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Second Powers of Cover Ideals of Paths

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Abstract. We show that the second power of the cover ideal of a path graph has linear quotients. To prove our result we construct a recursively defined order on the generators of the ideal which yields linear quotients. Our construction has a natural generalization to the larger class of chordal graphs. This generalization allows us to raise some questions that are related to some open problems about powers of cover ideals of chordal graphs.

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1 Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. We say that a monomial ideal I has *linear quotients* if there exists an order $u_1, \dots, u_r$ of its minimal monomial generators such that for each $i = 2, \dots, r$ there exists a subset of $\{x_1, \dots, x_n\}$ which generates the colon ideal $(u_1, \dots, u_{i-1}) : (u_i)$.

Ideals with linear quotients were introduced in [20]. Many interesting classes of ideals are known to have linear quotients. For example, stable ideals, squarefree stable ideals and (weakly) polymatroidal ideals all have linear quotients. Moreover, in the squarefree case, having linear quotients translates into the concept of shellability in combinatorial topology. Indeed, if I is the Stanley–Reisner ideal of a simplicial complex Δ, then I has linear quotients if and only if the Alexander dual of Δ is shellable.

If an ideal I has linear quotients, then I is componentwise linear, i.e., for each d, the ideal generated by all degree d elements of I has a linear resolution. In particular, if I is generated in single degree and has linear quotients, then it has a linear resolution. Herzog, Hibi and Zheng [18] proved that when I is a monomial ideal generated in degree 2, the ideal I has a linear resolution if and only if it has linear quotients. Moreover, they proved that if I has a linear resolution, then so does every power of I.

Given a finite simple graph G with vertices $x_1, \ldots, x_n$, the *edge ideal* of G, denoted by $I(G)$, is generated by the monomials $x_i x_j$ such that x_i and x_j are adjacent vertices. Edge ideals are extensively studied in the literature; see, for example, the survey papers [12, 24]. Since every edge ideal is generated in degree 2, having a linear resolution and having linear quotients are equivalent concepts for such ideals. In addition, according to a result of Fröberg [10], it is known that the edge ideal $I(G)$ of a graph G has a linear resolution if and only if the complement graph of G is chordal.

The Alexander dual of $I(G)$ is known as the (*vertex*) *cover ideal* of G, and it is denoted by $J(G)$. Note that $J(G)$ is defined by

$$J(G) = (x_{i_1} \cdots x_{i_k} : \{x_{i_1}, \ldots, x_{i_k}\} \text{ is a minimal vertex cover of } G).$$

By a result of Herzog and Hibi [14], it is known that the Stanley–Reisner ideal arising from a simplicial complex Δ is (componentwise) linear if and only if the Alexander dual of Δ is (sequentially) Cohen–Macaulay. This implies that for any graph, being (sequentially) Cohen–Macaulay is equivalent to having a (componentwise) linear cover ideal. Unlike the edge ideals, there is no combinatorial characterization of cover ideals with linear resolutions. In fact, the authors of [8] described an example of a graph in [8, Example 4.4] whose cover ideal has a linear resolution if and only if the characteristic of the ground field is not two. The problem of classifying all Cohen–Macaulay or sequentially Cohen–Macaulay graphs is considered to be intractable, and thus this problem is studied for special classes of graphs; for examples, see [7–9, 15, 19, 28, 30].

Powers of edge ideals were studied by many authors recently; see, for example, the survey article [2]. A main motivation to study powers is to understand the behavior of (Castelnuovo–Mumford) regularity in terms of graph properties. It is well known that if I is a homogeneous ideal, then $\operatorname{reg}(I^s)$, the regularity of I^s, is a linear function in s for sufficiently large s (see [4, 21]). Powers of cover ideals were relatively less explored than edge ideals in the literature. In [13], for a unimodular hypergraph $\mathcal{H}$ (bipartite graph in particular) the regularity of $J(\mathcal{H})^s$ was determined for s big enough. However, the regularity of $J(\mathcal{H})^s$ is unknown for small values of s. The reader can refer to [3, 6, 13, 23, 25–27] for some recent articles where powers of cover ideals were studied.

Van Tuyl and Villarreal [28] showed that the cover ideal of a chordal graph has linear quotients, extending the results in [9], where it was shown that such ideals are componentwise linear. In fact, Woodroofe [29] showed that the independence complex of a graph with no chordless cycles of length other than 3 or 5 is vertex decomposable and hence shellable. In [17] the authors studied powers of componentwise linear ideals and they showed that all powers of a Cohen–Macaulay chordal graph have linear resolutions. Their proof was based on the method of *x-condition*, which, when satisfied, guarantees that all powers of the ideal have linear resolutions. More generally, they proposed the following conjecture:

Conjecture 1.1. [17, Conjecture 2.5] All powers of the vertex cover ideal of a chordal graph are componentwise linear.

The x-condition method requires that the ideal be generated by monomials of the same degree, which is indeed the case for the cover ideal of a Cohen–Macaulay chordal graph. However, generators of the cover ideal of an arbitrary chordal graph can have different degrees. Therefore, the x-condition method cannot be applied in the general case. There has not been any progress on Conjecture 1.1 except a few classes of graphs. In addition to Cohen–Macaulay chordal graphs it is known that the conjecture holds for generalized star graphs [22]. Moreover, powers of cover ideals of Cohen–Macaulay chordal graphs are known to have linear quotients [22].

It is unknown whether Conjecture 1.1 is true for the second power of the cover ideal of a chordal graph. The following question arises naturally:

Question 1.2. Let G be a chordal graph.

(1) Does $J(G)^2$ have linear quotients?
(2) Does $J(G)^s$ have linear quotients for all s?

In this paper we address Question 1.2(1) for a path graph P_n. Our main result Theorem 5.1 states that the second power of cover ideal of a path has linear quotients. We construct a recursively defined order, which we call a *rooted order*, on the minimal generators of $J(P_n)^2$ which produces linear quotients. Our method is purely combinatorial and it is completely different from the x-condition method, which involves the study of Gröbner bases of defining ideals of Rees algebras.

We summarize the contents of this paper. In Section 2 we introduce the necessary definitions and notations. Section 3 is devoted to some technical results about the rooted lists as well as minimal generators of $J(P_n)$ and $J(P_n)^2$, which will be needed in the next section. In Section 4 we analyze some cases where the product of two generators of the cover ideal may not produce a minimal generator for the second power of the ideal. Note that if a monomial ideal I is generated in the same degree, and u and v are two minimal generators, then the 2-fold product uv is necessarily a minimal generator of I^2. Since the cover ideal of a path is generated in different degrees, describing generators of the second power of the cover ideal is not trivial as in the case of equigenerated ideals. The goal of Section 5 is to prove the main result that $J(P_n)^2$ has linear quotients. We also extend the concept of rooted order to chordal graphs and discuss Question 1.2.

2 Definitions and Notations

Let K be a field and $S = K[x_1, x_2, \ldots, x_n]$ the polynomial ring over K in n indeterminates. Let G be a finite simple graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and edge set $E(G)$. Then the *edge ideal* $I(G) \subset S$ of G is generated by all quadratic monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. A *vertex cover* C of G is a subset of $V(G)$ such that $C \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover of G is called *minimal* if it does not strictly contain any other vertex cover of G. Let $\mathcal{M}(G)$ be the set of all minimal vertex covers of G. Then the (*vertex*) *cover ideal* of G, denoted by $J(G)$, is generated by $x_{i_1} x_{i_2} \cdots x_{i_k}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \mathcal{M}(G)$. It is a well known fact that $J(G)$ is the Alexander dual of $I(G)$. Throughout this paper we will use a set of vertices C interchangeably with its corresponding monomial $\prod_{x_i \in C} x_i$.

A graph is called *chordal* if it has no induced cycles except triangles. Every chordal graph contains a *simplicial vertex*, i.e., a vertex whose neighbors form a complete graph. The graph G is called a *path* on $\{x_1, x_2, \ldots, x_n\}$ if

$$E(G) = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}.$$

We denote a path on n vertices by P_n. Note that every path is a chordal graph. Our main goal is to prove that $J(P_n)^2$ has linear quotients. If I is a monomial ideal, we denote by $G(I)$ the set of the minimal monomial generators of I. Recall that a monomial ideal I is said to have *linear quotients* if there exists a suitable order of the minimal generators $u_1, u_2, \ldots, u_m$ such that for all $2 \leq i \leq m$ the ideal $(u_1, \ldots, u_{i-1}) : (u_i)$ is generated by variables. Given two monomials u and v, we will use the notation $u : v$ for the monomial $u/\gcd(u, v)$.

In order to simplify the notation in the following text, we set $uA = \{ua : a \in A\}$, where u is a monomial in S and A is a subset of S. Similarly, if $A = a_1, \ldots, a_n$ is a list of elements of S, then uA is a new list defined by $uA := ua_1, \ldots, ua_n$. Note that we use a non-standard way to represent a list. Normally, one would write $A = (a_1, \ldots, a_n)$ but we will drop the parentheses to avoid possible confusion between lists and ideals. The following lemma gives the relation among $\mathcal{M}(P_n)$, $\mathcal{M}(P_{n-2})$ and $\mathcal{M}(P_{n-3})$, or equivalently, the minimal generators of $J(P_n)$, $J(P_{n-2})$ and $J(P_{n-3})$.

