compact sets which are uniformly bounded. We denote the sets

$$\liminf_{n\to\infty} A_n := \{z \in \mathbb{C} : \exists\{z_n\}, A_n \ni z_n \xrightarrow{n\to\infty} z\}, \tag{2.41}$$

and

$$\limsup_{n\to\infty} A_n := \{z \in \mathbb{C} : \exists\{n_k\}, n_k \nearrow \infty \text{ and } \exists\{z_{n_k}\}, A_{n_k} \ni z_{n_k} \xrightarrow{k\to\infty} z\}. \tag{2.42}$$

The sets $\liminf\limits_{n\to\infty} A_n$, and $\limsup_{n\to\infty} A_n$ are in $\mathscr{K}$:

Lemma 2.4.1 (Christiansen et al. (2019))**.** *Let $\{A_n\} \subset \mathscr{K}$ be a uniformly bounded sequence of compact sets. The complements of $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ are open and*

$$z_0 \in \mathbb{C} \setminus \liminf_{n\to\infty} A_n \iff \exists\delta_0 \ \exists\{n_k\}, n_k \nearrow \infty \quad s.t \ \ \forall k : d(z_0, A_{n_k}) > \delta_0 \tag{2.43}$$

and

$$z_0 \in \mathbb{C} \setminus \limsup_{n\to\infty} A_n \iff \exists\delta_0 \ \exists N \quad s.t \ \ \forall n \geq N : d(z_0, A_n) > \delta_0. \tag{2.44}$$

As a consequence, both $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ are compact.

It is easy to see that a uniformly bounded sequence $\{A_n\}_n \in \mathscr{K}$ is pre-compact in $(\mathscr{K}, \chi)$. Moreover, $\{A_n\}_n$ converges to A with respect to Hausdorff metric, denoted by $\lim\limits_{k\to\infty} A_n = A$, if and only if $\liminf\limits_{n\to\infty} A_n = \limsup\limits_{n\to\infty} A_n = A$.

Example 2.4.1. *Let $E_n := \{z \in \mathbb{C} : |z| \leq 1 + \dfrac{1}{n}\}$ and $D_1 = \{z \in \mathbb{C} : |z| \leq 1\}$. We can* *observe that*

$$\liminf_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = D_1. \tag{2.45}$$

Next, we denote the collection of all polynomially convex compact regular subsets of $\mathbb{C}$ by $\mathscr{R}$. M. Klimek defined a natural metric by using Green's functions in [Klimek (1995)]: for regular compact subsets E, F of $\mathbb{C}$, we let $g_{\Omega_E}, g_{\Omega_F}$ be Green's functions with the pole at infinity for $\widehat{\mathbb{C}} \setminus \mathrm{Pc}(E)$ and $\widehat{\mathbb{C}} \setminus \mathrm{Pc}(F)$ respectively. The *Klimek distance* between E and F is defined by

$$\Gamma(E,F) := \max(\| g_{\Omega_E} \|_F, \| g_{\Omega_F} \|_E) = \| g_{\Omega_E} - g_{\Omega_F} \|_{\mathbb{C}} . \tag{2.46}$$

The Klimek distance Γ induces a pseudo-metric on regular compact subsets of $\mathbb{C}$ and a metric on $\mathscr{R}$. In fact, the pair $(\mathscr{R}, \Gamma)$ forms a complete metric space [Klimek (1995)].

We remark that the topologies induced by Hausdorff and Klimek metrics on $\mathscr{R}$ are different. In particular, convergence in Klimek distance does not imply convergence in Hausdorff distance (see Example 4.2.1). On the other hand, there is a relation between these two metrics for some special cases. Klimek showed that if E_n, E are regular, polynomially convex, connected subsets of the complex plane containing the origin, and if $\{E_n\}_n$ converges to E in Hausdorff topology then $\{E_n\}_n$ converges to E in Klimek distance [Klimek (1995), Proposition 1]. We can prove similar relation for the filled Julia sets of asymptotically minimal polynomials.

Proposition 2.4.2. *Let E be a regular compact subset of $\mathbb{C}$ and $\{p_n\}_n$ be a sequence of asymptotically minimal polynomials whose filled Julia sets K_n's are uniformly bounded. If $\{K_n\}_n$ converges to $\mathrm{Pc}(E)$ in Hausdorff distance, then $\{K_n\}_n$ converges to $\mathrm{Pc}(E)$ in Klimek distance.*

Proof. Suppose that $\{K_n\}_n$ converges to $\mathrm{Pc}(E)$ in Hausdorff distance. Let $\epsilon > 0$.

Since g_Ω is continuous and K_n's are uniformly bounded, then g_Ω is uniformly continuous on $\overline{D(0,R)} \supseteq (\cup_n K_n) \cup Pc(E)$ for $R > 0$. Then, there exists $\delta > 0$ such that

$$|z - w| < \delta \Rightarrow |g_\Omega(z) - g_\Omega(w)| < \epsilon \tag{2.47}$$

for all $z, w \in \overline{D(0,R)}$.

On the other hand, since $(K_n)_n$ converges to $\mathrm{Pc}(E)$ in Hausdorff distance, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\chi(K_n, \mathrm{Pc}(E)) < \delta. \tag{2.48}$$

Lastly, by Lemma 3.1.5 , there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ and for all $z \in \mathbb{C}$,

$$g_n(z) \leq g_\Omega(z) + \epsilon. \tag{2.49}$$

Now, let $N = \max\{N_1, N_2\}$ and $n \geq N$.

Let $z \in \mathrm{Pc}(E)$. By inequality (2.49), we have that

$$g_n(z) \leq g_\Omega(z) + \epsilon. \tag{2.50}$$

Since $z \in \mathrm{Pc}(E)$, then $g_\Omega(z) = 0$. So we have that $g_n(z) \leq \epsilon$. Since this holds for all $z \in \mathrm{Pc}(E)$, then $\| g_n \|_{\mathrm{Pc}(E)} < \epsilon$.

Let $z \in K_n$. By inequality (2.48), there exists $w \in \mathrm{Pc}(E)$ such that $|z - w| < \delta$. Then, by inequality (2.47),

$$|g_\Omega(z) - g_\Omega(w)| < \epsilon. \tag{2.51}$$

Since $g_\Omega(w) = 0$, then $|g_\Omega(z)| < \epsilon$. Therefore $\| g_\Omega \|_{K_n} < \epsilon$.

Hence $\Gamma(K_n, \mathrm{Pc}(E)) = \max(\| g_n \|_{\mathrm{Pc}(E)}, \| g_\Omega \|_{K_n}) < \epsilon$. This means that $\{K_n\}_n$ converges to $\mathrm{Pc}(E)$ in Klimek distance. $\square$

Recall that for $E \in \mathscr{K}$ and $\delta > 0$, the *modulus of continuity* is defined by

$$\omega_E(\delta) = \sup\{g_{\Omega_E}(z) : \mathrm{dist}(z, E) \leq \delta\}.$$

In particular, $E \in \mathscr{K}$ is regular if and only if $\lim_{\delta \to 0^+} \omega_E(\delta) = 0$. In general convergence in Hausdorff metric does not imply convergence in Klimek metric but the following is known:

Proposition 2.4.3 (Siciak (1997)). *A subfamily $\mathcal{E} \subset \mathscr{R}$ is pre-compact with respect to Klimek topology if and only if the following hold*

(i) The family $\mathcal{E}$ is uniformly bounded that is there exists $R > 0$ such that $E \subset B(0, R)$ for all $E \in \mathcal{E}$.

(ii) The family $\mathcal{E}$ has equicontinuity property that is $\lim_{\delta\to 0^+}[\sup_{E\in\mathcal{E}}\omega_E(\delta)] = 0$.

In particular, if $E_n \in \mathscr{R}$ and $\chi(E_n, E) \to 0$ for some $E \in \mathscr{R}$ then $\Gamma(E_n, E) \to 0$ if and only if $\lim_{\delta\to 0^+}[\sup_n \omega_{E_n}(\delta)] = 0$.

