Anonymous Implementation[*]

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Abstract

We consider Nash implementation under complete information with the additional feature that planners have to obey anonymity when designing mechanisms and shaping individuals' unilateral deviation opportunities. Our notion of full implementation, *anonymous implementation*, demands the following: First, any socially optimal alternative at any one of the given states is attainable via a Nash equilibrium (NE) at that state, which provides the same opportunity set for all individuals. Second, any such NE at any one of the states must be socially optimal at that state. We identify the necessary and (almost) sufficient condition for anonymous implementation of social choice correspondences. Further, we show that there are collective goals that are anonymously implementable but fail to be Nash implementable. Therefore, anonymity provides society with additional decentralizable social choice rules that are otherwise not Nash implementable. Unfortunately, anonymity imposes a heavy burden when implementing efficiency: The Pareto social choice correspondence is not anonymously implementable in the full domain.

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1 Introduction

Implementation of collective goals in Nash equilibrium (NE) involves designing mechanisms that incentivize society members to chose outcomes aligned with the desired goal.[1] The seminal works Maskin (1999)[circulated since 1977], Hurwicz (1986), Saijo (1988), Moore and Repullo (1990), Dutta and Sen (1991), and de Clippel (2014) establish that designing mechanisms that provide incentives aligned with the collective goal involves the identification of choice sets corresponding to opportunities individuals can sustain through unilateral deviations within the mechanism. de Clippel (2014) shows that Nash implementation of collective goals is almost fully characterized by the existence of a collection of choice sets providing individuals incentives consistent with the goal at hand. Indeed, a consistent collection of sets of alternatives is a family of choice sets indexed for each individual, each state, and each socially optimal alternative at that state such that the following hold: A socially optimal alternative at a state is chosen by every individual at that state from the corresponding choice set; if an alternative is socially optimal at the first state but not at the second, then there is an individual who does not choose this alternative at the second state from her choice set corresponding to this alternative and the first state.

On the other hand, the nearly complete characterization of Nash implementable collective goals based on consistency reveals that planners have significant flexibility when designing mechanisms and shaping individuals' opportunity sets. However, in many interesting economic environments, planners often face binding restrictions.[2] In this study, we analyze Nash implementation in complete information environments with the feature that the planners are restricted exogenously when shaping individuals' opportunity sets. To address the issue more directly, we ask, what if the planner were restricted by anonymity and has to offer each individual the same set of opportunities when designing mechanisms?

Consequently, we propose the notion of *anonymous implementation*: A social choice correspondence is anonymously implementable if (*i*) any socially optimal alternative at any one of the given states is achievable via a NE at that state, which provides the same opportunity set for all individuals, and (*ii*) any such NE at any one of the states must be socially optimal at that state.

To see an intuitive example for the applicability of anonymous implementation, consider a council consisting of multidisciplinary team of specialized doctors treating a patient.[3] In this con-

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[1] For more on Nash implementation, please see Maskin and Sjöström (2002), Palfrey (2002), and Serrano (2004).

[2] These limitations may arise due to legal considerations, such as constitutional rights or gender-neutrality. Also, it may not be realistic to consider a small trustees meeting of a major conglomerate with choice sets exclusively custom-tailored to each member's preferences. These limitations may also arise due to practical considerations, e.g., when the design of individual specific choice sets and the resulting administration of implementation are complex and costly.

[3] We thank Atila Abdülkadiroğlu for suggesting this example.

text, requiring equilibrium play in the mechanism (the set of rules governing doctors' interactions) result in each expert facing the same set of treatment opportunities seems appealing: Each team member agrees on the treatment method as well as the admissible options. On the other hand, sustaining NE of mechanisms with experts facing different sets of treatment options may create objections and problems within the team. Notwithstanding, this example also displays why Nash implementation via a symmetric game-form may be excessive: The team members are doctors specialized in different areas of treatment, and demanding that they face the same set of opportunities with one another for each one of their choices may end up to be significantly restrictive.[4]

We provide a necessary and (almost) sufficient condition for anonymous implementation of social choice correspondences, namely, *anonymous consistency*. This condition coincides with de Clippel (2014)'s consistency with the restriction that choice sets are independent of individuals' identities. We prove that if a social goal is anonymously implementable, then there exists a collection of choice sets anonymous consistent with the goal at hand (*necessity*); if a unanimous social goal possesses an anonymous consistent profile of choice sets, then it is anonymously implementable whenever there are at least three individuals (*sufficiency*).

We show that anonymous implementation does not necessarily restrict the set of Nash implementable social goals: In Section 3, we describe an environment and a social choice correspondence that is anonymously implementable but is not Nash implementable. Indeed, anonymity enlarges society's opportunities by allowing society to decentralize social choice correspondences that are otherwise not implementable in NE.

On the other hand, we show that when dealing with efficiency, anonymity imposes a heavy burden: We identify a domain description which, if allowed for, implies that the Pareto social choice correspondence is not anonymously implementable. As the full domain of preferences includes this particular instance, we observe that the Pareto social choice correspondence is not anonymously implementable on the full domain.