Lemma 2.1. *For all $n \geq 5$, $G(J(P_n)) = x_{n-1}G(J(P_{n-2})) \sqcup x_n x_{n-2} G(J(P_{n-3}))$. Moreover, if $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ are the minimal generators of $J(P_{n-2})$ and $J(P_{n-3})$, respectively, written in linear quotients order, then $J(P_n)$ has linear quotients with respect to the order $x_{n-1}u_1, \ldots, x_{n-1}u_p, x_n x_{n-2} v_1, \ldots, x_n x_{n-2} v_q$.*

Proof. Since P_n is a chordal graph and x_n is a simplicial vertex, the result follows from [5, Theorem 3.1]. □

According to Lemma 2.1, we define a recursive order on the generators of $J(P_n)$.

Definition 2.2. (Rooted list, rooted order) Let P_n be the path with edge ideal $I(P_n) = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n)$. We recursively define the *rooted list*, denoted by $\mathcal{R}(P_n)$, of minimal generators of $J(P_n)$ as follows:

- $\mathcal{R}(P_2) = x_1, x_2$;
- $\mathcal{R}(P_3) = x_2, x_1x_3$;
- $\mathcal{R}(P_4) = x_1x_3, x_2x_3, x_2x_4$;
- for $n \geq 5$, if $\mathcal{R}(P_{n-2}) = u_1, \ldots, u_r$ and $\mathcal{R}(P_{n-3}) = v_1, \ldots, v_s$, then we define $\mathcal{R}(P_n) = x_{n-1}u_1, \ldots, x_{n-1}u_r, x_n x_{n-2} v_1, \ldots, x_n x_{n-2} v_s$.

We set $\mathcal{R}(P_1)$ as an empty list. Moreover, we define a total order $>_{\mathcal{R}}$, which we call a *rooted order* of the minimal generators of $J(P_n)$, as follows: if $\mathcal{R}(P_n) = w_1, \ldots, w_t$, then $w_i >_{\mathcal{R}} w_j$ for $i < j$.

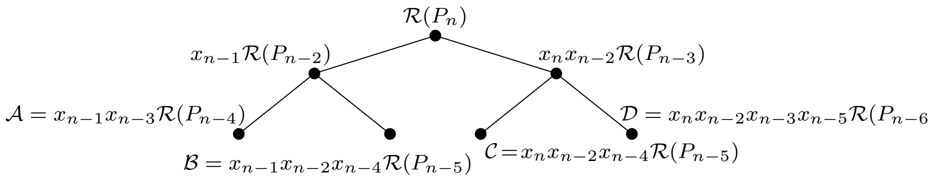

Figure 1 Branching of $\mathcal{R}(P_n)$

Remark 2.3. Let $\mathcal{R}(P_n) = u_1, \dots, u_q$ for $n \geq 2$. Then Lemma 2.1 together with the definition of rooted list implies that $J(P_n)$ has linear quotients with respect to the order $u_1, \dots, u_q$.

Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$ be two elements in $\mathbb{Z}^n$. Then we say that $\mathbf{a} >_{\text{lex}} \mathbf{b}$ if the first non-zero entry in $\mathbf{a} - \mathbf{b}$ is positive. In the following definition we adopt the same terminology used in [1, Discussion 4.1].

Definition 2.4. (2-fold product, maximal expression) Let $I = (u_1, \dots, u_q)$. We say that $M = u_1^{a_1} \cdots u_q^{a_q}$ is a *2-fold product* of minimal generators of I if $a_i \geq 0$ and $a_1 + \cdots + a_q = 2$. We write $u_1^{a_1} \cdots u_q^{a_q} >_{\text{lex}} u_1^{b_1} \cdots u_q^{b_q}$ if $(a_1, \dots, a_q) >_{\text{lex}} (b_1, \dots, b_q)$. We say that $M = u_1^{a_1} \cdots u_q^{a_q}$ is a *maximal expression* if $(a_1, \dots, a_q) >_{\text{lex}} (b_1, \dots, b_q)$ for any other 2-fold product $M = u_1^{b_1} \cdots u_q^{b_q}$.

Notation 2.5. For a monomial ideal I, we set $F(I^2) = \{uv : u, v \in G(I)\}$.

Note that for an arbitrary monomial ideal I, while $G(I^2) \subseteq F(I^2)$, not every 2-fold product is a minimal generator of I^2. However, if I is generated by monomials of the same degree, in particular if I is an edge ideal, then $G(I^2) = F(I^2)$.

Definition 2.6. (Rooted order on the second power) Let $\mathcal{R}(P_n) = u_1, \dots, u_q$ for $n \geq 2$. We define a total order $>_{\mathcal{R}}$ on $F(J(P_n)^2)$, which we call *rooted order*, as follows. For $M, N \in F(J(P_n)^2)$ with maximal expressions $M = u_1^{a_1} \cdots u_q^{a_q}$ and $N = u_1^{b_1} \cdots u_q^{b_q}$, we set $M >_{\mathcal{R}} N$ if $(a_1, \dots, a_q) >_{\text{lex}} (b_1, \dots, b_q)$.

Let $G(J(P_n)^2) = \{U_1, U_2, \dots, U_s\}$. Then we say that $U_1, U_2, \dots, U_s$ is a *rooted list* of generators of $J(P_n)^2$ if $U_1 >_{\mathcal{R}} U_2 >_{\mathcal{R}} \cdots >_{\mathcal{R}} U_s$. In this case, we denote the rooted list of generators by $\mathcal{R}(J(P_n)^2) = U_1, U_2, \dots, U_s$.

The following table shows the rooted list $\mathcal{R}(P_n)$ for $2 \leq n \leq 7$.

Table 1

n	$\mathcal{R}(P_n)$
2	$u_1 = x_1$, $u_2 = x_2$
3	$u_1 = x_2$, $u_2 = x_1x_3$
4	$u_1 = x_1x_3$, $u_2 = x_2x_3$, $u_3 = x_2x_4$
5	$u_1 = x_2x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_1x_3x_5$, $u_4 = x_2x_3x_5$
6	$u_1 = x_1x_3x_5$, $u_2 = x_2x_3x_5$, $u_3 = x_2x_4x_5$, $u_4 = x_2x_4x_6$, $u_5 = x_1x_3x_4x_6$
7	$u_1 = x_2x_4x_6$, $u_2 = x_1x_3x_4x_6$, $u_3 = x_1x_3x_5x_6$, $u_4 = x_2x_3x_5x_6$, $u_5 = x_1x_3x_5x_7$, $u_6 = x_2x_3x_5x_7$, $u_7 = x_2x_4x_5x_7$

Given the above labeling of elements of $\mathcal{R}(P_n)$ for $2 \le n \le 7$, Table 2 shows the rooted list of the minimal generators of $J(P_n)^2$, and the 2-fold products in $F(J(P_n)^2)\backslash G(J(P_n)^2)$.

Table 2

n	$\mathcal{R}(J(P_n)^2)$	$F(J(P_n)^2)\backslash G(J(P_n)^2)$
2	u_1^2, u_1u_2, u_2^2	
3	u_1^2, u_1u_2, u_2^2	
4	$u_1^2, u_1u_2, u_1u_3, u_2^2, u_2u_3, u_3^2$	
5	$u_1^2, u_1u_2, u_1u_3, u_1u_4, u_2^2, u_2u_3, u_3^2, u_3u_4, u_4^2$	u_2u_4 (divisible by u_1u_3)
6	$u_1^2, u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_2^2, u_2u_3, u_2u_4,$ $u_3^2, u_3u_4, u_4^2, u_4u_5, u_5^2$	u_2u_5, u_3u_5 (divisible by u_1u_4)
7	$u_1^2, u_1u_2, u_1u_3, u_1u_4, u_1u_5, u_1u_6, u_1u_7,$ $u_2^2, u_2u_3, u_2u_5, u_3^2, u_3u_4, u_3u_5, u_3u_6{=}u_4u_5,$ $u_4^2, u_4u_6, u_5^2, u_5u_6, u_5u_7, u_6^2, u_6u_7, u_7^2$	u_2u_4 (divisible by u_1u_3), u_2u_6, u_2u_7, u_3u_7 (divisible by u_1u_5), u_4u_7 (divisible by u_1u_6)

3 Some Properties of $G(J(P_n))$, $G(J(P_n)^2)$ and Rooted Lists

In this section, we will prove some technical results about properties of rooted lists. We will write $u \ge_{\mathcal{R}} v$ if either $u = v$ or $u >_{\mathcal{R}} v$. We start with some observations.

Remark 3.1. Let $n \ge 4$ and let $\mathcal{R}(P_{n-2}) = u_1, \dots, u_m$. Observe that by the definition of rooted order, for any $k, \ell \in \{1, \dots, m\}$ we have

$$u_k >_{\mathcal{R}} u_\ell \text{ in } \mathcal{R}(P_{n-2}) \iff x_{n-1}u_k >_{\mathcal{R}} x_{n-1}u_\ell \text{ in } \mathcal{R}(P_n).$$

Therefore, for any $i \le j$, the expression u_iu_j is maximal if and only if the expression $(x_{n-1}u_i)(x_{n-1}u_j)$ is maximal with $x_{n-1}u_i \ge_{\mathcal{R}} x_{n-1}u_j$.

