EQUIDISTRIBUTION OF ZEROS OF RANDOM BERNOULLI POLYNOMIAL SYSTEMS

by
ÇİĞDEM ÇELİK

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ABSTRACT

EQUIDISTRIBUTION OF ZEROS OF RANDOM BERNOULLI POLYNOMIAL SYSTEMS

ÇİĞDEM ÇELİK

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Keywords: random polynomials, angle discrepancy, radius discrepancy, directional resultant, Bernoulli distribution, equidistribution of zeros

In this thesis, we consider the full systems of random polynomials with independent ± 1-valued Bernoulli distributed coefficients.

In the first part of study, we examine the distribution of common solutions of random Bernoulli systems. In order to determine that whether the common solutions are discrete or not, we focus on the directional resultants of these systems. Using the results obtained from the computations of directional resultants, we prove that common solutions of Bernoulli polynomial systems are discrete outside of an exceptional set $\mathcal{E}_{n,d}$ which has small probability. Randomizing the deterministic results of D'Andrea, Galligo and Sombra, we prove that outside of $\mathcal{E}_{n,d}$, the zeros of Bernoulli polynomial systems are equidistributed towards the Haar measure on the unit torus.

In the second part, we focus on the expected zero measures of random Bernoulli systems. We study the angular discrepancies and radius discrepancies of sets of common solutions of random Bernoulli polynomial systems. We prove that the expected angular discrepancy and radius discrepancy approach to zero as the degree of polynomials approaches to infinity. Using these results and appyling the classical method in analysis, we prove that the expected zero measure of Bernoulli polynomial systems converges to Haar measure on the unit disc $(S^1)^n$ in $\mathbb{C}^n$.

Lastly, we generalize these results for the random Bernoulli systems on $\mathbb{C}^2$ for more

general supports.

ÖZET

BERNOULLİ KATSAYILI RASSAL POLİNOM SİSTEMLERİNİN SIFIRLARININ EŞİT DAĞILIMI

ÇİĞDEM ÇELİK

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Anahtar Kelimeler: rassal polinomlar, açı uyuşmazlığı, yarıçap uyuşmazlığı, yönlendirilmiş resültant, Bernoulli dağılımı, sıfırların eşit dağılımı

Bu tez çalışmasında bağımsız ve eş-dağılımlı ± 1 değerli Bernoulli katsayılara sahip rassal polinom sistemleri ele alınmıştır.

Çalışmanın ilk kısmında rassal Bernoulli sistemlerinin ortak sıfırlarının dağılımları incelenmiştir. Ortak çözüm kümesinin ayrık noktalardan oluşup-oluşmadığını belirleyebilmek amacıyla bu sistemlerin yönlendirilmiş resültantlarına dikkat verilmiştir. Elde edilen yönlendirilmiş resültant hesapları kullanılarak, yeterince küçük olasılığa sahip istisnai bir $\mathcal{E}$ kümesi dışında bağımsız Bernoulli katsayılı sistemlerin ortak çözümlerinin ayrık olduğu ispatlanmıştır. D'Andrea, Galligo ve Sombra tarafından deterministik (rastgele olmayan) katsayılara sahip polinom sistemler için verilen sonucu, rassal Bernoulli katsayılı polinom sistemleri için uygun olacak şekilde dönüştürerek, Bernoulli dağılımlı sistemlerin sıfırlarının eşit dağılımlı oldukları ispatlanmıştır.

Çalışmanın ikinci kısmında, Bernoulli katsayılı sistemlerinin ortak sıfırlarının beklenen ölçüsü üzerinde durulmuştur. Bu sistemlerin ortak çözümlerin oluşturduğu kümelerin açı uyuşmazlığı ve yarıçap uyuşmazlığı üzerine çalışılmıştır. Beklenen açı uyuşmazlığı ölçüsü ve beklenen yarıçap uyuşmazlığı ölçüsünün sistemi oluşturan polinomların derecesi büyüdükçe sıfıra yaklaştığı gösterilmiştir. Elde edilen sonuçlar, klasik analiz metotlarıyla birleştirilerek, Bernoulli polinom sistemlerinin ortak sıfırlarının beklenen ölçüsününün de Haar ölçüsüne yakınsadığı ispatlanmıştır.

Son olarak, bu sonuçlar $\mathbb{C}^2$ üzerinde tanımlı daha genel dayanaklara sahip polinom sistemleri için genellenmiştir.

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To my family

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1. INTRODUCTION

1.1 Motivation

Random polynomials arise in many disciplines and their behaviour is of the interest to mathematicians, physicist and probabilist, as well as, statisticians engineers and economists. Although some of their suprising and intriguing behaviour has been known for as long as a century, they still preserve many mysteries. One of the main interests on random polynomials is the asymptotic behaviour of their zeros and it has been studied by many authors [(Bloch & Pólya, 1932),(Littlewood & Offord, 1939),(Erdös & Turán, 1950),(Kac, 1943), (Rice, 1945),(Hammersley, 1956), etc.].

A classical result of Kac (Kac, 1943) and Hammersley (Hammersley, 1956) asserts that if the coefficients are independent standard Gaussian random variables, the normalized empirical measure $\delta_{Z(f)}$ associated with the zeros of $f(x) = \sum_{j=0}^{d} a_j x^j$, almost surely converges to Haar measure ν_{Haar} of the unit circle $S^1 := \{|x| = 1\}$ as the degree $d \to \infty$. In other words, the zeros of $f(x)$ accumulate around the unit circle S^1 almost surely when the degree $d \to \infty$. After a few years, in 1950, Erdös - Turán (Erdös & Turán, 1950) states that for a univariate polynomial over $\mathbb{C}$, the argument of its roots are approximately equidistributed, if the middle coefficients do not grow too faster than the constant term and the leading term. Also, again for the location of the zeros, in 2008, result of Hughes and Nikeghbali (Hughes & Nikeghbali, 2008) shows that for the polynomials having not necessarily independent coefficients the roots concentrate on the unit circumference.

The universality for univariate Kac ensemble is proven in (Ibragimov & Zeitouni, 1997) and it asserts that for the random polynomials $f(x)$ with nondegenerate independent and identically distributed coefficients a_j, the normalized empirical measure $\delta_{Z(f)}$ converges almost surely to Haar measure ν_{Haar} if and only if the random coefficients satisfy the condition $\mathbb{E}[\log(1+|a_0|)] < \infty$.

The asymptotic zero distribution of multivariate polynomial systems is another interest and has been studied by many authors. Recently, Bloom and Shiffman (Bloom & Shiffman, 2006) improved Hammersley's result to multivariable case, i.e., the common zeros of n complex polynomials in $\mathbb{C}^n$ for $k = 1, \dots, n$,

$$f_i(x) = \sum_{|J| \le d} c_J^i x_1^{j_1} \dots x_n^{j_n}$$

tends to concentrate on the product of the unit circles $|x_j| = 1$ as $d \to \infty$, when the coefficients are independent and identically distributed complex Gaussian variables. Besides that, Zelditch, Shiffman, Bloom, Levenberg and Bayraktar have many results on complex random polynomials on $(\mathbb{C}^*)^n$ with Gaussian coefficients by using pluripotential theoretical techniques (see (Bayraktar, 2017),(Bloom & Levenberg, 2015),(Shiffman & Zelditch, 2003), (Bloom, 2005), (Bloom, 2007), (Bayraktar, 2019), etc.).

The universality of systems of multivariate Kac ensembles is given by Bayraktar in (Bayraktar, 2016) for the independent and identically distributed continuous coefficients satisfying the tail decay condition. On the other hand, the distribution of zeros of random polynomial mappings with discrete coefficients is still quite a mystery. In this thesis, we achieved to give an equidistribution result for the systems of random polynomials with independent ± 1-valued Bernoulli coefficients.

1.2 Statement of the Results

Let $\mathcal{A} = d\Sigma_n \cap \mathbb{Z}^n$ for some positive integer d where

$$\Sigma_n = \left\{ \boldsymbol{t} \in \mathbb{R}^n_{\ge 0} : \sum_{i=1}^n t_i \le 1 \right\}$$

is the unit simplex in $\mathbb{R}^n$. Assume that $\{\alpha_{\boldsymbol{J}}\}$ be a family of independent and identically distributed ± 1-valued Bernoulli random variables for $\boldsymbol{J} = (j_1, \dots, j_n) \in \mathcal{A}$. Following (Kozma & Zeitouni, 2011), a *random Bernoulli polynomial* with support

$\mathcal{A}$ is of the form

$$f_{d,i}(\boldsymbol{x}) = \sum_{|\boldsymbol{J}|\le d} \alpha_{i,\boldsymbol{J}} \boldsymbol{x}^{\boldsymbol{J}} \in \mathbb{C}[x_1, \ldots, x_n]$$

where $\boldsymbol{x}^{\boldsymbol{J}} = x_1^{j_1} \ldots x_n^{j_n}$.

Throughout this thesis, we concentrate on systems $(f_{d,1}, \ldots, f_{d,n})$ of random Bernoulli polynomials. We write $\boldsymbol{f}_d = \Big(f_{d,1}, \ldots, f_{d,n}\Big)$ for short.

We denote the collection of systems consisting random Bernoulli polynomials in n variables and of degree d whose support is $\mathcal{A}$ by

$$Poly_{n,d}(\mathcal{A}) := \Big\{\boldsymbol{f}_d = \Big(f_{d,1}, \ldots, f_{d,n}\Big) : supp(f_{d,i}) = \mathcal{A}\Big\},$$

and endow with the product probability measure $Prob_d$.

If the simultaneous solutions of a system $\boldsymbol{f}_d$ are isolated, we denote the empricial measure corresponding the $Z(\boldsymbol{f}_d)$ by $\delta_{Z(\boldsymbol{f}_d)}$. Also, we let ν_{Haar} denote the Haar measure of $(S^1)^n$ of total mass 1. The main theorem of study is the following.

Theorem 1.2.1. *Let $\boldsymbol{f}_d = (f_{d,1}, \ldots, f_{d,n})$ be a system of random polynomials with independent ± 1-valued Bernoulli coefficients. Then there exists a dimensional constant $K = K(n) > 0$ and an exceptional set $\mathcal{E}_{n,d} \subset Poly_{n,d}$ with $Prob_d\{\mathcal{E}_{n,d}\} \le K/d$ such that for all $\boldsymbol{f}_d \in Poly_{n,d}(\mathcal{A}) \setminus \mathcal{E}_{n,d}$*

$$\lim_{d\to\infty} \delta_{Z(\boldsymbol{f}_d)} = \nu_{Haar}.$$

In particular, $\delta_{Z(\boldsymbol{f}_d)} \to \nu_{Haar}$ in probability as $d \to \infty$.

We define *the expected zero measure* by

$$\Big\langle \mathbb{E}[\widetilde{Z}(\boldsymbol{f}_d)], \varphi \Big\rangle = \int_{Poly_{n,d}\setminus\mathcal{E}_{n,d}} \sum_{\xi_i \in Z(\boldsymbol{f}_d)} \varphi(\xi_i)\ dProb_d(\boldsymbol{f}_d) \tag{1.1}$$

where φ is a continuous function with compact support in $\mathbb{C}^n$. We consider the measure valued random variables

$$\widetilde{Z}(\boldsymbol{f}_d) = \begin{cases} \sum_{\xi_i \in Z(\boldsymbol{f}_d)} \delta(\xi_i) & \text{for } \boldsymbol{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2.2. *Let $\boldsymbol{f}_d = (f_{d,1}, \ldots, f_{d,n})$ be a system of random polynomials with* *independent ± 1-valued Bernoulli coefficients. Then*

$$\lim_{d\to\infty} d^{-n}\mathbb{E}[Z(\boldsymbol{f}_d)] = \nu_{Haar}$$

in weak topology.

Theorem 1.2.1 and Theorem 1.2.2 are proven for special kind of supports because of some techniquel restrictions on higher dimensions caused by the tools we use in this study. However, we achieve to generalize the type of supports on $\mathbb{C}^2$.

A convex body P in $(\mathbb{R}_+)^n$ is called a *lower set* if for each $(x_1,\dots,x_n)\in P$, the vectors $(y_1,\dots,y_n)\in P$ for $0<y_i<x_i$ for $i=1,\dots,n$.

Theorem 1.2.3. *Let $f_{d,1}, f_{d,2}$ be two bivariate Bernoulli random polynomials of degree d as in* (6.1) *with support $\mathcal{A} = Q\cap\mathbb{Z}^2$ where Q is a lower set in $(\mathbb{R}_+)^2$. Then for all nonzero vector $\boldsymbol{v}\in\mathbb{Z}^2$, their directional resultant $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}}}(f^{\boldsymbol{v}}_{d,1}, f^{\boldsymbol{v}}_{d,2}) \neq 0$ with overwhelming probability. Morever, outside of a set that has probability at most K/d for a positive constant K, we have*

$$\delta_{Z(f_{d,1},f_{d,2})} \to \nu_{Haar} \tag{1.2}$$

weakly in probability as $d\to\infty$. Moreover, we have

$$\lim_{d\to\infty} d^{-2}\mathbb{E}[\widetilde{Z}(\boldsymbol{f}_d)] = \nu_{Haar} \tag{1.3}$$

weakly.

The content of this study as follows: Chapter 2 introduces the required background on elimination theory and the geometry of convex polytopes. Chapter 3 is dedicated to equidistribution results on the zeros of univariate random polynomials and the simultaneous zeros of systems of multivariate random polynomials with continuous distributed coefficients. Chapter 4 includes the proof of Theorem 1.2.1 and Chapter 5 contains the proof of Theorem 1.2.2. Lastly, Chapter 6 includes further results on bivariate random Bernoulli polynomial systems and the proof of Theorem 1.2.3.

2. The Resultant of Polynomial Systems

In this chapter, we will give the preliminary definitions and some important results that we will use throughout this thesis.

2.1 Preliminaries

Let S be a subset of $\mathbb{R}^n$. The smallest convex set containing S is called *convex hull* of S and denoted by $conv(S)$. A *polytope* is a convex hull of a finite subset of $\mathbb{R}^n$. Throughout this thesis, we concentrate on polytopes which are convex hulls of sets of points with integer coordinates. Such polytopes are called *integral polytopes* or *lattice polytopes* or *Newton polytopes.* Thus, a lattice polytope is a set of the form $Q = conv(\mathcal{A}) \subset \mathbb{R}^n$, where $\mathcal{A} \subset \mathbb{Z}^n$ is finite.

Now, let $Q_1, \ldots, Q_k$ be lattice polytopes in $\mathbb{Z}^n$. Their Minkowski sum is defined as

$$Q_1 + \cdots + Q_k := \{q_1 + \cdots + q_k : q_i \in Q_i\},$$

and for a nonzero real number λ, the scaled polytope λQ is of the form

$$\lambda Q = \{\lambda q : q \in Q\}.$$

Let Σ denote the standart unit simplex in $\mathbb{R}^n$, that is, $\Sigma = conv(0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ where $\boldsymbol{e}_i$ represents the standart basis elements in $\mathbb{R}^n$. We let Vol_n denote the normalized volume of a subset of $\mathbb{R}^n$ with respect to the Lebesgue measure such that $Vol_n(\Sigma) = \frac{1}{n!}$. One can see that a polytope Q in $\mathbb{R}^n$ has a positive n dimensional volume if and only if the dimension of Q is n. Minkowski and Steiner stated that $Vol_n(d_1 Q_1 + \cdots + d_k Q_k)$ is a homogeneous polynomial in variables $d_1, \ldots, d_k \in \mathbb{Z}_+$ of degree n. In particular, if $k = n$, then the coefficient of the monomial $d_1 \ldots d_n$ in the homogeneous polynomial $Vol_n(d_1Q_1+\cdots+d_nQ_n)$ is called the *mixed volume* of $Q_1,\ldots,Q_n$ and it is denoted by $MV_{\mathbb{R}^n}(Q_1,\ldots,Q_n)$. Using polarization formula, the mixed volume of the polytopes $Q_1,\ldots,Q_n$ can be computed as follows

$$MV_{\mathbb{R}^n}(Q_1,\ldots,Q_n)=\sum_{k=1}^{n}\sum_{1\leq j_1\leq\ldots\leq j_k\leq n}(-1)^{n-k}Vol_n(Q_{j_1}+\ldots+Q_{j_k}).$$

In particular, if $Q=Q_1=\ldots=Q_n$ then

$$MV_{\mathbb{R}^n}(Q):=MV_n(Q,\ldots,Q)=n!Vol_n(Q).$$

Let $Q\subset\mathbb{R}^n$ be a convex set. Its support function $s_Q:\mathbb{R}^n\to\mathbb{R}$ is defined by

$$s_Q(\boldsymbol{v}):=\inf_{\boldsymbol{q}\in Q}\langle\boldsymbol{q},\boldsymbol{v}\rangle \tag{2.1}$$

where $\langle .,.\rangle$ represents the Euclidean inner product in $\mathbb{R}^n$. Then the equation

$$\langle\boldsymbol{q},\boldsymbol{v}\rangle=s_Q(\boldsymbol{v})$$

is called a *supporting hyperplane* of Q and $\boldsymbol{v}$ is called an *inward pointing normal* of Q. The intersection of Q with the supporting hyperplane in the direction $\boldsymbol{v}\in\mathbb{R}^n$ is denoted by

$$Q^{\boldsymbol{v}}=\{\boldsymbol{q}\in Q:\langle\boldsymbol{q},\boldsymbol{v}\rangle=s_Q(\boldsymbol{v})\}. \tag{2.2}$$

$Q^{\boldsymbol{v}}$ is called the *face* of Q determined by $\boldsymbol{v}$. If $Q^{\boldsymbol{v}}$ has codimension 1, it is called a *facet* of Q.

2.2 Elimination Theory

In this section we give a brief about elimination theory which is used to solve systems of the polynomial equations. Using the methods of this theory, one can determine if a given polynomial system has a solution or convert it to one with less variables and/or less equations. There are various versions of the resultants, such as Sylvester resultant, Macaulay resultant, Dixon resultant, etc. The choice of the method basically depends on the number of the polynomials in the system, the number of the variable and also support of the polynomials. In order to understand how to choose the convenient resultant, one can check the table in (Stiller, 1996) (pg. 3). In this thesis, we mostly follow (Gelfand, Kapranov & Zelevensky, 1995), (Cox, Little & O'shea, 2006), (Busé, 2021) and we concentrate on the systems containing $n+1$ polynomials in n variables or $n+1$ homogeneous polynomials in $n+1$ variables. For more general versions and applications one can check (Cox, 2020), (Gelfand et al., 1995), etc.

2.2.1 Resultant of Two Polynomials in One Variable

In this section, we consider two univariate polynomials f and g in $\mathbb{C}[x]$ with $a_n, b_m \neq 0$ defined as

$$f(x) = a_0 + a_1x + \ldots + a_nx^n \quad \text{and} \quad g(x) = b_0 + b_1x + \ldots + b_mx^m, \tag{2.3}$$

of degree n and m.

