BERGMAN SPACES ON FINITELY CONNECTED DOMAINS

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ABSTRACT

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Keywords: Bergman spaces, Carleson measures, kernel estimate, Toeplitz operator, composition operator

In this thesis study, the weighted Bergman spaces on finitely connected planar domains are investigated. They are isomorphic to the product of weighted Bergman spaces on the unit disk. Using this argument, the Carleson embeddings are characterized and kernel estimates are proved. Bounded, compact and Schatten class composition and Toeplitz operators on these spaces are characterized. Main results generalize several recent ones in the unit disk or simply connected case.

ÖZET

SONLU BAĞLANTILI BÖLGELERDE AĞIRLIKLI BERGMAN UZAYLARI

SINEM YELDA SÖNMEZ

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Anahtar Kelimeler: Bergman uzayları, Carleson ölçüleri, çekirdek tahminleri, Toeplit operatör, bileşke operatör

Bu tez çalışmasında sonlu bağlantılı bölgeler üzerinde tanımlı ağırlıklı Bergman uzayları ele alınmıştır. Bu uzaylar birim diskte tanımlı ağırlıklı Bergman uzaylarının çarpımına izomorfiktir. Bu argümanı kullanarak Carleson gömmeleri ve çekirdek tahminleri karakterize edilmiştir. Bu uzaylar üzerinde tanımlı sınırlı ve Schatten snıfında olan bileşke ve Toeplitz operatörleri karakterize edilmiştir. Bu tezdeki ana sonuçlar, birim disk üzerinde verilen önceki sonuçları sınırlı, Dini-düzgün ve Jordan bölgeler üzerinde genelleştirmiştir.

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To Hypatia,
who is the first female mathematician.

TABLE OF CONTENTS

1. Introduction

Denote the Lebesgue measure on $\mathbb{C}$ by dA. Let Ω be a domain of the complex plane and let ω be a continuous, strictly positive function on Ω so that $\int_{\Omega}\omega dA < \infty$. Such a function will be called *a weight function* on Ω. We extend ω to the complex plane by setting $\omega(z) = 0$ if z is not in Ω.

Definition 1.0.1 (Weighted Bergman Space). For $0 < p < \infty$, the weighted Bergman space $A^p_\omega(\Omega)$ consists of all holomorphic functions on Ω such that

$$\|f\|^p_{A^p_\omega(\Omega)} := \int_\Omega |f(z)|^p \omega(z) dA(z) < \infty.$$

In particular, if the weight function $\omega \equiv 1$, then the space is called the classical Bergman space $A^p(\Omega)$ on Ω. Moreover, for the case when $p = 2$ the weighted Bergman space becomes a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{A^2_\omega(\Omega)} = \int_\Omega f(z)\overline{g(z)}\omega(z) dA(z),$$

for any $f, g \in A^2_\omega(\Omega)$.

Each point evaluation map is bounded on A^2_ω, which follows from Proposition 2.1.1. Hence, the Riesz representation theorem implies that there exists a unique function $K_z \in A^2_\omega$ so that

$$f(z) = \int_\Omega f(\zeta)\overline{K_z(\zeta)}\omega(\zeta) dA(z),$$

for every $f \in A^2_\omega$. The function $K(z, \zeta)$ defined on $\Omega \times \Omega$ by $K(z, \zeta) = \overline{K_z(\zeta)}$ is called the Bergman kernel or the reproducing kernel of A^2_ω.

It follows from a well-known fact that A^p_ω is a closed subspace of L^p_ω. Therefore, there exists an orthogonal projection P defined from L^2_ω onto A^2_ω, which is called the Bergman projection. As a result of the reproducing property of the Bergman kernel function and the point evaluation map we obtain that

$$Pf(z) = \int_\Omega f(\zeta)K(z, \zeta)\omega(\zeta) dA(\zeta),$$

for any $f \in L^2_\omega$ and $z \in \Omega$.

As an initial result, a decomposition theorem is provided. The notations will be explained in the next sections. Our decomposition result below will be important throughout the thesis. Similar decompositions will be obtained for the weighted Dirichlet spaces and the multipliers of the weighted Bergman space.

Theorem 1.0.2. *Let $1 \leq p < \infty$, ω be a positive continuous, integrable weight function on Ω. Then*

$$A^p_\omega(\Omega) = A^p_\omega(\Omega_0) + A^p_{\omega,0}(\Omega_1) + A^p_{\omega,0}(\Omega_2) + \cdots + A^p_{\omega,0}(\Omega_N),$$

where $A^p_{\omega,0}(\Omega_j) = H_0(\Omega_j) \cap A^p_\omega(\Omega_j)$. Moreover, every function $f \in A^p_\omega(\Omega)$ has a unique decomposition $f = F_0 + F_1 + \cdots + F_N$ and there exists a positive constant C such that the following inequalities hold:

$$\|F_j\|_{A^p_\omega(\Omega_j)} \leq C\|f\|_{A^p_\omega(\Omega)}, \;\; j = 0, \ldots, N.$$

Such a decomposition result is well-known, for instance, on the space H^∞ of bounded holomorphic functions (Chevreau & Shields, 1981) on finitely connected domains. However, the last estimates in Theorem 1.0.2 are new. Using these estimates, we establish an isomorphism between the Bergman space $A^p_\omega(\Omega)$ of Ω and a product space of weighted Bergman spaces on the unit disk (Corollary 2.2.2). This allows a characterization of Carleson embeddings for $A^p_\omega(\Omega)$ as in Theorem 3.0.8. Carleson embeddings is a vast subject; a recent work of Gonzales (Gonzales, 2020) studies the Carleson embeddings of the weighted Bergman spaces on simply connected domains, where the weight is a power of the distance function to the boundary of the domain.

We then focus on a specific weight function, which is described next. The distance from z to the boundary of Ω is denoted by $\rho(z) = \rho_\Omega(z)$. Let μ be any positive Borel measure on Ω (not necessarily finite) and let $s > 0$ so that

$$\int_\Omega \rho^s(z)\,d\mu(z) < \infty. \tag{1.1}$$

Define

$$U_{\mu,s}(z) = \int_\Omega G^s_\Omega(z,w)\,d\mu(w).$$

For the case when $s = 1$, the function $U_{\mu,1}$ is called the potential of the measure μ on Ω. In 90's the Dirichlet space with harmonic weights was introduced in (Richter, 1991) and the Dirichlet space with superharmonic weights was introduced in (Two-Isometries, Ale). Göğüş, Bao and Pouliasis considered the properties of composition operators on the

Dirichlet space with superharmonic weights $U_{\mu,1}$, (Bao, Göğüş & Pouliasis, 2017).

Let ν be a finite measure on the unit circle $\partial\Omega$. The Poisson integral P_ν of ν is the harmonic function defined on Ω via

$$P_\nu(z) = \int_{\partial\Omega} P_\Omega(z,\zeta)d\nu(\zeta),$$

where $P_\Omega(z,\zeta)$ denotes the Poisson kernel on Ω, (Ransford, 1995). It is easy to show (Armitage & Gardiner, 2001, p. 98) that $U_{\mu,s}(z) \not\equiv \infty$ if and only if (1.1) holds. Let $p > 0$, $s \geq 0$, $q + s > -1$, $q > -2$, and set

$$\omega_{\mu,q,s,\nu}(z) = \rho^q(z)U_{\mu,s}(z) + P_\nu(z) \tag{1.2}$$

for $z \in \Omega$. We will consider the space $A^p_{\omega_{\mu,q,s,\nu}}(\Omega)$.

Norm and pointwise estimates for the kernel function of the weighted Bergman space defined on the unit disk, where the weight function is harmonic, have been proved in (El-Fallah, Mahzouli, Marrhich & Naqos, 2018). In this thesis, norm and pointwise estimates for the case when the domain is a bounded, Dini-smooth, Jordan domain have been proved. The novelty of our approach is defining an equivalent norm on A^p_ω to obtain an isometry from A^p_ω to the product space of weighted Bergman spaces defined on the unit disk.

Afterwards, the methods in (El-Fallah, Mahzouli, Marrhich & Naqos, 2016) are used, in order to characterize the Schatten class Toeplitz (Theorem 4.4.5) and composition operators (Corollary 4.4.9). The authors in (El-Fallah et al., 2018) provide a characterization of the membership in the Schatten class of Toeplitz operator acting on harmonically weighted Bergman space defined on the unit disk. We extend their result to the case when the weight is as in (1.2) and to the case when the domain is bounded, Dini-smooth and Jordan in the complex plane.

Furthermore, it is proved in the last section that the composition operator between distinct weighted Bergman spaces over finitely connected domains is bounded, for general p, while (Li & Huang, 2020) proved that the composition operator from $A^2(\Omega)$ to $A^2(\tilde{\Omega})$, is bounded.

2. Weighted Functional Spaces on Finitely Connected Domains

2.1 Preliminaries

We will denote an open disk in the complex plane centered at z with radius r by $\mathbb{D}(z,r)$. The unit disk is denoted by $\mathbb{D}$. We denote the class of all holomorphic functions on a domain Ω by $H(\Omega)$. If φ is a holomorphic map from a domain $\tilde{\Omega}$ into a domain Ω, then the composition operator induced by φ is the linear operator which takes a function $f \in H(\Omega)$ to the function $f \circ \varphi \in H(\tilde{\Omega})$. We will write $a \lesssim b$ if there exists an absolute constant C such that $a \leq Cb$. Also, the symbol $a \approx b$ means that $a \lesssim b$ and $b \lesssim a$.

Let Ω be a bounded domain in the complex plane of C^∞ boundary. Then either Ω is simply connected or it is a finitely connected domain, (Berenstein & Gay, 1991, Proposition 1.4.7). If $\Omega \subset \mathbb{C}$ is a finitely connected domain bounded by $N+1$ disjoint closed curves of C^2 boundary, then it is biholomorphic to a domain of the form

$$U = \mathbb{D} \setminus \cup_{j=1}^{N} \overline{\mathbb{D}}(z_j, r_j), \tag{2.1}$$

where $z_j \in \mathbb{D}$ and $0 < r_j < 1$ for all $j \in \{1, \ldots, N\}$. Moreover, if φ is such a biholomorphism from Ω onto U, then φ extends continuously to the closure $\overline{\Omega}$ of Ω and there exist positive constants m, M so that $0 < m \leq |\varphi'(z)| \leq M$ on Ω.

Let ω be a weight function on Ω and set $u(z) = \frac{\omega \circ \varphi^{-1}(z)}{|\varphi'(z)|^2}$. A calculation shows that

$$\int_\Omega |f(z)|^p \omega(z) dA(z) = \int_U |f \circ \varphi^{-1}(z)|^p u(z) dA(z)$$

for any $f \in A^p_\omega(\Omega)$. Hence, $C_{\varphi^{-1}}$ is an isometric isomorphism from $A^p_\omega(\Omega)$ onto $A^p_u(U)$.

Let Ω be a domain as in (2.1). We set $\Omega_0 = \mathbb{D}$ and $\Omega_j = \mathbb{C} \setminus \overline{\mathbb{D}}(z_j, r_j)$ for $j = 1, \ldots, N$. If

$f \in H(\Omega)$, it is well-known that f has a unique decomposition

$$f = F_0 + F_1 + \ldots + F_N \tag{2.2}$$

such that $F_0 \in H(\mathbb{D})$ and $F_j \in H_0(\Omega_j)$, where $H_0(\Omega_j)$ is the space of holomorphic functions on Ω_j that vanish at infinity (Chevreau & Shields, 1981). In fact, the functions F_j are the Cauchy transforms of f and we have

$$\mathcal{C}_0 f(z) := F_0(z) := \lim_{r \to 1^-} \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w-z} dw \tag{2.3}$$

for $z \in \mathbb{D}$,

$$\mathcal{C}_j f(z) := F_j(z) := -\lim_{r \to r_j^+} \frac{1}{2\pi i} \int_{|w|=r} \frac{f(z_j + w)}{w-z} dw \tag{2.4}$$

for $z \in \Omega_j$, $j = 1, \ldots, N$.

Proposition 2.1.1. *Let Ω be a domain in the complex plane and $A^p_\omega(\Omega)$ be the weighted Bergman space, where the weight function ω is strictly positive and $\omega \equiv 0$ on $\mathbb{C} \setminus \Omega$. For each compact subset K of Ω, suppose that $m_K > 0$, where $m_K = \inf_K \omega$. Then for every compact subset K of Ω there exists a positive constant c depending on K such that for every point $z \in K$,*

$$|f(z)|^p \leq c(K) \|f\|^p_{A^p_\omega(\Omega)}$$

for every $f \in A^p_\omega(\Omega)$.

Proof. For any $z_0 \in \Omega$, let $0 < r < 1$ so that $\overline{\mathbb{D}}(z_0, r)$ is contained in Ω. Let $S := \overline{\mathbb{D}}(z_0, r)$. Hence,

$$\begin{aligned}
\int_\Omega |f(z)|^p \omega(z) dA(z) &\geq \int_S |f(z)|^p \omega(z) dA(z) \\
&\geq m_S \int_S |f(z)|^p dA(z) \\
&= m_S \frac{1}{2\pi} \int_0^r \int_0^{2\pi} s |f(z_0 + s e^{i\theta})|^p d\theta ds \\
&\geq \frac{1}{2} r^2 m_S |f(z_0)|^p,
\end{aligned}$$

where the last inequality follows from the subharmonicity of $|f|^p$. Therefore, by taking $r \leq \rho(z_0)$, we obtain the following

$$\frac{1}{2} \rho^2(z_0) m_S |f(z_0)|^p \leq \|f\|^p_{A^p_\omega(\Omega)}.$$

Let K be a compact subset of Ω. For each $z \in K$, let $S_z := \overline{\mathbb{D}}(z, r_z)$ be a neighborhood of

z for some $0 < r_z < 1$ such that $S_z \subset \Omega$. According to the above inequality, we have that

$$\frac{1}{3}\rho^2(z) m_{S_z} |f(z)|^p \leq \|f\|^p_{A^p_\omega(\Omega)}.$$

Moreover, $L := \cup_{z\in K}\mathbb{D}(z, r_z)$ is an open cover of K. Since K is compact, it has a finite subcover so that $L' = \cup_{n=1}^N \mathbb{D}(z_n, r_n) \supset K$. Hence,

$$\frac{1}{3}\rho_\Omega(K) m_K |f(z)|^p \leq \|f\|^p_{A^p_\omega(\Omega)}.$$

Consequently,

$$|f(z)|^p \leq c(K)\|f\|^p_{A^p_\omega(\Omega)},$$

where $c(K) = 3(m_K \rho_\Omega(K))^{-1}$. □

Proposition 2.1.2. *Let Ω be a bounded, Dini-smooth and Jordan domain in the complex plane. Let $f_k \in H(\Omega)$, $k = 1, 2, \ldots$ and suppose that f_k converges uniformly to a function f on compact subsets of Ω. Then $\mathcal{C}_j f_k$ converges to $\mathcal{C}_j f$ uniformly on compact subsets of Ω, for $j = 0, 1, \ldots, N$ where $\mathcal{C}_j$'s are the Cauchy transforms as described in* (2.3) *and* (2.4).

Proof. Without loss of generality we may assume that Ω is of the form (2.1). Let $\{f_k\}_{k\in\mathbb{N}}$ be any sequence in $H(\Omega)$ so that f_k converges uniformly to a function f on compact subsets of Ω. Hence, $f \in H(\Omega)$. Besides, the functions f and f_k's can be written as $f_k = F_{k,0} + F_{k,1} + \ldots + F_{k,N}$ and $f = F_0 + F_1 + \ldots + F_N$ uniquely, where $F_0, F_{k,0} \in H(\mathbb{D})$ and $F_j, F_{k,j} \in H(\Omega_j)$ for $j = 1, \ldots, N$. Notice that, $F_j = \mathcal{C}_j f$ and $F_{k,j} = \mathcal{C}_j f_k$ for every $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, N\}$.

For the case when $j = 0$, our aim is to show that $F_{k,0}$ converges to F_0 uniformly on compact subsets of $\mathbb{D}$. Recall that

$$F_{k,0}(z) - F_k(z) = \lim_{r\to 1^-} \frac{1}{2\pi}\int_{|\varepsilon|=r} \frac{f_k(\zeta) - f(\zeta)}{\zeta - z} d\zeta.$$

Notice that,

$$\frac{1}{2\pi}\int_{|\zeta|=r} \frac{f_k(\zeta) - f(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{2\pi}\int_{|\zeta|=r}(f_k - f)(\zeta)\zeta^{-n-1} d\zeta$. Let $0 < r_1 < r_2 < 1$ be any positive numbers such that $\overline{A} := \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\} \subset \Omega$. Then, we claim that

$$\int_{|\zeta|=r_1} \frac{f_k(\zeta) - f(\zeta)}{\zeta - z} d\zeta = \int_{|\zeta|=r_2} \frac{f_k(\zeta) - f(\zeta)}{\zeta - z} d\zeta,$$

for any $0 < r_1 < r_2 < 1$. Equivalently, the claim states that

$$\int_{|\zeta|=r_1} (f_k - f)(\zeta)\zeta^{-n-1} d\zeta = \int_{|\zeta|=r_2} (f_k - f)(\zeta)\zeta^{-n-1} d\zeta.$$

Observe that $f_k - f \in H(\overline{A})$ and $g \in H(\overline{A})$, where $g(z) := z^{-n-1}$. Thus, $(f_k - f)g \in H(\overline{A})$. The Cauchy's theorem yields that

$$\int_{\partial A} ((f_k - f)g)(\zeta) d\zeta = 0.$$

Hence,

$$\int_{|\zeta|=r_2} ((f_k - f)g)(\zeta) d\zeta - \int_{|\zeta|=r_1} ((f_k - f)g)(\zeta) d\zeta = 0,$$

which proves the claim.

The assumption in the statement provides that for any $\varepsilon > 0$ there exists an $M > 0$ so that

$$\frac{1}{2\pi} \int_{|\zeta|=r} \frac{|f_k(\zeta) - f(\zeta)|}{|\zeta - z|} d\zeta \le \frac{1}{2\pi} \frac{\varepsilon}{1 - |z|} \le \varepsilon C_K$$

for every $k \ge M$. By taking limit as r goes to 1, we obtain that

$$\max_{z \in K} |F_{k,0}(z) - F_k(z)| \le \varepsilon C_K.$$

Then it follows that $F_{k,0}$ converges to F_0 uniformly on K.

For the case when $j \in \{1, \ldots, N\}$, it follows from the same method that $\{F_{k,j} - F_j\}$ converges to zero uniformly on compact subsets of Ω_j. □

2.2 Weighted Bergman Spaces

The decomposition result given in the next theorem is also based on well-known arguments, but it will be important for the next sections.