Our results cover both the rational and behavioral environments.

Gavan and Penta (2023) proposes a new framework for implementation theory by requiring that any individual and group deviations (up to a fixed size) from the equilibrium must lead to acceptable outcomes, and hence, parallels the fault tolerant implementation of Eliaz (2002). Anonymous implementation aligns with the essence of Gavan and Penta's approach in that we require unilateral deviations from the equilibrium to lead to the same set of alternatives for every individual.

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[4] The standard symmetry notion for games is referred to as 'the ordinary symmetry' in Cao and Yang (2018), a study observing that "defining natural and useful classes of symmetric games is a nontrivial task." This notion implies that all individuals have a common set of messages in the mechanism and each message profile of $n - 1$ individuals generates a set of opportunities that is independent of the identity of the odd-man out. Further, there are two other symmetry notions that paper present: 'name-irrelevant symmetric games' and 'renaming symmetric games'.

In a related paper, Barlo and Dalkıran (2022b) considers the implementation problem where planners have to ensure that the mechanism results in desirable outcomes even when they have partial information about individuals' state-contingent preferences.[5]

The organization of the paper is as follows: Section 2 provides the preliminaries, and 3 the example discussed above. In Section 4, we deal with the necessity and sufficiency of anonymous implementation, while Section 5 provides our results concerning efficiency. Finally, Section 6 presents our concluding remarks.

2 Preliminaries

Let $N = \{1, ..., n\}$ denote a *society* with at least two individuals, X a set of *alternatives*, 2^X the set of all subsets of X, and $\mathcal{X}$ the set of all non-empty subsets of X.

We denote by Ω the set of all *possible states* of the world, capturing all the payoff-relevant characteristics of the environment. In *behavioral environments*, the choice correspondence of individual $i \in N$ at state $\omega \in \Omega$ maps 2^X to itself so that for all $S \in 2^X$, $C_i^{\omega}(S)$ is a (possibly empty) subset of S. In *rational environments*, every individual's choice correspondence at every state satisfies the weak axiom of revealed preferences (WARP) and are represented by *preferences* of individual $i \in N$ at state $\omega \in \Omega$ captured by a complete and transitive binary relation, a ranking, $R_i^{\omega} \subseteq X \times X$, while P_i^{ω} represents its strict counterpart.[6] In rational environments, for all $i \in N$, all $\omega \in \Omega$, and all $S \in \mathcal{X}$, $C_i^{\omega}(S) := \{x \in S \mid x R_i^{\omega} y \text{ for all } y \in S\}$, and $L_i^{\omega}(x) := \{y \in X \mid x R_i^{\omega} y\}$ denotes the *lower contour set of individual i at state ω of alternative x*.

We refer to any $\tilde{\Omega} \subset \Omega$ as a *domain*. A *social choice correspondence* (SCC) defined on a domain $\tilde{\Omega}$ is $f : \tilde{\Omega} \to \mathcal{X}$, a non-empty valued correspondence mapping $\tilde{\Omega}$ into X. Given $\omega \in \tilde{\Omega}$, $f(\omega)$, the set of *f-optimal* alternatives at ω, consists of alternatives that the planner desires to sustain at ω. SCC f on $\tilde{\Omega}$ is ***unanimous*** if for any $\omega \in \tilde{\Omega}$, $x \in \cap_{i \in N} C_i^{\omega}(X)$ implies $x \in f(\omega)$.

The environment is of complete information in the sense that the true state of the world is common knowledge among the individuals but unknown to the planner as in Maskin (1999).

A mechanism $\mu = (M, g)$ assigns each individual $i \in N$ a non-empty *message space* M_i and specifies an *outcome function* $g : M \to X$ where $M = \times_{j \in N} M_j$. Given a mechanism μ and

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[5] The implementation notion of Barlo and Dalkıran (2022b) rests on an ex-post approach under incomplete information; we refer to Barlo and Dalkıran (2023a, 2023b) for more on implementation under incomplete information.

[6] It is well-known that a choice correspondence satisfies WARP if and only if it satisfies the independence of irrelevant alternatives (IIA) and Sen's β. A choice correspondence C defined on $\mathcal{X}$ satisfies the IIA if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x \in C(T) \cap S$ implies $x \in C(S)$, and Sen's β if for all $S, T \in \mathcal{X}$ with $S \subset T$, $x, y \in C(S)$ implies $x \in C(T)$ if and only if $y \in C(T)$. Further, a binary relation $R \subseteq X \times X$ is *complete* if for all $x, y \in X$ either xRy or yRx or both; *transitive* if for all $x, y, z \in X$ with xRy and yRz implies xRz.

$m_{-i} \in M_{-i} := \times_{j \neq i} M_j$, the *opportunity set* of individual i pertaining to others' message profile m_{-i} in mechanism μ is $O_i^{\mu}(m_{-i}) := g(M_i, m_{-i}) = \{g(m_i, m_{-i}) \mid m_i \in M_i\}$.