Remark 3.2. Let $n \ge 5$ and let $\mathcal{R}(P_{n-3}) = u_1, \dots, u_m$. Observe that by the definition of rooted order, for any $k, \ell \in \{1, \dots, m\}$ we have

$$u_k >_{\mathcal{R}} u_\ell \text{ in } \mathcal{R}(P_{n-3}) \iff x_nx_{n-2}u_k >_{\mathcal{R}} x_nx_{n-2}u_\ell \text{ in } \mathcal{R}(P_n).$$

Therefore, for any $i \le j$, the expression u_iu_j is maximal if and only if the expression $(x_nx_{n-2}u_i)(x_nx_{n-2}u_j)$ is maximal with $x_nx_{n-2}u_i \ge_{\mathcal{R}} x_nx_{n-2}u_j$.

Lemma 3.3. *Let $n \ge 4$. Then U, V are in $F(J(P_{n-2})^2)$ if and only if x_{n-1}^2U, x_{n-1}^2V are in $F(J(P_n)^2)$. Moreover, in this case $U >_{\mathcal{R}} V$ if and only if $x_{n-1}^2U >_{\mathcal{R}} x_{n-1}^2V$.*

Proof. The first statement is clear by Lemma 2.1. Let $\mathcal{R}(P_{n-2}) = u_1, \dots, u_m$. Suppose that $U = u_iu_j$ and $V = u_su_t$ are maximal expressions, where $i \le j$ and $s \le t$. Then by Remark 3.1 it follows that $x_{n-1}^2U = (x_{n-1}u_i)(x_{n-1}u_j)$ and $x_{n-1}^2V = (x_{n-1}u_s)(x_{n-1}u_t)$ are maximal expressions, where $x_{n-1}u_i \ge_{\mathcal{R}} x_{n-1}u_j$ and $x_{n-1}u_s \ge_{\mathcal{R}} x_{n-1}u_t$. Keeping Remark 3.1 in mind, we observe that

$$\begin{aligned} U >_{\mathcal{R}} V &\iff \text{either } u_i >_{\mathcal{R}} u_s \text{ or } u_i = u_s \text{ and } u_j >_{\mathcal{R}} u_t \\ &\iff \text{either } x_{n-1}u_i >_{\mathcal{R}} x_{n-1}u_s \text{ or } x_{n-1}u_i = x_{n-1}u_s \\ &\qquad \text{and } x_{n-1}u_j >_{\mathcal{R}} x_{n-1}u_t \\ &\iff x_{n-1}^2U >_{\mathcal{R}} x_{n-1}^2V. \end{aligned}$$

□

Lemma 3.4. *Let $n \geq 5$. Then $U, V \in F(J(P_{n-3})^2)$ if and only if both $x_n^2 x_{n-2}^2 U$ and $x_n^2 x_{n-2}^2 V$ belong to $F(J(P_n)^2)$. Moreover, in this case $U >_{\mathcal{R}} V$ if and only if $x_n^2 x_{n-2}^2 U >_{\mathcal{R}} x_n^2 x_{n-2}^2 V$.*

Proof. The proof is almost identical to that of the previous lemma if one uses Remark 3.2 instead of Remark 3.1. $\square$

Lemma 3.5. *Let $n \geq 4$ and let $u \in G(J(P_n))$ such that $x_n | u$. Then there exists $v \in G(J(P_{n-2}))$ such that v divides u/x_n.*

Proof. If $x_n | u$, then u is not divisible by x_{n-1} because u is a minimal vertex cover of P_n. Then u/x_n contains a minimal vertex cover of P_{n-2}, which verifies the statement. $\square$

Lemma 3.6. *Let $u_1, \ldots, u_r$ be the rooted list of P_n. Let k be the smallest index such that $x_{n-1} \nmid u_k$. Then $(u_1, \ldots, u_{k-1}) : (u_k) = (x_{n-1})$. Moreover, if $i > k$, then $(u_1, \ldots, u_{i-1}) : (u_i) = (x_{n-1}) + (u_k, \ldots, u_{i-1}) : (u_i)$.*

Proof. The statement is clear when $n = 2$ or $n = 3$. Otherwise, it follows from Lemma 3.5. $\square$

Lemma 3.7. *Let $\mathcal{R}(P_n) = u_1, \ldots, u_m$.*
(1) *If $x_n | u_i$ for some i, then $x_n | u_j$ for all $j \geq i$.*
(2) *If $x_{n-2} | u_i$ for some i, then $x_{n-2} | u_j$ for all $j \geq i$.*
(3) *If $\mathcal{R}(P_{n-2}) = v_1, \ldots, v_k$ and $\mathcal{R}(P_{n-1}) = w_1, \ldots, w_\ell$, then*

$$\mathcal{R}(P_n) = x_{n-1}v_1, \ldots, x_{n-1}v_k, x_n w_1, \ldots, x_n w_\alpha \quad \textit{for some } \alpha < \ell.$$

Proof. (1) follows from the definition of rooted list.

(2) can be confirmed by applying (1) to P_{n-2} in the recursive definition of $\mathcal{R}(P_n)$.

To see (3), we will just compare the recursively defined lists of P_{n-1} and P_n. For $n \leq 5$, one can refer to Table 1 to confirm that the statement holds. Assume that $n \geq 6$. Let $\mathcal{R}(P_{n-4}) = y_1, \ldots, y_s$ and let $\mathcal{R}(P_{n-3}) = z_1, \ldots, z_t$. Then by the recursive definition of rooted list we have

$$\begin{aligned}\mathcal{R}(P_n) &= x_{n-1}v_1, \ldots, x_{n-1}v_k, x_n x_{n-2} z_1, \ldots, x_n x_{n-2} z_t,\\ \mathcal{R}(P_{n-1}) &= x_{n-2}z_1, \ldots, x_{n-2}z_t, x_{n-1}x_{n-3}y_1, \ldots, x_{n-1}x_{n-3}y_s.\end{aligned}$$

Therefore, $w_1 = x_{n-2}z_1, \ldots, w_t = x_{n-2}z_t$ and $\alpha = t$, as desired. $\square$

Lemma 3.8. *Let $n \geq 4$. Then $x_{n-1}^2 U \in G(J(P_n)^2)$ if and only if $U \in G(J(P_{n-2})^2)$.*

Proof. ($\Rightarrow$) Suppose that $x_{n-1}^2 U \in G(J(P_n)^2)$. Then $x_{n-1}^2 U = (x_{n-1}u_1)(x_{n-1}u_2)$ for some $u_1, u_2 \in G(J(P_{n-2}))$ by Lemma 2.1. Let $V = v_1 v_2$ for some v_1, v_2 in $G(J(P_{n-2}))$ such that $V | U$. Then we have $W = (x_{n-1}v_1)(x_{n-1}v_2) \in F(J(P_n)^2)$ by Lemma 3.3. Since $W | x_{n-1}^2 U$ and $x_{n-1}^2 U$ is a minimal generator, we get $W = x_{n-1}^2 U$. Therefore, $V = U$ and U is a minimal generator of $J(P_{n-2})^2$.

($\Leftarrow$) Let $U \in G(J(P_{n-2})^2)$. Then $U = u_1 u_2$ for some $u_1, u_2 \in G(J(P_{n-2}))$. Thus, by Lemma 3.3, $x_{n-1}^2 U = (x_{n-1}u_1)(x_{n-1}u_2) \in F(J(P_n)^2)$. Let $V \in G(J(P_n)^2)$ such that $V | x_{n-1}^2 U$. By Lemma 2.1 one can write $V = (x_{n-1}v_1)(x_{n-1}v_2)$ for some

$v_1, v_2 \in G(J(P_{n-2}))$. Hence, we get $v_1v_2|u_1u_2$. Since U is a minimal generator, $v_1v_2 = u_1u_2$. Therefore, $V = x_{n-1}^2 U \in G(J(P_n)^2)$, as desired. □

Lemma 3.9. *Let $n \geq 5$. Then we have $x_n^2 x_{n-2}^2 U \in G(J(P_n)^2)$ if and only if $U \in G(J(P_{n-3})^2)$.*

Proof. One can mimic the arguments in the proof of the previous lemma by using Lemma 3.4 instead of Lemma 3.3. □

Lemma 3.10. *Let $n \geq 7$ and $uv, u'v' \in F(J(P_n)^2)$ with $u >_{\mathcal{R}} v$ and $u' >_{\mathcal{R}} v'$. Suppose that $u'v'$ divides uv. In the notation of Figure 1, the following statements hold:*

(1) *If $u \in \mathcal{A}$ and $v \in \mathcal{C}$, then $u' \in \mathcal{A}$ and $v' \in \mathcal{C}$.*

(2) *If $u \in \mathcal{B}$ and $v \in \mathcal{C}$, then $u' \in \mathcal{B}$ and $v' \in \mathcal{C}$.*

Proof. (1) First note that if $u \in \mathcal{A}$ and $v \in \mathcal{C}$, then $u \in x_{n-1}\mathcal{R}(P_{n-2})$ and $v \in x_n x_{n-2}\mathcal{R}(P_{n-3})$. Since $u'v'|uv$ and $u' >_{\mathcal{R}} v'$, it shows that $u' \in x_{n-1}\mathcal{R}(P_{n-2})$ and $v' \in x_n x_{n-2}\mathcal{R}(P_{n-3})$.