Theorem 2.2.1. *Let $1 \le p < \infty$, ω be a positive continuous, integrable weight function on Ω and let $\omega \equiv 0$ on $\mathbb{C} \setminus \Omega$. Then*

$$A^p_\omega(\Omega) = A^p_\omega(\Omega_0) + A^p_{\omega,0}(\Omega_1) + A^p_{\omega,0}(\Omega_2) + \cdots + A^p_{\omega,0}(\Omega_N),$$

where $A^p_{\omega,0}(\Omega_j) = H^p_0(\Omega_j) \cap A^p_\omega(\Omega_j)$. Moreover, every function $f \in A^p_\omega(\Omega)$ has a unique *decomposition $f = F_0 + F_1 + \cdots + F_N$ and there exists a positive constant C such that the following inequalities hold:*

$$\|F_j\|_{A^p_\omega(\Omega_j)} \leq C\|f\|_{A^p_\omega(\Omega)}, \ \ j = 0, \ldots, N.$$

Proof. Firstly, for any $f \in A^p_\omega(\Omega)$ we want to show that $f \in A^p_\omega(\mathbb{D}) + A^p_{\omega,0}(\Omega_1) + A^p_{\omega,0}(\Omega_2) + \cdots + A^p_{\omega,0}(\Omega_N)$. Let $f \in A^p_\omega(\Omega)$ and $f = F_0 + F_1 + \cdots + F_N$ be the unique decomposition as in (2.2). We need to show that $f_0 \in A^p_\omega(\mathbb{D})$ and $f_j \in A^p_{\omega,0}(\Omega_j)$. Since $F_0 = f - \sum_{j=1}^N F_j$, we have that

$$\begin{aligned}\int_{\mathbb{D}} |F_0|^p \omega dA &= \int_\Omega |F_0|^p \omega dA \\ &\leq 2^{(p-1)(N-1)} \left(\int_\Omega |f(z)|^p \omega(z) dA(z) + \sum_{k=1}^N \int_\Omega |F_k(z)|^p \omega(z) dA(z) \right).\end{aligned}$$

The first integral on the right is finite, since $f \in A^p_\omega(\Omega)$. Now let $0 < r_0 < 1$ be close to 1, and let $A_0 := \{z \in \mathbb{C} : |z| > r_0\}$. Let $r_j < r_j' < 1$ for each $j \in \{1, ..., N\}$, and let $A_j = \{z \in \mathbb{C} : |z - z_j| \leq r_j'\}$. Let $K_j = \Omega_j \backslash A_j$, $j = 0, \ldots, N$. Since $F_0 \in H(\mathbb{D})$, F_0 is bounded on K_0 and since $F_j \in H_0(\Omega_j)$, for each $j \in \{1, \ldots, N\}$, there exists $M_j > 0$ such that $|F_j(z)| \leq M_j$ for all $z \in K_j$. Hence,

$$\begin{aligned}\int_{\Omega_0} |F_0|^p \omega dA &= \int_{K_0} |F_0|^p \omega dA + \int_{\Omega \cap A_0} |F_0|^p \omega dA \lesssim \|f\|^p_{A^p_\omega(\Omega)} + \sum_{j=0}^N \int_{\Omega \cap A_0} |F_j|^p \omega dA \\ &\leq \|f\|^p_{A^p_\omega(\Omega)} + \sum_{j=0}^N M_j^p \int_\Omega \omega dA < \infty.\end{aligned}$$

Hence, $F_0 \in A^p_\omega(\mathbb{D})$.

Fix $j \in \{1, \ldots, N\}$. A similar estimate shows that

$$\begin{aligned}\int_{\Omega_j} |F_j|^p \omega dA &= \int_{K_j} |F_j|^p \omega dA + \int_{A_j \cap \Omega_j} |F_j|^p \omega dA \\ &\lesssim \|f\|^p_{A^p_\omega(\Omega)} + \sum_{1 \leq k \neq j \leq N} \int_{A_k \cap \Omega_k} |F_k|^p \omega dA \\ &\leq \|f\|^p_{A^p_\omega(\Omega)} + \sum_{1 \leq k \neq j \leq N} M_k^p \int_\Omega \omega dA < \infty.\end{aligned}$$

Thus, $F_j \in A^p_{\omega,0}(\Omega_j)$. We have shown that

$$A^p_\omega(\Omega) \subset A^p_\omega(\mathbb{D}) + A^p_{\omega,0}(\Omega_1) + A^p_{\omega,0}(\Omega_2) + \ldots + A^p_{\omega,0}(\Omega_N).$$

For the converse inclusion, let $F_0 \in A^p_\omega(\mathbb{D})$ and $F_j \in A^p_{\omega,0}(\Omega_j)$ be given for each $j \in$

$\{1,\ldots,N\}$. Then from the inequality

$$|F_0+F_1+\ldots+F_N|^p \le 2^{(p-1)(N-1)}\left(|F_0|^p+|F_1|^p+\ldots+|F_N|^p\right),$$

it can be easily seen that the function $f=F_0+F_1+\ldots+F_N$ belongs to $A^p_\omega(\Omega)$.

Let Y be the product space $A^p_\omega(\Omega_0)\times\ldots\times A^p_{\omega,0}(\Omega_N)$. Let T be the linear mapping defined by $Tf=(\mathcal{C}_0 f,\ldots,\mathcal{C}_N f)$ for $f\in H(\Omega)$. Clearly, T is linear and injective. From Proposition 2.1.1, Proposition 2.1.2 and the closed graph theorem, it follows that $T: A^p_\omega(\Omega)\to Y$ is bounded. Hence,

$$\|F_j\|_{A^p_\omega(\Omega_j)} \le C\|f\|_{A^p_\omega(\Omega)}$$

for $j=0,\ldots,N$. □

Let φ_j be the biholomorphic mapping defined from Ω_j to $\mathbb{D}\backslash\{0\}$ given by

$$\varphi_j(z)=\frac{r_j}{z-z_j} \tag{2.5}$$

for every $z\in\Omega_j$, for $j=1,\ldots,N$. Using a change of variables formula we obtain that

$$\begin{aligned}\int_{\Omega_j}|F|^p\omega\,dA &= \int_{\mathbb{D}}|F\circ\varphi_j^{-1}|^p\frac{\omega\circ\varphi_j^{-1}}{|\varphi_j'|^2}dA \\ &= \int_{\mathbb{D}}|g|^p\eta_j\,dA,\end{aligned} \tag{2.6}$$

where $F\in H_0(\Omega_j)$, $\eta_j(z)=\frac{|z-z_j|^4}{r_j^2}\omega(\frac{r_j+z_jz}{z})$, $g(z)=F(\frac{r_j+z_jz}{z})$, $z\in\mathbb{D}$, $j=1,\ldots,N$.

Additionally, we observe that

$$\eta_j(z)\approx(\omega\circ\varphi_j^{-1})(z)$$

for $z\in\mathbb{D}$, $j=1,\ldots,N$. Let $\boldsymbol{A}^p(\omega,\Omega):=A^p_\omega(\mathbb{D})\times A^p_{\omega,0}(\Omega_1)\times\cdots\times A^p_{\omega,0}(\Omega_N)$ and $\mathcal{C}: A^p_\omega(\Omega)\to\boldsymbol{A}^p(\omega,\Omega)$ be the mapping given by $\mathcal{C}f=(\mathcal{C}_0f,\mathcal{C}_1f,\ldots,\mathcal{C}_Nf)$. The $\boldsymbol{A}^p(\omega,\Omega)$ norm has the property that

$$\|\boldsymbol{F}\|_{\boldsymbol{A}^p(\omega,\Omega)}=\left(\sum_{j=0}^N\|F_j\|^p_{A^p_{\omega,0}(\Omega_j)}\right)^{1/p}\approx\max_{0\le j\le N}\|F_j\|_{A^p_{\omega,0}(\Omega_j)}$$

for every $\boldsymbol{F}\in\boldsymbol{A}^p(\omega,\Omega)$, where we set $\Omega_0=\mathbb{D}$. Also let $\boldsymbol{A}^p(H,\mathbb{D}):=A^p_\omega(\mathbb{D})\times A^p_{\eta_1,0}(\mathbb{D})\times\cdots\times A^p_{\eta_N,0}(\mathbb{D})$ and $\boldsymbol{C}_\Phi:\boldsymbol{A}^p(\omega,\Omega)\to\boldsymbol{A}^p(H,\mathbb{D})$ be the mapping given by $\boldsymbol{C}_\Phi f=(C_{\varphi_0}f,C_{\varphi_1^{-1}}f,\ldots,C_{\varphi_N^{-1}}f)$, where we set $\varphi_0(z)=z$, and every member of the spaces $A^p_{\eta_j,0}(\mathbb{D})$ vanishes at zero, for each $j = 1,\ldots,N$. All these observations up to here lead to the following corollary, here, the symbol $\cong$ stands for the isomorphism of spaces with equivalent norms.

Corollary 2.2.2. *We have*

$$A^p_\omega(\Omega) \cong \boldsymbol{A}^p(\omega,\Omega) \cong \boldsymbol{A}^p(H,\mathbb{D}),$$

where the linear maps $\mathcal{C}$ and $\boldsymbol{C}_\Phi$ give the first and second isomorphisms, respectively.

2.3 Multiplier of Weighted Bergman Spaces

Definition 2.3.1 (Multiplication Operator). Let $\Omega \in \mathbb{C}$ be any domain. An analytic function $\varphi \in H(\Omega)$ is called a multiplier for $A^p_\omega(\Omega)$, if $\varphi f \in A^p_\omega(\Omega)$ for all $f \in A^p_\omega(\Omega)$. The multiplication operator denoted by M_φ defined from $A^p_\omega(\Omega)$ to $A^p_\omega(\Omega)$ such that $M_\varphi f = \varphi f$ for all $f \in A^p_\omega(\Omega)$. The space of all multipliers for $A^p_\omega(\Omega)$ denoted by $M(A^p_\omega(\Omega))$.

Proposition 2.3.2. *Assume that Ω is a finitely connected domain in the complex plane and ω is strictly positive such that $\omega \equiv 0$ on $\mathbb{C}\setminus\Omega$. Moreover, suppose that $m_K > 0$ for each compact subset $K \subset \Omega$. If $\varphi \in M(A^p_\omega(\Omega))$, then M_φ is a bounded linear operator on $A^p_\omega(\Omega)$.*

Proof. Clearly, M_φ is linear. We will show that M_φ is continuous, using the closed graph theorem. The graph of M_φ is

$$G(M_\varphi) = \{(f,\varphi f) : f \in A^p_\omega(\Omega)\}.$$

Let $\{(f_n,\varphi f_n)\}_{n\geq 1}$ be a convergent sequence in $G(M_\varphi)$ and assume that the sequence converges to some (f_0,g_0) in $A^p_\omega(\Omega)\times A^p_\omega(\Omega)$. Since $\lim_{n\to\infty}\|f_n - f_0\|_{A^p_\omega(\Omega)} = 0$, $\lim_{n\to\infty} f_n(z) = f_0(z)$ for $z \in \Omega$ by Proposition 2.1.1.

Similarly, since $\lim_{n\to\infty}\|\varphi f_n - g_0\| = 0$, we have $\lim_{n\to\infty}\varphi(z)f_n(z) = g_0(z) = \varphi(z)f_0(z)$ for $z \in \Omega$. Therefore, $(f_0,g_0) = (f_0,\varphi f_0) \in G(M_\varphi)$. Consequently, the graph of M_φ is closed. $\square$

Proposition 2.3.3. *Suppose that $\varphi \in H(\Omega)$. Then $M(A^p_\omega(\Omega)) = H^\infty(\Omega)$.*

Proof. Let $T_z f = f(z)$ be the point evaluation map on $A^p_\omega(\Omega)$. For any $f \in A_\omega(\Omega)$ and any $z \in \Omega$, we have the following estimate

$$|\varphi(z)f(z)| = |T_z M_\varphi f| \le \|T_z\| \|M_\varphi\| \|f\|_{A_\omega(\Omega)}.$$

By taking supremum over $f \in A^p_\omega(\Omega)$ such that $\|f\|_{A^p_\omega(\Omega)} \le 1$, we obtain that

$$|\varphi(z)| \|T_z\| \le \|T_z\| \|M_\varphi\|,$$

that is, $\|\varphi\|_\infty \le \|M_\varphi\|$. Besides, for any $f \in A^p_\omega(\Omega)$

$$\|M_\varphi f\|_{A^p_\omega(\Omega)} = \|\varphi f\|_{A^p_\omega(\Omega)} \le \|\varphi\|_\infty \|f\|_{A^p_\omega(\Omega)},$$

which implies that $\|M_\varphi\| \le \|\varphi\|_\infty$.

□

Theorem 2.3.4. *Let Ω be a domain as in* (2.1) *and $\omega \in L^1(\Omega)$ be a positive function. Assume that $\omega \equiv 0$ on $\mathbb{C} \setminus \Omega$ and $\Omega_i = \mathbb{C} \setminus \overline{\mathbb{D}}(z_j, r_j)$ for all $1 \le i \le N$. Then we have the following decomposition*

$$M(A^p_\omega(\Omega)) = M(A^p_\omega(\mathbb{D})) + M_0(A^p_\omega(\Omega_1)) + M_0(A^p_\omega(\Omega_2)) + \ldots + M_0(A^p_\omega(\Omega_N)),$$

where $M_0(A^p_\omega(\Omega_j)) = H_0(\Omega_j) \cap M(A^p_\omega(\Omega_j))$. Moreover, if $\varphi = \varphi_0 + \varphi_1 + \cdots + \varphi_N$ is the unique decomposition, then for $0 \le j \le N$,

$$\|\varphi_j\|_{M_0(A^p_\omega(\Omega_j))} \lesssim \|\varphi\|_{M(A^p_\omega(\Omega))}. \tag{2.7}$$

Proof. For simplicity, we consider the case $N = 2$, that is, the domain $\Omega = \mathbb{D} \setminus (\overline{\mathbb{D}}(z_1, r_1) \cup \overline{\mathbb{D}}(z_2, r_2))$. For any $\varphi \in M(A^p_\omega(\Omega))$ we know that $\varphi \in H^\infty(\Omega) \cap A^p_\omega(\Omega)$. According to Theorem 2.2.1, φ has a decomposition such that $\varphi = \varphi_0 + \varphi_1 + \varphi_2$ where $\varphi_0 \in H^\infty(\mathbb{D}) \cap A^p_\omega(\mathbb{D})$, and $\varphi_j \in H^\infty_0(\Omega_j) \cap A^p_\omega(\Omega_j)$, for each j. Proposition 2.3.3 implies that φ_j's are also bounded for each j. Hence, it follows from the H^∞ decomposition that $\varphi_j \in H^\infty(\Omega_j)$.

In order to prove the second statement, consider the following operator,

$$\begin{aligned} T : M(A^p_\omega(\Omega)) &\to M(A^p_\omega(\mathbb{D})) \times M_0(A^p_\omega(\Omega_1)) \times \ldots M_0(A^p_\omega(\Omega_N)) \\ \varphi &\longmapsto (\varphi_0, \ldots, \varphi_N), \end{aligned}$$

Recall that T is bounded as it is explained in Theorem 2.2.1. We set $Y := M(A^p_\omega(\mathbb{D})) \times M_0(A^p_\omega(\Omega_1)) \times \ldots \times M_0(A^p_\omega(\Omega_N))$. Thus,

$$\|\varphi_j\|_\infty \leq \max_{0\leq j\leq N} \|\varphi_j\|_\infty \approx \|T\varphi\|_Y \lesssim \|\varphi\|_{M(A^p_\omega(\Omega))} = \|\varphi\|_\infty,$$

which is derived from Proposition 2.3.3. □

Theorem 2.3.5. *Assume that $f = f_0 f_1 \ldots f_N \in A^p_\omega(\Omega)$ is the product function where $f_0 \in A^p_\omega(\mathbb{D})$ and $f_j \in A^p_\omega(\Omega_j)$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{A^p_\omega(\Omega)} \leq C\|f_0\|_{A^p_\omega(\mathbb{D})}\|f_1\|_{A^p_\omega(\Omega_1)} \cdots \|f_N\|_{A^p_\omega(\Omega_N)}.$$

Proof. For any j, we fix a point $z \in \Omega_j$. The point evaluation map is defined by

$$T_z : A^p_\omega(\Omega_j) \to, \quad T_z f_j = f_j(z).$$

Hence, there exists a positive integer $c_j(z)$ such that $\|T_z f_j\| = |f_j(z)| \leq C_j(z)\|f_j\|_{A^p_\omega(\Omega_j)}$. Furthermore, taking $C' := \max_{1\leq i\leq N} C_j$ implies that

$$\begin{aligned}
|f(z)|^p = \left|\prod_{0\leq j\leq N} f_j(z)\right|^p &= \left(\prod_{0\leq j\leq N} |f_j(z)|\right)^p \\
&\leq \left(\prod_{0\leq j\leq N} C'\|f_j\|_{A^p_\omega(\Omega_j)}\right)^p \\
&= (C')^p \prod_{1\leq j\leq N} \|f_j\|^p_{A^p_\omega(\Omega_j)}.
\end{aligned}$$

By Theorem 2.2.1, we obtain

$$\begin{aligned}
\int_\Omega |f(z)|^p \omega(z)dA(z) &\leq \sum_{j=0}^N \int_{\Omega_j} |f(z)|^p \omega(z)dA(z) \\
&\leq (C')^p \prod_{0\leq i\leq N} \|f_j\|^p_{A^p_\omega(\Omega_i)} \sum_{j=0}^N \int_{\Omega_j} \omega(z)dA(z) \\
&\leq (C')^p \prod_{0\leq j\leq N} \|f_j\|^p_{A^p_\omega(\Omega_j)} (N+1)\|\omega\|_{L^1}.
\end{aligned}$$

Thus, the proof is finished, by taking $C = (N+1)(C')^p\|\omega\|_{L^1}$. □

2.4 Weighted Dirichlet Spaces

Definition 2.4.1 (**Weighted Dirichlet Space**). Let Ω be a domain in the complex plane. For $0 < p < \infty$, the weighted Dirichlet space consists of the analytic functions on Ω such that

$$\int_{\Omega} |f'(z)|^p \omega(z) dA(z) < \infty.$$

The norm of the weighted Dirichlet space is defined by fixing a point $z_0 \in \Omega$ and for all $f \in D^p_\omega(\Omega)$,

$$\|f\|^p_{D_\omega(\Omega)} = |f(z_0)|^p + \int_{\Omega} |f'(z)|^p \omega(z) dA(z).$$

In particular, if $\omega \equiv 1$, the space is called the classical Dirichlet space.

Since the Bergman space and the Dirichlet space are isometric, as a consequence of Theorem 2.2.1 we have the following corollary.

Corollary 2.4.2. *Let Ω be a domain as in* (2.1) *and $\omega \in L^1(\Omega)$ be a positive function. Assume that $\omega \equiv 0$ on $\mathbb{C} \setminus \Omega$ and $\Omega_i = \mathbb{C} \setminus \overline{\mathbb{D}}(z_j, r_j)$ for all $1 \leq i \leq N$.*

$$D^p_\omega(\Omega) = D^p_\omega(\mathbb{D}) + D^p_{\omega,0}(\Omega_1) + D^p_{\omega,0}(\Omega_2) + \ldots + D^p_{\omega,0}(\Omega_N),$$

Moreover, every function $f \in A^p_\omega(\Omega)$ has a unique decomposition $f = F_0 + F_1 + \cdots + F_N$ and there exists a positive constant C such that the following inequalities hold:

$$\|F_j\|_{D^p_\omega(\Omega_j)} \leq C \|f\|_{D^p_\omega(\Omega)}, \;\; j = 0, \ldots, N.$$

The relation between the weighted Hardy space and the weighted Dirichlet space will be investigated. We will show that the weighted Hardy space is always included by the weighted Dirichlet space. However, it will be proved that the converse inclusion holds only for the Carleson measure.