3. Weak Limits of Measures of Maximal Entropy for Asymptotically Minimal Polynomials

In this chapter, we will give the proof of Theorem 1.0.1. The equidistribution of the weak limits of measures of maximal entropies for orthonormal polynomials and normalized Chebyshev polynomials has proved in [Petersen & Uhre (2021)] and [Christiansen et al. (2021)], respectively.

Let μ be a Borel probability measure on $\mathbb{C}$ with compact non-polar support and $\{p_n\}_n$ be the unique sequence of orthonormal polynomials with respect to μ. In [Petersen & Uhre (2021)], Petersen and Uhre showed that the sequence $\{w_n\}_{n\geq 2}$ of measure of maximal entropies of p_n's is pre-compact for the weak*-topology and for any limit measure ν for a weakly convergent sub-sequence $\{w_{n_k}\}_k$, the support of ν is contained in $\mathrm{Pc}(\mathrm{supp}(\nu))$. Moreover, if $\mu \in \mathbf{Reg}$ then

$$w_n \xrightarrow{w^*} w_{Supp(\mu)}. \tag{3.1}$$

More recently, this result was generalized to the case of normalized Chebyshev polynomials in [Christiansen et al. (2021)]. Let $E \subset \mathbb{C}$ be a non-polar compact set and let $\{p_n\}_n$ be the associated sequence of normalized Chebyshev polynomials. Christiansen, Henriksen, H. Petersen and C. Petersen proved that the corresponding sequence of filled Julia sets $\{K_n\}_n$ is pre-compact in $\mathscr{K}$ and for any limit point K_∞ of a convergent subsequence $\{K_{n_k}\}_k$, we have that

$$E \subset \mathrm{Pc}(K_\infty) \subset \mathrm{Pc}(\limsup_{n\to\infty} K_n) \subset \mathrm{Co}(E). \tag{3.2}$$

where $\mathrm{Co}(E)$ is the convex hull of E which is the smallest convex set that contains E. They also showed that the sequence $\{w_n\}_n$ of the unique measure of maximal entropies for p_n's converges weak* to the equilibrium measure on K:

$$w_n \xrightarrow{w^*} w_E. \tag{3.3}$$

3.1 Proof of Theorem 1.0.1

Throughout this section, $E, \Omega, \mathrm{Pc}(E), \omega_E$ and p_n are as given in the introduction. We also let g_Ω be the Green's function for Ω with pole at ∞. We denote $K_n := K_{p_n}$ the filled Julia set, $\Omega_n := \mathbb{C} \setminus K_n$ and $J_n := J_{p_n}$ denote the Julia set of p_n. We also denote the Brolin measures by $\omega_n := \omega_{J_{p_n}}$. Firstly, we need the following lemma on the behavior of $|p_n|^{\frac{1}{n}}$. The proof is analogous to that of (Lindsey & Younsi, 2019, Lemma 3.1 (i)).

Lemma 3.1.1. *Let $\{p_n\}_n$ be as in Theorem 1.0.1 with with bounded zeros and $F \subset \mathbb{C}$ be a polynomially convex compact set such that*

$$\cup_{n=1}^{\infty}\{z \in \mathbb{C} : p_n(z) = 0\} \cup E \subset F.$$

Assume that the sequence of counting measure of zeros $\{\mu_n\}_{n\geq 1}$ is weak-convergent to a measure ν. Then $|p_n|^{\frac{1}{n}} \to \exp(g_\Omega)$ locally uniformly on $\widehat{\mathbb{C}} \setminus F$.*

Proof. Let $\widetilde{p_n} := \dfrac{p_n}{|a_{n,n}|}$. First, we will show that $|\widetilde{p_n}|^{\frac{1}{n}} \to e^{U_\nu}$ locally uniformly on $\widehat{\mathbb{C}} \setminus F$. Let $z \in \mathbb{C} \setminus F$. Then the function $w \to \log|z-w|$ is continuous on F and

$$\int_F \log|z-w| d\mu_n(w) \longrightarrow \int_F \log|z-w| d\nu(w) \quad \text{as} \quad n \to \infty. \tag{3.4}$$

By Corollary 2.2.4 we have $\mathrm{supp}(\nu) \subseteq \mathrm{Pc}(E)$. Thus, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log|\widetilde{p_n}(z)| = U_\nu(z), \tag{3.5}$$

point-wise for all $z \in \mathbb{C} \setminus F$. Now, let D be a disk whose closure is contained in $\mathbb{C} \setminus F$. Since the sequence of analytic functions $\{\widetilde{p_n}^{\frac{1}{n}}\}_n$ is uniformly bounded on D, by Montel's Theorem every subsequence has a further subsequence converging uniformly to some analytic function f satisfying $|f| = e^{U_\nu}$ on D. This shows that

$$\lim_{n\to\infty} |\widetilde{p_n}|^{\frac{1}{n}} = e^{U_\nu} \tag{3.6}$$

locally uniformly on $\mathbb{C} \setminus F$. Note that uniform convergence near ∞ follows from the equation $U_\nu(z) = \log|z| + o(|z|^{-1})$ as $z \to \infty$. Finally, using (2.10) and by Theorem 2.2.4, we observe that

$$\lim_{n\to\infty} |p_n|^{\frac{1}{n}} = (1/\mathrm{cap}(E)) e^{U_\nu} = \exp(g_\Omega) \tag{3.7}$$

locally uniformly on $\widehat{\mathbb{C}} \setminus F$. $\square$

We can prove Lemma 3.1.1 for more general families of asymptotically minimal polynomials, for details see Section 3.2.

It is known that the filled Julia sets of the sequence of orthonormal polynomials with respect to a fixed regular measure on a compact set are uniformly bounded [Christiansen et al. (2019)] (see also [Christiansen et al. (2021)] for Chebyshev polynomials). We prove the analogue of this result for asymptotically minimal polynomials:

Lemma 3.1.2. *Let $\{p_n\}_n$ be as in Theorem 1.0.1 with*

$$\cup_{n=1}^{\infty}\{z \in \mathbb{C} : p_n(z) = 0\} \subset D(0,M).$$

Assume that the counting measure of zeros $\{\mu_n\}_{n\geq 1}$ of $\{p_n\}_n$ is weak-convergent to a measure ν. Then there exists $R > 0$ and $N \in \mathbb{N}$ such that*

$$K_n \subset p_n^{-1}(\overline{D(0,R)}) \subset D(0,R) \tag{3.8}$$

for all $n \geq N$.

Proof. By Lemma 3.1.1,

$$\lim_{n\to\infty} |p_n|^{\frac{1}{n}} = e^{U_\nu}\mathrm{cap}(E)^{-1} \tag{3.9}$$

locally uniformly on $\widehat{\mathbb{C}} \setminus (\mathrm{Pc}(E) \cup \overline{D(0,M)})$.

For each $\varepsilon > \log^+\big(\mathrm{cap}(E)\big)$ there exists $R > M$ such that $\mathrm{Pc}(E) \subset D(0,R)$ and $U_\nu(z) > 2\varepsilon$ for all $z \in \partial D(0,R)$. This implies that

$$U_\nu(z) - \log \mathrm{cap}(E) \geq \varepsilon \tag{3.10}$$

for all $z \in \partial D(0,R)$. Then by (3.9) and compactness of $\partial D(0,R)$, there exists N such that

$$|p_n(z)|^{\frac{1}{n}} \geq e^{\frac{\varepsilon}{2}} \tag{3.11}$$

for all $n \geq N$ and for all $z \in \partial D(0,R)$. By increasing N if necessary, we can assume that $\log(R) < N\frac{\varepsilon}{2}$. This gives $|p_n(z)| > R$ for all $z \in \partial D(0,R)$.
Since the zeros of p_n are contained in $D(0,R)$, the minimum modulus principle

implies that

$$p_n(\mathbb{C}\setminus D(0,R)) \subset \mathbb{C}\setminus \overline{D(0,R)} \tag{3.12}$$

for all $n \geq N$. Thus, the equality $K_n = \bigcap_{k\geq 0} p_n^{-k}(\overline{D(0,R)})$ yields

$$K_n \subset p_n^{-1}(\overline{D(0,R)}) \subset D(0,R) \tag{3.13}$$

for all $n \geq N$. □

We will need the following result in the sequel:

Lemma 3.1.3. *Let $\{p_n\}_n$ be as in Theorem 1.0.1. For any compact set $V \subseteq \mathbb{C}\setminus \mathrm{Pc}(E)$ we have*

$$\lim_{n\to\infty}\Big(\sup_{w\in D(0,R)}\Big(\frac{1}{n}\sum_{p_n(z)=w} 1_V(z)\Big)\Big) = 0 \tag{3.14}$$

where 1_V denotes the characteristic function of the set V.

