**Hankel Operators**

**by**

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## Abstract

Analytical problems are not solvable on a general domain in $\mathbb{C}^d$ when $d \geq 2$. It has been shown that in domains such as strongly psuedoconvex domains, finite type domains, bounded symmetric domains and convex domain some analytical problems are tractable. A common feature of these domains is that the Bergman metric has bouded geometry on them. This led to the definition of the domains with bounded intrinsic geometry which are introduced to the literature in the quest for the most general type of domain on which one can solve analytical problems. They include many of the well-known domains investigated in the literature including the aforementioned ones. Here in this thesis we made a review of conditions of compactness of Hankel operator on them.

Hankel Operatörü

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## Özet

Karmaşık uzay $\mathbb{C}^d$ içinde genel bir alanda $d \geq 2$ olduğunda analitik problemler her zaman çözülemez. Güçlü psuedokonveks alanlar, sonlu tip alanlar, sınırlı simetrik alanlar ve dışbükey alan gibi alanlarda bazı analitik problemlerin çözülebilir olduğu gösterilmiştir. Bu alanların ortak bir özelliği, üzerlerindeki Bergman metriğinin sınırlı geometriye sahip olmasıdır. Bu gözlem, analitik problemlerin çözülebileceği en genel alan türü arayışında literatüre tanıtılan sınırlı içsel geometriye sahip alanların tanımlanmasına yol açtı. Bunlar, yukarıda bahsedilenler de dahil olmak üzere literatürde araştırılan iyi bilinen alanların çoğunu içerir. İşte bu tezde, Hankel operatörünün üzerlerindeki kompaktlık koşullarının bir incelemesini yaptık.

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*To the memory of a decade full of love and friendship with Nasim*

# Contents

# Chapter 1

# Several Complex Variables

Several complex variables(SCV) is more than a generalization of complex analysis in one variable. Many core theorems in one dimensional complex analysis are no longer valid in SCV; for instance, one can mention the Riemann mapping theorem and discreteness of zero sets of holomorphic functions. The concept of *domain of holomorphy*, a domain that is the largest domain for some holomorphic function, is much more complicated in several variables. Here we discuss some of the most important perspectives which highlight the importance of SCV as an independent research topic from one dimensional complex analysis. Our discussion follows [1] closely; necessary background from one dimensional complex analysis is borrowed from [2] and [3] to make the material self-sufficient for a novice reader in complex analysis.

## 1.1 Domain of Holomorphy

**Definition 1.1.1.** *Assume that $\Omega \subset \mathbb{C}^d$ is open and connected, $f : \Omega \to \mathbb{C}$ is analytic on $\Omega$ if for every $P \in \Omega$ and every $1 \leq j \leq n$ the function $f(P_1, \ldots, P_{j-1}, P_j + z_j, P_{j+1}, \ldots, P_d)$ is a holomorphic function of $z_j$ in the sense of one variable when $|z_j|$ is small enough.*

**Definition 1.1.2.** *Open connected set $\Omega \subset \mathbb{C}^d$ is called a domain of holomorphy if there exists a holomorphic function $f : \Omega \to \mathbb{C}^d$ such that $f$ cannot be extended holomorphically to a domain strictly larger than $\Omega$.*

### 1.1.1 Weierstrass Theorem

We start our discussion here by proving the Weierstrass' theorem on zeros of one variable holomorphic functions. This will allow us to prove an interesting result which helps us to characterize domains of holomorphy in the complex plane.

**Lemma 1.1.3.** *If $u_1, \ldots, u_N$ are complex numbers, and if $p_N = \prod_{n=1}^{N}(1+u_n)$ and $p_N^* = \prod_{n=1}^{N}(1+|u_n|)$ then*

$$p_N^* \leq exp(|u_1| + \ldots + |u_N|) \tag{1.1}$$

$$|p_N - 1| \leq p_N^* - 1 \tag{1.2}$$

**Theorem 1.1.4.** *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of bounded complex functions defined on a set $S \subseteq \mathbb{C}$, such that $\Sigma|u_n(z)|$ converges uniformly on $S$. Then the product*

$$f(z) = \prod_{n=1}^{\infty}(1+u_n(z)) \tag{1.3}$$

*converges uniformly on $S$, and $f(z_0) = 0$ at some $z_0 \in S$ if and only if $u_n(z_0) = -1$ for some $n \in \mathbb{N}$*

*Proof.* The assumption implies that $\Sigma|u_n(z)|$ is bounded on $S$. Therefore, if $p_N$ is the partial product of (1.3) by lemma 1.1.4 we can find $C < \infty$ such that $|p_N(z)| \leq C$ for all $N$ and all $z \in S$. Given $\epsilon > 0$, choose $0 < \delta < \frac{1}{2}$ satisfying $\delta \leq \frac{\epsilon}{2C}$. We can find $N_0$ such that $\sum_{n=N_0}^{\infty} |u_n(z)| < \delta$. If $M > N \geq N_0$ then

$$|p_M - p_N| \leq |p_N|(e^{\delta} - 1) \leq 2C\delta \leq \epsilon \tag{1.4}$$

Where the first inequality is due to lemma 1.1.4. Using 1.4 we can see that $|p_M - P_{N_0}| \leq 2|P_{N_0}|\delta$ if $M > N_0$ so that $|p_M| \geq (1-2\delta)|p_{N_0}|$. Hence

$$|f(z)| \geq (1-2\delta)|p_{N_0}(z)| \qquad z \in S \tag{1.5}$$

and $f(z_0) = 0$ if and only if $p_{N_0}(z_0) = 0$ consequently.

**Corollary 1.1.5.** *Let $\{f_n\}$ be a sequence in $\mathcal{O}(\Omega)$, none of the $f_n's$ is identically zero in any connected part of $\Omega$ and $\sum_{n=1}^{\infty} |1 - f_n(z)|$ is uniformly convergent on compact subsets of $\Omega$. Then*

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{1.6}$$

*Converges uniformly on compact subsets of $\Omega$ and $f \in \mathcal{O}(\Omega)$, consequently. Moreover, order of multiplicity of the zero of $f$ at $z$ is the sum of order of multiplicity of zero of $f_n's$ at $z$.*

*Proof.* First part is an immediate result of the theorem 1.1.4. In the second part, one can see that each $z \in \Omega$ has a neighborhood $U$ where at most finitely many of the $f_n's$ has zero in it. □

**Definition 1.1.6 (Elementary Factors).** *Let $E_0(z) = 1 - z$ and for $p \geq 1$ $E_p(z) = (1 - z)\ exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p})$. These functions are called elementary factors and first introduced by Weierstrass.*

**Lemma 1.1.7.** $|1-E_p(z)| \leq |z|^{p+1}$ *for all non-negative integers p whenever,* $|z| \leq 1$.

**Theorem 1.1.8 (Weierstrass).** *Let $\Omega$ be a proper open subset of $S^2$, where $S^2$ is the Riemann sphere, and $A \subset \Omega$ be such that it has no limit point in $\Omega$. Assign to each $\alpha \in A$ a positive integer $m_\alpha$. Then there exists $f \in \mathcal{O}(\Omega)$ such that its zero set is precisely $A$ with order of multiplicity $m_\alpha$ at each $\alpha \in A$.*

*Proof.* We can assume that $\infty \in \Omega$ and $\infty \notin A$ without loss of generality (Using a linear fractional transformation the general case can be followed). Therefore $S^2 \setminus \Omega$ will be a nonempty compact subset of the plane, and $\infty$ is not a limit point of $A$. For $A$ being a finite set a rational function satisfies the statement. Note that $A$ cannot be uncountable, otherwise it would have a limit point in $\Omega$. Take $\{\alpha_n\}$ such that it covers each member of $A$ exactly $m_\alpha$ times. As $S^2 \setminus \Omega$ is compact, for each $\alpha_n \in A$ we can find $\beta_n \in S^2 \setminus \Omega$ such that $|\beta_n - \alpha_n| \leq |\beta - \alpha_n|$ for all $\beta \in S^2 \setminus \Omega$. Then $|\beta_n - \alpha_n| \to 0$ as $n \to \infty$, otherwise $A$ would have a limit point in $\Omega$. The claim is that $f(z) = \prod_{n=1}^{\infty} E_n(\frac{\alpha_n - \beta_n}{z - \beta_n})$ has the desired properties. Let $K$ be a compact subset of $\Omega$. As $|\beta_n - \alpha_n| \to 0$ we can find $N$ such that $|z - \beta_n| \geq 2|\alpha_n - \beta_n|$ for all $z \in K$ and $n \geq N$, or $|\frac{\alpha_n - \beta_n}{z - \beta_n}| \leq \frac{1}{2}$ equivalently. It follows from the lemma 1.1.7 that $|1 - E_n(\frac{\alpha_n - \beta_n}{z - \beta_n})| \leq (\frac{1}{2})^{n+1}$ for $z \in K$ and $n \geq N$. Now, the theorem 1.1.5 can be used to prove the claim. □

### 1.1.2 Domains of holomorphy in the Complex Plane

The following theorem plays an important role in characterizing domains of holomorphy in one variable.

**Theorem 1.1.9.** *Let $\Omega$ be an open subset of $\mathbb{C}$ bounded by a simple closed curve. Then there exists $f \in \mathcal{O}(\Omega)$ such that $f$ cannot be extended holomorphically to any open set $\Omega^{'}$ containing $\Omega$ properly.*

*Proof.* Consider the countable set $A = \{\alpha_j\}$ in a way that it has no limit points inside $\Omega$ and every point $P \in \partial\Omega$ is its limit point. The Weierstrass theorem now tells us that there exists a non-trivial $f \in \mathcal{O}(\Omega)$ such that its zero set is precisely $A$. Assume that $F$ is a holomorphic extension of $f$ on a strictly larger domain, therefore there exists a point $P \in \partial\Omega$ such that it is an interior point of the domain of $F$, but $P$ is a limit point of $A$ which results in $f$ being identically zero which is a contradiction. □

In proof of the theorem 1.1.9 we used Weierstrass' theorem which employs elementary factors to build the required function. Another interesting approach can be seen in the Hadamard's gap theorem which mentions a family of analytic functions written as a power series centered at the origin with $D(0,1)$ as its natural boundary. To achieve this result, we need some definitions and the theorem 1.1.11 due to Ostrowski.

**Definition 1.1.10 (Regular and Singular Points).** *Let $D$ be an open disk in the complex plane, and $f \in \mathcal{O}(D)$ a point $P$ located on the boundary of $D$ is called regular point of $f$ such that we can find an open disc $D^{'}$ around $P$ such that $f$ can be extended holomorphically to $D \cup D^{'}$. $P$ is called singular if it is not regular.*

**Theorem 1.1.11 (Ostrowski).** *Suppose, $\lambda$, $p_k$ and $q_k$ are positive integers, $p_k$ is monotone increasing, $\lambda q_k > (\lambda+1)p_k$ for all integer $k \geq 0$. Also, assume that $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$ has radius of convergence 1, and $a_n = 0$ if $p_k < n < q_k$ for some $k$. If $P$ is a regular point of $f$ on the unit circle and $s_p(z) = \sum\limits_{n=1}^{p} a_n z^n$, then $\{s_{p_k}(z)\}$ converges in some neighborhood of $P$.*

*Proof.* Without loss of generality, we can assume that $P = 1$. Then $f$ can be extended holomorphically to a domain $\Omega$ containing $B(0,1) \cup \{1\}$. Let $g(w) = \frac{1}{2}(w^{\lambda} + w^{\lambda+1})$ and $F(w) = f(g(w))$ where $w$ is such that $g(w) \in \Omega$. Note that if $|w| \leq 1$ but $w \neq 1$ then $|g(w)| < 1$ and $g(1) = 1$. Therefore $\epsilon > 0$ can be found such that $g(B(0,1+\epsilon)) \subset$ and the series $F(w) = \sum\limits_{n=1}^{\infty} b_m w^m$ converges if $|w| < 1+\epsilon$. If we look at the powers of $w$ in $(g(w))^n$ the lowest is $\lambda n$ and the highest is $(\lambda+1)n$. By our assumption the highest exponent of $w$ in $(g(w))^{p_k}$ will be less than the lowest exponent of $w$ in $(g(w))^{q_k}$. Using the gap condition of $a_n$'s we can see that

$$\sum_{n=1}^{p_k} a_n(g(w))^n = \sum_{m=1}^{(\lambda+1)p_k} b_m w^m \tag{1.7}$$

For $|w| < 1+\epsilon$, as $k \to \infty$ right hand side of (1.7) is convergent. Therefore, $\{s_{p_k}(z)\}$ converges for all $z \in g(B(0,1)+\epsilon)$ as desired. $\square$

**Theorem 1.1.12 (Hadamard).** *Suppose, $\lambda$ and $p_k$ are positive integers such that, $\lambda p_{k+1} > (\lambda+1)p_k$ for all integer $k \geq 0$. Also, assume that*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{p_k} \tag{1.8}$$

*has radius of convergence 1, then the unit circle $(T)$ is the natural boundary of $f$, i.e $f$ cannot be extended holomorphically to any domain larger than the unit disc.*

*Proof.* Assume the contrary, i.e. there exists a point $P$ which is a regular point of $f$. From the theorem 1.1.11 we conclude that the sum in 1.8 is convergent at some point out of closure of the unit disc which is a contradiction as the radius of convergent of the series is 1. $\square$

### 1.1.3 Domains of holomorphy in $\mathbb{C}^d$ $(d \geq 2)$

**Definition 1.1.13.** *The polydisc of radius $r > 0$ centered at $P \in \mathbb{C}^d$ is defined as follows:*

$$D^d(P,r) = \{z \ \in \mathbb{C}^d : |z_j - p_j| < rj = 1, \dots, d\} \tag{1.9}$$

*Similarly the open ball of radius $r > 0$ centered at $P \in \mathbb{C}^d$ is defined as follows:*

$$B(P,r) = \{z \ \in \mathbb{C}^d : |z_1 - p_1|^2 + \dots + |z_d - p_d|^2 < r^2\} \tag{1.10}$$

A simple observation is that every polydisc is a domain of holomorphy. Consider $D^2(0,1)$ for example. As if $f : D(0,1) \to \mathbb{C}$ is a non-extendable holomorphic function then $F : D^2(0,1) \to \mathbb{C}$ such that $F(z_1,z_2) = f(z_1)f(z_2)$ will also be a non-extendable holomorphic function. Also, in one dimension theorem 1.1.9 ensures us that $D(0,r) \setminus \bar{D}(0,r/2)$ is a domain of holomorphy. On the other hand, the next theorem due to Hartog shows that this is not the case for $D^d(0,r) \setminus \bar{D}^d(0,r/2)$ whenever $d \geq 2$.