Definition 2.2.1. *The resultant of f and g is an irreducible polynomial in the coefficients of f and g, that is a polynomial in the ring $\mathbb{Z}[a_0,\ldots,a_n,b_0,\ldots,b_m]$, denoted by $Res(f,g)$, which vanishes if and only if f and g has a common root in $\mathbb{C}$.*

The resultant of two polynomials can be denoted by $Res_{m,n}(f,g)$ if one needs to emphasize the degrees of polynomials, where f and g are defined as in (2.3). Now, if we consider the homogenization of f and g, which we denote by F and G respectively, and which are defined as

$$F(x,y) = a_0y^n + a_1xy^{n-1} + \ldots + a_nx^n, \;\; G(x,y) = b_0y^m + b_1xy^{m-1} + \ldots + b_mx^m,$$

the resultant $Res(F,G)$ which equals to $Res(f,g)$, vanishes if and only if F and G have a common solution other than $(0,0)$, i.e., a common solution in the projective space $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$.

There are various methods to compute the resultant of two univariate polynomials f and g (or resultant of their homogenizations F and G). Here, we mention two very well known formulas: The Sylvester formula and the method of Bézout-Cayley.

2.2.1.1 The Sylvester Resultant

Let f and g be two polynomials of degree n and m, respectively, as in the equation (2.3). Then their *Sylvester matrix* is defined as

$$(2.4) \qquad Syl(f,g) := \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} & a_n & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & a_3 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_{m-1} & b_m & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{m-2} & b_{m-1} & b_m & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_0 & b_1 & b_2 & b_3 & \dots & b_m \end{bmatrix}$$

which is a square matrix of size $(n+m)\times(n+m)$.

Theorem 2.2.1 (Cox et al. (2006))**.** *The resultant of f and g is defined as the determinant of the Sylvester matrix $Syl(f,g)$.*

Example 2.2.1. *Let $f(x) = a_2x^2 + a_1x + a_0$ and $g(x) = b_2x^2 + b_1x + b_0$, then their resultant, $Res(f,g)$ can be computed as*

$$Res(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}$$
$$= a_0^2b_2^2 + a_0a_2b_1^2 - a_0a_1b_1b_2 + a_1^2b_0b_2 - a_1a_2b_0b_1 + a_2^2b_0^2 - 2a_0a_2b_0b_2.$$

Example 2.2.2. *Let $f = ax^2 + bx + c$ and $g(x) = f^{'}(x) = 2ax + b$, then*

$$(2.5) \qquad Res_{2,1}(f,g) = \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix} = a(b^2 - 4ac).$$

Remark 2.2.1 (Gelfand et al. (1995))**.** *The discriminant $\Delta(f)$ of a polynomial $f(x) = a_nx^n + \ldots + a_1x + a_0$, $a_n \neq 0$, is the resultant of f and its derivative $f^{'}$. The exact relation is*

$$(2.6) \qquad \Delta(f) = \frac{1}{a_n} Res_{n,n-1}(f,f^{'})$$

which is a homogeneous polynomial of degree $2n-2$ in the $n+1$ variables $a_0,...,a_n$.

Also, if $a_n, b_m \neq 0$, the resultant can be defined in terms of the discriminant as follows

$$(Res(f,g))^2 = (-1)^{nm} \frac{\Delta(fg)}{\Delta(f)\Delta(g)}$$

where f and g are as in described in (2.3).

2.2.1.2 The Method of Bézout-Cayley

Suppose $f(x) = a_n x^n + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + \ldots + b_1 x + b_0$ and that $a_0 = 1$. Writing

$$\frac{g(x)}{f(x)} = c_0 + c_1 x + c_2 x^2 + \cdots,$$

we write the Bézout's method as in (Stiller, 1996)

$$Res(f,g) = \begin{vmatrix} c_m & \cdots & c_{m+n-1} \\ c_{m-1} & \cdots & c_{m+n-2} \\ \vdots & \ddots & \vdots \\ c_{m-n+1} & \cdots & c_m \end{vmatrix}$$

which is an $n \times n$ determinant. One can see that the condition $a_0 = 1$ can be relaxed as $a_0 \neq 0$ since we always can scale f such that $a_0 = 1$.

Example 2.2.3 (Stiller (1996)). *Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = 5x^2 + x - 6$. Then we compute*

$$\frac{g(x)}{f(x)} = (-6 + x + 5x^2)(1 + 3x + 7x^2 + 15x^3 + \cdots) = -6 - 17x - 34x^2 - 68x^3 + \cdots$$

Using Bezout formula, we have

$$Res(f,g) = \begin{vmatrix} c_2 & c_3 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} -34 & -68 \\ -17 & -34 \end{vmatrix} = 0$$

which is expected since $x = 1$ is a common root.

2.2.2 The Multipolynomial Resultant

In this part, we introduce the resultant of multivariate homogeneous polynomials which is a direct generalization of the resultant of two homogeneous polynomials on $\mathbb{P}^1$ to the resultant of $n+1$ homogeneous polynomials on $\mathbb{P}^n$.

Following (Cox et al., 2006), we consider the 'universal' homogeneous polynomials of degree d_i as

$$\mathbf{F}_i = \sum_{|\alpha|=d_i} u_{i,\alpha} t^\alpha$$

for $i = 0, \ldots, n$ where α is a multi-index $(\alpha_0, \ldots, \alpha_n)$ and t^α indicates the monomial $t^{\alpha_1} \ldots t^{\alpha_n}$ which is of degree $|\alpha| = \sum_{i=1}^n \alpha_i$. One can see that the homogeneous polynomials of degree d_i form an affine space by identifying $\sum_{|\alpha|=d_i} u_{i,\alpha} t^\alpha$ with point $(u_{i,\alpha})_{|\alpha|=d_i} \in \mathbb{C}^{N(d_i)}$, where $N(d_i) = \binom{n+d_i-1}{n-1}$.

Define the *incidence variety* $W \subset \prod_{i=0}^n \mathbb{C}^{N(d_i)} \times \mathbb{P}^n$,

$$\mathcal{W} = \left\{ (c_{i,\alpha}, t_0, \ldots, t_n) \in \prod_{i=0}^n \mathbb{C}^{N(d_i)} \times \mathbb{P}^n : \boldsymbol{F}_i(c_{i,\alpha}, t_0, \ldots, t_n) = 0 \quad \text{for each } i = 0, \ldots, n \right\}.$$

$\mathcal{W}$ is an irreducible variety of dimension $(\sum_{i=0}^n N(d_i)) - 1$. Consider the canonical projection onto first factor $\pi : \prod_{i=0}^n \mathbb{C}^{N(d_i)} \times \mathbb{P}^n \to \prod_{i=0}^n \mathbb{C}^{N(d_i)}$ defined as $\pi(c_{i,\alpha}, t_0, \ldots, t_n) = (c_{i,\alpha})$. The image of the incidence variety under this projection $\pi(\mathcal{W})$ has the same dimension as $\mathcal{W}$ (Busé (2021)). Hence $\pi(\mathcal{W})$ forms an irreducible hypersurface in $\prod_{i=0}^n \mathbb{C}^{N(d_i)}$.

Theorem 2.2.2 (Cox et al. (2006))**.** *The set $\pi(W)$ is defined by a single irreducible equation $Res_{d_0,\ldots,d_n} = 0$ which is called the multipolynomial resultant. The expression $Res_{d_0,\ldots,d_n}(F_0, \ldots, F_n)$ is evaluation of this polynomial at the coefficients of the polynomials $F_0, \ldots, F_n$.*

In 1902, Macaulay proposed an efficient formula for computing the multipolynomial resultant in the article (Macaulay (1902)). The multipolynomial resultant is also known as *Macaulay resultant* and *classical resultant.*

Theorem 2.2.3 (Cox et al. (2006))**.** *Given $d_0, \ldots, d_n \in \mathbb{N}$, there exists a unique polynomial $Res_{d_0,\ldots,d_n} \in \mathbb{Z}[u_{i,\alpha}]$ which satisfies*

- *If $F_0, \ldots, F_n \in \mathbb{C}[t_0, \ldots, t_n]$ are homogenous polynomials then the system*

$$F_0 = \cdots = F_n = 0$$

has a nontrivial solution in $\mathbb{P}^n$ *if and only if* $Res_{d_0,\ldots,d_n}(F_0,\ldots,F_n)=0$.

- $Res_{d_0,\ldots,d_n}(t_0^{d_0},\ldots,t_n^{d_n})=1$.

In a more general point of view, each homogeneous polynomial F_i can be considered as a section of the hyperplane bundle $\mathcal{O}(d_i)$ on $\mathbb{P}^n$. Hence $Res_{d_0,\ldots,d_n}$ vanishes means these sections have a common root in $\mathbb{P}^n$.

Proposition 2.2.4 (Busé (2021)). *The multipolynomial resultant has the following properties:*

- ***Multi-degree of resultant.*** *Let* $F_0,\ldots,F_n$ *are generic homogeneous polynomials of degree* $d_0,\ldots,d_n$. *Then* $Res_{d_0,\ldots,d_n}(F_0,\ldots,F_n)$ *is a homogeneous polynomial in the coefficients of* F_i *of degree* $d_0d_1\ldots d_n/d_i$ *for each* $i=0,\ldots,n$.

- ***Additive property.*** *Let* $F_0,\ldots,F_i,F_i^{'},\ldots,F_n$ *be* $n+2$ *homogeneous polynomials in* $\mathbb{C}[t_0,\ldots,t_n]$ *of positive degrees. Then*

$$Res(F_0,\ldots,F_{i-1},F_iF_i^{'},F_{i+1},\ldots,F_n)=$$

$$Res(F_0,\ldots,F_{i-1},F_i,F_{i+1},\ldots,F_n)Res(F_0,\ldots,F_{i-1},F_i^{'},F_{i+1},\ldots,F_n).$$

- ***Invariance under elementary transformation.*** *Let* $F_0,\ldots,F_n\in\mathbb{C}[t_0,\ldots,t_n]$ *be homogeneous polynomials of positive degree. Then*

$$Res(F_0,\ldots,F_{n-1},F_i+\sum_{j\neq i}h_{i,j}F_j,F_{i+1},\ldots,F_n)=Res(F_0,\ldots,F_{n-1},F_i,F_{n+1},\ldots,F_n)$$

for any $i=0,\ldots,n$ *and for any homogeneous polynomials* $h_{i,j}$ *such that the polynomial* $F_i+\sum_{j\neq i}h_{i,j}F_j$ *is homogeneous of the same degree as* F_i.

- ***The base change formula.*** *Let* $F_0,\ldots,F_n\in\mathbb{C}[t_0,\ldots,t_n]$ *be homogeneous polynomials of positive degrees* $d_0,\ldots,d_n$, *respectively. Also, consider* $n+1$ *homogeneous polynomials* $G=(G_0,\ldots,G_n)$ *where each* G_j *is of degree* $d\geq 1$ *for* $j=0,\ldots,n$. *Then*

$$Res(F_0\circ G,\ldots,F_n\circ G)=Res(G_0,\ldots,G_n)^{d_0d_1\ldots d_n}Res(F_0,\ldots,F_n)^{d^n}.$$

- ***Reduction by one variable.*** *Let* $F_0,\ldots,F_{n-1}$ *be homogeneous polynomials of positive degree in* $\mathbb{C}[t_0,\ldots,t_n]$ *for* $n\geq 2$. *Set*

$\bar{F}_i(t_0,\ldots,t_{n-1}) := F_i(t_0,\ldots,t_{n-1},0) \in \mathbb{C}[t_0,\ldots,t_{n-1}]$. *Then,*

$$Res(F_0,\ldots,F_{n-1},t_n) = Res(\bar{F}_0,\ldots,\bar{F}_{n-1}) \in \mathbb{C}.$$

- ***Permutation of variables.*** *Let* $F_0,\ldots,F_n \in \mathbb{C}[t_0,\ldots,t_n]$ *be homogeneous polynomials of positive degree and let* σ *represent a permutation of the group of* $n+1$ *elements. Then*

  $$Res(F_{\sigma(0)},F_{\sigma(1)},\ldots,F_{\sigma(n)}) = \varepsilon(\sigma)^{d_0d_1\ldots d_n} Res(F_0,\ldots,F_n),$$

  where $\varepsilon(\sigma)$ *denotes the signature of the permutation* σ*.*

2.2.3 The Sparse Eliminant

In this part, following (Gelfand et al., 1995) and (Cox et al., 2006), we mention the sparse eliminant as a generalization of the classical resultant. Let $\mathcal{A}_0,\ldots,\mathcal{A}_n$ be a non-empty finite subsets of $\mathbb{Z}^n$, and let $u_i = \{u_{i,a}\}_{a\in\mathcal{A}_i}$ be a group of $\#\mathcal{A}_i$ variables, $i = 0,\ldots,n$ and set $\boldsymbol{u} = \{u_0,\ldots,u_n\}$. For each i, the general Laurent polynomial f_i with support $\mathcal{A}_i$ is defined as

$$f_i = \sum_{\boldsymbol{a}\in\mathcal{A}_i} u_{i,\boldsymbol{a}}\boldsymbol{x}^{\boldsymbol{a}} \in \mathbb{C}[\boldsymbol{u}][x_1^{\pm 1},\ldots,x_n^{\pm 1}].$$

Such polynomials are called *sparse polynomials* in the literature.

Put $\mathcal{A} = (\mathcal{A}_0,\ldots,\mathcal{A}_n)$ and consider the incidence variety,

$$W_{\mathcal{A}} = \left\{(\boldsymbol{u},\boldsymbol{x}) \in \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \times (\mathbb{C}^*)^n : f_0(\boldsymbol{u}_0,\boldsymbol{x}) = \cdots = f_n(\boldsymbol{u}_n,\boldsymbol{x}) = 0\right\}. \tag{2.7}$$

Consider the canonical projection on the first coordinate

$$\pi : \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \times (\mathbb{C}^*)^n \to \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$$

and let $\overline{\pi(W_{\mathcal{A}})}$ denote the Zariski closure of the $W_{\mathcal{A}}$ under the projection π. Following (Gelfand et al., 1995), we define the *sparse eliminant* as follows

Definition 2.2.2 (Gelfand et al. (1995))**.** *The sparse eliminant, denoted by* $Elim_{\mathcal{A}}$*, is defined as follows: if the variety* $\overline{\pi(W_{\mathcal{A}})}$ *has* *codimension 1, then the sparse eliminant is the unique (up to sign) irreducible polynomial in* $\mathbb{Z}[\boldsymbol{u}]$ *which is the defining equation of* $\overline{\pi(W_{\mathcal{A}})}$ *, i.e., it vanishes on* $\overline{\pi(W_{\mathcal{A}})}$ *. If* $codim(\overline{\pi(W_{\mathcal{A}})}) \geq 2$*, then* $Elim_{\mathcal{A}}$ *is defined to be constant 1. The expression*

$$Elim_{\mathcal{A}}(f_0,\dots,f_n)$$

is the evaluation of $Elim_{\mathcal{A}}$ *at the coefficients of* $f_0,\dots,f_n$*.*

The projective variety $\overline{\pi(W_{\mathcal{A}})}$ is irreducible and its codimension in $\prod_{i=0}^{n}\mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ is the maximum of $\#(I) - rank(I)$ where I runs over all subsets of $\{0,1,\dots,n\}$. The variety $\overline{\pi(W_{\mathcal{A}})}$ has codimension 1 if and only if there exists an essential family $\{\mathcal{A}_i\}_{i\in I}$ where $I \subset \{0,1,\dots,n\}$, see (Gelfand et al., 1995) and (Sturmfels, 1994). For example, if each $Q_i = conv(\mathcal{A}_i)$ is of dimension n, then the family $\{\mathcal{A}_i\}_{i\in I}$ is essential for $I = \{0,1,\dots,n\}$ and hence $\overline{\pi(W_{\mathcal{A}})}$ defines a hypersurface in the product projective space $\prod_{i=0}^{n}\mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$.

In general, the sparse eliminant is called *sparse resultant* by many authors, as in (Gelfand et al., 1995),(Cox et al., 2006), etc. However, we prefer to use the name *sparse eliminant*, since in the next subsection, following (D'Andrea, Galligo & Sombra, 2014) and (D'Andrea & Sombra, 2015), we introduce another type of resultant which will be called *sparse resultant*. The first efficient method was introduced by Sturmfels in (Sturmfels, 1991) and (Sturmfels, 1994) for computing sparse eliminants. Also, Canny and Emiris introduced algorithm for the same purpose in (elimination theory, CE1) and (Canny & Emiris, 2000). In 2002, D'Andrea succeed to propose Macaulay type formula for computing sparse eliminant in (D'Andrea, 2002).

The classical resultant $Res_{d_0,\dots,d_n}$ is the special case of the sparse eliminant. Here, let $\mathcal{A}_i$ be the set of all integer points in the d_i-simplex, i.e., $\mathcal{A}_i = d_i\Sigma_n \cap \mathbb{Z}^n$ and Σ_n is the standard unit simplex:

$$d_i\Sigma_n := \{(a_0,\dots,a_n) \in \mathbb{R}^{n+1} : a_j \geq 0, \quad \sum_j a_j \leq d_i\}.$$

Following (Cox et al., 2006) and (Gelfand et al., 1995), for simplicity let all the sparse polynomials $f_0,\dots,f_n$ have the same support $\mathcal{A}_d = d\Sigma_n \cap \mathbb{Z}^n$ for some positive integer d and consider the system

$$\begin{cases} f_0 = u_{01}\boldsymbol{x}^{\alpha_1} + \ldots + u_{0d}\boldsymbol{x}^{\alpha_n} = 0 \\ \vdots \\ f_n = u_{n1}\boldsymbol{x}^{\alpha_1} + \ldots + u_{nd}\boldsymbol{x}^{\alpha_n} = 0 \end{cases} \tag{2.8}$$

Also consider the homogeneous coordinates $t_0, \dots, t_n$ which are related to $x_1, \dots, x_n$ via the change of variable $x_i = t_i/t_0$ for $i = 1, \dots, n$. Then we homogenize the sparse system (2.8) by defining

$$F_i(t_0, \dots, t_n) = t_0^d f_i(t_1/t_0, \dots, t_n/t_0) = t_0^d f_i(x_1, \dots, x_n), \tag{2.9}$$

for $0 \leq i \leq n$. This method gives n+1 homogeneous polynomials of total degree d in the variables $t_0, \dots, t_n$ and this definition is independent of the choice of homogeneous coordinates.

The following proposition gives the relation between the multipolynomial resultant and the sparse eliminant.

Proposition 2.2.5 (Cox et al. (2006))**.** *Let $\mathcal{A}_d = d\Sigma_n \cap \mathbb{Z}^n$ and consider the systems of polynomials $\boldsymbol{F}$ and $\boldsymbol{f}$ as above. Then*

$$Elim_{\mathcal{A}}(f_0, \dots, f_n) = \pm Res_{d,\dots,d}(F_0, \dots, F_n),$$

where $\mathcal{A} = (\mathcal{A}_d, \dots, \mathcal{A}_d)$.

