MULTI-MODE CAPACITATED LOT SIZING PROBLEM WITH PERIODIC CARBON EMISSION CONSTRAINTS

by
GÜNIZ IRMAK KÖKSALAN

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ABSTRACT

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GÜNIZ IRMAK KÖKSALAN

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In this thesis, we investigate the single item capacitated multi-mode lot sizing problem with periodic carbon emission constraints where the carbon emission constraints define an upper bound for average emission per product produced in any period. The uncapacitated version of this problem was discussed in Absi et al. (2013) and solved in polynomial time. We prove that this generalization of the problem is NP-Hard and discuss important structural properties of optimal solutions. We develop algorithms to construct the piecewise linear total production cost functions for each period when the number of modes is fixed where mode represents the number of machines available for usage in production. This allows us to solve the problem using existing dynamic programming algorithms developed for the lot sizing problem with piecewise concave production cost functions. Additionally, we examine an extension of the problem where at most two resources can be used at any period, and produce a polynomial time algorithm to solve it when the number of resources, the cost and emission parameters, and the capacities of the resources are time-invariant.

ÖZET

ÇOK MODLU KAPASİTELİ VE DÖNEMSEL KARBON EMİSYON KISITLI KAFİLE BÜYÜKLÜĞÜ PROBLEMİ

GÜNIZ IRMAK KÖKSALAN

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Tez Danışmanı: Dr. Öğretim Üyesi Esra Koca Paç

Anahtar Kelimeler: Kapasite kısıtlı kafile büyüklüğü, Dönemsel karbon emisyon kısıtları, Çok modlu, Karmaşıklık analizi, Dinamik programlama

Bu tezde, çok modlu kapasiteli ve dönemsel karbon emisyon kısıtlı kafile büyüklüğü problemini araştırıyoruz. Dönemsel karbon emisyon kısıtlamaları bir dönemde üretilen ürün başına ortalama emisyon için bir üst sınır tanımlanması anlamına gelmektedir. Bu problemin kapasite kısıtsız versiyonu Absi ve ark. (2013) tarafından ortaya atılmış ve polinom zamanda çözülmüştür. Problemin bu genellemesinin NP-Zor olduğunu kanıtlıyoruz ve optimal çözümlerin karmaşıklık ve yapısal özellikleri tartışılmıştır. Mod sayısı sabit olduğunda her bir dönem için parçalı doğrusal toplam üretim maliyeti fonksiyonlarını oluşturan algoritmalar geliştirilmiştir. Bu algoritma sayesinde, parçalı içbükey üretim maliyeti fonksiyonları ile kafile büyüklüğü problemi için geliştirilmiş mevcut dinamik programlama algoritmalarını kullanarak problemi çözülmüştür. Ek olarak, genel problemin herhangi bir dönemde en fazla iki kaynağın kullanılabileceği uzantısı incelenmiştir ve kaynak sayısı, maliyet ve emisyon parametreleri ve kaynakların kapasitelerinin zamanla değişmez olduğu özel durum için polinom zamanlı bir algoritma geliştirilmiştir.

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To my loving parents Ayşe and Mehmet

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1. INTRODUCTION

Today, climate change and global warming are one of the greatest issues influencing our lives. Indeed, the last ten years (2010-2019) are the warmest decade on human history (United Nations, 2020). Under the supervision of United Nations there is a consensus among scientists that human activities are the main reason of global warming and the increase in temperature (NASA, 2018). It is important to realize that the reason behind the temperature increase is the rise of carbon dioxide levels in atmosphere which adversely affects not only social but also economic life. The work of Burke, Davis & Diffenbaugh (2018) links the temperature with gross domestic product (GDP). The study demonstrates a possible scenario under the preventive measures of Paris Climate Agreement: if the temperature continue to increase by up to 4 degrees till the end of century, the worldwide GDP per capita in 2100 will decrease worse than the Great Depression times, more than 30% (Burke et al., 2018).

Alarmed by economic and environmental concerns, countries worldwide begin to include this topic in their agenda. Governments enforce firms and companies to take serious action towards reducing greenhouse gases. Using public awareness governments administrated new laws limiting the carbon emission levels of businesses. For this reason, companies introduce carbon emission constraints to their production planning and transportation systems designs. This marks a need to understand the various perceptions of carbon emission constraints that exist among production planning problems. As a result, researchers have shown an increased interest in green production planning problems.