A message profile $m^* \in M$ is a ***Nash equilibrium of mechanism*** μ ***at state*** $\omega \in \Omega$ if $g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$.[7] Given mechanism μ, the correspondence $NE^{\mu} : \Omega \twoheadrightarrow 2^X$ identifies ***Nash equilibrium outcomes of mechanism*** μ ***at state*** $\omega \in \Omega$ and is defined by $NE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*)) \text{ and } g(m^*) = x\}$. A mechanism μ ***implements SCC*** f ***on domain*** $\tilde{\Omega}$***,*** $f : \tilde{\Omega} \to \mathcal{X}$***, in Nash equilibrium*** if $NE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$.

Thanks to the necessity result for Nash implementability of an SCC by Maskin (1999), we know that if $f : \Theta \to \mathcal{X}$ is Nash implementable, then it is ***Maskin-monotonic***: $x \in f(\omega)$ and $L_i^{\omega}(x) \subset L_i^{\tilde{\omega}}(x)$ for all $i \in N$ implies $x \in f(\tilde{\omega})$. de Clippel (2014) generalizes Maskin's results on Nash implementation to behavioral domains. The resulting necessary condition behavioral implementation is equivalent to Maskin-monotonicity in the rational domain (Barlo & Dalkıran, 2022a) and calls for the existence of a profile of sets that are *consistent* with this SCC at hand: We say that a profile of sets $\mathbf{S} := (S_i(x, \theta))_{i \in N, \omega \in \tilde{\Omega}, x \in f(\theta)}$ is ***consistent*** with a given SCC $f : \tilde{\Omega} \to \mathcal{X}$ if

(i) if $x \in f(\omega)$, then $x \in \cap_{i \in N} C_i^{\omega}(S_i(x, \omega))$, and

(ii) if $x \in f(\omega) \setminus f(\tilde{\omega})$, then $x \notin \cap_{i \in N} C_i^{\tilde{\omega}}(S_i(x, \omega))$.

The current study aims to restrict the planner to anonymity when designing the mechanism and its choice sets. That is why we introduce the notion of anonymous implementation:

Definition 1. *A mechanism* μ ***anonymously implements SCC*** f ***on domain*** $\tilde{\Omega}$***,*** $f : \tilde{\Omega} \to \mathcal{X}$***,*** *if*

(i) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$*, there is* $m^{(x,\omega)} \in M$ *such that* $g(m^{(x,\omega)}) = x \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^{(x,\omega)}))$*, and* $O_i^{\mu}(m_{-i}^{(x,\omega)}) = O_j^{\mu}(m_{-j}^{(x,\omega)})$ *for all* $i, j \in N$*; and*

(ii) *if* $m^* \in M$ *is such that* $g(m^*) \in \cap_{i \in N} C_i^{\tilde{\omega}}(O_i^{\mu}(m_{-i}^*))$ *and* $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ *for all* $i, j \in N$*, then* $g(m^*) \in f(\tilde{\omega})$*.*

A practical shortcut to formalizing anonymous implementation involves the introduction of the following refinement of NE:[8] A message profile $m^* \in M$ is an ***anonymous Nash equilibrium of mechanism*** μ ***at state*** $\omega \in \Omega$ if $g(m^*) \in \cap_{i \in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ and $O_i^{\mu}(m_{-i}^*) = O_j^{\mu}(m_{-j}^*)$ for all $i, j \in N$. So, a mechanism μ anonymously implements SCC f on domain $\tilde{\Omega}$ if and only if $ANE^{\mu}(\omega) = f(\omega)$ for all $\omega \in \tilde{\Omega}$, where $ANE^{\mu} : \Omega \twoheadrightarrow 2^X$, the set of ANE outcomes of mechanism μ at state $\omega \in \Omega$, is given by $ANE^{\mu}(\omega) := \{x \in X \mid \exists m^* \in M \text{ s.t. } m^* \in M \text{ is an } ANE \text{ of } \mu \text{ at } \omega\}$. We wish to emphasize that ANE implementation of an SCC does not necessitate a symmetric mechanism.

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[7] The notion of NE in behavioral domains, the behavioral Nash equilibrium, is introduced by Korpela (2012).

[8] We thank Kemal Yıldız for suggesting this approach.

3 An Example

In what follows, we present an example in the rational domain involving an SCC that is anonymously implementable but is not implementable in NE. We have two agents, Ann and Bob, and three alternatives, a, b, c. The domain $\tilde{\Omega}$ equals $\{\omega^{(1)}, \omega^{(2)}, \omega^{(3)}\}$, and individuals' state-contingent rankings are as in Table 1. The planner aims to implement SCC $f : \tilde{\Omega} \rightarrow \mathcal{X}$ given by $f(\omega^{(1)}) = \{a\}$, $f(\omega^{(2)}) = \{b\}$, and $f(\omega^{(3)}) = \{c\}$. Consider the mechanism in Table 2.

$\omega^{(1)}$		$\omega^{(2)}$		$\omega^{(3)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_B^{\omega^{(2)}}$	$R_A^{\omega^{(3)}}$	$R_B^{\omega^{(3)}}$
b	a	a, b	c	c	c
a	b	c	a, b	a	a
c	c			b	b

Table 1: Individuals' state-contingent rankings.