Corollary 2.2.6. *Let $\boldsymbol{f} = (f_1, \dots, f_n)$ be a system of polynomials with supp(f_i) $= \mathcal{A}_d$ for $i = 1, \dots, n$. Assume that the system $\boldsymbol{F} = (F_0, \dots, F_n)$ consists the homogenizations of f_i according to process in (2.9) and denote the set of simultaneous solutions of $\boldsymbol{F}$ by $Z(\boldsymbol{F})$. Suppose that $Z(\boldsymbol{F}) \cap H^{\infty}(t_0) = \emptyset$ where $H^{\infty}(t_0)$ is the hyperplane at infinity for $t_0 = 0$. Then the system of polynomials $\boldsymbol{f}$ has no common solution if and only if $Elim_{\mathcal{A}_d}(f_0, \dots, f_n) \neq 0$.*

Proof. If $\mathrm{Elim}_{\mathcal{A}_d}(f_0, \dots, f_n) \neq 0$, then by definition the system

$$f_0 = \ldots = f_n = 0$$

has no solution. On the other hand, let $\boldsymbol{x} \notin Z(\boldsymbol{f})$, then there exists an $i \in \{0, \dots, n\}$ such that $f_i(\boldsymbol{x}) \neq 0$. Suppose that F_i is the homogenization of f_i as described above process and assume that for corresponding variable $\boldsymbol{t} = (t_0, \dots, t_n)$, i.e., $F_i(\boldsymbol{t}) = t_0^d f_i(\boldsymbol{x})$. In this case $F_i = 0$ if only $t_0 = 0$ and this cause that $Z(\boldsymbol{F}) \cap H^{\infty}(t_0) \neq \emptyset$ which contradicts to our assumption. Hence $F_i(\boldsymbol{t}) \neq 0$ which means $\mathrm{Res}_d \boldsymbol{F} = \mathrm{Elim}_{\mathcal{A}_d} \boldsymbol{f} \neq 0$. □

2.2.4 The Sparse Resultant

Beside being a generalization of the multipolynomial resultant and involving considerable large amount of the system of polynomials, the sparse eliminant does not satisfy some essential properties which is necessary in many applications, such as additivity property and Poisson formula. In 2015, D'Andrea and Sombra introduced the following definition for the sparse resultant in (D'Andrea & Sombra, 2015) which satisfies many of the desired features.

Consider the nonempty finite subsets $\mathcal{A}_0, \ldots, \mathcal{A}_n$ of $\mathbb{Z}^n$ and the incidence variety

$$\mathcal{W}_{\mathcal{A}} = \left\{ (\boldsymbol{u}, \boldsymbol{x}) \in \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \times (\mathbb{C}^*)^n : f_0(\boldsymbol{u}_i, \boldsymbol{x}) = \cdots = f_n(\boldsymbol{u}_n, \boldsymbol{x}) = 0 \right\}. \tag{2.10}$$

The direct image of $\mathcal{W}_{\mathcal{A}}$ under the canonical projection

$$\pi : \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \times (\mathbb{C}^*)^n \to \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$$

is the Weil divisor of $\prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ given by

$$\pi_*(\mathcal{W}_{\mathcal{A}}) = \begin{cases} \mathit{deg}(\pi|_{\mathcal{W}_{\mathcal{A}}}) \overline{\pi(\mathcal{W}_{\mathcal{A}})} & \text{if } \operatorname{codim}(\overline{\pi(\mathcal{W}_{\mathcal{A}})}) = 1, \\ 0 & \text{if } \operatorname{codim}(\overline{\pi(\mathcal{W}_{\mathcal{A}})}) \geq 2 \end{cases} \tag{2.11}$$

where $\mathit{deg}(\pi|_{\mathcal{W}_{\mathcal{A}}})$ represents the degree of the restriction of the canonical map π to the incidence variety $\mathcal{W}_{\mathcal{A}}$.

Definition 2.2.3 (D'Andrea & Sombra (2015),D'Andrea et al. (2014))**.** *The* sparse resultant*, denoted by $\mathcal{R}es_{\mathcal{A}}$, is defined as any primitive polynomial in $\mathbb{Z}[\boldsymbol{u}]$ of this Weil divisor $\pi_*(\mathcal{W}_{\mathcal{A}})$. The expression*

$$\mathcal{R}es_{\mathcal{A}}(f_0, \ldots, f_n)$$

is the evaluation of $\mathcal{R}es_{\mathcal{A}}$ at the coefficients of $f_0, \ldots, f_n$.

In the next proposition, we see the relation between the sparse eliminant and the sparse resultant.

Proposition 2.2.7 (D'Andrea & Sombra (2015))**.** *The sparse resultant $\mathcal{R}es_{\mathcal{A}} \neq 1$ if and only if the sparse eliminant $Elim_{\mathcal{A}} \neq 1$ and, in this case*

$$\mathcal{R}es_{\mathcal{A}} = \pm Elim_{\mathcal{A}}^{\mathit{deg}(\pi|_{W_{\mathcal{A}}})}.$$

Example 2.2.4. *Let* $A_0 = A_1 = A_2 = \{(0,0),(2,0),(0,2)\}$. *Then* $Elim_{\mathcal{A}} = \det(u_{i,j})$ *and* $\mathcal{R}es_{\mathcal{A}} = \pm[\det(u_{i,j})]^4$.

2.2.4.1 The Directional Resultant

For a subset $\mathcal{B} \subset \mathbb{Z}^n$ and a polynomial $f = \sum_{\boldsymbol{b}\in\mathcal{B}} \beta_{\boldsymbol{b}} \boldsymbol{x}^{\boldsymbol{b}}$ with support $\mathcal{B}$, we write

$$\mathcal{B}^{\boldsymbol{v}} := \Big\{\boldsymbol{b} \in \mathcal{B} : \langle \boldsymbol{b}, \boldsymbol{v}\rangle = s_{conv(\mathcal{B})}(\boldsymbol{v})\Big\} \tag{2.12}$$

and

$$f^{\boldsymbol{v}} = \sum_{\boldsymbol{b}\in\mathcal{B}^{\boldsymbol{v}}} \beta_{\boldsymbol{b}} \boldsymbol{x}^{\boldsymbol{b}} \tag{2.13}$$

where $\boldsymbol{v} \in \mathbb{R}^n$ and $s_{conv(\mathcal{B})}(\boldsymbol{v})$ is defined as equation (2.1).

Definition 2.2.4. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$ *be a family of* n *non-empty finite subsets,* $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$, *and* $\boldsymbol{v}^{\perp} \subset \mathbb{R}^n$ *be the orthogonal subspace. Then, for* $i = 1, \ldots, n$, *there exists some* $\boldsymbol{b}_{i,\boldsymbol{v}} \in \mathbb{Z}^n$ *such that* $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}} \subset \mathbb{Z}^n \cap \boldsymbol{v}^{\perp}$. *The resultant of* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *in the direction of* $\boldsymbol{v}$, *denoted* $\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}$ *is defined as the sparse resultant of the family of the finite subsets* $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}}$.

Let $f_i \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ *be Laurent polynomials with support* $supp(f_i) \subset \mathcal{A}_i$, $i = 1, \ldots, n$. *For each* $i = 1, \ldots, n$, *we write* $f_i^{\boldsymbol{v}} = \boldsymbol{x}^{\boldsymbol{b}_{i,\boldsymbol{v}}} g_{i,\boldsymbol{v}}$ *for a Laurent polynomial* $g_{i,\boldsymbol{v}} \in \mathbb{C}[\mathbb{Z}^n \cap \boldsymbol{v}^{\perp}] \simeq \mathbb{C}[y_1^{\pm 1}, \ldots, y_{n-1}^{\pm 1}]$ *with* $supp(g_{i,\boldsymbol{v}}) \subset \mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}}$. *The expression*

$$\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \ldots, f_n^{\boldsymbol{v}})$$

is defined as the evaluation of this resultant at the coefficients of the $g_{i,\boldsymbol{v}}$.

One can check that for every nonzero vector $\boldsymbol{v} \in \mathbb{Z}^n$, it is always possible to find a vector $\boldsymbol{b}_{i,\boldsymbol{v}} \in \mathbb{Z}^n$ such that $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}} \subset \mathbb{Z}^n \cap \boldsymbol{v}^{\perp}$ because of the fact that the finite integer valued set $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}}$ is a subset of a supporting hyperplane of the convex hull of the Minkowski sum which is obtained by the family $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$. Further, this procedure is independent of the choice of the vector $\boldsymbol{b}_{i,\boldsymbol{v}}$ since the resultant is invariant under translations (D'Andrea & Sombra, 2015, Proposition 3.3).

If the direction vector $\boldsymbol{v}$ is an inward point normal to a facet of the Minkowski sum $\sum_{i=1}^{n} conv(\mathcal{A}_i)$, then the directional resultant $\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}} \neq 1$ and it is the only case when we have a nontrivial directional resultant by Definition 2.2.2.

2.2.4.2 Comparison of Sparse Eliminant and Sparse Resultant

As a generalization of the classical resultant, the sparse eliminant preserves some properties such as irreducibility, homogenenities and determinantial formulas. However, it does not satisfy some crucial properties such as as Poisson or additivity formulas. The reason of the emerge of the sparse resultant is the lack of such important features. Despite of the fact that the sparse resultant is not irreducible anymore, we acquire the following properties by virtue of the new definition.

Proposition 2.2.8 (D'Andrea & Sombra (2015),D'Andrea et al. (2014))**.**

- ***Additivity formula.*** *Suppose that* $\mathcal{A}_0,\ldots,\mathcal{A}_n,\mathcal{A}_i^{'}$ *are nonempty finite subsets of* $\mathbb{Z}^n$ *for* $i=0,\ldots,n$. *Assume that* f_j *are Laurent polynomials with* $supp(f_j)\subset\mathcal{A}_j$ *and let* $f_i^{'}$ *be a further Laurent polynomials with* $supp(f_i^{'})=\mathcal{A}_i^{'}$. *Then,*
$$\mathcal{R}es_{\mathcal{A}_0,\ldots,\mathcal{A}_i+\mathcal{A}_i^{'},\ldots,\mathcal{A}_n}(f_0,\ldots,f_if_i^{'},\ldots,f_n)=$$
$$\pm\mathcal{R}es_{\mathcal{A}_0,\ldots,\mathcal{A}_i,\ldots,\mathcal{A}_n}(f_0,\ldots,f_i,\ldots,f_n)\mathcal{R}es_{\mathcal{A}_0,\ldots,\mathcal{A}_i^{'},\ldots,\mathcal{A}_n}(f_0,\ldots,f_i^{'},\ldots,f_n).$$

- ***Poisson formula*** *Suppose that* $\mathcal{A}=(\mathcal{A}_0,\ldots,\mathcal{A}_n)$ *is a family of nonempty finite subsets of* $\mathbb{Z}^n$ *and consider the Laurent polynomials* f_i *with* $supp(f_i)\subset\mathcal{A}_i$ *for* $i=0,\ldots,n$. *Let* $\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\ldots,f_n^{\boldsymbol{v}})\neq 0$ *for all nonzero vector* $\boldsymbol{v}\in\mathbb{Z}^n$. *Then*
$$\mathcal{R}es_{\mathcal{A}}(f_0,f_1,\ldots,f_n)=\pm\left(\prod_{\boldsymbol{v}}\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\ldots,f_n^{\boldsymbol{v}})^{-s_{\mathcal{A}_0(\boldsymbol{v})}}\right)\prod_{\boldsymbol{\xi}\in V(\boldsymbol{f})}f_0(\boldsymbol{\xi})^{mult(\boldsymbol{\xi}|\boldsymbol{f})}$$
where $V(\boldsymbol{f})$ *denote the set of isolated solution for the system* $f_1=\ldots=f_n=0$ *and* $s_{\mathcal{A}_0}(.)$ *is the support function of* $\mathcal{A}_0$ *as described in (2.1).*

2.2.5 The Bernstein Theorem

In this part, as a generalization of the Bézout's theorem, we mention some versions of Bernstein-Kushnirenko theorem (or Bernstein-Kushnirenko-Khovanskii theorem (BKK)) which gives the upper bound for the number of the solution of Laurent polynomial systems. We first recall the classical Bézout's bound for a system of n polynomials in n variables.

Theorem 2.2.9. *(Cox et al., 2006, The Bézout's Theorem) Let $g_1,\dots,g_n \in \mathbb{C}[x_1,\dots,x_n]$ be polynomial equations of positive degrees $d_1,\dots,d_n$, respectively. Then the system of equations*

$$g_1(x_1,\dots,x_n) = \dots = g_n(x_1,\dots,x_n) = 0 \tag{2.14}$$

has either an infinite number of solutions (including the solutions at infinity) or the number of the complex solutions cannot exceed the number $d = d_1 \dots d_n$. Furthermore, if the solutions at infinity are counted and with appropriate multiplicity, the exact number of solutions is $d = d_1 \dots d_n$ in the complex projective space $\mathbb{P}^n(\mathbb{C})$.

First, we introduce the Kushnirenko's theorem following (Gelfand et al., 1995).

Theorem 2.2.10 (Gelfand et al. (1995))**.** *Consider $f_1,\dots,f_n$ Laurent polynomials and let $supp(f_i) = \mathcal{A}$ be a nonempty finite subset of $\mathbb{Z}^n$ with $Q = conv(\mathcal{A})$. Then the number of common zeros of the f_i in the algebraic torus $(\mathbb{C}^*)^n$ is at most the volume $n!Vol_n(Q)$. Furthermore, for generic polynomials (for generic choice of the coefficients in the f_i), the number of common zeros is exactly $n!Vol_n(Q)$.*

Example 2.2.5. *Consider a univariate Laurent polynomial $f(x) = a_i x^i + \dots + a_n x^n$. Assuming that $a_i, a_n \neq 0$, the number of nonzero roots of f is $n - i$. Observe that the Newton polytope of f is the line segment $[i, n]$ and the length of the segment is $n - i$.*

The Bernstein theorem generalizes the Kouchnirenko's theorem to the case of the systems of equations where each equation might have different supports. We give the following version of Bernstein's theorem which is given in (Gelfand et al., 1995).

Theorem 2.2.11 (Gelfand et al. (1995))**.** *Let $\mathcal{A}_1,\dots,\mathcal{A}_n$ be nonempty finite subsets of $\mathbb{Z}^n$ and Q_i be the convex hull of $\mathcal{A}_i$. Assume that $\mathbb{C}^{\mathcal{A}_i}$ be space of Laurent polynomials in $x_1,\dots,x_n$ with monomials from $\mathcal{A}_i$. Then there is a dense Zariski open subset $\Omega \subset \prod \mathbb{C}^{\mathcal{A}_i}$ satisfying the property that for any choice of $(f_1,\dots,f_n)$, the number of the common zeros in $(\mathbb{C}^*)^n$ equals the mixed volume $MV_{\mathbb{R}^n}(Q_1,\dots,Q_n)$.*

We also introduce the most common version of Bernstein-Kushnirenko-Khovanskii theorem that can be found in many books. Here, we follow (Cox et al., 2006).

Theorem 2.2.12. *(Cox et al., 2006, BKK theorem) Suppose that given Laurent polynomials $f_1, \dots, f_n$ over $\mathbb{C}$ have finitely many common zeros in $(\mathbb{C}^*)^n$. Let $Q_i = conv(\mathcal{A}_i)$ be the Newton polytope of f_i in $\mathbb{R}^n$. Then the number of common zeros of f_i in $(\mathbb{C}^*)^n$ is at most the mixed volume $MV_{\mathbb{R}^n}(Q_1, \dots, Q_n)$. Moreover, for the generic choices of f_i, the number of common zeroes is exactly $MV_{\mathbb{R}^n}(Q_1, \dots, Q_n)$.*

In this thesis, we use the original version of Bernstein's theorem as mentioned in (Bernshtein, 1975) which is stated as follows:

Theorem 2.2.13 (Bernshtein (1975))**.** *Let $\boldsymbol{f} = (f_1, \dots, f_n)$ be a system of Laurent polynomials with supports $\mathcal{A}_1, \dots, \mathcal{A}_n$, respectively. Then if for any nonzero vector $\boldsymbol{v}$, the directed system $\boldsymbol{f}^{\boldsymbol{v}} = (f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}})$ has no common zero in $(\mathbb{C}^*)^n$ then all common zeros of the system $\boldsymbol{f}$ are isolated. Further, the exact number of the solutions is $MV_{\mathbb{R}^n}(Q_1, \dots, Q_n)$ where $Q_i = conv(\mathcal{A}_i)$ for $i = 1, \dots, n$.*

The sparse resultant techniques performance a new bound which bounds the number of solutions of a system like (2.14) in $(\mathbb{C}^*)^n$. One can see that the torus $(\mathbb{C}^*)^n$ can be obtained by substractiong all coordinate hyperplanes $x_i = 0$ from the complex projective space $\mathbb{P}^n(\mathbb{C})$ for all $i = 0, \dots, n$. Since $x_i = 0$ is the hyperplane at infinity, the BKK bound counts solutions to the sytem (2.14) in $\mathbb{C}^n$ which have no zero coordinate and it is computed from the mixed volumes of a sum of polytopes obtained by the sytem (2.14).

For a given Laurent polynomial system $\boldsymbol{f}$, Huber and Sturmfels introduced the following result which determines the number of solutions using the sparse eliminant.

Theorem 2.2.14. *(Huber & Sturmfels, 1995, Theorem 6.1) Let $\boldsymbol{f} = (f_1, \dots, f_n)$ be a Laurent polynomial system with $supp(f_i) = \mathcal{A}_i$ for $i = 1, \dots, n$ and $Q_i = conv(\mathcal{A}_i)$. The number simultaneous solutions in $(\mathbb{C}^*)^n$ of the system $\boldsymbol{f}$ is $MV(Q_1, \dots, Q_n)$, counting multiplicities, if and only if for all facet inner normal vectors $\boldsymbol{v}$ of $Q_1 + \cdots + Q_n$, the sparse eliminant $Elim_{\mathcal{A}^{\boldsymbol{v}}} \boldsymbol{f}^{\boldsymbol{v}}$ is a nonzero complex number.*

Using the relation in between sparse eliminant and the sparse resultant as in Proposition 2.2.7, we have the following corollary.

Corollary 2.2.15. *The system $\boldsymbol{f}$ has $MV(Q_1, \dots, Q_n)$ simultaneous zeros in $(\mathbb{C}^*)^n$, counting multiplicities, if and only if for all facet inner normal vectors $\boldsymbol{v}$ of $Q_1 + \cdots + Q_n$, the sparse resultant $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}}} \boldsymbol{f}^{\boldsymbol{v}}$ is a nonzero complex number.*

3. Equidistributions of Zeros of Random Polynomial Systems

In this chapter, we introduce some of the very well known results on the equidistribution of zeros of random polynomials and of simultaneous zeros of random multivariate polynomial systems.

3.1 Distribution of Zeros of Random Univariate Polynomials

We mention the univariate random polynomials and introduce the related results in the literature. Following Tao & Vu (2015) we define univariate *random polynomials* as follows:

Definition 3.1.1. *Let d be a positive integer and $c_0, \ldots, c_d$ be deterministic complex numbers and $a_0, \ldots, a_d$ be nondegenerate independent and identically distributed (i.i.d.) random variables of mean zero and finite nonzero variance. A random polynomial $f_d : \mathbb{C} \to \mathbb{C}$ associated to c_i and a_i is defined as*

$$f_d(x) = \sum_{i=0}^{d} c_i a_i x^i. \tag{3.1}$$

Here, a random variable is called nondegenerate if the supports of its probability law contains at least two points.

Example 3.1.1.