Now we show that $u' \in \mathcal{A}$ and $v' \in \mathcal{C}$. Indeed, if $u' \in \mathcal{B}$, then $x_{n-2}^2|u'v'$ but $x_{n-2}^2 \nmid uv$. Therefore, $u' \in \mathcal{A}$. Furthermore, if $v' \in \mathcal{D}$ then $x_{n-3}^2|u'v'$ because $u' \in \mathcal{A}$. But, again $x_{n-3}^2 \nmid uv$. Hence, $v' \in \mathcal{C}$, as required.

(2) First note that if $u \in \mathcal{B}$ and $v \in \mathcal{C}$, then we obtain $u \in x_{n-1}\mathcal{R}(P_{n-2})$ and $v \in x_n x_{n-2}\mathcal{R}(P_{n-3})$. Since $u'v'|uv$ and $u' >_{\mathcal{R}} v'$, it follows that $u' \in x_{n-1}\mathcal{R}(P_{n-2})$ and $v' \in x_n x_{n-2}\mathcal{R}(P_{n-3})$. Accordingly, to show $u' \in \mathcal{B}$ and $v' \in \mathcal{C}$, note that if $u' \in \mathcal{A}$ or $v' \in \mathcal{D}$, then $x_{n-3}|u'v'$ but $x_{n-3} \nmid uv$. □

Remark 3.11. Let $\mathcal{R}(P_{n-2}) = u_1, \ldots, u_m$. Furthermore, let u_i, u_j be monomials such that $x_{n-1}u_i \in \mathcal{A}$ and $x_n u_j \in \mathcal{C}$, where $\mathcal{A}, \mathcal{C}$ are as in Figure 1. Let u_iu_j be a maximal expression in $F(J(P_{n-2})^2)$ with $i \leq j$. Then $(x_{n-1}u_i)(x_n u_j)$ is a maximal expression in $F(J(P_n)^2)$. Indeed, otherwise, from Lemma 3.10 we see that the maximal expression of $x_{n-1}x_n u_i u_j$ is of the form $(x_{n-1}u_p)(x_n u_q)$ with $x_{n-1}u_p \in \mathcal{A}$ and $x_n u_q \in \mathcal{C}$. From $x_{n-1}x_n u_i u_j = x_{n-1}x_n u_p u_q$ we see that $u_iu_j = u_pu_q$. Also, we have $x_{n-1}u_p >_{\mathcal{R}} x_{n-1}u_i$ or $x_{n-1}u_p = x_{n-1}u_i$ and $x_n u_q >_{\mathcal{R}} x_n u_j$ in $\mathcal{R}(P_n)$. This shows that $u_p >_{\mathcal{R}} u_i$ or $u_p = u_i$ and $u_q >_{\mathcal{R}} u_j$ in $\mathcal{R}(P_{n-2})$, which gives a contradiction of the fact that the expression u_iu_j is maximal in $F(J(P_{n-2})^2)$.

Lemma 3.12. *Let u and v be monomials such that $x_{n-1}u \in \mathcal{A}$ and $x_n v \in \mathcal{C}$, where $\mathcal{A}$, $\mathcal{C}$ are as in Figure 1. If $uv \in G(J(P_{n-2})^2)$, then $x_{n-1}x_n uv \in G(J(P_n)^2)$.*

Proof. Let $uv \in G(J(P_{n-2})^2)$. Assume on the contrary that $x_{n-1}x_n uv \notin G(J(P_n)^2)$. Then there exists some $U \in G(J(P_n)^2)$ such that U strictly divides $x_{n-1}x_n uv$. Thus, from Lemma 3.10 we see that $U = (x_{n-1}u')(x_n v')$ for some $x_{n-1}u' \in \mathcal{A}$ and $x_n v' \in \mathcal{C}$. Consequently, $u'v' \in F(J(P_{n-2})^2)$ and $u'v'$ strictly divides uv, which contradicts the hypothesis that $uv \in G(J(P_{n-2})^2)$. □

Lemma 3.13. *Let u, u', v, v' be monomials such that $x_{n-1}u, x_{n-1}u' \in \mathcal{A}$ and $x_n v, x_n v' \in \mathcal{C}$, where $\mathcal{A}$, $\mathcal{C}$ are as in Figure 1. If $uv >_{\mathcal{R}} u'v'$ in $F(J(P_{n-2})^2)$, then $x_{n-1}x_n uv >_{\mathcal{R}} x_{n-1}x_n u'v'$ in $F(J(P_n)^2)$.*

Proof. The branching of $\mathcal{R}(P_n)$ in Figure 1 shows that if $x_{n-1}u, x_{n-1}u' \in \mathcal{A}$ and $x_nv, x_nv' \in \mathcal{C}$, then $u, u' \in x_{n-3}\mathcal{R}(P_{n-4})$ and $v, v' \in x_{n-2}x_{n-4}\mathcal{R}(P_{n-5})$. Hence, by the definition of $>_{\mathcal{R}}$ it follows that $u >_{\mathcal{R}} v$ and $u' >_{\mathcal{R}} v'$ in $\mathcal{R}(P_{n-2})$.

Note that because of Lemma 3.10(1) we may assume that uv and $u'v'$ are maximal expressions in $F(J(P_{n-2})^2)$. Then Remark 3.11 implies that the expressions $(x_{n-1}u)(x_nv)$ and $(x_{n-1}u')(x_nv')$ are maximal in $F(J(P_n)^2)$.

Let $uv >_{\mathcal{R}} u'v'$ in $F(J(P_{n-2})^2)$. Then by the definition of $>_{\mathcal{R}}$, we have either $u >_{\mathcal{R}} u'$ or $u = u'$ and $v >_{\mathcal{R}} v'$ in $\mathcal{R}(P_{n-2})$. If $u >_{\mathcal{R}} u'$, then by Remark 3.1 $x_{n-1}u >_{\mathcal{R}} x_{n-1}u'$ in $\mathcal{R}(P_n)$. If $v >_{\mathcal{R}} v'$ in $\mathcal{R}(P_{n-2})$, then $x_{n-1}v$ appears before $x_{n-1}v'$ in the sublist $\mathcal{B}$. This implies that x_nv appears before x_nv' in the sublist $\mathcal{C}$, and thus $x_nv >_{\mathcal{R}} x_nv'$, as desired. □

Lemma 3.14. *Let $u, v \in G(J(P_{n-5}))$. If $uv \in G(J(P_{n-5})^2)$, then*

$$(x_{n-1}x_{n-2}x_{n-4}u)(x_nx_{n-2}x_{n-4}v) \in G(J(P_n)^2).$$

Proof. A similar argument to that in Lemma 3.12 gives the desired result. □

Remark 3.15. Let $\mathcal{R}(P_{n-5}) = u_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_p$. If u_iu_j is a maximal expression with $i \leq j$ in $F(J(P_{n-5})^2)$, then together with Lemma 3.10 and a similar explanation to that in Remark 3.11, we see that $(x_{n-1}x_{n-2}x_{n-4}u_i)(x_nx_{n-2}x_{n-4}u_j)$ is a maximal expression in $F(J(P_n)^2)$.

Lemma 3.16. *Let $\mathcal{R}(P_{n-5}) = u_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_p$. If $u_iu_k >_{\mathcal{R}} u_ju_l$ in $F(J(P_{n-5})^2)$ for some i, j, k and l, then we have $x_nx_{n-1}x_{n-2}^2x_{n-4}^2u_iu_k >_{\mathcal{R}} x_nx_{n-1}x_{n-2}^2x_{n-4}^2u_ju_l$ in $F(J(P_n)^2)$.*

Proof. We may assume that u_iu_k and u_ju_l are maximal expressions with $i \leq k$ and $j \leq l$. By Remark 3.15, the expressions $(x_{n-1}x_{n-2}x_{n-4}u_i)(x_nx_{n-2}x_{n-4}u_k)$ and $(x_{n-1}x_{n-2}x_{n-4}u_j)(x_nx_{n-2}x_{n-4}u_l)$ are both maximal in $F(J(P_n)^2)$. Given that $u_iu_k >_{\mathcal{R}} u_ju_l$ in $F(J(P_{n-5})^2)$, we have either $i < j$ or $i = j$ and $k < l$.

If $i < j$, then $x_{n-1}x_{n-2}x_{n-4}u_i$ appears before $x_{n-1}x_{n-2}x_{n-4}u_j$ in the sublist $\mathcal{B}$. Therefore, we get $x_{n-1}x_{n-2}x_{n-4}u_i >_{\mathcal{R}} x_{n-1}x_{n-2}x_{n-4}u_j$ in $\mathcal{R}(P_n)$.

If $k < l$, then $x_nx_{n-2}x_{n-4}u_k$ appears before $x_nx_{n-2}x_{n-4}u_l$ in the sublist $\mathcal{C}$. Therefore, $x_nx_{n-2}x_{n-4}u_k >_{\mathcal{R}} x_nx_{n-2}x_{n-4}u_l$ in $\mathcal{R}(P_n)$. Thus, the result follows by the definition of rooted order. □

4 2-Fold Products of $J(P_n)$ Versus Minimal Generators of $J(P_n)^2$

In order to prove our main result, we need to filter out those 2-fold products which are not in $G(J(P_n)^2)$. The next lemma gives a sufficient condition for a 2-fold product to be a non-minimal generator. We advise the reader to keep in mind, while reading the proof of this lemma, that a minimal vertex cover of a path cannot contain three consecutive vertices.