**Theorem 1.1.14 (Hartog).** *If $f$ is a holomorphic function on $\Omega = D^2(0,r) \setminus \bar{D}^2(0,r/2)$ where $r > 0$ then we may find a holomorphic function $F$ on $D^2(0,r)$ that $F|_\Omega = f$.*

*Proof.* Fix $z_1$ such that $|z_1| < r$ f as a function of $z_2$ can be expanded as a Laurent series as follows

$$f(z_1, z_2) = \sum_{k=-\infty}^{+\infty} a_k(z_1) z_2^k \tag{1.11}$$

If $r/2 < |z_2| < r$ the series in 1.11 converges regardless of the value of $z_1$ and the formula $a_k(z_1) = \dfrac{1}{2\pi i} \oint_\gamma \dfrac{f(z_1,\zeta)}{\zeta^{k+1}} d\zeta$ shows that $a_k(z_1)$ is holomorphic. Similarly, if $r/2 < |z_1| < r$ then $f(z_1, z_2)$ is a holomorphic function of $z_2$ on $D(0,r)$. Thus $a_k(z_1) = 0$ whenever $k < 0$ and $r/2 < |z_1| < r$, and the analytic continuation shows that $a_k \equiv 0$ for $k < 0$. Therefore the function $F(z_1, z_2) = \sum\limits_{k=0}^{+\infty} a_k(z_1) z_2^k$ satisfies the desired properties. $\square$

**Theorem 1.1.15.** *$B(0,1) \in \mathbb{C}^2$ is a domain of holomorphy.*

*Proof.* Consider point $P$ on the boundary of $B(0,1)$, define

$$\phi_P(z) = \frac{1}{2} < z + P, P > \tag{1.12}$$

Where $< ., . >$ denotes the inner product. Obviously, $\phi_P(P) = 1$ and $|\phi_P|$ is strictly less than 1 everywhere else in $\bar{B}(0,1) \setminus P$. Let $\tilde{\mathcal{A}} = \{\tilde{\alpha_j}\}_{j\in\mathbb{N}}$ be a set which has all of the point in $\partial B(0,1)$ as limit point. We construct the sequence $\mathcal{A} = \{\alpha_j\}_{j\in\mathbb{N}}$ as follows

$$\tilde{\alpha_1}, \tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\alpha_3}, \ldots \tag{1.13}$$

This way, $\mathcal{A}$ cover each element of $\tilde{\mathcal{A}}$ infinite many times. Assign $r_j > 0$ to each $\alpha_j$ such that it is as large as possible and $D_j \equiv D^2(\alpha_j, r_j) \subset B(0,1)$. Define $K_j = \bar{B}(0, 1-1/j)$ and take $z_j \in D_j \setminus K_j$. Let $p_j = \dfrac{z_j}{|z_j|}$. This way, $\phi_{p_j}(z)$ has larger modulus on $z_j$ than it does on $K_j$. Define $h_j(z) = \dfrac{\phi_{p_j}(z)}{\phi_{p_j}(z_j)}$ then we will have

$$h_j(z_j) = 1 \quad and \quad |h_j|\Big|_{K_j} < t_j < 1 \tag{1.14}$$

For some $t_j$. Take $N_j$ large enough such that $|m_j|\Big|_{K_j} < \frac{1}{2^j}$ where $m_j(z) = (h_j(z))^{N_j}$. The theorem 1.1.4 ensures us that $f(z) = \prod\limits_{j=1}^{\infty} (1 - m_j(z)^j)$ converges uniformly on each $K_j$ and has a zero of at least $j$ at $z_j$. Each $\tilde{\alpha_j}$ is repeated infinitely often so at each $D_j$ there are points at which $f$ is zero with an arbitrary high order. For any holomorphic extension of $f$ to an open neighborhood of $P \in \partial B(0,1)$ will have a limit point $z_0$ of a sequence of zeros of $f$ with increasing order, so $f$ has a zero of order infinity at $P$ and $f \equiv 0$ which is a contradiction. □

The problem of geometric description of domains of holomorphy in $\mathbb{C}^d$ was an open problem for a long time, called Levi Problem. It was solve by Oka, Bremermann et al. this geometric notion is called *pseudoconvexity*.

## 1.2 Zeros of Holomorphic Functions

In one-dimensional complex analysis, the zero set of a holomorphic function is discrete. Also, in Weierstrass theorem we saw that zero set of a holomorphic function can be any discrete set if it is not constant. A surprising consequence of Hartog theorem is that this is no longer true for more than one variable.

**Theorem 1.2.1.** *A holomorphic dunction $f$ defined on a domain $\Omega \subseteq \mathbb{C}^d$ and $d \geq 2$ cannot have an isolated zero.*

*Proof.* Let $P$ be an isolated zero of $f$. Take the polydisc $D^d(P,r)$ such that $f$ has not any zero in it other than $P$. Hence, the function $g(z) = 1/f(z)$ will be holomorphic on $D^d(P,r) \setminus D^d(P,r/2)$. By Hartog's theorem g can be extended to a holomorphic function on $D^d(P,r)$, more specifically it is defined at $P$, which is a contradiction.

The next result reveals another interesting fact on zeros of holomorphic functions in several variables, however to prove this we need two classical results in one variable due to Weierstrass and Hurwitz.

**Theorem 1.2.2 (Weierstrass).** *Take $\{f_n\}_{n\in\mathbb{N}}$ as a sequence in $\tilde{\mathcal{A}}(\Omega)$, which converges uniformly on every compact subset of $\Omega$ to a function $f$. Then $f \in \mathcal{O}(\Omega)$, in additional $\{f_n'\}$ converges uniformly on every compact subset of $\Omega$ and its limit is $f'$.*

*Proof.* For $a \in \Omega$, take $r > 0$ such that $\bar{D}(a,r) \subset \Omega$. If $\gamma$ is the boundary of $D(a,r)$, $0 < \rho < r$ and $w \in \Omega$ be such that $|w - a| \leq \rho$ we will have

$$f_n(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f_n(w)}{z-w} dz \tag{1.15}$$

and

$$\frac{1}{|z-w|} \leq \frac{1}{r-\rho} \tag{1.16}$$

Using the assumption, $f_n$ converges to $f$ uniformly on $\gamma$, therefore

$$f(w) = \lim_{n\to\infty} f_n(w) = \lim_{n\to\infty} \frac{1}{2\pi i} \oint_\gamma \frac{f_n(w)}{z-w} dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{z-w} dz \tag{1.17}$$

So, $f$ is holomorphic on $D(a,\rho)$ and $f \in \mathcal{O}(\Omega)$ consequently. To prove the second part, note that by a similar argument to the first part we have

$$\frac{1}{|z-w|^2} \leq \frac{1}{(r-\rho)^2} \tag{1.18}$$

Therefore

$$\lim_{n\to\infty} f_n'(w) = \lim_{n\to\infty} \frac{1}{2\pi i} \oint_\gamma \frac{f_n(w)}{(z-w)^2} dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(z-w)^2} dz = f'(w) \tag{1.19}$$

So $f_n'$ converges to $f'$ uniformly on $\bar{D}(a,\rho)$ as desired. □

**Theorem 1.2.3 (Hurwitz).** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\tilde{\mathcal{A}}(\Omega)$, and $f_n$ converges uniformly on every compact subset of $\Omega$ to a function $f$, where $\Omega$ is a connected open subset of $\mathbb{C}$. Assume that each $f_n$ is none-zero on $\Omega$. Then, either $f$ is non-zero on $\Omega$ or $f \equiv 0$.*

*Proof.* First, note that by theorem 1.2.2 $f$ is holomorphic and $f_n' \to f'$ uniformly on compact subsets of $\Omega$. If $f \not\equiv 0$ and $f$ vanishes on $a \in \Omega$. Then we can find a closed disc $\bar{D}(a,r)$ on which $f$ has no zeros other than $a$ and $\gamma$ is the boundary of this disc. We will have

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} dz = m \tag{1.20}$$

Where $m \geq 1$ denotes order of multiplicity of the zero of $f$ at $a$. Also, $f_n'/f_n \to f'/f$ uniformly on $\gamma$, but $\frac{1}{2\pi i} \oint_\gamma \frac{f_n'}{f_n} dz = 0$ for all $n$ as all of the $f_n$'s are non-zero everywhere on $\Omega$. This is a contradiction as

$$0 = \frac{1}{2\pi i} \oint_\gamma \frac{f_n'}{f_n} dz \to \frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} dz \neq 0 \tag{1.21}$$

**Theorem 1.2.4.** *If $f$ is a holomorphic function on a bounded domain $\Omega \subset \mathbb{C}^d$, $d \geq 2$ and $\mathcal{Z}$ is its zero set. Then $\mathcal{Z}$ is cannot be compact if it is nonempty $\Omega$.*

*Proof.* Assume that $\mathcal{Z}$ is nonempty and compact, take $X \in \mathbb{C} \setminus \Omega$ and choose $P \in \mathcal{Z}$ such that it is as far as possible from $X$. Define $\vec{v} = \dfrac{\vec{XP}}{|\vec{XP}|}$, and let $\vec{w}$ be a vector with magnitude one and normal to $\vec{v}$. Let $r > 0$ be small enough and integer $j > 0$ sufficiently large such that we can define

$$\phi_j(\zeta) = f(P + (1/j)\vec{v} + r\zeta\vec{v}) \quad \zeta \in D(0,1) \tag{1.22}$$

With this definition all $\phi_j$'s are non-zero in $D(0,1)$ and $\phi_j \to \phi$ uniformly on compact subsets of the unit disc, where $\phi(\zeta) = f(P + r\zeta\vec{w})$. One can immediately see that $\phi(0) = 0$ and therefore $\phi \equiv 0$ using the theorem 1.2.3, but this is a contradiction as we have chosen $P$ and $\vec{w}$ such that $P + r\zeta\vec{w} \notin \mathcal{Z}$ whenever $\zeta \neq 0$ □

**Corollary 1.2.5.** *For a holomorphic $f$ on a bounded domain $\Omega \subset \mathbb{C}^d$, $d \geq 2$, then every level set of it escapes to the boundary.*

Another interesting difference between one and sevaral variable complex analysis will be revealed in the next theorem.

**Theorem 1.2.6.** *Let $\Omega \subset \mathbb{C}^d$, $d \geq 2$ be a bounded domain and $f$ is a holomorphic function on it and it is continuous on $\bar{\Omega}$. Then $f(\partial\Omega) = f(\bar{\Omega})$.*

*Proof.* If $P \in \Omega$ and $f(P) = w \notin f(\partial\Omega)$. Then $f^{-1}(w)$ is a bounded closed set disjoint from the $\partial\Omega$, therefore it is compact which is a contradiction by theorem 1.2.4. □

Obviously the statement in the theorem 1.2.6 is not valid in one variable (one can take $f(z) = z$ on $D(0,1)$ as a counter example).

## 1.3 Inner Functions

**Definition 1.3.1.** *Function $f$ defined on $B \equiv B(0,1) \subset \mathbb{C}^d$ is called inner, if its modulus is 1 almost everywhere on $\partial B$.*

**Example 1.3.2 (Blaschke Factor and Product).** *In one dimensional complex plane, Blaschke factor $B_a(z) = \dfrac{z-a}{1-\bar{a}z}$ where $a \in B$ and every Blaschke product $\prod_{j=1}^{\infty} -\dfrac{\bar{a}_j}{|a_j|} B_{a_j}(z)$ are inner.*
*Every Blaschke factor, is an automorphism of the unit disc. In fact, every automorphism of the unit disc can be written as a composition of a rotation around the origin and a Blaschke factor. Also, one can easily show that $(B_a)^{-1} = B_{-a}$.*

**Lemma 1.3.3.** *An inner function $f$ on $B \subset \mathbb{C}$ which is not constant and bounded from zero does not exist.*

*Proof.* Consider $f$ to be an inner function such that it is not constant and

$$|f(z)| \geq c > 0 \quad z \in B \tag{1.23}$$

Then $g(z) = \dfrac{1}{f(z)}$ is holomorphic, bounded on $B$. $f$ on $B$ is the Poisson integral of the boundary and therefore $|f(z)| < 1$ on $B$. Similarly, $|g(z)| < 1$ on $B$. Which is a contradiction. $\square$

**Lemma 1.3.4.** *Range of a non-constant inner function $f$ on $B \subset \mathbb{C}$ is everywhere dense in $B$.*

*Proof.* Assume that range of $f$ does not cover a disc having its center at $a \in B$. The function $B_a \circ f$ will be an inner function bounded from zero, which is a contradiction by lemma 1.3.3. $\square$

**Theorem 1.3.5.** *The cluster set $\mathcal{C}(P)$ of a non-constant inner function $f$ on $B \subset \mathbb{C}^2$ is the entire $\bar{D}$.*

$$\mathcal{C}(P) \equiv \{w \in \mathbb{C}|\ \textit{the sequence } z_j \to P \textit{ existss.t } f(z_j) \to w\} \tag{1.24}$$

*Proof.* Let $P$ be a point in the boundary, and $\mathcal{P}_j$ be a sequence of hyperplanes which approach $P$ from inside of the ball. Then by 1.3.4 $f(\mathcal{P}_j \cup B)$ is dense in $D$ for each $j$ as desired. $\square$

Unlike one dimensional inner fucntions which are elementary, construction of higher dimesional inner functions is very technical . In fact due to their odd behaviour, a dense collection of level sets escapes to all boundary points, their existence was a surprising discovery in 1981. (reference needed)

## 1.4 Holomorphic Mappings

In one dimensional complex analysis a simply connected domain of the complex plane is biholomorphic to the disc, this result is known as Riemann Mapping Theorem, but this is no longer true for higher dimensions. First we prove the Riemann Mapping Theorem and show that why it is not valid in SCV. To prove Riemann Mapping Theorem several classical results from complex analysis are needed which are brought here.

**Theorem 1.4.1** (**Maximum Modulus Principle**). *Assume that $f \in \mathcal{O}(\Omega)$, $f$ is non-constant and $\Omega$ is an open connected subset of the complex plane. If $\bar{D}(a,r) \subset \Omega$ for some $a \in \Omega$ and $r > 0$. Then, $|f(a)| < \max_{\theta} |f(a + re^{i\theta})|$.Consequently, $|f|$ is without local maximum on $\Omega$.*

*Proof.* If $|f(a + re^{i\theta})| \leq |f(a)|$ for all $0 \leq \theta < 2\pi$, then the Parseval relation yields us

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq |f(a)| = |a_0|^2 \tag{1.25}$$

Where $\sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic expansion of $f$ at $a$. (1.27) implies that $a_n = 0$ for all $n \geq 1$. Therefore, $f$ is a constant function which is a contradiction. □

**Theorem 1.4.2** (**Schwarz's Lemma**). *Consider $f \in \mathcal{O}(D)$ such that, $f(0) = 0$, $f \not\equiv 0$ and $|f(z)| \leq M$ for all $z \in D$, where $M > 0$. Then $|f(z)| \leq M|z|$ for $z \in D$. Moreover, if $|f(z_0)| = M|z_0|$ for some $z_0 \in D \setminus \{0\}$, then $f(z) = Me^{i\theta} z$ and $0 \leq \theta < 2\pi$.*

Define $g(z) = \dfrac{f(z)}{z}$ if $z \neq 0$ and $g(0) = f'(0)$. Then $g \in \mathcal{O}(D)$. If $a \in \partial D$, we have

*Proof.*

$$\limsup_{z \to a} |g(z)| = \limsup_{z \to a} |f(z)| \leq \tag{1.26}$$

Where the equality comes from the fact that $|a| = 1$. Maximum modulus principle now tells us that $|g(z)| \leq M$. □

**Corollary 1.4.3.** *We have $|f'(0)| < M$, whenever the assumptions of the Schwarz's lemma are valid.*

**Theorem 1.4.4** (**Open Mapping Theorem**). *Let $f \in \mathcal{O}(\Omega)$ where $\Omega$ is a connected open subset of the complex plane, and $f$ is not constant. Then $f : \Omega \to \mathbb{C}$ is an open map.*

*Proof.* Let $a \in \Omega$, without loss of generality we can assume that, $f(a) = 0$. Note that it will be enough to show that $f(\Omega)$ is a neighborhood of 0, as we can substitute $\Omega$ by $U$, a connected open set containing 0, and apply the results on $f\big|_U$. Choose $r > 0$ such that $\bar{D}(a,r) \subset \Omega$ and $f$ is none-zero on $\bar{D}(a,r) \backslash a$. Then $\delta = inf\{|f(z)| \big| |z-a| = r\}$ will be positive. Now the claim is that if $w \in \mathbb{C} \setminus f(\Omega)$ then $|w| \geq \frac{1}{2}\delta$. To prove this claim, let $g(z) = \dfrac{1}{f(z)-w}$. From the definition $g \in \mathcal{O}(\Omega)$. By maximum modulus principle we will have

$$\frac{1}{|w|} = |g(a)| \leq \sup_{|z-a|=r} |g(z)| = \frac{1}{\inf |f(z)-w|} \tag{1.27}$$

If $|w| < \delta$, then $|f(z) - w| \geq |f(z)| - || \geq \delta - |w|$ for $|z - a| = r$. So by (1.27) $\dfrac{1}{|w|} \leq \dfrac{1}{\delta - |w|}$ or $|w| \geq \frac{1}{2}\delta$. Thus, either $|w| \geq \frac{1}{2}\delta$ or $|w| \geq \delta$. And $D(0, \frac{1}{2}\delta) \subset f(\Omega)$ as desired. □