- *Kac polynomials are polynomials associated to the coefficients $c_i = 1$.*
- *Weyl polynomials are polynomials associated to the coefficients $c_i = \sqrt{\frac{1}{i!}}$.*
- *Elliptic polynomials are polynomials associated to the coefficients $c_i = \sqrt{\binom{n}{i}}$.*

- *Hyperbolic polynomials are polynomials associated to the coefficients* $c_i = \sqrt{\frac{M(M+1)...(M+i-1)}{i!}}$ *for some parameter* $M > 0$.

One can see that the Kac polynomial is a special case of hyperbolic polynomials for $M = 1$.

Let $Poly_d$ denote the space of polynomials of degree at most d. Identifying a polynomial $f_d \in Poly_d$ with $(a_0, \ldots, a_d) \in \mathbb{C}^{d+1}$, we endow the complex vector space $\mathbb{C}^{d+1}$ with a $(d+1)$-fold product probability measure $\mathbf{P}_d$ which is obtained from the individual probability laws of the random coefficients. Pulling back this operation, we define the d-th stage product probability space as $(Poly_d, \mathbf{P}_d)$. Since we can operate this for each $d \in \mathbb{N}$, we define the product probability space $\prod_{d=0}^{\infty}(Poly_d, \mathbf{P}_d)$ which contains the sequence of random polynomials $\{f_d\}_d$ with increasing degree.

Also, if we let $Z(f_d) := \{\xi_i : f_d(\xi_i) = 0\}$ be the set of zeros of f_d in $\mathbb{C}$, then a polynomial $f_d \in Poly_d$ can be expressed as

$$f_d = \sum_{i=0}^{d} a_i x^i = a_d \prod_{i=1}^{d}(x - \xi_i).$$

We define a random valued measure $\delta_{Z(f_d)}$ associated to the zeros of f_d as

$$Poly_d \to \mathcal{M}(\mathbb{C}) \tag{3.2}$$

$$f_d \mapsto \delta_{Z(f_d)} := \frac{1}{d}\sum_{i=1}^{d} \delta_{\xi_i} \tag{3.3}$$

where δ_{ξ_i} is the Dirac mass function with support $Z(f_d)$. We define *the expected zero measure* of f_d as

$$\left\langle \mathbb{E}[\delta_{Z(f_d)}], \varphi \right\rangle := \int_{Poly_d} \sum_{i=1}^{d} \varphi(\xi_i) d\mathbf{P}_d \tag{3.4}$$

for a compactly supported continuous function $\varphi \in C_c(\mathbb{C})$.

Our interest is the asymptotic behaviour of the zeros of f_d as $d \to \infty$. First, we state a very well known equidistribution result for Kac ensembles that has been studied extensively for many authors.

Theorem 3.1.1 (Kac (1943),Hammersley (1956),Shepp & Vanderbei (1995))**.** *Let* $f_d(x) = \sum_{i=0}^{d} a_i x^i$ *be a Kac polynomial. Assume that* a_i *are independent and* *identically distributed real or complex valued Gaussian random variables of mean zero and variance one. Then almost surely*

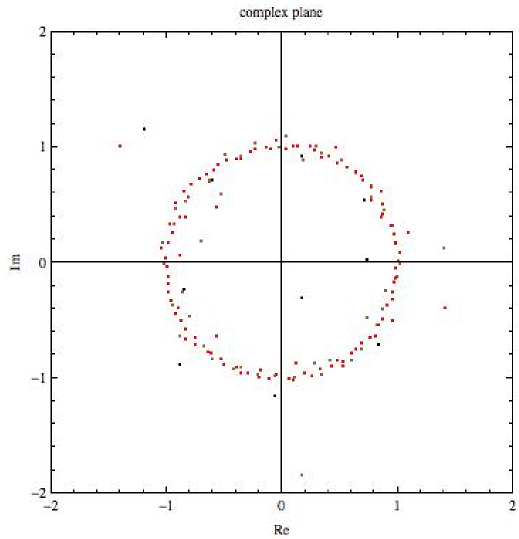
$$\delta_{Z(f_d)} \to \frac{1}{2\pi} d\theta \tag{3.5}$$

weakly as $d \to \infty$.

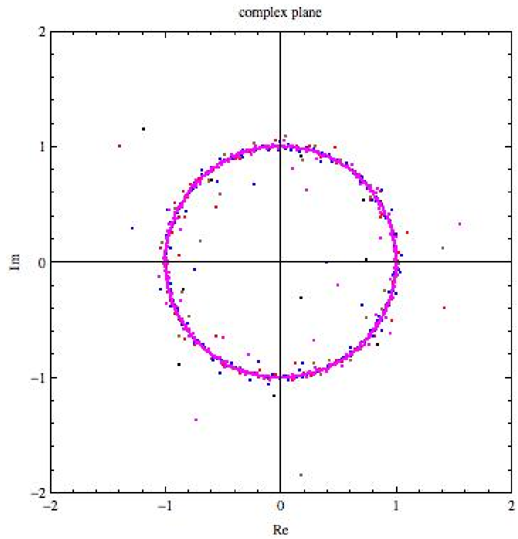
Also,

$$\frac{1}{d}\mathbb{E}[\delta_{Z(f_d)}] \to \frac{1}{2\pi} d\theta$$

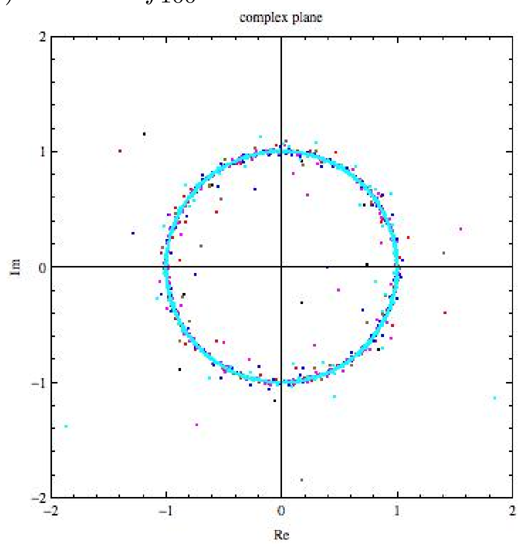
as $d \to \infty$.



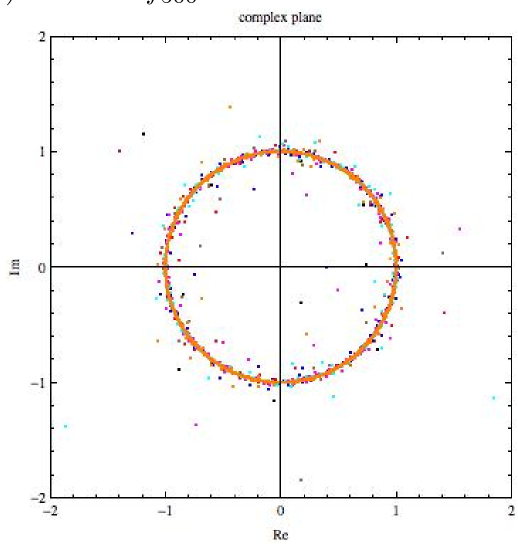
(a) Zeros of f_{100}



(b) Zeros of f_{500}



(c) Zeros of f_{1000}



(d) Zeros of f_{2000}

Figure 3.1 Distributions of Zeros of Standard Gaussian Random Polynomials

In other words, the zeros of a random polynomial tend towards the unit circle $S^1 = \{|z| = 1\}$ mostly as degree $d \to \infty$ if the coefficients are i.i.d. complex Gaussian random variables of mean zero and variance one. The variance condition in the statement is not strict. In fact, one can replace it as nonzero finite variance as in the definition of random polynomials. On the other hand, the normalization of the random variables to have unit variance does not affect the zeros of f_d.

Shiffman and Zelditch generalized this result in (Shiffman & Zelditch (2003)) for any simply connected bounded domain Ω in $\mathbb{C}$ with analytic boundary. They considered the monomials $\{z^i\}_i$ as an orthonormal basis with respect to the inner product on S^1, i.e.,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{S^1} f(z)\overline{g(z)} d\theta \tag{3.6}$$

for the polynomials $f, g \in L^2\left(\frac{1}{2\pi} d\theta\right)$.

In general, any inner product on $Poly_d$ of the form

$$\langle f, g \rangle_\mu := \int_{\mathbb{C}} f(z)\overline{g(z)} d\mu$$

induces a Gaussian measure P_d as follows: Let $\{A_i(z)\}_{i=0}^d$ denote an orthonormal basis of $Poly_d$ with respect to the inner product $\langle ., . \rangle_\mu$ and express a polynomial $f_d \in Poly_d$ as

$$f_d(z) = \sum_{i=0}^{d} a_i A_i(z).$$

If the coefficients are independent and identically distributed Gaussian random variables with mean zero and variance one, then the Gaussian measure is $\pi^{-d-1} e^{|a|^2} da$ in terms of the coefficients.

Let Ω be a bounded simply connected C^ω domain in $\mathbb{C}$ and ϱ is a positive C^ω density on the boundary $\partial\Omega$. Suppose that the inner product (3.6) on S^1 is replaced by an inner product on $\partial\Omega$ of the form

$$\langle f, g \rangle_{\partial\Omega, \varrho} := \int_{\partial\Omega} f \overline{g} \varrho d\theta \tag{3.7}$$

for $\varrho \in C^\omega(\partial\Omega)$. The Gaussian measure induced by this inner product on $Poly_d$ will be denoted by $P_{d,\partial\Omega}$. Hence, the d-th stage probability space is $(Poly_d, P_{d,\partial\Omega})$.

Following (Ransford, 1995), we introduce some terminology from potential theory in order to complete the approach of Shiffman and Zelditch. Let $K \subset \mathbb{C}$ be a compact set and let $\mathcal{M}(K)$ denote the the set of all positive unit Borel measures supported on K. For a measure $\mu \in \mathcal{M}(K)$, the *energy of* μ is defined as

$$I(\mu) := \int\int \log \frac{1}{|z-w|} d\mu(z) d\mu(w),$$

and the *equilibrium measure* of K is the unique probability measure μ_K which minimizes the energy, that is,

$$I(\mu_K) := \inf_{\mu \in \mathcal{M}(K)} I(\mu). \tag{3.8}$$

A set $K \subset \mathbb{C}$ is called *polar* if its energy $I(\mu) = -\infty$ for all $\mu \in \mathcal{M}(K)$.

Theorem 3.1.2 (Shiffman & Zelditch (2003))**.** *Suppose that Ω is a bounded simply connected C^ω domain. Let $P_{d,\partial\Omega}$ be the Gaussian measure on $Poly_d$ induced by the inner product in (3.7) and let $\prod_{d=0}^{\infty} P_{d,\partial\Omega}$ be the product probability measure on $Poly = \prod_{d=0}^{\infty}(Poly_d, P_{d,\partial\Omega})$. Then for almost all sequences of random polynomials with i.i.d. Gaussian coefficients $\{f_d\}_d$,*

$$\lim_{d\to\infty} \delta_{Z(f_d)} = \mu_\Omega.$$

Using the approach of Shiffman and Zelditch, Bloom generalized Theorem 3.1.2 for the compact subsets K of $\mathbb{C}$ with some mild conditions in (Bloom (2007)). Again, following Ransford (1995), for a compact set $K \subset \mathbb{C}$ we define the *logarithmic capacity* of K as

$$cap(K) := e^{-\gamma(K)},$$

where

$$\gamma(K) := \inf_{\mu \in \mathcal{M}(K)} I(\mu).$$

The number $\gamma(K)$ is called the *Robin's constant* of K. Hence, for a compact set $K \subset \mathbb{C}$, its logarithmic capacity $cap(K)$ is positive whenever the Robin's constant $\gamma(K)$ is finite.

In general, the capacity of an arbitrary set $E \subset \mathbb{C}$ is defined as

$$cap(E) := \sup\{cap(K) : K \subset E, \quad K \text{ is compact}\}.$$

Assume that $cap(K) > 0$. Then, by definition, the Robin's constant $\gamma(K)$ attains its infimum at the equilibrium measure of K, i.e.,

$$I(\mu_K) = \gamma(K).$$

Let V_K denote the Green's function of the unbounded component of $\mathbb{C} \setminus K$ with pole at infinity and assume that V_K is defined on $\mathbb{C}$ by setting $V_K = 0$ on K and on the bounded components of K. Suppose that K is a regular set in the potential theory sense such that the function V_K is continuous and the equilibrium measure of K can be expressed as

$$d\mu_K := \frac{1}{2\pi} dd^c V_K.$$

For a compact set K and a measure $\mu \in \mathcal{M}(K)$, we say that the pair (K, μ) satisfies the Bernstein-Markov property, if for an $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon)$ such that

$$\|f_d\|_K \leq C(1+\varepsilon)^d \|f_d\|_{L^2(\mu)},$$

for all $f_d \in Poly_d$.

The result of Bloom is deduced from the following deterministic result of Blatt, Saff and Simkani for regular compact sets.

Theorem 3.1.3 (Blatt, Saff & Simkani (1988))**.** *Suppose K is a nonpolar and regular compact set in the complex plane $\mathbb{C}$ and $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies the Bernstein-Markov inequality. Let $f_d(x) = \sum_{i=0}^{d} a_i^d x^i$ be a sequence of polynomials satisfying*

- $\overline{\lim}_{d \to \infty} ||f_d||_K^{1/d} \leq 1$,
- $\lim_{d \to \infty} d^{-1} \log |a_d^d| = -\log(cap(K))$,
- *for each bounded connected component in $\mathbb{C} \setminus K$ there exists a point x_0 such that* $\lim_{d \to \infty} |f_d(x_0)|^{1/d} = 1$.

Then,

$$\lim_{d \to \infty} \delta_{Z(f_d)} = d\mu_K,$$

weakly on $\mathbb{C} \cup \{\infty\}$.

Theorem 3.1.4 (Bloom (2007))**.** *Let K be a compact set in the complex plane and $\{A_i(z)\}_i$ be an orthonormal basis with respect to a regular measure μ supported on K and suppose that (K, μ) satisfies the Bernstein-Markov property. Consider the* *polynomials of the form*

$$f_d(z) = \sum_{i=0}^{d} a_i A_i(z) \tag{3.9}$$

where the coefficients a_i are i.i.d. complex Gaussian random variables. Then

$$\delta_{Z(f_d)} \to \mu_K$$

almost surely as $d \to \infty$, and

$$d^{-1}\mathbb{E}[\delta_{Z(f_d)}] \to \mu_K \tag{3.10}$$

weakly, where μ_K is the equilibrium measure of the compact set K.

Another interest is examining the zeros of random polynomials with more general coefficients. The universality result on the coefficients for Kac ensembles is given by Ibragimov and Zaporozhets which generalizes Theorem 3.1.1 for a quite large family of random variables.

Theorem 3.1.5 (Ibragimov & Zaporozhets (2013))**.** *Let $f_d(x) = \sum_{i=0}^{d} a_i x^i$ be a Kac polynomial. If the coefficients a_i are nondegenerate i.i.d. random variables, then $\mathbb{E}[\log(1+|a_0|)] < \infty$ is the necessary and sufficient condition for*

$$\delta_{Z(f_d)} \to \frac{1}{2\pi} d\theta$$

almost sure weakly as $d \to \infty$.

Generalization of Theorem 3.1.5 for more general type of random polynomials is studied in (Kabluchko & Zaporozhets, 2014), (Bloom & Dauvergne, 2019), (Pritsker, 2018) and (Pritsker & Ramachandran, 2017), etc. But recently, the most comprehensive version is given by Dauvergne.

Definition 3.1.2. *Let K be a compact set on the complex plane. A sequence of degree d polynomials $\left\{p_d = \sum_{i=0}^{d} c_{d,i} z^i : n \in \mathbb{N}\right\}$ is called asymptotically minimal on K if there exists a regular measure μ of support K and a $p \in (0,\infty]$ satisfying*

$$\lim_{d\to\infty} \frac{1}{d} \log|c_{d,d}| = -\log cap(K) \quad \textit{and} \quad \lim_{d\to\infty} \frac{1}{d} \log\|p_d\|_{L^p(\mu)} = 0.$$

Theorem 3.1.6 (Dauvergne (2019))**.** *Let μ be a regular measure with nonpolar compact support $K \subset \mathbb{C}$. Let $\{p_i\}$ be a sequence of asymptotically minimal polynomials on K and $\{a_i\}$ be a sequence of i.i.d. non-degenerate complex random* *variables. Consider the random polynomials of the form*

$$f_d(z) = \sum_{i=0}^{d} a_i p_i(z).$$

Then the measure $\delta_{Z(f_d)}$ converges to the equilibrium measure μ_K almost surely if and only if

$$\mathbb{E}[\log(1+|a_0|)] < \infty. \tag{3.11}$$

Furthermore, the measure $\delta_{Z(f_d)}$ converges weakly to μ_K in probability if and only if

$$\mathbb{P}\left(|a_0| > e^d\right) = o(d^{-1}). \tag{3.12}$$

In particular, if the condition (3.11) and (3.12) does not occur, then the sequence $\delta_{Z(f_d)}$ does not have almost sure limit, and limit in probability in the space of probability measures on $\mathbb{C}$, respectively.

Another approach for the distribution of zeros of random or deterministic polynomials is given by Erdös-Turán and Hughes-Nikeghbali in terms of the angle discrepancy and radius discrepancy of the roots of f_d, respectively.

Let $f_d(x) = a_d x^d + \ldots + a_0$ be a deterministic or random complex polynomial of degree d and let $Z(f_d)$ denote the set of zeros of f_d in $\mathbb{C}$. For each $-\pi \leq \alpha < \beta \leq \pi$, consider the set

$$Z_{\alpha,\beta}(f_d) = \{\xi \in Z(f_d) : \alpha < \arg(\xi) \leq \beta\}$$

where $\arg(\xi)$ denotes the argument of ξ. Then the *angle discrepancy* of $Z(f_d)$ is defined as

$$\Delta_{ang}(Z(f_d)) = \sup_{-\pi \leq \alpha < \beta \leq 2\pi} \left| \frac{|Z_{\alpha,\beta}(f_d)|}{d} - \frac{\beta - \alpha}{2\pi} \right|.$$

Also consider

$$Z_\varepsilon(f_d) = \left\{\xi \in Z(f_d) : 1 - \varepsilon < |\xi| < (1-\varepsilon)^{-1}\right\}$$

for $0 < \varepsilon < 1$. The *radius discrepancy* of $Z(f_d)$ is

$$\Delta_{rad}(Z(f_d), \varepsilon) = 1 - \frac{|Z_\varepsilon(f_d)|}{d}$$

For example, if $f_d(x) = x^d - 1$ then $\Delta_{ang}(Z(f_d)) = 1/d$ and $\Delta_{rad}(Z(f_d), \varepsilon) = 0$ for each ε.

Theorem 3.1.7 (Erdös & Turán (1950), Hughes & Nikeghbali (2008))**.** *Let $f_d = a_d x^d + \ldots + a_0$ be a complex polynomial with $d \geq 1$ and $a_0 a_n \neq 0$. Then*

$$\Delta_{ang}(Z(f_d)) \leq 16\sqrt{\frac{1}{d}\log\left(\frac{\|f\|_{\sup}}{\sqrt{a_0 a_d}}\right)}, \quad \Delta_{rad}(Z(f_d),\varepsilon) \leq \frac{2}{\varepsilon d}\log\left(\frac{\|f\|_{\sup}}{\sqrt{a_0 a_d}}\right),$$

for $0 < \varepsilon < 1$, where the supremum norm is defined as $\|f\|_{\sup} = \sup_{|w|=1}|f(w)|$.