Definition 2.4.3. Let Ω be a bounded, Dini-smooth and Jordan domain. The weighted Hardy space defined on Ω is

$$H^2_\mu := \{f \in H^2(\Omega) : \int_{\partial\Omega} |f^*(\zeta)|^2 S_\mu(\zeta) d\sigma(\zeta) < \infty\},$$

where σ is the arc length measure on the boundary of Ω, f^* is the almost everywhere boundary value of f (Duren, 1970) and

$$S_\mu(\zeta) := \int_{\Omega} P_\Omega(\zeta, z) d\mu(z),$$

for $\zeta \in \partial\Omega$.

Definition 2.4.4. The weighted Dirichlet space defined on Ω is

$$D_\mu := \{f \in H^2(\Omega) : \int_\Omega |f'(w)|^2 U_\mu(w) dA(w) < \infty\},$$

where

$$U_\mu(w) := \int_\Omega G_\Omega(w,z) d\mu(z),$$

for $w \in \Omega$.

Let X be a Banach space of holomorphic functions on a domain Ω. A positive Borel measure on Ω is said to be a q-Carleson measure for X if

$$\left(\int_\Omega |f|^q d\mu\right)^{1/q} \lesssim \|f\|_X \tag{2.8}$$

for every $f \in X$.

Theorem 2.4.5. *Let Ω be a finitely connected domain whose boundary consists of disjoint Jordan curves Γ_j, $0 \le j \le N$, with rectifiable boundary so that Γ_0 surrounds Ω, and μ be a positive Borel measure on Ω. Then*

$$H^2_\mu \subset D_\mu \subset H^2.$$

Moreover, $H^2_\mu = D_\mu$ if and only if μ is a Carleson measure for D_μ.

Proof. Let $f \in H^2$. Then $\Delta|f(z)|^2 = 4|f'(z)|^2$ for $z \in \Omega$. The least harmonic majorant of the subharmonic function $|f|^2$ on Ω is given by

$$h_f(z) = \frac{1}{2\pi}\int_{\partial\Omega} P_\Omega(z,\zeta)|f(\zeta)|^2 d\sigma(\zeta),$$

for $z \in \Omega$, from (Duren, 1970). Since $|f|^2$ is subharmonic and Ω is a regular domain, the Poisson Jensen formula implies that

$$\begin{aligned}|f(z)|^2 &= h_f(z) - \frac{1}{2\pi}\int_\Omega g_\Omega(z,w)\Delta|f(w)|^2 dA(w)\\ &= h_f(z) - \frac{2}{\pi}\int_\Omega g_\Omega(z,w)|f'(w)|^2 dA(w)\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\int_{\partial\Omega}|f(\zeta)|^2 S_\mu(\zeta)d\sigma(\zeta) &= \int_{\partial\Omega}|f(\zeta)|^2\frac{1}{2\pi}\int_\Omega P_\Omega(z,\zeta)d\mu(z)d\sigma(\zeta)\\ &= \int_\Omega h_f(z)d\mu(z)\\ &= \int_\Omega |f(z)|^2 d\mu(z) + \int_\Omega \frac{2}{\pi}\int_\Omega g_\Omega(z,w)|f'(w)|^2 dA(w)d\mu(z)\\ &= \int_\Omega |f(z)|^2 d\mu(z) + \frac{2}{\pi}\int_\Omega |f'(w)|^2 U_\mu(w)dA(w),\end{aligned}$$

which follows from Fubini's theorem and the above equality. Consequently, we obtain that,

$$\|f\|^2_{H^2_\mu} = \|f\|^2_{L^2(\mu)} + \frac{2}{\pi}\|f\|^2_{D_\mu} - \frac{2}{\pi}|f(z_0)|^2.$$

Hence, $H^2_\mu = D^2_\mu \cap L^2_\mu$. Then, $H^2_\mu = D_\mu$ if and only if $D_\mu \subset L^2_\mu$. It follows from the closed graph theorem that $D_\mu \subset L^2_\mu$ if and only if μ is a Carleson measure for D_μ.

□

3. Carleson Measures

In this chapter, we investigate some Carleson measure characterizations when the domain is a bounded Dini-smooth Jordan domain. Its description is obtained from (Selmi, 2000) as follows: A curve γ is called a closed Jordan curve if and only if $\gamma(0) = \gamma(2\pi)$ and $\gamma(t_1) \neq \gamma(t_2)$ for any $t_1, t_2 \in (0, 2\pi)$. A domain is called a closed Jordan domain if and only if it is bounded and its boundary consists of finitely many disjoint Jordan curves. For $\delta \in [0, 2\pi]$, the modulus of continuity of γ is defined by

$$u(\delta) = \sup\{|\gamma(t_1) - \gamma(t_2)| : t_1, t_2 \in [0, 2\pi], |t_1 - t_2| \leq \delta\}.$$

The function γ is called Dini continuous provided

$$\int_0^\pi u(t)dt < \infty.$$

If a curve γ has a parametrization $\gamma(t)$, $0 \leq t \leq 2\pi$ so that the derivative $\gamma'(t)$ is Dini-continuous and nonzero, then γ is called a Dini-smooth curve. A domain Ω is called Dini-smooth if and only if Ω is bounded and $\partial\Omega$ consists of finitely many disjoint closed Dini-smooth Jordan curves. In this chapter, Ω will always denote a bounded, Dini-smooth Jordan domain. The following proposition states that the distance is preserved under certain conditions in the domain.

Proposition 3.0.1. *(Selmi, 2000)*

(i) Let Ω be a bounded, simply connected, Dini-smooth Jordan domain. Let φ be a conformal map from Ω onto $\mathbb{D}$. Then there exists a constant $c = c(\Omega) > 0$ so that for all $z \in \Omega$,

$$\frac{1}{c}\rho_\Omega(z) \leq \rho_{\mathbb{D}}(\varphi(z)) \leq c\rho_\Omega(z).$$

(ii) Let Ω be a bounded, multiply connected, Dini-smooth Jordan domain. Then there exist a conformal mapping φ from Ω onto a bounded domain U in the complex plane of C^∞ boundary and a constant $c = c(\Omega) > 0$ so that for all $z \in \Omega$,

$$\frac{1}{c}\rho_\Omega(z) \leq \rho_U(\varphi(z)) \leq c\rho_\Omega(z).$$

(iii) In both cases, φ extends continuously to the closure $\overline{\Omega}$ of Ω and there exists a constant $c > 0$ such that for all z, $w \in \Omega$,

$$\frac{1}{c}|z-w| \leq |\varphi(z)-\varphi(w)| \leq c|z-w|.$$

We denote the (positive) Green function on Ω by $G_\Omega(z,w)$ (cf. (Ransford, 1995)).

Proposition 3.0.2. *(Selmi, 2000) Let Ω be a bounded, Dini-smooth Jordan domain. There exists a constant $c > 0$ such that for all z, $w \in \Omega$,*

$$\frac{1}{c}\log\left(1+\frac{\rho_\Omega(z)\rho_\Omega(w)}{|z-w|^2}\right) \leq G_\Omega(z,w) \leq c\log\left(1+\frac{\rho_\Omega(z)\rho_\Omega(w)}{|z-w|^2}\right).$$

Let μ be a positive Borel measure on Ω and $\varphi : \Omega \to U$ be a one-to-one holomorphic map. We denote by $d\mu^*$ the push-forward measure under φ, that is, $\mu^*(E) = \mu(\varphi^{-1}(E \cap \varphi(\Omega)))$ for any measurable set $E \subset U$. Define

$$v_{\mu,q,s}(z) = \rho_\Omega^q(z)\int_\Omega \left(\frac{\rho_\Omega(z)\rho_\Omega(w)}{|z-w|^2+\rho_\Omega(z)\rho_\Omega(w)}\right)^s d\mu(w).$$

If $\Omega = \mathbb{D}$, then $v_{\mu,q,s}(z) = (1-|z|)^q \int_\mathbb{D}(1-|\sigma_z(w)|^2)^s d\mu(w)$, where $\sigma_z(w) = \frac{z-w}{1-\overline{w}z}$ for z, $w \in \mathbb{D}$.

Proposition 3.0.3. *Let Ω be a bounded, Dini-smooth Jordan domain. Let μ be a positive Borel measure on Ω, $p > 0$, $s \geq 0$, $q+s > -1$, $q > -2$. Then*

$$\int_\Omega |f(z)|^p \omega_{\mu,q,s,0}(z)dA(z) \approx \int_\Omega |f(z)|^p v_{\mu,q,s}(z)dA(z)$$

for $f \in A^p_{\omega_{\mu,q,s,0}}(\Omega)$.

Proof. If Ω is the unit disk, it follows from a well-known argument, (Xiao, 2001). We outline the proof for completeness. Firstly, notice that

$$\begin{aligned}\omega_{\mu,q,s,0}(z) &= \rho_\mathbb{D}^q(z)U_{\mu,s}(z)\\ &= (1-|z|)^q\int_\mathbb{D} G_\mathbb{D}^s(z,\zeta)d\mu(\zeta)\\ &= (1-|z|)^q\int_\mathbb{D}\log^s\frac{1}{|\sigma_z(\zeta)|}d\mu(\zeta),\end{aligned}$$

and

$$v_{\mu,q,s}(z) = (1-|z|)^q\int_\mathbb{D}(1-|\sigma_z(\zeta)|^2)^s d\mu(\zeta).$$

We have the following inequalities; for the case when $t \in (0,1]$

$$-2\log t = \log \frac{1}{t^2} \geq 1 - t^2,$$

and for the case when $t \in (\frac{1}{4}, 1]$

$$-\log t = \log \frac{1}{t} \leq 4(1-t^2).$$

Therefore, we obtain the following estimates;

$$\log \frac{1}{|\sigma_z(\zeta)|} \geq \frac{1}{2}(1 - |\sigma_z(\zeta)|^2),$$

for $z, \zeta \in \mathbb{D}$ and

$$\log \frac{1}{|\sigma_z(\zeta)|} \leq 4(1 - |\sigma_z(\zeta)|^2),$$

for $|\sigma_z(\zeta)| \in (\frac{1}{4}, 1)$. It remains to consider the case when $|\sigma_z(\zeta)| \in (0, \frac{1}{4}]$.

Using polar coordinates and the fact that $\int_{|\zeta|=r} |f(r\zeta)|^p d\sigma(\zeta)$ is an increasing function of r, we obtain the following result,

$$\begin{aligned}
\int_{\mathbb{D}(0,\frac{1}{4})} |f(z)|^p \log^s \frac{1}{|z|} dA(z) &= \int_0^{\frac{1}{4}} \int_{|\zeta|=1} r|f(r\zeta)|^p \log^s \frac{1}{r} d\sigma(\zeta) dr \\
&\leq \int_0^{\frac{1}{4}} \log 4 \int_{|\zeta|=1} |f(\zeta/4)|^p d\sigma(\zeta) dr \\
&\lesssim \int_{|\zeta|=1} |f(\zeta/4)|^p d\sigma(\zeta) dr \\
&\lesssim \int_{\mathbb{D}\setminus\mathbb{D}(0,\frac{1}{4})} |f(z)|^p (1-|z|^2)^s dA(z) \\
&\leq \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^s dA(z).
\end{aligned}$$

Hence,

$$\int_{\mathbb{D}(0,\frac{1}{4})} |f(z)|^p \log^s \frac{1}{|z|} dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^s dA(z).$$

Replace f by $\frac{f \circ \sigma_w}{(\sigma_w')^{2/p}}$, and use a change of variable formula to obtain the following result,

$$\int_{\{|\sigma_w(z)| \leq 1/4\}} |f|^p \log^s \frac{1}{|\sigma_w|} dA \lesssim \int_{\mathbb{D}} |f|^p (1 - |\sigma_w|^2)^s dA.$$

Therefore, we conclude that

$$\int_{\mathbb{D}}|f(z)|^p\omega_{\mu,q,s,0}(z)dA(z)\approx\int_{\mathbb{D}}|f(z)|^p v_{\mu,q,s}(z)dA(z)$$

for $f\in A^p_{\omega_{\mu,q,s,0}}(\mathbb{D})$.

If Ω is simply connected, the result follows by Proposition 3.0.1 and Proposition 3.0.2.

Thanks to Proposition 3.0.1, we can assume without loss of generality that Ω is a domain of the form (2.1). In view of Theorem 2.2.1, it is enough to estimate the $A^p_{\omega_{\mu,q,s,0}}(\Omega_j)$ norm of each $F_j=\mathcal{C}_j f$ for $j=0,\dots,N$. For $j=0$, $\Omega_0=\mathbb{D}$. Let $h_0=F_0$. For other j, let $h_j=F_j\circ\varphi_j^{-1}$ and $\eta_j=\omega_{\mu,q,s,0}\circ\varphi_j^{-1}$, where φ_j are defined as in (2.5). Then $\rho_{\mathbb{D}}(a)\approx\rho_{\Omega_j}(\varphi_j^{-1}(a))$ for $a\in\mathbb{D}\backslash\{0\}$. If $d\mu_j^*$ denotes the push-forward of $d\mu$ under φ_j, we have

$$\begin{aligned}
\|F_j\|^p_{A^p_{\omega_{\mu,q,s,0}}(\Omega_j)}&\approx\|h_j\|^p_{A^p_{\eta_j}(\mathbb{D})}\\
&\approx\int_{\mathbb{D}}|h_j(a)|^p\rho^q_{\mathbb{D}}(a)\int_{\Omega_j}\log^s\left(1+\frac{\rho_{\Omega_j}(\varphi_j^{-1}(a))\rho_{\Omega_j}(w)}{|\varphi_j^{-1}(a)-w|^2}\right)d\mu(w)dA(a)\\
&\approx\int_{\mathbb{D}}|h_j(a)|^p\rho^q_{\mathbb{D}}(a)\int_{\mathbb{D}}G^s_{\mathbb{D}}(a,b)d\mu_j^*(b)dA(a)\\
&\approx\int_{\mathbb{D}}|h_j(a)|^p\rho^q_{\mathbb{D}}(a)\int_{\mathbb{D}}\left[\frac{\rho_{\mathbb{D}}(a)\rho_{\mathbb{D}}(b)}{|a-b|^2+\rho_{\mathbb{D}}(a)\rho_{\mathbb{D}}(b)}\right]^s d\mu_j^*(b)dA(a)\\
&\approx\int_{\Omega_j}|F_j(z)|^p v_{\mu,q,s}(z)dA(z).
\end{aligned}$$

The proof is finished. □

Let us restate all that have been observed so far as a corollary.

Corollary 3.0.4. *Let Ω be a bounded, Dini-smooth Jordan domain. Let μ and ν be positive Borel measures on Ω and $\partial\Omega$, respectively. Let $p>0$, $s\geq 0$, $q+s>-1$, $q>-2$. Then*

$$\|f\|_{A^p_{\omega_{\mu,q,s,\nu}}(\Omega)}\approx\max_{0\leq j\leq N}\|h_j\|_{A^p_{\omega_{\mu_j^*,q,s,\nu_j^*}}(\mathbb{D})}$$

for $f\in A^p_{\omega_{\mu,q,s,0}}(\Omega)$, where $d\mu_j^*$ and $d\nu_j^*$ denote the push-forward of $d\mu$ and $d\nu$ under $\varphi_j\circ\varphi$, respectively.

We will make use of the following integral estimates frequently throughout the thesis (Zhang, Li, Shang & Guo, 2018, Theorem 3.1).

Lemma 3.0.5. *For $r \geq 0$, $t \geq 0$, $\delta > -1$, let*

$$J(w,a) = \int_{\mathbb{D}} \frac{(1-|z|^2)^{\delta}}{|1-\overline{w}z|^t|1-\overline{a}z|^r} dA(z),$$

$w \in \mathbb{D}$, $a \in \overline{\mathbb{D}}$.

Then the following estimate holds

$$J(w,a) \approx \frac{1}{(1-|w|^2)^{t-\delta-2}|1-a\overline{w}|^r},$$

if $t > \delta + 2 > r$.

We now choose suitable test functions for $A^p_{\omega_{\mu,q,s,\nu}}$.

Lemma 3.0.6. *Let μ be a positive Borel measure on $\mathbb{D}$, $p > 0$, $s > 0$, $q+s > -1$, $q > -2$ and $s < q+2$. For $z \in \mathbb{D}$, $w \in \mathbb{D}$, $pr > \max\{2+q+s, 3\}$ and $t > \max\{q+s, 1\}$, define*

$$f_{w,t}(z) = \left(\frac{(1-|w|^2)^t}{\omega_{\mu,q,s,\nu}(w)} \right)^{1/p} \frac{1}{(1-\overline{w}z)^{(2+t)/p}}. \tag{3.1}$$

Then $\sup_{w \in \mathbb{D}} \|f_{w,t}\|_{A^p_{\omega_{\mu,q,s,\nu}}} < \infty$.

Proof. We estimate using Lemma 3.0.5 and Fubini's theorem

$$\begin{aligned}
\|f_{w,t}\|^p_{A^p_{\omega_{\mu,q,s,\nu}}} &= \frac{(1-|w|^2)^t}{\omega_{\mu,q,s,\nu}(w)} \int_{\mathbb{D}} \frac{(1-|z|^2)^q U_{\mu,s}(z) + P_\nu(z)}{|1-\overline{w}z|^{2+t}} dA(z) \\
&\leq \frac{(1-|w|^2)^t}{(1-|w|^2)^q U_{\mu,s}(w)} \int_{\mathbb{D}} (1-|b|^2)^s \times \\
&\times \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{s+q}}{|1-\overline{w}z|^{2+t}|1-\overline{b}z|^{2s}} dA(z) \right) d\mu(b) \\
&+ \frac{(1-|w|^2)^t}{P_\nu(w)} \int_{\partial\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{w}z|^{2+t}|1-\overline{\zeta}z|^2} dA(z) \right) d\nu(\zeta) \\
&\lesssim \frac{1}{U_{\mu,s}(w)} \int_{\mathbb{D}} \frac{(1-|w|^2)^s(1-|b|^2)^s}{|1-\overline{b}w|^{2s}} d\mu(b) \\
&+ \frac{1}{P_\nu(w)} \int_{\partial\mathbb{D}} \frac{1-|w|^2}{|\zeta-w|^2} d\nu(\zeta) = 2.
\end{aligned}$$

$\square$

Let's consider the case when Ω is the unit disk $\mathbb{D}$. Let $E_w(r)$ denote the pseudo-hyperbolic disk

$$E_w(r) = \left\{ z \in \mathbb{D} : \left| \frac{z-w}{1-\overline{w}z} \right| < r \right\}.$$

Since $|1-\overline{w}z| \approx 1-|w|^2 \approx \sqrt{A(E_w(r))} \approx r$, $P_\nu(z) \approx P_\nu(w)$ and $U_{\mu,s}(z) \approx U_{\mu,s}(w)$ for every $z \in E_w(r)$, we have for every $w \in \mathbb{D}$ the estimate

$$\|f\|^p_{A^p_{\omega_{\mu,q,s,\nu}}} \geq \int_{E_w} |f(z)|^p \omega_{\mu,q,s,\nu}(z) dA(z) \\ \gtrsim |f(w)|^p (1-|w|^2)^2 \omega_{\mu,q,s,\nu}(w). \tag{3.2}$$

We denote the class of nonnegative subharmonic functions on the unit disk by sh_+. We will need a beautiful method in (Luecking, 1983)).