**Definition 1.4.5** (**Holomorphic Mapping**). *Function $F$ defined below is called a holomorphic mapping from $\Omega \subseteq \mathbb{C}^d$ to $\Omega' \subseteq \mathbb{C}^m$*

$$F(z_1, \dots, z_d) = (f_1(z_1, \dots, z_d), \dots, f_m(z_1, \dots, z_d)) \tag{1.28}$$

*Where each $f_j$ is a holomorphic function.*

**Definition 1.4.6** (**Biholomorphic Mapping**). *A holomorphic mapping of a domain $\Omega \subseteq \mathbb{C}^d$ to a domain $\Omega' \subseteq \mathbb{C}^d$ that is bijective is called a biholomorphic mapping and $\Omega$ and $\Omega'$ are called biholomorphic to eachother.*

**Definition 1.4.7** (**Normal Families**). *$\mathcal{F} \subset \mathcal{O}(\Omega)$ is called a normal family if every sequence of members of $\mathcal{F}$ has a uniformly convergent subsequence on compact subsets of $\Omega$. Note that the limit of that subsequence is not necessarily is in $\mathcal{F}$.*

**Lemma 1.4.8.** *If $\mathcal{F} \subset \mathcal{O}(\Omega)$, $\Omega$ is an open connected subset of $\mathbb{C}$ and $\mathcal{F}$ is uniformly bounded on each compact subset of $\Omega$. Then $\mathcal{F}$ is a normal family.*

**Theorem 1.4.9** (**Riemann Mapping Theorem**). *Let $\Omega$ be a simply connected open set in the complex plane and is not the plane itself. Then $\Omega$ is biholomorphic with $D$.*

*Proof.* Take $z_0 \in \mathbb{C} \setminus \Omega$ and $\mathcal{F}$ as the collection of all injective holomorphic mappings from $\Omega$ to $D$. Our first claim is that $\mathcal{F}$ is nonempty. $\Omega$ is simply connected therefore there exists $h \in \mathcal{O}(\Omega)$ in a way that $h(z) = z - w_0$. It is a simple observation that $h$ is injective. Open mapping theorem now can be used to show that $h(\Omega)$ contains a disc $D(a, r)$ such that $0 < r < |a|$. Also, there exists no $z_1$ and $z_2$ in $\Omega$ such that $h(z_1) = -h(z_2)$. Therefore, the disc $D(-a, r)$ does not intersects with $h(\Omega)$ and the function $\tilde{h} = \dfrac{r}{\phi + a}$ will be a member of $\mathcal{F}$.

Our second claim is that if $f_0 \in \mathcal{F}$ is not surjective and $z_1 \in \Omega$, then there exists

$f_1 \in \mathcal{F}$ such that $|f_1'(z_1)| > |f_0'(z_1)|$. Assume, $f_0 \in \mathcal{F}$, $w_0 \in D$ and $w_0 \notin f_0(\Omega)$. Then $B_{w_0} \circ f_0 \in \mathcal{F}$ and $B_{w_0} \circ f_0$ has no zero in $\Omega$, so we can find $g \in \mathcal{O}(\Omega)$ such that $g^2 = B_{w_0} \circ f_0$. One may observe that $g$ is injective and $g \in \mathcal{F}$ consequently. If $w_1 = g(z_1)$, define $f_1 = B_{w_1} \circ g$ and let $s(z) = z^2$. We will have

$$f = B_{-w_0} \circ s \circ g = B_{-w_0} \circ s \circ B_{-w_1} \circ f_1 \tag{1.29}$$

Note that $f_1(z_0) = 0$ and let $F = B_{-w_0} \circ s \circ B_{w_1}$. Using the chain rule yields

$$f'(z_1) = F'(0) f_1'(z_1) \tag{1.30}$$

The function $F$ has the property that $F(D) \subset D$, applying Schwarz lemma gives us $|F'(0)| < 1$, and $|f_0'(z_1)| < |f_1'|$ as desired.
Fix $z_1 \in \Omega$ and let $\eta = \sup |f'(z_1)| \big| f \in \mathcal{F}$. The third claim is that we may find $f \in \mathcal{F}$ such that $|f'(z_1)| = \eta$, existence of $f$ finishes the proof as $f$ has to be surjective otherwise we will be able to find $f_1$ such that $|f_1'(z_1)| > |f'(z_1)|$ which is a contradiction. As $|f(z)| < 1$ for every $f \in \mathcal{F}$ and $z \in \Omega$, we can conclude from lemma 1.4.8 that $\mathcal{F}$ is a normal family. By definition of $\eta$, there exists a sequence $\{f_n\}$ in $\mathcal{F}$ that $|f_n'(z_1)| \to \eta$ and normality of $\mathcal{F}$ ensures us that we may find a subsequence that is uniformly convergent on compact subsets of $\Omega$ rename $\{f_n\}$ to be that subsequence. Let $h$ be limit of $\{f_n\}$ by 1.2.2 $h \in \Omega$ and $|h'(z_1)| = \eta$. As $f_n(\Omega) \subset D$, we will have $h(D) \subset \bar{D}$, but open mapping theorem shows that actually $h(\Omega) \subset D$. We should now prove that $h$ is also injective. Fix distinct $w_1$ and $w_2$ in $\Omega$, put $\alpha = h(w_1)$ and $\alpha_n = f_n(w_1)$ for $n \geq 1$. Consider $\bar{D_0}$ as a closed disc with center at $w_2$ in a way that $w_1 \notin \bar{D_0}$ and $h - \alpha$ is non-zero on the boundary of $D_0$, this is possible since the zero set of $h - \alpha$ has no limit point in $\Omega$. The sequence $f_n - \alpha_n$ converges to $h - \alpha$ uniformly on $\bar{D_0}$ they have no zeros in $D_0$, as they are injective and have their zero on $w_1$, from Hurwitz's theorem it follows that $h - \alpha$ has no zero in $D_0$, particularly $h(w_1) \neq h(w_2)$ and $h \in \mathcal{F}$ consequently. □

**Theorem 1.4.10.** *The polydisc $D^2 \equiv D^2(0,1)$ and the ball $B \equiv B(0,1)$ in $\mathbb{C}^2$ are not biholomorphic to each other.*

*Proof.* Assume the contrary that $B$ and $D^2$ are biholomorphic and $F : B \to D^2$ is a biholomorphic map between tham. Without loss of generality we can assume that $F(0) = 0$. We define

$$X(B,0) = \{w \in \mathbb{C}^2 | \exists \textit{ holomorphic mapping } h : D \to B \textit{ s.t } h(0) = 0,\ h'(0) = w\} \tag{1.31}$$

$$X(D^2,0) = \{w \in \mathbb{C}^2 | \exists \text{ holomorphic mapping } h : D \to D^2 \text{ s.t } h(0) = 0,\ h'(0) = w\} \tag{1.32}$$

Then, the Jacobian matrix $F'(0) = [\frac{\partial f_i}{\partial z_j}(0)]$ will be a biholomorphic map (as it is linear) between $X(B,0)$ and $X(D^2,0)$, for if $w \in X(B,0)$ and $h$ be the corresponding holomorphic function such that $h(0) = 0$ and $h'(0) = w$, then $(F \circ h)(0) = 0$ and $(F \circ h)'(0) = (F'(0))(w)$ (Note that $det(F'(0)) \neq 0$ as $F$ is a biholomorphy between $B$ and $D^2$) Now, the claim is that $X(B,0) = \bar{B}$ and $X(D^2,0) = \bar{D^2}$. Note that this will be a contradiction as $\bar{B}$ has smooth boundary and cannot be biholomorphic to $\bar{D^2}$ which has edges.
To prove that $X(B,0) = \bar{B}$, take any $w \in \bar{B}$ then $g(z) = wz$ maps $D$ into $B$ and has the properties $g(0) = 0$ and $g'(0) = w$. Therefore, $\bar{B} \subseteq X(B,0)$. On the other hand, if $w \in X(B,0)$. Let $g : D \to B$ be such that $g(0) = 0$ and $g'(0) = w$. Take $\pi_1 : B \to \{(z_1,0)|\ |z_1| < 1\}$ and $\pi_1(z_1,z_2) = z_1$, along with any unitary rotation $\sigma$. The map $\pi_1 \circ \sigma \circ g$ sends $D$ to $D$ and 0 to 0. We may now apply Schwarz's lemma to conclude that $|(\pi_1 \circ \sigma \circ g)'(0)| \leq 1$ and the chain rule yields $|(\pi_1 \circ \sigma)(w)| \leq 1$. Since $\sigma$ was arbitrary we will have $|w| \leq 1$ as desired. Thus $X(B,0) = \bar{B}$. It remains us to show that $X(D^2,0) = \bar{D^2}$. In a similar fashion to the previous part we can show that $\bar{D^2} \subseteq X(D^2,0)$. Now if $w \in X(D^2,0)$ and $g : D \to D^2$ be the function such that $g(0) = 0$ and $g'(0) = w$ then both $\pi_1 \circ g$ and $\pi_2 \circ g$ have the conditions of applying Schwarz's lemma. So, $|\pi_1(w)| \leq 1$ and $|\pi_2(w)| \leq 1$ i.e. $w \in \bar{D^2}$. Whence, $X(D^2,0) = \bar{D^2}$ and we are done. □

## 1.5 Bergman Kernel

Here we will assume that $\Omega$ is a bounded subset of $\mathbb{C}^d$, which is not a necessity for many of the following materials but it is a useful simplification. In this section we will follow the approach brought in [4].

**Definition 1.5.1** (**Defining Function**). *Defining function of $\Omega$ is a $\mathcal{C}^2$ function $\rho$ such that $\Omega = \{z \in \mathbb{C}^d : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.*

**Definition 1.5.2** (**Complex Tangent Vector**). *If $P \in \partial\Omega$, $w$ is a complex tangent vector at $P$ denoted by $w \in \mathcal{T}_P(\partial\Omega)$ if $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(P)w_j = 0$.*

**Definition 1.5.3** (**Psuedoconvexity**). *$\partial\Omega$ is called weakly psuedoconvex at $P$ if for every $w \in \mathcal{T}_P(\partial\Omega)$ we have $\sum_{j,k=1}^{d} \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k w_j \bar{w}_k} \geq 0$. It is called strongly psuedoconvex* *at $P$ if the inequality is strict for every $w \neq 0$ and $w \in \mathcal{T}_P(\partial\Omega)$. Then $\Omega$ is a psuedoconvex (strongly psuedoconvex) if $\partial\Omega$ is psuedoconvex (strongly psuedoconvex) at every point $P \in \partial\Omega$.*

**Definition 1.5.4** (**Bergman Space**). *The Bergman space at $\Omega$ is defined as follows*

$$A^2(\Omega) = \{f \in \mathcal{O}(\Omega) | (\int\limits_{\Omega} |f(z)|^2 dV(z))^{\frac{1}{2}} \equiv ||f||_{A^2} < \infty\} \tag{1.33}$$

**Lemma 1.5.5.** *If $K$ be a compact subset of $\Omega$, $C > 0$ can be found in a way $\sup\limits_{z \in K} \leq C_K ||f||_{A^2(\Omega)}$ for every $f \in A^2(\Omega)$.*

*Proof.* Compactness of $K$ implies that we may find $r > 0$ such that for every $z \in K$ $B(z,r) \subseteq \Omega$. Then we can write that
$|f(z)| = |\frac{1}{V(B(z,r))} \int\limits_{B(z,r)} f(t)dV(t)| = |\frac{1}{V(B(z,r))} \int\limits_{\mathbb{C}^d} f(t)\mathcal{X}_{B(z,r)} dV(t)| \leq$
$\frac{1}{V(B(z,r))} ||f||_{L^2} ||\mathcal{X}_{B(z,r)}||_{L^2} = \frac{||f||_{A^2}}{V(B(z,r))^{\frac{1}{2}}}$

**Lemma 1.5.6.** *$A^2(\Omega)$ with the inherited inner product from $L^2$ is a Hilbert space*

*Proof.* If $\{f_j\} \subset A^2(\Omega)$ is a Cauchy sequence then it has a limit $f$ in $L^2(\Omega)$. By lemma 1.5.6 $f_j \to f$ uniformly on compact subsets of $\Omega$. By a generalization of the Weierstrass theorem (theorem 1.2.2) to several variables we have $f \in \mathcal{O}(\Omega)$. □

**Lemma 1.5.7.** *For every $z \in \Omega$, $\phi_z : f \to f(z)$ is in the continuous dual of $A^2(\Omega)$.*

*Proof.* Put $K = z$ in the lemma 1.5.5. □

**Definition 1.5.8** (**Bergman Kernel**). *Note that if we apply Riesz Representation Theorem to $\phi_z$ we can conclude that there exists a unique $K_z \in A^2(\Omega)$ such that $\phi_z(f) = f(z) =< f, K_z >$ for every $f \in A^2(\Omega)$. We then define the Bergman Kernel as $K_\Omega(z,w) = \overline{K_z(w)}$ so $f(z) = \int\limits_{\Omega} K_\Omega(z,w)f(w)dV(w)$ for every $f \in A^2(\Omega)$.*

**Proposition 1.5.9.** *$K_\Omega(z,w) = \overline{K_\Omega(w,z)}$.*

**Proposition 1.5.10.** *Bergman Kernel with the following properties is unique:*
*1. $K_\Omega(z,w)$ is an element of $A^2(\Omega)$ in $z$.*
*2. Is conjugate symmetric.*
*3. Reproduces $A^2(\Omega)$.*

*Proof.* if $K'(z,w)$ be another such kernel then we will have
$K_\Omega(z,w) = \overline{K_\Omega(w,z)} = \int_\Omega K'(z,t)\overline{K_\Omega(w,t)}dV(t) = \overline{\int_\Omega K_\Omega(w,t)\overline{K'(z,t)}dV(t)} = \overline{\overline{K'(z,w)}} = K'(z,w)$ □

**Remark 1.5.11.** *$A^2(\Omega)$ as a subspace of the separable Hilbert space $L^2(\Omega)$ is separable and therefore it has a countable, total orthonormal basis.*

**Proposition 1.5.12.** *If $L$ is a compact subset of $\Omega$ and $\{\phi_j\}_{j\in\mathbb{N}} \subset A^2(\Omega)$, then $\sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}$ converges uniformly on $L \times L$ to the Bergman Kernel.*

*Proof.* $\sup\limits_{z\in L}(\sum_{j=1}^{\infty} |\phi_j(z)|^2)^{\frac{1}{2}} = \sup\limits_{z\in L} ||(\phi_j(z))_{j\in\mathbb{N}}||_{l^2} = \sup\limits_{z\in L \ \& \ ||(a_j)_{j\in\mathbb{N}}||_{l^2}=1} |\sum_{j=1}^{\infty} a_j\phi_j(z)| = \sup\limits_{z\in L \ \& \ ||f||_{A^2}=1} |f(z)| \leq C$

So we we have $\sum_{j=1}^{\infty} |\phi_j(z)\overline{\phi_j(w)}| \leq (\sum_{j=1}^{\infty} |\phi_j(z)|^2)^{\frac{1}{2}}(\sum_{j=1}^{\infty} |\phi_j(w)|^2)^{\frac{1}{2}} \leq C^2$ and it is uniformly convergent in $L \times L$ consequently. If put $K'(z,w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}$, then one can easily check that $K'(z,w)$ has the three properties in the proposition 1.5.10 so it will be equal to $K_\Omega(z,w)$. □

**Definition 1.5.13** (**Bergman Projection**). *The mapping $P_\Omega(f) = \int_\Omega K_\Omega(.,w)f(w)d\mu(w)$ where $f \in L^2(\Omega)$ is the normal projection of $L^2(\Omega)$ to $A^2(\Omega)$. $P$ is called the Bergman Projection.*

**Definition 1.5.14.** *If $\Omega \subseteq \mathbb{C}^d$ is a domain and $f : \Omega \to \mathbb{C}^d$ is holomorphic. Take $w_j = f_j(z)$, $1 \leq j \leq n$. We define the holomorphic Jacobian matrix of $f$ as follows*

$$J_{\mathbb{C}} f = \frac{\partial(w_1, \ldots, w_d)}{\partial(z_1, \ldots, z_d)} \tag{1.34}$$