		Bob		
		L	M	R
Ann	U	$\textcircled{a}$	c	a
	M	c	◇c	a
	D	a	a	$\boxed{b}$

Table 2: The mechanism.

The message profile (U, L) (shown as circled) is an ANE of μ at state $\omega^{(1)}$ as $a \in C_A^{\omega^{(1)}}(O_A^{\mu}(L)) \cap C_B^{\omega^{(1)}}(O_B^{\mu}(U))$ and $O_A^{\mu}(L) = O_B^{\mu}(U) = \{a, c\}$. Moreover, $NE^{\mu}(\omega^{(1)}) = \{a\}$ and hence $ANE^{\mu}(\omega^{(1)}) = \{a\} = f(\omega^{(1)})$. On the other hand, $b \in C_A^{\omega^{(2)}}(O_A^{\mu}(R)) \cap C_B^{\omega^{(2)}}(O_B^{\mu}(D))$ and $O_A^{\mu}(R) = O_B^{\mu}(D) = \{a, b\}$ enables us to conclude that (D, R) (depicted with a square around it) $b \in ANE^{\mu}(\omega^{(2)})$. Meanwhile, the other NE are given by (D, L) and (D, M). As $O_A^{\mu}(L) = O_A^{\mu}(M) = \{a, c\}$ and $O_B^{\mu}(D) = \{a, b\}$, we conclude that $ANE^{\mu}(\omega^{(2)}) = \{b\} = f(\omega^{(2)})$. Similarly, $c \in C_A^{\omega^{(3)}}(O_A^{\mu}(M)) \cap C_B^{\omega^{(3)}}(O_B^{\mu}(M))$ and $O_A^{\mu}(M) = O_B^{\mu}(M) = \{a, c\}$ implies that (M, M) (depicted with a diamond around it) $c \in ANE^{\mu}(\omega^{(3)})$; $NE^{\mu}(\omega^{(3)}) = \{c\}$ we see that $ANE^{\mu}(\omega^{(3)}) = \{c\} = f(\omega^{(3)})$.

Therefore, μ anonymously implements SCC f.

To illustrate how NE that are not ANE may constitute grounds for objection based on justified envy, let us consider the message profile (D, M), an NE at state $\omega^{(2)}$ resulting in alternative a, which is not desirable at that state according to the given SCC. Then, only Ann (but not Bob) has alternative c as an additional opportunity while c is Bob's top choice. That is why Bob envies

Ann's equilibrium opportunities in NE (D, M) at state $\omega^{(2)}$ even though the mechanism itself is symmetric.[9]

Meanwhile, (D, M) being NE at $\omega^{(2)}$ also shows that μ does not implement f in NE since $NE^{\mu}(\omega^{(2)}) = \{a, b\} \neq \{b\} = f(\omega^{(2)})$.

One may wonder if there is another mechanism that implements SCC f in NE. In what follows, we establish that in this example, the answer is negative: f is not Nash implementable.

To achieve a contradiction, suppose that SCC $f : \tilde{\Omega} \to \mathcal{X}$ were implementable in NE. Then, thanks to de Clippel's necessity result, we know there is a profile of sets $\mathbf{S} = (S_i(x, \omega))_{i \in N, \omega \in \tilde{\Omega}, x \in f(\omega)}$ consistent with f. In particular, for any $i \in N$, $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $S_i(x, \omega)$ is given by $O^{\mu}_{i}(m^{(x,\omega)}_{-i})$ where $m^{(x,\omega)} \in M$ is a NE sustaining x, i.e., $g(m^{(x,\omega)}) = x \in \cap_{i \in N} C^{\omega}_{i}(O^{\mu}_{i}(m^{(x,\omega)}_{-i}))$. So, $f(\omega^{(2)}) = \{b\}$ and (i) of consistency implies $S_B(b, \omega^{(2)})$ equals either $\{b\}$ or $\{a, b\}$. If $S_B(b, \omega^{(2)}) = \{b\}$, then the mechanism μ has a NE $m^{(b,\omega^{(2)})} \in M$ such that $O^{\mu}_{B}(m^{(b,\omega^{(2)})}_{A}) = \{b\}$ (i.e., b constitutes Bob's only choice) and hence for all messages $m_B \in M_B$ we have $g(m^{(b,\omega^{(2)})}_{A}, m_B) = b$. So, $b \in O^{\mu}_{A}(m_B)$ for all $m_B \in M_B$. As b is Ann's top-ranked alternative at $\omega^{(1)}$ and $O^{\mu}_{B}(m^{(b,\omega^{(2)})}_{A}) = \{b\}$, we observe that $(m^{(b,\omega^{(2)})}_{A}, m_B)$ is a NE of μ at $\omega^{(1)}$ since $b \in C^{\omega^{(1)}}_{A}(O^{\mu}_{A}(m_B)) \cap C^{\omega^{(1)}}_{B}(O^{\mu}_{B}(m^{(b,\omega^{(2)})}_{A}))$. But, $b \notin f(\omega^{(1)}) = \{a\}$. Thus, $S_B(b, \omega^{(2)}) = \{a, b\}$ as $S_B(b, \omega^{(2)})$ cannot equal $\{b\}$. So, $S_B(b, \omega^{(2)}) = O^{\mu}_{B}(m^{(b,\omega^{(2)})}_{A}) = \{a, b\}$ and hence there exists $\tilde{m}_B \in M_B$ such that $g(m^{(b,\omega^{(2)})}_{A}, \tilde{m}_B) = a$; ergo, $a \in O^{\mu}_{A}(\tilde{m}_B)$. Then, because $a \in C^{\omega^{(2)}}_{B}(S_B(b, \omega^{(2)})) = C^{\omega^{(2)}}_{B}(\{a, b\}) = \{a, b\}$ and a is Ann's top-ranked alternative at $\omega^{(2)}$, a emerges as a Nash equilibrium outcome (and message profile $(m^{(b,\omega^{(2)})}_{A}, \tilde{m}_B)$ as a NE) at $\omega^{(2)}$ because $a \in C^{\omega^{(2)}}_{A}(O^{\mu}_{A}(\tilde{m}_B)) \cap C^{\omega^{(1)}}_{B}(O^{\mu}_{B}(m^{(b,\omega^{(2)})}_{A}))$. But, $a \notin f(\omega^{(2)}) = \{b\}$. Hence, we cannot have $S_B(b, \omega^{(2)})) = \{a, b\}$ as well, which implies the desired contradiction.