Lemma 4.1. *Let $n \geq 5$. Let u and v be minimal generators of $J = J(P_n)$ such that $x_{n-1}x_{n-4}|u$ and $x_nx_{n-3}|v$. Then uv is not a minimal generator of J^2. Moreover, there exists a 2-fold product $pw \in F(J^2)$ such that $pw|uv$ and $pw >_{\mathcal{R}} uv$.*

Proof. Let $X = x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4}$. Since v is a minimal generator, $x_{n-1} \nmid v$, which implies $x_{n-2} | v$. Then we get $x_{n-4} \nmid v$ because v is minimal. Thus, we have $\gcd(v, X) = x_n x_{n-2} x_{n-3}$. By the minimality of u we get $x_n \nmid u$. Since x_{n-4} divides u, we see that u is divisible by either x_{n-2} or x_{n-3}, but not both. Therefore, we obtain $\gcd(u, X) = x_{n-1} x_{n-2} x_{n-4}$ or $\gcd(u, X) = x_{n-1} x_{n-3} x_{n-4}$.

Let $u'v'$ be a maximal expression of uv for some $u' >_{\mathcal{R}} v'$. We claim that $\gcd(u, X) = \gcd(u', X)$ and $\gcd(v, X) = \gcd(v', X)$. First notice that since $u' >_{\mathcal{R}} v'$, the variable x_{n-1} divides u' but not v'. This implies that x_n divides v' but not u' because both u' and v' are minimal generators. Consider the following cases.

Case 1. Suppose $\gcd(u, X) = x_{n-1} x_{n-2} x_{n-4}$. Since $uv = u'v'$, we see that x_{n-2} divides both u' and v'. Now by the minimality of u' we must have $x_{n-3} \nmid u'$. This implies $x_{n-3} | v'$. The minimality of v' requires $x_{n-4} \nmid v'$. Then $x_{n-4} | u'$. Hence, $\gcd(v', X) = x_n x_{n-2} x_{n-3}$ and $\gcd(u', X) = x_{n-1} x_{n-2} x_{n-4}$, as desired.

Case 2. Suppose $\gcd(u, X) = x_{n-1} x_{n-3} x_{n-4}$. Since $x_{n-3}^2 | uv = u'v'$, we see that x_{n-3} divides both u' and v'. By the minimality of u' we observe that $x_{n-2} \nmid u'$. Since $x_{n-2} | uv = u'v'$, we get $x_{n-2} | v'$. Now by the minimality of v' we get $x_{n-4} \nmid v'$, which implies $x_{n-4} | u'$. Hence, $\gcd(v', X) = x_n x_{n-2} x_{n-3}$ and $\gcd(u', X) = x_{n-1} x_{n-3} x_{n-4}$, which completes the proof of our claim.

Observe that $w = (v' x_{n-1})/(x_n x_{n-2})$ is a minimal vertex cover of P_n. Again, we consider two cases: (a) Suppose $x_{n-3} | u'$. Observe that $p = (u' x_{n-2} x_n)/(x_{n-1} x_{n-3})$ is a minimal vertex cover of P_n and $p w x_{n-3} = u'v'$. (b) Suppose $x_{n-2} | u'$. Observe that $p = (u' x_n)/x_{n-1}$ is a minimal vertex cover of P_n and $p w x_{n-2} = u'v'$. In either case, $w >_{\mathcal{R}} p$. Since both w and u' are in $x_{n-1}\mathcal{R}(P_{n-2})$, applying Lemma 3.7(2) to P_{n-2}, we see that $w >_{\mathcal{R}} u'$. Therefore, $pw >_{\mathcal{R}} u'v'$. □

We will need the next result to detect some of the 2-fold products which yield non-minimal generators or non-maximal expressions.

Proposition 4.2. *Let $\mathcal{R}(P_n) = u_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_k$, where $n \geq 2$. Let $1 < i < j \leq k$. Suppose that u_j contains a variable from $(u_1, \ldots, u_{i-1}) : (u_i)$. Then either $u_i u_j$ is not a minimal generator of $J(P_n)^2$ or $u_i u_j$ is not a maximal 2-fold expression.*

Proof. We proceed by induction on n. The statement holds for $n \leq 7$; see Table 2 for verification. Assume that $n \geq 8$. Observe that if $u_i u_j$ is divisible by x_{n-1}^2, then the result follows from Lemmas 3.3 and 3.8 and the induction assumption on P_{n-2}. If $u_i u_j$ is divisible by x_n^2, then the result follows from Lemmas 3.4, 3.6 and 3.9 and the induction assumption on P_{n-3}. Therefore, we may assume that $x_{n-1} | u_i$ and $x_n | u_j$. Now consider the following rooted lists:

$$\begin{aligned} \mathcal{R}(P_{n-4}) &: \ v_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} v_\ell, \\ \mathcal{R}(P_{n-5}) &: \ w_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} w_m, \\ \mathcal{R}(P_{n-6}) &: \ p_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} p_q. \end{aligned}$$

Note that $\mathcal{R}(P_n)$ is the join of the lists $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in Figure 1 in the given order. We consider the following cases.

Case 1. Suppose that $u_j \in \mathcal{D}$. If $x_{n-4} | u_i$, then the result follows from Lemma 4.1. Assume that $x_{n-4} \nmid u_i$ as well. Observe now that $u_i \in \mathcal{A}$. Note that by Lemma

3.7(3) we get $\mathcal{R}(P_{n-4}) = x_{n-5}p_1, \dots, x_{n-5}p_q, x_{n-4}w_1, \dots, x_{n-4}w_\alpha$ for some $\alpha \le m$. Therefore, $u_i = x_{n-1}x_{n-3}x_{n-5}p_i$ and $u_j = x_nx_{n-2}x_{n-3}x_{n-5}p_\beta$ for some $\beta \le q$. Clearly we have $(u_1, \dots, u_{i-1}) : (u_i) = (p_1, \dots, p_{i-1}) : (p_i)$ and $\beta \neq i$. Thus, p_β contains a variable generator of $(p_1, \dots, p_{i-1}) : (p_i)$.

Subcase 1.1. If $\beta < i$, then observe that we can produce a new expression

$$u_iu_j = \underbrace{(x_{n-1}x_{n-3}x_{n-5}p_\beta)}_{u_\beta}\underbrace{(x_nx_{n-2}x_{n-3}x_{n-5}p_i)}_{u_\gamma}$$

for some γ. Then $u_\beta >_{\mathcal{R}} u_\gamma$ and $u_\beta >_{\mathcal{R}} u_i$, which implies that the expression u_iu_j is not maximal.

Subcase 1.2. Let $\beta > i$. Then by induction assumption, either p_ip_β is not a minimal generator of $J(P_{n-6})^2$ or the expression p_ip_β is not maximal. Any minimal generator of $J(P_{n-6})^2$ which divides p_ip_β or any 2-fold expression which is greater than p_ip_β can be multiplied by the appropriate variables to obtain the desired conclusion for u_iu_j.

Case 2. Suppose $u_j \in \mathcal{C}$ so that $u_j = x_nx_{n-2}x_{n-4}w_s$ for some $s \ge 1$.

Subcase 2.1. Suppose $u_i \in \mathcal{B}$ so that $u_i = x_{n-1}x_{n-2}x_{n-4}w_t$ for some t. Observe that since $u_i \in \mathcal{B}$, we have $i \ge \ell + 1$. We claim that $i > \ell + 1$. Assume for a contradiction that $i = \ell + 1$. Then by Lemma 3.6, $(u_1, \dots, u_{i-1}) : (u_i) = (x_{n-3})$, which implies $x_{n-3}|u_j$, a contradiction. Hence, $i > \ell + 1$ indeed. Applying Lemma 3.6, we obtain $(u_1, \dots, u_{i-1}) : (u_i) = (u_{\ell+1}, \dots, u_{i-1}) : (u_i) + (x_{n-3})$.

Now since $(u_{\ell+1}, \dots, u_{i-1}) : (u_i) = (w_1, \dots, w_{t-1}) : (w_t)$, there exists a variable generator of this ideal dividing u_j and thus dividing w_s. Clearly $s \neq t$. If $s > t$, then by induction assumption on P_{n-5}, either w_tw_s is a non-minimal generator or w_tw_s is not a maximal expression, and the result follows as in Case 1. Lastly, suppose that $s < t$. Then we obtain a different expression for u_iu_j as follows:

$$\begin{aligned} u_iu_j &= (x_{n-1}x_{n-2}x_{n-4}w_t)(x_nx_{n-2}x_{n-4}w_s) \\ &= (x_{n-1}x_{n-2}x_{n-4}w_s)(x_nx_{n-2}x_{n-4}w_t) \\ &= u_{s+|\mathcal{A}|}u_{t+|\mathcal{A}|+|\mathcal{B}|} \text{ (by Figure 1)} = u_{s+\ell}u_{t+\ell+m}. \end{aligned}$$

Since $i = t + \ell$, we have $u_{s+\ell} >_{\mathcal{R}} u_i$ and the expression u_iu_j is not maximal.

Subcase 2.2. Suppose $u_i \in \mathcal{A}$ so that $u_i = x_{n-1}x_{n-3}v_i$. Observe that by Lemma 3.7(1) the variable x_{n-4} is not a generator of the ideal

$$(u_1, \dots, u_{i-1}) : (u_i) = (v_1, \dots, v_{i-1}) : (v_i)$$

and w_s is divisible by a variable generator of $(v_1, \dots, v_{i-1}) : (v_i)$.