In the context of production planning, lot sizing is a crucial planning process that directly impacts the system performance. For this reason the right lot sizing decision is very critical for a company to be competitive in the market (Karimi, Fatemi Ghomi & Wilson, 2003). A considerable amount of literature has been published on both static inventory and dynamic lot sizing problems with carbon emission constraints. These studies focused on uncapacitated lot sizing problems with single item and multi-item extensions. Additionally, few focused on the multi-mode case where each mode represents machines used in production. However, far too little attention has been paid to the capacitated single item green lot sizing problem with dynamic inventory models. This thesis intends to tackle this issue. This study aims to fill the gap of capacitated green lot sizing problem by introducing a new problem: capacitated multi-mode single item lot sizing problem with periodic carbon emission constraints (CLS-PC).

The overall structure of the thesis takes the form of eight chapters, including this introductory chapter. Chapter 2 consists of the literature review on the lot sizing problem. In Chapter 3, the CLS-PC problem is introduced, and its complexity and structural properties are discussed in detail. Chapter 4 considers the special case of CLS-PC problem with exactly two resources, Chapter 5 examines the problem without setup costs which helps us to construct the structure of the total production cost function for the general case in Chapter 6. Chapter 7 illustrates an extension of the problem where at most two resources can be used at any period. Finally, Chapter 8 gives a brief summary and critique of the findings.

2. LITERATURE REVIEW

This chapter begins by a brief overview of the capacitated, uncapacitated and multi-source lot sizing problems. Then, the related literature on the green lot sizing problems will be reviewed. A systematic literature review of the lot sizing problem and extensions can be found in Drexl & Kimms (1997), Yves & Wolsey (2006), Jans & Degraeve (2008).

First, we will shortly discuss the classical lot sizing problem. In the classical lot sizing problem, we determine a minimum cost production plan over a planning horizon of T periods. For each period t, let d_t be the demand, p_t be the unit production cost, f_t be the fixed production cost and h_t be the unit inventory holding cost. x_t be the amount produced at period t, y_t be the binary variable representing whether there is production or not in period and s_t be the inventory (stock) on hand at the end of period t. Using these definitions we can formulate the classical lot sizing problem as LS:

$$\min \quad \sum_{t=1}^{T}(p_t x_t + f_t y_t) + \sum_{t=0}^{T} h_t s_t \tag{2.1a}$$

$$\text{s.t.} \quad s_{t-1} + x_t = d_t + s_t \qquad t = 1, \ldots, T \tag{2.1b}$$

$$x_t \leq M_t y_t \qquad t = 1, \ldots, T \tag{2.1c}$$

$$s_0 = 0 \tag{2.1d}$$

$$x_t, s_t \geq 0 \qquad t = 1, \ldots, T \tag{2.1e}$$

$$y_t \in \{0, 1\} \qquad t = 1, \ldots, T \tag{2.1f}$$

The objective function (2.1a) minimizes the production costs and the total inventory holding cost over the planning horizon T. Constraints (2.1b) are the inventory balance equations ensuring for each period t, inventory at the beginning of a period plus the amount produced in that period should be sufficient to satisfy the demand of the period, and the remaining items will be kept in the inventory at the end of the period. Constraints (2.1c) guarantee that when there is no production $y_t = 0$, the quantity of production is equal to zero $x_t = 0$. The value of M_t determines whether the problem is *uncapacitated* or *capacitated*. We call the problem *uncapacitated* when the M_t value in (2.1c) is a very big number, so large that one can produce in sufficiently large quantities in each period. We call the problem *capacitated* when Big M_t value is equal to the capacity of period t. Constraints (2.1d) assumes there are no initial inventory. Constraints (2.1e) and (2.1f) define the types and ranges of the decision variables.

The lot sizing problem can be classified into single item and multi-item based on the number of items produced. Moreover, in terms of the number of machines used in the problem one can define two subgroups: single machine CLSP and multi machine CLSP. In literature the terms, multi-mode, multi-machine, multi-source and multi-resource are frequently used interchangeably. In 2006 and 2017 Brahimi et al. reviewed the single item lot sizing problem literature and examined the mathematical programming formulations, complexity and extensions.

