QUALITATIVE ANALYSIS FOR THE DISPERSION GENERALIZED CAMASSA-HOLM EQUATION

by
NESİBE AYHAN

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ABSTRACT

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NESİBE AYHAN

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In this thesis, we establish local well-posedness of the Cauchy problem for a dispersion generalized Camassa-Holm equation by using Kato's semigroup approach for quasi-linear evolution equations. We show that for initial data in the Sobolev space $H^s(\mathbb{R})$ with $s > \frac{7}{2} + p$, the Cauchy problem is locally well-posed, where p is a positive real number determined by the order of the differential operator L corresponding to the dispersive effect added to the Camassa-Holm equation. We first explain Kato's semigroup approach on the Camassa-Holm equation and then give the proofs for the dispersion generalized Camassa-Holm equation. Finally, we compare the results of both equations and propose open problems related to the dispersion generalized Camassa-Holm equation.

ÖZET

GENELLEŞTİRİLMİŞ DİSPERSİYON CAMASSA-HOLM DENKLEMİNİN MATEMATİKSEL ANALİZİ

NESİBE AYHAN

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Anahtar Kelimeler: Genelleştirilmiş Camassa-Holm denklemi, Kato'nun yarıgrup yaklaşımı, Yerel varlık

Bu tezde, genelleştirilmiş dispersiyon Camassa-Holm denklemi için yazılmış Cauchy probleminin yerel olarak iyi konulmuş olduğu Kato'nun yarıgrup yaklaşımı kullanılarak gösterilmistir. Başlangıç verileri $s > \frac{7}{2} + p$ için $H^s(\mathbb{R})$ Sobolev uzayında alınarak ilgili Cauchy probleminin yerel olarak iyi konulmuş olduğu ispatlanmıştır. Burada, bahsedilen p sayısı pozitif bir reel sayı olmakla birlikte Camassa-Holm denklemine dispersif etki olarak eklenen L türev operatörünün mertebesidir. İlk olarak Kato'nun yarıgrup yaklaşımı Camassa-Holm denklemi üzerinde açıklanmış, daha sonra genelleştirilmiş dispersiyon Camassa-Holm denklemi için gerekli kanıtlar verilmiştir. Son olarak, her iki denklemin sonuçları karşılaştırılmış ve genelleştirilmiş dispersiyon Camassa-Holm denklemiyle ilgili açık problemler önerilmiştir.

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to my family

&

to whom supported my study

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1. Introduction

The nonlinear dispersive wave equation

$$u_t + u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

was introduced by Camassa and Holm [Camassa & Holm (1993)] to model the unidirectional propagation of shallow water waves over a flat bottom. $u(x,t)$ represents the fluid velocity at time t and in the spatial direction x [Wang, Li & Qiao (2018)]. The equation is known as the Camassa–Holm (CH) equation, whose various generalizations have appeared in the literature in recent years. These generalizations are mostly based on the mathematical structure of the equation; making comments about the physical meaning and derivation of these equations is considered as another study subject.

To understand the concept of dispersion, let's assume $u(x,t)$ is a function that satisfies the one dimensional single linear partial differential equation with constant coefficients for $-\infty < x < \infty$ and $t > 0$,

$$P\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u(x,t) = 0,$$

where P is a polynomial, and t and x are independent time and space variables. We seek a plane wave solution of the equation in the form

$$u(x,t) = Ae^{i(kx-\omega t)},$$

where A is the amplitude, k is the wave number, and ω is the frequency. When we substitute the solution into the partial differential equation, we get an algebraic equation of the form

$$P(ik, -i\omega) = 0.$$

Assume that this equation can be solved explicitly given by $\omega = \omega(k)$. Then, the phase velocity of the waves can be defined by the relation

$$c = \frac{\omega}{k},$$

where it defines the velocity at which a surface of constant phase moves. Thus, according to this relation, the phase velocity c depends on the wave number k. In other words, different waves propagate with different phase velocities, and such waves are called *dispersive*. On the other hand, waves are called *non-dispersive* if the phase velocity c does not depend on the wave number k [Debnath & Debnath (2005)]. In this case, the Camassa-Holm equation (1.1) has a phase velocity $c = \frac{1}{1+k^2}$, which makes it a dispersive wave equation.

When the terms in the CH equation are examined in detail, the uu_x term denotes nonlinear steeping, the u_{xxt} term denotes the linear dispersion effect, and the $2u_x u_{xx} + uu_{xxx}$ terms denote the nonlinear dispersion effect. When the momentum density $m = (1-\partial_x^2)u$ is defined, the CH equation becomes

$$m_t + m_x u + 2mu_x = 0. \tag{1.2}$$

There are generalizations of the CH equation for different momentum density forms in the literature. Important examples of these can be given as follows:

(*i*) Hunter and Saxton in [Hunter & Saxton (1991)] considered the Camassa-Holm equation with $m = -\partial_x^2 u$.

(*ii*) Holm et al. in [Holm, Náraigh & Tronci (2009)] considered the following Camassa-Holm equation

$$m_t + 2mu_x + m_x u = 0, \qquad m = (1-\alpha^2\partial_x^2)u,$$

where α is a constant, and the fluid velocity u is a function of time t and position x on the real line.

(*iii*) Khesin et al. in [Khesin, Lenells & Misiołek (2008)] introduced a μ-version of Camassa-Holm equation as follows

$$m_t + 2mu_x + m_x u = 0, \qquad m = (\mu - \partial_x^2)u,$$

where $u(x,t)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u dx$ denotes its mean. This equation describes evolution of rotators in liquid crystals with external magnetic field and self-interaction. It is also studied in [Fu, Liu & Qu (2012), Deng & Chen (2020), Yamane (2020), Wang, Luo, Fu & Qu (2022)].

For the periodic case, Wang [Wang et al. (2018)] considered $m = \mu(u) - u_{xx} + u_{xxxx}$. Moreover, Wang studied the modified μ-version of Camassa-Holm equation in [Wang

& Fu (2016)] as follows

$$m_t + (2\mu(u)u - u_x^2)m)_x = 0.$$

(iv) Wang considered with $m = \mu(u) + u_{xxxx}$ in [Wang, Li & Qiao (2017)].

(v) Ding et al. considered with $m = u - u_{xx} + u_{xxxx}$ in [Ding & Zhang (2017)].

(vi) There are also higher order forms of the CH equation, where $m = (1-\partial_x^2)^k u$ for positive integer k [Constantin & Kolev (2003)], which describe exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane. Studies for $k = 2$ can be found in [McLachlan & Zhang (2009), Mu, Zhou & Zeng (2011), Reyes, Zhu & Qiao (2021)].

(vii) For $r \geq 1$, Camassa-Holm system with two components, where $m = (1-\partial_x^2)^r u$ is studied in [Chen & Zhou (2017)].

The common feature of the examples in the literature is that they put different effects on linear and non-linear dispersive terms to observe the results. In this thesis, our main aim is to study

$$m_t + u_x + bm_x u + amu_x = 0, \quad a, b > 0, \tag{1.3}$$

with the following form of momentum density:

$$m = (1 - L\partial_x^2)u. \tag{1.4}$$

Note that for L as an identity operator, $a = 2$ and $b = 1$, it becomes Camassa-Holm equation.

Here, L is a general differential operator in spatial variable x whose order is a positive real number p. With this momentum density, the dispersive effect is increased in (1.3), since new dispersion relation will be $c = \frac{1}{1+k^{2+p}}$. Note that choosing L as the identity operator corresponds to the CH equation given by (1.1), choosing as $(1-\alpha^2\partial_x^2)$ corresponds to the example (ii), choosing $(1-\partial_x^2+\partial_x^4)$ to the example (v), and choosing $(1-\partial_x^2)^2$ to the example (vi).