Theorem 3.0.7. *(Luecking, 1983) Let u be a nonnegative function and ν be a positive measure on $\mathbb{D}$ so that there exist constants $c_1 > 0$, $c_2 > 0$ with $u(z) < c_1 u(w)$ whenever z belongs to the set E_w, and*

$$\nu(E_w) \leq c_2 \int_{E_w} u dA$$

for all $w \in \mathbb{D}$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} g d\nu \leq C \int_{\mathbb{D}} g u dA$$

for all $g \in sh_+$.

A recent work of Gonzales (Gonzales, 2020) studies the Carleson embeddings of the weighted Bergman spaces on simply connected domains, where the weight is a power of the distance function to the boundary of the domain. Our result generalizes the characterization of Gonzales in two ways: the domain is finitely connected and the weight is more general. We use Luecking's method to describe Carleson measures for $A^p_{\omega_{\mu,q,s,\nu}}(\Omega)$.

Theorem 3.0.8. *Let Ω be a bounded, Dini-smooth Jordan domain. Let γ be a positive measure on $\mathbb{D}$. Then the following are equivalent.*

(i) γ is a $\tilde{p}$-Carleson measure for $A^p_\omega(\Omega)$.

(ii) γ is a $\tilde{p}$-Carleson measure for $A^p_\omega(\Omega^*_j)$ for $j = 0, 1, \ldots, N$, where $\Omega^*_j = \varphi^{-1}(\Omega_j)$.

(iii) γ^*_j is a $\tilde{p}$-Carleson measure for $A^p_{\eta_j}(\mathbb{D})$ for $j = 0, 1, \ldots, N$, where $\eta_j = \omega \circ (\varphi_j \circ \varphi)^{-1}$ and γ^*_j denotes the push-forward measure of γ under $\varphi_j \circ \varphi$ on $\mathbb{D}$.

Let $\tilde{p} \geq p > 0$, $s > 0$, $q+s > -1$, $q > -2$ and $s < q+2$. Let ν be a finite positive measure on $\partial\Omega$ and μ be a positive measure on Ω. If $\omega = \omega_{\mu,q,s,\nu}$, then any of the statements above is equivalent to the following.

(iv) Let $d\mu^*_j$ and $d\nu^*_j$ denote the push-forward of $d\mu$ and $d\nu$ under $\varphi_j \circ \varphi$, respectively. Let

$D_a^j = (\varphi_j \circ \varphi)^{-1}(E_a)$ *for* $a \in \mathbb{D}$. *We have*

$$\sup_{a\in\mathbb{D}} \frac{\gamma(D_a^j)}{[\omega_{\mu_j^*,q,s,\nu_j^*}(a)]^{\tilde{p}/p}(1-|a|^2)^{2\tilde{p}/p}} < \infty \tag{3.3}$$

for $j = 0, 1, \ldots, N$.

Proof. γ is a $\tilde{p}$-Carleson measure for $A^p_\omega(\Omega)$ if and only if

$$\int_\Omega |f|^{\tilde{p}} d\gamma \lesssim \|f\|^{\tilde{p}}_{A^p_\omega(\Omega)}$$

for all $f \in A^p_\omega(\Omega)$. In particular, this inequality holds for restrictions to Ω of functions F in $A^p_\omega(\Omega_j^*)$. Hence, (i) implies (ii). Suppose (ii) holds. Any function $f \in A^p_\omega(\Omega)$ is of the form $f = F_0 + F_1 + \ldots + F_N$. Since $\Omega \subset \Omega_j^*$, γ can be regarded as a measure on Ω_j^* supported in Ω. By Theorem 2.2.1,

$$\int_\Omega |f|^{\tilde{p}} d\gamma \lesssim \sum_{j=0}^N \int_{\Omega_j^*} |F_j|^{\tilde{p}} d\gamma \lesssim \sum_{j=0}^N \|F_j\|^{\tilde{p}}_{A^p_\omega(\Omega_j^*)} \approx \|f\|^q_{A^p_\omega(\Omega)}.$$

This means that γ is a $\tilde{p}$-Carleson measure for $A^p_\omega(\Omega)$. Hence, (ii) implies (i).

Notice that (ii) is equivalent to the fact that γ is a $\tilde{p}$-Carleson measure for $A^p_{\omega,0}(\Omega_j^*)$ for $j = 0, 1, \ldots, N$. By Corollary 2.2.2, $A^p_{\omega,0}(\Omega_j^*) \cong A^p_{\eta_j,0}(\mathbb{D})$. Thus, (ii) is equivalent to the fact that γ_j^* is a $\tilde{p}$-Carleson measure for $A^p_{\eta_j,0}(\mathbb{D})$ for $j = 0, 1, \ldots, N$. Since $A^p_{\eta_j}(\mathbb{D})$ norms of h and $h - h(0)$ are comparable for $h \in A^p_{\eta_j}(\mathbb{D})$, we get the equivalence of (ii) and (iii).

We will start proving the equivalence of (iii) and (iv). Assume for the moment that $\Omega = \mathbb{D}$. Let $f_{a,t}$ be the functions defined as in (3.1), where t is chosen as described in Lemma 3.0.6. If γ is a $\tilde{p}$-Carleson measure for $A^p_{\omega_{\mu,q,s,\nu}}$, then we apply (2.8) to the functions $f_{a,t}$ to get

$$\begin{aligned}
\frac{\gamma(E_a)}{[\omega_{\mu,q,s,\nu}(a)]^{\tilde{p}/p}(1-|a|^2)^{2\tilde{p}/p}} &\lesssim \frac{(1-|a|^2)^{t\tilde{p}/p}}{[\omega_{\mu,q,s,\nu}(a)]^{\tilde{p}/p}} \int_{E_a} \frac{d\gamma(z)}{|1-\bar{a}z|^{(2+t)\tilde{p}/p}} \\
&\leq \int_{\mathbb{D}} |f_{a,t}(z)|^{\tilde{p}} d\gamma(z) \\
&\lesssim \|f_{a,t}\|^{\tilde{p}}_{A^p(\mu,q,s)} \lesssim 1.
\end{aligned}$$

Hence, we get (3.3).

Conversely, suppose (3.3) holds. Let $f \in A^p_{\omega_{\mu,q,s,\nu}}$. We have

$$\begin{aligned}\gamma(E_a) &\le C[\omega_{\mu,q,s,\nu}(a)]^{\tilde{p}/p}(1-|a|^2)^{2\tilde{p}/p} \\ &\le C_0 \int_{E_a} [\omega_{\mu,q,s,\nu}(a)]^{\tilde{p}/p}(1-|w|^2)^{2\tilde{p}/p-2} dA(z)\end{aligned} \tag{3.4}$$

for all $a \in \mathbb{D}$. By Luecking's result,

$$\int_{\mathbb{D}} |f(z)|^{\tilde{p}} d\gamma(z) \lesssim \int_{\mathbb{D}} |f(z)|^{\tilde{p}} (1-|z|^2)^{2(\tilde{p}/p)-2} [\omega_{\mu,q,s,\nu}(z)]^{\tilde{p}/p} dA(z). \tag{3.5}$$

If $\tilde{p} \ge p$, from the estimate (3.2), we get that

$$|f(z)|^{\tilde{p}-p} \lesssim (1-|z|^2)^{2\frac{p-\tilde{p}}{p}} [\omega_{\mu,q,s,\nu}(z)]^{\frac{p-\tilde{p}}{p}} \tag{3.6}$$

for all $z \in \mathbb{D}$. Combining (3.5) and (3.6) we get

$$\int_{\mathbb{D}} |f(z)|^{\tilde{p}} d\gamma(z) \lesssim \int_{\mathbb{D}} |f(z)|^{p} \omega_{\mu,q,s,\nu}(z) dA(z) = \|f\|^p_{A^p_{\omega_{\mu,q,s,\nu}}}.$$

Hence, γ is a $\tilde{p}$-Carleson measure for $A^p_{\omega_{\mu,q,s,\nu}}$. Now, for general Ω, in (iii) and (iv), $\gamma(D_a^j) = \gamma_j^*(E_a)$ for $a \in \mathbb{D}$. Hence, (iii) and (iv) are equivalent. $\square$

In the case of simply connected domain, the statement of the above theorem simplifies to the following corollary.

Corollary 3.0.9. *Let Ω be a simply connected domain and γ be a positive measure on $\mathbb{D}$. Let $\tilde{p} \ge p > 0$, $s > 0$, $q + s > -1$, $q > -2$ and $s < q + 2$. Let ν be a finite positive measure on $\partial\Omega$ and μ be a positive measure on Ω. If $\omega = \omega_{\mu,q,s,\nu}$, then the following statements are equivalent.*

(i) γ is a $\tilde{p}$-Carleson measure for $A^p_\omega(\Omega)$.

(ii) Let $d\mu^*$ and $d\nu^*$ denote the push-forward of $d\mu$ and $d\nu$ under φ. Let $D_a = (\varphi)^{-1}(E_a)$ for $a \in \mathbb{D}$. We have

$$\sup_{a \in \mathbb{D}} \frac{\gamma(D_a)}{[\omega_{\mu^*,q,s,\nu^*}(a)]^{\tilde{p}/p}(1-|a|^2)^{2\tilde{p}/p}} < \infty. \tag{3.7}$$

4. Kernel Estimates

A pointwise estimate for the kernel function of the weighted Bergman space defined on the unit disk, where the weight function is harmonic, has been proved in (El-Fallah et al., 2018). In this chapter, our aim is to adapt the results in (El-Fallah et al., 2018) to our case. Let $\omega = \omega_{\mu,q,s,\nu}$. We denote the reproducing kernel for A^2_ω by K^ω_z or simply by K_z for $z \in \Omega$.

4.1 Norm Estimate for Kernel Function

Theorem 4.1.1. *Let Ω be a bounded, Dini-smooth Jordan domain. Let $1 < p \le 2$ and $q = \frac{p}{p-1}$ be the conjugate of p. Then*

$$\|K_z\|^q_{A^p_\omega} \gtrsim \frac{1}{(\rho_\Omega(z))^2\omega(z)}, \quad \textit{and} \quad \|K_z\|^p_{A^p_\omega} \lesssim \frac{1}{(\rho_\Omega(z))^2\omega(z)} \qquad z \in \Omega.$$

In particular, we have

$$\|K_z\|^2_{A^2_\omega} \approx \frac{1}{(\rho_\Omega(z))^2\omega(z)}, \qquad z \in \Omega.$$

Proof. Without loss of any generality, we may assume that Ω is a domain of the form (2.1). We use the same notation in the paragraph before Corollary 2.2.2. For each $j \in \{0,\dots,N\}$ and $z \in \Omega$, let $K^{\eta_j}_{\varphi_j(z)}$ denote the reproducing kernel for $A^2_{\eta_j}(\mathbb{D})$. Then $\mathbf{K}_{\Phi(z)} = \left(K^{\eta_0}_{\varphi_0(z)},\dots,K^{\eta_N}_{\varphi_N(z)}\right)$ is the reproducing kernel for $\mathbf{A}^2(H,\mathbb{D})$. We define an equivalent norm $\|\cdot\|_2$ on the space $A^2_\omega(\Omega)$ so that $\mathbf{C}_\Phi \circ \mathcal{C}$ gives an isometry. Let $K^{2,\omega}_z$ or simply K^2_z denote the reproducing kernel of the Hilbert space $(A^2_\omega(\Omega), \|\cdot\|_2)$. It is easy to see that

$K^{\eta_j}_{\varphi_j(z)} = C_{\varphi_j^{-1}} \mathcal{C}_j K^{2,\omega}_z$ and

$$\|K^{\omega}_z\|_{A^2_\omega(\Omega)} \approx \|K^{2,\omega}_z\|_{(A^2_\omega(\Omega),\|\cdot\|_2)} = \left(\sum_{j=0}^{N} \|K^{\eta_j}_{\varphi_j(z)}\|^2_{A^2_{\eta_j}(\mathbb{D})} \right)^{1/2}$$
$$\approx \max_{0\le j\le N} \|K^{\eta_j}_{\varphi_j(z)}\|_{A^2_{\eta_j}(\mathbb{D})}.$$

Hence, it is enough to prove the statement when $\Omega = \mathbb{D}$. We assume that Ω is the unit disk.

By the usual pairing, the dual $(A^p_\omega)^*$ of A^p_ω is isomorphic to A^q_ω. Therefore, we have

$$\|K_z\|_{A^p_\omega} = \sup\left\{ \left| \int_{\mathbb{D}} \overline{K_z} g\omega dA \right| : \|g\|_{A^q_\omega} = 1 \right\}.$$

Thus, by using the test functions as in Lemma 3.0.6,

$$\|K_z\|^q_{A^p_\omega} \gtrsim |\langle f^{p/q}_{t,z}, K_z\rangle|^q = |f^{p/q}_{t,z}(z)|^q = \frac{1}{(1-|z|^2)^2\omega(z)}.$$

By using the subharmonicity of $|K_z|^p$, we have that

$$\omega(a)|K_z(a)|^p \lesssim (1-|a|^2)^{-2} \int_{E_a} |K_z(b)|^p \omega(b) dA(b),$$

which implies that

$$\frac{|K_z(a)|^p}{\|K_z\|^p_{A^p_\omega}} \lesssim \frac{1}{\omega(a)(1-|a|^2)^2}.$$

By taking $a = z$, we obtain that

$$\frac{(K_z(z))^p}{\|K_z\|^p_{A^p_\omega}} \lesssim \frac{1}{\omega(z)(1-|z|^2)^2}.$$

Since $(K_z(z))^p = \|K_z\|^{2p}_{A^2_\omega} \ge \|K_z\|^{2p}_{A^p_\omega}$, we get

$$\|K_z\|^p_{A^p_\omega} \lesssim \frac{1}{\omega(z)(1-|z|^2)^2}.$$

The proof is finished. $\square$

We produce an example of a case where the norm equivalence does not provide the inner product equivalence, but the kernel equivalence.

Example 1. Let $\omega_1(z) = 1$, $\omega_2(z) = |z|$, for $z \in \mathbb{D}$. Let $\mathcal{H}_1 := A^2(\mathbb{D}) = A^p_{\omega_1}(\mathbb{D})$, and $\mathcal{H}_2 := A^2_{\omega}(\mathbb{D})$. Observe that as a set $\mathcal{H}_1$ is $\mathcal{H}_2$. The norms and inner products are described as follows:

$$\|f\|_1 = \int_{\mathbb{D}} |f(z)|^2 dA(z), \quad \langle f, g\rangle_1 = \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z)$$

for $f, g \in \mathcal{H}_1$ and

$$\|f\|_2 = \int_{\mathbb{D}} |f(z)|^2 |z| dA(z), \quad \langle f, g\rangle_2 = \int_{\mathbb{D}} f(z)\overline{g(z)} |z| dA(z)$$

for $f, g \in \mathcal{H}_2$. Then by (Luecking, 1981)

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) \approx \int_{\mathbb{D}} |f(z)|^2 |z| dA(z)$$

for every $f \in \mathcal{H}_2$. Hence, $\|.\|_1$ and $\|.\|_2$ are equivalent.

Let $f(z) = z + \frac{2}{3}$ and $g(z) = z - 1$ for $z \in \mathbb{D}$. Then, it is concluded that

$$\begin{aligned}
\langle f, g\rangle_1 &= \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z) \\
&= \int_0^1 \left(\int_0^{2\pi} (re^{i\theta} + \frac{2}{3})(re^{-i\theta} - 1) d\theta \right) r dr \\
&= 2\pi \int_0^1 \left(r^4 - \frac{2r^2}{3} \right) dr \\
&= -\frac{2\pi}{45},
\end{aligned}$$

and

$$\begin{aligned}
\langle f, g\rangle_2 &= \int_{\mathbb{D}} f(z)\overline{g(z)} |z| dA(z) \\
&= \int_0^1 \left(\int_0^{2\pi} (re^{i\theta} + \frac{2}{3})(re^{-i\theta} - 1) d\theta \right) r^2 dr \\
&= 2\pi \int_0^1 \left(r^5 - \frac{2r^3}{3} \right) dr = 0,
\end{aligned}$$

which means that these two inner products can not be equivalent.

Recall that, a reproducing kernel can be obtained by using an orthonormal basis. Consider two orthonormal basis $\{e_n(z)\}_{n\in\mathbb{N}}$ and $\{E_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, where $e_n(z) = \frac{\sqrt{2n+2}}{\sqrt{2\pi}} z^n$, and $E_n(z) = \frac{\sqrt{2n+3}}{\sqrt{2\pi}} z^n$. Then the kernels can be found as following

$$K^1(z,w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)},$$

and

$$K^2(z,w) = \sum_{n=0}^{\infty} E_n(z)\overline{E_n(w)}.$$

A computation yields that,

$$K^1(z,w) = \frac{1}{2\pi(1-z\overline{w})^2}, \quad \text{and} \quad K^2(z,w) = \frac{3-z\overline{w}}{2\pi(1-z\overline{w})^2},$$

for $z,w \in \mathbb{D}$. Hence, the following connection is obtained,

$$2|K^1(z,w)| \leq |K^2(z,w)| \leq 4|K^1(z,w)|,$$

for $z,w \in \mathbb{D}$.

4.2 Pointwise Estimate for Kernel Function

In this section, we proved pointwise estimates by Berndtsson's method.

Lemma 4.2.1. *There exists a positive constant C so that for $f \in A^2_\omega(\Omega)$*

$$|f(z)-f(w)| \leq C\frac{|z-w|}{\rho_\Omega(z)}\|K_z\|_{A^2_\omega}\|f\|_{A^2_\omega}$$

for $z,w \in \Omega$ such that $|z-w| \leq \frac{\rho_\Omega(z)}{4}$.

Proof. Let $z,w \in \Omega$ with the property that $|z-w| \leq \frac{\rho_\Omega(z)}{4}$. We set $r = \frac{\rho_\Omega(z)}{4}$ and $D_r = \mathbb{D}(z,r)$. From the fundamental theorem of calculus we have that

$$|f(z)-f(w)| = \left|\int_w^z f'(\zeta)d\zeta\right| \leq \int_w^z |f'(\zeta)|d\zeta \leq |z-w| \sup_{\zeta \in D_r}|f'(\zeta)|.$$

Moreover, the Cauchy's estimate implies that

$$|f'(\zeta)| \lesssim \frac{1}{\rho_\Omega(z)} \sup_{\partial D_r}|f(\zeta)|.$$

Therefore, we conclude that

$$
\begin{aligned}
|f(z)-f(w)| &\lesssim \frac{|z-w|}{\rho_\Omega(z)} \sup_{\zeta\in D_{2r}} |f(\zeta)| \le \frac{|z-w|}{\rho_\Omega(z)} \sup_{\zeta\in D_{2r}} \|K_\zeta\|_{A^2_\omega}\|f\|_{A^2_\omega} \\
&\le C\frac{|z-w|}{\rho_\Omega(z)}\|K_z\|_{A^2_\omega}\|f\|_{A^2_\omega}.
\end{aligned}
$$

The proof is finished. $\square$

Corollary 4.2.2. *There exists a constant* $0<\alpha<1$ *such that*

$$
|K_z(w)| \approx \|K_z\|_{A^2_\omega}\|K_w\|_{A^2_\omega},
$$

if $|z-w|\le \alpha\rho_\Omega(z)$.