2.1 Capacitated & Uncapacitated Lot Sizing Problem

The first serious discussions and analyses of lot sizing problem emerged during the late 1950s with seminal work of Wagner & Whitin (1958). Henceforth, the lot sizing problem has been studied extensively. Wagner & Whitin introduce the uncapacitated single-item lot sizing problem and solve the problem by a $O(T^2)$ dynamic programming algorithm A number of studies (Aggarwal & Park, 1993; Federgruen & Tzur, 1991; Wagelmans, Hoesel & Kolen, 1992) decrease time complexity of Wagner & Whitin to $O(TlogT)$.

Florian & Klein (1971) introduced the capacitated lot sizing problem (CLSP). CLSP is NP-Hard (Bitran & Yanasse, 1982; Florian, Lenstra & Rinnooy Kan, 1980) in general. Since the problem is proven to be NP-Hard, a number of studies have examined heuristic approaches. For instance, Diaby, Bahl, Karwan & Zionts (1992) introduced a Lagrangian Relaxation based heuristic to solve the CLSP.

In 1982, Bitran & Yanasse published a paper in which they described polynomially solvable special cases of CLSP. Bitran & Yanasse showed that the non-increasing setup and production cost, constant capacity single item CLSP can be solved in $O(T^3)$, fixed setup and production cost, zero inventory cost single item CLSP can be solved in $O(TlogT)$, non-decreasing setup and production cost, zero inventory cost and non increasing capacity single item CLSP can be solved in $O(T)$ time.

Wagelmans et al. (1992) developed an $O(T^3)$ complexity algorithm to solve the constant capacity CLSP.

A number of studies have examined the lot sizing problem extensions: setup times, backlog and lost sales. The studies on these extensions can be seen in the book by Yves & Wolsey (2006).

Zangwill (1969) showed that the extension of ULSP with backlogging is still solvable using Wagner & Whitin algorithm. Aksen, Altinkemer & Chand (2003) modeled the single item uncapacitated lot sizing problem with lost sales by converting the objective to profit maximization. Moreover, they introduced a Wagner & Whitin variant dynamic programming algorithm with $O(T^2)$ complexity. Absi, Detienne & Dauzère-Pérès (2013) examine metaheuristics to solve the multi item capacitated lot sizing problem with lost sales. Absi et al. (2013) highlight the Lagrangian Relaxation based metaheuristic since it provides good lower bounds.

2.2 Multi - Mode Lot Sizing Problem

The multi-mode problem is often called as multi-machine and multi-source in the literature. The multi-machine term is generally classified into two types: identical machines also known as parallel machines, and non-identical machines or multi-machine or alternative machines. Very little was found in the literature on the multi-source lot sizing problem, with non-identical machines. There is a large volume of published studies focusing on lot sizing problem with parallel machines. Cheng & Sin (1990) provides a general review of the parallel machine scheduling problems. Furthermore, the current literature concentrates on multi-item multi-source lot sizing problems (Wu, Xiao, Zhang, He & Liang, 2018).

Erenguc & Tufekci (1987) studied the single item multi-source dynamic capacitated lot sizing problem with backlogging and considered production and subcontracting as the sources of supply. The authors propose a branch and bound algorithm to solve this problem.

Several studies investigated the capacitated lot sizing problem with multi-sources, where each machine had different efficiencies (Belvaux & Wolsey, 2000; Kang, Malik & Thomas, 1999; Ozdamar & Barbarosoglu, 1999; Ozdamar & Birbil, 1998). Kang et al. (1999) focused on non-identical machines with sales and sequence dependent setup costs. They proposed a column generation based algorithm to solve the problem.

Belvaux & Wolsey (2001) formulated the multi-machine lot sizing problem and extensions using mixed integer programming. Belvaux & Wolsey studied reformulations and cutting planes, emphasizing the need for heuristics to solve large problem instances. Jans (2009) different from Belvaux & Wolsey (2001) analysed the parallel machine problem, used branch and bound to solve the resulting problem. Additionally, Jans (2009), pointed out the fact that little attention has been paid to non-identical multi-resource lot sizing problem.