In the literature, the dispersion generalized CH equation with (1.4) has not been studied yet. Although it seems that there are similar forms, it should be emphasized that in these studies—for example, Zhu et al. 2017 [Zhu & Wang (2017)], Darwich et al. 2021 [Darwich & Israwi (2021)]—only the linear dispersion effect has changed. In the equation that is considered in this thesis, unlike those, both the linear and the non-linear dispersion effects have changed. In terms of qualitative analysis, this makes a big difference since L operator is given in closed form and applies also on nonlinear terms in the equation.

This thesis presents the dispersion generalized Camassa-Holm equation given by (1.3)-(1.4) and proves local well-posedness of the solutions for the corresponding Cauchy problem. It can be considered as a non-local and nonlinear dispersive partial differential equation and mathematical generalization of Camassa-Holm equation rather than a physical generalization. We prove that the Cauchy problem is locally well-posed for the initial data in H^s, $s > \frac{7}{2} + p$. The positive real number p is the order of the general differential operator L which appears in closed form.

Our proof relies on Kato's semigroup approach for quasilinear equations. For this reason, in Section 2, we present a short review of the definitions and theorems we need. Some special function spaces, Sobolev spaces, Sobolev embedding theorems, commutator estimates, and Kato's semigroup approach are briefly given. We also give some useful inequalities at the end of Section 2.

Before discussing the local well-posedness of the solutions of Cauchy problem corresponding to the dispersion generalized Camassa-Holm equation, we present the Cauchy problem for the Camassa-Holm equation in Section 3. In the literature, it is studied in [Constantin & Escher (1998)] for the periodic case, and local results are improved in [Rodríguez-Blanco (2001)]. We want to show how Kato's semigroup approach applies to a problem in which the form of the equation and the assumed initial data are appropriate to prove local well-posedness results. It is also necessary to observe the results for CH equation in order to enlighten the effects of the changes in the proposed form.

In Section 4, we establish local well-posedness for the dispersion generalized Camassa-Holm equation. We make use of commutator operators to obtain a suitable form of the equation to use Kato's semigroup approach. Main reason is that L is in closed form among the nonlinear terms in the equation as well and usual differentiation rules do not apply. We follow the approach and prove the assumptions to show that the Cauchy problem is well-posed.

As a conclusion, in Section 5, we compare the results we found in Sections 3 and 4 for Camassa-Holm and dispersion generalized Camassa-Holm equations. The changes in the dispersive effect, the differences in their non-local forms, the initial data classes chosen for the corresponding Cauchy problems are discussed.

We end the thesis with Section 6 in which we provide open problems to be discussed. According to mathematical analysis questions for the Camassa-Holm equation appearing in the literature, we can say that further qualitative analysis is also possible for the dispersion generalized Camassa-Holm equation, such as global well-posedness and finite time blow-up.

2. Preliminaries

2.1 Some Special Function Spaces

In this subsection, we give some special function spaces that will be needed throughout this thesis. We will use them while introducing some definitions and inequalities later. We refer to Evans (2010) for the definitions given below.

Let Ω be an open subset of $\mathbb{R}^n$, and $f(x) : \Omega \to \mathbb{R}$ be a measurable function.

Definition 2.1.1 [Evans (2010)]
The space $C(\Omega)$ denotes all continuous functions on Ω, and $C_b(\Omega)$ denotes all continuous, bounded functions on Ω with the sup-norm

$$||f||_\infty = sup_{x\in\Omega}|f(x)|.$$

Definition 2.1.2
The space $C_b^k(\Omega)$ denotes all bounded, continuous functions on Ω whose derivatives up to order k also belong to $C_b(\Omega)$ with the norm

$$||f||_{C_b^k(\Omega)} = \sum_{n=0}^{k} \Big|\Big|\frac{d^n f}{dx^n}(x)\Big|\Big|_\infty.$$

Definition 2.1.3
Let $1 \le p < \infty$. The space $L^p(\Omega)$ consists of all measurable functions with the norm

$$||f||_{L^p} = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}.$$

Similarly, $L^p_{loc}(\Omega)$ denotes the space of all locally integrable functions instead of integrable ones.

Definition 2.1.4
A function is essentially bounded on Ω if it is measurable and there exists a real number $M > 0$ such that $|u(x)| \leq M$ for almost all $x \in \Omega$. The infumum of all such numbers M is called *essential supremum* of f and is denoted by

$$\text{ess } sup_{x\in\Omega}|f(x)|.$$

Definition 2.1.5
The space $L^{\infty}(\Omega)$ consists of all measurable functions that are essentially bounded on Ω with the norm

$$||f||_{L^{\infty}} = \text{ess } sup_{x\in\Omega}|f(x)|.$$

We see that the given spaces are Banach Spaces with the chosen norms [Evans (2010)].

Let X be a Banach space with norm $||.||$.

Definition 2.1.6
Let $1 \leq p < \infty$, $-\infty \leq a < b \leq \infty$. Then, the space $L^p((a,b);X)$ consists of all measurable functions with the norm

$$||f||_{L^p((a,b);X)} = \Big(\int_a^b (||f(t)||_X)^p dt\Big)^{1/p}.$$

Definition 2.1.7
Let $p = \infty$, $-\infty \leq a < b \leq \infty$. Then, the space $L^{\infty}((a,b);X)$ consists of all measurable functions that are essentially bounded with the norm

$$||f||_{L^{\infty}((a,b);X)} = \text{ess } sup_{t\in(a,b)}||f(t)||_X.$$

Definition 2.1.8
Let $-\infty < a < b < \infty$. The space $C([a,b];X)$ consists of continuous Banach-valued functions with the norm

$$||f||_{C([a,b];X)} = max_{t\in[a,b]}||f(t)||_X.$$

Definition 2.1.9
Let $-\infty < a < b < \infty$. The space $C^k([a,b];X)$ consists of all bounded, continuous functions whose derivatives up to order k also belong to X with the norm

$$||f||_{C^k([a,b];X)} = \sum_{n=0}^{k} max_{t\in[a,b]}\Big|\Big|\frac{d^n f}{dx^n}(t)\Big|\Big|_X.$$

2.2 Sobolev Spaces

In this subsection, we introduce Sobolev spaces and related concepts, Evans (2010).

Definition 2.2.1 [Evans (2010)]
Let u be a function in $L^1([a,b])$. A function v in $L^1([a,b])$ is a *weak derivative* of u if

$$\int_a^b u\phi'(x)dx = -\int_a^b v\phi(x)\ dx,$$

for all infinitely integrable functions ϕ with $\phi(a) = \phi(b) = 0$.

Generalizing to n dimension, if u and v are in $L^1_{loc}(\Omega)$ for some Ω be an open set in $\mathbb{R}^n$, and $\alpha = (\alpha_1, ..., \alpha_n)$ be a multiindex, it is said that v is the α^{th}*-weak derivative* of u if

$$\int_\Omega uD^\alpha\phi\ dx = (-1)^{|\alpha|}\int_\Omega v\phi\ dx,$$

where

$$D^\alpha\phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\phi,$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, that is, for all infinitely differentiable functions ϕ with compact support in Ω.

Remark 2.2.2
The weak derivative is unique up to a set of measure zero. So, if u has a weak derivative, then it is generally denoted by $D^\alpha u$.

Definition 2.2.3
Suppose that Ω is an open set in $\mathbb{R}^n$, $1 \le p \le \infty$ and k is a nonnegative integer. The *Sobolev space* $W^{k,p}(\Omega)$ consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α,

$$D^\alpha u \in L^p(\Omega), \quad \text{for } 0 \le |\alpha| \le k.$$

Similarly, $W^{k,p}_{loc}(\Omega)$ denotes the space of locally integrable functions instead of integrable ones.