Proof. Let $\alpha=\min\{\frac{1}{4},\frac{1}{2C}\}$. By taking $f=K_z$ in Lemma 4.2.1 we obtain that

$$
\begin{aligned}
|K(z,z)-K(z,w)| &\le \frac{C}{\rho_\Omega(z)}|z-w|\|K_z\|^2_{A^2_\omega} = C\frac{|z-w|}{\rho_\Omega(z)}K(z,z) \\
&\le \frac{1}{2}K(z,z)
\end{aligned}
$$

if $|z-w|\le\alpha\rho_\Omega(z)$. Hence, $\frac{1}{2}K(z,z)\le|K(z,w)|$. Besides, it follows from the Cauchy Schwarz inequality that

$$
|K(z,w)| = |\langle K_z,K_w\rangle| \le \|K_z\|_{A^2_\omega}\|K_w\|_{A^2_\omega} \approx \|K_z\|_{A^2_\omega} = |K(z,z)|.
$$

Therefore, we obtain that

$$
|K(z,w)| \approx |K(z,z)| \approx \|K_z\|_{A^2_\omega}\|K_w\|_{A^2_\omega}.
$$

$\square$

Theorem 4.2.3. *Let* μ *be a positive Borel measure on the unit disk,* ν *be a positive finite measure on* $\partial\mathbb{D}$*, and let* $\omega=\omega_{\mu,q,s,\nu}$ *on the unit disk. Then there exists a constant* $\alpha>0$ *such that*

$$
|K_z(w)|^2 \approx \frac{1}{(1-|z|^2)^2(1-|w|^2)^2\omega(z)\omega(w)},
$$

where $|z-w|\le\alpha(1-|z|^2)$*. Moreover, for all* $t\in(0,1)$*, there exists a positive constant* C_t *so that*

$$
|K_z(w)|^2 \le C_t\|K_z\|^2_{A^2_\omega}\|K_w\|^2_{A^2_\omega}\left(\frac{(1-|z|^2)(1-|w|^2)}{|w-z|^2}\right)^t, \quad z,w\in\mathbb{D}.
$$

Proof. The first statement of the theorem follows from Corollary 4.2.2 and Theorem 4.1.1, that is,

$$|K_z(w)|^2 \approx \|K_z\|_{A^2_\omega}\|K_w\|_{A^2_\omega} \approx \frac{1}{(1-|z|^2)^2(1-|w|^2)^2\omega(z)\omega(w)}, \quad z,w\in\mathbb{D},$$

if $|z-w|\le\alpha(1-|z|)$.

We define the distance function ρ on the unit disk as $\rho(z)=1-|z|^2$ for $z\in\mathbb{D}$. Let D_z be the Euclidean disk centered at z with radius $\frac{\rho(z)}{2}$. Assume that $t\in(0,1)$ and $z,w\in\mathbb{D}$. We will examine this part by dividing into two cases.

Case 1: If $D_z\cap D_w\neq\emptyset$, where $D_z=D(z,\frac{\rho(z)}{2})$ and $D_w=D(w,\frac{\rho(w)}{2})$, then there is at least an element $x\in D_z\cap D_w$ such that $|z-x|\le\frac{\rho(z)}{2}$ and $|w-x|\le\frac{\rho(w)}{2}$, which implies the following result

$$|z-w|\le|z-x|+|w-x|\le\frac{(1-|z|^2)}{2}+\frac{(1-|w|^2)}{2}\approx 1-|z|^2,$$

similarly, we have that

$$|z-w|\lesssim 1-|w|^2.$$

After the combination of the above results we obtain that there is some positive constant c_t so that

$$(|z-w|^2)^t\le c_t[(1-|z|^2)(1-|w|^2)]^t.$$

From the Cauchy-Schwarz inequality, we conclude that

$$|K(z,w)|^2\le\|K_z\|^2_{A^2_\omega}\|K_w\|^2_{A^2_\omega}\le\|K_z\|^2_{A^2_\omega}\|K_w\|^2_{A^2_\omega}c_t\left(\frac{(1-|z|^2)(1-|w|^2)}{|z-w|^2}\right)^t.$$

Case 2: If $D_z\cap D_w=\emptyset$. Let χ be a smooth real function such that $0\le\chi\le1$ and $\chi=1$ on $D(w,\frac{\rho(w)}{4})$, $supp(\chi)\subset D_w$ and $|\overline{\partial}\chi|^2\lesssim\frac{\chi}{\rho^2}$. Since $|K_z|^2$ is subharmonic we have that

$$\begin{aligned}
|K_z(w)|^2\omega(w) &\lesssim \frac{1}{\rho^2(w)}\int_{D(w,\frac{\rho(w)}{2})}|K_z(\zeta)|^2\omega(\zeta)dA(\zeta)\\
&\le \frac{1}{\rho^2(w)}\int_{\mathbb{D}}|K_z(\zeta)|^2\chi(\zeta)\omega(\zeta)dA(\zeta)\\
&\lesssim \frac{1}{\rho^2(w)}\|K_z\|^2_{L^2(\mathbb{D},\chi\omega dA)}=\frac{1}{\rho^2(w)}\sup_{f\in B}|\langle f,K_z\rangle_{L^2(\mathbb{D},\chi\omega dA)}|^2,
\end{aligned}$$

where $B=\{f\in H(\mathbb{D}):\|f\|_{L^2(\mathbb{D},\chi\omega dA)}=1\}$. Note that $f\chi\in L^2(\mathbb{D},\omega dA)$. Then $\langle f,K_z\rangle_{L^2(\mathbb{D},\chi\omega dA)}=P(f\chi)(z)$, where P is the orthogonal projection from $L^2(\mathbb{D},\omega dA)$ to A^2_ω. Hence, $u_f=f\chi-P(f\chi)$ is the solution of the equation $\overline{\partial}u_f=\overline{\partial}(f\chi)=f\overline{\partial}\chi$, with minimal norm in $L^2(\mathbb{D},\omega dA)$. We have that

$$|\langle f,K_z\rangle_{L^2(\mathbb{D},\chi\omega dA)}|=|P(f\chi)(z)|=|u_f(z)|,$$

for all $z\notin D(w,\frac{\rho(w)}{2})$. Thanks to the combination of above results we achieve that

$$|K_z(w)|^2\omega(w)\lesssim\frac{1}{\rho^2(w)}\sup_{f\in B}|\langle f,K_z\rangle_{L^2(\mathbb{D},\chi\omega dA)}|^2=\frac{1}{\rho^2(w)}\sup_{f\in B}|u_f(z)|^2,$$

for all $z\notin D(w,\frac{\rho(w)}{2})$. Moreover, because of the fact that $supp(\chi)\subset D_w$, we have $u_f\in H(D(z,\frac{\rho(z)}{2}))$. Hence, $|u_f|^2$ is subharmonic. Then we obtain that

$$|u_f(w)|^2\omega(w)\lesssim\frac{1}{\rho^2(w)}\int_{D(w,\frac{\rho(w)}{4})}|u_f(\zeta)|^2\omega(\zeta)dA(\zeta).$$

Let $t\in(0,1)$. We define the following functions

$$\phi(\zeta):=t\log\frac{|1-\overline{w}\zeta|^2}{1-|\zeta|^2}\quad\text{and}\quad\psi(\zeta)=-\log\omega(\zeta).$$

Then we have that

$$e^{\phi(\zeta)}=\left(\frac{|1-\overline{w}\zeta|^2}{1-|\zeta|^2}\right)^t\quad\text{and}\quad e^{\psi(\zeta)}=\frac{1}{\omega(\zeta)}.$$

Then we also obtain the following

$$\begin{aligned}|u_f(z)|^2e^{-\psi(z)+\phi(z)}&\lesssim\frac{1}{\rho^2(z)}\int_{D(z,\frac{\rho(z)}{4})}|u_f(\zeta)|^2e^{-\psi(\zeta)+\phi(\zeta)}dA(\zeta)\\&\le\frac{1}{\rho^2(z)}\int_{\mathbb{D}}|u_f(\zeta)|^2e^{-\psi(\zeta)+\phi(\zeta)}dA(\zeta),\end{aligned}$$

since $|1-\overline{w}\zeta|\approx|1-\overline{w}z|$ and $1-|\zeta^2|\approx1-|z|^2$ for $\zeta\in D(z,\frac{\rho(z)}{4})$. Furthermore,

$$\frac{\partial}{\partial\zeta}\phi(\zeta)=t\left(\frac{\overline{w}}{1-\overline{w}\zeta}-\frac{\overline{\zeta}}{1-|\zeta|^2}\right),$$

and

$$\Delta\phi(\zeta)=\frac{\partial^2}{\overline{\partial}\zeta\partial\zeta}\phi(\zeta)=\frac{t}{(1-|\zeta|^2)^2}.$$

Recall that $\overline{\partial}u_f=f\overline{\partial}\chi$, and $|\overline{\partial}\chi|^2\lesssim\frac{\chi}{\rho^2}$ where $supp(\chi)\subset D_w$. Besides, by a direct calculation we conclude that

$$\frac{\left|\frac{\partial\phi(\zeta)}{\partial\zeta}\right|^2}{\Delta\phi(\zeta)}=t\left|\frac{w-\zeta}{1-\overline{w}\zeta}\right|^2\le t.$$

Therefore, the condition (2.3) in (Berndtsson, 2017) is satisfied. Then, by Berndtsson's theorem (Berndtsson, 2017, page 46), we have

$$\begin{aligned}\int_{\mathbb{D}}|u_f(\zeta)|^2e^{-\psi(\zeta)+\phi(\zeta)}dA(\zeta) &\lesssim \frac{1}{(1-t)^2}\int_{\mathbb{D}}\frac{|\overline{\partial}u_f(\zeta)|^2}{\Delta\phi(\zeta)}e^{-\psi(\zeta)+\phi(\zeta)}dA(\zeta)\\ &\lesssim \frac{1}{(1-t)^2}\int_{\mathbb{D}}|f(\zeta)|^2|\overline{\partial}\chi(\zeta)|^2(1-|\zeta|^2)^2\frac{|1-\overline{w}\zeta|^{2t}}{(1-|\zeta|^2)^t}\omega(\zeta)dA(\zeta)\\ &\lesssim \frac{1}{(1-t)^2}\int_{D_w}|f(\zeta)|^2\frac{\chi(\zeta)}{\rho^2(\zeta)}(1-|\zeta|^2)^2\frac{|1-\overline{w}\zeta|^{2t}}{(1-|\zeta|^2)^t}\omega(\zeta)dA(\zeta)\\ &\lesssim \frac{1}{(1-t)^2}(1-|w|^2)^t\int_{D_w}|f(\zeta)|^2\omega(\zeta)dA(\zeta)\\ &\lesssim \frac{1}{(1-t)^2}(1-|w|^2)^t.\end{aligned}$$

Hence, we achieve that

$$\begin{aligned}|u_f(z)|^2e^{-\psi(\zeta)+\phi(z)} &\lesssim \frac{1}{\rho^2(z)}\int_{\mathbb{D}}|u_f(\zeta)|^2e^{-\psi(\zeta)+\phi(\zeta)}dA(\zeta)\\ &\lesssim \frac{1}{\rho^2(z)}\frac{(1-|w|^2)^t}{(1-t)^2},\end{aligned}$$

that is,

$$|u_f(z)|^2 \lesssim \frac{1}{\rho^2(z)}\frac{(1-|w|^2)^t}{(1-t)^2}\frac{1}{\omega(z)}\left(\frac{|1-\overline{w}\zeta|^2}{1-|\zeta|^2}\right)^{-t}.$$

As a consequence, we have

$$|K_z(w)|^2 \lesssim \frac{1}{(1-t)^2}\frac{1}{\rho^2(w)\omega(w)\rho^2(z)\omega(z)}\frac{(1-|z|^2)^t(1-|w|^2)^t}{|1-\overline{w}z|^{2t}}.$$

Then from Theorem 4.1.1 and the results obtained above, there exists a positive constant C so that

$$|K_z(w)|^2 \le \frac{C}{(1-t)^2}\left(\frac{\rho(z)\rho(w)}{|z-w|^2}\right)^t\|K_z\|^2_{A^2_\omega}\|K_w\|^2_{A^2_\omega},$$

where $|1-\overline{w}z| \approx |z-w|$ for the case that $D_z\cap D_w=\emptyset$. $\square$

4.3 Some Weight Classes

Definition 4.3.1 ($\mathcal{W}_0$ **Class**). (El-Fallah et al., 2016) Let Ω be a bounded domain in the complex plane and ω be a weight function defined on Ω. Let K_z be the reproducing kernel of the space $A^2_\omega(\Omega)$ (with possibly an equivalent norm). We say that the couple (ω, K_z) is a member of $\mathcal{W}_0$ if the following satements are satisfied:

$$\lim_{z \to \partial\Omega} \|K_z\| = \infty. \tag{4.1}$$

There exists $\alpha \in (0,1)$ and a positive constant C so that if $z, \zeta \in \Omega$ with $|z - \zeta| \leq \alpha\rho(z)$, then we have

$$\|K_z\|_{A^2_\omega(\Omega)} \|K_\zeta\|_{A^2_\omega(\Omega)} \leq C|K(\zeta, z)|, \tag{4.2}$$

and

$$\frac{1}{C} K(\zeta,\zeta)\omega(\zeta) \leq K(z,z)\omega(z) \leq CK(\zeta,\zeta)\omega(\zeta). \tag{4.3}$$

Furthermore, we say that (ω, K_z) is a member of $\mathcal{W}$ if the couple (ω, K_z) also satisfies the following property:

$$|K(\zeta, z)| = o(\|K_z\|) \tag{4.4}$$

as z goes to the boundary, for every $\zeta \in \Omega$.

Note that, Lemma 4.2.1 and Corollary 4.2.2 are also valid for the function K^2_z defined in the proof of Theorem 4.1.1. Let $\mathcal{H} := A^2_\omega(\Omega)$ and $\mathcal{H}_2 := (A^2_\omega(\Omega), \|.\|_2)$.

Proposition 4.3.2. *Let $\omega_{\mu,q,s,\nu}$ be a weight as in* (1.2). *Then the couple (ω, K_z) is in the class $\mathcal{W}_0$ and the couple (ω, K^2_z) is in the class $\mathcal{W}$, where K^2_z is the reproducing kernel for $\mathcal{H}_2$.*

Proof. Briefly, we set $\omega = \omega_{\mu,q,s,\nu}$. Recall the equivalence $\|K^\omega_z\|_\mathcal{H} \approx \|K^{2,\omega}_z\|_{\mathcal{H}_2}$. The statement (4.1) for the case when K^2_z is considered as the kernel function is satisfied if and only if $\omega(z)\rho^2(z)$ goes to zero as z goes to the boundary, thanks to Theorem 4.1.1. Since $U_{\mu,s}$ is bounded and $q + 2 > 0$, $\rho^{q+2}(z)U_{\mu,s}(z)$ goes to zero as z goes to the boundary.

Hence, we obtain that

$$\begin{aligned}\lim_{z\to\partial\Omega}\omega(z)\rho^2(z) &= \lim_{z\to\partial\Omega}\Big(\rho^{q+2}(z)U_{\mu,s}(z)+\rho^2(z)P_\nu(z)\Big)\\ &= \lim_{z\to\partial\Omega}\rho^2(z)P_\nu(z)\\ &= \lim_{z\to\partial\Omega}\int_{\partial\Omega}P_\Omega(z,\zeta)\rho^2(z)d\nu(\zeta).\end{aligned}$$

We refer to the following well-known estimate (Krantz, 2005),

$$P_\Omega(z,\zeta)\approx\frac{\rho(z)}{|z-\zeta|^2}.$$

Let $z\in\Omega$ and $\zeta\in\partial\Omega$ with $|z-\zeta|\geq\rho(z)$. Then we have

$$\rho^2(z)P_\Omega(z,\zeta)\approx\rho^2(z)\frac{\rho(z)}{|z-\zeta|^2}\leq\rho(z).$$

Therefore, we conclude that

$$\begin{aligned}\lim_{z\to\partial\Omega}\omega(z)\rho^2(z) &= \lim_{z\to\partial\Omega}\int_{\partial\Omega}P_\Omega(z,\zeta)\rho^2(z)d\nu(\zeta)\\ &\lesssim \lim_{z\to\partial\Omega}\rho(z)\nu(\partial\Omega)=0,\end{aligned}$$

since ν is a finite measure. This proves (4.1) for K_z^2.

We desire to demonstrate that there exists $\alpha\in(0,1)$ and $C>0$ such that

$$\|K_z^2\|_2\|K_\zeta^2\|_2\leq C|K^2(\zeta,z)|,$$

for $z,\zeta\in\Omega$ with $|z-\zeta|\leq\alpha\rho(z)$. The above inequality arises from the adaptation of Corollary 4.2.2 for the case when K_z^2 is considered as the kernel function. Hence, (4.2) holds for K_z^2.

Recall the fact that $K^2(\zeta,\zeta)=\langle K_\zeta^2,K_\zeta^2\rangle_2=\|K_\zeta^2\|_2^2$. By Theorem 4.1.1, (4.3) for the case when K_z^2 is considered as the kernel function is satisfied if and only if there exists a positive constant C so that

$$\frac{1}{C}\rho(\zeta)\leq\rho(z)\leq C\rho(\zeta).$$

Let $\Delta_\alpha(z)=\mathbb{D}(z,\alpha\rho(z))$ for some $\alpha\in(0,1)$. Thus, $\zeta\in\Delta_\alpha(z)$ if and only if $|z-\zeta|<\alpha\rho(z)$. For some $z_0,\zeta_0\in\partial\Omega$ we have that $\rho(z)=|z-z_0|$ and $\rho(\zeta)=|\zeta-\zeta_0|$. Thus,

$$\rho(z)=|z-z_0|\leq|z-\zeta_0|\leq|z-\zeta|+|\zeta-\zeta_0|=|z-\zeta|+\rho(\zeta),$$

which implies that

$$|z-\zeta| \lesssim \alpha\rho(z) \leq \alpha|z-\zeta| + \alpha\rho(\zeta),$$

hence,

$$|z-\zeta| \leq \frac{\alpha}{(1-\alpha)}\rho(\zeta).$$

Then we conclude the following inequalities;

$$\rho(z) \leq |z-\zeta| + \rho(\zeta) \leq \left(\frac{\alpha}{1-\alpha}+1\right)\rho(\zeta) = \frac{1}{1-\alpha}\rho(\zeta),$$

and

$$\rho(\zeta) \leq |z-\zeta| + \rho(z)\alpha\rho(z) + \rho(z) = (1+\alpha)\rho(z),$$

which imply that $\rho(\zeta)$ and $\rho(z)$ are comparable. Hence, (4.3) is satisfied, for the function K_z^2.

For any $\zeta \in \Omega$, the statement (4.4) indicates that

$$\lim_{z\to\partial\Omega} \frac{|K^2(\zeta,z)|}{\|K_z^2\|_2} = 0.$$

In order to prove the statement we need to explain some properties which will be utilized. Since $\|K_z^2\|_2 \approx \max_{0\leq j\leq N} \|K_{\varphi_j(z)}^{\eta_j}\|_{A^2_{\eta_j}(\mathbb{D})}$, we have

$$\frac{1}{\|K_z^2\|} \lesssim \frac{1}{\|K_{\varphi_{j_0}(z)}^{\eta_{j_0}}\|_{A^2_{\eta_{j_0}}(\mathbb{D})}} \approx \rho_{\mathbb{D}}^2(\varphi_{j_0}(z))\eta_{j_0}(\varphi_{j_0}(z)) \approx \rho_{\mathbb{D}}^2(\varphi_{j_0}(z))\omega(z),$$

for any $j_0 \in \{0,\cdots,N\}$, which follows from Theorem 4.1.1 and the fact that $\eta_j \circ \varphi_j \approx \omega$ for each j.