*Also if, $z_j = x_j + iy_j$ and $w_j = \zeta_j + i\eta_j$ $1 \leq j \leq n$. The real Jacobian matrix of $f$ is as follows*

$$J_{\mathbb{R}} f = \frac{\partial(\zeta_1, \eta_1, \ldots, \zeta_n, \eta_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)} \tag{1.35}$$

**Theorem 1.5.15.** $\det J_{\mathbb{R}} f = |\det J_{\mathbb{C}} f|^2$.

*Proof.* $d\zeta_1 \wedge d\eta_1 \wedge \ldots \wedge d\zeta_n \ldots d\eta_n = (\det J_{\mathbb{R}f})\ dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$.
Also, we have
$d\zeta_1 \wedge d\eta_1 \wedge \ldots \wedge d\zeta_n \ldots d\eta_n = \frac{1}{(2i)^n} d\bar{w}_1 \wedge dw_1 \wedge \ldots d\bar{w}_n \wedge dw_n = \frac{1}{(2i)^n}\overline{(\det J_{\mathbb{R}f})}(\det J_{\mathbb{R}f}) d\bar{z}_1 \wedge dz_1 \wedge \ldots d\bar{z}_n \wedge dz_n = |\det J_{\mathbb{C}} f|^2\ dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$. □

**Theorem 1.5.16 (Implicit Function Theorem).** *If the functions $f_j(w,z)$, $1 \leq j \leq m$ are holomorphic near $(w^0, z^0)$ where $(w,z) \in \mathbb{C}^m \times \mathbb{C}^n$. Also suppose that $f_j(w^0, z^0) = 0$ for every $1 \leq j \leq m$ and $\det[\frac{\partial f_j}{\partial w_k}(w^0, z^0)] \neq 0$. Then, for the equation $f_j(w,z) = 0$ a solution $w(z)$ in a neighborhood of $z^0$ exists which holomorphic, unique and satisfies $w(z^0) = w^0$.*

*Proof.* Think of this system of equation as the $2m$ system $\Re f_j(w,z) = 0$ and $\operatorname{Im} f_j(w,z) = 0$, then by theorem 1.5.15 determinant of Jacobian of this $2m$ system is non-zero at $(w^0, z^0)$. Implicit function theorem implies that there is a solution $w(z)$ such that $w(z^0) = w^0$ and $f_j(w(z), z) = 0$ where $w \in \mathcal{C}^1$. Hence, we will have $\sum_{k=1}^{m} \frac{\partial f_j}{\partial w_k} dw_k + \sum_{k=1}^{n} \frac{\partial f_j}{\partial z_k} dz_k$. Note that as $f_j$'s are holomorphic we do not have any multiples of $d\bar{w}_j$ or $d\bar{z}_j$, so in the previous equation we can solve for $dw_k$ in terms of $dz_j$ and $w_k$'s are holomorphic as desired. □

**Theorem 1.5.17.** *If $\Omega_1$ and $\Omega_2$ are domains in $\mathbb{C}^d$ and $f : \Omega_1 \to \Omega_2$ is a biholomorphy between them. Then we will have*

$$\det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(\zeta)) \overline{\det J_{\mathbb{C}} f(\zeta)} = K_{\Omega_1}(z, \zeta) \tag{1.36}$$

*Proof.* Let $\phi \in A^2(\Omega_1)$, then we can write that

$$\int_{\Omega_1} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), f(\zeta)) \overline{\det J_{\mathbb{C}} f(\zeta)} \phi(\zeta) dV(\zeta) =$$

$$\int_{\Omega_2} \det J_{\mathbb{C}} f(z) K_{\Omega_2}(f(z), \tilde{\zeta} \overline{\det J_{\mathbb{C}} f(f^{-1}(\tilde{\zeta}))} \phi(f^{-1}(\tilde{\zeta})) \det J_{\mathbb{R}} f^{-1}(\tilde{\zeta}) dV(\tilde{\zeta}) = \phi(z)$$

Now theorem 1.5.10 yields us the desired conclusion.

**Theorem 1.5.18.** *For every $z \in \Omega \subset \mathbb{C}^d$ we have $K_\Omega(z,z) > 0$.*

*Proof.* If $\{\phi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for $A^2(\Omega)$ then $K_\Omega(z,z) = \sum_{j=1}^{\infty} |\phi_j(z)|$ and therefore $K_\Omega(z,z)$ is non-negative. Also if $K_\Omega(z,z) = 0$ for some $z \in \Omega$ then we will have $f(z) = 0$ for every $f \in A^2(\Omega)$ which is a contradiction. □

**Definition 1.5.19 (Bergman Metric).** *Assume that $\Omega \subseteq \mathbb{C}^d$ is a domain and it is bounded we define $g_\Omega(z) = \sum_{j,k=1}^{d} g_{jk}^{\Omega}(z) dz_j \wedge d\bar{z}_k = \sum_{j,k=1}^{d} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_\Omega(z,z) dz_j \wedge d\bar{z}_k$ and let square length of a tangent vector $\zeta = (\zeta_1, \ldots, \zeta_d)$ at a point $z \in \Omega$ be*

$|\zeta|^2_{B,z} = \sum\limits_{j,k} g_{jk}(z)\zeta_j\bar{\zeta}_k$. *So the length of a* $\mathcal{C}^1$ *curve* $\gamma : [0,1] \to \Omega$ *can be defined as* $l(\gamma) = \int\limits_0^1 (\sum\limits_{j,k} g_{jk}(z)\gamma'_j(t)\bar{\gamma}'_k(t))^{\frac{1}{2}} dt$. *For* $P$ *and* $Q$ *as two points of* $\Omega$ *distance with respect to Bergman Metric* $d_\Omega(P,Q)$ *is defined as the infimum of length of all piecewise* $\mathcal{C}^1$ *curves which connect them. Here in this thesis, we will show the open ball with center at* $\zeta \in \Omega$ *and radius* $r$ *with respect to Bergman distance by* $\mathbb{B}_\Omega(\zeta, r)$ *and by* $dV_\Omega$ *we mean the volume form induced by Bergman metric or* $|det[g^\Omega_{jk}]|d\mu$ *where* $\mu$ *is the Lebesgue measure.*

**Remark 1.5.20.** $[g_{jk}(z)]$ *is positive definite for every* $z \in \Omega$ *and therefore the Bergman metric is well-defined.*

**Theorem 1.5.21.** *Let* $f : \Omega_1 \to \Omega_2$ *be a biholomorphy where* $\Omega_1$ *and* $\Omega_2$ *are domains in* $\mathbb{C}^d$. *Then* $f$ *is an isometry of Bergman metric between the two spaces i.e.* $|\zeta|_{B,z} = |(J_{\mathbb{C}}f)\zeta|_{B,f(z)}$ *for every* $z \in \Omega_1$ *and* $\zeta \in \mathbb{C}^d$.

*Proof.* From theorem 1.5.17 we have that

$$\begin{aligned} g^{\Omega_1}_{jk} = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_{\Omega_1}(z,z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log\{|\det J_{\mathbb{C}}f(z)|^2 K_{\Omega_2}(f(z), f(z))\} = \\ \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_{\Omega_2}(f(z), f(z)) = \sum_{l,m} g^{\Omega_2}_{l,m}(f(z)) \frac{\partial f_l(z)}{\partial z_j} \overline{\frac{\partial f_m(z)}{\partial \bar{z}_k}} \end{aligned} \tag{1.37}$$

Where the third equality comes from the fact that

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log|\det J_{\mathbb{C}f(z)}|^2 = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} (\log \det J_{\mathbb{C}}f(z) + \log \overline{\det J_{\mathbb{C}}f(z)}) = 0 \tag{1.38}$$

From (1.37) the statement follows.

**Theorem 1.5.22.** *Let* $\Omega \subset \mathbb{C}^d$ *be a domain and* $z \in \Omega$, *then* $K_\Omega(z,z) = \sup\limits_{||f||_{A^2(\Omega)}=1} |f(z)|^2$.

*Proof.* If $\phi_z$ is the point evaluation functional at $z$, then the definition of Bergman kernel and Riesz's representation theorem imply that $||\phi_z||_{(A^2(\Omega))^*} = ||K_z||_{A^2(\Omega)}$ and the statement follows immediately. □

## 1.6 $\bar{\partial}$-Equation and the Hörmander's Solution

If $\Omega \subseteq \mathbb{C}^d$ is a domain, and $f$ is a $(p, q+1)$ differential form on $\Omega$ $\bar{\partial}$-Problem is about finding a $(p,q)$ form $g$ on $\Omega$ such that $\bar{\partial}g = f$. In section 1.6.1 using a result from unbounded operator's theory we show that the existence of a solution can be reduced to the problem of finding an estimate. In section 1.6.2 some technical preliminaries brought which are necessary to understand Hörmander's approach to the problem [5] in case of $\Omega$ being psuedoconvex and $f$ having $L^2_{loc}$ coefficients. Finally, the general idea of Hörmander discussed in section 1.6.3.

### 1.6.1 Unbounded Operators on Hilbert Spaces

**Definition 1.6.1.** *If $H_1$ and $H_2$ are two Hilbert spaces, $D$ is a dense subset of $H_1$ and $T : D \to H_2$ is a linear operator which is not necessarily bounded. We call $T$ a densely defined operator from $H_1$ to $H_2$ and write that $T : H_1 \to H_2$. We also show the domain of $T$ by $D_T$.*

**Definition 1.6.2.** *A linear operator $T : H_1 \to H_2$ is called closed if its graph $\mathcal{G}_T = \{(x, Tx)\big| x \in D_T\} \subseteq H_1 \times H_2$ is a closed set.*

**Definition 1.6.3.** *If $T : H_1 \to H_2$ is a linear operator, We say $\psi \in H_2$ is in the domain of the adjoint of $T$ and show by $\psi \in D_{T^*}$ if there exists a constant $C > 0$ which only depends on $\psi$ such that*

$$| < T\phi, \psi >_{H_2} | \leq C||\phi||_{H_1} \qquad \forall \phi \in D_T \tag{1.39}$$

**Theorem 1.6.4.** *If $\psi \in D_{T^*}$, then there exists a unique member of $H_1$ named $T^*\psi$ such that*

$$< T\phi, \psi >_{H_2} = < \phi, T^*\psi >_{H_1} \qquad \forall \phi \in D_T \tag{1.40}$$

*Proof.* By the definition of $D_{T^*}$ the functional $\phi \mapsto < T\phi, \psi >_{H_2}$ is densely defined and bounded, so we can extend it to a unique continuous linear functional on $H_1$. Riesz representation theorem ensures us that a unique member of $H_1$ with the desired property for $T^*\psi$ exists. □

**Definition 1.6.5.** *If $H_1$ and $H_2$ are two Hilbert spaces we equip $H_1 \times H_2$ with the inner product $< (h_1, h_2), (h'_1, h'_2) >_{H_1 \times H_2} = < h_1, h'_1 >_{H_1} + < h_2, h'_2 >_{H_2}$ and with this definition $H_1 \times H_2$ will be a Hilbert space.*

**Definition 1.6.6.** *We define $J : H_2 \times H_1 \to H_1 \times H_2$ with $J(h_2, h_1) = (-h_1, h_2)$. One can easily show that $J$ and $J^{-1}$ map closed space to closed spaces.*

**Theorem 1.6.7.** *If $T : H_1 \to H_2$ is a linear operator, then $(\mathcal{G}_T)^\perp = J(\mathcal{G}_{T^*})$.*

*Proof.* If $(-T^*y, y) \in J(\mathcal{G}_{T^*})$, $(x, Tx) \in \mathcal{G}_T$, then $<(-T^*y, y), (x, Tx)>=<-T^*y, x>$ $+<y, Tx>=0$ so $J(\mathcal{G}_{T^*}) \subseteq (\mathcal{G}_T)^\perp$. Conversely ,if $(a, b) \in (\mathcal{G}_T)^\perp$, then for any $x \in D_T$ we have $0=<(a, b), (x, Tx)>=<a, x>+<b, Tx>$. So $|<b, Tx>|=|<a, x>| \leq ||a||||x||$ and $b \in D_{T^*}$ consequently and we can write that $0=<a, x>+<b, Tx>=<a+T^*b, x>$ or simply $T^*b=-a$, therefore $(a, b) \in J(\mathcal{G}_{T^*})$. $\square$

**Corollary 1.6.8.** *$T^*$ is a closed operator.*

**Lemma 1.6.9.** *Let $T : H_1 \to H_2$ be a closed densely defined operator and $F \subseteq H_2$ be a closed subspace such that $RangeT \subseteq F$. Moreover, assume that $D_{T^*}$ is dense. Then $F = RangeT$ if and only if a constant $C > 0$ exists such that*

$$||y||_{H_2} \leq C||T^*y||_{H_1} \qquad \forall y \in F \cap D_{T^*} \tag{1.41}$$

*Proof.* If (1.41) holds. Assume $z \in F$, if $w \in T^*(F \cap D_{T^*})$ and $w = T^*y$, we define $\phi(w) =<y, z>$. $\phi$ will be bounded on $T^*(F \cap D_{T^*})$ by $C||z||$. By Hahn-Banach theorem we can extend $\phi$ to all of $H_2$ and by Riesz representation theorem it has a unique representative $x$. Therefore, $<y, z>_{H_2}= \phi(w) =<w, x>_{H_1}=<T*y, x>_{H_1}$ for every $y \in F \cap D_{T^*}$. By closedness of $T$, we may write $T^{**} = T$ so $z = Tx$. On the other hand, if $F = RangeT$ then we set for each $z \in F$ an $x_z \in H_1$ such that $Tx_z = z$. So for any $f \in F \cap D_{T^*}$ we have $|<f, z>_{H_2}|=|<f, Tx_z>_{H_2}|=|<T^*f, x_z>_{H_1}$ $| \leq ||T^*f||_{H_1}||x_z||_{H_1}$. Therefore, if we think of the set $F \cap D_{T^*} \cap f \in H_2 \big| ||T^*f|| \leq 1$ as linear functionals on $H_2$ the uniform boundedness principle yields that $y$ satisfies (1.41). $\square$

### 1.6.2 Technical Preliminaries

**Definition 1.6.10.** *If $\Omega \subset \mathbb{C}^d$ and $p, q$ are two integers from 0 to 1 then $L^2_{(p,q)}(\Omega)$ is the space of $(p, q)$ forms on $\Omega$ with $L^2(\Omega)$ coefficients. If $f \in L^2_{(p,q)}(\Omega)$ we write that*

$$f = \sum_{|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J \tag{1.42}$$

*The sum is taken over increasing multi-indices $I, J$. We also define the inner product of two members $f$ and $g$ to be $<f, g>= \sum\limits_{I,J} \int\limits_{\Omega} f_{IJ}\bar{g}_{IJ}dV$. $L^2_{(p,q)}(\Omega)$ with this inner product will be a Hilbert space. Also if $\phi : \Omega \to \mathbb{R}$ is a continuous function then $L^2_{(p,q)}(\Omega, \phi)$ is the space of $(p, q)$ forms with square integrable coefficients in the measure $e^{-\phi}dV$.*

**Definition 1.6.11.** *If* $\Omega \subset \mathbb{C}^d$ *,* $D = (\frac{\partial}{\partial z})^I (\frac{\partial}{\partial \bar{z}})^J$ *and* $f, g \in L^1(\Omega)$*. We say that* $Df = g$ *in the weak sense (or weakly in the sense of distributions) if for all* $\phi \in \mathcal{C}_c^\infty$ *we have* $\int\limits_\Omega f D\phi dV = (-1)^{|I|+|J|} \int\limits_\Omega g\phi dV$*.*

If $\Omega \subseteq \mathbb{C}^d$ is a domain, $H_1 = L^2_{(p,q)}(\Omega, \phi_1)$, $H_2 = L^2_{(p,q+1)}(\Omega, \phi_2)$ $H_3 = L^2_{(p,q+2)}(\Omega, \phi_3)$, $T \equiv \bar{\partial} : H_1 \to H_2$, $S \equiv \bar{\partial} : H_2 \to H_3$ and $F = \ker S \subseteq H_2$. Also, $f \in D_T$ means that $f \in H_1$ and $\bar{\partial} f$ exists in the weak sense. We will prove that if $\phi_1,\ \phi_2,\ \phi_3$ are chosen properly and $\Omega$ is psuedoconvex we have $C > 0$ such that.

$$||f||^2_{H_2} \leq C^2\{||T^*f||^2_{H_1} + ||Sf||^2_{H_3}\} \quad \forall f \in D_{T^*} \cap D_S \tag{1.43}$$

Note that this means that $f$ also satisfies (1.41) whenever $f \in D_{T^*} \cap F$ and by lemma 1.6.9 $F = \ker S$ is exactly the set for its members $f$ the equation $\bar{\partial} g = f$ has a solution $g \in D_T$. Lemma 1.6.15 simplifies the existence problem even further as it proves for suitably chosen $\phi_i$ $i = 1, 2, 3$ it is enough to prove that the inequality (1.41) holds whenever $f \in D_{(p,q+1)}(\Omega)$.