In the proof of Lemma 3.1.3, we utilize the following result:

Lemma 3.1.4. *((Stahl & Totik, 1992, Lemma 1.3.2)) Let $V, E \subseteq \mathbb{C}$ be two compact sets. If $V \subseteq \mathbb{C}\setminus \mathrm{Pc}(E)$ then there exists $b < 1$ and $N = N(E,V,b) \in \mathbb{N}$ such that for arbitrary N points $x_1, \cdots, x_N \in V$ there exists N points $y_1, \cdots, y_N \in \mathbb{C}$ for which the rational function*

$$r_N(z) := \prod_{j=1}^{N} \frac{(z-y_j)}{(z-x_j)} \tag{3.15}$$

has the sup norm $\|r_N\|_E \leq b$.

Proof of Lemma 3.1.3. We prove the case $p \in (0,\infty)$ for a regular measure τ. For $w \in D(0,R)$, let $x_{n,1}, \cdots, x_{n,l(n)}$ be the roots of $p_n - w$ in V. Then by Lemma 3.1.4 there exist $b < 1$, $N = N(E,V,b) \in \mathbb{N}$ and points $y_{n,1}, \cdots, y_{n,l(n)} \in \mathbb{C}$ such that

$$\| \, (p_n(z)-w) \prod_{j=1}^{N\lfloor \frac{l(n)}{N} \rfloor} \frac{z-y_{n,j}}{z-x_{n,j}} \, \|_{L^p(\tau)} \leq \| \, p_n - w \, \|_{L^p(\tau)} \, b^{\lfloor \frac{l(n)}{N} \rfloor}. \tag{3.16}$$

Now, let q_n be the monic polynomial of degree n with minimal $L^p(\tau)$ norm. Then by (3.16) we obtain

$$\|a_{n,n} q_n\|_{L^p(\tau)} \leq \|p_n - w\|_{L^p(\tau)} b^{\lfloor \frac{l(n)}{N} \rfloor} \leq (\|p_n\|_{L^p(\tau)} + R\|\tau\|^{\frac{1}{p}}) b^{\lfloor \frac{l(n)}{N} \rfloor}$$

which in turn implies that

$$0 \leq \frac{1}{n}\lfloor \frac{l(n)}{N} \rfloor \log \frac{1}{b} \leq \frac{1}{n} \log \left(\frac{\|p_n\|_{L^p(\tau)} + R\|\tau\|^{\frac{1}{p}}}{\|a_{n,n} q_n\|_{L^p(\tau)}} \right). \tag{3.17}$$

The right hand side is independent of w and by asymptotic minimality of p_n it tends to zero. Hence, the assertion follows.
The proof for the case $p = \infty$ is almost identical and omitted. □

The following lemma gives a relation between Green's functions for Ω_n and Ω with pole at infinity.

Lemma 3.1.5. *Let $\{p_n\}_n$ be as in Theorem 1.0.1. Assume that the counting measures $\{\mu_n\}_{n \geq 1}$ of $\{p_n\}_n$ is weak*-convergent to a measure ν. Then*

$$\limsup_{n \to \infty} g_n(z) \leq g_\Omega(z) \tag{3.18}$$

locally uniformly on $\mathbb{C}$ where g_n (resp. g_Ω) is the Green's function for Ω_n (resp. Ω) with pole at infinity.

Proof. By (2.10) and Theorem 2.3.1, we have $\lim_{n\to\infty} \text{cap}(K_n) = \text{cap}(E) > 0$. Thus, there exists $C \in (0,1)$ such that $\text{cap}(K_n) \geq C$ for n sufficiently large. Moreover, by Lemma 3.1.2 there exist $R > 0$ and $N \in \mathbb{N}$ such that $K_n \subset p_n^{-1}(\overline{D(0,R)}) \subset D(0,R)$ for all $n \geq N$. Hence by (Christiansen et al., 2021, Proposition 3.5), there exists $N \in \mathbb{N}$ and $M > 0$ such that

$$\| \, g_n(z) - \frac{1}{n} \log^+ |p_n(z)| \, \|_\infty \leq \frac{M}{n} \tag{3.19}$$

for all $n \geq N$.

Now, let $p \in (0, \infty)$. Then by (Stahl & Totik, 1992, Theorem 3.2.1) and (Stahl & Totik, 1992, Remark 3.2.2)

$$\limsup_{n \to \infty} \left(\frac{|p_n(z)|}{\| \, p_n \, \|_{L^p(\tau)}} \right)^{\frac{1}{n}} \leq e^{g_\Omega(z)} \tag{3.20}$$

locally uniformly for $z \in \mathbb{C}$. This implies that

$$\limsup_{n \to \infty} (|p_n(z)|)^{\frac{1}{n}} \leq e^{g_\Omega(z)} \tag{3.21}$$

locally uniformly on $\mathbb{C}$.

For $p = \infty$, using (2.11) and Bernstein's Lemma (see (Ransford, 1995, Theorem

5.5.7)), we obtain

$$\limsup_{n\to\infty}(|p_n(z)|)^{\frac{1}{n}} \le e^{g_\Omega(z)} \tag{3.22}$$

uniformly on $\mathbb{C}$. Hence, the assertion follows by combining (3.19), (3.21) and (3.22). □

Proof of the Theorem 1.0.1. First, we will show that $\{\omega_n\}_n$ is sequentially pre-compact with respect to the weak*-topology. Indeed, for each subsequence $\{\omega_{n_k}\}_k$, since support of μ_{n_k}'s are uniformly bounded, by Helly's Theorem there is a further subsequence such that the empirical measure of zeros $\mu_{n_{k_l}}$ weak* converges to a probability measure ν. Then by Lemma 3.1.2, $K_{n_{k_l}}$'s are uniformly bounded and hence the Brolin measures $\{\omega_{n_{k_l}}\}_{l\ge 1}$ has a weak* convergent subsequence.

Next, by Brolin's Theorem [Brolin (1965)], for any measurable function $f:\mathbb{C}\to\mathbb{C}$ and for all $n\in\mathbb{N}$,

$$\int_{\mathbb{C}} f(z)d\omega_n(z) = \frac{1}{n}\int_{\mathbb{C}}\left(\sum_{p_n(w)=z} f(w)\right)d\omega_n. \tag{3.23}$$

Now, let σ be a weak* limit of a subsequence $\{\omega_{n_k}\}_k$. We will show that $\text{supp}(\sigma)\subset J=\partial\text{Pc}(E)$. Passing to a further subsequence if necessary, we may and we do assume that the normalized measure of zeros μ_{n_k} are weak* convergent to a measure ν. Let $F\subset\mathbb{C}$ be a compact set with $F\cap\text{Pc}(E)=\emptyset$. Then, by Lemma 3.1.2 the filled Julia sets K_n are uniformly bounded and by Lemma 3.1.3 we have

$$\omega_{n_k}(F) = \int_{\mathbb{C}} 1_F(z)d\omega_{n_k}(z) = \int_{\mathbb{C}}\left(\frac{1}{n_k}\sum_{p_{n_k}(w)=z} 1_F(w)\right)d\omega_{n_k}(z) \xrightarrow{k\to\infty} 0. \tag{3.24}$$

Hence, $\sigma(F)=0$. Since this is true for all compact set disjoint from $\text{Pc}(E)$, we deduce that $\text{supp}(\sigma)\subset\text{Pc}(E)$.