4 Necessity and Sufficiency

We proceed with the key condition for anonymous implementation. In what follows, we show that this condition is necessary and almost sufficient for anonymous implementation of SCCs. We note that the following condition applies both to the rational and the behavioral domains.

Definition 2. *Given an environment* $\langle N, X, \Omega, (C^{\omega}_{i})_{i \in N} \rangle$ *and SCC* f *on domain* $\tilde{\Omega}$, $f : \tilde{\Omega} \to \mathcal{X}$, *a profile of sets* $\mathbf{S} := (S(x, \omega))_{\omega \in \tilde{\Omega},\ x \in f(\omega)}$ *is* ***anonymous consistent*** *with* f *on domain* $\tilde{\Omega}$ *if*

- (i) *for all* $\omega \in \tilde{\Omega}$ *and all* $x \in f(\omega)$, $x \in \bigcap_{i \in N} C^{\omega}_{i}(S(x, \omega))$; *and*
- (ii) $x \in f(\omega) \setminus f(\tilde{\omega})$ *for any* $\omega, \tilde{\omega} \in \tilde{\Omega}$ *implies that* $x \notin \bigcap_{i \in N} C^{\tilde{\omega}}_{i}(S(x, \omega))$.

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[9] Similarly, Ann envies Bob's equilibrium opportunities in NE (D, L) at state $\omega^{(1)}$: This NE results in alternative a, and in equilibrium only Bob has b as an additional opportunity while it is Ann's top ranked alternative at $\omega^{(1)}$.

Next, we present our result, providing a full characterization of SCCs that are anonymously implementable (both in the rational and behavioral domains):

Theorem 1. *Given an environment* $\langle N, X, \Omega, (C_i^{\theta})_{i\in N}\rangle$,

- (i) *if SCC* $f : \tilde{\Omega} \to \mathcal{X}$ *is anonymously implementable on domain* $\tilde{\Omega}$*, then there is a profile of sets anonymous consistent with* f *on domain* $\tilde{\Omega}$*.*
- (ii) *if there is a profile of sets anonymous consistent with a unanimous SCC* $f : \tilde{\Omega} \to \mathcal{X}$*, then* f *is anonymously implementable on domain* $\tilde{\Omega}$ *whenever* $n \geq 3$*.*

Proof of (i) **of Theorem 1.** To prove (i) of Theorem 1, suppose that $f : \tilde{\Omega} \to \mathcal{X}$ is anonymously implementable in NE on domain $\tilde{\Omega}$. So, for all ω and all $x \in f(\omega)$, there is $m^{x,\omega} \in M$ such that $O_i^{\mu}(m_{-i}^{x,\omega}) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$ and $g(m^{x,\omega}) = x \in \cap_{i\in N} C_i^{\omega}(O^{\mu}(m_{-i}^{x,\omega}))$. Define $\mathbf{S}$ as follows: for all ω and $x \in f(\omega)$, $S(x,\omega) := O_i^{\mu}(m_{-i}^{x,\omega})$ for any $i \in N$. Then $\mathbf{S}$ satisfies (i) of anonymous consistency as for all $\omega \in \tilde{\Omega}$, and $x \in f(\omega)$, $g(m^{x,\omega}) = x \in \cap_{i\in N} C_i^{\omega}(O^{\mu}(m_{-i}^{x,\omega}))$ and $O_i^{\mu}(m_{-i}^{x,\omega}) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$. To show that $\mathbf{S}$ satisfies (ii) of anonymous consistency, suppose for some $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega) \backslash f(\tilde{\omega})$ and $x \in \cap_{i\in N} C_i^{\tilde{\omega}}(S(x,\omega))$. Then, $x \in \cap_{i\in N} C_i^{\tilde{\omega}}(O^{\mu}(m_{-i}^{x,\omega}))$. Since, $O_i^{\mu}(m_{-i}^{x,\omega}) = S(x,\omega) = O_j^{\mu}(m_{-j}^{x,\omega})$ for all $i, j \in N$, $m^{x,\omega}$ is an ANE at $\tilde{\omega}$ as $x = g(m^{x,\omega})$. Thus, we obtain the desired contradiction as $x \in f(\tilde{\omega})$ (as μ implements f anonymously on $\tilde{\Omega}$). ■