**Lemma 1.6.12.** *Consider* $\mathcal{X}$ *to be a function in* $\mathcal{C}_0^\infty(\mathbb{R}^N)$ *such that* $\int \mathcal{X} dV = 1$ *and* $\mathcal{X}_\epsilon(x) = \epsilon^{-N}\mathcal{X}(x/\epsilon),\ x \in \mathbb{R}^N$*. If* $g \in L^2(\mathbb{R}^N)$ *then* $g * \mathcal{X}_\epsilon(x) = \int g(y)\mathcal{X}_\epsilon(x-y)dy = \int g(x-\epsilon y)\mathcal{X}_\epsilon(y)dy$ *is a* $\mathcal{C}^\infty$ *in a way that* $||g * \mathcal{X}_\epsilon - g||_{L^2} \to 0$ *whenever* $\epsilon \to 0$*. Distance of the support of* $g * \mathcal{X}_\epsilon$ *from the support of* $g$ *is not greater than* $\epsilon$ *if the support of* $\mathcal{X}$ *is inside* $\mathbb{B}(0,1)$*.*

**Lemma 1.6.13.** *if* $f = \sum\limits_{|I|=p,|J|=q+1} f_{IJ} dz^I \wedge d\bar{z}^J \in D_{T^*}$ *then*

$$T^*f = (-1)^{p-1} \sum_{|I|=p,|K|=q} \sum_{j=1}^n e^{\phi_1} \frac{\partial(e^{-\phi_2} f_{I,jK})}{\partial z_j} dz^I \wedge d\bar{z}^K \tag{1.44}$$

*Where* $f_{I,jK} = \sum\limits_{|J|=q+1} f_{IJ}\epsilon^J_{jk}$ *and* $\epsilon^J_{jk}$ *is the sign of permutation between* $J$ *and* $jK$ *if it exists and* $0$ *otherwise.*

**Corollary 1.6.14.**

$$\begin{aligned} e^{\phi_2-\phi_1}T^*f &= \\ (-1)^{p-1} \sum_{|I|=p,|K|=q} \sum_{j=1}^n \frac{\partial f_{I,jK}}{\partial z_j} dz^I \wedge d\bar{z}^K &- \\ (-1)^{p-1} \sum_{|I|=p,|K|=q} \sum_{j=1}^n \frac{\partial \phi_2}{\partial z_j} f_{I,jK} dz^I \wedge d\bar{z}^K & \\ &\equiv \mathcal{T}^*f + \mathcal{A}f \end{aligned} \tag{1.45}$$

*Where $\mathcal{A}$ is an operator obtained by multiplication to a $\mathcal{C}^\infty$ function and $\mathcal{T}^*$ is a first order partial differential operator.*

**Lemma 1.6.15.** *Let $\eta_m$ $m \in \mathbb{N}$ be a sequence in $\mathcal{C}_0^\infty(\Omega)$ in a way that $0 \leq \eta_m \leq 1$ and $\eta_m = 1$ on any compact subset of $\Omega$ when $m$ is large enough and each $\eta_m$ has compact support. Assume that $\phi_2 \in \mathcal{C}^1(\Omega)$ and that $e^{-\phi_{j+1}} \sum_{k=0}^{n} |\frac{\partial \eta_m}{\partial \bar{z}_k}|^2 \leq e^{-\phi_j}$ $j = 1, 2;$ $m \in \mathbb{N}$. Then $D_{(p,q+1)}(\Omega)$ is dense in $D_{T^*} \cap D_S$ with the graph norm $||f||_{\mathcal{G}} = ||f||_{H_2} + ||T^* f||_{H_1} + ||Sf||_{H_3}$.*

*Proof.* If $f \in D_S$ then $S(\eta_m f) - \eta_m Sf = \bar{\partial}\eta_m \wedge f$ by using the assumption we have $e^{\phi_3}|S(\eta_m f) - \eta_m Sf|^2 \leq |f|^2 e^{-\phi_2}$. Now the dominated convergence theorem combined with the fact that $\eta_m = 1$ when $m$ is large enough yield that $||S(\eta_m f) - \eta_m Sf||_{H_3} \to 0$ whenever $m \to 0$. Now we claim that if $f \in D_{T^*}$ and $\eta \in \mathcal{C}_0^\infty(\Omega)$ then $\eta f \in D_{T*}$ as we have $< \eta f, Tu >_{H_2} =< f, T(\bar{\eta} u) >_{H_2} + < f, \bar{\eta} Tu - T(\bar{\eta} u) >_{H_2} =< \eta T^* f, u >_{H_1} + < \eta f, -\bar{\partial} \wedge u >_{H_2}$ where $u \in D_T$ and the second equality tells us that $< \eta f, Tu >_{H_2}$ can be written in form of a anti-linear functional in terms of $u$ which is continuous with respect to the norm $||u||_{H_1}$ so by Riesz representation theorem there exists a $v \in H_1$ such that $< v, u >_{H_1} =< \eta f, Tu >_{H_2}$ and the claim is proved. Using the fact we just proved we can write that $< T^*(\eta_m f) - \eta_m T^* f, u >_{H_1} =< f, \bar{\eta}_m Tu - T(\bar{\eta}_m u) >_{H_2}$ and again by using the lemma's assumption $| < T^*(\eta_m f) - \eta_m T^* f, u >_{H_1} | \leq \int |f| e^{-\phi_2/2} |u| e^{-\phi_1/2} dV$ and consequently $|T^*(\eta_m f) - \eta_m T^* f|^2 e^{-\phi_1} \leq |f|^2 e^{-\phi_2}$. By applying the dominated convergence theorem again we reach to the conclusion that $||T^*(\eta_m f) - \eta_m T^* f||_{H_1} \to 0$ as $m \to \infty$. Therefore if $f \in D_{T^*} \cap D_S$ $\eta_m f \to f$ in the graph norm, note that this means that to prove the statement we only need to prove that for every $f \in D_{T^*} \cap D_S$ with compact support we can find a sequence $f_n$ of members of $D_{(p,q+1)}$ which is convergent to $f$ in the graph norm. So let $f \in D_{T^*} \cap D_S$ and have compact support, define $f_n = f * \mathcal{X}_{\frac{1}{n}}$ as in lemma 1.6.12 by performing the convolution on each coefficient of $f$. Lemma 1.6.12 also tells us that for large enough $n$ support of all $f_n$'s lie in a fixed compact set and so $||f - f_n||_{H_2} \to 0$. Also it is easy to check that $Sf_n = (Sf) * \mathcal{X}_{\frac{1}{n}}$ and therefore $||Sf - Sf_n|| \to 0$ as $n \to \infty$. With the notation of equation (1.45) it is easy to show that $(\mathcal{T}^* + \mathcal{A}) f_n = (\mathcal{T}^* + \mathcal{A}) f * \mathcal{X}_{\frac{1}{n}} + \mathcal{A} f_n - (\mathcal{A} f) * \mathcal{X}_{\frac{1}{n}}$ but the right hand side converges to $(\mathcal{T}^* + \mathcal{A}) f$ as $n \to \infty$ and $||T^* f_n - T^* f||_{H_1} \to 0$ consequently which means that $f_n \to f$ in the graph norm and the proof is complete. □

**Remark 1.6.16.** *If we let $\psi \in \mathcal{C}^\infty$ to be such that $|\bar{\partial}\eta_m|^2 \leq e^\psi$ for all $m \in \mathbb{N}$ then for any choice of $\phi \in \mathcal{C}^\infty$, $\phi_1 = \phi - 2\psi, \phi_1 = \phi - \psi, \phi_3 = \phi$ satisfy conditions of the lemma 1.6.15.*

### 1.6.3 Hörmander's Existence Theorem

Based on what we have proved in the last section we only need to find a proper choice for $\phi$ which we can show the existence of solution for the $\bar{\partial}$-Problem. Here the main idea is that for a psuedoconvex domain we can find such a $\phi$.

**Lemma 1.6.17.** *If $\Omega \subset \mathbb{C}^d$ is a psuedoconvex domain then a $\mathcal{C}^\infty$, $\phi : \Omega \to \mathbb{R}$ exists in a way that $\sum_{j,k=1}^{d} \frac{\partial^2 \phi}{\partial z_j \bar{\partial}} w_j \bar{w}_k \geq 2(|\bar{\partial}|^2 + e^{\psi}) \sum_{j=1}^{d} |w_j|^2$ for every $w \in \mathbb{C}^d$.*

**Lemma 1.6.18.** *For a psuedoconvex $\Omega \subseteq \mathbb{C}^d$ if we choose $\phi$ as in lemma 1.6.17 and $\psi, \phi_1, \phi_2, \phi_3$ as in the remark 1.6.16 then for every $f \in D_(p, q+1)(\Omega)$ we have $||f||^2_{H_2} \leq C(||T^* f||^2_{H_1} + ||Sf||^2_{H_3})$ where $C$ is a positive constant.*

**Theorem 1.6.19.** *Let $f$ be a $(p, q+1)$ form with $L^2_{loc}$ coefficients satisfying $\bar{\partial} f = 0$ in the weak sense, then a $(p, q)$ form $u$ on $\Omega$ with $L^2_{loc}$ coefficient exists in a way that $\bar{\partial} u = f$ in the weak sense.*

*Proof.* It implies from the assumptions of the theorem that $\tilde{\phi} : \Omega \to \mathbb{R}$ exists in a way that $f \in L^2_{(p,q+1)}(\Omega, \tilde{\phi})$. We can find a $\phi$ with the condition of the lemma 1.6.17 and large enough such that $\phi - \psi \geq \tilde{\phi}$, where in the notation of remark 1.6.16 this means that $f \in L_(p, q+1)^2(\Omega, \phi_2)$ as well. Applying lemmas 1.6.18 and 1.6.9 we can find a solution $u \in L^2_{(p,q)}(\Omega, \phi_1)$ to the problem which also means that coefficients of u are in $L^2_{loc}$ as well. $\square$

# Chapter 2

# Hankel Operators on Domains with Bounded Intrinsic Geometry

Domains with bounded intrinsic are first introduced to the literature by A. Zimmer in [6], later in [7] he derived a necessary and sufficient condition on the symbol function for compactness of the Hankel operator. Here we brought an explanatory review of this work.

## 2.1 Preliminaries

**Definition 2.1.1** (**Approximate Inequalities**). *Consider $f, g : \Omega \to \mathbb{R}$, we denote $f \lesssim g$ or equivalently $g \gtrsim f$ if a constant $C > 0$ exists such that $f(z) \leq Cg(z)$ for every $z \in \Omega$. Also if $f \gtrsim g$ and $g \gtrsim f$ we will write $f \asymp g$.*

**Definition 2.1.2** (**Levi form**). *Assume that $\Omega$ is a domain in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{R}$ is $\mathcal{C}^2$. Levi form of $f$ would be defined as follows*

$$\mathcal{L}(f) = \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \tag{2.1}$$

**Remark 2.1.3.** *$f$ is plurisubharmonic if $\mathcal{L}(f)$ is a semi-positive $(1, 1)$ form.*

**Remark 2.1.4.** *It implies from the definition that $\mathcal{L}(\log K_\Omega(z, z)) = g_\Omega$.*

**Definition 2.1.5** (**Functions with Self Bounded Gradient**). *We say that a $\mathcal{C}^2$ plurisubharmonic function $\lambda : \Omega \to \mathbb{R}$ has self bounded gradient if $||\partial\lambda(z)||_{\mathcal{L}(\lambda(z))} =$*

$\sup\{|(\partial\lambda(z))(X)|\big|X \in \mathbb{C}^d, \mathcal{L}(\lambda(z))(X,X) \le 1\}$ *is uniformly bounded on* $\Omega$. *Or equivalently, there exists a constant* $C > 0$ *such that* $|\sum\limits_{j=1}^{d} \frac{\partial\lambda}{\partial z_j} X_j|^2 \le C \sum\limits_{j,k=1}^{d} \frac{\partial^2\lambda}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k$ *for all* $X \in \mathbb{C}^d$.

**Definition 2.1.6 (Hankel Operator).** *For* $\Omega \subset \mathbb{C}^d$ *being a bounded domain and* $\phi \in L^2(\Omega)$ *as the symbol function its associated Hankel operator is defined as* $H_\phi(f) = \phi.f - P_\Omega(\phi.f) = (i - P_\Omega)(\phi.f)$ *where* $f \in dom(H_\phi) = \{f \in A^2(\Omega)\big|\phi.f \in L^2(\Omega)\}$.

**Definition 2.1.7 (Domain with Bounded Intrinsic Geometry).** *Domain* $\Omega \subset \mathbb{C}^d$ *is called a domain with bounded intrinsic geometry if there exists Kahler metric* $g$ *on it such that*

1. $g$ *has bounded sectional curvature and positive injectivity radius.*
2. *There exists* $\lambda : \Omega \to \mathbb{R}$ *which is* $\mathcal{C}^2$*, its Levi form is bi-Lipschitz to* $g$ *and* $||\partial\lambda||_g$ *is bounded on* $\Omega$.

The following theorem is an immediate corollary of the theorem 4.5 of [8] and the definition of a domain with bounded intrinsic geometry.

**Theorem 2.1.8.** *Assume that* $\Omega \in \mathbb{C}^d$ *is bounded domain with bounded intrinsic geometry, then one may find* $C > 0$ *such that for plurisubharmonic* $\phi_2 : \Omega \to \{-\infty\} \cup \mathbb{R}$ *and* $\alpha \in L^{2,loc}_{(0,1)}(\Omega)$ *with* $\bar{\partial}\alpha = 0$, $u \in L^{2,loc}(\Omega)$ *exists such that* $\bar{\partial}u = \alpha$ *and*

$$\int\limits_{\Omega} |u|^2 e^{-\phi_2} d\mu \le C \int\limits_{\Omega} ||\alpha||^2_{g_\Omega} e^{-\phi_2} d\mu < +\infty \tag{2.2}$$

**Theorem 2.1.9.** *Assume that* $\Omega \in \mathbb{C}^d$ *is a bounded domain having bounded intrinsic geometry. Then for every* $r > 0$ *we can find a sequence* $(\zeta_m)_{m\in\mathbb{N}}$ *in* $\Omega$ *such that:*
*(1) If* $i \neq j$ *then* $d_\Omega(\zeta_i, \zeta_j) \ge r$.
*(2)* $\Omega = \bigcup\limits_{m\in\mathbb{N}} \mathbb{B}_\Omega(\zeta_m, r)$.
*(3) For any* $R > 0$ *we have* $\sup\limits_{z\in\Omega} \#\{m\big|\zeta_m \in \mathbb{B}_\Omega(z,R)\} < +\infty$.

We will accept the following theorem, which is a deep result of combining [9],[10] and theorem 3.2 of [6], without proof.