Then, we have the following sequence of inclusions of Sobolev spaces:

$$L^p(\Omega) = W^{0,p}(\Omega) \supset W^{1,p}(\Omega) \supset W^{2,p}(\Omega) \supset ...$$

A sequence $(u_k) \to u$ in $W^{k,p}(\Omega)$ if and only if $D^\alpha u_k \to D^\alpha u$ in $L^p(\Omega)$ as $k \to \infty$ for all multiindices α such that $|\alpha| \leq k$.

Theorem 2.2.4
For $1 \leq p \leq \infty$ and $k = 1, 2, ..,$ the Sobolev space $W^{k,p}(U)$ is a Banach space with the norm

$$||u||_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha| \leq k} |D^\alpha u|^p \Big)^{1/p}, \qquad \text{for } 1 \leq p < \infty,$$

and

$$||u||_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \text{ess } sup_\Omega |D^\alpha u|, \quad \text{for } p = \infty.$$

Remark 2.2.5
For $p = 2$, we have $W^{k,2}(\Omega)$, which is a Hilbert space. Therefore, we use the notation

$$H^k(U) = W^{k,2}(U), \text{ where } k = 0, 1, ...$$

Also, note that $H^0(U) = L^2(U)$.

Definition 2.2.6
The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. In other words, $f \in H^{-1}(\Omega)$ means that f is a bounded linear functional of $H_0^1(\Omega)$, where

$$f : H_0^1(\Omega) \to \mathbb{R}.$$

Definition 2.2.7
If $f \in H^{-1}(\Omega)$, we define the norm as

$$||f||_H^{-1}(\Omega) = sup\Big\{ < f, u > | u \in H_0^1(\Omega), ||u||_{H_0^1(\Omega)} \leq 1 \Big\}.$$

2.3 Sobolev Embedding Theorems

In this subsection, we give Sobolev embedding theorems, Evans (2010).

Let X and Y be Banach spaces with norms $||.||_x$ and $||.||_y$, respectively.

Definition 2.3.1 [Evans (2010)]
A Banach space X is *continuously embedded* into Y if

• $X \subset Y$,
• $\forall x \in X$, there exists $C > 0$ such that

$$||x||_Y \leq C||x||_X.$$

The notation $X \hookrightarrow Y$ is used for continuous embedding.

Definition 2.3.2-
A Banach space X is compactly embedded into Y if
• $X \subset Y$,
• Each bounded set $M \subset X$ is compact in Y.
The notation $X \subset\subset Y$ is used for compact embedding.

Theorem 2.3.3
If $\Omega \in \mathbb{R}^n$ is a bounded domain, then $H^1(\Omega) \subset\subset L^2(\Omega)$.

Lemma 2.3.4
Suppose that $\Omega \in \mathbb{R}^n$ is an open domain, $1 \leq p \leq q \leq r$, and $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$, where $\theta \in (0,1)$. If $u \in L^p(\Omega) \cap L^r(\Omega)$, then $u \in L^q(\Omega)$, and the following inequality holds

$$||u||_q \leq ||u||_p^{\theta} ||u||_r^{1-\theta}.$$

Theorem 2.3.5
Let $\Omega \in \mathbb{R}^n$ be a bounded domain and $p > 1$. Then, $W_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$.

Theorem 2.3.6 *(Rellich-Kondrashov)*
Suppose that $\Omega \in \mathbb{R}^n$ is a bounded domain and $1 \leq q \leq p^* = \frac{np}{n-p}$, $1 \leq p < n$. Then, $W_0^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$.

Theorem 2.3.7
If $\Omega \in \mathbb{R}^n$ is a bounded domain, then $H^1(\Omega) = W^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega)$.

Theorem 2.3.8
Assume that $\Omega \in \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, $1 < p \leq q < \infty$, $m, k \in \mathbb{Z}_+$, $m \geq k$, and the following inequality holds

$$m - \frac{n}{p} \geq k - \frac{n}{q} \tag{2.1}$$

Then,

$$W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega).$$

If the inequality 2.1 is strict, then

$$W^{m,p}(\Omega) \subset\subset W^{k,q}(\Omega).$$

Theorem 2.3.9
Assume that $\Omega \in \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, $1 \leq p \leq \infty$, $m \in \mathbb{Z}_+$, and $k \leq m - \frac{n}{p}$. Then,

$$W^{m,p}(\Omega) \hookrightarrow C^k(\Omega).$$

If the above inequality is strict, then

$$W^{m,p}(\Omega) \subset\subset C^k(\Omega).$$

2.4 Commutator Estimates

In this subsection, we give some commutator estimates used in various steps while proving the assumptions of Kato's semigroup approach.

Definition 2.4.1 [Jørgensen & Moore (2012)]
The *commutator* of two operators A and B is defined as

$$[A,B] = AB - BA.$$

Proposition 2.4.2 [Jørgensen & Moore (2012)]
The following properties are useful while working with commutators:
- $[A,B] = -[B,A]$
- $[A,B+C] = [A,B] + [A,C]$
- $c[A,B] = [cA,B] = [A,cB]$

Proposition 2.4.3 [Kato (1983))]
Let $f \in H^s, s > \frac{3}{2}$, and M_f be the operator of multiplication by f. Then, $\Lambda^{-r}[\Lambda^{r+t+1}, M_f]\Lambda^{-t} \in \mathcal{L}(L^2(\mathbb{R}))$ if $|r|,|t| \leq s-1$. Moreover,

$$||\Lambda^{-r}[\Lambda^{r+t+1}, M_f]\Lambda^{-t}||_{\mathcal{L}(L^2(\mathbb{R}))} \leq c||f||_s,$$

where c is a constant depending only on r,t, and $\mathcal{L}(L^2(\mathbb{R}))$ denotes the bounded linear operators in $L^2(\mathbb{R})$.

Proposition 2.4.4 [Lannes (2013))]
Let $n > 0$, $s \geq 0$, and $3/2 < s+n \leq \sigma$. Then, for all $f \in H^{\sigma}$ and $g \in H^{s+n-1}$, one has

$$||[\Lambda^n,\ f]g||_s \leq c||f||_{\sigma}||g||_{s+n-1},$$

where c is a constant which is independent of f and g.

2.5 Kato's semigroup approach

In this subsection, first we give some definitions and related lemmas about C_0-semigroup approach. Then, Kato's semigroup approach and the related theorem is given. We refer to Pazy (2012) for the definitions given below.

Let X be a Banach space with norm $||.||$.

Definition 2.5.1 [Pazy (2012)]
A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X to X is a *semigroup* of bounded linear operator on X if

(i) $T(0) = I$, I is the identity operator on X.
(ii) $T(s+t) = T(s)T(t)$, for all $s,t \geq 0$, the semigroup property.

Definition 2.5.2
The linear operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$\mathcal{D}(A) = \{x \in X : \lim_{t\to 0} \tfrac{1}{t}(T(t)x - x)\text{exists}\}$$

and

$$Ax = \lim_{t\to 0} \tfrac{1}{t}(T(t)x - x),\ \ x \in \mathcal{D}(A).$$

is called *the infinitesimal generator* of the given semigroup, where $\mathcal{D}(A)$ is the domain of A.

Definition 2.5.3
A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a *strongly*

continuous semigroup of bounded linear operators if

$$\lim_{t\to 0}||T(t)x - x|| = 0, \text{ for all } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X is simply denoted by C_0-*semigroup*.

Lemma 2.5.4
Let T be a C_0-semigroup on X. There exists a constant $M \geq 1$ and $\mu \in \mathbb{R}$ such that

$$||T(t)|| \leq Me^{\mu t}, \text{ for all } t \geq 0.$$

Lemma 2.5.5
Let T be a C_0-semigroup on X with generator A. Then,

(i) $\mathcal{D}(A)$ is dense in X.

(ii) A is a closed operator.