By induction assumption, $(x_{n-3}v_i)(x_{n-2}x_{n-4}w_s)$ is either a non-minimal generator of $J(P_{n-2})^2$ or a non-maximal expression. If it is not a minimal generator, then it is divisible by some $(x_{n-3}v_\alpha)(x_{n-2}x_{n-4}w_\beta)$ and $v_\alpha w_\beta | v_iw_s$. In this case, multiplying $v_\alpha w_\beta$ by the appropriate variables, one can see that u_iu_j is not a minimal generator. Lastly, observe that if $(x_{n-3}v_i)(x_{n-2}x_{n-4}w_s)$ is a non-maximal expression, then so is u_iu_j. $\square$

Lemma 4.3. *Let I be a squarefree monomial ideal and let u be a minimal generator of I. Then u^s is a minimal generator of I^s for all s.*

Proof. Suppose that $v = v_1^{a_1} \cdots v_q^{a_q} \in G(I^s)$, where $v_1, \ldots, v_q$ are some minimal generators of I, $a_1 + \cdots + a_q = s$ and $a_i > 0$ for all i. Suppose that v divides u^s. Then each v_i divides u since v_i is squarefree. Hence, by the minimality of u we get $u = v_i$ for all $i = 1, \ldots, q$. $\square$

Remark 4.4. Note that in the above lemma the squarefreeness assumption cannot be omitted. For example, if $I = (a^2bc, b^2, c^2)$, then $(a^2bc)^2 \notin G(I^2)$.

The following lemma is of crucial importance to prove the main result stated in Theorem 5.1.

Lemma 4.5. *Let $U \in F(J(P_n)^2) \backslash G(J(P_n)^2)$. Then there exists $V \in G(J(P_n)^2)$ such that $V >_{\mathcal{R}} U$ and $V|U$.*

Proof. We will prove the assertion by applying induction on n. The statement holds for $n \leq 7$; see Table 2 for verification. Assume that $n \geq 8$.

Let $\mathcal{R}(P_n) = u_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} u_f$. Because of Lemma 4.3, we may assume that $U = u_i u_j$ is a maximal expression for some $i < j$. From Figure 1, which describes the branching of rooted order of minimal generators of $J(P_n)$, we see that we have the following three possibilities:

(1) $u_i, u_j \in x_{n-1}\mathcal{R}(P_{n-2})$;
(2) $u_i, u_j \in x_n x_{n-2}\mathcal{R}(P_{n-3})$;
(3) $u_i \in x_{n-1}\mathcal{R}(P_{n-2})$ and $u_j \in x_n x_{n-2}\mathcal{R}(P_{n-3})$.

Since $U \notin G(J(P_n)^2)$, there exists $U' \in G(J(P_n)^2)$ such that U' strictly divides U. Let $U' = u_p u_q$ be a maximal expression for some $p \leq q$. Now we discuss each of the above possibilities separately. Let $\mathcal{R}(P_{n-2}) = v_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} v_d$ and $\mathcal{R}(P_{n-3}) = l_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} l_e$.

(1) Let $u_i, u_j \in x_{n-1}\mathcal{R}(P_{n-2})$. Then $u_p, u_q \in x_{n-1}\mathcal{R}(P_{n-2})$ because $U'|U$. Also, in this case we have $U = (x_{n-1}v_{i'})(x_{n-1}v_{j'})$ and $U' = (x_{n-1}v_{p'})(x_{n-1}v_{q'})$ for some $v_{i'}, v_{j'}, v_{p'}, v_{q'} \in \mathcal{R}(P_{n-2})$. Thus, the monomial $v_{p'}v_{q'}$ strictly divides $v_{i'}v_{j'}$. By the induction hypothesis on P_{n-2}, there exists $v_{r'}v_{s'} \in G(J(P_{n-2})^2)$ such that $v_{r'}v_{s'}|v_{i'}v_{j'}$ and $v_{r'}v_{s'} >_{\mathcal{R}} v_{i'}v_{j'}$. Let $V = (x_{n-1}v_{r'})(x_{n-1}v_{s'})$. By Lemma 3.8, we see that $V \in G(J(P_n)^2)$. Note that $V|U$, and by following Lemma 3.3 we get $V >_{\mathcal{R}} U$, as required.

(2) Let $u_i, u_j \in x_n x_{n-2}\mathcal{R}(P_{n-3})$. Then $u_p, u_q \in x_n x_{n-2}\mathcal{R}(P_{n-3})$ because $U'|U$. Also, in this case $U = (x_n x_{n-2} l_{i'})(x_n x_{n-2} l_{j'})$ and $U' = (x_n x_{n-2} l_{p'})(x_n x_{n-2} l_{q'})$ for some $l_{i'}, l_{j'}, l_{p'}, l_{q'} \in \mathcal{R}(P_{n-3})$. Thus, the monomial $l_{p'}l_{q'}$ strictly divides $l_{i'}l_{j'}$. By the induction hypothesis on P_{n-3}, there exists $l_{r'}l_{s'} \in G(J(P_{n-3})^2)$ such that $l_{r'}l_{s'}|l_{i'}l_{j'}$ and $l_{r'}l_{s'} >_{\mathcal{R}} l_{i'}l_{j'}$. Let $V = (x_n x_{n-2} l_{r'})(x_n x_{n-2} l_{s'})$. By Lemma 3.9, we see that $V \in G(J(P_n)^2)$. Note that $V|U$, and by following Lemma 3.4 we get $V >_{\mathcal{R}} U$, as required.

(3) If $u_i \in x_{n-1}\mathcal{R}(P_{n-2})$ and $u_j \in x_n x_{n-2}\mathcal{R}(P_{n-3})$, then again from Figure 1 we see that either $u_i \in \mathcal{A}$ or $u_i \in \mathcal{B}$, and either $u_j \in \mathcal{C}$ or $u_j \in \mathcal{D}$. We list these four cases in the following way:

(3.a) $u_i \in \mathcal{A}$ and $u_j \in \mathcal{C}$;
(3.b) $u_i \in \mathcal{A}$ and $u_j \in \mathcal{D}$;
(3.c) $u_i \in \mathcal{B}$ and $u_j \in \mathcal{C}$;

(3.d) $u_i \in \mathcal{B}$ and $u_j \in \mathcal{D}$.
We define the following rooted lists:

$$\begin{aligned}\mathcal{R}(P_{n-4}) &= a_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} a_g,\\ \mathcal{R}(P_{n-5}) &= b_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} b_h,\\ \mathcal{R}(P_{n-6}) &= c_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} c_k.\end{aligned}$$

Case (3.a). If $u_i \in \mathcal{A}$ and $u_j \in \mathcal{C}$, then $U = (x_{n-1}x_{n-3}a_{i'})(x_nx_{n-2}x_{n-4}b_{j'})$ for some $a_{i'} \in \mathcal{R}(P_{n-4})$ and $b_{j'} \in \mathcal{R}(P_{n-5})$. Since $U'|U$, by Lemma 3.10 we get $u_p \in \mathcal{A}$ and $u_q \in \mathcal{C}$. Then $U' = (x_{n-1}x_{n-3}a_{p'})(x_nx_{n-2}x_{n-4}b_{q'})$ for some $a_{p'} \in \mathcal{R}(P_{n-4})$ and $b_{q'} \in \mathcal{R}(P_{n-5})$. Moreover, $U'|U$ gives $a_{p'}b_{q'}|a_{i'}b_{j'}$.

Note that $x_{n-3}a_{i'}, x_{n-3}a_{p'}, x_{n-2}x_{n-4}b_{j'}, x_{n-2}x_{n-4}b_{q'} \in \mathcal{R}(P_{n-2})$ and the monomial $(x_{n-3}a_{p'})(x_{n-2}x_{n-4}b_{q'})$ strictly divides $Y = (x_{n-3}a_{i'})(x_{n-2}x_{n-4}b_{j'})$, which shows $Y \in F(J(P_{n-2})^2)\backslash G(J(P_{n-2})^2)$. Then by the induction hypothesis on P_{n-2}, we know that there exists $Y' \in G(J(P_{n-2})^2)$ such that $Y'|Y$ and $Y' >_{\mathcal{R}} Y$. Observe that $Y' = (x_{n-3}a_{i''})(x_{n-2}x_{n-4}b_{j''})$ for some $a_{i''} \in \mathcal{R}(P_{n-4})$ and $b_{j''} \in \mathcal{R}(P_{n-5})$. Let $V = x_nx_{n-1}Y'$, and note that from Lemma 3.12 we have $V \in G(J(P_n)^2)$. Clearly V divides U. From Lemma 3.13 it follows that $V >_{\mathcal{R}} U$, as desired.

Case (3.b). If $u_i \in \mathcal{A}$ and $u_j \in \mathcal{D}$, then $U = (x_{n-1}x_{n-3}a_{i'})(x_nx_{n-2}x_{n-3}x_{n-5}c_{j'})$ for some $a_{i'} \in \mathcal{R}(P_{n-4})$ and $c_{j'} \in \mathcal{R}(P_{n-6})$. Since $U'|U$ and $U' = u_pu_q$, one can see that $u_p \in \mathcal{A}$. Also, $u_q \in \mathcal{C}$ or $u_q \in \mathcal{D}$.