**Theorem 2.1.10.** *Assume that* $\Omega \in \mathbb{C}^d$ *is a bounded domain with bounded intrinsic geometry and* $\zeta \in \Omega$ *an embedding* $\mathfrak{N}_\zeta : \mathbb{B} \to \Omega$ *exists which is holomorphic in a way that* $\mathfrak{N}_\zeta(0) = \zeta$ *and* $g_{EUC} \asymp \mathfrak{N}^*_\zeta g_\Omega$ *on* $\mathbb{B}$.

**Corollary 2.1.11.** *$C_1 > 0$ exists such that $\mathbb{B}_\Omega(\zeta, \frac{r}{C_1}) \subset \mathfrak{N}_\zeta(r\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta, C_1 r)$ whenever $r < \frac{1}{C_1}$.*

**Theorem 2.1.12.** *The function $\beta_\zeta : \mathbb{B} \times \mathbb{B} \to \mathbb{C}$, with the definition $\beta_\zeta(u,w) = K_\Omega(\mathfrak{N}_\zeta(u), \mathfrak{N}_\zeta(w)) \det \mathfrak{N}'_\zeta(u) \overline{\det \mathfrak{N}'_\zeta(w)}$ has the following properties.*
*(1) $C_2 > 1$ exists in a way that $\frac{1}{C_2} \leq \beta_\zeta(w,w) \leq C_2$ whenever $\zeta \in \Omega$ and $w \in \mathbb{B}$.*
*(2) For every $0 < \delta < 1$ and multi-indices $I, J$ there exists $C_{\delta,I,J} > 0$ in a way that $|\frac{\partial^{|I|+|J|}\beta_\zeta(u,w)}{\partial u^I \partial \bar{w}^J}| \leq C_{\delta,I,J}$ for all $\zeta \in \Omega$ and $u, w \in \delta\mathbb{B}$.*

**Theorem 2.1.13.** *There exists $r_0 > 0$ and $C_3 > 1$ such that, if $\zeta \in \Omega$ and $z \in \mathfrak{N}_\zeta(r_0\mathbb{B})$ then*

$$\frac{1}{C_3} K_\Omega(z,z) K_\Omega(\zeta,\zeta) \leq |K_\Omega(z,\zeta)|^2 \leq C_3 K_\Omega(z,z) K_\Omega(\zeta,\zeta).$$

*Proof.* From the first part of the theorem 2.1.12 we have that

$$\frac{1}{C_2} \leq \beta_\zeta(u,u) \leq C_2$$

For all $u \in \mathbb{B}$ and $u = 0$ in particular. Also fixing $0 < \delta < 1$ we will have $|\frac{\partial \beta_\zeta}{\partial u}(u,0)| < K$ when $u \in \delta\mathbb{B}$ for some $K > 0$ which only depends on $\delta$, as a result of second part of theorem 2.1.12. Then for $r_0 < \delta$ and $u \in r_0\mathbb{B}$ we can write that $\frac{1}{C_2} - r_0 K < |\beta_\zeta(u,0)| < C_2 + r_0 K$. So an appropriate choice of $r_0$ will lead us to the inequality

$$\frac{1}{2C_2^2} \leq |\beta_\zeta(u,0)|^2 \leq 2C_2^2 \qquad u \in r_0\mathbb{B}$$

Now let $z = \mathfrak{N}_\zeta(u)$ where $u \in r_0\mathbb{B}$, we can write that

$$|K_\Omega(z,\zeta)|^2 = |\beta_\zeta(u,0)|^2 |\det \mathfrak{N}'_\zeta(u)|^{-2} |\det \mathfrak{N}'_\zeta(0)|^{-2}$$

Now the choice of $2C_2^4$ for $C_3$, yields us the desired result. □

**Theorem 2.1.14.** *Let $u : \Omega \to [0\ \infty)$ is such that $\log(u)$ is plurisubharmonic and $r < C_1$, then there exists $C_4$ such that $u(\zeta) \leq \frac{C_4}{r^{2n}} K_\Omega(\zeta,\zeta) \int_{\mathbb{B}(\zeta,r)} u d\mu$ for all $\zeta \in \Omega$.*

*Proof.* By theorem 2.1.10 we have that $\mathfrak{N}_\zeta(\frac{r}{C_1}\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta, r)$, therefore

$$\int_{\frac{r}{C_1}\mathbb{B}} (u \circ \mathfrak{N}_\zeta)(w) |\det \mathfrak{N}'_\zeta(w)|^2 d\mu(w) \leq \int_{\mathbb{B}_\Omega(\zeta,r)} u d\mu$$

Plurisubharmonicity of $\log(u)$ along with holomorphicity of $\mathfrak{N}_\zeta$ now tells us that $\log(u)\circ\mathfrak{N}_\zeta+\log|\det\mathfrak{N}'_\zeta|^2$ is plurisubharmonic so is its exponential $(u\circ\mathfrak{N}_\zeta).|\det\mathfrak{N}'_\zeta|^2$. Mean-value theorem now can be used to deduce that

$$u(\zeta)|\det\mathfrak{N}'_\zeta(0)|^2 \leq \frac{C_1^{2n}}{r^{2n}\mu(\mathbb{B})}\int\limits_{\mathbb{B}_\Omega(\zeta,r)} ud\mu$$

Now by theorem 2.1.12 we will have:

$$u(\zeta) \leq \frac{C_1^{2n}C_2}{r^{2n}\mu(\mathbb{B})}K_\Omega(\zeta,\zeta)\int\limits_{\mathbb{B}_\Omega(\zeta,r)} ud\mu$$

And therefore $C_4 = \dfrac{C_1^{2n}C_2}{\mu(\mathbb{B})}$ works for our purpose. □

**Theorem 2.1.15.** *For $r < C_1$ and $\zeta \in \Omega$ we have*

$$\int\limits_\Omega |K_\Omega(z,\zeta)|^2 d\mu(z) \leq \frac{C_4}{r^{2n}}\int\limits_{\mathbb{B}_\Omega(\zeta,r)} |K_\Omega(z,\zeta)|^2 d\mu(z)$$

*Proof.* In theorem 2.1.14 let $u = |K_\Omega(.,\zeta)|^2$ and note that $K_\Omega(\zeta,\zeta) = \int\limits_\Omega |K_\Omega(z,\zeta)|^2 d\mu(z)$.

□

**Theorem 2.1.16.** *For every $z \in \Omega$ we have $K_\Omega(z,z)d\mu(z) \asymp dV_\Omega(z)$ i.e we can find $C_5 > 1$ such that*

$$\frac{1}{C_5}K_\Omega(z,z)d\mu(z) \leq dV_\Omega(z) \leq C_5 K_\Omega(z,z)d\mu(z)$$

*for all $z \in \Omega$.*

*Proof.* Theorem 2.1.10 implies that

$$\frac{1}{C_1^{2d}} \leq |\det[(\mathfrak{N}_\psi^* g_\Omega)]| \leq C_1^{2d}$$

And

$$\frac{1}{C_1^{2d}}|\det\mathfrak{N}'_\psi(0)|^{-2} \leq |\det[(\mathfrak{N}_\psi^* g_\Omega)]| \leq C_1^{2d}|\det\mathfrak{N}'_\psi(0)|^{-2}$$

Now if we let $C_5 = C_1^{2d}C_2$ by theorem 2.1.12 we will have

$$\frac{1}{C_5}K_{\Omega(z,z)}d\mu(z) \leq dV_\Omega(z) \leq C_5 K_{\Omega(z,z)}d\mu(z)$$

as desired. □

**Definition 2.1.17.** *Let $\Omega \subseteq \mathbb{C}^d$ for every $\zeta \in \Omega$ we define $s_\zeta(z) = \dfrac{1}{\sqrt{K_\Omega(\zeta,\zeta)}} K_\Omega(z,\zeta)$. Note that $s_\zeta \in A^2(\Omega)$ and $||s_\zeta||_2 = 1$.*

**Theorem 2.1.18.** *If $\zeta_m \to \partial\Omega$, then $s_{\zeta_m}$ converges locally uniformly to the zero function.*

*Proof.* Let $\{\phi_j\}$ be an orthonormal basis for $A^2(\Omega)$ then by Holder's inequality and 1.5.12 we may write that

$$||s_{\zeta_m}(z)|| \leq (\sum_{j=0}^{\infty} |\phi_j(z)|^2)^{\frac{1}{2}} = \sqrt{K_\Omega(z,z)} \quad \forall z \in \Omega$$

Applying Montel's theorem we may pass to a subsequence $s_{\zeta_m}$ locally uniformly convergent to a holomorphic function $f$. Let us assume that $f$ is nonzero. Fatou's Lemma implies that $\int\limits_\Omega |f|^2 d\mu \leq 1$. Let $K_m$ be a sequence of compact sets in $\Omega$ in a way that $\int\limits_{K_m} |f|^2 d\mu \to \int\limits_\Omega |f|^2 d\mu$ again by passing to a subsequence of $\{s_{\zeta_m}\}$ we may assume that $\int \lim\limits_{K_m} |s_{\zeta_m} - f|^2 d\mu = 0$ so we will have

$$\limsup_{m\to\infty} ||s_{\zeta_m} - f||_2 = \limsup_{m\to\infty} ||(s_{\zeta_m} - f)\mathcal{X}_{\Omega\setminus K_m}||_2 \leq ||s_{\zeta_m}\mathcal{X}_{\Omega\setminus K_m}||_2 + ||f\mathcal{X}_{\Omega\setminus K_m}||_2 = 1 - ||f||_2$$

Choosing $r < C_1$, we may apply theorem 2.1.14 on $|f|^2$ to write that

$$|f(\zeta_m)| \leq \frac{C_4}{r^{2d}} \sqrt{K_\Omega(\zeta_m,\zeta_m)} (\int\limits_{\mathbb{B}(\zeta_m,r)} |f|^2 d\mu)^{\frac{1}{2}} \quad \forall m \geq 1$$

As the Bergman metric is proper and $\zeta_m \to \partial\Omega$ for any compact $K \subset \Omega$ the set $\mathbb{B}_\Omega(\zeta_m, r) \cap K$ will be empty for large enough $m$, hence $\lim\limits_{m\to\infty} \int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |f|^2 d\mu = 0$ and $\lim\limits_{m\to\infty} \dfrac{|f(\zeta_m)|}{\sqrt{K_\Omega(\zeta_m,\zeta_m)}} = 0$. This means that for large $m$ we have $||h_m(\zeta_m)||_2 < 1$ and $|h_m(\zeta_m)| > \sqrt{K_\Omega(\zeta_m,\zeta_m)}$ which is a contradiction due to the theorem 1.5.22.

**Definition 2.1.19.** *Let $\phi \in L^2(\Omega)$, we define the multiplication operator $M_\phi$ as $M_\phi(f) = \phi.f$. Note that its domain would be $dom(M_\phi) = \{f \in A^2(\Omega) : \phi.f \in L^2(\Omega)\}$.*

## 2.2 Multiplication Operator

**Theorem 2.2.1.** *Let $\Omega \subset \mathbb{C}^d$ be a domain with bounded intrinsic geometry and $\phi \in L^2(\Omega)$. Then:*

1. *The following are equivalent:*

   (a) $\exists r > 0$ *such that* $\sup_{\zeta\in\Omega} \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi|^2 dV_\Omega < +\infty$

   (b) $dom(M_\phi) = A^2(\Omega) and M_\phi : A^2(\Omega) \to L^2(\Omega)$ *is bounded.*

2. *The following are equivalent:*

   (a') $\exists r > 0$ *such that* $\lim_{\zeta\to\partial\Omega} \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi|^2 dV_\Omega = 0$

   (b') $dom(M_\phi) = A^2(\Omega) and M_\phi : A^2(\Omega) \to L^2(\Omega)$ *is compact.*

*Proof.*

(i) $(b') \Rightarrow (a')$:

By theorems 2.1.13 and 2.1.16 every $\zeta \in \Omega$ we can find $r > 0$ and $C > 0$ such that

$$\frac{1}{C} dV_\Omega \leq |s_\zeta|^2 d\mu \leq C dV_\Omega$$

on $\mathbb{B}_\Omega(\zeta, r)$. Let $\zeta_m$ be a sequence such that $\zeta_m \to \partial\Omega$ and

$$\lim_{m\to\infty} \int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega = \limsup_{\zeta\to\partial\Omega} \int\limits_{\mathbb{B}_\Omega(\zeta,r)} |\phi|^2 dV_\Omega$$

By theorem 2.1.18 $s_{\zeta_m}$ converges locally uniformly to 0, so it converges weakly to 0. By compactness of $M_\phi$ we may conclude that

$$\lim_{m\to\infty} \int\limits_\Omega |\phi . s_{\zeta_m}|^2 d\mu = 0$$

And

$$\lim_{m\to\infty} \int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega \leq C \lim_{m\to\infty} \int\limits_\Omega |\phi . s_{\zeta_m}|^2 d\mu = 0$$

As desired

(ii) $(b) \Rightarrow (a)$:

Let $\zeta_m$ be a sequence such that

$$\lim_{m\to\infty} \int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega = \sup_{\zeta\in\Omega} \int\limits_{\mathbb{B}_\Omega(\zeta,r)} |\phi|^2 dV_\Omega$$

By a similar argument from the previous part we have

$$\lim_{m\to\infty}\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)}|\phi|^2 dV_\Omega \leq C\lim_{m\to\infty}\int\limits_{\Omega}|\phi.s_{\zeta_m}|^2 d\mu < \infty$$

As $M_\phi$ is bounded and $||s_{\zeta_m}||_2 = 1$ for all $m \geq 1$.

(iii) $(a) \Rightarrow (b)$:

Let $f \in A^2(\Omega)$ and $\{\zeta_m\}$ be a sequence of distinct points in $\Omega$ satisfying conditions in 2.1.9 i.e.

(1) If $i \neq j$ then $d_\Omega(\zeta_i, \zeta_j) \geq r$.

(2) $\Omega = \bigcup\limits_{m\in\mathbb{N}} \mathbb{B}_\Omega(\zeta_m, r)$.

(3) $L = \sup\limits_{z\in\Omega} \#\{m : \zeta_m \in \mathbb{B}_\Omega(z, 2r)\} < +\infty$.

If we apply 2.1.14 on $|f|^2$ we will have that for $z \in \mathbb{B}_\Omega(\zeta_m, r)$

$$|f(z)|^2 \lesssim K_\Omega(z,z)\int\limits_{\mathbb{B}_\Omega(z,r)}|f|^2 d\mu \lesssim K_\Omega(z,z)\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)}|f|^2 d\mu$$

Theorem 2.1.16 implies:

$$\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)}|\phi.f|^2 d\mu \lesssim (\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)}|\phi|^2 K_\Omega(z,z)d\mu)(\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)}|f|^2 d\mu) \lesssim$$
$$(\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)}|\phi|^2 dV_\Omega)(\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)}|f|^2 d\mu) \lesssim (\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)}|f|^2 d\mu)$$

Therefore

$$\int\lim_{\Omega}|\phi.f|^2 d\mu \lesssim \sum\lim_{m}\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)}|\phi.f|^2 d\mu \lesssim \sum\lim_{m}\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)}|f|^2 d\mu \leq L\int\limits_{\Omega}|f|^2 d\mu$$

So $dom(M_\phi) = A^2(\Omega)$ and $M_\mathfrak{N}$ is bounded

(iv) $(a') \Rightarrow (b')$:

Let $\{f_n\}$ be a sequence of weakly convergent to 0 unit vectors in $A^2(\Omega)$. We will show that $M_\phi(f_n)$ converges strongly to 0 which is a sufficient condition for compactness of $M_\phi$. Same as the previous part we may take a sequence $\{\zeta_m\}$ of distinct points in $\Omega$ such that:

(1) If $i \neq j$ then $d_\Omega(\zeta_i, \zeta_j) \geq r$.

(2) $\Omega = \bigcup\limits_{m \in \mathbb{N}} \mathbb{B}_\Omega(\zeta_m, r)$.

(3) $L = \sup\limits_{z \in \Omega} \#\{m : \zeta_m \in \mathbb{B}_\Omega(z, 2r)\} < +\infty$.