Proof of (ii) **of Theorem 1.** Suppose SCC $f : \tilde{\Omega} \to \mathcal{X}$ is unanimous and the profile $\mathbf{S} = (S(x,\omega))_{\omega\in\Omega,\ x\in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$. Consider the canonical mechanism given as follows: $M_i = X \times \tilde{\Omega} \times X \times \mathbb{N}$ where $m_i = (x^i, \omega^i, y^i, k^i)$ with $x^i \in f(\omega^i)$, $y^i \in X$, $\omega^i \in \tilde{\Omega}$, and $k^i \in \mathbb{N}$ for all $i \in N$; the outcome function $g : M \to X$ defined by

Rule 1: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N$, then $g(m) = x$;

Rule 2: If $m_i = (x, \omega, \cdot, \cdot)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $m_j \neq m_i$ with $m_j = (x', \omega', y', \cdot)$, then

$$g(m) = \begin{cases} x & \text{if } y' \notin S(x,\omega), \\ y' & \text{if } y' \in S(x,\omega). \end{cases}$$

Rule 3: In all other cases, $g(m) = y^{i^*}$ where $i^* = \max\{i \in N \mid k^i \geq k^j\ \forall j \in N\}$.

The result holds thanks to the following two claims.

Claim 1. *For all* $\omega \in \tilde{\Omega}$ *and* $x \in f(\omega)$*,* $m^{(x,\omega)}$ *defined by* $m_i^{(x,\omega)} = (x, \omega, x, 1)$ *is an ANE of* μ *at* ω *s.t.* $g(m^{(x,\omega)}) = x$*.*

Proof. Let $\omega \in \tilde{\Omega}$, $x \in f(\omega)$, and $m^{(x,\omega)}$ be as in the statement of the claim. Then, Rule 1 holds under $m^{(x,\omega)}$. So, $g(m^{(x,\omega)}) = x$, and due to Rules 1 and 2, $O_i^{\mu}(m_{-i}^{(x,\omega)}) = S(x,\omega)$ for all $i \in N$. By (i) of anonymous consistency, $g(m^{(x,\omega)}) = x \in \cap_{i\in N} C_i^{\omega}(S(x,\omega))$. So, $m^{x,\omega}$ is an ANE of μ at ω. ■

Claim 2. *If m^* is an ANE of μ at $\omega \in \tilde{\Omega}$, then $g(m^*) \in f(\omega)$.*

Proof. Suppose m^* is an ANE of μ at $\omega \in \tilde{\Omega}$.

Suppose additionally that Rule 1 holds under m^*. So, let $m_i^* = (x', \omega', \cdot, \cdot)$ with $\omega' \in \tilde{\Omega}$ and $x' \in f(\omega')$ for all $i \in N$. By Rules 1 and 2, $O_i^{\mu}(m_{-i}^*) = S(x', \omega')$ for all $i \in N$ and $g(m^*) = x'$. If $x' \notin f(\omega)$, then $x' \notin \cap_{i\in N} C_i^{\omega}(S(x', \omega'))$ (by (*ii*) of anonymous consistency); this is equivalent to $x' \notin \cap_{i\in N} C_i^{\omega}(O_i^{\mu}(m_{-i}^*))$ thanks to Rule 1; i.e., m^* is not an ANE of μ at ω. This delivers the desired contradiction and establishes that $g(m^*) = x' \in f(\omega)$ when Rule 1 holds under m^*.

If Rule 2 holds under m^*, then (by Rules 1,2, and 3) for all $i \in N \setminus \{j\}$ for some $j \in N$, $O_i^{\mu}(m_{-i}^*) = X$ and $O_j^{\mu}(m_{-j}^*) = S(x, \omega)$. Thus, $S(x, \omega) = X$ as m^* is an ANE. Then, as f is unanimous, $g(m^*) \in \cap_{i\in N} C_i^{\omega}(X)$ implies $g(m^*) \in f(\omega)$.