On the other hand, by Lemma 3.1.5 we have that $\lim_{n\to\infty} g_n = 0$ uniformly on any compact subset of the interior of $\text{Pc}(E)$ since $g_\Omega\equiv 0$ in the interior of $\text{Pc}(E)$.

Let L be a compact subset of the interior $\text{Int}(\text{Pc}(E))$ and U be a compact neighborhood of L contained in $\text{Int}(\text{Pc}(E))$ and φ be a C^2 function with support in supp(φ) $\subset U$ satisfying $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on L. Then we have

$$\begin{aligned}\sigma(L) &\le \int \varphi(z) d\sigma(z) \\ &= \lim_{k\to\infty} \int \varphi(z) d\omega_{n_k}(z) \\ &= \lim_{k\to\infty} \frac{1}{2\pi} \int \Delta\varphi(z) g_{n_k}(z) dA(z) = 0\end{aligned}$$

where $dA(z)$ is the standard Euclidean area element.

On the other hand, by [(Ransford, 1995, Lemma A.3.3)]

$$\sigma(\mathrm{Int}(\mathrm{Pc}(E))) = \sup\{\sigma(L) : L \subset \mathrm{Int}(\mathrm{Pc}(E)) \text{ and } L \text{ is compact}\}$$

Hence, we conclude that $\sigma(\mathrm{Int}(\mathrm{Pc}(E)) = 0$ and $\mathrm{supp}(\sigma) \subset \partial\mathrm{Pc}(E)$.

Finally, we will show that $\omega_n \xrightarrow{weak^*} \omega_E$ as $n \to \infty$. By (Ransford, 1995, Lemma 3.3.3), we have

$$\limsup_{k\to\infty} I(\omega_{n_k}) \le I(\sigma). \tag{3.25}$$

On the other hand,

$$\limsup_{n\to\infty} I(\omega_{n_k}) = I(\omega_E) \tag{3.26}$$

by asymptotic minimality 2.2.1 and (2.3.1) this in turn implies that $I(\omega_E) \le I(\sigma)$. Since ω_E is the unique measure of maximal energy, we deduce that $\sigma = \omega_E$. Since this is true for all weak*-convergent subsequences of $\{\omega_n\}_{n\ge1}$ we conclude that $\omega_n \to \omega_E$ in the weak* topology. □

3.2 Further Results on Asymptotically Minimal Polynomials

In the previous section, we showed that n-th roots of absolute value of asymptotically minimal polynomials with bounded zeros converge uniformly to Green's function on $\widehat{\mathbb{C}} \setminus F$. We can prove this convergence for asymptotically minimal polynomials with unbounded zeros:

Proposition 3.2.1. *Let E be a compact non-polar subset of $\mathbb{C}$ and $\{p_n\}_n$ be an asymptotically minimal on E such that the sequence of counting measure of zeros*

$\{\mu_n\}_n$ *is weak***-convergent to a measure* ν*. Assume that there exists* $k_0 > 0$ *and a compact set* E_0 *such that all the zeros of each* p_n *except at most* k_0 *many of them are in* E_0*. Then* $|p_n|^{\frac{1}{n}} \to e^{g_\Omega}$ *locally uniformly on* $\widehat{\mathbb{C}} \setminus (\mathrm{Pc}(E) \cup E_0)$*.*

To prove this proposition, we need the following lemma:

Lemma 3.2.2. *Let* E *be a compact non-polar subset of* $\mathbb{C}$ *and* $\{p_n\}_n$ *be asymptotically minimal on* E*. Assume that there exists* $k_0 > 0$ *and compact set* E_0 *such that all the zeros of each* p_n *except at most* k_0 *many of them are in* E_0 *: if* $p_n(z) = 0$*,* $z \notin E_0$*, then* $z = z_{n,k}$ *for some* $k = 1, 2, \cdots, l(n)$ *and* $l(n) \le k_0$*. Write*

$$p_n^* := p_n(z) / \prod_{j=1}^{l(n)} (z - z_{n,j}).$$

Then, $\min\{|z_{n,j}| : j = 1, 2, \cdots, l(n)\} \to \infty$ *and* $\{p_n^*\}_n$ *is asymptotically minimal on* E*.*

Proof. By the assumption on the zeros of p_n's, we observe that

$$\min\{|z_{n,j}| : j = 1, 2, \cdots, l(n)\} \to \infty. \tag{3.27}$$

This gives us

$$\lim_{n\to\infty} \frac{1}{n} \log \Big\| \prod_{j=1}^{l(n)} (z - z_{n,j}) \Big\|_{L^p(\tau)} \ge 0. \tag{3.28}$$

Now, we will show that $\{p_n^*\}_n$ is an asymptotically minimal sequence on E. Since leading coefficients of p_n's and p_n^*'s are same, we have (2.10). So it is enough to prove (2.11). We have that;

$$\begin{aligned} \log \mathrm{cap}(E) &\le \lim_{n\to\infty} \frac{1}{n} \log \Big\| \frac{p_n^*}{a_{n,n}} \Big\|_{L^p(\tau)} \\ &\le \lim_{n\to\infty} \frac{1}{n} \log \Big\| \frac{p_n}{a_{n,n}} \Big\|_{L^p(\tau)} - \lim_{n\to\infty} \frac{1}{n} \log \Big\| \prod_{j=1}^{l(n)} (z - z_{n,j}) \Big\|_{L^p(\tau)} \\ &\le \log \mathrm{cap}(E). \end{aligned} \tag{3.29}$$

This yields

$$\lim_{n\to\infty} \frac{1}{n} \log \|p_n^*\|_{L^p(\tau)} = 0. \tag{3.30}$$

Hence, $\{p_n^*\}_n$ is asymptotically minimal on E. □

Proof of Proposition 3.2.1. Let $z_{n,k}$'s are zeros of p_n's which are not contained in E_0 for some $k = 1, 2, \cdots, l(n)$ and $l(n) \le k_0$ and define

$$p_n^*(z) := p_n(z) / \prod_{j=1}^{l(n)} (z - z_{n,j}). \tag{3.31}$$

Then $\{p_n^*\}_n$ is asymptotically minimal on E by Lemma 3.2.2. Note that zeros of p_n^*'s are contained in E_0 and the sequence of counting measures of zeros of p_n^*'s is weak*-convergent to ν. Then by Lemma 3.1.1 and Corollary 2.2.5, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log |p_n^*(z)| = g_\Omega. \tag{3.32}$$

locally uniformly on $\widehat{\mathbb{C}} \setminus (\mathrm{Pc}(E) \cup E_0)$. By (3.21),

$$\limsup_{n\to\infty} \frac{1}{n} \log |p_n(z)| \le e^{g_\Omega} \tag{3.33}$$

locally uniformly on $\mathbb{C}$. On the other hand, we can observe that,

$$\begin{aligned} g_\Omega(z) &= \lim_{n\to\infty} \frac{1}{n} \log |p_n^*(z)| \\ &= \lim_{n\to\infty} \frac{1}{n} \log \frac{|p_n(z)|}{\prod_{j=1}^{l(n)} |(z - z_{n,j})|} \\ &\le \lim_{n\to\infty} \frac{1}{n} \log |p_n(z)| - \lim_{n\to\infty} \frac{1}{n} \log \prod_{j=1}^{l(n)} |(z - z_{n,j})| \\ &\le \lim_{n\to\infty} \frac{1}{n} \log |p_n(z)| \end{aligned} \tag{3.34}$$

locally uniformly on $\widehat{\mathbb{C}} \setminus (\mathrm{Pc}(E) \cup E_0)$. Hence, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log |p_n(z)| = g_\Omega(z) \tag{3.35}$$

locally uniformly on $\widehat{\mathbb{C}} \setminus (\mathrm{Pc}(E) \cup E_0)$. □

In the first part of Theorem 1.0.1, we showed that the filled Julia sets of asymptotically minimal polynomials are uniformly bounded. We used boundedness of zeros of these polynomials in this part. So if we can prove this part for asymptotically minimal polynomials with unbounded zeros, the rest of the proof of Theorem 1.0.1 will be same. Lemma 3.1.1 is the main part of the proof of lemma which states that the filled Julia sets of asymptotically minimal polynomials are uniformly bounded. This motivates us to show uniform boundedness for the polynomials in Proposition 3.2.1. However, we observed that, this may not be true, see Example 2.2.5. For all $n \in \mathbb{N}$, we have $c_n \in K_n$ and hence $\{K_n\}_{n\geq 1}$ is not pre-compact in neither Hausdorff topology nor Klimek topology.