Here note that $\zeta_m \to \partial\Omega$ as a consequence of above properties. By a similar argument to the previous part

$$\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi.f_n|^2 d\mu \lesssim (\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega)(\int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)} |f_n|^2 d\mu)$$

Given $\epsilon > 0$, as $\lim\limits_{\zeta \to \partial\Omega} \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi|^2 dV_\Omega = 0$ we may find $M > 0$ such that

$$\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega < \epsilon \quad \forall m > M$$

$\{f_n\}$ converges to 0 weakly so it does converge to 0 locally uniformly as well. This allows us to write

$$\lim_{n\to\infty} \sum_{m \leq M} \int\limits_{\mathbb{B}_\Omega(\zeta_m,2r)} |f_n|^2 d\mu = 0$$

So

$$\limsup_{n\to\infty} \int\limits_{\Omega} |\phi.f_n|^2 d\mu \leq \limsup_{n\to\infty} \int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi.f_n|^2 d\mu \lesssim$$

$$\limsup_{n\to\infty} \sum_m (\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega)(\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |f_n|^2 d\mu)$$

$$= \limsup_{n\to\infty} \sum_{m>M} (\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi|^2 dV_\Omega)(\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |f_n|^2 d\mu) \leq \limsup_{n\to\infty} \sum_{m>M} \epsilon(\int\limits_{\mathbb{B}_\Omega(\zeta_m,r)} |f_n|^2 d\mu) \leq \epsilon L.$$

So $M_\phi(f_n)$ strongly converges to 0 as desired.

## 2.3 Hankel Operator with $\mathcal{C}^1$-Smooth Symbol

**Theorem 2.3.1.** *Let $\Omega \subset \mathbb{C}^d$ be a domain with bounded intrinsic geometry and $\phi \in \mathcal{C}(\Omega) \cap \mathcal{S}(\Omega)$ then:*

*(1) If* $r > 0$ *exists in a way that*

$$\sup_{\zeta \in \Omega} \int\limits_{\mathbb{B}_\Omega(\zeta,r)} ||\bar{\partial}\phi||^2_{g\Omega} dV_\Omega < +\infty$$

$H_\phi$ *extends to a bounded operator on* $A^2(\Omega)$.

*(2) If there exists* $r > 0$ *such that*

$$\lim_{\zeta \to \partial\Omega} \int\limits_{\mathbb{B}_\Omega(\zeta,r)} ||\bar{\partial}\phi||^2_{g\Omega} dV_\Omega = 0$$

$H_\phi$ *extends to a compact operator on* $A^2(\Omega)$.

*Proof.* Define $M : A^2(\Omega) \to L^2(\Omega)$ as $M(f) = ||\bar{\partial}\phi||_{g\Omega}.f$. Let $f \in dom(H_\phi)$ it follows from the definition that

$$||H_\phi(f)||_2 = \min_{h \in A^2(\Omega)} ||f\phi - h||_2$$

Theorem 2.1.8 also implies that $C > 0$ and some $u \in L^2(\Omega)$ exist such that $\bar{\partial}u = f\bar{\partial}\phi$ and

$$\int\limits_\Omega |u|^2 d\mu \leq C \int\limits_\Omega |f|^2 ||\bar{\partial}\phi||^2_{g\Omega} = C||M(f)||_2^2$$

Take $h = f\phi - u \in A^2(\Omega)$ we may write that

$$||H_\phi(f)||_2 \leq ||f\phi - h||_2 = ||u||_2 \leq \sqrt{C}||M(f)||_2$$

And the result follows immediately using theorem 2.2.1. $\square$

## 2.4 Hankel Operator

**Theorem 2.4.1.** *Let* $\Omega \subset \mathbb{C}^d$ *be a bounded domain with bounded intrinsic geometry and* $\phi \in \mathcal{S}(\Omega)$. *Then the following are equivalent:*

*(1)* $H_\phi$ *extends to a compact operator*

*(2) There exists* $r > 0$ *such that:*

$$\liminf_{\zeta \to \partial\Omega} \{ \int\limits_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 dV_\Omega : h \in \mathcal{O}(\mathbb{B}_\Omega(\zeta, r))\} = 0$$

*Proof.* (i) (1) $\Rightarrow$ (2): Let $\{\zeta_m\}$ be a sequence converging to $\partial\Omega$. Theorem 2.1.18 implies that $s_{\zeta_m}$ converges weakly to zero. $H_\phi$ is compact so

$$\lim_{m\to\infty} ||H_\phi(s_{\zeta_m})|| = 0$$

Theorem 2.1.13 ensures us that we can find $r > 0$ and $C > 1$ in a way that for every $\zeta \in \Omega$ and $z \in \mathbb{B}_\Omega(\zeta, r)$ we have

$$\frac{1}{C}|K_\Omega(z,\zeta)|^2 \leq K_\Omega(z,z)K_\Omega(\zeta,\zeta) \leq C|K_\Omega(z,\zeta)|^2$$

Perhaps by increasing $C$ and using theorem 2.1.16 we may also assume that on every $\mathbb{B}_\Omega(\zeta, r)$.

$$\frac{1}{C}dV_\Omega \leq |s_\zeta(z)|^2 d\mu \leq C dV_\Omega$$

So we may conclude that $s_{\zeta_m}$ is non-vanishing on $\overline{\mathbb{B}_\Omega(\zeta_m, r)}$. As $dom(H_\phi)$ is dense in $A^2(\Omega)$ for all $m$ we may find a sequence $\{f_{m,k}\}_{k\in\mathbb{N}}$ in $dom(H_\phi)$ converging to $s_{\zeta_m}$. We can also choose $k_m$ such that

$$\frac{1}{C}|f_{m,k_m}| \leq |s_{\zeta_m}| \leq C|f_{m,k_m}|$$

as $s_{\zeta_m}$ is non-vanishing on $\overline{\mathbb{B}_\Omega(\zeta_m, r)}$. So by changing $k_m$ to a larger enough value we can also assume that

$$\lim_{m\to\infty} ||H_\phi(f_{m,k_m})||_2 = 0$$

Put $f_m = f_{m,k_m}$. The function $h_m = f_m^{-1}P_\Omega(\phi f_m)$ would be in $\mathcal{O}(\mathbb{B}_\Omega(\zeta_m, r))$. Now we may write that

$$\lim_{m\to\infty}\int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi - h_m|^2 dV_\Omega = \lim_{m\to\infty}\int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi f_m - P_m(\phi f_m)|^2 |f_m|^{-2} dV_\Omega \leq$$
$$\lim_{m\to\infty} C^2 \int_{\mathbb{B}_\Omega(\zeta_m,r)} |H_\phi(f_m)|^2 d\mu \leq \lim_{m\to\infty} C^2 ||H_\phi(f_m)||_2^2 = 0$$

as desired.

(ii) (2) $\Rightarrow$ (1):

(I) Step 1: Let $C_1$ and $\mathfrak{N}_\zeta$ be as in theorem 2.1.10. If $r_1 < \dfrac{r}{C_1^2}$ by theorem 2.1.9 we may find a sequence $\{\zeta_m\}$ of distinct points in $\Omega$ such that

(1) If $i \neq j$ then $d_\Omega(\zeta_i, \zeta_j) \geq r_1$.

(2) $\Omega = \bigcup_{m\in\mathbb{N}} \mathbb{B}_\Omega(\zeta_m, r_1)$.

(3) $L = \sup_{z\in\Omega} \#\{m : \zeta_m \in \mathbb{B}_\Omega(z, 3r/2)\} < +\infty$.

By completeness of Bergman Metric we have $\zeta_m \to \partial\Omega$. So for each $m$ there exists $h_m \in \mathcal{O}(\mathbb{B}_\Omega(\zeta_m, r))$ such that

$$\lim_{m\to\infty} \epsilon_m = \lim_{m\to\infty} \int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi - h_m|^2 dV_\Omega = 0$$

Assume that $\mathcal{X} : \mathbb{B} \to [0, 1]$ is a smooth function with compact support in a way that $\mathcal{X} \equiv 1$ on $C_1 r_1 \mathbb{B}$ and $supp(\mathcal{X}) \subset \frac{r}{C_1}\mathbb{B}$. Define $\mathcal{X}_m = \mathcal{X} \circ \mathfrak{N}_{\zeta_m}^{-1}$. By theorem 2.1.10 we have

$$\mathbb{B}(\zeta_m, r_1) \subset \mathfrak{N}_{\zeta_m}(C_1 r_1 \mathbb{B}) \subset \mathcal{X}_m^{-1}$$
$$supp(\mathcal{X}_m) \subset \mathfrak{N}_{\zeta_m}(\frac{r}{C_1}\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta_m, r)$$

It also implies

$$||\bar{\partial}\mathcal{X}_m||_{g_\Omega} = ||\bar{\partial}\mathcal{X}_m||_{\mathfrak{N}_\zeta^* g_\Omega} \le C_1 ||\bar{\partial}\mathcal{X}||_2 \lesssim 1$$

Defining $\hat{\mathcal{X}_m} = \frac{1}{\sum_n \mathcal{X}_n}\mathcal{X}_m$ we have

$$||\bar{\partial}\hat{\mathcal{X}_m}||_{g_\Omega} = ||\frac{1}{\sum_n \mathcal{X}_n}\bar{\partial}\mathcal{X}_m - \frac{\mathcal{X}_m}{(\sum_n \mathcal{X}_n)^2}\sum_n \bar{\partial}\mathcal{X}_n||_{g_\Omega} \le (L+1)\sup_{n\ge1} ||\bar{\partial}\mathcal{X}_n||_{g_\Omega} \lesssim 1$$

Now let $\phi_1 = \sum_m \hat{\mathcal{X}_m} h_m$ and $\phi_2 = \phi - \phi_1$.

(II) Step 2: We claim that $\lim_{\zeta\to\partial\Omega} \int_{min} |\phi_2|^2 dV_\Omega = 0$. Note that by 2.2.1 this means that

(1) $dom(M_{\phi_2}) = A^2(\Omega)$ and $M_{\phi_2}$ is a compact operator

(2) $dom(H_{\phi_2}) = A^2(\Omega)$ and $H_{\phi_2}$ is a compact operator

(3) $dom(H_{\phi_1}) = dom(H_\phi)$

Take $\zeta \in \Omega$ and let $\{m_1, \ldots, m_k\} = \{m : supp(\mathcal{X}_m) \cap \mathbb{B}_\Omega(\zeta, r/2) \ne \emptyset\} \subset \{m : \zeta_m \in \mathbb{B}_\Omega(\zeta, 3\zeta/2)\}$. Note that $k$ has to be less than or equal to $L$. So

$$\int_{\mathbb{B}_\Omega(\zeta,r/2)} |\phi_2|^2 dV_\Omega = \int_{\mathbb{B}_\Omega(\zeta,r/2)} |\sum_{j=1}^{k} \hat{\mathcal{X}_{m_j}}(\phi - h_{m_j})|^2 dV_\Omega$$
$$\le \sum_{j=1}^{k} \int_{\mathbb{B}_\Omega(\zeta_{m_j},r)} |\phi - h_{m_j}|^2 dV_\Omega \le L max\{\epsilon_m : \zeta \in supp(\zeta_m)\}$$

And the claim follows immediately.

(III) Step 3: Let $r_2 < \frac{r}{2}$ be small enough such that if $w \in supp(\mathcal{X})$ then $\mathbb{B}(w, C_1 r_2) \subset \frac{1}{C_1}\mathbb{B}$. Now the claim is that if $\zeta \in supp(\mathcal{X}_n) \cap supp(\mathcal{X}_m)$ then

$$\int_{\mathbb{B}_\Omega(\zeta, r_2)} |h_n - h_m|^2 dV_\Omega \le \epsilon_m + \epsilon_n$$

For $\zeta \in supp(\mathcal{X}_n) \cap supp(\mathcal{X}_m)$ we have

$$\mathbb{B}_\Omega(\zeta, r_2) \subset \mathbb{B}_\Omega(\zeta_n, r) \cap \mathbb{B}_\Omega(\zeta_m, r)$$

So

$$\int_{\mathbb{B}_\Omega(\zeta, r_2)} |h_n - h_m|^2 dV_\Omega \le \int_{\mathbb{B}_\Omega(\zeta, r_2)} |h_n - f|^2 dV_\Omega + \int_{\mathbb{B}_\Omega(\zeta, r_2)} |f - h_m|^2 dV_\Omega \le \epsilon_m + \epsilon_n$$

And we have the claim proved.

(IV) Step 4: $\lim\limits_{\zeta \to \partial\Omega} \int_{\mathbb{B}_\Omega(\zeta, r_2)} ||\bar{\partial}\phi_1||^2_{g_\Omega} dV_\Omega = 0$.

Note that it suffice to prove that

$$\int_{\mathbb{B}_\Omega(\zeta, r_2)} ||\bar{\partial}\phi_1||^2_{g_\Omega} dV_\Omega \lesssim max\{\epsilon_m : \zeta \in supp(\mathcal{X}_m)\}$$

For $\zeta \in \Omega$ let $\{m_1, \ldots, m_k\} = \{m : supp(\mathcal{X}_m) \cap \mathbb{B}_\Omega(\zeta, r_2) \neq \emptyset\} \subset \{m : \zeta_m \in \mathbb{B}_\Omega(\zeta, r + r_2)\}$. Note that $k \le L$ as $r_2 < \frac{r}{2}$. And $\bar{\partial}\phi_1(\zeta) = \sum_{j=1}^{k} h_{m_j} \bar{\partial}\hat{\mathcal{X}}_{m_j}$ on $\mathbb{B}_\Omega(\zeta, r_2)$. Also as $\{\hat{\mathcal{X}_m}\}$ is a partition of unity we have $\sum_{j=1}^{k} h_{m_j} \bar{\partial}\hat{\mathcal{X}}_{m_j} = 0$ on $\mathbb{B}_\Omega(\zeta, r_2)$. Hence on $\mathbb{B}_\Omega(\zeta, r_2)$ we have

$$\bar{\partial}\phi_1 = \sum_{j=2}^{k} (h_{m_j} - h_{m_1}) \bar{\partial}\hat{\mathcal{X}}_{m_j} \tag{2.3}$$

So it follows from previous steps that

$$\int_{\mathbb{B}_\Omega(\zeta, r_2)} ||\bar{\partial}\phi_1||^2_{g_\Omega} dV_\Omega \le \sum_{j=2}^{k} \int_{\mathbb{B}_\Omega(\zeta, r_2)} |h_{m_j} - h_{m_1}|^2 dV_\Omega \le \sum_{j=2}^{k} (\epsilon_{m_j} + \epsilon_{m_1}) \lesssim max\{\epsilon_m : \zeta \in supp(\mathcal{X}_m)\}$$

Now we may argue that $H_\phi = H_{\phi_1} + H_{\phi_2}$ is extendable to a compact operator on $A^2(\Omega)$

**Definition 2.4.2.** *Let $\Omega \subset \mathbb{C}^d$ be a domain we say that the function $\phi$ is holomorphic on every analytic variety in $\partial\Omega$ if for every holomorphic map $F : \mathbb{D} \to \partial\Omega$ the function $\phi \circ F$ is holomorphic.*

**Lemma 2.4.3.** *Let $\phi$ be holomorphic on every analytic variety in $\partial\Omega$, then $r > 0$ exists in a way that*

$$\liminf_{\zeta \to \partial\Omega} \{ \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 dV_\Omega : h \in \mathcal{O}(\mathbb{B}_\Omega(\zeta, r))\} = 0$$

*Proof.* Let $r < \dfrac{1}{2C_1}$. Then $\mathbb{B}_\Omega(\zeta, r) \subset \mathfrak{N}_\zeta(\frac{1}{2}\mathbb{B})$ for every $\zeta \in \Omega$. Consider $\{\zeta_m\}$ as a sequence in $\Omega$ such that $\zeta_m \to \partial\Omega$ in a way that

$$\limsup_{\zeta \to \partial\Omega} \inf \{ \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 dV_\Omega : h \in \mathcal{O}(\mathbb{B}_\Omega(\zeta, r))\} =$$
$$\lim_{m \to \infty} \{ \int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi - h|^2 dV_\Omega : h \in \mathcal{O}(\mathbb{B}_\Omega(\zeta_m, r))\}$$