Akbalik & Penz (2009) introduced an extension of the capacitated single item lot sizing problem, where the production occurs on multi-machines (alternative machines). Akbalik & Penz (2009) proved that the problem is NP-Hard. Each resource (machine) has different stepwise production cost and a different capacity, making the computation of total production and inventory cost to be harder. Akbalik & Penz (2009) suggested a pseudo-polynomial time dynamic programming algorithm to solve the single-item multi-source capacitated lot sizing problem.

2.3 Lot Sizing Problem with Carbon Emission Constraints

The carbon emission constraints are acknowledged on both static inventory and dynamic lot sizing problems. This study concentrates on dynamic lot sizing models. For further information on Economic Order Quantity (EQQ) problems with carbon emission constraints we recommend readers to check the literature review by Christata & Daryanto (2020).

There are two basic approaches on combining carbon emission restrictions with lot sizing problems. One is keeping the carbon emission restrictions in the form of constraints. And the other is penalizing the carbon emissions with a coefficient and "penalizing" in the objective function. The former approach is more popular in the literature.

The first systematic study of lot sizing problem with carbon emission constraints was reported by Benjaafar, Li & Daskin (2013). This paper provided a mathematical formulation of lot sizing problems with a global carbon emission limit (constraint). This new global emission constraint limits all carbon emissions related to production and storage activities over the planning horizon, while giving the producers a chance of creating large carbon emissions at the beginning of the planning horizon by producing large amounts and balancing the total emission by producing nothing at the end of the planning horizon. The authors considered this constraint under four policies: carbon cap, carbon cap and trade, carbon cap and offset, and carbon taxing. Carbon cap policy imposes an upper bound on production quantities, enforces a limit on the carbon emission level. Carbon cap and trade policy gives the company the opportunity of buying and selling carbon units in the carbon market in case of need or surplus. Carbon offset policy enables the company to purchase carbon units from independent suppliers and/or invest into projects where one can purchase carbon units while satisfying carbon limitations. Carbon taxing policy forces the company to pay taxes for each carbon units emitted. The authors explored the effects of these practices on the management decisions, provide in depth analysis of problem parameters and concentrate on mathematical formulation. However, they did not examine solution methods.

In a conference paper, Absi, Dauzère-Pérès, Kedad-Sidhoum, Penz & Rapine (2012) presented a different approach where instead of a global carbon emission limit (Benjaafar et al., 2013), a unit dependent maximum average emission allowance is permitted. Absi et al. (2012) introduced a new problem, uncapacitated multi-sourcing lot sizing problem with carbon emission constraints, analyzed its' complexity and provided experimental results.

Absi et al. extended the conference paper and discussed the problem in depth in Absi, Dauzère-Pérès, Kedad-Sidhoum, Penz & Rapine (2013). The authors analysed the carbon emission constraint for periodic, cumulative, global and rolling carbon scenarios under the assumption that only production activities generate carbon emission. Note that, the formulation of the emission constraints do not violate the supply capacities, since one can always select another resource to satisfy the emission constraints. The periodic carbon emission is the most restrictive one since it forces the quantity of carbon emission to be lower than or equal than the maximum environmental allowance for each period. The cumulative carbon emission constraint is weaker. It allows the amount of unused carbon emission of a given period to be used in future without exceeding the cumulative capacity. Global carbon emission constraint is also weaker than periodic carbon emission. The formulation of this constraint is identical to Benjaafar et al. (2013). The rolling carbon emission constraint assumes that carbon emissions can be compensated only for a rolling horizon of R periods which seems like a more realistic approach compared to cumulative and global emission constraints. This formulation makes the problem less dependent on planning horizon. When $R = 1$ we get the periodic carbon emission, and $R = T$ we get the global emission constraint. The authors showed that the problem with periodic carbon emission constraints reduces to the uncapacitated multi-sourcing lot sizing problem, and the latter one can be solved by a polynomial time dynamic programming algorithm. The remaining three multi-sourcing lot sizing problem with the cumulative, global and rolling carbon emission were proved to be NP-Hard. Since this work was the first study to model lot sizing problem with carbon emission constraints, these findings provide insights for a further study with new models on lot sizing problem with carbon emission constraints.