On the other hand, if Rule 3 holds under m^*, then for all $i \in N$, $O_i^{\mu}(m_{-i}^*) = X$. As m^* is an ANE, $g(m^*) \in \cap_{i\in N} C_i^{\omega}(X)$. This implies that $g(m^*) \in f(\omega)$ since f is unanimous. ■ ■

Before proceeding further with the efficiency analysis, we wish to display the relation of anonymous consistency with Maskin-monotonicity in the following lemma:

Lemma 1. *Given a rational environment $\langle N, X, \Omega, (C_i^{\omega})_{i\in N}\rangle$ and SCC $f : \tilde{\Omega} \to \mathcal{X}$, there exists a profile of sets anonymous consistent with f on domain $\tilde{\Omega}$ if and only if f satisfies the following (anonymous Maskin-monotonicity) condition on domain $\tilde{\Omega}$: For any $\omega, \tilde{\omega} \in \tilde{\Omega}$,*

$$x \in f(\omega) \text{ and } \cap_{i\in N}\, L_i^{\omega}(x) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x) \text{ implies } x \in f(\tilde{\omega}).$$

Proof of Lemma 1. Suppose that the environment $\langle N, X, \Omega, (C_i^{\omega})_{i\in N}\rangle$ is rational and SCC f defined on domain $\tilde{\Omega}$ is given by $f : \tilde{\Omega} \to \mathcal{X}$.

For the necessity direction of the lemma, suppose that $\mathbf{S} := (S(x, \omega))_{\omega\in\tilde{\Omega},\ x\in f(\omega)}$ is anonymous consistent with f on domain $\tilde{\Omega}$ and adopt the hypothesis that $\omega, \tilde{\omega} \in \tilde{\Omega}$, $x \in f(\omega)$, and $\cap_{i\in N} L_i^{\omega}(x) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x)$. Hence, by (*i*) of anonymous consistency, we see that $S(x, \omega) \subset \cap_{i\in N} L_i^{\omega}(x)$. Ergo, $S(x, \omega) \subset \cap_{i\in N} L^{\tilde{\omega}}(x)$. If $x \notin f(\tilde{\omega})$, then by (*ii*) of anonymous consistency, there is $j \in N$ such that $x \notin C_i^{\tilde{\omega}}(S(x, \omega))$. So, there is $j \in N$ and $y^* \in S(x, \omega)$ such that $y^* P_j^{\tilde{\omega}} x$; i.e., $y^* \notin L_j^{\tilde{\omega}}(x)$. But, $y^* \in S(x, \omega)$ and $y^* \notin L_j^{\tilde{\omega}}(x)$ contradicts $S(x, \omega) \subset \cap_{i\in N} L_i^{\tilde{\omega}}(x)$.

To establish the sufficiency direction, define $\mathbf{S}$ so that for any $\omega \in \tilde{\Omega}$ and $x \in f(\omega)$, we have $S(x, \omega) := \cap_{i\in N} L_i^{\omega}(x)$. Then, $\mathbf{S}$ satisfies (*i*) of anonymous consistency trivially due to the definition of lower contour sets. To obtain (*ii*) of anonymous consistency, suppose that $x \in f(\omega) \setminus f(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in \tilde{\Omega}$. So, $S(x, \omega) = \cap_{i\in N} L_i^{\omega}$ is not a subset of $\cap_{i\in N} L_i^{\tilde{\omega}}(x)$. Thus, there is $j \in N$ and $y^* \in S(x, \omega)$ with $y^* \notin L_j^{\tilde{\omega}} x$; i.e. $y^* P_j^{\tilde{\omega}} x$. Ergo, $x \notin C_j^{\tilde{\omega}}(S(x, \omega))$. ■

5 Efficiency

In rational environments, the Pareto SCC on the full domain Ω, $PO : \Omega \rightarrow \mathcal{X}$, is defined by

$$PO(\omega) := \{x \in X \mid \nexists y^* \in X \text{ s.t. } y^* P_i^\omega x \; \forall i \in N\}$$

for any $\omega \in \Omega$. On the other hand, in behavioral environments, we consider the efficiency SCC introduced by de Clippel (2014), $E^{\text{eff}} : \Omega \rightarrow \mathcal{X}$, which is defines as follows

$$E^{\text{eff}}(\omega) := \{x \in X \mid \exists (S_i)_{i \in N} \in \mathcal{X}^N \text{ s.t. } x \in \cap_{i \in N} C_i^\omega(S_i) \text{ and } \cup_{i \in N} S_i = X\}$$

for any $\omega \in \Omega$. We know that when $\tilde{\Omega}$ is a subset of the rational domain, then these two notions coincide, and hence efficiency SCC is an extension of the Pareto SCC to behavioral domains (de Clippel, 2014). Moreover, as choices are nonempty-valued, so are these SCCs: We observe that for all ω (in rational or behavioral domains) $x \in C_1^\omega(X)$ implies $x \in E^{\text{eff}}(\omega)$ by setting $S_1 = X$ and $S_j = \{x\}$ for all $j \neq 1$.

Below, we report bad news about the anonymous implementation of these efficiency notions.

We observe that PO is not anonymously implementable in the full rational domain whenever choices are non-empty valued due to the following: Suppose PO were anonymously implementable on the full rational domain and consider two states $\omega, \tilde{\omega}$ such that $L_1^\omega(x) = X$, $L_2^\omega(x) = \{x\}$, and $\cup_{i \in N} L_i^{\tilde{\omega}}(x) \neq X$. Then, $x \in PO(\omega) \setminus PO(\tilde{\omega})$. Further, $L_2^\omega(x) = \{x\}$ implies $O_i^\mu(m_{-i}^{\omega,x}) = \{x\}$ for all $i \in N$ where $m^{\omega,x} \in M$ is an ANE sustaining x at ω. But then, $m^{\omega,x}$ is also an ANE at state $\tilde{\omega}$ as $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(\{x\})$.