4. Geometric Limit of Filled Julia Sets

4.1 The Julia Sets of Asymptotically Minimal Polynomials

Julia sets of complex polynomials have been studied in many aspects. In 2013, one line of research was initiated by K. Lindsey [Lindsey (2015)]. She proved that if $E \subset \mathbb{C}$ is a closed Jordan domain, then there exists a sequence of polynomials such that E is totally approximable by the collection of the filled Julia sets of these polynomials. Recall that a nonempty proper subset E of the plane is *totally approximable* by a collection $\mathcal{A}$ of nonempty proper subsets of the plane is for any ε, there exists $F \in \mathcal{A}$ such that

$$\chi(S,F) < \varepsilon, \quad \chi(\partial S, \partial F) < \varepsilon. \tag{4.1}$$

A compact subset E of the plane is *uniformly perfect* if there exists a real number $r > 0$ such that for any $z \in E$ and for any $0 < d < \operatorname{diam}(E)$, there is a point $w \in E$ with $rd \leq |z - w| \leq d$. Let $E \subset \mathbb{C}$ be a uniformly perfect compact set with connected complement with $0 \in \operatorname{int}(E)$ and $\{q_n\}_n$ be sequence of monic polynomials of degree n having all zeros in ∂E such that counting measures converge weak* to the equilibrium measure for E. In [Lindsey & Younsi (2019)], K. Lindsey and Younsi showed that E is totally approximable by the filled Julia sets of the polynomials $P_{n,s}$ where

$$P_{n,s}(z) := z \frac{e^{-ns/2}}{\operatorname{cap}(E)^n} q_n(z). \tag{4.2}$$

For this, they proved that for any bounded neighborhood U of E, there exist s and n such that

$$E \subset \operatorname{int}(K(P_{n,s})) \subset U. \tag{4.3}$$

Moreover, they gave a characterization of the sets which can be approximated arbitrarily well by filled Julia sets of polynomials:

Theorem 4.1.1 (Lindsey & Younsi (2019))**.** *A nonempty proper subset E of the complex plane $\mathbb{C}$ is totally approximable by polynomial filled Julia sets if and only if E is bounded and $\mathbb{C}\setminus\mathrm{int}(E)$ is connected.*

In addition to this, they showed that if E is a uniformly perfect compact set with nonempty interior and connected complement, then there exists a real number $c = c(E)$ depending only on E such that

$$s_n(E) \le c\frac{\log n}{\sqrt{n}} \quad (n \ge 1) \tag{4.4}$$

$s_n(E) := \inf\{s > 0 : \exists p_n$ of degree n such that $E \subset K(p_n) \subset E \cup E_s\}$ where $E_s := \{z \in \Omega : g_\Omega(z,\infty) \le s\}$. In 2018, L. Bialas-Ciez, M. Kosek and M. Stawiska examined the rate of approximation by Julia sets and provided examples of sequences of polynomials that guarantee a better rate of approximation than the one in [Lindsey & Younsi (2019)]. In [Bialas-Ciez et al. (2018)], they approximate regular compact sets by using Klimek distance. For this, they used a similar method in [Lindsey & Younsi (2019)]. They started with Lagrange polynomials and defined new polynomials by using them. More precisely, they used the polynomials $P_n(z) := ze^{-ns/3}Q_n(z)$ where Q_n's are Lagrange polynomials.

We can prove the following theorem. The proof is analogous to that ([Lindsey & Younsi (2019)], Theorem 3.2)

Theorem 4.1.2. *Let E be a compact nonpolar subset of $\mathbb{C}$ with connected complement and $0 \in E$. Then E is totally approximable by the filled Julia sets of the polynomials*

$$P_{n,s}(z) := ze^{-ns/2}p_n(z) \tag{4.5}$$

where $s > 0$ and p_n's are asymptotically minimal polynomials whose zeros are contained in E.

Proof. Let $s > 0$. For sufficiently large n, we have that

(i) $(R/r)^{1/n}\|p_n\|_E^{1/n} < e^{s/2}$ for all $z \in E$,

(ii) $|\frac{1}{n}\log|p_n(z)| - g_\Omega(z)| \le \frac{s}{4}$ for all $z \in \Omega_s$,

(iii) $re^{ns/4} > R$

where $r, R > 0$ are such that $\overline{D}(0,r) \subset E$ and $E_s \subset D(0,R)$.

By (i), for $z \in E$, we can observe that;

$$|P_{n,s}(z)| = |ze^{-ns/2}p_n(z)| < Re^{-ns/2}e^{ns/2}\frac{r}{R} = r. \tag{4.6}$$

This yields us $P_{n,s}(E) \subset D(0,r) \subset E$ and hence $E \subset \text{int}(K_{P_{n,s}})$.

Now, let $z \in \Omega_s$. Then, by (ii)

$$|P_{n,s}(z)| > |z|e^{-ns/2}e^{ns/4} = |z|e^{ns/4}. \tag{4.7}$$

On the other hand, by (iii), $|P_{n,s}(z)| > |z|e^{ns/4} > re^{ns/4} > R$. So $P_{n,s}(z) \in \Omega_s$ if $z \in \Omega_s$. Hence, we have $K_{P_{n,s}} \subset E_s$. This means that E is totally approximable by the filled Julia sets of $P_{n,s}$'s. □

As a conclusion of this theorem, we can show that the polynomial $P_{n,s}$ is hyperbolic. Recall that the polynomial p is *hyperbolic* on the Julia set J_p if there exists $a > 0$ and $A > 1$ such that $|(p^n)'| \geq aA^n$ on J_p for all $n \geq 1$. Moreover, the rational function p is hyperbolic on J_p if and only if every critical point belongs to $\widehat{\mathbb{C}} \setminus J_p$ and is attracted to an attracting cycle. A Jordan curve is called a *quasicircle* if it is the image of a circle under a quasiconformal homeomorphism of the sphere. It is known that if $\widehat{\mathbb{C}} \setminus J_p$ of a rational function p has exactly two components and that p is hyperbolic on J_p, then J_p is a quasicircle [Carleson & Gamelin (1993)]. The proof of the following theorem is same with the proof of Theorem 7.1 in [Lindsey & Younsi (2019)].

Theorem 4.1.3. *Let E be a connected compact set with connected complement, and assume that 0 is an interior point of E. Then for any $s > 0$ and $n \in \mathbb{N}$ such that $E \subset \text{int}(K_{P_{n,s}})$, where $P_{n,s}$ is as defined in Theorem 4.1.2, the Julia set $J_{P_{n,s}}$ is a Jordan curve. Moreover, the polynomial $P_{n,s}$ is hyperbolic and $J_{P_{n,s}}$ is a quasicircle.*

Hyperbolic polynomials are important in complex analysis. We know exact formula for the Hausdorff dimension of Julia set of hyperbolic polynomials. Let E be an arbitrary subset of $\mathbb{C}$. For $\alpha > 0$, the *α-dimensional Hausdorff measure* of E is defined by

$$m_\alpha(E) = \lim_{\delta \to 0}(\inf\{\sum_i (\text{diam}U_i)^\alpha : E \subset \cup_i U_i, \text{diam}U_i < \delta\}) \tag{4.8}$$

the infimum being taken over all countable coverings of E by subsets U_i of diameter $< \delta$. This limit always exists, though it may be infinite. In fact, there is always a number $\theta \in [0,2]$ such that

$$m_\alpha(E) = \begin{cases} \infty & \text{if } \alpha < \theta \\ 0 & \text{if } \alpha > \theta \end{cases}. \tag{4.9}$$

This θ is called the *Hausdorff dimension* of E, denoted by $\dim(E)$.