If $u_p \in \mathcal{A}$ and $u_q \in \mathcal{C}$, then $U' = (x_{n-1}x_{n-3}a_{p'})(x_nx_{n-2}x_{n-4}b_{q'})$ for some $a_{p'} \in \mathcal{R}(P_{n-4})$ and $b_{q'} \in \mathcal{R}(P_{n-5})$. Thus, $U'|U$ gives

$$(x_{n-1}x_{n-3}a_{p'})(x_nx_{n-2}x_{n-4}b_{q'})|(x_{n-1}x_{n-3}a_{i'})(x_nx_{n-2}x_{n-3}x_{n-5}c_{j'}),$$

which implies $x_{n-4}a_{p'}b_{q'}|x_{n-3}x_{n-5}a_{i'}c_{j'}$. Therefore, $x_{n-4}|a_{i'}$ because $c_{j'} \in \mathcal{R}(P_{n-6})$. This shows that $x_{n-1}x_{n-4}|u_i$. Then by Lemma 4.1 we get the desired result.

If $u_p \in \mathcal{A}$ and $u_q \in \mathcal{D}$, then $U' = (x_{n-1}x_{n-3}a_{p'})(x_nx_{n-2}x_{n-3}x_{n-5}c_{q'})$ for some $a_{p'} \in \mathcal{R}(P_{n-4})$ and $c_{q'} \in \mathcal{R}(P_{n-6})$.

Keeping Figure 2 in mind, one can check that either x_{n-4} divides $a_{i'}$ or $a_{i'}$ is in $x_{n-5}\mathcal{R}(P_{n-6})$. If x_{n-4} divides $a_{i'}$, then u_i is divisible by $x_{n-1}x_{n-4}$ and the result follows from Lemma 4.1.

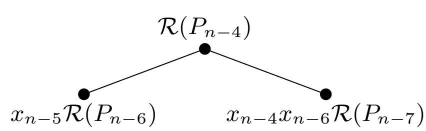

Figure 2 Branching of $\mathcal{R}(P_{n-4})$

Therefore, let us assume that $a_{i'} \in x_{n-5}\mathcal{R}(P_{n-6})$. Then it is not hard to show $a_{p'} \in x_{n-5}\mathcal{R}(P_{n-6})$ as well. Thus, a similar argument to that in Case (3.c) shows that we can find $V \in G(J(P_n)^2)$ such that $V|U$ and $V >_{\mathcal{R}} U$.

Case (3.c). If $u_i \in \mathcal{B}$ and $u_j \in \mathcal{C}$, then we have $u_i = x_{n-1}x_{n-2}x_{n-4}b_{i'}$ and $u_j = x_nx_{n-2}x_{n-4}b_{j'}$ for some $b_{i'}, b_{j'} \in \mathcal{R}(P_{n-5})$. Since $U'|U$, it follows from Lemma 3.10(2) that $U' = (x_{n-1}x_{n-2}x_{n-4}b_{p'})(x_nx_{n-2}x_{n-4}b_{q'})$ for some $b_{p'}, b_{q'} \in \mathcal{R}(P_{n-5})$. Further, $b_{p'}b_{q'}$ strictly divides $b_{i'}b_{j'}$. This shows $b_{i'}b_{j'} \in F(J(P_{n-5})^2)\backslash G(J(P_{n-5})^2)$.

Then by induction hypothesis, we know that there exists $Y \in G(J(P_{n-5})^2)$ such that $Y|b_{i'}b_{j'}$ and $Y >_{\mathcal{R}} b_{i'}b_{j'}$. Accordingly, by Lemma 3.16 we get

$$x_n x_{n-1} x_{n-2}^2 x_{n-4}^2 Y >_{\mathcal{R}} x_n x_{n-1} x_{n-2}^2 x_{n-4}^2 b_{i'} b_{j'} \quad \text{in } F(J(P_n)^2).$$

Now set $V = x_n x_{n-1} x_{n-2}^2 x_{n-4}^2 Y$, and the line above becomes $V >_{\mathcal{R}} U$ in $F(J(P_n)^2)$. Clearly, $V|U$. Because of Lemma 3.14 we get $V \in G(J(P_n)^2)$, as desired.

Case (3.d). If $u_i \in \mathcal{B}$ and $u_j \in \mathcal{D}$, by Lemma 4.1 we get the desired result. □

5 Linear Quotients of the Second Power of $J(P_n)$

We are now ready to prove our main theorem.

Theorem 5.1. *Let $G(J(P_n)^2) = \{U_1, \dots, U_p\}$. Then $J(P_n)^2$ has linear quotients with respect to $U_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} U_p$.*

Proof. We will prove the assertion by applying induction on n. The statement holds for $n \leq 5$; see Table 2 for verification. Suppose that $n \geq 5$. We need to show that $(U_1, \dots, U_{r-1}) : (U_r)$ is generated by variables for all $2 \leq r \leq p$. Let $\mathcal{R}(P_{n-2}) = m_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} m_a$ and $\mathcal{R}(P_{n-3}) = l_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} l_b$.

Case 1. Suppose that x_n^2 divides U_r. Let us assume that U_r has the maximal expression $U_r = (x_n x_{n-2} l_i)(x_n x_{n-2} l_j)$ for some $l_i, l_j \in \mathcal{R}(P_{n-3})$ with $i \leq j$. First, we claim that x_{n-1} is a generator of $(U_1, \dots, U_{r-1}) : (U_r)$.

In fact, by Lemma 3.5 there exits $m_q \in \mathcal{R}(P_{n-2})$ such that $m_q | x_{n-2} l_i$. Let

$$V = (x_{n-1} m_q)(x_n x_{n-2} l_j).$$

Notice that $V : U_r = x_{n-1}$. Moreover, if $V \in G(J(P_n)^2)$, then $V >_{\mathcal{R}} U_r$. Otherwise, by Lemma 4.5 there exists U_k with $1 \leq k \leq r-1$ such that $U_k | V$ and $U_k >_{\mathcal{R}} V$. Then $U_k : U_r = x_{n-1}$, which proves the claim.

Let $\mathcal{R}(J(P_{n-3})^2) = L_1 >_{\mathcal{R}} L_2 >_{\mathcal{R}} \cdots >_{\mathcal{R}} L_s$ and $L_t = l_i l_j$. Now we will show that $(U_1, \dots, U_{r-1}) : (U_r) = (x_{n-1}) + (L_1, L_2, \dots, L_{t-1}) : (L_t)$. Observe that the proof will be complete once we prove the equality above because of the induction assumption on P_{n-3}. Combining Lemmas 3.4 and 3.9 and the claim that has been proved, we obtain $(x_{n-1}) + (L_1, L_2, \dots, L_{t-1}) : (L_t) \subseteq (U_1, \dots, U_{r-1}) : (U_r)$.

It remains to show that the reverse inclusion holds. Note that for each $\ell \leq r-1$, the monomial U_ℓ is divisible by either $(x_{n-2}x_n)^2$ or x_{n-1} by the definition of rooted order. If $x_{n-1}|U_\ell$, then it is easy to see that in this case $U_\ell : U_r \in (x_{n-1})$. If $(x_{n-2}x_n)^2 | U_\ell$, then by Lemma 3.9 we have $U_\ell / (x_{n-2}x_n)^2 = L_k$ for some k. Clearly we have $U_\ell : U_r = L_k : L_t$. Furthermore, since $U_\ell >_{\mathcal{R}} U_r$, by Lemma 3.4 we get $L_k >_{\mathcal{R}} L_t$, which completes the proof in this case.

Case 2. Suppose that x_{n-1}^2 divides U_r. Let $U_r = (x_{n-1} m_i)(x_{n-1} m_j)$ be the maximal expression for some $m_i, m_j \in \mathcal{R}(P_{n-2})$ with $i \leq j$. Then the monomial $m_i m_j$ is also in its maximal expression by Remark 3.1. Thus, Lemma 3.8 implies $m_i m_j \in G(J(P_{n-2})^2)$. Let $\mathcal{R}(J(P_{n-2})^2) = M_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} M_s$. Then $m_i m_j = M_t$ for some $1 < t \leq s$. Note that $1 < t$, because if $t = 1$ then $r = 1$, which is not true. By induction hypothesis, $(M_1, \dots, M_{t-1}) : (M_t)$ is generated by variables. We claim that $(M_1, \dots, M_{t-1}) : (M_t) = (U_1, \dots, U_{r-1}) : (U_r)$.

Indeed, by Remark 3.1 and Lemma 3.8, it is clear that

$$(M_1, \ldots, M_{t-1}) : (M_t) \subseteq (U_1, \ldots, U_{r-1}) : (U_r).$$

We need to show the reverse inclusion. Observe that for every $\ell \leq r-1$, the monomial U_ℓ is divisible by either x_{n-1}^2 or $x_{n-1}x_n$ by the definition of rooted order. If x_{n-1}^2 divides U_ℓ, then again by Lemma 3.8 we get $U_\ell/x_{n-1}^2 = M_k$ for some k. Clearly, $U_\ell : U_r = M_k : M_t$. Therefore, it remains to show $k < t$. Lemma 3.3 together with $U_\ell >_{\mathcal{R}} U_r$ implies $M_k >_{\mathcal{R}} M_t$, as desired.