Absi et al. continued to work on another extension of the lot sizing problem with carbon emission constraints. In 2016, they studied the effect of fixed carbon emission with periodic carbon emission constraint (ULS-FPC) with the same approach: limiting average carbon emission value for each product without including production capacity each period. Absi et al. (2016) showed that even the simplest case where there is no inventory holding cost with two available resources the problem is NP-Hard. Two dynamic programming algorithms (one polynomial and one pseudo-polynomial) are developed to solve the ULS-FPC extension where the number of resources are fixed.

Helmrich, Jans, Van den Heuvel & Wagelmans (2015) similar to Benjaafar et al. study the global carbon emission constraint. In their case production activities and inventory holding generate carbon emission and the problem is uncapacitated. Helmrich et al. (2015) demonstrated that the resulting uncapacitated lot sizing problem with global carbon emission constraints is NP-Hard. Moreover, Helmrich et al. (2015) point out when holding cost is not included as a carbon emission activity (only production generates emission) and two resources are available for production with a linear emission values the problem is still NP-Hard. To solve the problem two heuristic algorithms (Lagrangian heuristic and Fully Polynomial Time Approximation Scheme (FPTAS)) are presented.

Velázquez-Martínez, Fransoo, Blanco & Mora-Vargas (2014) also discussed a global emission constraint for the lot sizing problem but in their approach they focused on carbon emission generated by transportation activities. They assumed carbon emission generated by production activities are linear functions of the production amount. Additionally, Velázquez-Martínez et al. (2014) highlighted the fact that little is known regarding capacitated lot sizing problem with carbon emission constraints.

Using the carbon cap and trade policy, Akbalik & Rapine (2014) formulated the problem by assuming that it is possible to buy or sell the carbon units in a carbon market. As a result, Akbalik & Rapine polynomially solved the problem of Helmrich et al. (2015) by reducing the problem into ULSP. However, their study show that including a capacity constraint on carbon emission levels make the problem NP-Hard.

Hong, Chu & Yu (2016) constructed a two resource (green and regular) ULSP with periodic cap and trade carbon emission policy constraints. Carbon emission is only generated from production activities. In contrast to Absi et al. (2013) and Helmrich et al. (2015), Hong et al. (2016) used a fixed carbon emission cap for each period of production. To solve the resulting problem a dynamic programming algorithm is proposed.

Wu et al. (2018) introduced a multi-item multi-machine lot sizing problem with periodic carbon emission constraints. Wu et al. (2018) used the mathematical formulation of Absi et al. (2013) and discussed Lagrangian relaxation, Dantzig-Wolfe decomposition and column generation methods to enhance lower bounds and solve the problem heuristically.

So far, there has been little discussion about including carbon emission cost in the objective function. A number of authors have reported about static inventory models (for example: Bouchery, Ghaffari, Jemai & Dallery (2012); Chen, Benjaafar & Elomri (2013); Konur & Schaefer (2014). However, few writers (Palak, Ekşioğlu & Geunes, 2014; Romeijn, Morales & Van den Heuvel, 2014) studied on the dynamic lot sizing problems with carbon emission costs in the literature.

Romeijn et al. (2014) solved a bilevel lot sizing problem with global and periodic carbon emission constraints where the second objective is minimizing emission costs. Romeijn et al. (2014) provided the Pareto efficient solutions of the problem and presented detailed complexity analysis for different parameter settings.

Palak et al. (2014) concentrated on modelling a real life case taking carbon emission costs in the objective function. Carbon is generated by transportation activities and holding inventory. Carbon cap, carbon cap and trade, carbon offset and carbon taxing policies are considered. Carbon cap,and carbon cap and trade models are polynomially solvable, whereas the remaining two policies are proved to be NP-Hard.

In this thesis, we consider the periodic carbon emission constraints proposed by Absi et al. (2013) in a capacitated multi-mode lot sizing problem.

In Figure 2.1, 6 mostly related studies are compared with the current study in terms of number of items, production modes, resource capacities, carbon emission policies, carbon emission types and solution methods. It is a brief summary of the contribution of our work in to the field of interest.