Example 4.1.1. *The Hausdorff dimension of a nonempty open subset of* $\mathbb{C}$ *is always* 2 *and the Hausdorff dimension of a countable subset of* $\mathbb{C}$ *is always* 0.

The Hausdorff dimension of a set is hard to calculate. For the Hausdorff dimension of Julia sets J of polynomials, we know that

$$\dim(J) \geq \frac{\log d}{\log(\sup_J |q'|)}. \tag{4.10}$$

There is one case where we have an exact formula for $\dim(J)$, namely when q is hyperbolic. In this case, the Hausdorff dimension of the Julia set is given by the Bowen-Ruelle-Manning formula:

$$\dim(J) = \sup_\mu \Big(\frac{e_\mu(q)}{\int_J \log|q'| d\mu} \Big), \tag{4.11}$$

where the supremum is taken over all Borel probability measures μ on J which are q-invariant, and $e_\mu(q)$ denotes the entropy of q with respect to μ.

4.2 Proof of Theorem 1.0.2

For $s > 0$ and a compact set $E \subset \mathbb{C}$ we denote by $E^s := \{z \in \mathbb{C} : \operatorname{dist}(z, E) < s\}$. Clearly, E^s forms a neighborhood base of the set E in $\mathbb{C}$. Furthermore, if the set E is regular, we define $E_s := \{z \in \mathbb{C} : g_\Omega(z) \leq s\}$ and $\Omega_s := \{z \in \mathbb{C} : g_\Omega(z) > s\}$. It follows that $\{E_s\}_{s>0}$ also forms a neighborhood base of $\operatorname{Pc}(E)$ (see [Klimek (1995)]). Now, we prove Theorem 1.0.2.

Proof of Theorem 1.0.2. First, we will show that $\{K_n\}_n$ is sequentially pre-compact in $(\mathscr{R}, \Gamma)$. Indeed, passing to a subsequence $\{K_{n_j}\}_j$ we may assume that the counting measures of zeros of p_{n_j}'s are weak*-convergent. Then by Lemma 3.1.2 the collection $\{K_{n_j}\}_j$ is uniformly bounded. Next, we show that $\{K_{n_j}\}_j$ has the equicontinuity property. Indeed, for $s > 0$ by assumption all zeros of p_n are contained in $E_{\frac{s}{2}}$ for sufficiently large n_j. Then it follows from Lemma 3.1.1 that

$$\Big|\frac{1}{n_j}\log|p_{n_j}(z)| - g_\Omega(z)\Big| < \frac{s}{2} \text{ for } z \in \partial\Omega_s. \tag{4.12}$$

for sufficiently large n_j. This in turn implies that

$$|p_{n_j}(z)| > e^{\frac{sn_j}{2}} \quad \text{for} \quad z \in \partial\Omega_s. \tag{4.13}$$

Since all zeros of p_{n_j} are contained in $\mathbb{C}\setminus\Omega_s$ by applying the minimum modulus principle on the domain Ω_s we deduce that $|p_{n_j}(z)| > e^{\frac{sn_j}{2}}$ for all $z \in \Omega_s$ and sufficiently large n_j. Next, by (3.19) there exists $M > 0$ such that

$$|g_{n_j}(z) - \frac{1}{n_j}\log|p_{n_j}(z)|| \le \frac{M}{n_j} \quad \text{for} \quad z \in \Omega_s \tag{4.14}$$

which implies that $|g_{n_j}(z)| > \frac{s}{4}$ for $z \in \Omega_s$ and sufficiently large n_j. This in turn yields

$$K_{n_j} \subset \mathbb{C}\setminus\Omega_s = E_s \tag{4.15}$$

for sufficiently large n_j. Since $(E_s)_{s>0}$ form a neighborhood basis for $\mathrm{Pc}(E)$ in $\mathbb{C}$ for each $\epsilon > 0$ we can find $0 < s < \epsilon$ such that $E_s \subset \mathrm{Pc}(E)^\epsilon$. On the other hand, by Lemma 3.1.5, we have

$$g_{n_j}(z) \le g_\Omega(z) + s \tag{4.16}$$

for all $z \in \mathrm{Pc}(E)^\epsilon$ and sufficiently large j. Then from (4.15) and (4.16) we deduce that

$$\lim_{\epsilon\to 0^+}[\sup_{n_j}\omega_{K_{n_j}}(\epsilon)] = 0.$$

Thus, by Proposition 2.4.3 the family $\{K_n\}_n$ is sequentially pre-compact in $(\mathscr{R},\Gamma)$. Moreover, $\Gamma(K_{n_j},\mathrm{Pc}(E)) \to 0$. Indeed, for each $s > 0$ by (4.15)

$$\|g_\Omega\|_{K_{n_j}} \le s \tag{4.17}$$

for sufficiently large n_j. Moreover, since E is regular $g_\Omega(z) = 0$ on $\mathrm{Pc}(E)$ and by (4.16) we conclude that $\| g_{n_j} \|_{\mathrm{Pc}(E)} \le s$. Hence, combining (4.17) and (4.16) we deduce that

$$\Gamma(K_{n_j},\mathrm{Pc}(E)) = \max(\| g_{n_j} \|_{\mathrm{Pc}(E)}, \| g_\Omega \|_{K_{n_j}}) \le s.$$

Since $s > 0$ arbitrary we conclude that $\Gamma(K_{n_j},\mathrm{Pc}(E)) \to 0$ as $j \to \infty$.

For the general case, since the zeros of $\{p_n\}_n$ are bounded for each Klimek convergent subsequence of $\{K_n\}_n$ it has a further subsequence $\{K_{n_j}\}$ such that the counting measures of zeros of corresponding p_n's are weak*-convergent. Then by above argument $\Gamma(K_{n_j}, \mathrm{Pc}(E)) \to 0$. Since this holds for all convergent subsequences we conclude that $\{K_n\}_n$ converges to $\mathrm{Pc}(E)$ in the Klimek distance. □

As a corollary we obtain the following:

Corollary 4.2.1. *The collection of all filled Julia sets of asymptotically minimal polynomials associated with regular planar compact sets is a proper dense subset of* $(\mathscr{R}, \Gamma)$.

Proof. Note that the filled Julia set of a polynomial of degree $d \geq 2$ has Hölder property [Carleson & Gamelin (1993)]. Hence, the collection of filled Julia sets is a proper subset of $\mathscr{R}$. The density follows from Theorem 1.0.2. □

The next example illustrates that for a sequence of asymptotically minimal polynomials $\{p_n\}_n$ associated with a regular compact set $E \subset \mathbb{C}$ as in Theorem 1.0.2, their filled Julia sets need not to converge in Hausdorff topology. It was stated as an open problem in [Christiansen et al. (2021)] that for the sequence of normalized Chebyshev polynomials and a limit K_∞ set of $\{K_n\}_n$ in $(\mathscr{K}, \chi)$ whether the difference $E \setminus K_\infty$ is a polar set. For asymptotically minimal polynomials this difference could be quite large:

Example 4.2.1. *For fixed* $c \in \mathbb{C}$, *we let* $p_n(z) = z^n + c$ *for* $n \in \mathbb{N}$. *Then it is easy to see that* $\{p_n\}_{n \in \mathbb{N}}$ *satisfy hypotheses of Theorem 1.0.2 on the unit circle* $E = S^1$. *Then, the filled Julia sets of* p_n*'s converge to* S^1 *in the Klimek distance. On the other hand, by [(Boyd & Schulz, 2012, Theorem 1.2)] if* $|c| < 1$ *then the filled Julia sets* K_{p_n} *converges to the closed unit disc* $\overline{\mathbb{D}}$; *however if* $|c| > 1$ *then the filled Julia sets converges to* S^1 *with respect to Hausdorff topology. Finally, for almost every* $c \in S^1$ *the filled Julia sets do not converge to any compact set [Kaschner, Romero & Simmons (2015)]. Moreover, again by [(Boyd & Schulz, 2012, Theorem 1.2)] for any limit set* K_∞ *of the filled Julia sets* K_{p_n} *in the Hausdorff topology we have* $\mathrm{Pc}(K_\infty) = \overline{\mathbb{D}}$.