Definition 2.5.6 [Kato (2013)]
An operator $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A)$ is its domain, is called *accretive* if

$$||u - v|| \leq ||u - v + \lambda(A(u) - A(v))||,$$

for all $u, v \in \mathcal{D}(A)$ and $\lambda \in \mathbb{R}^+$. If $-A$ is accretive, then it is called *dissipative*.

We say that A is *quasi-accretive* if $A + \alpha$ is accretive for some scalar α.

If, in addition, $Range(I + \lambda A) = X$ for every $\lambda > 0$, then A is called *m-accretive*, where I is the identity operator. Similarly, we say that, A is *quasi-m-accretive* if $A + \alpha$ is m-accretive for some scalar α.

Remark 2.5.7
The preceding definitions are for the operator $A(u)$, which appears in the form $u_t = A(u)u + f(u)$. However, we take into account the quasi-linear equations of the form $u_t + A(u)u = f(u)$. Therefore, the definitions should be considered accordingly.

Now, consider the Cauchy problem for the quasi-linear equation of evolution:

$$\frac{du}{dt} + A(u)u = f(u), \quad t \geq 0, \quad u(0) = u_0. \tag{2.2}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Assume that

$(A1)$ Let $B_r(0)$ be an open ball centered in the origin in Y with radius $r > 0$. The linear operator $A(u) : X \to X$ generates a C_0-semigroup $T(t)$ on X which satisfies

$$||T(t)||_{\mathcal{L}(X)} \le e^{\beta t}, \text{ for all } t \in [0,\infty),$$

for a constant $\beta > 0$.

$(A2)$ For any $u \in Y$, $A(u)$ is a bounded linear operator from Y to X, i.e, $A(u) \in \mathcal{L}(Y,X)$ with

$$||(A(u)-A(v))w||_X \le \lambda_1 ||u-v||_X ||w||_Y, \quad u,v,w \in X.$$

$(A3)$ For any $u \in Y$, there exists a bounded linear operator $B(u) \in \mathcal{L}(X)$ satisfying $B(u) = QA(u)Q^{-1} - A(u)$, where $B : Y \to \mathcal{L}(X)$ is uniformly bounded sets in Y. Moreover,

$$||(B(u)-B(v))w||_X \le \lambda_2 ||u-v||_Y ||w||_X, \quad u,v \in Y,\ w \in X.$$

$(A4)$ For all $t \in [0,\infty)$, f is uniformly bounded on bounded sets in Y. Moreover, the map $f : Y \to Y$ is locally X-Lipschitz continuous in the sense that there exists a constant $\lambda_3 > 0$ such that

$$||f(u)-f(v)||_X \le \lambda_3 ||u-v||_X, \text{ for all } u,v \in B_r(0) \subseteq Y,$$

and locally Y-Lipschitz continuous in the sense that there exists a constant $\lambda_4 > 0$ such that

$$||f(u)-f(v)||_Y \le \lambda_4 ||u-v||_Y, \text{ for all } u,v \in B_r(0) \subseteq Y.$$

Here, for all $r > 0$,$\lambda_1, \lambda_2, \lambda_3$, and λ_4 depend only on r.

Theorem 2.5.8 (*Kato's Semigroup Approach*) [Kato (1975)]
Assume that $(A1)-(A4)$ hold. For given $u_0 \in Y$, there is a $T > 0$, depending on u_0, and a unique solution u to (2.2) such that

$$u = u(u_0,.) \in C([0,T),Y) \cap C^1([0,T),X).$$

Moreover, the solution depends continuously on the initial data, i.e, the map $u_0 \to u(u_0,.)$ is continuous from Y to $C([0,T),Y) \cap C^1([0,T),X)$.

2.6 Some Useful Inequalities

In this subsection, we give some useful inequalities used in various steps while proving the assumptions of Kato's semigroup approach.

Lemma 2.6.1 *(Minkowski's Inequality)* [Evans (2010)]
Let $u.v \in L^p(\Omega)$ and $1 \leq p \leq \infty$. Then

$$||u+v||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)} + ||v||_{L^p(\Omega)}.$$

Lemma 2.6.2 *(Sobolev Inequalities)* [Evans (2010)]
• Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < n$. Then, for each function $u \in W_0^{1,p}(\Omega)$,

$$||u||_{L^{p^*}(\Omega)} \leq C(n,p)||Du||_{L^p(\Omega)},$$

where $p^* = \frac{np}{n-p}$, and $C(n,p) = \frac{p(n-1)}{n-p}$.

• For any $0 < s_1 \leq s_2 < \infty$, the followings hold:

$$H^{s_2}(\mathbb{R}) \subset H^{s_1}(\mathbb{R}),$$

and

$$||f||_{H^{s_1}(\mathbb{R})} \leq ||f||_{H^{s_2}(\mathbb{R})}.$$

• Let $f,g \in H^s(\mathbb{R})$ and $s \geq 0$. Then,

$$||fg||_{H^s(\mathbb{R})} \leq C\Big(||f||_{H^s(\mathbb{R})}||g||_{L^\infty(\mathbb{R})} + ||f||_{L^\infty(\mathbb{R})}||g||_{H^s(\mathbb{R})}\Big).$$

• Let $n = 1, p = 2, s > \frac{1}{2}$. Then,

$$H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}),$$

and there exists a constant d depends on s such that

$$||f||_{L^\infty} \leq d||f||_{H^s}.$$

3. Cauchy Problem for the Camassa-Holm Equation

In this subsection, we apply Kato's semigroup approach to establish local well-posedness for the Cauchy problem associated to the Camassa-Holm equation:

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.1}$$

Observe that

$$(1-\partial_x^2)(uu_x) = uu_x - 3u_x u_{xx} - uu_{xxx}.$$

By using this equality, we can write

$$3uu_x - 2u_x u_{xx} - uu_{xxx} = (1-\partial_x^2)(uu_x) + 2uu_x + u_x u_{xx}.$$

Then, the CH equation becomes

$$(1-\partial_x^2)(u_t) + (1-\partial_x^2)(uu_x) + 2uu_x + u_x u_{xx} = 0.$$

Notice that $2uu_x = (u^2)_x$ and $u_x u_{xx} = (\frac{1}{2}u_x^2)_x$. Then, we can rewrite CH as

$$u_t + uu_x = -\partial_x(1-\partial_x^2)^{-1}(\tfrac{1}{2}u_x^2 + u^2).$$

Note that $-\partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}u_x^2 + u^2)$ can be written as

$$-\partial_x p * (\tfrac{1}{2}u_x^2 + u^2),$$

where $p(x) = \frac{1}{2}e^{-|x|}$ is the Green's function for the operator $(1-\partial_x^2)$, and $(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ being the convolution operator.

Then, the equation takes the form of a quasi-linear equation:

$$u_t + A(u)u = f(u),$$

where

$$A(u) = u\partial_x,$$

and

$$f(u) = -\partial_x (1-\partial_x^2)^{-1}(\frac{1}{2}u_x^2 + u^2).$$

Kato's semigroup approach gives the following result provided that (A1)-(A4) are verified:

Theorem 3.1
Assume that $(A1)-(A4)$ hold. Let $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ be given. Then, there exists a maximal time of existence $T > 0$, depending on u_0, such that there is a unique solution u to (3.1) satisfying

$$u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1}).$$

Moreover, the solution depends continuously on the initial data, i.e, the map $u_0 \in H^s \to u \in C([0,T), H^s)$.

In order to prove this result, we will apply Kato's semigroup approach with $X = H^{s-1}$, $Y = H^s$, and $Q = (1-\partial_x^2)^{1/2}$, where Q is an isomorphism from H^{s-1} to H^s. Now, we need to verify that $A(u)$ and $f(u)$ satisfy the assumptions $(A1)-(A4)$.

Proof of Assumption (A1):

Below, you will find the needed lemmas to be used in the proof of assumption $(A1)$.