Notice that, because of the equality $\rho_\Omega = \min_{0\leq j\leq N} \rho_{\Omega_j}$, we obtain that $\frac{1}{\rho_{\Omega_j}(z)} \leq \frac{1}{\rho_\Omega(z)}$ for each j and any $z \in \Omega$. Moreover, we will use the fact $K_z^2 = \sum_{j=0}^N C_j K_z^2$. We set $K_{j,z}^2 := C_j K_z^2$ for each j. Now we obtain the following

$$
\begin{aligned}
\frac{|K^2(\zeta,z)|^2}{\|K_z^2\|_2^2} &\lesssim \frac{|K^2(\zeta,z)|^2}{\|K^{\eta_{j_0}}_{\varphi_{j_0}(z)}\|^2_{A^2_{\eta_{j_0}}(\mathbb{D})}} \approx |K^2(\zeta,z)|^2\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\eta_{j_0}(\varphi_{j_0}(z)) \\
&\approx |K^2(\zeta,z)|^2\omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z)) \leq \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}|K_j^2(\zeta,z)|^2 \\
&= \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}|\langle K^2_{j,\zeta},K^2_{j,z}\rangle_{A^2_\omega(\Omega_j)}|^2 \\
&\approx \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}|\langle C_{\varphi_j^{-1}}K^2_{j,\zeta},C_{\varphi_j^{-1}}K^2_{j,z}\rangle_{A^2_{\eta_j}(\mathbb{D})}|^2 \\
&= \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}|\langle K^{\eta_j}_{\varphi_j(\zeta)},K^{\eta_j}_{\varphi_j(z)}\rangle_{A^2_{\eta_j}(\mathbb{D})}|^2 \\
&= \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}|K^{\eta_j}(\varphi_j(\zeta),\varphi_j(z))|^2.
\end{aligned}
$$

It follows from Theorem 4.2.3 that for any $t \in (0,1)$, there exists a positive constant C_t so that

$$
\begin{aligned}
|K^{\eta_j}(\varphi_j(\zeta),\varphi_j(z))|^2 \leq C_t\|K^{\eta_j}_{\varphi_j(\zeta)}\|^2_{A^2_{\eta_j}(\mathbb{D})}\|K^{\eta_j}_{\varphi_j(z)}\|^2_{A^2_{\eta_j}(\mathbb{D})} \\
\times\left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z)|^2)}{|\varphi_j(\zeta)-\varphi_j(z)|^2}\right)^t.
\end{aligned}
$$

Thus, we obtain the following

$$
\begin{aligned}
\frac{|K^2(\zeta,z)|^2}{\|K_z^2\|_2^2} &\lesssim \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}C_t\|K^{\eta_j}_{\varphi_j(\zeta)}\|^2_{A^2_{\eta_j}(\mathbb{D})}\|K^{\eta_j}_{\varphi_j(z)}\|^2_{A^2_{\eta_j}(\mathbb{D})} \\
&\times\left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z)|^2)}{|\varphi_j(\zeta)-\varphi_j(z)|^2}\right)^t \\
&\lesssim \omega(z)\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))\sum_{j=0}^{N}C_t\|K^{\eta_j}_{\varphi_j(\zeta)}\|^2_{A^2_{\eta_j}(\mathbb{D})}\frac{1}{\rho^2_{\mathbb{D}}(\varphi_j(z))\eta_j(\varphi_j(z))} \\
&\times\left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z)|^2)}{|\varphi_j(\zeta)-\varphi_j(z)|^2}\right)^t \\
&\lesssim \frac{\rho^2_{\mathbb{D}}(\varphi_{j_0}(z))}{\rho^2_{\Omega}(z)}C_t\|K^2_\zeta\|_2\sum_{j=0}^{N}\left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z)|^2)}{|\varphi_j(\zeta)-\varphi_j(z)|^2}\right)^t,
\end{aligned}
$$

by using Theorem 4.1.1 and the fact that $\|K^{\eta_j}_{\varphi_j(\zeta)}\|_{A^2_{\eta_j}(\mathbb{D})} \leq \|K^2_\zeta\|_2$, for each j.

Furthermore, note that the equivalence relation $\rho_{\mathbb{D}}(\varphi_j(z)) \approx \rho_{\Omega_j}(z)$ is valid for every

$z \in \Omega$. Now let $z_k \in \Omega$ such that $z_k \to \partial\Omega$ as $k \to \infty$. Hence,

$$\frac{|K^2(\zeta, z_k)|^2}{\|K^2_{z_k}\|_2^2} \lesssim \frac{\rho^2_{\Omega_{j_0}}(z_k)}{\rho^2_{\Omega}(z_k)} C_t \|K^2_\zeta\|_2 \sum_{j=0}^{N} \left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z_k)|^2)}{|\varphi_j(\zeta)-\varphi_j(z_k)|^2} \right)^t .$$

Now for every k, choose $j_k = j_0$ so that $\rho_{\Omega_{j_k}}(z_k) = \min_j \rho_{\Omega_j}(z_k) = \rho_\Omega(z_k)$, which implies that

$$\frac{\rho^2_{\Omega_{j_0}}(z_k)}{\rho^2_{\Omega}(z_k)} = 1.$$

Consequently, we obtain the following

$$\frac{|K^2(\zeta, z_k)|^2}{\|K^2_{z_k}\|_2^2} \lesssim C_t \|K^2_\zeta\|_2 \sum_{j=0}^{N} \left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z_k)|^2)}{|\varphi_j(\zeta)-\varphi_j(z_k)|^2} \right)^t .$$

Thus, $\lim_{k\to\infty} |\varphi_j(z_k)| = 1$, since $z_k \to \partial\Omega$. Hence, the above finite sum goes to zero. We conclude that the statement (4.4) is satisfied, for the case when K^2_z is considered as the kernel function. $\square$

As a consequence of the proof of the previous proposition, we obtain following corollary.

Corollary 4.3.3. *For any $t \in (0,1)$ there exists a positive constant C_t such that*

$$|K^2(\zeta, z)|^2 \lesssim C_t \|K^2_\zeta\|_2 \|K^2_z\|_2^2 \sum_{j=0}^{N} \left(\frac{(1-|\varphi_j(\zeta)|^2)(1-|\varphi_j(z)|^2)}{|\varphi_j(\zeta)-\varphi_j(z)|^2} \right)^t ,$$

for any $\zeta, z \in \Omega$. Furthermore, there exists $\alpha > 0$ so that

$$|K_z(w)|^2 \approx \frac{1}{\rho^2_\Omega(z)\rho^2_\Omega(w)\omega(z)\omega(w)}, \qquad z, w \in \Omega.$$

where $|z-w| \leq \alpha\rho_\Omega(z)$.

Proof. The first statement follows from the proof of Proposition 4.3.2 and the second statement follows from Theorem 4.1.1 and Corollary 4.2.2. $\square$

Definition 4.4.1 (**Toeplitz Operator**)**.** Let P be a Bergman projection defined from L^2 to A^2. For any $\varphi \in L^\infty$, define an operator T_φ by $T_\varphi f = P(\varphi f)$ for any $f \in L^2$. The operator T_φ is said to be the Toeplitz operator with symbol φ.

Definition 4.4.2 (**Berezin Transform**)**.** Let T be a linear operator on A^2. We define a transform B of T by

$$B(T)(z) = \langle Tk_z, k_z \rangle_{A^2_\omega}, \tag{4.5}$$

where k_z is the normalized reproducing kernel of A^2_ω.

If T is a compact operator on a separable Hilbert space $\mathcal{H}$, then there exist orthonormal sets $\{a_n\}$ and $\{b_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_n \lambda_n \langle x, a_n \rangle b_n, \quad x \in \mathcal{H},$$

where λ_n is the nth singular value of T, (Zhu, 1990). The sequence $\{\lambda_n\}_{n \in \mathcal{N}}$ is called the singular value sequence of T.

Definition 4.4.3 (**Schatten Class**)**.** Let $\mathcal{H}$ be a separable Hilbert space. For $0 < p < \infty$, the space of all compact operators on $\mathcal{H}$, whose singular value sequence belong to l^p, is called the Schatten p-class of H.

Definition 4.4.4 (**Generalized Schatten Class**)**.** [(El-Fallah et al., 2018)] Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function such that $h(0) = 0$. Let T be a compact operator on a complex Hilbert space $\mathcal{H}$. The operator T is an element of $S_h(\mathcal{X})$ if there exists a positive constant c such that

$$\sum_{n=1}^{\infty} h(c(s_n(T))) < \infty,$$

where $s_n(T)$ denotes the sequence of singular values of T.

In order to comprehend the notion which is resulted from defining a norm to obtain an isometry as we did in Theorem 4.1.1, we consider the following facts. Firstly, recall that $\mathcal{H}_2$ is a Hilbert space with the norm $\|.\|_2$, which can be expressed as follows

$$\|f\|_2 = \left(\sum_{j=0}^{N} \|C_{\varphi_j^{-1}}(\mathcal{C}_j f)\|^2_{A^2_{\eta_j}(\mathbb{D})} \right)^{\frac{1}{2}},$$

for any $f \in \mathcal{H}_2$. Let $T := \boldsymbol{C}_\Phi \circ \mathcal{C} : \mathcal{H}_2 \to \boldsymbol{A}^2(H,\mathbb{D})$. Then for $z \in \Omega$

$$TK_z^2 = (K^{\eta_0}_{\varphi_0^{-1}(z)}, \cdots, K^{\eta_N}_{\varphi_N^{-1}(z)}).$$

Hence, the reproducing kernel for the product space $\boldsymbol{A}^2(H,\mathbb{D})$ is $\boldsymbol{K}_{\Phi(z)} = TK_z^2$. Let $A := T^*T$. Then for any $z \in \Omega$, we obtain that $K_z^2 = A^{-1}K_z$. Then

$$\langle f, g\rangle_2 = \langle f, Ag\rangle_{\mathcal{H}},$$

for any $f, g \in \mathcal{H}_2$.

Let η be a positive finite Borel measure on a domain Ω which is as in (2.1). The Toeplitz operator with the symbol η is given by

$$T_\eta(f(z)) = \int_\Omega f(\zeta)K(z,\zeta)\omega(\zeta)d\eta(\zeta),$$

for $f \in A^2_\omega$. We define the Topelitz operator with the symbol η associated with K^2 by

$$T^2_\eta(f(z)) = \int_\Omega f(\zeta)K^2(z,\zeta)\omega(\zeta)d\eta(\zeta),$$

for $f \in (A^2_\omega, \|.\|_2)$.

The Berezin transform of the Toeplitz operator T_η is defined by

$$B(T_\eta)(z) = \langle T_\eta k_z, k_z\rangle_{\mathcal{H}},$$

where $k_z = \frac{K(z,.)}{\|K_z(z)\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathcal{H}$, for $z \in \Omega$. Moreover, we define the Berezin transform of the Toeplitz operator T^2_η by

$$B_2(T^2_\eta)(z) = \langle T^2_\eta k^2_z, k^2_z\rangle_2,$$

where k^2_z is the normalized reproducing kernel of $\mathcal{H}_2$, for $z \in \Omega$. As a result of Theorem 4.1.1, it can be obtained that

$$\begin{aligned} K^2(z,z) = \langle K^2_z, K^2_z\rangle_2 = \|K^2_z\|^2_2 &\approx \frac{1}{(\rho_\Omega(z))^2\omega(z)} \\ &\approx \|K_z\|^2_{\mathcal{H}} = \langle K_z, K_z\rangle_{\mathcal{H}} = K(z,z), \end{aligned}$$

for any $z \in \Omega$. Hence,

$$\begin{aligned}
B_2T_\eta^2(z) = \langle T_\eta^2 k_z^2, k_z^2\rangle_2 &= \frac{1}{K^2(z,z)}\langle T_\eta^2 K_z^2, K_z^2\rangle_2 \\
&= \frac{1}{K^2(z,z)}\langle T_\eta^2 A^{-1}K_z, A^{-1}K_z\rangle_2 \\
&= \frac{1}{K^2(z,z)}\langle T_\eta^2 A^{-1}K_z, K_z\rangle_{\mathcal{H}} \\
&\approx \frac{1}{K(z,z)}\langle T_\eta^2 A^{-1}K_z, K_z\rangle_{\mathcal{H}} \\
&= \langle T_\eta^2 A^{-1}k_z, k_z\rangle_{\mathcal{H}} \\
&= BT_\eta^2 A^{-1}(z),
\end{aligned}$$

for $z \in \Omega$. Therefore, we obtain a relation between the following Berezin transforms

$$B_2T_\eta^2(z) \approx BT_\eta^2 A^{-1}(z), \quad z \in \Omega. \tag{4.6}$$

Moreover, Fubini's theorem yields that

$$\begin{aligned}
\langle T_\eta f, f\rangle_{A^2_\omega(\Omega)} &= \int_\Omega T_\eta f(\zeta)\overline{f(\zeta)}\omega(\zeta)dA(\zeta) \\
&= \int_\Omega \left(\int_\Omega f(z)\overline{K_\zeta(z)}\omega(z)d\eta(z)\right)\overline{f(\zeta)}\omega(\zeta)dA(\zeta) \\
&= \int_\Omega f(z)\omega(z)\left(\int_\Omega \overline{f(\zeta)}K_z(\zeta)\omega(\zeta)dA(\zeta)\right)d\eta(z) \\
&= \int_\Omega f(z)\overline{f(z)}\omega(z)d\eta(z) \\
&= \int_\Omega |f(z)|^2\omega(z)d\eta(z),
\end{aligned}$$

for any $f \in A^2_\omega(\Omega)$. Therefore, we obtain that

$$\langle T_\eta f, f\rangle_{A^2_\omega(\Omega)} = \int_\Omega |f(z)|^2\omega(z)d\eta(z), \tag{4.7}$$

for any $f \in A^2_\omega(\Omega)$.

In the light of the previous information, for $f \in \mathcal{H}$,

$$\begin{aligned}
\langle T_\eta^2 f, f\rangle_2 &= \langle T_\eta^2 f, Af\rangle_{\mathcal{H}} \\
&= \int_\Omega \overline{Af(b)} \left(\int_\Omega f(a)\overline{K_b^2(a)}\omega(a)d\eta(a) \right) \omega(b)dA(b) \\
&= \int_\Omega f(a)\omega(a) \left(\int_\Omega \overline{Af(b)} K_a^2(b)\omega(b)dA(b) \right) d\eta(a) \\
&= \int_\Omega f(a)\omega(a)\langle k_a^2, Af\rangle_{\mathcal{H}} d\eta(a) \\
&= \int_\Omega f(a)\overline{f(a)}\omega(a)d\eta(a) \\
&= \int_\Omega |f(a)|^2\omega(a)d\eta(a) \\
&= \langle T_\eta f, f\rangle_{\mathcal{H}}.
\end{aligned}$$

Furthermore, it follows from Theorem 4.1.1 and Corollary 4.2.2 that if $\omega \in \mathcal{W}_0$, then there exists an $\alpha \in (0,1)$ such that $\omega(z) \approx \omega(\zeta)$ where $\zeta \in \Delta_\alpha(z)$. Explicitly, there exists such an α so that

$$\begin{aligned}
\omega(\zeta) &\approx \frac{1}{\rho^2(\zeta)\|K_\zeta\|^2} \approx \frac{1}{\rho^2(z)\|K_\zeta\|^2} \\
&\approx \frac{\|K_z\|^2}{\rho^2(z)|K_\zeta(z)|^2} \approx \frac{1}{\rho^2(z)\|K_z\|^2} \\
&\approx \omega(z),
\end{aligned}$$

for $\zeta \in \Delta_\alpha(z)$.

In the proof of Theorem 4.4.5, the equivalence of (i) and (iii) is due to (El-Fallah et al., 2016), we include the proof here for completeness.

Theorem 4.4.5 (Theorem 6.2,(El-Fallah et al., 2016)). *Let $\omega \in \mathcal{W}$. Let η be a positive Borel measure on Ω and h be an increasing convex function from $\mathbb{R}^+$ to itself. Then the following statements are equivalent:*

(i) $T_\eta \in S_h(\mathcal{H})$.

(ii) $T_\eta^2 \in S_h(\mathcal{H}_2)$.

(iii) $\int_\Omega h(cBT_\eta(z)))\rho_\Omega^{-2}(z)dA(z) < \infty$.

Proof. Assume that $\{e_n\}_{n\geq 1}$ is an orthonormal basis for $\mathcal{H} := A^2_\omega(\Omega)$ and $\{e_{n,2}\}_{n\geq 1}$ is an orthonormal basis $\mathcal{H}_2 = (A^2_\omega(\Omega), \|.\|_2)$. Then,

$$T_\eta = \sum_{n\geq 1} \lambda_n\langle ., e_n\rangle e_n \quad \text{and} \quad T_\eta^2 = \sum_{n\geq 1} \lambda_{n,2}\langle ., e_{n,2}\rangle_2 e_{n,2},$$

where λ_n and $\lambda_{n,2}$ are the singular values of the Toeplitz operators T_η and T^2_η, respectively. Hence,

$$\lambda_n(T_\eta) = \langle T_\eta e_n, e_n\rangle \quad \text{and} \quad \lambda_{n,2}(T^2_\eta) = \langle T^2_\eta e_{n,2}, e_{n,2}\rangle_2.$$

By (4.7) we have that

$$\begin{aligned}\lambda_{n,2}(T^2_\eta) &= \langle T^2_\eta e_{n,2}, e_{n,2}\rangle_2 = \langle T_\eta e_{n,2}, e_{n,2}\rangle_{\mathcal{H}} = \langle T_\eta A^{-\frac{1}{2}} e_n, A^{-\frac{1}{2}} e_n\rangle_{\mathcal{H}} \\ &= \langle A^{-\frac{1}{2}} T_\eta A^{-\frac{1}{2}} e_n, e_n\rangle_{\mathcal{H}} = \lambda_n(A^{-\frac{1}{2}} T_\eta A^{-\frac{1}{2}}).\end{aligned}$$

Hence, $T^2_\eta \in S_h(\mathcal{H}_2)$ if and only if $A^{-\frac{1}{2}} T_\eta A^{-\frac{1}{2}} \in S_h(\mathcal{H})$. Furthermore, notice that $A^{-\frac{1}{2}} T_\eta A^{-\frac{1}{2}} \in S_h(\mathcal{H})$ if and only if $T_\eta \in S_h(\mathcal{H})$, which is obtained by the fact that $S_h(\mathcal{H})$ is a both sided ideal. Thus, $T^2_\eta \in S_h(\mathcal{H}_2)$ if and only if $T_\eta \in S_h(\mathcal{H})$. Therefore, (i) is satisfied if and only if (ii) is satisfied.