Reference Name	Items	Production Mode	Resource Capacity	Carbon Emission Policy	Carbon Emission Type	Solution Method
Absi et al. (2013)	single	multi	uncap	carbon cap	global, cumulative, rolling & ***periodic***	DP
Benjafaar et al. (2013)	multi	single	uncap	carbon cap, carbon tax, carbon cap and	global	
Akbalik & Rapine (2014)	single	single	uncap	carbon cap and trade	global	
Velázquez-Martínez et al. (2014)	single	single	uncap	carbon cap	global	
Helmrich et al. (2015)	single	single	uncap	carbon cap	global	FPTAS
Wu et al. (2018)	multi	single	uncap	carbon cap	periodic	PS Heuristic
This Study	single	multi	cap	carbon cap	periodic	

Figure 2.1 Comparison of our study with the related literature

3. MULTI-MODE CAPACITATED LOT SIZING PROBLEM WITH PERIODIC CARBON EMISSION CONSTRAINTS (CLS-PC)

3.1 Problem Definition

We consider a multi-mode lot sizing problem with carbon emission constraints, called capacitated multi-mode lot sizing problem with periodic carbon emission constraints (CLS-PC). Assume that there are M modes (resources or machines) with different carbon emissions (allowance), e_t^m. We must determine how much to produce using which resource in order to satisfy a deterministic time dependent demand of a single item while satisfying a periodic carbon emission constraint over a finite planning horizon of T periods.

Our problem CLS-PC is the generalization of both the multi-mode lot sizing problem (MMLS) and the uncapacitated single-item problem with periodic carbon emission constraint, called ULS-PC in Absi et al. (2013). We need at least one of these resources $m = 1, \ldots, M$ to produce the product. But the products produced by different resources will have different carbon emission levels, and different setup and production costs. Let p_t^m be the unit production costs for the product produced by resource $m = 1, \ldots, M$ in period $t = 1, \ldots, T$. Let f_t^m be the setup cost for period t when resource m is used. We assume that we know the values of these parameters. Let resource capacity, C_t^m be the production capacity for the resource m in period t.

To formulate the problem, we define the following decision variables:

- y_t^m : 1 if resource m is used in period t to produce the product, 0 o/w for $t = 1, \ldots, T$, $m = 1, \ldots, M$

- x_t^m : amount produced with resource m at period t for $t = 1, \dots, T$, $m = 1, \dots, M$
- s_t : inventory on hand at the end of period t for $t = 1, \dots, T$

The periodic carbon emission constraint aims to ensure that at any period the average carbon emitted does not exceed the periodic carbon allowance of $\bar{e}_t^m$ can be formulated as:

$$\frac{\sum_{m=1}^{M} e_t^m x_t^m}{\sum_{m=1}^{M} x_t^m} \le \bar{e}_t^m \qquad t = 1, \dots, T \tag{3.1}$$

Note that our version of periodic carbon emission constraints (3.1) does not impose an upper bound on the quantities that can be produced in any period. Constraints (3.1) do not limit the production quantity in a resource if the unit carbon emission of the resource is less than or equal to the periodic carbon allowance unless the resource has additional capacity restrictions. When constraint (3.1) is tight for a period, the average carbon emission of the period is equal to $\bar{e}_t^m$, meaning that product is produced at maximum carbon emission level.

The capacitated multi-mode single item lot sizing problem with periodic carbon emission constraints (CLS-PC) can be formulated as follows:

$$\min \quad \sum_{t=1}^{T} \sum_{m=1}^{M} (f_t^m y_t^m + p_t^m x_t^m) + \sum_{t=1}^{T} h_t s_t \tag{3.2a}$$

$$\text{s.t.} \quad s_{t-1} + \sum_{m=1}^{M} x_t^m = d_t + s_t \qquad t = 1, \dots, T \tag{3.2b}$$

$$x_t^m \le C_t^m y_t^m \qquad t = 1, \dots, T,\ m = 1, \dots, M \tag{3.2c}$$

$$\sum_{m=1}^{M} (e_t^m - \bar{e}_t^m) x_t^m \le 0 \qquad t = 1, \dots, T \tag{3.2d}$$