Lemma 3.2
The operator $A(u) = u\partial_x$ in H^{s-1}, with $u \in H^s$, $s > \frac{3}{2}$ is quasi-m-accretive.

Proof
It can be found in [Rodríguez-Blanco (2001)].

Similar arguments will be discussed in Section 4 for the dispersion generalized Camassa-Holm equation.

Proof of Assumption (A2):

Below, you will find the needed lemmas to be used in the proof of assumption $(A2)$.

Lemma 3.3
Let the operator $A(u) = u\partial_x$, with $u \in H^s,\ s > \frac{3}{2}$. Then, $A(u) \in \mathcal{L}(H^s, H^{s-1})$, for any $u \in H^s$. Moreover,

$$||(A(u) - A(v))w||_{s-1} \leq \lambda_1 ||u - v||_{s-1} ||w||_s, \quad u, v, w \in H^s.$$

Proof
Let $u, v, w \in H^s,\ s > \frac{3}{2}$, and note that H^{s-1} is a Banach algebra. Then, we have

$$\begin{aligned} ||(A(u) - A(v))w||_{s-1} &= ||(u\partial_x - v\partial_x)w||_{s-1} \\ &= ||((u - v)\partial_x)w||_{s-1} \\ &\leq c||u - v||_{s-1} ||\partial_x w||_{s-1} \\ &\leq \lambda_1 ||u - v||_{s-1} ||w||_s, \end{aligned}$$

where λ_1 is a constant. Then, we get

$$||(A(u) - A(v))w||_{s-1} \leq \lambda_1 ||u - v||_{s-1} ||w||_s.$$

If we take $v = 0$ in the above inequality, we will get $||A(u)w||_{s-1} \leq c||w||_s$, which implies that $A(u) \in \mathcal{L}(H^s, H^{s-1})$.

Proof of Assumption (A3):

Below, you will find the needed lemmas to be used in the proof of assumption $(A3)$.

Lemma 3.4
For any $u \in H^s$, there exists a bounded linear operator $B(u) \in \mathcal{L}(H^{s-1})$ satisfying $B(u) = \Lambda A(u) \Lambda^{-1} - A(u)$, where $B : H^s \to \mathcal{L}(H^{s-1})$ is uniformly bounded sets in H^s. Moreover,

$$||(B(u) - B(v))w||_{s-1} \leq \lambda_2 ||u - v||_s ||w||_{s-1}, \quad u, v \in H^s,\ w \in H^{s-1}.$$

Proof
Note that we can rewrite $B(u)$ as

$$B(u) = \Lambda A(u)\Lambda^{-1} - A(u) = \Lambda u\partial_x\Lambda^{-1} - u\partial_x = [\Lambda, u\partial_x]\Lambda^{-1}.$$

Then, for $u, v \in H^s$, $s > \frac{3}{2}$, $w \in H^{s-1}$,

$$\begin{aligned}
||(B(u) - B(v))w||_{s-1} &= ||[\Lambda,\ (u-v)\partial_x]\Lambda^{-1}w||_{s-1} \\
&= ||\Lambda^{s-1}[\Lambda,\ (u-v)\partial_x]\Lambda^{-1}w||_0 \\
&= ||\Lambda^{s-1}[\Lambda,\ (u-v)]\Lambda^{-1}\partial_x w||_0 \\
&= ||\Lambda^{s-1}[\Lambda,\ (u-v)]\Lambda^{1-s}\Lambda^{s-2}\partial_x w||_0 \\
&\leq ||\Lambda^{s-1}[\Lambda(u-v)]\Lambda^{1-s}||_0||\Lambda^{s-2}\partial_x w||_0 \\
&\leq ||\Lambda^{s-1}[\Lambda(u-v)]\Lambda^{1-s}||_{L(L^2)}||\Lambda^{s-2}\partial_x w||_0 \\
&\leq \lambda_2||u-v||_s||w||_{s-1},
\end{aligned}$$

where λ_2 is a constant, and we use the commutator estimate (Proposition 2.4) with $r = 1 - s$ and $t = s - 1$. Also, if we take $v = 0$ in the above inequality, we get $||B(u)w||_{s-1} \leq \lambda_2||w||_{s-1}$, which implies that $B(u) \in \mathcal{L}(H^{s-1})$.

Proof of Assumption (A4):

Below, you will find the needed lemmas to be used in the proof of assumption $(A4)$.

Lemma 3.5
For all $t \in [0, \infty)$, f is uniformly bounded on bounded sets in H^s. Moreover, the map $f : H^s \to H^s$ is locally H^{s-1}-Lipschitz continuous in the sense that there exists a constant $\lambda_3 > 0$ such that

$$||f(u) - f(v)||_{s-1} \leq \lambda_3||u - v||_{s-1}, \text{ for all } u, v \in B_r(0) \subseteq H^s,$$

and locally H^s-Lipschitz continuous in the sense that there exists a constant $\lambda_4 > 0$

such that

$$||f(u) - f(v)||_s \leq \lambda_4 ||u - v||_s, \text{ for all } u, v \in B_r(0) \subseteq H^s.$$

Proof

Recall that we have $f(u) = -\partial_x(1 - \partial_x^2)^{-1}(\frac{1}{2}u_x^2 + u^2)$. We will start with the first inequality:

Let $u, v \in H^s, s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then, we have

$$\begin{aligned}
||f(u) - f(v)||_{s-1} &= ||\partial_x(1 - \partial_x^2)^{-1}(\frac{1}{2}(u_x^2 - v_x^2)) + (u^2 - v^2)||_{s-1} \\
&\leq ||(u_x^2 - v_x^2) + (u^2 - v^2)||_{s-2} \\
&\leq ||(u_x^2 - v_x^2)||_{s-2} + ||(u^2 - v^2)||_{s-2} \\
&\leq ||(u_x + v_x)(u_x - v_x)||_{s-2} + ||(u + v)(u - v)||_{s-2} \\
&\leq ||\partial_x(u + v)||_{s-2}||\partial_x(u - v)||_{s-2} + ||(u + v)||_{s-2}||(u - v)||_{s-2} \\
&\leq c\Big(||u||_{s-1} + ||v||_{s-1}\Big)||u - v||_{s-2} \\
&\leq \lambda_3 ||u - v||_{s-1},
\end{aligned}$$

where λ_3 is a constant depending on $||u||_{s-1}$ and $||v||_{s-1}$. We use the Cauchy-Schwartz inequality and the fact $||.||_{s-2} \leq ||.||_{s-1}$. This proves H^{s-1}-Lipschitz continuity.

Now, we will show that the second inequality also holds for our $f(u)$:

Let $u, v \in H^s, s > \frac{3}{2}$. Also, note that H^s is a Banach algebra. Then, we have

$$||f(u)-f(v)||_s = ||\partial_x(1-\partial_x^2)^{-1}(\frac{1}{2}(u_x^2-v_x^2))+(u^2-v^2)||_s$$

$$\leq ||(u_x^2-v_x^2)+(u^2-v^2)||_{s-1}$$

$$\leq ||(u_x^2-v_x^2)||_{s-1}+||(u^2-v^2)||_{s-1}$$

$$\leq ||(u_x+v_x)(u_x-v_x)||_{s-1}+||(u+v)(u-v)||_{s-1}$$

$$\leq ||\partial_x(u+v)||_{s-1}||\partial_x(u-v)||_{s-1}+||(u+v)||_{s-1}||(u-v)||_{s-1}$$

$$\leq c\Big(||u||_s+||v||_s\Big)||u-v||_s$$

$$\leq \lambda_4||u-v||_s,$$

where λ_4 is a constant depending on $||u||_s$ and $||v||_s$. Again, we used the Cauchy-Schwartz inequality and the fact $||.||_{s-1} \leq ||.||_s$. This proves H^s-Lipschitz continuity.