Suppose that $T_\eta \in S_h(\mathcal{H})$. So, for some positive constant c we have $\sum_{n\geq 1} h(c\lambda_n) < \infty$, where λ_n is the singular value sequence of T_η. Since T_η is a positive compact operator, it has a decomposition $T_\eta = \sum_{n\geq 1} \lambda_n \langle ., E_n\rangle_{\mathcal{H}} E_n$, where $\{E_n\}_{n\geq 1}$ is an orthonormal basis of $\mathcal{H}$. Then

$$\begin{aligned}\int_\Omega h(cBT_\eta(z)))\rho_\Omega^{-2}(z)dA(z) &\approx \int_\Omega h(cBT_\eta(z))\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z) \\ &= \int_\Omega h(c\langle T_\eta k_z, k_z\rangle_{\mathcal{H}})\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z) \\ &= \int_\Omega h\bigg(\sum_{n\geq 1} c\lambda_n |\langle k_z, E_n\rangle_{\mathcal{H}}|^2\bigg)\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z).\end{aligned}$$

Note that

$$\begin{aligned}\sum_{n\geq 1} |\langle k_z, E_n\rangle_{\mathcal{H}}|^2 &= \frac{1}{K(z,z)} \sum_{n\geq 1} |\langle K_z, E_n\rangle_{\mathcal{H}}|^2 \\ &= \frac{1}{K(z,z)} \sum_{n\geq 1} |E_n(z)|^2 = 1.\end{aligned}$$

Therefore, it follows from Jensen's inequality that

$$
\begin{aligned}
&\int_\Omega h\bigg(\sum_{n\geq 1} c\lambda_n |\langle k_z, E_n\rangle_{\mathcal{H}}|^2\bigg)\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z)\\
&\quad\leq \int_\Omega \bigg(\sum_{n\geq 1} h(c\lambda_n)|\langle k_z, E_n\rangle_{\mathcal{H}}|^2\bigg)\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z)\\
&\quad= \int_\Omega \sum_{n\geq 1} h(c\lambda_n)\frac{|E_n(z)|^2}{\|K_z\|^2_{\mathcal{H}}}\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z)\\
&\quad= \sum_{n\geq 1} h(c\lambda_n)\int_\Omega |E_n(z)|^2\omega(z) dA(z)\\
&\quad= \sum_{n\geq 1} h(c\lambda_n).
\end{aligned}
$$

Since $T_\eta \in S_h(\mathcal{H})$,

$$\int_\Omega h(cBT_\eta(z))\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z) < \infty.$$

Hence, (i) implies (iii).

We assume that

$$\int_\Omega h(cBT_\eta(z))\omega(z)\|K_z\|^2_{\mathcal{H}} dA(z) < \infty.$$

It follows from (4.7), the sub-mean property, and Theorem 4.1.1 that for some $\delta \in (0,1)$

$$
\begin{aligned}
\langle T_\eta E_n, E_n\rangle_{\mathcal{H}} &= \int_\Omega |E_n(z)|^2\omega(z) d\eta(z)\\
&\leq \int_\Omega \bigg(\frac{1}{|\Delta_\delta(z)|}\int_{\Delta_\delta(z)} |E_n(\zeta)|^2\omega(\zeta) dA(\zeta)\bigg) dA(z)\\
&= \int_\Omega \bigg(\frac{1}{\delta^2\rho^2_\Omega(z)}\int_{\Delta_\delta(z)} |E_n(\zeta)|^2\omega(\zeta) dA(\zeta)\bigg) dA(z)\\
&\approx \int_\Omega \bigg(\int_{\Delta_\delta(z)} \frac{1}{\rho^2_\Omega(\zeta)}|E_n(\zeta)|^2\omega(\zeta) dA(\zeta)\bigg) dA(z)\\
&\approx \int_\Omega \bigg(\int_{\Delta_\delta(z)} |E_n(\zeta)|^2\|K_\zeta\|^2_{\mathcal{H}}\omega(z)\omega(\zeta) dA(\zeta)\bigg) dA(z)\\
&\leq \int_\Omega \bigg(\int_{\Delta_\delta(z)} \frac{|K_z(\zeta)|^2}{\|K_z\|^2_{\mathcal{H}}}|E_n(\zeta)|^2\omega(\zeta) dA(\zeta)\bigg)\omega(z) d\eta(z)\\
&\leq \int_\Omega \bigg(\int_\Omega |k_\zeta(z)|^2\omega(z) d\eta(z)\bigg)|E_n(\zeta)|^2\omega(\zeta) dA(\zeta)\\
&= \int_\Omega BT_\eta(\zeta)|E_n(\zeta)|^2\omega(\zeta) dA(\zeta).
\end{aligned}
$$

Set $\nu(\zeta) := |E_n(\zeta)|^2\omega(\zeta) dA(\zeta)$. By applying Jensen's inequality to the function

$c\|\nu\|BT\eta$ with the probability measure $\frac{\nu}{\|\nu\|}$, we obtain that

$$h(C\langle T_\eta E_n, E_n\rangle_{\mathcal{H}}) \lesssim \int_\Omega h(cBT_\eta(\zeta))|E_n(\zeta)|^2\omega(\zeta)dA(\zeta).$$

Since $c > 0$ and h is increasing, we have

$$\begin{aligned}\sum_{n\geq 1} h(C\langle T_\eta E_n, E_n\rangle_{\mathcal{H}}) &\leq \int_\Omega h(cBT_\eta(\zeta)) \sum_{n\geq 1} |E_n(\zeta)|^2\omega(\zeta)dA(\zeta)\\ &= \int_\Omega h(cBT_\eta(\zeta)\|K_\zeta\|^2_{\mathcal{H}}\omega(\zeta)dA(\zeta)\end{aligned}$$

where the last quantity is finite because of the assumption. Therefore, $T_\eta \in S_h(\mathcal{H})$. Hence, (iii) implies (i). $\square$

For any $z \in \Omega$ and $\alpha \in (0,1)$, let $\Delta_\alpha(z) := \mathbb{D}(z, \alpha\rho_\Omega(z))$. Notice that $\rho_\Omega(w)$ and $\rho_\Omega(z)$ are comparable for any $w \in \Delta_\alpha(z)$, as it has been shown in the proof of Proposition 4.3.2. Which means that there is a positive constant C so that

$$\frac{\rho_\Omega(z)}{C} \leq \rho_\Omega(w) \leq C\rho_\Omega(z).$$

Let $C' = C + 1$ and $\delta \leq \frac{\alpha,}{C'}$. Then it follows from Proposition 3.1 in (El-Fallah et al., 2016) that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ which satisfies the following properties

$$\Omega \subset \cup_{n\geq 1}\Delta_\delta(z_n), \quad \Delta_{\frac{\delta}{2C}}(z_n) \cap \Delta_{\frac{\delta}{2C}}(z_m) = \emptyset,$$

for $n \neq m$ and $\Delta_\delta(z) \subset \Delta_{C'\delta}(z_n)$ for $z \in \Delta_\delta(z_n)$. Moreover, $(\Delta_{C'\delta}(z_n))_n$ is a covering of Ω of finite multiplicity.

Such a sequence is called a (ρ,δ)-lattices of Ω. Let $\mathcal{L}_\delta(\Omega, \rho_\Omega)$ be the set of (ρ,δ)-lattices of Ω.

The average function $\hat{\eta}_\delta$ is defined by

$$\hat{\eta}_\delta(z) = \frac{\eta(\Delta_\delta(z))}{|\Delta_\delta(z)|}$$

for some $\delta \in (0,1)$ and $z \in \Omega$.

The proof of Theorem 4.4.6 is due to (El-Fallah et al., 2016), we include the proof here for completeness.

Theorem 4.4.6. *(El-Fallah et al., 2016) Let $\omega \in \mathcal{W}$ and $\{z_n\}_{n\in\mathbb{N}} \in \mathcal{L}_\delta(\Omega, \rho_\Omega)$ with $\delta \leq \frac{\alpha}{C+1}$. Let η be a positive Borel measure on Ω and h be an increasing convex function from $\mathbb{R}^+$ to itself. Then the following statements are equivalent:*

$$\begin{aligned} &(1) \quad \textit{There exists } C_1 \textit{ so that} \quad \int_\Omega h(C_1 BT_\eta(z)) \frac{dA(z)}{\rho_\Omega^2(z)} < \infty, \\ &(2) \quad \textit{There exists } C_2 \textit{ so that} \quad \sum_{n\geq 1} h(C_2 \hat{\eta}_\delta(z_n)) < \infty. \end{aligned}$$

Proof. Assume that (1) is satisfied. Let $\delta \in (0, 1)$ and $z \in \Omega$. By (4.2) we obtain the following inequality

$$|k_z(\zeta)|^2 = \frac{|K(z,\zeta)|^2}{\|K_z\|^2} \gtrsim \|K_\zeta\|^2,$$

for any $\zeta \in \Delta_\delta(z)$. Hence, for any $w \in \Delta_\delta(z)$ we have that

$$\begin{aligned} BT_\eta(w) = \langle T_\eta k_w, k_w \rangle_{\mathcal{H}} &= \int_\Omega |k_w(\zeta)|^2 \omega(\zeta) d\eta(\eta) \\ &\geq \int_{\Delta_\delta(z)} |k_w(\zeta)|^2 \omega(\zeta) d\eta(\zeta) \\ &\gtrsim \int_{\Delta_\delta(z)} \|K_\zeta\|^2_{\mathcal{H}} \omega(\zeta) d\eta(\zeta) \\ &\approx \int_{\Delta_\delta(z)} \frac{1}{|\Delta_\delta(z)|} d\eta(\zeta) = \hat{\eta}_\delta(z), \end{aligned}$$

where Theorem 4.1.1 is used in the last approximation. Thus, for any $z \in \Delta_\delta(z_n)$ we obtain that $\hat{\eta}_\delta(z_n) \lesssim BT_\eta(z)$. Consequently,

$$\begin{aligned} \int_\Omega h(C_1 BT_\eta(z)) \frac{dA(z)}{\rho_\Omega^2(z)} &\approx \sum_n \int_{\Delta_\delta(z_n)} h(C_1 BT_\eta(z)) \frac{dA(z)}{\rho_\Omega^2(z)} \\ &\geq \sum_n \int_{\Delta_\delta(z_n)} h(C_1 c \hat{\eta}_\delta(z_n)) \frac{dA(z)}{\rho_\Omega^2(z)} \\ &\geq \sum_n h(C_1 c \hat{\eta}_\delta(z_n)), \end{aligned}$$

since $\rho_\Omega(z) \approx \rho_\Omega(z_n)$ for which $z \in \Delta_\delta(z_n)$. Therefore, (2) is satisfied.

Assume that (2) is satisfied. The sub-mean value property and Fubini's theorem provide that

$$
\begin{aligned}
BT_\eta(z) = \langle T_\eta k_z, k_z\rangle_2 &= \int_\Omega |k_z(\zeta)|^2 \omega(\zeta) d\eta(\zeta) \\
&\approx \sum_n \int_{\Delta_\delta(z_n)} |k_z(\zeta)|^2 \omega(\zeta) d\eta(\zeta) \\
&\lesssim \sum_n \int_{\Delta_\delta(z_n)} \left(\frac{1}{\rho_\Omega^2(\zeta)} \int_{\Delta_\delta(\zeta)} |k_z(w)|^2 \omega(w) dA(w) \right) d\eta(\zeta) \\
&\lesssim \sum_n \int_{\Delta_\delta(z_n)} \left(\frac{1}{\rho_\Omega^2(z_n)} \int_{\Delta_{B\delta}(z_n)} |k_z(w)|^2 \omega(w) dA(w) \right) d\eta(\zeta) \\
&= \sum_n \int_{\Delta_{B\delta}(z_n)} \left(\frac{1}{\rho_\Omega^2(z_n)} \int_{\Delta_\delta(z_n)} d\eta(\zeta) \right) |k_z(w)|^2 \omega(w) dA(w) \\
&\lesssim \sum_n \left(\int_{\Delta_{B\delta}(z_n)} |k_z(w)|^2 \omega(w) dA(w) \right) \hat{\eta}_\delta(z_n).
\end{aligned}
$$

where the constant B is obtained in [Proposition 3.1, (El-Fallah et al., 2016)]. Since

$$
\begin{aligned}
\sum_n \left(\int_{\Delta_{B\delta}(z_n)} |k_z(w)|^2 \omega(w) dA(w) \right) &\approx \int_\Omega |k_z(w)|^2 \omega(w) dA(w) \\
&= \|k_z\|_{\mathcal{H}} = 1,
\end{aligned}
$$

it follows from Jensen's inequality that for some positive constants C_1 and C_2

$$
h(C_1 BT_\eta(z)) \lesssim \sum_n \left(\int_{\Delta_{B\delta}(z_n)} |k_z(w)|^2 \omega(w) dA(w) \right) h(C_2 \hat{\eta}_\delta(z_n)).
$$

Furthermore, Theorem 4.1.1 implies the following result

$$
\begin{aligned}
\int_\Omega |k_z(w)|^2 \frac{dA(z)}{\rho_\Omega^2(z)} &= \int_\Omega \frac{|K(z,w)|^2}{\|K_z\|_{\mathcal{H}}^2} \frac{dA(z)}{\rho_\Omega^2(z)} \\
&\approx \int_\Omega |K(z,w)|^2 \omega(z) dA(z) = \|K_w\|_{\mathcal{H}}^2.
\end{aligned}
$$

Hence, by integrating both sides and using Fubini's theorem we obtain that

$$
\begin{aligned}
\int_\Omega h(C_1 BT_\eta(z)) \frac{dA(z)}{\rho_\Omega^2(z)} &\lesssim \int_\Omega \sum_n \left(\int_{\Delta_{B\delta}(z_n)} |k_z^2(w)|^2 \omega(w) dA(w) \right) \\
&\quad \times h(C_2 \hat{\eta}_\delta(z_n)) \frac{dA(z)}{\rho_\Omega^2(z)} \\
&\approx \sum_n \left(\int_{\Delta_{B\delta}(z_n)} \|K_w\|_{\mathcal{H}}^2 \omega(w) dA(w) \right) h(C_2 \hat{\eta}_\delta(z_n)) \\
&\approx \sum_n \left(\int_{\Delta_{B\delta}(z_n)} \frac{1}{\omega(w)\rho_\Omega^2(w)} \omega(w) dA(w) \right) h(C_2 \hat{\eta}_\delta(z_n)) \\
&= \sum_n h(C_2 \hat{\eta}_\delta(z_n).
\end{aligned}
$$

Consequently, (2) implies (1). $\square$

Corollary 4.4.7 ((El-Fallah et al., 2016)). *Let $\omega \in \mathcal{W}$ and $\{z_n\}_{n\in\mathbb{N}} \in \mathcal{L}_\delta(\Omega, \rho_\Omega)$. Let η be a positive Borel measure on Ω and h be an increasing convex function from $\mathbb{R}^+$ to itself. Then $T_\eta \in S_h(A^2_\omega)$ if and only if there exists a positive constant c such that*

$$\sum_n h(c\hat{\eta}_\delta(z_n)) < \infty,$$

where $\hat{\eta}_\delta(z) = \frac{\eta(\Delta_\delta(z_n))}{|\Delta_\delta(z_n)|}$ is the average function with $\Delta_\delta(z_n) := \mathbb{D}(z_n, \delta\rho_\Omega(z_n))$ for some $\delta > 0$.

Let φ be an analytic self map of Ω. The Nevanlinna counting function of φ corresponding to ω is given by

$$\mathcal{N}_{\varphi,\omega}(\zeta) = \begin{cases} \sum_{\varphi(z)=\zeta} \omega(z) & \text{if } \zeta \in \varphi(\Omega), \\ 0 & \text{if } \zeta \notin \varphi(\Omega). \end{cases}$$

Let $d\omega_\varphi$ denote the following measure

$$\omega_\varphi(E) = \int_E \frac{\mathcal{N}_{\varphi,\omega}(z)}{\omega(z)} dA(z),$$

for every Borel set $E \subset \Omega$. Therefore, the Toeplitz operator with symbol ω_φ is given by

$$T_{\omega_\varphi}(f(z)) = \int_\Omega f(\zeta)K(z,\zeta)\omega(\zeta)d\omega_\varphi(\zeta),$$

for $f \in A^2_\omega(\Omega)$

We denote the weighted Dirichlet space by $D^2_\omega(\Omega)$, which is the space of all analytic functions defined on Ω whose derivatives are square integrable with the weight function ω.

Lemma 4.4.8. *Let μ and η be positive Borel measures on Ω and ν be a positive finite Borel measure on $\partial\Omega$. Let $\omega = \omega_{\mu,q,s\nu}$. Assume that φ is a self map of Ω and $h : \mathcal{R}^+ \to \mathcal{R}^+$ is an increasing function so that $h(0) = 0$. Then*

$$C_\varphi \in S_h(D^2_\omega(\Omega)) \quad \iff \quad T_{\omega_\varphi} \in S_{h\circ\sqrt{\cdot}}(A^2_\omega(\Omega)).$$

Proof. Let L be a differential operator from $D^2_\omega(\Omega)$ to $A^2_\omega(\Omega)$ so that $L(f) = f'$ for every $f \in D^2_\omega(\Omega)$. It suffice to prove that $C_\varphi \in S_h(D^2_\omega(\Omega))$ if and only if $D_\varphi = LC_\varphi L^* \in S_h(A^2_\omega(\Omega))$. Moreover, the change of variable formula yields that

$$\begin{aligned} D^*_\varphi D_\varphi(f(z)) = \langle D_\varphi(f), D_\varphi(K_z)\rangle_{A^2_\omega(\Omega)} &= \int_\Omega f(\varphi(a))\overline{K_z(\varphi(a))}|\varphi'(a)|^2\omega(a)dA(a) \\ &= \int_\Omega f(a)\overline{K_z(a)}\omega(a)\frac{\mathcal{N}_{\varphi,\omega}(a)}{\omega(a)}dA(a) = T_{\omega_\varphi}(f(z)), \end{aligned}$$

which completes the proof. $\square$

Corollary 4.4.9. *Let μ and η be positive Borel measures on Ω and ν be a positive finite Borel measure on $\partial\Omega$. Assume that φ is a self map of Ω. Then $C_\varphi \in S_p(D^2_\omega(\Omega))$ if and only if the Berezin transfrom of $T_{\mathcal{N}_{\omega,\varphi}}$ is in $L^{p/2}(\Omega, \rho_\Omega^{-2} dA)$.*

Proof. It follows from Lemma 4.4.8 and Theorem 4.4.5 that $C_\varphi \in S_p(D^2_\omega(\Omega))$ if and only if $T_{\omega_\varphi} \in S_{p/2}(A^2_\omega(\Omega))$ if and only if

$$\int_\Omega (cBT_{\omega_\varphi}(z))^{p/2} \rho_\Omega^{-2}(z) dA(z) < \infty.$$

Moreover, by setting $c_1 = c^{p/2}$ we have

$$\begin{aligned}
\int_\Omega c_1 [BT_{\omega_\varphi}(z)]^{p/2} \rho_\Omega^{-2}(z) dA(z) &= c_1 \int_\Omega \left[\int_\Omega |k_z(\zeta)|^2 \omega(\zeta) d\omega_\varphi(\zeta) \right]^{p/2} \rho_\Omega^{-2}(z) dA(z) \\
&= c_1 \int_\Omega \left[\int_\Omega |k_z(\zeta)|^2 \mathcal{N}_{\omega,\varphi}(\zeta) dA(\zeta) \right]^{p/2} \rho_\Omega^{-2}(z) dA(z) \\
&= c_1 \int_\Omega (B(T_{\mathcal{N}_{\omega,\varphi}})(z))^{p/2} \rho_\Omega^{-2}(z) dA(z),
\end{aligned}$$

which completes the proof. $\square$

Proposition 4.4.10. $C_\varphi \in S_2(D^2_\omega(\Omega))$ *if and only if*

$$\int_\Omega \frac{\omega(\zeta)}{\rho_\Omega(\varphi(\zeta))^2 \omega(\varphi(\zeta))} |\varphi'(\zeta)|^2 dA(\zeta) < \infty.$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $A^2_\omega(\Omega)$. Due to Lemma 4.4.8 we have that $C_\varphi \in S_2(D^2_\omega(\Omega))$ if and only if $T_{\omega_\varphi} \in S_1(A^2_\omega(\Omega))$ if and only if

$$\|T_{\omega_\varphi}\|_{S_1(A^2_\omega(\Omega))} = \sum_n \langle T_{\omega_\varphi} e_n, e_n \rangle_{A^2_\omega(\Omega)} < \infty.$$

Furthermore,

$$\begin{aligned}
\sum_n \langle T_{\omega_\varphi} e_n, e_n\rangle_{A^2_\omega(\Omega)} &= \sum_n \int_\Omega |e_n(z)|^2 \omega(z) d\omega_\varphi(z) \\
&= \int_\Omega \Big(\sum_n |e_n(z)|^2\Big)\omega(z) d\omega_\varphi(z) \\
&= \int_\Omega \|K_z\|^2_{A^2_\omega(\Omega)} \omega(z) d\omega_\varphi(z) \\
&\approx \int_\Omega \frac{1}{\rho_\Omega(z)^2 \omega(z)} \omega(z) d\omega_\varphi(z) \\
&= \int_\Omega \frac{1}{\rho_\Omega(z)^2} d\omega_\varphi(z) \\
&= \int_\Omega \frac{1}{\rho_\Omega(z)^2} \frac{\mathcal{N}_{\varphi,\omega(z)}}{\omega(z)} dA(z) \\
&= \int_\Omega \frac{1}{\rho_\Omega(\varphi(\zeta))^2 \omega(\varphi(\zeta))} \omega(\zeta)|\varphi'(\zeta)|^2 dA(\zeta),
\end{aligned}$$

which completes the proof. □

5. Weighted Composition Operators

In this section, as an application of the second inequality in Theorem 2.2.1, boundedness and compactness of (weighted) composition on weighted Bergman spaces defined on finitely connected domains are investigated.