$$s_0 = 0 \tag{3.2e}$$

$$x_t^m, s_t \ge 0 \qquad t = 1, \dots, T,\ m = 1, \dots, M \tag{3.2f}$$

$$y_t^m \in \{0, 1\} \qquad t = 1, \dots, T,\ m = 1, \dots, M \tag{3.2g}$$

The objective function (3.2a) minimizes the fixed and production costs and the total inventory holding cost over the planning horizon. Constraints (3.2b) are the inventory balance equations ensuring for each period t, inventory at the beginning of a period plus the amount produced in that period should be sufficient to satisfy the demand of period, and the remaining items will be kept in the inventory at the end of that period. Constraints (3.2c) are the production capacity constraints, relating the continuous variables x with the binary variables y. Constraints (3.2d) are the periodic carbon emission constraints, linearized version of (3.1). Constraint (3.2e) assumes there is no initial inventory (initial inventory is zero). Constraints (3.2f) and (3.2g) define the lower bounds and the types of the decision variables.

To make the presentation easier to follow, we define special names for the resource's depending on their carbon emission levels:

- *Green Resources*: Resources with carbon emission level e_t^m, less than or equal to average carbon emission level $\bar{e}_t^m$, e.g. $e_t^m \leq \bar{e}_t^m$
- *Regular (Standard) Resources*: Resources with carbon emission level e_t^m, greater than average carbon emission level $\bar{e}_t^m$, e.g. $e_t^m > \bar{e}_t^m$

We indicate the number of green resources with M_g and the number of regular resources with M_r. Assume that $M_g \geq 1$ and $M_r \geq 0$. Note that the summation of the number of green resources with the number of regular resources give the total number of resources available for production ($|M| = |M_g| + |M_r|$).

In the next section, we study the complexity of CLS-PC and discuss several important optimal solution properties.

3.2 Complexity and Structural Properties of Optimal Solutions

Absi et al. (2013) show that the ULS-PC is polynomially solvable by introducing optimal solution properties. However, these properties do not hold in CLS-PC in general. To put it another way, the complexity of the CLS-PC problem is open. This section intends to analyze this problem's complexity under different settings and prove that CLS-PC is NP-Hard in general.

In the following, we prove that the problem is NP-Hard even if there exists a single period and the emission constraints (3.2d) are redundant.

Theorem 3.1. *Single period CLS-PC is NP-Hard.*

Proof. To prove that single period CLS-PC is NP-hard, we will show that there is an NP-complete problem: Knapsack Decision Problem (KDP) proven by Garey & Johnson (1990), such that KDP is reducible to CLS-PC in polynomial time.

We formulate an instance and decision version of KDP in the following:

Instance: N items with values b_i and weights w_i for $i = 1, \ldots, N$, capacity of the knapsack W and an integer B.
Question: Is there a subset S of items such that $\sum_{i \in S} b_i \geq B$ and $\sum_{i \in S} w_i \leq W$?

Given the above instance of KDP, the corresponding decision version of CLS-PC can be stated:

Instance: N machines with no carbon emissions, single period capacitated lot sizing problem: $T = 1$, $M = N$, $C^i = b_i$, $f^i = w_i$, $p^i = 0$, $e^i = 0$ for $i = 1, \ldots, M$, $h_1 = 0$, $d_1 = B$.
Question: Does there exist a subset of machines $S \subseteq \{1, \ldots, M\}$ such that the total capacity is sufficient to satisfy the demand of the single period, and the total cost is at most W, i.e. $\sum_{i \in S} C_1^i = \sum_{i \in S} b_i \geq d_1 = B$ and $\sum_{i \in S} f_1^i = \sum_{i \in S} w_i \leq W$?

As a result, the answer to the CLS-PC decision problem is "yes" if and only if the answer to KDP is also "yes". □

Note that Theorem 3.1 shows that the single period multi-mode capacitated lot sizing problem, which is a special case of CLS-PC where $e_t^m = 0$ for all t and m, is also NP-Hard.