Recall that in Chapter 3, we showed that if the counting measures of zeros of $\{p_n\}_n$ are weak*-convergent, then the filled Julia sets K_n's are uniformly bounded. This yields us $\{K_n\}_n$ is sequentially pre-compact in $\mathscr{K}$ with respect to Hausdorff topology. Motivated by Example 4.2.1 we prove Proposition 1.0.3. First, adapting the argument in [(Christiansen et al., 2021, Proposition 4.3)] we can prove the following:

Lemma 4.2.2. *For any limit point K_∞ of a convergent subsequence $\{K_{n_k}\}_k$ with respect to Hausdorff topology we have that*

$$E \subset \mathrm{Pc}(K_\infty) \subset \mathrm{Pc}(\limsup_{n\to\infty} K_n). \tag{4.18}$$

Proof. Passing to a subsequence if necessary we may assume that the counting measures of zeros of $\{p_n\}_n$ are weak*-convergent. The rest of the proof follows from [(Christiansen et al., 2021, Proposition 4.3)] and Lemma 3.1.5. □

Now, we prove Proposition 1.0.3:

Proof of Proposition 1.0.3. Note that E_s is polynomially convex for all $s > 0$ [Siciak (1981)]. Passing to a subsequence if necessary we may assume that the counting measures of zeros of $\{p_n\}_n$ is weak*-convergent. Then, by Lemma 4.2.2 and the proof of Theorem 1.0.2, for $s > 0$ small we have

$$E \subset \mathrm{Pc}(K_\infty) \subset \mathrm{Pc}(\limsup_{n\to\infty} K_n) \subset E_s. \tag{4.19}$$

Letting $s \to 0$ we deduce that

$$\mathrm{Pc}(K_\infty) = \mathrm{Pc}(E).$$

□

Finally, we focus on the Hausdorff limit of the Julia sets of asymptotically minimal polynomials. Let E be a compact non-polar subset of $\mathbb{C}$ and Ω be the unbounded component of $\mathbb{C} \setminus E$. We denote the outer boundary of E by $J_E := \partial\Omega$. We also denote the exceptional set (see [Ransford (1995)] for definition) for the Green's function g_Ω by F_E. This means that $F_E = \{z \in E : g_\Omega(z) > 0\}$. We adopt the argument in [(Christiansen et al., 2019, Theorem 1.3(ii))] to our setting to prove that the limit of Julia sets of asymptotically minimal polynomials contain the regular points of the outer boundary:

Sketch of proof of Theorem 1.0.4. Since we mainly follow the argument in the proof of [(Christiansen et al., 2019, Theorem 1.3(ii))] we only give the main differences that require clarification. Assume that there exists $z_0 \in J \setminus F$ such that $z_0 \notin \liminf_{n\to\infty} J_n$. Then $g_\Omega(z_0) = 0$ and there exists $\delta > 0$ and $(n_k)_k$ with $n_k \nearrow \infty$ such that for all k, $D(z_0, \delta) \cap J_{n_k} = \emptyset$. By passing to a further subsequence if necessary we may assume that the counting measures of zeros of p_{n_k}'s are weak*-convergent. By using Lemma 3.1.1 and inequality (3.19), we can observe that for every compact set $V \subseteq \Omega$ and every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \operatorname{cap}(\{z \in V : g_{\Omega_{n_k}}(z) < g_{\Omega}(z) - \varepsilon\}) = 0. \tag{4.20}$$

Then following the argument in the proof of [(Christiansen et al., 2019, Theorem 1.3(ii))] one can show that there exist $\varepsilon > 0$ and $k \in \mathbb{N}$ such that for all $k \geq N$, there exists $z_k \in D(z_0, \delta)$ with $g_{n_k}(z_k) \geq \epsilon$ and $D(z_0, \delta) \subset \Omega_{n_k}$. Then by Harnack's inequality, we obtain

$$g_{n_k}(z_0) \geq g_{n_k}(z_k) \frac{1-\frac{1}{2}}{1+\frac{1}{2}} \geq \frac{\epsilon}{3} > 0. \tag{4.21}$$

On the other hand, by Lemma 3.1.5, we have that

$$\limsup_{k\to\infty} g_{n_k}(z_0) \leq g_{\Omega}(z_0) = 0 \tag{4.22}$$

which is a contradiction. Hence, we deduce that

$$\overline{J \setminus F} \subseteq \liminf_{n\to\infty} J_n. \tag{4.23}$$

□

5. Fractal Approximation in $\mathbb{C}^n$

In this chapter, we will give the basic concepts of pluripotential theory and some theorems which show us a way to approximate compact regular subset of $\mathbb{C}^k$. In Chapter 3, we proved that the measures of maximal entropies of asymptotically minimal polynomials with bounded zeros converge weakly to the equilibrium measure of E and in Chapter 4, we approximated regular compact sets by using the filled Julia sets of asymptotically minimal polynomials. In one variable case, these polynomials have various useful properties. We know that for any subsequential limit of zero measures of these polynomials is supported in polynomially convex hull $\mathrm{Pc}(E)$. Moreover, the balayage of this limit is equal to the balayage of the equilibrium measure of E, (see Section 2). In addition to these, logarithm of $n-$th roots of these polynomials converges locally uniformly to the Green's function of E in the outside of $\mathrm{Pc}(E)$ and the bounded set containing all zeros of these poynomials.

For higher dimensions, we want to define similar polynomial mappings $F : \mathbb{C}^k \to \mathbb{C}^k$, which give us the approximation to regular polynomially convex compact subsets of $\mathbb{C}^k$. For this, we expect that zero measures of these polynomial mappings to have similar properties like in one variable case. Moreover, we are interested in finding a relation between the equilibrium measure of this compact sets and measure of maximal entropies of these polynomial mappings.

Now, we will give the basic definitions of pluripotential theory. Let $E \subset \mathbb{C}^k$ be a polynomially convex regular compact subset. We define the *pluricomplex Green function* of the set E by:

$$V_E(z) = \sup\{u(z) : \ u \in \mathcal{L}, u \leq 0 \text{ on } E\} \qquad z \in \mathbb{C}^k,$$

where $\mathcal{L} := \{u \in PSH(\mathbb{C}^k) : u(z) - \log\|z\| \leq O(1) \text{ as } \|z\| \to \infty\}$. We denote by g_E the upper regularization V_E^* of V_E. Since E is regular, V_E is continuous. Hence,

$g_E = V_E$. If E is a non-pluripolar subset of $\mathbb{C}^k$, then $g_E \in \mathcal{L}_+$, where

$$\mathcal{L}_+ := \{u \in PSH(\mathbb{C}^k) : u(z) - \log\|z\| = O(1) \text{ as } \|z\| \to \infty\}.$$

In this case $w_E = (dd^c g_E)^k$ is called the *(complex) equilibrium measure* for E where $d = \partial + \overline{\partial}$, $d^c = (i/2\pi)(\overline{\partial} - \partial)$ and $(dd^c)^k$ is the complex Monge-Ampère operator. Note that, for $k = 1$, it is equal the measure $(1/2\pi)\Delta g_E$ for E. See [Bedford & Taylor (1987)] or [Klimek (1991)] for more information about complex equilibrium measure for the case $k > 1$.