Furthermore, the famous Littlewood's Subordination Principle states that the composition operator C_φ defined from $A^p(\mathbb{D})$ to itself is bounded, for general p, (Littlewood, 1925). Li and Huang extended Littlewood's result to the case when domain is bounded and of $\mathcal{C}^2$ boundary, for $p = 2$, (Li & Huang, 2020). We have extended their result to general $p \geq 1$.

Let $\Omega, \tilde{\Omega} \subset \mathbb{C}$ be bounded domains. Assume that φ is a holomorphic function from $\tilde{\Omega}$ to Ω, and ψ is an holomorphic function from $\tilde{\Omega}$ to $\mathbb{C}$. The weighted composition operator $C_{\varphi,\psi}$ is defined from $H(\Omega)$ to $H(\tilde{\Omega})$ by $C_{\varphi,\psi}(f) = (f \circ \varphi)\psi$ for all $f \in H(\Omega)$. Note that if $\psi \equiv 1$, then we have that $C_{\varphi,\psi} = C_\varphi$, which is the classical composition operator.

Let $\tilde{\omega} \in L^1(\tilde{\Omega})$ be a continuous, strictly positive function on $\tilde{\Omega}$ and let $0 < \tilde{p} < \infty$. Recall that for any $f \in A^p_\omega(\Omega)$ there is a unique decomposition of the form $f = F_0 + F_1 + \ldots + F_N$, by Theorem 2.2.1. Therefore, we have

$$C_{\varphi,\psi}(f) = \sum_{j=0}^{N} C_{\varphi,\psi,j} F_j, \tag{5.1}$$

where $C_{\varphi,\psi,j}$ is defined from $A^p_\omega(\Omega_j)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$.

Assume that P_k is a projection map from $\tilde{X} := A^{\tilde{p}}_{\tilde{\omega}}(\mathbb{D}) \times A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_1) \times \ldots \times A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_N)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_k)$, and $\tilde{T}$ is an isomorphism defined from $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ to $\tilde{X}$ such that $\tilde{T}(f) = (f_0, f_1, \ldots, f_n)$ for each $f \in A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, where $f = f_0 + f_1 + \ldots + f_n$ is the unique decomposition of f.

5.1 Boundedness of Composition Operator

Proposition 5.1.1. *Let $\Omega = \mathbb{D} \setminus \cup_{j=1}^{N} \overline{\mathbb{D}}(z_j, r_j)$ be as in* (2.1) *and $\tilde{\Omega} \subset \mathbb{C}$ be a bounded domain with smooth boundary. Let φ be a holomorphic function from $\tilde{\Omega}$ to Ω, and ψ be a holomorphic function on $\tilde{\Omega}$. Then the following statements are equivalent:*

(i) *The weighted composition operator $C_{\varphi,\psi}$ is bounded from $A^p_\omega(\Omega)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$.*

(ii) *$C_{\varphi,\psi,j}$ is bounded from $A^p_\omega(\Omega_j)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ for each $j \in \{0, \ldots, N\}$, where $\Omega_j = \mathbb{C} \setminus \overline{\mathbb{D}}(z_j, r_j)$.*

(iii) *The operator $C_{\tilde{\varphi}_k^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}$ from $A^p_{\eta_j}(\mathbb{D})$ to $A^{\tilde{p}}_{\tilde{\eta}_k}(\mathbb{D})$ is bounded.*

Proof. First suppose that $C_{\varphi,\psi} : A^p_\omega(\Omega) \to A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ is bounded. Let R_j be the restriction map from $H(\Omega_j)$ onto $H(\Omega)$. Then for any $g \in A^p_\omega(\Omega_j)$, we have

$$C_{\varphi,\psi,j}(g) = (g \circ \varphi)\psi = ((R_j(g)) \circ \varphi)\,\psi = C_{\varphi,\psi} R_j(g),$$

and also $\|R_j(g)\|_{A^p_\omega(\Omega)} = \|g\|_{A^p_\omega(\Omega_j)}$. The latter equation implies that the restriction map R_j is bounded, and $\|R_j\| = 1$. Hence,

$$\|C_{\varphi,\psi,j}\| = \|C_{\varphi,\psi} R_j\| \leq \|C_{\varphi,\psi}\|,$$

that is $C_{\varphi,\psi,j}$ is bounded for each j. For the converse, suppose that $C_{\varphi,\psi,j}$ is bounded for each j. For any $f \in A^p_\omega(\Omega)$,

$$\|C_{\varphi,\psi}(f)\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})} \leq C \left(\sum_{j=0}^{N} \|C_{\varphi,\psi,j}\| \right) \|f\|_{A^p_\omega(\Omega)},$$

which follows from Theorem 2.2.1. Hence, $\|C_{\varphi,\psi}(f)\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})} \leq C' \|f\|_{A^p_\omega(\Omega)}$, which implies that $C_{\varphi,\psi}$ is bounded.

It is obtained from the definition that (i) implies (iii). In order to prove that (iii) implies (i), we suppose that $C_{\tilde{\varphi}_k^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}$ is bounded for each $0 \leq j \leq N$ and $0 \leq k \leq \tilde{N}$. Recall that φ_j is defined from Ω_j to $\mathbb{D}$ so that $\varphi_j(z) = \frac{r_j}{z - z_j}$ and $\tilde{\varphi}_k$ is defined from $\tilde{\Omega}_k$ to $\mathbb{D}$ so that $\tilde{\varphi}_k(z) = \frac{\tilde{r}_k}{z - \tilde{z}_k}$. Hence, for every $g \in A^p_\omega(\Omega_j)$ we have that

$$\|g\|^p_{A^p_\omega(\Omega_j)} = \int_{\Omega_j} |g(z)|^p \omega(z) dA(z) \approx \int_{\mathbb{D}} |(g \circ \varphi_j^{-1})(z)|^p (\omega \circ \varphi_j^{-1})(z) dA(z) = \|h\|_{A^p_{\eta_j}(\mathbb{D})},$$

where $h = g \circ \varphi_j^{-1}$ and $\eta_j = \omega \circ \varphi_j^{-1}$. Hence, $P_k \tilde{T} C_{\varphi,\psi,j}$ is also bounded. Then for any $f \in A^p_\omega(\Omega)$,

$$\|P_k(\tilde{T}[(f \circ \varphi)\psi])\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_k)} \leq c_k \|f\|_{A^p_\omega(\Omega_j)},$$

which implies that

$$\max_{0 \leq k \leq \tilde{N}} \|P_k(\tilde{T}[(f \circ \varphi)\psi])\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_k)} \leq c \|f\|_{A^p_\omega(\Omega_j)},$$

where $c = \max_{0 \leq k \leq \tilde{N}} c_k$. Furthermore,

$$\max_{0 \leq k \leq \tilde{N}} \|P_k(\tilde{T}[(f \circ \varphi)\psi])\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega}_k)} \approx \|\tilde{T}[(f \circ \varphi)\psi]\|_{\tilde{X}} \approx \|C_{\varphi,\psi}(f)\|_{A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})},$$

Consequently, $C_{\varphi,\psi}$ is bounded. □

Li and Huang proved that the composition operator C_φ from $A^2(\Omega)$ to $A^2(\tilde{\Omega})$ is bounded (Li & Huang, 2020). In the next theorem, we extend their result to the $A^p(\Omega)$ case for $p \geq 1$.

Theorem 5.1.2. *Let Ω and $\tilde{\Omega}$ be any finitely connected domains. Assume that φ is a holomorphic map from $\tilde{\Omega}$ to Ω, and $1 \leq \tilde{p} \leq p \leq \infty$. Then the composition operator C_φ, which is defined from $A^p(\Omega)$ to $A^{\tilde{p}}(\tilde{\Omega})$, is bounded.*

Proof. Let φ_j and η_j be the functions defined in (2.5) and (2.6), respectively. Since in this case $\omega \equiv 1$, η_j is bounded on $\mathbb{D}$ and it is away from zero on the boundary $\partial\mathbb{D}$ of the unit disk. Hence, $A^p_{\eta_j}(\mathbb{D}) = A^p(\mathbb{D})$ and $C_{\varphi_j^{-1}} : A^p(\Omega_j) \to A^p(\mathbb{D})$ is an isometric isomorphism. Note that for any $f \in A^p(\Omega_j)$,

$$C_\varphi(f) = C_{\varphi_j \circ \varphi} C_{\varphi_j^{-1}}(f).$$

Hence, in order to investigate the composition operator between $A^p(\Omega_j)$ and $A^{\tilde{p}}(\tilde{\Omega})$, it suffices to consider the composition operator between $A^p(\mathbb{D})$ and $A^{\tilde{p}}(\tilde{\Omega})$. Let ψ be a holomorphic map from $\tilde{\Omega}$ to $\mathbb{D}$. Then $\tilde{\Omega}$ is contained in a union of finitely many simply connected subdomains $\{\tilde{S}_j\}_{1 \leq j \leq K}$ of $\tilde{\Omega}$ with C^∞ boundary, and a compact subset L. So for any $f \in A^p(\mathbb{D})$ we have the following

$$\begin{aligned}
\|C_\psi(f)\|^{\tilde{p}}_{A^{\tilde{p}}(\tilde{\Omega})} &= \int_{\tilde{\Omega}} |f(\psi(z))|^{\tilde{p}} dA(z) \\
&\leq \int_L |f(\psi(z))|^{\tilde{p}} dA(z) + \sum_{j=1}^{K} \int_{\tilde{S}_j} |f \circ \psi(z)|^p dA(z) \\
&= \int_L |f(\phi(z))|^{\tilde{p}} dA(z) + \sum_{j=1}^{K} \int_{\mathbb{D}} |f \circ \psi \circ \psi_j^{-1}(z)|^p / |\psi_j'(z)|^2 dA(z),
\end{aligned}$$

where ψ_j is a biholomorphism from $\tilde{S}_j$ to $\mathbb{D}$. Since $|f(\psi(z))|^{\tilde{p}}$ is bounded on the compact set L, the first integral is finite. We use the fact that both $|\psi_j'(z)|$ and $1/|\psi_j'(z)|$ are bounded on $\mathbb{D}$. Littlewood's subordination principle implies that

$$\int_{\tilde{S}_j} |f \circ \psi(z)|^p dA(z) < \infty$$

for each j. Therefore, $C_\psi(f) \in A^{\tilde{p}}(\tilde{\Omega})$ for $f \in A^p(\Omega)$. By the closed graph theorem, we obtain that $C_\psi : A^p(\mathbb{D}) \to A^{\tilde{p}}(\tilde{\Omega})$ is bounded. Then by Proposition 5.1.1, $C_\varphi : A^p(\Omega) \to A^{\tilde{p}}(\tilde{\Omega})$ is bounded and the proof is finished. □

5.2 Compactness of Composition Operators

Proposition 5.2.1. *Let $\Omega, \tilde{\Omega} \subset \mathbb{C}$ be bounded and finitely connected domains with smooth boundaries. Assume that φ is an analytic function from $\tilde{\Omega}$ to Ω, and ψ be an analytic function on $\tilde{\Omega}$. Then we have the following statements;*

(i) *The weighted composition operator $C_{\varphi,\psi}$, which is defined from $A^p_\omega(\Omega)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, is compact if and only if $C_{\varphi,\psi,j}$, which is defined from $A^p_\omega(\Omega_j)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, is compact for each $j \in \{0, \ldots, N\}$.*

(ii) *The weighted composition operator $C_{\varphi,\psi}$, which is defined from $A^p_\omega(\Omega)$ to $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, is compact if and only if the operator $C_{\tilde{\varphi}_k^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}$ from $A^p_{\eta_j}(\mathbb{D})$ to $A^{\tilde{p}}_{\tilde{\eta}_k}(\mathbb{D})$ is compact.*

Proof. Proof of (i): Firstly, we assume that $C_{\varphi,\psi}$ is compact. For any bounded sequence $\{g_n\}_{n\in\mathbb{N}} \in A^p_\omega(\Omega_j)$, consider the sequence $\{C_{\varphi,\psi,j}(g_n)\}_{n\in\mathbb{N}}$. Note that $C_{\varphi,\psi,j}(g_n) = C_{\varphi,\psi} R_j(g_n)$, where R_j is a restriction map from $A^p_\omega(\Omega_j)$ to $A^p_\omega(\Omega)$. Moreover, the sequence $\{R_j(g_n)\}_{n\in\mathbb{N}}$ is also bounded in $A^p_\omega(\Omega)$, since the restriction map is bounded. Hence, by using the compactness of $C_{\varphi,\psi}$ we conclude that $\{C_{\varphi,\psi}(R_j(g_n))\}_{n\in\mathbb{N}}$ has a convergent subsequence in $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, say $\{C_{\varphi,\psi}(R_j(g_{n,k}))\}_{k\in\mathbb{N}}$. Which implies that $C_{\varphi,\psi,j}$ is compact, because of the fact that $\{C_{\varphi,\psi}(R_j(g_{n,k}))\}_{k\in\mathbb{N}} = \{C_{\varphi,\psi,j}(g_{n,k})\}_{k\in\mathbb{N}}$.

Suppose that $C_{\varphi,\psi,j}$ is compact for each j. For any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$, consider the sequence $\{C_{\varphi,\psi}(f_n)\}_{n\in\mathbb{N}}$. Recall that f_n has a unique decomposition such that $f_n = f_{n,0} + \ldots + f_{n,N}$, for each n. Moreover, for some positive constant C we know that

$$\|f_{n,j}\|_{A^p_\omega(\Omega)} \leq C\|f_n\|_{A^p_\omega(\Omega)},$$

by Theorem 2.2.1, which implies the sequences $\{f_{n,j}\}_{n\in\mathbb{N}}$ are bounded. Because of the assumption that $C_{\varphi,\psi,0}$ is compact and $\{f_{n,0}\}_{n\in\mathbb{N}}$ is bounded, there exists an infinite subset $S_0 \subset \mathbb{N}$ such that $f_n \circ \varphi$ converges to some f_0 in $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ as n goes to infinity, for $n \in S_0$. In addition, since $\{f_{n,1}\}_{n\in\mathbb{N}}$ is bounded and $C_{\varphi,\psi,1}$ is compact, there exists an infinite subset $S_1 \subset S_0$ such that $f_{n,1} \circ \varphi$ converges to some f_1 in $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ as n goes to infinity, for $n \in S_1$. By an induction argument, we construct subsets $S_N \subset S_{N-1} \subset \ldots \subset S_0 \subset \mathbb{N}$ so that $f_{n,j} \circ \varphi$ converges f_j as n goes to infinity, for $n \in S_j$. Therefore, we have the following

$$\lim_{n\to\infty} f_n \circ \varphi = \lim_{n\to\infty} \Big(\sum_{j=0}^{N} f_{n,j}\Big) \circ \varphi = \Big(\sum_{j=0}^{N} f_j\Big) \circ \varphi,$$

in $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ for $n \in S_N$. Furthermore, we conclude that

$$\lim_{n\to\infty} C_{\varphi,\psi}(f_n) = \lim_{n\to\infty} (f_n \circ \varphi)\psi = \Big[\Big(\sum_{j=0}^{N} f_j\Big) \circ \varphi\Big]\psi,$$

in $A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$, for $n \in S_N$. Hence, $C_{\varphi,\psi}$ is compact.

Proof of (ii): Suppose that $C_{\tilde{\varphi_k}^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}$ is compact for each $0 \le j \le N$ and $0 \le k \le \tilde{N}$. Since C_{φ_j} and $C_{\tilde{\varphi_k}^{-1}}$ are isomorphsims, $P_k \tilde{T} C_{\varphi,\psi,j}$ is also compact. Moreover, the compactness of $P_k \tilde{T} C_{\varphi,\psi,j}$ implies the compactness of $C_{\varphi,\psi,j}$, because P_k and $\tilde{T}$ are isomorphisms. Hence, from (i) we conclude the desired statement.

We assume that $C_{\varphi,\psi}$ is compact. As a notation, say $K_{k,j} := C_{\tilde{\varphi_k}^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}$ For any bounded sequence $\{f_n\}_{n\in\mathbb{N}} \subset A^p_{\eta_i}(\mathbb{D})$, we want to show that $\{K_{k,j}(f_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence. Firstly, note that $\{C_{\varphi_j}(f_n)\}_{n\in\mathbb{N}}$ is a bounded sequence in $A^p_\omega(\Omega_i)$. Because of the assumption and the part (i), we conclude that $C_{\varphi,\psi,j} : A^p_\omega(\Omega_j) \to A^{\tilde{p}}_{\tilde{\omega}}(\tilde{\Omega})$ is compact. Therefore, $\{C_{\varphi,\psi,j}(C_{\varphi_j}(f_n))\}_{n\in\mathbb{N}}$ has a convergent subsequence in $A^{\tilde{\omega}}_{\tilde{\omega}}(\tilde{\Omega})$, say $\{C_{\varphi,\psi,j}(C_{\varphi_j}(f_{n,l}))\}_{l\in\mathbb{N}}$. Hence, we obtain the following

$$\{C_{\tilde{\varphi_k}^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}(f_{n,l})\}_{l\in\mathbb{N}} \subset \{C_{\tilde{\varphi_k}^{-1}} P_k \tilde{T} C_{\varphi,\psi,j} C_{\varphi_j}(f_n)\}_{n\in\mathbb{N}},$$

which implies that $K_{k,j}$ is compact.

□

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