The constant 16 in the theorem above is replaced by Ganelius. He showed that the constant c has to be at most $\sqrt{\frac{2\pi}{K}}$ where $K = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^2}$ is the Catalan's constant in (Ganelius (1954)). On the other hand, Amoroso and Mignotte improved that the constant c cannot be less than $\sqrt{2}$ in (Amoroso & Mignotte (1996)).

As a consequence of the Theorem 3.1.5., roughly speaking, it can be seen that for a univariate random polynomial f_d, if the distributions of middle coefficients do not grow too faster than the extreme ones, the leading term and the constant term, then the set of zeros $Z(f_d)$ tends to unit circle $\{|z| = 1\}$. More precisely:

Corollary 3.1.8. *Let $\{f_d\}_{d\geq 1}$ be a sequence of polynomials of degree d with $\log\left(\frac{\|f\|_{\sup}}{\sqrt{a_0 a_d}}\right) = o(d)$. Then*

$$\lim_{d\to\infty}\frac{|Z_{\alpha,\beta}(f_d)|}{d} = \frac{\beta-\alpha}{2\pi} \qquad \textit{and} \qquad \lim_{d\to\infty}\frac{|Z_{\varepsilon}(f_d)|}{d} = 1.$$

A further result related to the discrepancy estimates is given by Pritsker and Sola. They give a quantitative estimate for the expected discrepancy of the random polynomials for annular sectors. More precisely, let $0 \leq \alpha < \beta < 2\pi$ and $0 < \varepsilon < 1$. Define the *annular sectors* as

$$A_{\varepsilon}(\alpha,\beta) := \{\xi \in \mathbb{C} : \varepsilon < |\xi| < \varepsilon^{-1},\ \alpha \leq \arg(\xi) < \beta\}.$$

Theorem 3.1.9 (Pritsker & Sola (2014))**.** *Let $f_d(x) = \sum_{i=0}^{d} a_i x^i$ be random polynomial such that the coefficients a_i are i.i.d. complex random variables with absolutely continuous distribution. If $\mathbb{E}[|a_0|^t] < \infty$, for some $t > 0$, then*

$$\begin{aligned} &\mathbb{E}\left[\left||A_{\varepsilon}(\alpha,\beta)| - \frac{\beta-\alpha}{2\pi}\right|\right] \\ &\leq \left(\sqrt{\frac{2\pi}{K}} - \frac{2}{1-\varepsilon}\right)\sqrt{\frac{\frac{t+2}{2t}\log(d+1) + \frac{1}{t}\log\mathbb{E}[|a_0|^t] + \frac{1}{2e} - \mathbb{E}[\log|a_0|]}{d}}, \end{aligned} \tag{3.13}$$

where K denotes the Catalan's constant.

The generalization of the discrepancies of the zero sets of polynomials on Jordan domains can be found in (Andrievskii & Blatt, 1999, 2001, 1997; Erdélyi, 2008).

3.2 Distribution of Zeros of Random Polynomial Systems

In this part, as a generalization of the univariate case, we mention the distribution of simultaneous zeros of random polynomial systems and introduce the related results in the literature.

In order to state the next result as an analogue of Theorem 3.1.4, we list some result of pluripotential theory following (Klimek, 1991).

Let $D \subset \mathbb{C}^n$ be an open set. A function $u : D \to [-\infty, \infty)$ is called *plurisubharmonic* on D if

- u is uppersemicontinuous,
- for each $a \in D$ and $b \in \mathbb{C}^n$, the function $\lambda \mapsto u(a+\lambda b)$ is subharmonic on the set $\{\lambda \in \mathbb{C} : a+\lambda b \in D\}$.

We denote the collection of plurisubharmonic functions on D by $\mathcal{PSH}(D)$. A set $E \subset \mathbb{C}^n$ is *pluripolar* if there exists a nonconstant plurisubharmonic function u satisfying $E \subset \{u = -\infty\}$. Pluripolar sets are Lebesgue measure zero since plurisubharmonic functions are locally integrable. Let $\mathcal{L}(\mathbb{C}^n)$ denote the Lelong class which is the set of plurisubharmonic functions of logarithmic growth on $\mathbb{C}^n$, i.e.,

$$\mathcal{L}(\mathbb{C}^n) := \{u \in \mathcal{PSH}(\mathbb{C}^n) : u(x) \leq \log^+ ||x|| + O(1)\},$$

where $||x|| = (\sum_{i=1}^n |x_i|^2)^{1/2}$ and $\log^+ ||x|| = \max(0, \log ||x||)$. Let $\mathcal{P}_d$ denotes the collection of the polynomials on $\mathbb{C}^n$ and of degree at most d. Note that if $p \in \mathcal{P}_d$, then $\frac{1}{deg(p)} \log |p| \in \mathcal{L}(\mathbb{C}^n)$.

For a compact subset K of $\mathbb{C}^n$, its *pluricomplex Green function* or the *extremal function* $V_K(x)$ is defined as

$$V_K(x) = \sup\{u(x) : u \in \mathcal{L}(\mathbb{C}^n),\ u \leq 0 \quad \text{on} \quad K\} \tag{3.14}$$

or thank to results of Siciak and Zaharyuta (see Klimek (1991) and references therein)

(3.15)
$$V_K(x) := \sup\left\{\frac{1}{deg(p)}\log|p(x)| : p \text{ is a nonconstant polynomial, and } ||p||_K \leq 1\right\}.$$

Example 3.2.1. *If* $K = (S^1)^n = \{(x_1, \ldots, x_n) : |x_i| = 1 \quad for \quad i = 1 \ldots = n\}$, *then*

$$V_K(x_1, \ldots, x_n) = \max\{0, \log|x_1|, \ldots, \log|x_n|\}.$$

If K is non-pluripolar, then the uppersemicontinuous regularization of its Green function $V_K^* \in \mathcal{L}(\mathbb{C}^n)$. We suppose that K is regular that is its Green function V_K is continuous on $\mathbb{C}^n$, i.e., $V_K = V_K^*$. The function V_K is a locally bounded function on $\mathbb{C}^n$ and it satisfies

$$V_K = \log^+||x|| + O(1). \tag{3.16}$$

It is very well-known that the complex Monge-Ampère operator $(dd^c)^n = (2i\partial\overline{\partial})^n$ is defined for locally bounded plurisubharmonic functions on $\mathbb{C}^n$ (Bedford & Taylor, 1982), in particular for V_K. Also, by (Klimek, 1991, Cor. 5.5.3), the *equilibirium measure* of a regular compact set K is defined by

$$\mu_K := \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^n. \tag{3.17}$$

Let μ be a unit Borel measure on a nonpluripolar compact set $K \subset \mathbb{C}^n$. We say that the measure μ satisfies the Bernstein-Markov inequality, if for each $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon)$ satisfying

$$||p||_K \leq Ce^{\varepsilon deg(p)}||p||_{L^2(\mu)} \tag{3.18}$$

for each polynomial $p \in \mathcal{P}_d$, where $||.||_K$ denotes the supnorm on K.

Suppose that for a polynomial $f \in \mathcal{P}_d$, we write $f = \sum_{i=1}^{d_n} a_i p_i$, where $\{p_i\}_i$ forms an orthonormal basis for $\mathcal{P}_d$ with respect to the inner product

$$\langle f, g\rangle = \int_K f\overline{g}d\mu \tag{3.19}$$

where $d_n = dim\mathcal{P}_d = \binom{n+d}{d}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector in $\mathbb{N}^n$ and let $|\alpha| = \sum_{i=1}^{n} \alpha_i$ represents the lenght of the vector α. We define a random polynomial of degree d on $\mathbb{C}^n$ as

$$f_{d,i}(x) = \sum_{|J| \leq d} a_{i,J} P_i(x)$$

where $\{a_{i,J}\}_{|J|\leq d}$ is a sequence of independent and identically distributed random variables and $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$ for $i = 1, \ldots, n$.

Similar to the univariate case, if we identify a random polynomial $f_d \in \mathcal{P}_d$ with its random coefficients $\{a_{i,J}\}_{|J|\leq d}$, then $\mathcal{P}_d$ is identified with $\mathbb{C}^{d_n}$. Assume that the coefficients $\{a_{i,J}\}_{|J|\leq d}$ are independent and identically distributed Gaussian random variables with mean zero and variance 1. The Guassian measure $\boldsymbol{P}_d$ induced by the inner product (3.19) is

$$\boldsymbol{P}_d = \left(\frac{1}{\pi^{d_n}}\right)^{e^{-|a|^2}} d\lambda_{d_n},$$

where $d\lambda_{d_n}$ represents the $2d_n$-dimensional Lebesgue measure on $\mathbb{C}^{d_n}$ and $|a|^2 = \sum_{|J|\leq d} |a_J|^2$. We define the product probabiliy space

$$\mathcal{P} := \prod_{d=1}^{\infty} \mathcal{P}_d,$$

where $|J| = d$ and it contains the sequence of random polynomials of increasing degree. Hence, the product measure $\boldsymbol{P} := \prod_{d=0}^{\infty} \boldsymbol{P}_d$ defines a probability measure on $\mathcal{P}$.

Let $\boldsymbol{f}_d^k = (f_{d,1}, \ldots, f_{d,k})$ be a polynomial system containing k many random polynomials on $\mathbb{C}^n$ for $1 \leq k \leq n$. If $\boldsymbol{f}_d^k$ contains less than n polynomials, it is called *k-system* and when $k = n$, $\boldsymbol{f}_d^n := \boldsymbol{f}_d$ is called a *full system.* Our main focus is to state the results on asymptotic behaviors of the simultaneous zeros of such systems. Consider the zero locus of a system $\boldsymbol{f}_d^k$, that is,

$$Z(\boldsymbol{f}_d^k) := \{z \in \mathbb{C}^n : f_{d,1} = \cdots = f_{d,k} = 0\}, \tag{3.20}$$

and define the normalized zero currents

$$\widetilde{Z}_{\boldsymbol{f}_d^k} := \frac{1}{d^k}[Z_{\boldsymbol{f}_d^k}] \tag{3.21}$$

where $[Z_{\boldsymbol{f}_d^k}]$ represents the current of integration along the variety $Z(\boldsymbol{f}_d^k)$. Then the *expected zero current* is defined by

$$\left\langle \mathbb{E}[\tilde{Z}_{\boldsymbol{f}_d^k}], \varphi \right\rangle := \int_{Z(\boldsymbol{f}_d^k)} \left\langle \widetilde{Z}_{\boldsymbol{f}_d^k}, \varphi \right\rangle d\boldsymbol{P}_d, \tag{3.22}$$

where φ is a bidegree $(n-k, n-k)$ test form on $\mathbb{C}^n$. By Bertini's theorem, whenever the coefficients $\{a_{i,J}\}_{|J|\leq d}$ have continuous probability law, the complex hypersurfaces $Z(\boldsymbol{f}_d^k)$ are smooth and intersect transversely. Therefore $Z(\boldsymbol{f}_d^k)$ is a codimension k subvariety of $\mathbb{C}^n$. In particular, if $k = n$, the set of simultaneous solutions of the full system $Z(\boldsymbol{f}_d^n) := Z(\boldsymbol{f}_d)$ is of codimension n, i.e., a set of isolated points.

Theorem 3.2.1 (Shiffman (2008))**.** *Suppose that $K \subset \mathbb{C}^n$ is a regular compact set and μ be a unit Borel measure on K such that the pair (K, μ) satisfies the Bernstein-Markov inequality. Let $(\mathcal{P}_d^k, \boldsymbol{P}_d^k)$ denote the ensemble of $k-$sytems of independent and identically distributed Gaussian random polynomials of degree at most d with the Gaussian measure $d\boldsymbol{P}_d$ induced by $L^2(\mu)$, for $1 \leq k \leq n$. Then for the sequences of the systems $\{\boldsymbol{f}_d^k\} \in \prod_{d=1}^{\infty} \mathcal{P}_d^k$,*

$$\widetilde{Z}_{\boldsymbol{f}_d^k} \to \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^k := \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right) \wedge \cdots \wedge \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right) \tag{3.23}$$

weak almost surely, where V_K is the pluricomplex Green function of K with pole at infinity. Moreover, if $k = n$, then*

$$\widetilde{Z}_{\boldsymbol{f}_d} \to \mu_K = \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^n, \tag{3.24}$$

weak almost surely as the degree $d \to \infty$.*

The following example generalizes the Theorem 3.1.1 for the multivariate Kac polynomials.

Example 3.2.2. *Let $K = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| \leq 1\}$ be the unit polydisc in $\mathbb{C}^n$. The pair (K, μ_K) satisfies the Bernstein-Markov inequality, and the monomials $\{x^J\}_{|J|\leq d}$ form an orthonormal basis for $\mathcal{P}_d$ with respect to $L^2(\mu_K)$ and hence a random polynomial $f_d \in \mathcal{P}_d$ can be written as*

$$f_d(x) = \sum_{|J|\leq d} a_J x_1^{j_1} \cdots x_n^{j_n}. \tag{3.25}$$

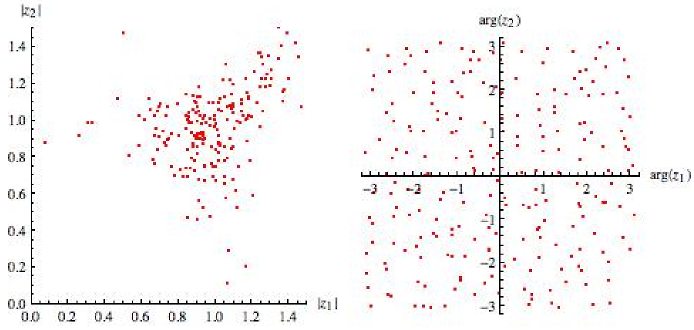
Its Green function $V_K = \max \log^+ |x_i|$, and the equilibrium measure

$$(dd^c V_K)^n = \frac{1}{2\pi} d\theta_1 \cdots \frac{1}{2\pi} d\theta_n$$

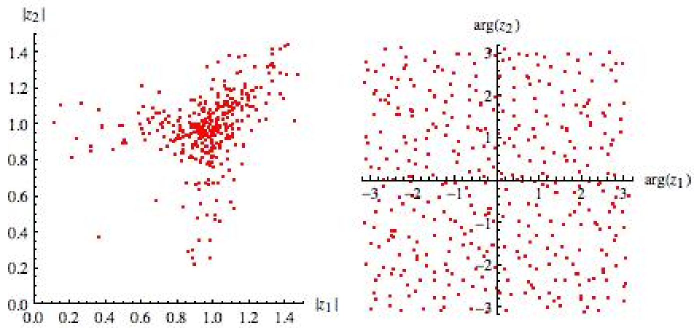
where $d\theta$ represents the arc length measure on the unit circle S^1. Then for almost all sequences of full system polynomials $\boldsymbol{f}_d \in \mathcal{P}$, we have

$$\widetilde{Z}_{\boldsymbol{f}_d^k} \to \frac{1}{2\pi} d\theta_1 \cdots \frac{1}{2\pi} d\theta_n \tag{3.26}$$

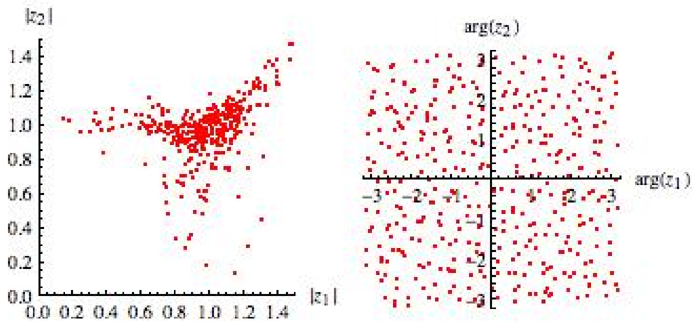
weakly as $d \to \infty$.



(a) Zeros of $(f_{(10,1)}, f_{(10,2)})$



(b) Zeros of $(f_{(15,1)}, f_{(15,2)})$



(c) Zeros of $(f_{(20,1)}, f_{(20,2)})$

Figure 3.2 Distributions of Zeros of Standard Gaussian Random Polynomials

Theorem 3.2.2 (Bloom & Shiffman (2006)). *Let μ be a Borel probability measure on a regular compact set $K \subset \mathbb{C}^n$, and assume that (K,μ) satisfies the Bernstein-Markov inequality. Let $(\mathcal{P}_d^k, \boldsymbol{P}_d^k)$ denote the ensemble of $k-$sytems of independent and identically distributed Gaussian random polynomials of degree at most d with the Gaussian measure $d\boldsymbol{P}_d$ induced by $L^2(\mu)$, for $1 \leq k \leq n$. Then for the sequences of the systems $\{\boldsymbol{f}_d^k\} \in \prod_{d=1}^{\infty} \mathcal{P}_d^k$,*

$$d^{-k}\mathbb{E}[\widetilde{Z}_{\boldsymbol{f}_d^k}] \to \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^k \tag{3.27}$$

weak as $d \to \infty$. In particular, when $k = n$,*

$$d^{-n}\mathbb{E}[\widetilde{Z}_{\boldsymbol{f}_d^k}] \to \mu_K \tag{3.28}$$

weakly as $d \to \infty$.

In the next theorem, we introduce another result on the asymptotics of $\widetilde{Z}_{\boldsymbol{f}_d}$ for full systems, but before we need some concepts from weighted pluripotential theory. We refer the reader to (Saff & Totik, 2013, Appendix B) for background and details.

A set $K \subset \mathbb{C}^n$ is called *locally regular* in the sense of pluripotential theory if for all $r > 0$ and for all $x \in K$ the intersection $K \cap B(x,r)$ is regular where $B(x,r)$ denotes the ball with radius r and centered at x.

Let K be a locally regular compact set and $w \geq 0$ be continuous function on K satisfying $\{z \in K : w > 0\}$ is nonpluripolar. For a nonpluripolar regular compact set $K \subset \mathbb{C}^n$ define $Q := -\log w$. Then the *weighted pluricomplex Green function $V_{K,Q}$* of K is defined as

$$V_{K,Q} := \sup\{u : u \in \mathcal{L},\ u \leq Q \text{ on } K\}. \tag{3.29}$$

Since K is locally regular and w is continuous, it is known that the weighted Green function $V_{K,Q}$ is a continuous, locally bounded plurisubharmonic function . Hence, by (Bedford & Taylor, 1982), the operator $(dd^cV_{K,Q})$ is well defined and is a Borel measure with support in K.

Suppose that μ be a Borel probability measure supported on K. We say that (K,w,μ) satisfies the weighted Bernstein-Markov inequality if for all $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that

$$||w^d f_d||_K \leq C(1+\varepsilon)^d ||w^d f_d||_{L^2(\mu)} \tag{3.30}$$

for all $f_d \in \mathcal{P}_d$.