As we verify that all the assumptions $(A1)-(A4)$ hold in Theorem (3.1), local well-posedness for the Camassa-Holm equation is established.

4. Cauchy Problem for the Dispersion Generalized Camassa-Holm Equation

In this subsection, we apply Kato's semigroup approach to establish local well-posedness for the Cauchy problem associated to the generalized Camassa-Holm equation:

$$
\begin{cases} u_t - L\partial_x^2 u_t + (a+b)uu_x - au_x L\partial_x^2 u - buL\partial_x^2 u_x = 0, & t>0,\ x\in\mathbb{R}, \\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{4.1}
$$

where a and b are positive constants, L is a positive operator with positive even order p. To construct a non-local form of this equation, we use commutator operator:

$$[L\partial_x^2, u]u_x = L\partial_x^2(uu_x) - uL\partial_x^2 u_x.$$

Then,

$$
\begin{aligned}
(a+b)[L\partial_x^2, u]u_x = &\ (a+b)L\partial_x^2(uu_x) - (a+b)uL\partial_x^2 u_x \\
= &\ (a+b)L\partial_x^2(uu_x) - auL\partial_x^2 u_x - buL\partial_x^2 u_x \qquad .
\end{aligned}
$$

So, we can write

$$(a+b)uu_x - buL\partial_x^2 u_x = (a+b)(1-L\partial_x^2)(uu_x) + (a+b)[L\partial_x^2, u]u_x + auL\partial_x^2 u_x.$$

Then, we have

$$(1-L\partial_x^2)u_t + (a+b)(1-L\partial_x^2)(uu_x) + (a+b)[L\partial_x^2, u]u_x + auL\partial_x^2 u_x - au_x L\partial_x^2 u = 0.$$

Also, it can be seen that

$$\begin{aligned}auL\partial_x^2u_x - au_xL\partial_x^2u &= a(uL\partial_x^2u_x - u_xL\partial_x^2u)\\ &= a(uL\partial_x^2u_x - (L\partial_x^2u)u_x)\\ &= a(uL\partial_x^2 - (L\partial_x^2u))u_x\\ &= -a[L\partial_x^2, u]u_x.\end{aligned}$$

Then, the equation (4.1) becomes

$$(1-L\partial_x^2)u_t + (a+b)(1-L\partial_x^2)(uu_x) + b[L\partial_x^2, u]u_x = 0.$$

With $\Gamma^s = (1-L\partial_x^2)^{s/p+2}$, the equation takes the form of a quasi-linear:

$$u_t + A(u)u = f(u),$$

where

$$A(u) = (a+b)u\partial_x + b\Gamma^{-(p+2)}[L\partial_x^2, u]\partial_x = r(u)\partial_x,$$

and

$$f(u) = 0.$$

Here, you may notice that it looks different from the Camassa-Holm equation. The reason is that the operator L is in a closed form. For example, the usual partial derivative operators and rules in the Camassa-Holm equation are clear, and to use Kato's semigroup approach, the non-local form of the equation is easily obtained. However, since L is in closed form, similar derivative operators and rules cannot be applied. Since there are more than one possible ways of writing non-local form of an equation, the form where we collect nonlinear effects in the operator $A(u)$, as above, holds and serves for our purpose.

Now, we recall Section 2.5 and give the main theorem about local well-posedness:

Theorem 4.1
Assume that assumptions $(A1)-(A4)$ hold. Let $u_0 \in H^s(\mathbb{R}),\ s > \frac{7}{2}+p$ be given. Then, there exists a maximal time of existence $T > 0$, depending on u_0, such that there is a unique solution u to (4.1) satisfying

$$u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1}).$$

Moreover, the solution depends continuously on the initial data, i.e, the map $u_0 \to u(u_0,.)$ is continuous from H^s to $C([0,T),H^s)\cap C^1([0,T),H^{s-1})$.

In order to prove this result, we will apply Kato's semigroup approach with $Q = \Gamma = (1 - L\partial_x^2)^{1/p+2}$, where Q is an isomorphism from H^{s-1} to H^s. Since $f(u) = 0$ in our Cauchy problem, we only need to verify assumptions (A1)-(A3).

Note that for L as an identity operator, $a = 2$ and $b = 1$, we get the Camassa-Holm equation. Considering this, the steps in the proof can be followed clearly.

Proof of Assumption (A1):

Below, you will find the lemmas to be used in the proof of assumption $(A1)$.

Lemma 4.2
The operator $A(u) = r(u)\partial_x$ in L^2, with $u \in H^s, s > \frac{7}{2} + p$ is quasi-m-accretive.

Proof
Since L^2 is a Hilbert space, $A(u)$ is quasi-m-accretive if and only if there is a real number β such that

(a) $(A(u)w,w)_0 \geq -\beta||w||_0^2$,
(b) The range of $A(u)+\lambda$ is all of X for some (or all) $\lambda > \beta$.

First, we will prove part (a). By using integration by parts, the left-hand side of the equality can be written as follows:

$$(A(u)w,w)_0 = (r(u)\partial_x w,w)_0 = \frac{-1}{2}((r(u))_x w,w)_0$$

since if we let

$$K = (r(u)w_x,\ w)_0 \ = -((r(u)w)_x,\ w)_0$$

$$= -(r(u)w_x,\ w)_0 \ - \ ((r(u))_x w,\ w)_0,$$

where $-(r(u)w_x,\ w)_0 = -K$. Then,

$$2K = -((r(u))_x w,\ w)_0$$

$$K = -\frac{1}{2}((r(u))_x w,\ w)_0.$$

Then, it follows that

$$\begin{aligned}
|(r(u)\partial_x w, w)_0| &= |\frac{-1}{2}((r(u))_x w, w)_0| \\
&\leq c||(r(u))_x w||_0 ||w||_0 \\
&\leq c||(r(u))_x||_{L^\infty} ||w||_0^2 \\
&\leq c||(r(u)||_s ||w||_0^2 \\
&\leq \beta ||w||_0^2,
\end{aligned}$$

where $||r(u)||_s < \infty$ and $\beta = c||r(u)||_s$. Since $u \in H^s$ with $s > \frac{7}{2} + p$, it follows that $||u_x||_{L^\infty} \leq ||u||_s$. We show that the operator A is dissipative for all $\lambda > \beta$. That means the operator $-A$ is accretive, where we have $(A(u)w, w)_0 \geq -\beta||w||_0^2$ as required.

Now, we will prove part (b). Note that if A is a closed operator, then $A(u) + \lambda$ has closed range in X for all $\lambda > \beta$. So, it is enough to show that $A(u) + \lambda$ has dense range in x for all $\lambda > \beta$.

First, we will show that A is a closed operator in L^2. Let (v_n) be a sequence in $\mathcal{D}(A)$ which converges to $v \in L^2$ and Av_n converges to $w \in L^2$. Then, since $v_n \in \mathcal{D}(A)$ and $\mathcal{D}(A) = \{w \in L^2 \mid r(u)w \in H^1\} \subset L^2$, we can conclude that $rv_n \in H^1$. Also, by the continuity of the multiplication $H^r \times L^2 \to L^2$ for $r > \frac{1}{2}$, both $rv_n \to rv$ and $r_x v_n \to r_x v$ in L^2, which implies that $(rv_n)_x \to w + r_x v$ in L^2. Then, we have the sequences (rv_n) and $(rv_n)_x$ converges in L^2. Then we can conclude that (rv_n) converges to rv in H^1, which implies that $v \in \mathcal{D}(A)$. Moreover, by continuity of $\partial_x : H^1 \to L^2$ implies that $\lim_{n\to\infty}(rv_n)_x = (rv)_x$, thus we get $w = (rv)_x - r_x v = Av$. Hence, by definition, we showed that A is a closed operator.