Let $F = (p_1, \ldots, p_k) : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial mapping such that $\deg p_1 = \ldots = \deg p_k = n$ and

$$\widetilde{p_1}^{-1}(0) \cap \cdots \cap \widetilde{p_k}^{-1}(0) = \{0\}$$

where $\widetilde{p_1}, \ldots, \widetilde{p_k}$ are homogeneous parts of $p_1, \ldots, p_k$, respectively. Such a mapping F is called *regular polynomial mapping.* Let

$$K_F = \{z \in \mathbb{C}^k : F^l(z) \nrightarrow \infty \quad \text{as} \quad l \to \infty\}$$

be the *filled Julia set* of F. The *escape rate function of* F is defined by

$$g_F(z) = \lim_{m \to \infty} \frac{1}{n^m} \log^+ \|F^m(z)\|. \tag{5.1}$$

The function g_F is continuous and coincide with the pluricomplex Green function of the filled Julia set K_F. The complex equilibrium measure $(dd^c g_{K_F})^k$ for K_F is a measure of maximal entropy for F [Fornaess & Sibony (1995)].

For a compact non-pluripolar set K, we denote for any positive real number R, $D(R)$ the bounded open sublevel set:

$$D(R) = \{z \in \mathbb{C}^k : g_K(z) < R\}.$$

In [Nivoche (2009)], S. Nivoche aproximated polynomially convex compact regular subsets E of $\mathbb{C}^k$ by using special polynomials polyhedron defined by proper polynomial mappings from $\mathbb{C}^k$ to $\mathbb{C}^k$. She proved that for any $\epsilon > 0$ sufficiently small, there exist an integer $d_\epsilon \geq 1$ and a proper polynomial mapping $F_\epsilon = (p_1, \ldots, p_k)$ of degree d_ϵ such that $\|p_j\|_K \leq 1$ for $1 \leq j \leq k$, and

$$K \subset \overline{D(\epsilon)} \subset \mathcal{P}_\epsilon \subset D(\epsilon + \epsilon^2); \tag{5.2}$$

where $\mathcal{P}_\epsilon$ is the *special polynomial polyhedron* which is the finite union of the connected components of the open set

$$\{z \in \mathbb{C}^k : \sup_{1 \le l \le k} \frac{1}{d_\epsilon} \log |p_l(z)| < \epsilon + \beta(\epsilon)\}$$

that meets the compact set $\overline{D(\epsilon)}$ and $0 < \beta(\epsilon) \le \epsilon^2/2$. To find this polynomial mapping, she used the following theorem:

Theorem 5.0.1 (Siciak (1982)). *Let u be a continuous plurisubharmonic function in $\mathcal{L}_+$.*

(i) For any $\epsilon > 0$ and for any compact set E in $\mathbb{C}^k$, there exist two integers $d = d(\epsilon) \ge 1$ and $N = N(\epsilon) \ge 1$ and there exist N polynomials $p_1, \ldots, p_N$ of degree less or equal to d such that

$$u(z) - \epsilon \le \sup_{1 \le l \le N} \frac{1}{d} \log |p_l(z)| \le u(z) \quad in \quad E. \tag{5.3}$$

(ii) If in addition $\lim_{\lambda \in \mathbb{C}, |\lambda| \to \inf,} (u(\lambda z) - \log |\lambda|)$ exists for any $z \in \mathbb{C}^k \setminus \{O\}$, then the previous approximation is uniform in all $\mathbb{C}^k$.

Note that, N may be bigger than k in this theorem. S. Nivoche constructed new polynomial mapping by using part (i) of this theorem and showed that this new polynomial mapping satisfies (5.2). By using (5.2), she proved that

$$\frac{\text{card}(Z_\epsilon \cap \mathcal{P}_\epsilon)}{d_\epsilon^k} \to 1 \tag{5.4}$$

when ϵ tends to 0, d_ϵ tends to infinity where Z_ϵ is the set containing the zeros of F_ϵ. This gives us some informations about counting measures of zeros of F_ϵ's. So, we aim to define a sequence of asymptotically minimal polynomial mappings $\{F_n = (p_n^1, \ldots, p_n^k)\}_n$ in $\mathbb{C}^k$ such that (5.2) holds for $\epsilon = 1/n$ and $d_\epsilon = n$. This may give us similar properities for the counting measures of zeros of such polynomial mappings like in one variable case. After that, we can consider the following questions:

Question 5.0.1. *What is the relation between the (complex) equilibrium measure of compact sets in $\mathbb{C}^k$ and measures of maximal entropies of asymptotically minimal polynomial mappings associated with the same compact set?*

Question 5.0.2. *For a given polynomially convex regular compact set $E \in \mathbb{C}^k$, is it true that under suitable assumptions for an asymptotically minimal sequence of polynomial mappings $F_n : \mathbb{C}^k \to \mathbb{C}^k$, we have $K_n \to E$ as $n \to \infty$ in Klimek topology? Or more specifically can we find a sequence of polynomial mappings from $\mathbb{C}^k$ to $\mathbb{C}^k$ which gives us similar convergence in Klimek topology?*

5.1 The Composite Julia Sets of Polynomial Mappings

In [Klimek (2001)], Klimek approximated to regular compact sets in $\mathbb{C}^k$ by using composite Julia sets with respect to Klimek distance. Let $F_1,\ldots,F_m$ be regular polynomial mappings of degree at least 2. We denote the collection of all polynomially convex compact regular subsets of $\mathbb{C}^k$ by $\mathcal{R}$. We know that $F_j^{-1}(E)\in\mathcal{R}$ for any $E\in\mathcal{R}$ and that

$$V_{F_j^{-1}(E)} = \frac{1}{\deg F_j}(V_E\circ F_j), \quad j=1,\ldots,m.$$

For more information, see [Klimek (1982)],[Klimek (1991)]. Consequently, the mapping

$$E \to (\bigcup_{j=1}^{m} F_j^{-1}(E))^{\wedge},$$

where $\wedge$ denotes the polynomial hull, is a contraction of the complete space $(\mathcal{R},\Gamma)$ [Klimek (1995)]. The unique fixed point of this contraction is denoted by $K^+[F_1,\ldots,F_m]$ and is called the *composite Julia set* of the mappings $F_1,\ldots,F_m$.

On the other hand, we can describe the set $K^+[F_1,\ldots,F_m]$ in terms of orbits. Define

$$\sum_k = \{\sigma=(\sigma_1,\sigma_2,\ldots) : \sigma_j\in\{1,\ldots,m\}\}.$$

If $z\in\mathbb{C}^k$ and $\sigma\in\sum_k$, we define the $\sigma-$*orbit* of z as the sequence $(F_{\sigma_n}\circ\cdots\circ F_{\sigma_1})(z)$, where $n\geq 1$. Let $S_\sigma^+[F_1,\ldots,F_m]$ be the set of all points in $\mathbb{C}^k$ whose $\sigma-$orbits are bounded and let

$$S^+[F_1,\ldots,F_m] = \bigcup_{\sigma\in\sum_k} S_\sigma^+[F_1,\ldots,F_m].$$

$S^+[F_1,\ldots,F_m]$ is compact and the composite Julia set $K^+[F_1,\ldots,F_m]$ is equal to the polynomially convex hull of $S^+[F_1,\ldots,F_m]$ (see [Kosek (1998)]).

In the case of a single regular mapping F, the set $K^+[F]=S^+[F]$ is simply the filled-in Julia set of F. Klimek proved the following theorem;

Theorem 5.1.1 (Klimek (2001))**.** *The family of all composite Julia sets in $\mathbb{C}^k$ is a proper dense subset of the metric space* $(\mathcal{R},\Gamma)$.

In this theorem, Klimek used composite Julia sets instead of filled Julia sets of polynomial mappings in $\mathbb{C}^k$. This gives us another aspect on approximating regular polynomially convex subsets of $\mathbb{C}^k$. Like in Klimek's theorem, we can use composite

Julia sets of asymptotically minimal polynomial mappings after finding suitable definition.

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