Theorem 3.2.3 (Bloom (2007)). *Let K be a locally regular compact set, $w \geq 0$ be a continuous function on K and μ be a Borel probability measure on K such that (K, w, μ) satisfies the weighted Bernstein-Markov inequality. Then for the full systems $\boldsymbol{f}_d$ of n random polynomials with independent and identically distributed Gaussian coefficients, we have*

$$\lim_{d\to\infty} \frac{\mathbb{E}[\widetilde{Z}(\boldsymbol{f}_d)]}{d^n} = \Big(\frac{i}{\pi} dd^c V_{K,Q}\Big)^n \tag{3.31}$$

weakly.

Next, we mention a result on asymptotic distributions of the zeros of random polynomial such that the distributions of the coefficients might be chosen from a wide class of distributions including complex Gaussians. Let $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ be random polynomials for $i = 1, \ldots, n$. Suppose that the array of the coefficients $\{a_{i,d}\}$ are real or complex independent and identically distributed random variables with distribution law $P := \varphi(x) d\lambda(x)$ satisfying $0 \leq \varphi(x) \leq C$ for some $C > 0$ and

$$P\{x \in \mathbb{C} : \log|x| > R\} = O(R^\rho) \tag{3.32}$$

where λ is the Lebesgue measure on $\mathbb{C}$ and the constant $\rho > n+1$.

Theorem 3.2.4 (Bayraktar (2016)). *Let K be a regular compact set in $\mathbb{C}^n$ and $Q : K \to \mathbb{R}$ be a continuous weight function. Suppose that $f_{d,1}, \ldots, f_{d,k}$ are random polynomials with coeffients as described as above for $1 \leq k \leq n$ Then,*

$$d^{-k}\mathbb{E}[\widetilde{Z}_{\boldsymbol{f}_d^k}] \to \Big(\frac{i}{\pi}\partial\bar{\partial} V_{K,Q}\Big)^k \tag{3.33}$$

in the sense of currents as $d \to \infty$. Furhermore, almost surely,

$$\widetilde{Z}_{\boldsymbol{f}_d^k} \to \Big(\frac{i}{\pi}\partial\bar{\partial} V_{K,Q}\Big)^k \tag{3.34}$$

as $d \to \infty$ in the sense of currents.

Note that when $k = n$, the limiting distribution $\mu_{K,q} := \left(\frac{i}{\pi}\partial\bar{\partial}V_{K,q}\right)^n$ is a probability measure and when $q \equiv 0$, it is the equilibrium measure μ_K of K.
So far we mention the results on classical random polynomials. In what follows, we mention asymptotic distributions of zeros of random polynomials with prediscribed Newton polytopes. Recall that a *Laurent polynomials* $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is of the form

$$f(x) = \sum_J a_J x^J \tag{3.35}$$

where $a_J \in \mathbb{C}$. The set of integer vectors $supp(f) := \{J \in \mathbb{Z}^n : a_J \neq 0\}$ is called the *support* of f, and the convex hull of $supp(f)$ is called the *Newton polytope* of f. Let P be an integral polytope which is defined as convex hull of a finite subset of $\mathbb{Z}^n$ and denote the set of polynomials whose supports lie in P as

$$Poly(P) = \{f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] : supp(f) \subset P\}.$$

In literature, such polynomials are called *sparse polynomials*. The following result is on the asymptotic behavior of the distribution of zeros of random sparse polynomial systems $(f_1^M, \dots, f_k^M)$ such that $supp(f_i^M) \subset MP_i$, as $M \to \infty$ for $i = 1, \dots, k$ and $1 \leq k \leq n$ and where MP denotes to M-dilate of P.

Let K be a regular nonpluripolar compact set in $\mathbb{C}^n$ and $Q : (\mathbb{C}^*)^n \to \mathbb{R}$ be a weight function for K. The weighted Green function of K is defined as

$$V_{P,K,Q} := \sup\{\varphi \in \mathcal{PSH}((\mathbb{C}^*)^n) : \varphi(x) \leq \max_{J \in P} \log|x^J| + C_\varphi \text{ on } (\mathbb{C}^*)^n \text{ and } \varphi \leq Q \text{ on } K\}.$$

A Laurent polynomial can be written as $f^M = \sum_{j=1}^{d_M} a_j P_j^M$, where d_M is the dimension of the polynomial space $Poly(MP)$ and where the orthonormal basis $\{P_j\}_{j=1}^{d_M}$ is fixed with respect to the inner product

$$\langle f, g \rangle = \int_K f(x)\overline{g(x)} e^{-2MQ(x)} d\mu(x) \tag{3.36}$$

for a unit Borel measure μ supported on K.

Similar to the previous cases, let us identify the polynomial space $Poly(MP)$ with $\mathbb{C}^{d_M}$ and endow it with a probability measure $\boldsymbol{P}$ such that $\boldsymbol{P}$ does not put any mass on pluripolar sets. Consider a random sparse polynomial system $\boldsymbol{f}^k = (f_1^M, \dots, f_k^M)$ for $1 \leq k \leq n$. By Bertini's theorem, it is known that in general position their zero locuses are smooth and intersect transversaly. Hence,

$$Z(\boldsymbol{f}^k) = \{x \in (\mathbb{C}^*)^n : f_1^M(x) = \cdots = f_k^M(x) = 0\}$$

is a smooth and of codimension k variety in $(\mathbb{C}^*)^n$.

In the next theorem, Bayraktar introduce a result on the distribution of zeros of random sparse systems whose distributions of coefficients might be chosen from a wide class of continuous distributions, including complex Gaussian.

Theorem 3.2.5 (Bayraktar (2017)). *Let $P_i \subset \mathbb{R}^n_{\geq 0}$ be an integral polytope with nonempty interior for each $i = 1, \dots, n$ and (K, Q) be a regular weighted compact* *set. If, for every* $\varepsilon > 0$

$$\sum_{M=1}^{\infty} \boldsymbol{P}\{a \in \mathbb{C}^{d_M} : \log ||a|| > M\varepsilon\} < \infty \tag{3.37}$$

and, for every $u \in S^{2d_M - 1}$ *and* $t > 0$

$$\sum_{M=1}^{\infty} \boldsymbol{P}\{a \in \mathbb{C}^{d_M} : \log |\langle a, u\rangle| < -Mt\} < \infty, \tag{3.38}$$

then, almost surely

$$\widetilde{Z}_{(f_1^M, \ldots, f_k^M)} \to dd^c(V_{P_1,K,Q}) \wedge \cdots \wedge dd^c(V_{P_k,K,Q}) \tag{3.39}$$

weakly on $(\mathbb{C}^*)^n$ *as* $M \to \infty$.

Lastly we introduce a result for the distribution of zeros of random polynomials on $\mathbb{C}^2$ including discrete coefficients. Let $\{a_{i,j} : (i,j) \in \mathbb{N}^2\}$ be an array of independent and identically distributed random variables and let bivariable Kac polynomial of degree d be of the form

$$f_d(x_1, x_2) = \sum_{0 \le i+j \le d} a_{i,j} x_1^i x_2^j. \tag{3.40}$$

Theorem 3.2.6 (Bloom & Dauvergne (2019))**.** *Let* f_d *be a bivariate Kac polynomial and suppose that the random variables satisfies the property*

$$\mathbb{E}[\log(1 + |a_0|)]^2 < \infty. \tag{3.41}$$

Then ,

$$\lim_{d \to \infty} \widetilde{Z}_{f_d} = dd^c V_K \tag{3.42}$$

almost surely, where $V_K(x_1, x_2) = \max\{0, \log|x_1|, \log|x_2|\}$.

4. An Equidistribution Result for Random Bernoulli Polynomials

Recall that a *random Bernoulli polynomial* of degree d is defined as

$$f_{d,i}(\boldsymbol{x}) = \sum_{|\boldsymbol{J}|\leq d} \alpha_{i,\boldsymbol{J}} \boldsymbol{x}^{\boldsymbol{J}} \in \mathbb{C}[x_1,\ldots,x_n]$$

where $\alpha_{i,J}$ are ±1-valued Bernoulli random variables. We consider full systems $(f_{d,1},\ldots,f_{d,n})$ of random Bernoulli polynomials on $\mathbb{C}^n$ with independent coefficients and we write $\boldsymbol{f}_d = \left(f_{d,1},\ldots,f_{d,n}\right)$ for short. We denote the collection of all systems of polynomials in n variables and of degree d by $Poly_{n,d}$ that is

$$Poly_{n,d}(\mathcal{A}) := \left\{\boldsymbol{f}_d = \left(f_{d,1},\ldots,f_{d,n}\right) : supp(f_{d,i}) = \mathcal{A}\right\},$$

which is endowed with the product probability measure $Prob_d$ and $\mathcal{A} = d\Sigma_n \cap \mathbb{Z}^n$. For a system $\boldsymbol{f}_d \in Poly_{n,d}$, if the simultaneous zeros $Z(\boldsymbol{f}_d)$ are isolated we denote the corresponding normalized empirical measure by $\delta_{Z(\boldsymbol{f}_d)}$. We also let ν_{Haar} denote the Haar measure of $(S^1)^n$ of total mass 1.

4.1 Proof of Theorem 1.2.1

Our proof is based on randomization of the following theorem which states an equidistribution result for the deterministic systems with integer coefficient polynomials.

Theorem 4.1.1 (D'Andrea et al. (2014))**.** *Let $\mathcal{A}_{d,1},\ldots,\mathcal{A}_{d,n}$ be nonempty finite subsets of $\mathbb{Z}^n$ and where $Q_i = conv(\mathcal{A}_i)$ and $\mathcal{A}_{d,i} = dQ_i \cap \mathbb{Z}^n$ for $i = 1,\ldots,n$. Suppose that $\boldsymbol{f}_d = (f_{d,1},\ldots,f_{d,n})$ is system of polynomials in $\mathbb{Z}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ of degree $d \geq 1$. Assume that $supp(f_{d,i}) \subseteq dQ_i$ and $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}}} \boldsymbol{f}_d^{\boldsymbol{v}} \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$*

and $\log \|f_{d,i}\|_{\sup} = o(d)$. *Then*

$$\lim_{d\to\infty} \delta_{Z(\boldsymbol{f}_d)} = \nu_{Haar}. \tag{4.1}$$

Proof of Theorem 1.2.1. Let $\boldsymbol{f}_d = (f_{d,1},\dots,f_{d,n})$ be a system of random Bernoulli polynomial system. Since all the coefficients are either 1 or -1, for $i=1,\dots,n$ we have

$$\|f_{d,i}\| = \sup_{|w_1|=\cdots=|w_n|=1} |f_{d,i}(w_1,\dots,w_n)| \le \binom{n+d}{d} = O(d^n) \tag{4.2}$$

by triangle inequality. Hence,

$$\log\|f_{d,i}\| \le \log d^n = n\log d \tag{4.3}$$

and this leads $\log\|f_{d,i}\| = o(d)$, which is desired.

The next step is to determine that for a system of Bernoulli polynomials $\boldsymbol{f}_d$ whether its directional resultant $\mathcal{R}es_{\mathcal{A}_d^{\boldsymbol{v}}} \boldsymbol{f}^{\boldsymbol{v}} \ne 0$ for each nonzero vector $\boldsymbol{v}\in\mathbb{Z}^n$.

Lemma 4.1.2. *Let* $\boldsymbol{f}_d = (f_{d,1},\dots,f_{d,n})$ *be a polynomial system of random Bernoulli polynomials for* $d\ge 1$. *Then there exist a constant* C_1 *which is independent of d such that*

$$Prob\{\boldsymbol{f}_d \in Poly_{n,d}(\mathcal{A}) : \mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}}} \boldsymbol{f}_d^{\boldsymbol{v}} = 0 \ \ \textit{for some nonzero} \ \ \boldsymbol{v}\in\mathbb{Z}^n\} \le \frac{C_1}{d}.$$

Proof. Recall that for a polynomial system $\boldsymbol{f}$, its directional resultant is other than one only for the inward pointing normal of the facets of the convex hull of Minkowski sum which is obtained by supports. Therefore, in our case we have to check merely the vectors $\boldsymbol{v}_m = \boldsymbol{e}_m$ for $m=1,\dots,n$ and $\boldsymbol{v}_{m+1} = -\sum_{m=1}^n \boldsymbol{e}_m$ where $\{\boldsymbol{e}_m\}$ is the standard basis of $\mathbb{Z}^n$, since the convex set $nd\Sigma_n$ has $n+1$ facets. We start with the vectors $\boldsymbol{v}_m = \boldsymbol{e_m}$ for $m=1,\dots,n$. Then the intersection of the support $\mathcal{A}$ with the supporting hyperplane in the direction $\boldsymbol{e_m}$ is of the form

$$\mathcal{A}^{\boldsymbol{v}_m} = \left\{(j_1,\dots,j_{m-1},0,j_{m+1},\dots,j_n)\in\mathcal{A} : \sum_{i=1}^n j_i \le d\right\}$$

$m=1,\dots,n$. Hence polynomials with support $\mathcal{A}^{\boldsymbol{v}_m}$ are represented as

$$f_i^{\boldsymbol{v}_m} := \sum_{\boldsymbol{J}\in\mathcal{A}^{\boldsymbol{v}_m}} \alpha_{i,\boldsymbol{J}} \boldsymbol{x}^{\boldsymbol{J}} \tag{4.4}$$

for $i=1,\dots,n$.

Following the Definition 2.2.4, if we choose the vector $\boldsymbol{b}_{i,\boldsymbol{v}_m} = \boldsymbol{0}$ such that $\mathcal{A}^{\boldsymbol{v}_m} - \boldsymbol{b}_{i,\boldsymbol{v}_m} \subset \mathbb{Z}^n \cap \boldsymbol{v}_m^{\perp}$, we see that the functions $f_i^{\boldsymbol{v}_m} = g_{i,\boldsymbol{v}_m}$ satisfy the equation $f_i^{\boldsymbol{v}_m} = \boldsymbol{x}^{\boldsymbol{b}_{i,\boldsymbol{v}_m}} g_{i,\boldsymbol{v}_m}$ for each $i = 1,\ldots,n$.
For two univariate polynomials $h_1, h_2 \in \mathbb{C}[x]$, their resultant $\mathcal{R}es(h_1,h_2)$ is nonzero if and only if h_1 and h_2 have no common solution in $\mathbb{C}$. Therefore, if $n = 2$ the necessary and sufficient condition for $g_{1,\boldsymbol{v}_m}$ and $g_{2,\boldsymbol{v}_m}$ have zero resultant is that they have a common solution. In order to determine the occuring probability of it, we use the following theorem which is stated by Kozma and Zeutoni.

Theorem 4.1.3 (Kozma & Zeitouni (2011))**.** *Let $f_1,\ldots,f_{n+1} \in \mathbb{Z}[x_1,\ldots,x_n]$ be $n+1$ independent random Bernoulli polynomials of degree d. Let*

$$\mathcal{P}(d,n) := Prob\{\exists \boldsymbol{x} \in \mathbb{C}^n : f_i(\boldsymbol{x}) = 0, \quad i = 1\ldots,n+1\}$$

denote the probability that the system $f_1 = \ldots = f_{n+1} = 0$ has a common solution. Then there exists a constant $K = K(n) < \infty$ satisfying that

$$\mathcal{P}(d,n) \leq K/d$$

for all $d \in \mathbb{Z}_+$.

Hence we can say that if $n = 2$, there exists a constant c_m so that the directional resultant of the system (f_1, f_2) in the direction of $\boldsymbol{v}_m$ is nonzero with probability at least $1 - c_m/d$ for $m = 1,\ldots,n$.
On the other hand, if $n \geq 3$, we need to use homogenization process as described in (2.9).

We obtain the homogeneous polynomials $G_{i,\boldsymbol{v}_m}$ of the form

$$G_{i,\boldsymbol{v}_m}(t,\boldsymbol{x}) = \sum_{\boldsymbol{J} \in \mathcal{A}^{\boldsymbol{v}_m}} \alpha_{i,\boldsymbol{J}} t^{\beta} \boldsymbol{x}^{\boldsymbol{J}} \tag{4.5}$$

such that $|J| + \beta = d$. In order to compare the sparse resultant of the polynomials $g_{i,\boldsymbol{v}_m}$ and the multipolynomial resultant of the homogeneous polynomials $G_{i,\boldsymbol{v}_m}$, we check the conditions of Corollary 2.2.6. Let $Z(\boldsymbol{G})$ be the set of nontrivial solutions of the system $\boldsymbol{G} = (G_{1,\boldsymbol{v}_m},\ldots,G_{n,\boldsymbol{v}_m})$ and suppose that $\boldsymbol{G}$ has a solution $\boldsymbol{\xi} = (t,\xi_2,\ldots,\xi_n)$ at hyperplane at infinity. If we evaluate these homogeneous polynomials at $t = 0$, we obtain the homogeneous part of the polynomials $g_{i,\boldsymbol{v}_m}$ for $i = 1,\ldots,n$. Since $\boldsymbol{\xi} \in H^{\infty}(t)$, it has a nonzero coordinate ξ_k for some $k \in \{2,\ldots,n\}$. For simplicity, let us assume $k = 2$ and define the new variables $z_i := \xi_{i+2}/\xi_2$ for $i = 1, \ldots, n-2$. Applying the change of variables, we obtain

$$\widetilde{G}_{i,\boldsymbol{v}_m}(z_1, \ldots, z_{n-2}) = \sum_{|J| \le d} \alpha_{i,\boldsymbol{J}} \boldsymbol{z}^{\varphi(\boldsymbol{J})} \tag{4.6}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}^{n-2}$, $(j_1, \ldots, j_n) \mapsto (j_3, \ldots, j_n)$. In this setting, there are n random Bernoulli polynomials of degree d in $n-2$ variables. In order to determine of the existence of solutions we again use Theorem 4.1.3. Hence by the above theorem, there exists a positive constant C_i, depending only the dimension such that the probability that the overdetermined system of these Bernoulli polynomials have a common solution is less than C_i/d. In the light of this observation, we see that the system of homogenized polynomials $G_{i,\boldsymbol{v}_m}$ has no common zero at hyperlane at infinity except a set that has probability at most C_i/d. Then by Corollary 2.2.6, outside of that small set, whenever the system of polynomials consisting $g_{i,\boldsymbol{v}_m}$ has a common solution, their resultant is zero. Since the system of Bernoulli polynomials $g_{i,\boldsymbol{v}_m}$ contains n polynomials in $n-1$ variables, by Theorem 4.1.3, there is a dimensional constant $\tilde{C}_i$ so that the probability that this system has common solution is at most $\tilde{C}_i/d$. Hence outside of a set that has probability $K_i/d := c_i/d + \tilde{C}_i/d$, the directional resultant $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}_m}} \boldsymbol{f}_d^{\boldsymbol{v}_m} \neq 0$ for all inward normals of facets $\boldsymbol{v}_m$, $m = 1, \ldots, n$.