Now, we need to show that $(A(u)+\lambda)$ has dense range in L^2 for all $\lambda > \beta$. Note that the adjoint operator of the $A(u) = r(u)\partial_x$ can be written as

$$A^*(u) = -r_x(u) - r(u)\partial_x.$$

Then,

$$A^*(u)w = -r_x(u)w - r(u)w_x = -(r(u)w)_x,$$

where $r_x(u)w \in L^2$ since $u_x \in L^\infty$ and $w \in L^2$, and $r(u)w_x = A(u) \in L^2$ for $w \in \mathcal{D}(A)$. Hence, we can obtain that

$$\mathcal{D}(A^*) = \{w \in L^2 \mid A^*(u)w \in L^2\}.$$

On the contrary, assume that the range of $(A(u)+\lambda)$ is not all of L^2. Then, there exists $z \neq 0 \in L^2$ such that

$$(A(u)w, z)_0 = 0, \ \ \forall w \in \mathcal{D}(A(u)).$$

Since $H^1 \subset \mathcal{D}(A)$, $\mathcal{D}(A)$ is dense in L^2. Then, due to $\mathcal{D}(A^*)$ is closed, $z \in \mathcal{D}(A^*)$. Then, by using the fact that $\mathcal{D}(A) = \mathcal{D}(A^*)$, we can write

$$((A(u)+\lambda)w, z)_0 = (w, (A(u)+\lambda)^* z)_0 = 0,$$

which implies that $(A(u)^* + \lambda)z = 0$ in L^2. After multiplying this equality by z, we can rewrite it as

$$\begin{aligned}
0 = ((A^*(u)+\lambda)z, z)_0 &= (A^*(u)z, z)_0 + (\lambda z, z)_0 \\
&= (z, A(u)z)_0 + (\lambda z, z)_0 \\
&\geq (\lambda - \beta)||z||_0^2, \quad \forall \lambda > \beta.
\end{aligned}$$

Since for all $\lambda > \beta$, the term $(\lambda - \beta) > 0$. Therefore, $z = 0$. However, it contradicts with the assumption $z \neq 0$, which completes the proof of Lemma (4.2).

Lemma 4.3
The operator $A(u) = r(u)\partial_x$ in H^{s-1}, with $u \in H^s, s > \frac{7}{2} + p$ is quasi-m-accretive.

Proof
Since H^{s-1} is a Hilbert space, $A(u) = r(u)\partial_x$ is quasi-m-accretive if and only if there is a real number β such that

(a) $(A(u)w, w)_{s-1} \geq -\beta ||w||_{s-1}^2$,

(b) -$A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}, for some (or all) $\lambda > \beta$.

First, we will prove part (a). Since $u \in H^s$, with $s > \frac{7}{2}$, we can say that u and u_x belong to L^∞. Then, it follows that $||u_x||_{L^\infty} \leq ||u||_s$. Note that

$$\Gamma^{s-1}(r(u)\partial_x w) = [\Gamma^{s-1}, r(u)]\partial_x w + r(u)\Gamma^{s-1}(\partial_x w)$$

$$= [\Gamma^{s-1}, r(u)]\partial_x w + r(u)\partial_x \Gamma^{s-1} w.$$

Then, we have

$$(A(u)w, w)_{s-1} = (r(u)\partial_x w,\ w)_{s-1}$$

$$= (\Gamma^{s-1} r(u)\partial_x w,\ \Gamma^{s-1} w)_0$$

$$= ([\Gamma^{s-1}, r(u)]\partial_x w,\ \Gamma^{s-1} w)_0 + (r(u)\partial_x \Gamma^{s-1} w,\ \Lambda^{s-1} w)_0.$$

For the first term $([\Gamma^{s-1}, r(u)]\partial_x w,\ \Gamma^{s-1} w)_0$, use the commutator estimate (Proposition 2.4) with $n = s-1$, and $\sigma = s$. Then, we get

$$|([\Gamma^{s-1}, r(u)]\partial_x w,\ \Gamma^{s-1} w)_0| \leq c||(r(u)||_s ||\partial_x w||_{s-2} ||w||_{s-1}$$

$$\leq \tilde{c}||w||_{s-1}^2,$$

where $\tilde{c}$ is a constant depending on $||u||_s$.

For the second term $(r(u)\partial_x \Gamma^{s-1} w,\ \Gamma^{s-1} w)_0$, use the integration by parts to get

$$
\begin{aligned}
|(r(u)\partial_x\Gamma^{s-1}w,\ \Gamma^{s-1}w)_0| = & \ \left|-\frac{1}{2}(r(u)_x,\ \Gamma^{s-1}w)^2)_0\right| \\
\leq & \ c||r(u)_x||_{L^\infty}||\Gamma^{s-1}w||_0^2 \\
\leq & \ c||r(u)_x||_{L^\infty}||w||_{s-1}^2 \\
\leq & \ \tilde{c}||w||_{s-1}^2,
\end{aligned}
$$

where $\tilde{c}$ is a constant depending on $||u||_s$.

Set $\beta = \tilde{c}||u||_s$. Then, we get $(A(u)w, w)_{s-1} \geq -\beta||w||_{s-1}^2$, as required.

Now, we will prove part (b). Let $Q = \Gamma^{s-1}$, note that Q is an isomorphism of H^{s-1} to L^2, and H^{s-1} is continuously and densely embedded into L^2 as $s > \frac{3}{2}$. Define

$$
\begin{aligned}
A_1(u) := QA(u)Q^{-1} = & \ \Gamma^{s-1}A(u)\Gamma^{1-s} \\
= & \ \Gamma^{s-1}r(u)\partial_x\Gamma^{1-s} \\
= & \ \Gamma^{s-1}r(u)\Gamma^{1-s}\partial_x
\end{aligned}
$$

and $B_1 = \ A_1(u) + \ A(u)$.
Let $w \in L^2$ and $u \in H^s$, where $s > \frac{7}{2} + p$. Then, we have

$$
\begin{aligned}
||B_1(u)||_0 = & \ ||[\Gamma^{s-1},\ A(u)]\Gamma^{1-s}w||_0 \\
= & \ ||[\Gamma^{s-1},\ r(u)]\Gamma^{1-s}\partial_x w||_0 \\
\leq & \ c||r(u)||_s||\Gamma^{1-s}\partial_x w||_{s-2} \\
\leq & \ c||r(u)||_s||w||_0 \qquad ,
\end{aligned}
$$

where we again use the commutator estimate (Proposition 2.4) with $n = s - 1$, and

$\sigma = s$. Therefore, we obtain $B_1(u) \in \mathcal{L}(L^2)$.

Lemma 4.4 [Pazy (2012)]
Let X and Y be two Banach spaces such that Y is continuously and densely embedded in X. Let -A be the infinitesimal generator of the C_0-semigroup $T(t)$ on X and let Q be an isomorphism from Y onto X. Then Y is -A-admissible (i.e. $T(t)Y \subset Y$ for all $t \geq 0$, and the restriction of $T(t)$ to Y is a C_0-semigroup on Y) if and only if -A_1 = - QAQ^{-1} is the infinitesimal generator of the C_0-semigroup $T_1(t) = QT(t)Q^{-1}$ on X. Moreover, if Y is -A-admissible, then the part of -A in Y is the infinitesimal generator of the restriction $T(t)$ to Y.

We show that $A(u)$ is quasi-m-accretive in L^2, i.e, -$A(u)$ is the infinitesimal generator of C_0-semigroup on H^{s-1} by Lemma (4.3). Thus, by using the perturbation theorem for semigroup [Pazy (2012)], we can say that $A_1(u) = A(u) + B_1(u)$ is also the infinitesimal generator of C_0-semigroup on L^2. Then, we can conclude that H^{s-1} is -A-admissible. Hence, -$A(u)$ is the infinitesimal generator of C_0-semigroup on H^{s-1} by Lemma 4.4 with $X = L^2$, $Y = H^{s-1}$, and $Q = \Gamma^{s-1}$. This completes the proof of Lemma 4.3, and thus assumption $(A1)$.