If $x_{n-1}x_n$ divides U_ℓ, then we may assume that $U_\ell = (x_{n-1}m_h)(x_n x_{n-2} l_q)$ is the maximal expression for some $m_h \in \mathcal{R}(P_{n-2})$ and $l_q \in \mathcal{R}(P_{n-3})$. Then by Lemma 3.5 there exists $m_v \in \mathcal{R}(P_{n-2})$ such that $m_v | x_{n-2} l_q$.

Note that since $U_\ell >_{\mathcal{R}} U_r$, we must have $m_h >_{\mathcal{R}} m_i, m_j$ in $\mathcal{R}(P_{n-2})$ by Lemma 3.3. Now consider

$$P = (x_{n-1}m_h)(x_{n-1}m_v).$$

If $P \in G(J(P_n)^2)$, then $P >_{\mathcal{R}} U_r$ and $P : U_r \in (M_1, \ldots, M_{t-1}) : (M_t)$. Since $P : U_r$ divides $U_\ell : U_r$, it follows that $U_\ell : U_r \in (M_1, \ldots, M_{t-1}) : (M_t)$.

If $P \notin G(J(P_n)^2)$, then by Lemma 4.5 there exists $U_\alpha \in G(J(P_n)^2)$ such that $U_\alpha | P$ and $U_\alpha >_{\mathcal{R}} P$. Thus, $U_\alpha > U_r$ and $U_\alpha : U_r \in (M_1, \ldots, M_{t-1}) : (M_t)$. Since $U_\alpha : U_r$ divides $P : U_r$ and $P : U_r$ divides $U_\ell : U_r$, we see that $U_\alpha : U_r$ divides $U_\ell : U_r$ and $U_\ell : U_r \in (M_1, \ldots, M_{t-1}) : (M_t)$, as desired.

Case 3. Suppose that $x_n x_{n-1}$ divides U_r. Let $U_r = (x_{n-1}m_i)(x_n x_{n-2} l_j)$ be the maximal expression for some $m_i \in \mathcal{R}(P_{n-2})$ and $l_j \in \mathcal{R}(P_{n-3})$.

Claim 1: $x_{n-1} \in (U_1, \ldots, U_{r-1}) : (U_r)$.

Indeed, by Lemma 3.5 there exists $m_k \in \mathcal{R}(P_{n-2})$ such that $m_k | x_{n-2} l_j$. Take $M = (x_{n-1}m_k)(x_{n-1}m_i) \in F(J(P_n)^2)$. If $M \in G(J(P_n)^2)$, then $M >_{\mathcal{R}} U_r$ and $M : U_r = x_{n-1}$, which proves the claim. If $M \notin G(J(P_n)^2)$, then by Lemma 4.5 there exists $U_s \in G(J(P_n)^2)$ such that $U_s | M$ and $U_s >_{\mathcal{R}} M$. Thus, $U_s >_{\mathcal{R}} U_r$ and $U_s : U_r = x_{n-1}$, which proves our claim.

Claim 2: If $i \geq 2$, we have $(m_1, \ldots, m_{i-1}) : (m_i) \subseteq (U_1, \ldots, U_{r-1}) : (U_r)$.

Indeed, by Remark 2.3 the ideal $(m_1, \ldots, m_{i-1}) : (m_i)$ is generated by variables. To prove our claim, let $t < i$ such that $m_t : m_i = x_z$ for some variable x_z. Consider $M = (x_{n-1}m_t)(x_n x_{n-2} l_j)$. If $M \in G(J(P_n)^2)$, then $M >_{\mathcal{R}} U_r$ and $M : U_r = x_z$. Otherwise, by Lemma 4.5 there is $U_k \in G(J(P_n)^2)$ such that $U_k | M$ and $U_k >_{\mathcal{R}} M$. Thus, $U_k >_{\mathcal{R}} U_r$ and $U_k : U_r = x_z$, which proves our claim.

Claim 3: If $j \geq 2$, then $(l_1, \ldots, l_{j-1}) : l_j \subseteq (U_1, \ldots, U_{r-1}) : (U_r)$.

In fact, by Remark 2.3 we know that the ideal $(l_1, \ldots, l_{j-1}) : (l_j)$ is generated by variables. Let $t < j$ such that $l_t : l_j = x_z$ for some variable x_z. Consider $M = (x_{n-1}m_i)(x_n x_{n-2} l_t)$. If $M \in G(J(P_n)^2)$, then we obtain $M >_{\mathcal{R}} U_r$ and $M : U_r = x_z$. Otherwise, by Lemma 4.5 there exists $U_k \in G(J(P_n)^2)$ such that $U_k | M$ and $U_k >_{\mathcal{R}} M$. Thus, $U_k >_{\mathcal{R}} U_r$ and $U_k : U_r = x_z$, which proves our claim.

Let $t < r$. By Claims 1 and 3, we may assume that $U_t = (x_{n-1}m_p)(x_n x_{n-2} l_q)$ and $p < i$. By Remark 2.3 there exists a variable $x_z \in (m_1, \ldots, m_{i-1}) : (m_i)$ such that x_z divides $m_p : m_i$. Observe that by Lemma 3.7(1) we get $x_z \neq x_{n-2}$. Proposition 4.2 implies that the monomial l_j is not divisible by x_z as $U_r \in G(J(P_n)^2)$ and the expression $U_r = (x_{n-1}m_i)(x_n x_{n-2} l_j)$ is maximal. Thus, x_z divides $U_t : U_r$, and the result follows from Claim 2. □

Lemma 5.2. *Let a_n denote the maximum degree of a minimal monomial generator of $J(P_n)$. For any $n \geq 5$ we have $a_n = \max\{a_{n-2}+1,\, a_{n-3}+2\}$. For any $n \geq 2$,*

$$a_n = \begin{cases} 2k & \text{if } n = 3k+1 \text{ or } n = 3k, \\ 2k+1 & \text{if } n = 3k+2. \end{cases}$$

Proof. The result follows from Lemma 2.1. □

If an ideal has linear quotients, then its regularity is equal to the highest degree of a generator in a minimal set of generators; see [16, Corollary 8.2.14]. Therefore, as a consequence of Theorem 5.1, we obtain the following result.

Corollary 5.3. *For any $n \geq 2$,*

$$\operatorname{reg}(J(P_n)^2) = \begin{cases} 4k & \text{if } n = 3k+1 \text{ or } n = 3k, \\ 4k+2 & \text{if } n = 3k+2. \end{cases}$$

Proof. If $u \in G(J(P_n))$, then $u^2 \in G(J(P_n)^2)$ by Lemma 4.3. The result follows from Lemma 5.2. □

Concluding remarks. We can generalize the concept of rooted list to chordal graphs as follows. First, let us introduce some notation. If v is a vertex of G, then the set of neighbors of v is denoted by $N(v)$. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. If A is a subset of vertices of G, then $G \backslash A$ denotes the graph which is obtained from G by removing the vertices in A.

Let G be a chordal graph with a simplicial vertex v_1 such that $N[v_1] = \{v_1, \ldots, v_r\}$ for some $r \geq 2$. Suppose that for each $i = 1, \ldots, r$, the list $\mathcal{R}(H_i)$ is a rooted list of the subgraph $H_i = G \backslash N[v_i]$. Then we say that

$$\mathcal{R}(H_1)N(v_1), \quad \mathcal{R}(H_2)N(v_2), \quad \ldots, \quad \mathcal{R}(H_r)N(v_r)$$

is a rooted list of G. Note that this list indeed consists of the minimal generators of $J(G)$; see [5, Theorem 3.1].

Observe that a path graph has only two simplicial vertices, namely the vertices at both ends of the path. However, a chordal graph in general can have many simplicial vertices. Therefore, one can construct rooted lists of chordal graphs recursively in different ways. We give below an example of how to construct a rooted list for a chordal graph.

Example 5.4. Consider the graph in Figure 3. Observe that a is a simplicial vertex with $N(a) = \{b, c\}$. We use the following rooted lists:

$$\mathcal{R}(G \backslash N[a]) = de, ef, df, \quad \mathcal{R}(G \backslash N[b]) = \emptyset, \quad \mathcal{R}(G \backslash N[c]) = d, f.$$

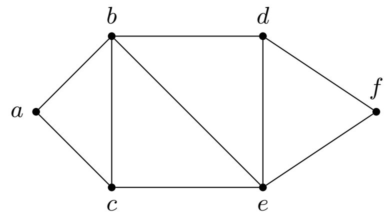

Figure 3 A chordal graph

Moreover, since we have $N(a) = bc$, $N(b) = acde$ and $N(c) = abe$, we can get the following rooted list for G:

$$u_1, u_2, u_3, u_4, u_5, u_6 := bcde,\ bcef,\ bcdf,\ acde,\ abed,\ abef.$$

Notice that since $J(G)$ is generated in single degree, every 2-fold product $u_i u_j$ is a minimal generator of $J(G)^2$. There is only one minimal generator of $J(G)^2$ which has multiple expressions, namely

$$ab^2cde^2f = u_1u_6 = u_2u_5.$$

Using Macaulay2 [11], we list the minimal generators of $J(G)^2$ in the rooted order as in Definition 2.6 and we confirm that such an order yields linear quotients.

It would be interesting to know whether the following question has a positive answer because it would settle the case of second powers for Conjecture 1.1.

Question 5.5. If G is a chordal graph, then does $J(G)^2$ have linear quotients with respect to a rooted list of minimal generators?

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