Next, for the inward normal vector $v_{n+1} = -\sum_{i=1}^n \boldsymbol{e}_i$, we find the minimal weight in this direction as $\mathcal{A}^{\boldsymbol{v}_{n+1}} = \{\boldsymbol{J} \in \mathcal{A} : |\boldsymbol{J}| = d\}$. Hence the polynomials in this directions are of the form

$$f_i^{\boldsymbol{v}_{n+1}} = \sum_{|\boldsymbol{J}|=d} \alpha_{i,\boldsymbol{J}} \boldsymbol{x}^{\boldsymbol{J}}. \tag{4.7}$$

In this case $\mathcal{A}^{\boldsymbol{v}_{n+1}}$ is not a subspace of $\mathbb{Z}^n \cap \boldsymbol{v}_{n+1}^{\perp}$, hence we need to translate it by substracting a suitable vector $\boldsymbol{b}_{i,\boldsymbol{v}_{n+1}}$. For Laurent polynomial systems, the sparse resultant is invariant under translations of supports (see D'Andrea & Sombra (2015), Proposition 3.3). Since the polynomials $f_{d,i}$ are not Laurent, we need to determine the effect of this translation.

Lemma 4.1.4. *Let $\boldsymbol{f}_d = (f_{d,1}, \ldots, f_{d,n})$ be a system of random Bernoulli polynomials of degree d and let $Z(\boldsymbol{f}_d)$ denote the set of their common solutions in $\mathbb{C}^n$. Then there exists a positive constant $C_1 = C_1(n)$ depending only the dimension n with the* *property that*

$$\text{(4.8)} \qquad Prob\{\boldsymbol{f}_d \in Poly_{n,d}(\mathcal{A}) : \exists\ \boldsymbol{x} \in Z(\boldsymbol{f}_d)\ \ni \prod_{i=1}^{n} x_i = 0\} \leq \frac{C}{d}.$$

Proof. Consider the system of Bernoulli polynomials $\boldsymbol{f}_d$ and set of its simultaneous zeros $Z(\boldsymbol{f}_d)$. For a solution $\boldsymbol{x} = (x_1, \ldots, x_n) \in Z(\boldsymbol{f}_d)$ assume that $x_1 = 0$. In order to examine the incidence of this case, we evaluate the the system $\boldsymbol{f}_d$ at $x_1 = 0$ and we obtain a new system of n Bernoulli polynomials with $n-1$ variables. By Theorem 4.1.3, there exists constant C_1 which is independent of d so that this system has a common solution with probability at most C_1/d. Therefore the probability of the event that $x_1 = 0$ is less than C_1/d. Hence there is no harm of translation of supports outside of a set that has probability at most C/d, where $C := \sum_i^n C_i/d$. □

Choosing the vector $\boldsymbol{b}_{i,\boldsymbol{v}_{n+1}} = (d, 0, \ldots, 0)$ so that $\mathcal{A}^{\boldsymbol{v}_{n+1}} - \boldsymbol{b}_{i,\boldsymbol{v}_{n+1}} \subset \mathbb{Z}^n \cap \boldsymbol{v}_{n+1}^{\perp}$, we obtain the polynomials of the form

$$\text{(4.9)} \qquad g_{i,\boldsymbol{v}_{n+1}} = \sum_{\boldsymbol{J} \in \mathcal{A}^{\boldsymbol{v}_{n+1}} - \boldsymbol{b}_{i,\boldsymbol{v}_{n+1}}} \alpha_{i,\boldsymbol{J}} \boldsymbol{x}^{\omega(\boldsymbol{J})}$$

with $\omega : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $(j_1, j_2, \ldots, j_n) \mapsto (-d + j_1, j_2, \ldots, j_n)$. We substitute the new variables $y_i := x_{i+1}/x_1$ into $g_{i,\boldsymbol{v}_{n+1}}$, $i = 1, \ldots, n-1$, and obtain

$$\text{(4.10)} \qquad g_{i,\boldsymbol{v}_{n+1}}(\boldsymbol{y}) = \sum_{|\boldsymbol{J}| \leq d} \alpha_{i,\boldsymbol{J}} \boldsymbol{y}^{\sigma(\boldsymbol{J})}$$

for $\boldsymbol{y} \in \mathbb{C}^{n-1}$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, $(j_1, j_2, \ldots, j_n) \mapsto (0, j_2, \ldots, j_n)$. The system containing the polynomials $g_{i,\boldsymbol{v}_{n+1}}(\boldsymbol{y})$, contains n random Bernoulli polynomials with $n-1$ random variable as in the cases $\boldsymbol{v}_i = \boldsymbol{e}_i$. By applying the same steps, it can be shown that $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}_{n+1}}}(\boldsymbol{f}_d^{\boldsymbol{v}_{n+1}}) \neq 0$ outside of a set that has probability K_{i+1}/d. □

Now, define the exceptional set $\mathcal{E}_{n,d}$ as a subset of $Poly_{n,d}$ which contains the systems $\boldsymbol{f}_d$ that has a zero directional resultant for some nonzero primitive vector $\boldsymbol{v}$ or the systems $\boldsymbol{f}_d$ have a common solution $\boldsymbol{x} \in \mathbb{C}^n$ with $x_i = 0$ for some $i = 1, \ldots, n$. More precisely,

$$\text{(4.11)} \qquad \begin{aligned} \mathcal{E}_{n,d} := \{\boldsymbol{f}_d \in Poly_{n,d} : \mathcal{R}es_{\boldsymbol{A}^{\boldsymbol{v}}} \boldsymbol{f}_d^{\boldsymbol{v}} = 0,\ \boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}\} \\ \bigcup \{\boldsymbol{f}_d \in Poly_{n,d} : \exists \boldsymbol{x} \in Z(\boldsymbol{f}_d) \ni \prod x_i = 0\}. \end{aligned}$$

By computations above, we can say that for a Bernoulli polynomial system $\boldsymbol{f}_d$ there exists a positive constant K which is independent of d so that $Prob\{\mathcal{E}_{n,d}\} \leq d^{-1}K$ where $K := \sum_{i=1}^{n+1} K_i + C/d$. Therefore outside of the exceptional set $\mathcal{E}_{n,d}$, we can guarantee that the directional resultant $\mathcal{R}es_{\boldsymbol{A}^{\boldsymbol{v}}} \boldsymbol{f}_d^{\boldsymbol{v}} \neq 0$ for all nonzero primitive vector $\boldsymbol{v} \in \mathbb{Z}^n$ and by Theorem 4.1.1, for all sequences $\{\boldsymbol{f}_d\}_d \subset Poly_{n,d} \setminus \mathcal{E}_{n,d}$,

$$\lim_{d \to \infty} \delta_{Z(\boldsymbol{f}_d)} = \nu_{\text{Haar}}$$

weakly. In particular, since $Prob\{\mathcal{E}_{n,d}\} \to_d 0$, $\delta_{Z(\boldsymbol{f}_d)} \to_d \nu_{\text{Haar}}$ in probability as $d \to \infty$. $\square$

5. Expected Zero Measure of Random Bernoulli Polynomial Systems

5.1 Proof of Theorem 1.2.2

In this section, first we mention the ingredients related to the subject and introduce the proof of Theorem 1.2.2.

Let Z be a non-empty finite collection of points $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and $m_{\boldsymbol{\xi}} \in \mathbb{N}$ denote the multiplicity of $\boldsymbol{\xi}$ in Z. The degree of Z is defined by $deg(Z) = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}$ which is a positive number.

Definition 5.1.1. *Let Z be a non-empty finite collection of points in $\mathbb{C}^n$. For each $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ with $-\pi \le \alpha_j < \beta_j \le \pi$, $j = 1, \dots, n$ consider the subset of Z*

$$Z_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \{\boldsymbol{\xi} \in Z : \alpha_j < \arg(\xi_j) \le \beta_j\}. \tag{5.1}$$

The angle discrepancy of Z is defined as

$$\Delta_{ang}(Z) = \sup_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left| \frac{deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{deg(Z)} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \right|. \tag{5.2}$$

Let $0 < \varepsilon < 1$ and consider the subset

$$Z_{\varepsilon} := \{\boldsymbol{\xi} \in Z : 1 - \varepsilon < |\xi_j| < (1-\varepsilon)^{-1}\}. \tag{5.3}$$

The radius discrepancy of Z with respect to ε is defined as

$$\Delta_{rad}(Z, \varepsilon) := 1 - \frac{deg(Z_{\varepsilon})}{deg(Z)}. \tag{5.4}$$

Note that $0 < \Delta_{\text{ang}}(Z)_{\boldsymbol{\alpha},\boldsymbol{\beta}} \leq 1$ and $0 \leq \Delta_{\text{rad}}(Z,\varepsilon) \leq 1$. One can see that the angle discrepancy and the radial discrepancy are defined as direct generalization of their one dimensional versions.

Let $\mathcal{A}_1,\ldots,\mathcal{A}_n \subset \mathbb{Z}^n$ be a collection of nonempty finite sets and let $Q_i = conv(\mathcal{A}_i)$ for each $i = 1,\ldots,n$. For a vector $\boldsymbol{w} \in S^{n-1}$ in the unit sphere in $\mathbb{R}^n$, let $\boldsymbol{w}^\perp$ be its orthogonal subspace and $\pi_{\boldsymbol{w}^\perp} : \mathbb{R}^n \to \boldsymbol{w}^\perp$ be the corresponding orthogonal projection. The mixed volume of the convex bodies of $\boldsymbol{w}^\perp$ induced by the Euclidean measure on $\boldsymbol{w}^\perp$ is denoted by $MV_{\boldsymbol{w}^\perp}$. Set

$$D_{\boldsymbol{w},i} = MV_{\boldsymbol{w}^\perp}\left(\pi_{\boldsymbol{w}}(Q_1),\ldots,\pi_{\boldsymbol{w}}(Q_{i-1}),\pi_{\boldsymbol{w}}(Q_{i+1}),\ldots,\pi_{\boldsymbol{w}}(Q_n)\right). \tag{5.5}$$

The the *Erdös-Turán size* of $\boldsymbol{f} = (f_1,\ldots,f_n)$ is defined as

$$\eta(\boldsymbol{f}) = \frac{1}{D}\sup_{\boldsymbol{w}\in S^{n-1}} \log\left(\frac{\prod_{i=1}^{n} ||f||_{sup}^{D_{\boldsymbol{w},i}}}{\prod_{\boldsymbol{v}} |\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\ldots,f_n^{\boldsymbol{v}})|^{\frac{|\langle \boldsymbol{v},\boldsymbol{w}\rangle|}{2}}}\right), \tag{5.6}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$ and the product in the denumerator is over all primitive vectors $\boldsymbol{v} \in \mathbb{Z}^n$. It can be seen that the Erdös-Turán size of a polynomial system $\boldsymbol{f}$ corresponds exacly to the bound in the Erdös-Turán theorem for univariate polynomials [see Erdös & Turán (1950),Hughes & Nikeghbali (2008),D'Andrea et al. (2014)]. In the next proposition, an upper bound is given for the Erdös-Turán size of polynomial systems $\boldsymbol{f}$ with integer coefficients. For the general case and for more properties of $\eta(\boldsymbol{f})$, see (D'Andrea et al., 2014, Proposition 3.15).

Proposition 5.1.1 (D'Andrea et al. (2014))**.** *Let $\mathcal{A}_1,\ldots,\mathcal{A}_n$ be a non-empty finite subsets of $\mathbb{Z}^n$ and set $Q_i = conv(\mathcal{A}_i)$ with $MV_{\mathbb{R}^n}(Q_1,\ldots,Q_n) \geq 1$. Let $d_i \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{b}_i \in \mathbb{Z}^n$ so that $d_i\Sigma_n + \boldsymbol{b}_i$, $i = 1,\ldots,n$. Suppose that $f_1,\ldots,f_n \in \mathbb{Z}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ with supp$(f_i) \subseteq \mathcal{A}_i$ and such that $\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\ldots,f_n^{\boldsymbol{v}}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Then*

$$\eta(\boldsymbol{f}) \leq \frac{1}{MV_{\mathbb{R}^n}(Q_1,\ldots,Q_n)}\left(\left(n+\sqrt{n}\right)\left(\prod_{i=1}^{n} d_i\right)\sum_{i=1}^{n}\frac{\log\|f_i\|_{\sup}}{d_i}\right). \tag{5.7}$$

The following theorem gives bounds for angle discrepancy and radius discrepancy of $Z(\boldsymbol{f})$ in terms of the Erdös-Turán size of $\boldsymbol{f}$. For one dimensional version see for instance (Erdös & Turán, 1950),(Hughes & Nikeghbali, 2008).

Theorem 5.1.2 (D'Andrea et al. (2014))**.** *Let $\mathcal{A}_1,\ldots,\mathcal{A}_n$ be a nonempty finite subsets of $\mathbb{Z}^n$ such that $MV_{\mathbb{R}^n}(Q_1,\ldots,Q_n) \geq 1$ with $Q_i = conv(\mathcal{A}_i)$ for $n \geq 2$. Let $f_1,\ldots,f_n \in \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ with supp$(f_i) \subseteq \mathcal{A}_i$ and such that*

$\mathcal{R}es_{\mathcal{A}_1^{\boldsymbol{v}},\ldots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\ldots,f_n^{\boldsymbol{v}}) \neq 0$ *for all* $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. *Then*

$$\Delta_{ang}(Z(\boldsymbol{f})) \leq 66n2^n(18+\log^+(\eta(\boldsymbol{f})^{-1}))^{\frac{2}{3}(n-1)}\eta(\boldsymbol{f})^{\frac{1}{3}} \tag{5.8}$$

and, for $0 < \varepsilon < 1$,

$$\Delta_{rad}(Z(\boldsymbol{f}),\varepsilon) \leq \frac{2n}{\varepsilon}\eta(\boldsymbol{f}). \tag{5.9}$$

For a random Bernoulli polynomial mapping $\boldsymbol{f}_d$ we let $Z(\boldsymbol{f}_d)$ be the set of simultaneous zeros of $\boldsymbol{f}_d$. We define the angle discrepancy $\Delta_{\text{ang}}(Z(\boldsymbol{f}))$ and the radius discrepancy $\Delta_{\text{rad}}(Z(\boldsymbol{f}),\varepsilon)$ as above whenever $Z(\boldsymbol{f}_d)$ is a discrete set of points. Otherwise, we set $\Delta_{\text{rad}}(Z(\boldsymbol{f}),\varepsilon) = \Delta_{\text{ang}}(Z(\boldsymbol{f})) = 1$. Note that as our probability space $(Poly_{n,d}, Prob)$ is discrete, measurability of these random variables is not an issue in this setting. Next, we estimate the asymptotic expected discrepancies:

Proposition 5.1.3. *For* $d \geq 1$, *let* $\boldsymbol{f}_d = (f_{d,1},\ldots,f_{d,n})$ *be a random Bernoulli system. Then*

$$\lim_{d\to\infty} \mathbb{E}[\Delta_{ang}(Z(\boldsymbol{f}_d))] = 0 \quad \textit{and} \quad \lim_{d\to\infty} \mathbb{E}[\Delta_{rad}(Z(\boldsymbol{f}_d))] = 0. \tag{5.10}$$

Proof. The proof is analogous to that [D'Andrea et al. (2014), Theorem 4.9]. Consider the expected value of the angular discrepancy which is

$$\mathbb{E}[\Delta_{\text{ang}}(Z(\boldsymbol{f}_d))] = \int_{Poly_{n,d}} \Delta_{\text{ang}}(Z(\boldsymbol{f}_d))dProb(\boldsymbol{f}_d). \tag{5.11}$$

Let $\mathcal{E}_{n,d}$ be the exceptional set which contains all the systems in $Poly_{n,d}$ with zero directional resultants for some nonzero primitive vector $\boldsymbol{v} \in \mathbb{Z}^n$ as described in equation 4.11. Then, there exist a constant K_1 which is independent of d so that

$$0 \leq \int_{\mathcal{E}_{n,d}} \Delta_{\text{ang}}(Z(\boldsymbol{f}_d))dProb(\boldsymbol{f}_d) \leq K_1 Prob\{\mathcal{E}_{n,d}\} \leq Kd^{-1} \tag{5.12}$$

since $0 < \Delta_{\text{ang}}(Z(\boldsymbol{f}_d)) \leq 1$. Hence $\int_{\mathcal{E}_{n,d}} \Delta_{\text{ang}}(Z(\boldsymbol{f}_d))dProb(\boldsymbol{f}_d) \to 0$ as $d \to \infty$.

Let $\boldsymbol{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$, then Proposition 5.1.1 implies that

$$\eta(\boldsymbol{f}_d) \leq \frac{1}{d^n}\left(d^{n-1}(n+\sqrt{n})\sum_{i=1}^{n}\log||f_{d,i}||_{\text{sup}}\right) \tag{5.13}$$

$$\leq \frac{1}{d^n}\left(d^{n-1}(n^2+n)\sum_{i=1}^{n}\log d\right) \tag{5.14}$$

$$\leq K_2\frac{\log d}{d} \tag{5.15}$$

for a constant K_2 which is independent of d.

Now, by Theorem 5.1.2 for $\boldsymbol{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$, for constants K_3, K_4, K_5 and K_6,

$$\Delta_{\text{ang}}(Z(\boldsymbol{f}_d)) \leq K_3 \eta(\boldsymbol{f}_d)^{\frac{1}{3}} \log\left(\frac{K_4}{\eta(\boldsymbol{f}_d)}\right)^{\frac{2}{3}(n-1)} \tag{5.16}$$

$$\leq K_5 \left(\frac{\log d}{d}\right)^{\frac{1}{3}} \log\left(\frac{d}{\log d}\right)^{\frac{2}{3}(n-1)} \leq K_6 \frac{\log d^{\frac{2n}{3}-\frac{1}{3}}}{d^{\frac{1}{3}}} \tag{5.17}$$

since the function $t^{\frac{1}{3}} \log(\frac{a}{t})^{\frac{n-1}{3}}$ is increasing for small values of $t > 0$. Combining the equations (5.15) and (5.17), we finish the first part of the proof. For the second part, in a very similar fashion for $0 < \varepsilon < 1$, there exists a constant K_7 so that

$$0 \leq \int_{\mathcal{E}_{n,d}} \Delta_{\text{rad}}(Z(\boldsymbol{f}_d)) dProb(\boldsymbol{f}_d) \leq K_7 d^{-1} \tag{5.18}$$

since $0 \leq \Delta_{\text{rad}}(Z(\boldsymbol{f}_d)) \leq 1$. Theorem 5.1.2 implies that for a constant C_8,

$$\Delta_{\text{rad}}(Z(\boldsymbol{f}_d)) \leq K_8 \eta(\boldsymbol{f}_d) \leq K_8 \frac{\log d}{d}. \tag{5.19}$$

Combining (5.15) and (5.19) completes the proof. $\square$

Proof of Theorem 1.2.2. We adapt the argument in (D'Andrea et al., 2014, Theorem 1.8) to our setting. Let $\nu_d = \frac{\mathbb{E}[\widetilde{Z}(\boldsymbol{f}_d)]}{d^n}$, where $\mathbb{E}[\widetilde{Z}(\boldsymbol{f}_d)]$ is the expected zero measure of $\boldsymbol{f}_d$ and ν_{Haar} be the Haar measure on $(S^1)^n$. We need to show that for each continuous function φ with compact support in $\mathbb{C}^n$ we have $\int \varphi d\nu_d \to \int \varphi d\nu_{\text{Haar}}$ as $d \to \infty$. It is enough to prove this for characteristic functions φ_U of the open sets

$$U := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : r_{1,j} < |z_j| < r_{2,j}, \alpha_j < \arg(z_j) < \beta_j \text{ for all } j\}, \tag{5.20}$$

where $0 \leq r_{1,j} < r_{2,j} \leq \infty$, $r_{i,j} \neq 1$ for $i = 1, 2$ and $-\pi < \alpha_j < \beta_j \leq \pi$.