Proof of Assumption (A2):

Below, you will find the needed lemmas to be used in the proof of assumption $(A2)$.

Lemma 4.5
Let the operator $A(u) = r(u)\partial_x$, with $u \in H^s$, $s > \frac{7}{2} + p$. Then, $A(u) \in \mathcal{L}(H^s, H^{s-1})$, for any $u \in H^s$. Moreover,

$$||(A(u) - A(v))w||_{s-1} \leq \lambda_1 ||u - v||_{s-1} ||w||_s, \qquad u, v, w \in H^s.$$

Proof
Let $u, v, w \in H^s$ with $s > \frac{7}{2} + p$, and note that H^{s-1} is a Banach algebra.

Then, we have

$$||(A(u)-A(v))w||_{s-1} = ||((a+b)(u-v)\partial_x + b\Gamma^{-(p+2)}[L\partial_x^2,(u-v)]\partial_x)w||_{s-1}$$

$$\leq ||(a+b)(u-v)\partial_x w||_{s-1} + ||b\Gamma^{-(p+2)}[L\partial_x^2,(u-v)]\partial_x w||_{s-1}$$

$$\leq ||(u-v)||_{s-1}||\partial_x w||_{s-1} + ||[L\partial_x^2,(u-v)]\partial_x w||_{s-p-3} \qquad .$$

We will use the commutator estimate (Proposition 2.4) with $n = p+2$, $s = s-p-3$, and $\sigma = s-1$, which implies $s+n-1 = s-2$. Then, for $f = u-v$ and $g = \partial_x w$, we get

$$||[L\partial_x^2,(u-v)]\partial_x w||_{s-p-3} \leq c||(u-v)||_{s-1}||\partial_x w||_{s-2}$$

$$\leq c||u-v||_{s-1}||w||_{s-1}$$

$$\leq \lambda_1||u-v||_{s-1}||w||_s,$$

where λ_1 is a constant. Then, we get

$$||(A(u)-A(v))w||_{s-1} \leq \lambda_1||u-v||_{s-1}||w||_s.$$

Take $v = 0$ in the above inequality to obtain $A(u) \in \mathcal{L}(H^s,\ H^{s-1})$. This completes the proof of Lemma 4.5, and thus assumption $(A2)$.

Proof of Assumption (A3):

Below, you will find the needed lemmas to be used in the proof of assumption $(A3)$.

Lemma 4.6

For any $u \in H^s$, there exists a bounded linear operator $B(u) \in \mathcal{L}(H^{s-1})$ satisfying $B(u) = \Gamma A(u)\Gamma^{-1} - A(u)$, where $B: H^s \to \mathcal{L}(H^{s-1})$ is uniformly bounded sets in H^{s-1}. Moreover,

$$||(B(u)-B(v))w||_{s-1} \leq \lambda_2||u-v||_s||w||_{s-1}, \quad u,v \in H^s,\ w \in H^{s-1}.$$

Proof

Note that since ∂_x and Γ commute, we can rewrite $B(u)$ as

$$B(u) = \Gamma A(u)\Gamma^{-1} - A(u) = \Gamma r(u)\partial_x\Gamma^{-1} - r(u)\partial_x = [\Gamma, r(u)]\Gamma^{-1}\partial_x.$$

First, we will show that $B(u)$ is bounded. To do that again we will use the commutator estimate (Prop. 2.4.4) with $n = 1$, $s = s-1$, and $\sigma = s-1$, which implies $s+n-1 = s-1$. Then, for $f = r(u)$ and $g = \Gamma^{-1}\partial_x w$, where $w \in H^{s-1}$, we can write

$$\begin{aligned} ||B(u)w||_{s-1} &= ||[\Gamma, r(u)]\Gamma^{-1}\partial_x w||_{s-1} \\ &\leq ||r(u)||_s ||\Gamma^{-1}\partial_x w||_{s-1} \\ &\leq ||r(u)||_s ||w||_{s-1} \\ &\leq c||w||_{s-1}, \end{aligned}$$

where c depends on $||u||_s$.

Moreover,

$$\begin{aligned} ||(B(u) - B(v))w||_{s-1} &= ||[\Gamma, r(u) - r(v)]\Gamma^{-1}\partial_x w||_{s-1} \\ &\leq ||r(u) - r(v)||_s ||\Gamma^{-1}\partial_x w||_{s-1} \\ &\leq ||r(u) - r(v)||_s ||w||_{s-1} \\ &\leq ||(a+b)(u-v) + a\Gamma^{-(p+2)}[L\partial_x^2, u-v]||_s ||w||_{s-1} \\ &\leq \Big(||u-v||_s + ||[L\partial_x^2, u-v]_{s-p-2}\Big)||w||_{s-1} \\ &\leq \Big(||u-v||_s + ||u-v||_s\Big)||w||_{s-1} \\ &\leq \lambda_2 ||w||_{s-1}, \end{aligned}$$

where λ_2 is a constant depending on $||u||_s$ and $||v||_s$. Take $v = 0$ in the above inequality to obtain $B(u) \in \mathcal{L}(H^{s-1})$. This completes the proof of Lemma 4.6, and thus assumption $(A3)$.

Proof of Assumption (A4):

Since $f(u) = 0$, it is trivial.

As we verify all the assumptions $(A1) - (A4)$ in Theorem 4.1, local well-posedness for the dispersion generalized Camassa-Holm equation is established.

5. Conclusion

In this section, we briefly summarize the results of Section 3 and 4.

In Section 3, we establish the local well-posedness of the Camassa-Holm equation by using Kato's semigroup approach. There are various studies in the literature for Camassa-Holm equation since it is a dispersive equation which can model wave breaking in shallow water wave theory. The 2:1 ratio of the coefficients corresponding to nonlinear terms enables to write non-local form in a simple manner. Afterwards, we show that $A(u)$ and $f(u)$ satisfy the assumptions $(A1)-(A4)$ of the theorem. At the end, we see that choosing the initial data u_0 from H^s, where $s > \frac{3}{2}$, the local well-posedness is established.

In Section 4, we prove the local well-posedness of the dispersion generalized Camassa-Holm equation by using the Kato's semigroup approach again. We use an operator L with a positive order p in a closed form. This operator represents the increment of the dispersive effect. Choosing L as the identity operator and constants $a = 2,\ b = 1$, the equation reduces to Camassa-Holm equation. Quasi-linear form is not easily obtained as in the Camassa-Holm equation. Since L is in a closed form in the nonlinear terms, we use the commutator operators to get the required form. We obtain the results by making assumptions only on the order of L. So, getting those results without writing the operator L explicitly will offer the chance to evaluate the results of the equation in a very wide class. After trying many ways, we need to write the quasi-linear form of the equation by collecting all nonlinear terms in the operator $A(u)$. Therefore, in this case, f becomes zero. Thus, it is enough to show that $A(u)$ satisfies the $(A1)-(A3)$ of the theorem. At the end, we see that choosing the initial data u_0 from H^s, where $s > \frac{7}{2} + p$, the local well-posedness is established. One can observe that the initial data class needs to be more regular compared to Camassa-Holm equation.

6. Future Work

Natural questions arising after this thesis study are the following:

1. Can we extend the maximal existence time to infinity so that we get global well-posedness?

2. Is the energy conserved as it is for Camassa-Holm equation?

3. Is there a time at which there is finite-time blow-up? In which form does it occur? Is it similar to the Camassa-Holm equation or, since the equation is more dispersive, does the blow-up occur later or never?

These are all open questions since this equation will appear in the literature for the first time. As we proceed, there can be more qualitative properties to analyze according to the results found, such as stability analysis.

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