We show that the failure of the anonymous implementability of efficiency extends to the behavioral domain whenever there are two states ω and $\tilde{\omega}$ in the domain $\tilde{\Omega}$ on which efficiency SCC is defined and an alternative $x \in X$ with $x \in E^{\text{eff}}(\omega) \setminus E^{\text{eff}}(\tilde{\omega})$ such that for any $S \in \mathcal{X}$, x is chosen from a set S at ω by all individuals implies x continues to be chosen from S at $\tilde{\omega}$ by all agents.

Proposition 1. *Given an environment $\langle N, X, \Omega, (C_i^\theta)_{i \in N} \rangle$, efficiency SCC $E^{eff} : \tilde{\Omega} \rightarrow \mathcal{X}$ is not anonymously implementable on domain $\tilde{\Omega}$ whenever there are $\omega, \tilde{\omega} \in \tilde{\Omega}$ and $x \in E^{eff}(\omega) \setminus E^{eff}(\tilde{\omega})$ such that for all $S \in \mathcal{X}$, $x \in \cap_{i \in N} C_i^\omega(S)$ implies $x \in \cap_{i \in N} C_i^{\tilde{\omega}}(S)$.*

Proof of Proposition 1. Let $\tilde{\Omega} \subset \Omega$ be a domain such that there are $\omega^{(1)}, \omega^{(2)} \in \tilde{\Omega}$ and $x^* \in E^{\text{eff}}(\omega^{(1)}) \setminus E^{\text{eff}}(\tilde{\omega}^{(2)})$ such that for any $S \in \mathcal{X}$, $x^* \in \cap_{i \in N} C_i^{\omega^{(1)}}(S)$ implies $x^* \in \cap_{i \in N} C_i^{\omega^{(2)}}(S)$. Then, from the above we know that $m^{x^*,\omega^{(1)}}$ is such that $g(m^{x^*,\omega^{(1)}}) = x^*$ and $O_i^\mu(m_i^{x^*,\omega^{(1)}}) = S^*$ for all $i \in N$; and $x^* \in \cap_{i \in N} C_i^{\omega^{(2)}}(S^*)$. But then, $m^{x^*,\omega^{(1)}}$ is also an ANE of μ at $\omega^{(2)}$ which implies (thanks to μ anonymouysly implementing E^{eff}) $x^* \in E^{\text{eff}}(\omega^{(2)})$, a contradiction. ■

Notwithstanding, anonymous implementation of the Pareto SCC on rational subdomains can be achieved as the following example demonstrates: Let us refer to two individuals as Ann and Bob, $X = \{a, b, c\}$, $\tilde{\Omega} = \{\omega^{(1)}, \omega^{(2)}\}$, where individuals' strict rankings are as in Table 3. Pareto SCC PO on $\tilde{\Omega}$ is given by $PO(\omega^{(1)}) = \{a, b\}$ and $PO(\omega^{(2)}) = \{b, c\}$. One can verify that the mechanism in Table 4 anonymously implements the Pareto SCC on domain $\tilde{\Omega}$ (where we depict ANE at $\omega^{(1)}$ by circling the corresponding cells and those at $\omega^{(2)}$ by using squares):

$\omega^{(1)}$		$\omega^{(2)}$	
$R_A^{\omega^{(1)}}$	$R_B^{\omega^{(1)}}$	$R_A^{\omega^{(2)}}$	$R_2^{\omega^{(2)}}$
a	b	b	c
b	a	c	b
c	c	a	a

Table 3: Anonymous implementation of Pareto SCC on a rational subdomain.

		Bob			
		L	M_1	M_2	R
Ann	U	$\textcircled{a}$	c	c	a
	C_1	c	$\textcircled{b}$	c	b
	C_2	c	c	$\boxed{c}$	a
	D	a	b	a	$\boxed{b}$

Table 4: The mechanism implementing SCC PO on a rational subdomain.

6 Concluding Remarks

Implementing SCCs anonymously requires the planner to adhere to anonymity during mechanism design. This entails ensuring that any socially optimal alternative at any given state is achievable through an ANE at that state, and that any ANE at any given state must be socially optimal at that state. We identify *anonymous consistency* as the necessary and (almost) sufficient condition for anonymous implementation. This condition mirrors de Clippel (2014)'s consistency, with the additional constraint that choice sets are independent of individuals' identities. We demonstrate that anonymous implementation does not necessarily restrict the set of Nash-implementable social goals: In our example in Section 3, we present an SCC that is anonymously implementable but not Nash implementable. Our observation confirms that anonymity can expand society's range of decentralizable SCCs beyond those attainable through Nash implementation. Notwithstanding, we show that anonymity imposes a heavy burden when dealing with efficiency: The Pareto SCC cannot be anonymously implemented on the full domain.

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