Consider the first case when $U \cap (S^1)^n = \emptyset$. Then there exists $0 < \varepsilon < 1$ such that U is disjoint from the set

$$\{(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n : 1 - \varepsilon < |\xi_j| < (1-\varepsilon)^{-1} \text{ for all } j\}. \tag{5.21}$$

Let $\boldsymbol{f}_d \in Poly_{n,d} \setminus \mathcal{E}_{n,d}$ where $\mathcal{E}_{n,d}$ as in the proof of Theorem 1.2.1. Then $Z(\boldsymbol{f}_d)$ is discrete and

$$deg(Z(\boldsymbol{f}_d)|_U) \leq deg(Z(\boldsymbol{f}_d)) \Delta_{\text{rad}}(\boldsymbol{f}_d, \varepsilon) \leq d^n \Delta_{\text{rad}}(\boldsymbol{f}_d, \varepsilon). \tag{5.22}$$

Also, if $\boldsymbol{f}_d \in \mathcal{E}_{n,d}$, then $deg(\widetilde{Z}(\boldsymbol{f}_d)|_U) = 0$. Hence $\nu_d(U) \leq \mathbb{E}[\Delta_{\text{rad}}(Z(\boldsymbol{f}_d, \varepsilon))]$. Then by Proposition 5.1.3, $\lim_{d\to\infty} \nu_d(U) = \nu_{\mathrm{Haar}}(U)$.

On the other hand, assume $U \cap (S^1)^n \neq \emptyset$. Set

$$\widetilde{U} = \{\boldsymbol{z} : \alpha_j \leq \arg(z_j) \leq \beta_j \quad \text{for all } j \}. \tag{5.23}$$

Then, we obtain

$$\begin{aligned}\left|\nu_d(\widetilde{U}) - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi}\right| &= \int_{Poly_{n,d} \setminus \mathcal{E}_{n,d}} \left|\frac{deg(Z(\boldsymbol{f}_d)_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{d^n} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi}\right| dProb(\boldsymbol{f}_d) + \frac{Kn}{d} \\ &\leq \int_{Poly_{n,d}} \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_d)) dProb(\boldsymbol{f}_d) + \frac{Kn}{d}.\end{aligned}$$

Furthermore, since

$$\nu_d(U) - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} = \left(\nu_d(\widetilde{U}) - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi}\right) - \nu_d(\widetilde{U} \setminus U)$$

we need to consider $\nu_d(\widetilde{U} \setminus U)$. But, the set $\widetilde{U} \setminus U$ is a union of a finite number of subsets U_m of the form 5.20 so that $U_m \cap (S^1)^n = \emptyset$ for all m. By previous consideration, $\lim_{d\to\infty} \nu_d(U_m) = 0$ and hence $\lim_{d\to\infty} \nu_d(\widetilde{U} \setminus U) = 0$. Therefore by Proposition 5.1.3,

$$\lim_{d\to\infty} \nu_d(U) = \lim_{d\to\infty} (\widetilde{U}) = \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} = \nu_{Haar}(U), \tag{5.24}$$

which concludes the proof. $\square$

6. Further results on $\mathbb{C}^2$

6.1 Proof of Theorem 1.2.3

Theorem 1.2.1 and Theorem 1.2.2 are proven for the polynomials with special kind of supports. The restriction for the case $n \geq 3$ occurs since Proposition 2.4 is effective only for the subsets $\mathcal{A} = \{J \in \mathbb{Z}_+^n : |\boldsymbol{J}| \leq d\}$ for some positive integer d. However, for bivariate polynomials it is valid for the polynomials with more general supports since the existence of solutions of two polynomials g_1, g_2 in one variable can be determined by their resultant $Res(g_1, g_2)$. Therefore, if $f_1, f_2 \in \mathbb{C}[x_1, x_2]$, then we can alter the simplex type support with rectangular and trapezium types etc. Intuitively, we anticipate that the support condition also can be relaxed for $n \geq 3$ by some other techniques.

Lemma 6.1.1. *Let*

$$f_1(x_1, x_2) = \sum_{|J| \leq d} \alpha_{\boldsymbol{J}} x_1^{j_1} x_2^{j_2} \qquad \textit{and} \qquad f_2(x_1, x_2) = \sum_{|J| \leq d} \beta_{\boldsymbol{J}} x_1^{j_1} x_2^{j_2} \tag{6.1}$$

be two random Bernoulli polynomials in $\mathbb{C}[x_1, x_2]$ *with support*

$$\mathcal{A} = \{(j_1, j_2) \in \mathbb{Z}_{\geq 0}^2 : 0 \leq j_1, j_2 \leq d\}.$$

Then their directional resultants $\mathcal{R}es_{\mathcal{A}}(f_1^{\boldsymbol{v}}, f_2^{\boldsymbol{v}}) \neq 0$ *with overwhelming probability.*

Proof. Recall that the directional resultant of f_1 and f_2 is other than 1 only for the inward pointing normals of $conv(\mathcal{A}+\mathcal{A})$. We can specify these vectors as $\boldsymbol{v}_1 = \boldsymbol{e}_1$, $\boldsymbol{v}_2 = \boldsymbol{e}_2$, $\boldsymbol{v}_3 = -\boldsymbol{e}_1$ and $\boldsymbol{v}_4 = -\boldsymbol{e}_2$ where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are the standard basis elements in $\mathbb{R}^2$. For the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, the analysis works as in the proof of the main theorem. For the inward normal $\boldsymbol{v}_3$, following the Definition 2.2.4, the minimal weight in this direction is the set of integer points on the facet $\mathcal{F}_3$. More precisely,

$$\mathcal{A}^{\boldsymbol{v}_3} = \{(d, j_2) \in \mathcal{A} : 0 \le j_2 \le d\}.$$

Choosing $\boldsymbol{b}_{i,\boldsymbol{v}_3} = (d,0)$, we have $\mathcal{A}^{\boldsymbol{v}_3} - \boldsymbol{b}_{i,\boldsymbol{v}_3} \subset \boldsymbol{v}_3^{\perp} \cap \mathbb{Z}^2$. Hence, the corresponding polynomials

$$g_{1,\boldsymbol{v}_3}(x_2) = \alpha_{d,0} + \alpha_{d,1}x_2 + \ldots + \alpha_{d,d}x_2^d \tag{6.2}$$

and

$$g_{2,\boldsymbol{v}_3}(x_2) = \beta_{d,0} + \beta_{d,1}x_2 + \ldots + \beta_{d,d}x_2^d \tag{6.3}$$

are two univariate random Bernoulli polynomials. In consequence of the Theorem 4.1.3, we guarentee that there exists a positive constant K such that the probability that the polynomials $g_{1,\boldsymbol{v}_3}$ and $g_{2,\boldsymbol{v}_3}$ have a common zero in $\mathbb{C}$ is less than K/d. This leads that their resultant, hence the resultant of f_1 and f_2 in the direction of $\boldsymbol{v}_3$ is nonzero by Definition 2.2.1.

For the inward normal $\boldsymbol{v}_4$, we machinery works in a very similar fashion. $\square$

Lemma 6.1.2. *Let Q be the trapezium with corners $(0,0),(d,0),(d,d)$ and $(2d,0)$ and let $\mathcal{A}$ is the set of integer points in Q, i.e., $\mathcal{A} := Q \cap \mathbb{Z}^2$. Suppose that $f_1, f_2 \in \mathbb{C}[x_1,x_2]$ be as in the equation (6.1) with supp$(f_i) = \mathcal{A}$ for $i = 1,2$. Then $\mathcal{R}es_{\mathcal{A}}(f_1^{\boldsymbol{v}}, f_2^{\boldsymbol{v}}) \neq 0$ outside of a set with small probability.*

Proof. The convex set Q has four facets $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ with inward normals $\boldsymbol{v}_1 = \boldsymbol{e}_1$, $\boldsymbol{v}_2 = \boldsymbol{e}_2$, $\boldsymbol{v}_3 = -\boldsymbol{e}_2$ and $\boldsymbol{v}_4 = -\boldsymbol{e}_1 - \boldsymbol{e}_2$, respectively. For the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, the result follows from the Lemma 6.1.1. For the vector $\boldsymbol{v}_4$, the directed set can be found as

$$\mathcal{A}^{\boldsymbol{v}_4} = \{(2d - j_2, j_2) \in \mathcal{A} : 0 \le j_2 \le d\}. \tag{6.4}$$

In order to move $\mathcal{A}^{\boldsymbol{v}_4}$ to be a subset of $\boldsymbol{v}_4^{\perp} \cap \mathbb{Z}^2$, we choose the vector $\boldsymbol{b}_{4,\boldsymbol{v}_4} = (d,d)$. Then we obtain the polynomials

$$g_{1,\boldsymbol{v}_4} = \alpha_{d,d} + \alpha_{d+1,d-1}x_1^1x_2^{-1} + \cdots + \alpha_{2d,0}x_1^dx_2^{-d} \tag{6.5}$$

$$g_{2,\boldsymbol{v}_4} = \beta_{d,d} + \beta_{d+1,d-1}x_1^1x_2^{-1} + \cdots + \beta_{2d,0}x_1^dx_2^{-d}. \tag{6.6}$$

Applying the change of variable $y = x_1/x_2$, we attain

$$g_{1,\boldsymbol{v}_4}(y) = \alpha_{d,d} + \alpha_{d+1,d-1}y + \cdots + \alpha_{2d,0}y^d \tag{6.7}$$

and

$$g_{2,\boldsymbol{v}_4}(y) = \beta_{d,d} + \beta_{d+1,d-1}y + \cdots + \beta_{2d,0}y^d \tag{6.8}$$

two random univariate Bernoulli polynomials. Again by Theorem 4.1.3, we say that there exists a dimensional constant K such that they have a common zero and hence their resultant is zero with probability less than K/d. Therefore the directional resultant $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}_4},\mathcal{A}^{\boldsymbol{v}_4}}(f_1^{\boldsymbol{v}_4}, f_2^{\boldsymbol{v}_4}) \neq 0$ with probability at least $1 - K/d$. $\square$

Definition 6.1.1. *Let $P \in (\mathbb{R}_+)^n$ be a* convex body *that is a compact, convex set with nonempty interior. A convex body P is called a lower set if for all $(x_1, \ldots, x_n) \in P$, the vectors $(y_1, \ldots, y_n) \in P$ for $0 < y_i < x_i$ for $i = 1, \ldots, n$.*

Proof of Theorem 1.2.3. Let Q be a lower set, then the inward normals of its facets satisfy

- $\boldsymbol{v} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_1, -\boldsymbol{e}_2\}$, or
- $\boldsymbol{v} = (\lambda_1\boldsymbol{e}_1, \lambda_2\boldsymbol{e}_2)$ for some integers $\lambda_1, \lambda_2 < 0$.

If the inward normal $\boldsymbol{v}$ is one of the first kind, then the result follows from Lemma 6.1.1. Otherwise, we apply the method in the proof of Lemma 6.1.2. $\square$

The condition that the convex hull of supports Q has to be a lower set in $(\mathbb{R}_+)^d$ is a necessary condition. However, it is not sufficient since we can find some support $\mathcal{A}$ so that its convex hull Q is not a lower set, but still two bivariate random Bernoulli polynomials with support $\mathcal{A}$ has nonzero directional resultant with high probability.

Example 6.1.1. *Let Q be the convex hull of the points $(0,0), (d,d)$ and $(2d,0)$ and $\mathcal{A}$ is the set of integers in Q. Suppose that $f_1, f_2 \in \mathbb{C}[x_1, x_2]$ be two random Bernoulli polynomials with support $\mathcal{A}$. Then for all nonzero vectors $\boldsymbol{v} \in \mathbb{Z}^2$, $\mathcal{R}es_{\mathcal{A}}(f_1^{\boldsymbol{v}}, f_2^{\boldsymbol{v}}) \neq 0$ with high probability.*

Proof. Let $Q = conv((0,0),(d,d),(2d,0))$ and $\mathcal{A} = Q \cap \mathbb{Z}^2$. Then the inward normals of the facets of Q are $\boldsymbol{v}_1 = \boldsymbol{e}_2$, $\boldsymbol{v}_2 = -\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\boldsymbol{v}_3 = \boldsymbol{e}_1 - \boldsymbol{e}_2$ where $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ are standart normal basis elements of $\mathbb{R}^2$.

For $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$, the result follows from the proof of Theorem 1.2.3. For the inward normal $\boldsymbol{v_3}$, we have $\mathcal{A}^{\boldsymbol{v}_3} = \{(j,j) : 0 \leq j \leq d\}$ which is already a subset of $\boldsymbol{v}_3^{\perp} \cap \mathbb{Z}^2$.

Then the corresponding polynomials are

$$g_{1,\boldsymbol{v}_3} = \alpha_{0,0} + \alpha_{1,1}x_1x_2 + \cdots + \alpha_{d,d}x_1^dx_2^d \tag{6.9}$$

and

$$g_{2,\boldsymbol{v}_4} = \beta_{0,0} + \beta_{1,1}x_1x_2 + \cdots + \beta_{d,d}x_1^dx_2^d \tag{6.10}$$

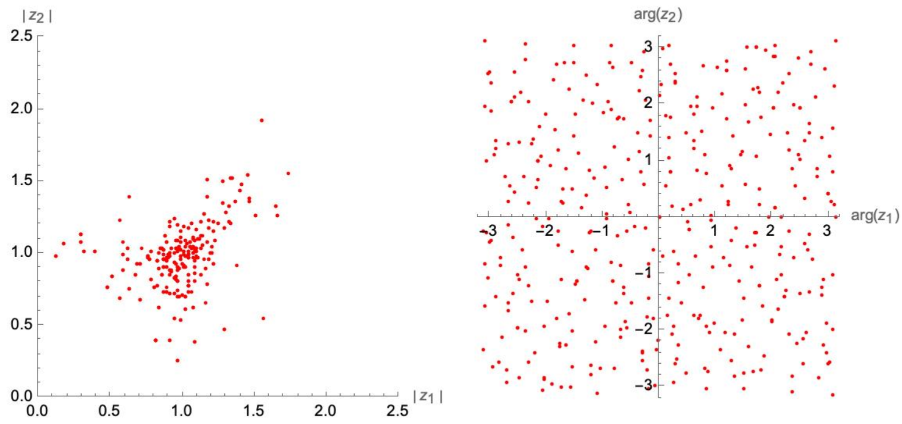
Applying the change of variable $y = x_1x_2$, we obtain

$$g_{1,\boldsymbol{v}_3}(y) = \alpha_{0,0} + \alpha_{1,1}y + \cdots + \alpha_{d,d}y^d. \tag{6.11}$$

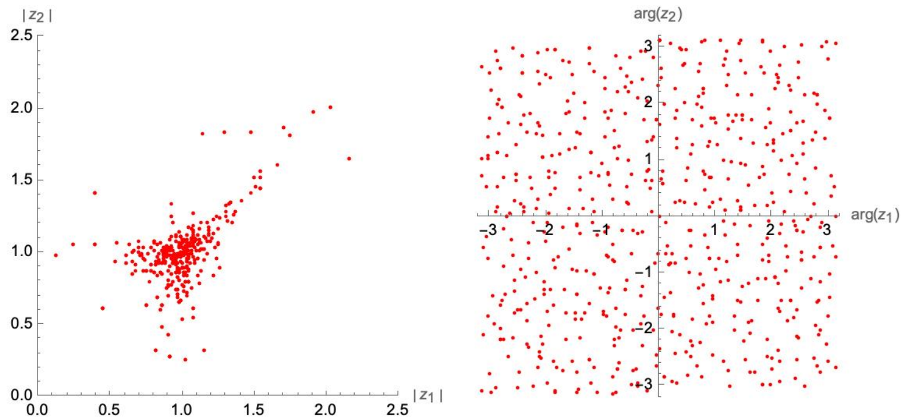
and

$$g_{2,\boldsymbol{v}_3}(y) = \beta_{0,0} + \beta_{1,1}y + \cdots + \beta_{d,d}y^d \tag{6.12}$$

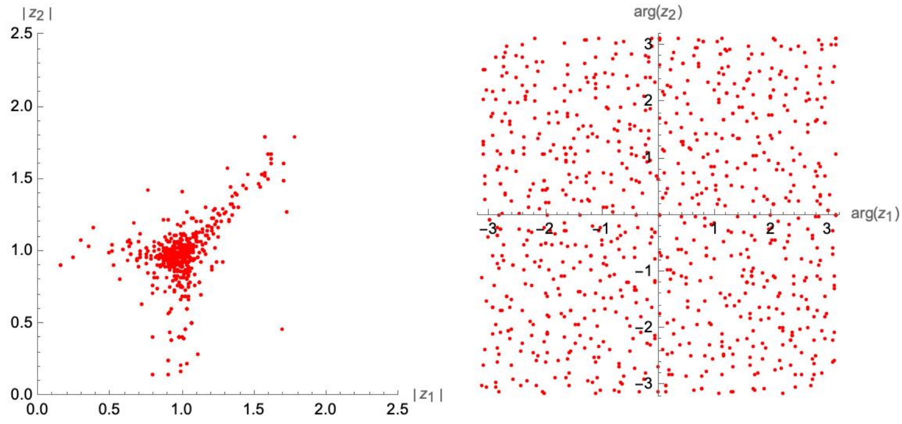
which are univariate random Bernoulli polynomials. Using the Theorem 4.1.3, we have that there exists a constant K such that $\mathcal{R}es_{\mathcal{A}^{\boldsymbol{v}_3},\mathcal{A}^{\boldsymbol{v}_3}}(f_1^{\boldsymbol{v}_3}, f_2^{\boldsymbol{v}_3}) \neq 0$ with probability at least 1-K/d. $\square$



(a) Zeros of $(f_{(20,1)}, f_{(20,2)})$



(b) Zeros of $(f_{(25,1)}, f_{(25,2)})$



(c) Zeros of $(f_{(30,1)}, f_{(30,2)})$

Figure 6.1 Distributions of Zeros of Random Bernoulli Polynomials

6.2 Conclusion

Asymptotic zero distribution of random polynomial systems $\boldsymbol{f}^k = (f_1, \ldots, f_k)$ with discrete random coefficients has many open parts, for $k = 1, \ldots, n$, including Bernoulli distribution.

On $\mathbb{C}^2$, in the light of the result of Bloom and Dauvergne, Theorem 3.2.6, we say that for a bivariate random Bernoulli polynomial f_d, the limit distribution of zeros converges to $dd^c V_K$, where V_K is the Green function of K with pole at infinity. On the other hand, our main result Theorem 1.2.1 introduces an equidistibution result for full systems of bivariate polynomials. Hence for the polynomials with ± 1-valued Bernoulli coefficients, we have all results for the zeros of $\boldsymbol{f}_k$ for $k = 1, 2$. However, on $\mathbb{C}^n$, $n \geq 3$, Theorem 1.2.1 is effective only for full systems, i.e., $k = n$. The cases $k = 1, \ldots, n-1$ still remain open.

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