By probably passing to a subsequence we may also assume that $\mathfrak{N}_{\zeta_m}$ converges locally uniformly to $\mathfrak{N} : \mathbb{B} \to \bar{\Omega}$. By properness of Bergman distance and the fact that $\mathfrak{N}_\zeta(\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta, C_1)$ we have $\mathfrak{N}(\mathbb{B}) \subset \partial\Omega$. By assumption $h_0 = \phi \circ \mathfrak{N}$ would be holomorphic. Let us define $h_m = h_0 \circ \mathfrak{N}_{\zeta_m}^{-1}\big|_{\mathbb{B}_\Omega(\zeta_m,r)}$ we will have

$$\lim_{m \to \infty} \int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi - h_m|^2 dV_\Omega \leq \limsup_{m \to \infty} \int_{\frac{1}{2}\mathbb{B}} |\phi \circ \mathfrak{N}_{\zeta_m} - h_0|^2 \mathfrak{N}_{\zeta_m}^* dV_\Omega \leq$$
$$C_1^{2d} \limsup_{m \to \infty} \int_{\frac{1}{2}\mathbb{B}} |\phi \circ \mathfrak{N}_{\zeta_m} - h_0|^2 d\mu = 0$$

□

**Lemma 2.4.4.** *Let $F : \mathbb{D} \to \partial\Omega$ be holomorphic and $z_0 \in \mathbb{D}$, then there exist $\delta_0 > 0$ and sequence $\{\zeta_m\}$ such that $\mathfrak{N}_{\zeta_m}$ locally uniformly converges to a holomorphic function $\mathfrak{N} : \mathbb{B} \to \partial\Omega$ where $\mathfrak{N}(0) = F(z_0)$ and $F(\mathbb{D}(z_0, \delta_0)) \subset \mathfrak{N}(\mathbb{B})$*

*Proof.* $\partial\Omega$ is $\mathcal{C}^0$ so we may find unit vector $v \in \mathbb{C}^d$ and $\delta_0 > 0$ in a way that $tv + F(\mathbb{D}(z_0, \delta_0)) \subset \Omega$ for all $t \in (0, \delta_0)$. Define $\zeta_m = \frac{\delta_0}{m} v + F(z_0)$, and let $d_\Omega^K$ and $d_\mathbb{D}^K$ denote Kobayashi distances on $\Omega$ and $\mathbb{D}$. Distance decreasing property of Kobayashi metric ensures us that for all $w \in \mathbb{D}(z_0, \delta_0)$ we have

$$d_\Omega^K(\zeta_m, \frac{\delta_0}{m} v + F(w)) \leq d_\mathbb{D}^K(0, \frac{w - z_0}{\delta_0})$$

Theorem 1.8 in [6] implies that there exists $\delta > 0$ such that

$$\frac{\delta_0}{m}v + \Psi(\mathbb{D}(z_0,\delta)) \subset \mathbb{B}_\Omega(\zeta_m, \frac{1}{2C_1}) \subset \mathfrak{N}_{\zeta_m}(\frac{1}{2}\mathbb{B})$$

By perhaps passing to a subsequence we may assume that $\mathfrak{N}_{\zeta_m}$ converges locally uniformly to $\mathfrak{N} : \mathbb{B} \to \bar{\Omega}$ so $\mathfrak{N}(0) = \lim_{m\to\infty} \mathfrak{N}_m(0) = F(z_0)$. Properness of Bergman distance and the fact that $\mathfrak{N}_\zeta(\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta, C_1)$ now tell us that $\mathfrak{N}(\mathbb{B}) \subset \partial\Omega$. And $F(\mathbb{D}(z_0,\delta)) \subset \mathfrak{N}(\mathbb{B})$ consequently. □

**Lemma 2.4.5.** *Consider that $r > 0$ exists in a way that*

$$\liminf_{\zeta\to\partial\Omega}\{\int_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 dV_\Omega : h \in \mathcal{O}(\mathbb{B}_\Omega(\zeta,r))\} = 0$$

*then $\phi$ is holomorphic on every analytic variety in $\partial\Omega$.*

*Proof.* By Lemma 2.4.4 we need to show that for a sequence $\{\zeta_m\}$ in $\Omega$ such that $\zeta_m \to \partial\Omega$ and $\mathfrak{N}_{\zeta_m}$ locally uniformly converges to $\mathfrak{N} : \mathbb{B} \to \partial\Omega$ which is holomorphic and we have $\phi \circ \mathfrak{N}$ is holomorphic in a neighborhood of 0. By assumption for each $m \in \mathbb{N}$ there exists $h_m \in \mathcal{O}(\mathbb{B}_\Omega(\zeta_m, r))$ such that

$$\lim_{m\to\infty} \epsilon_m = \lim_{m\to\infty} (\int_{\mathbb{B}_\Omega(\zeta_m,r)} |\phi - h_m|^2 dV_\Omega)^{1/2} = 0$$

Choose $r_1 < \min \frac{r}{C_1}, 1$. Then $\mathfrak{N}_{\zeta_m}(r_1\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta_m, r)$ and $\hat{h}_m = h_m \circ \mathfrak{N}_{\zeta_m}$ is well defined on $r_1\mathbb{B}$. Also define $\hat{\phi}_m = \phi_m \circ \mathfrak{N}_{\zeta_m}$ which converges uniformly on $r_1\mathbb{B}$ to $\hat{\phi} = \phi \circ \mathfrak{N}_{\zeta_m}$. Theorem 2.1.10 implies

$$\int_{r_1\mathbb{B}} |\hat{\phi}_m - \hat{h}_m|^2 d\mu \leq C_1^{2d} \int_{r_1\mathbb{B}} |\hat{\phi}_m - \hat{h}_m|^2 \mathfrak{N}^*_{\zeta_m} dV_\Omega = C_1^{2d} \int_{\mathfrak{N}_{\zeta_m}(r_1\mathbb{B})} |\phi_m - h_m|^2 dV_\Omega \leq C_1^{2d}\epsilon_m^2$$

$\hat{\phi}_m$ converges uniformly to $\phi$ which implies that $\int_{r_1\mathbb{B}} |\hat{h}_m|^2 d\mu$ is uniformly bounded. Perhaps by replacing $\hat{h}_m$ with a subsequence we may consider it as a locally uniformly convergent sequence to a holomorphic function $\hat{h}$ on $r_1\mathbb{B}$. Now we may use Fatou's lemma to write

$$\int_{r_1\mathbb{B}} |\hat{\phi} - \hat{h}| d\mu \leq \liminf_{m\to\infty} \int_{r_1\mathbb{B}} |\hat{\phi}_m - \hat{h}_m|^2 d\mu = 0$$

Which is the desired result. □

**Theorem 2.4.6.** *Assume that $\Omega \in \mathbb{C}^d$ is a domain with bounded intrinsic geometry and $\{\mathfrak{N}_\zeta : \zeta \in \Omega\}$ satisfies the conditions in theorem 2.1.10. Then the following are equivalent:*

*(1)* $\log K_\Omega(z,z)$ *has self bounded gradient*

*(2) For every* $r>0$ *there exists* $C>1$ *such that if* $d_\Omega(z,\zeta)\le r$ *then*

$$\frac{1}{C}\le\frac{K_\Omega(z,z)}{K_\Omega(\zeta,\zeta)}\le C$$

*(3) For every* $r>0$, $C>1$ *exists in a way that if* $\zeta\in\Omega$ *then*

$$\frac{1}{C}\frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))}\le K_\Omega(\zeta,\zeta)\le C\frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))}$$

*(4) For every* $r>0$, $C>1$ *exists such that if* $\zeta\in\Omega$ *then on* $\mathbb{B}_\Omega(\zeta,r)$ *we have*

$$\frac{1}{C}\frac{d\mu}{\mu(\mathbb{B}_\Omega(\zeta,r))}\le dV_\Omega\le C\frac{d\mu}{\mu(\mathbb{B}_\Omega(\zeta,r))}$$

*(5)*

$$\sup_{\zeta\in\Omega}||\partial_w\log|\det\mathfrak{N}'_\zeta|\big|_{w=0}||_2<+\infty$$

*Proof.* (i) (1) $\Rightarrow$ (2): Define $Q=\sup_{z\in\Omega}||\partial\log K_\Omega(z,z)||_{g_\Omega}$ which is finite by assumption. $\log K_\Omega(z,z)$ is real valued so $\bar\partial\log K_\Omega(z,z)=\overline{\partial\log K_\Omega(z,z)}$. Therefore

$$\sup_{z\in\Omega}||d\log K_\Omega(z,z)||_{g_\Omega}\le 2Q$$

And

$$\exp(-2Qd_\Omega(z,\zeta))\le\frac{K_\Omega(z,z)}{K_\Omega(\zeta,\zeta)}\le\exp(2Qd_\Omega(z,\zeta))$$

for all $z,\zeta\in\Omega$ consequently.

(ii) (2) $\Rightarrow$ (3): Consider $C_1$ to be the constant in theorem 2.1.10. We prove the statement in 2 steps

(i) If $r<\frac{1}{C_1}$. Let $\zeta\in\Omega$ then by theorem 2.1.10 we may write that

$$\mathfrak{N}_\zeta(\frac{r}{C_1}\mathbb{B})\subset\mathbb{B}_\Omega(\zeta,r)\subset\mathfrak{N}_\zeta(C_1r\mathbb{B})$$

Therefore,

$$\int_{\frac{r}{C_1}\mathbb{B}}|\det\mathfrak{N}'_\zeta(w)|^2d\mu(w)\le\mu(\mathbb{B}_\Omega(\zeta,r))\le\int_{C_1r\mathbb{B}}|\det\mathfrak{N}'_\zeta(w)|^2d\mu(w)$$

Now it follows from theorem 2.1.12 and the assumption that

$$|\det \mathfrak{N}'_\zeta(w)|^2 \asymp \frac{1}{K_\Omega(\mathfrak{N}_\zeta(w), \mathfrak{N}_\zeta(w))} \asymp \frac{1}{K_\Omega(\zeta,\zeta)}$$

for $w \in \mathbb{B}$. Note that $K_\Omega(\zeta,\zeta)$ is independent of $w$ and integrating with respect to $w$ would tell us that $C > 1$ exists in a way that if $\zeta \in \Omega$ then

$$\frac{1}{C}\frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))} \le K_\Omega(\zeta,\zeta) \le C\frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))}$$

(ii) If $r \ge \dfrac{1}{C_1}$. Choose $r_0 < \dfrac{1}{C_1}$ By theorem 2.1.9 we may find $\{\zeta_m\}$ a sequence of distinct points in $\Omega$ such that

(1) If $i \neq j$ then $d_\Omega(\zeta_i,\zeta_j) \ge r_0$.

(2) $\Omega = \bigcup_{m\in\mathbb{N}} \mathbb{B}_\Omega(\zeta_m, r_0)$.

(3) $L = \sup_{z\in\Omega} \#\{m : \zeta_m \in \mathbb{B}_\Omega(z, r+r_0)\} < +\infty$.

By the previous step if $\zeta \in \Omega$ then

$$\mu(\mathbb{B}_\Omega(\zeta,r)) \ge \mu(\mathbb{B}_\Omega(\zeta,r_0)) \gtrsim \frac{1}{K_\Omega(\zeta,\zeta)}$$

It also implies that

$$\mu(\mathbb{B}_\Omega(\zeta,r)) \le \sum_{\zeta_j\in\mathbb{B}_\Omega(\zeta,r+r_0)} \mu(\mathbb{B}_\Omega(\zeta_j,r_0)) \lesssim L \max_{\zeta_j\in\mathbb{B}_\Omega(z,r+r_0)} \frac{1}{K_\Omega(\zeta_j,\zeta_j)} \lesssim \frac{1}{K_\Omega(\zeta,\zeta)}$$

and the assertion follows immediately

(iii) (3) $\Rightarrow$ (2): Choose $r > 0$ if $d_\Omega(z,\zeta) < r$ then

$$\frac{K_\Omega(z,z)}{K_\Omega(\zeta,\zeta)} \lesssim \frac{\mu(\mathbb{B}_\Omega(\zeta,r))}{\mu(\mathbb{B}_\Omega(z,2r))} \le 1$$

and

$$\frac{K_\Omega(z,z)}{K_\Omega(\zeta,\zeta)} \gtrsim \frac{\mu(\mathbb{B}_\Omega(\zeta,2r))}{\mu(\mathbb{B}_\Omega(z,r))} \ge 1$$

(iv) (2 *and* 3) $\Rightarrow$ (4): Choose $r > 0$ by theorem 2.1.16

$$dV_\Omega(z) \asymp K_\Omega(z,z)d\mu(z)$$

By the assumption we have that on $\mathbb{B}_\Omega(\zeta,r)$

$$dV_\Omega(z) \asymp K_\Omega(z,z)d\mu(z) \asymp \frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))}d\mu(z)$$

(v) (4) $\Rightarrow$ (2): Choose $r > 0$ by theorem 2.1.16

$$dV_\Omega(z) \asymp K_\Omega(z,z)d\mu(z)$$

So if $z \in \mathbb{B}_\Omega(\zeta, r)$ we have

$$K_\Omega(z,z) \asymp \frac{1}{\mu(\mathbb{B}_\Omega(\zeta, r))}$$

and

$$K_\Omega(z,z) \asymp K_\Omega(\zeta,\zeta)$$

consequently.

(vi) (2) $\Rightarrow$ (5): Let $\{\zeta_m\}$ be a sequence such that:

$$\sup_{\zeta\in\Omega} ||\partial_w \log|\det \mathfrak{N}'_\zeta(w)||_{w=0}||_2 = \lim_{m\to\infty} ||\partial_w \log|\det \mathfrak{N}'_{\zeta_m}(w)||_{w=0}||$$

Define $f_m : \mathbb{B} \to \mathbb{C}$ as $f_m = \dfrac{\det \mathfrak{N}'_{\zeta_m}(w)}{\det \mathfrak{N}'_{\zeta_m}(0)}$. Theorem 2.1.10 ensures us that there exists $C_1 > 0$ such that for all $\zeta \in \Omega$

$$\mathfrak{N}_\zeta(\mathbb{B}) \subset \mathbb{B}_\Omega(\zeta, C_1)$$

Applying theorem 2.1.12 we may write that

$$|f_m(w)|^2 \asymp \frac{K_\Omega(\zeta,\zeta)}{K_\Omega(\mathfrak{N}_\zeta(w), \mathfrak{N}_\zeta(w))} \asymp 1$$

By Montel's theorem and perhaps by replacing the sequence with a subsequence we may assume that $f_m$ locally uniformly converges to $f : \mathbb{B} \to \mathbb{C}$ which is holomorphic. Then

$$\sup_{\zeta\in\Omega} ||\partial_w \log|\det \mathfrak{N}'_\zeta(w)||_{w=0}||_2 = \lim_{m\to\infty} ||\partial_w \log|\det \mathfrak{N}'_{\zeta_m}(w)||_{w=0}||_2 = \lim_{m\to\infty} ||\partial f_m(0)||_2 =$$

$$||\partial f(0)||_2 < +\infty$$

(vii) (1) $\Longleftrightarrow$ (5): By theorem 2.1.10

$$||\partial_z \log K_\Omega(z,z)|_{z=\zeta}||_{g_\Omega} = ||\partial_w \log K_\Omega(\mathfrak{N}_\zeta(w), \mathfrak{N}_\zeta(w))|_{w=0}||_{\mathfrak{N}*_z g_\Omega} \asymp$$

$$||\partial_w \log K_\Omega(\mathfrak{N}_\zeta(w), \mathfrak{N}_\zeta(w))|_{w=0}||_2$$

Moreover by theorem 2.1.12 $||\partial_w \log \beta_\zeta(w,w)|_{w=0}||_2$ is uniformly bounded and by definition

$$\partial_w \log K_\Omega(\mathfrak{N}_\zeta(w), \mathfrak{N}_\zeta(w))|_{w=0} = (\partial_w \log \beta_\zeta(w,w) - \partial_w \log|\det \mathfrak{N}'_\zeta(w)|^2)|_{w=0}$$

which proves the assertion.

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