As a result of the periodic carbon emission constraints (3.2d), in any period the resources might not be fully used. Due to problem formulation, the actual production quantities of periods depend on the resources' emission and capacity values. Furthermore, the number of breakpoints of the total production cost function is build upon the relation between the cost parameters of the resources (this will be examined in the next chapter).

If the resources have time-dependent capacities (C_t^m), or the cost (p_t^m) or emission parameters (e_t^m) of the resources are time-dependent, then the breakpoints of the production cost function will be also time-dependent. Namely, CLS-PC is NP-Hard in general if:

- if the capacities are time-dependent, or
- if the cost parameters for the resources are time-dependent, or
- if the emission parameters for the resources are time-dependent, or

Consequently, for the rest of this study we consider CLS-PC where the number of resources M is fixed, the capacities, the cost and the emission parameters of the resources are time-invariant, i.e. $C_t^m = C^m$, $f_m^t = f^m$, $p_m^t = p^m$, $e_m^t = e^m$ for all t and m, and $\bar{e}_t^m = \bar{e}^m$ for all t.

In the following part, we will discuss several critical results for the optimal solutions for CLS-PC.

Property 3.1. *In any feasible solution for CLS-PC, at least one green resource should be used in any production period.*

Proof. The proof follows from the periodic carbon emission constraints (3.2d). □

Property 3.2. *Zero inventory ordering (ZIO) policy of ULS-PC (Theorem 3 of Absi et al. (2013)) does not hold in CLS-PC, in general.*

Proof. ZIO policy means that the initial inventories of the production periods are equal to zero. i.e. $s_{t-1}\sum_{m=1}^{M} x_t^m = 0$ for $t = 1,...,T$. In fact, ZOI policy holds true for the uncapacitated lot sizing problem when there is no capacity restriction on production. This can be seen in the case of Wagner & Whitin (1958).

This policy does not hold for CLS-PC. For instance: $T = 2$, $M = 2$, $d = \left[25,15\right]$, $h_t = 1$ for $t = 1,2$; $f_t^1 = 50$, $C_t^1 = 20$, $p_t^1 = 10$, $e_t^1 = 0$ for $t = 1,2$; $f_t^2 = 100$, $C_t^2 = 20$, $p_t^2 = 5$, $e_t^2 = 0$ for $t = 1,2$; $\bar{e}_t = 0$ for $t = 1,2$. For this problem, the unique optimal solution is $y_1^1 = y_1^2 = 1$, $x_1^1 = x_1^2 = 20$, $s_1 = 15$, $y_2^1 = y_2^2 = 0$, $x_2^1 = x_2^2 = 0$. The objective function value is 465. The problem instance demonstrated above is actually an example of capacitated MMLS without the periodic carbon emission constraints (since all emission parameters are equal to zero). In fact, this instance shows that the resource capacities are the relevant parameters of the problem that result in the rejection of ZIO policy. □

Property 3.3. *More than two resources can be used in all optimal solutions for CLS-PC. In other words, Theorem 1 of Absi et al. (2013) does not hold for CLS-PC, i.e. there might not be an optimal solution for CLS-PC that uses at most two resources in a period.*

Proof. Theorem 1 of Absi et al. (2013) is invalid for the CLS-PC. For example: $T = 1$, $M = 3$, $d_1 = 200$, $\bar{e}_1 = 3$, $h_1 = 1$, $e_1 = \left[1,2,4\right]$, $f_1 = \left[1000,500,100\right]$, $C_1 = \left[100,100,50\right]$, $p_1 = \left[15,10,5\right]$. For this problem, the unique optimal solution is $x_1 = \left[50,100,50\right]$, with $y_1^m = 1$ for all 3 resources $m = 1,2,3$. The objective function is 3600. As shown above, there might not always be an optimal solution for the CLS-PC problem where at most two resources are used. □

According to Absi et al. (2013) with the help of property Property 3.3 it is possible to reduce an ULS-PC instance to an uncapacitated MMLS with M^2 modes where at any period there are no carbon emission constraints. Using the dynamic programming algorithm of Wagelmans et al. (1992) it is possible to solve the latter problem. However, as proved above these results are limited to the uncapacitated case and does not hold in CLS-PC. Although all optimal solutions of CLS-PC might require using more than two resources at any period, in the following theorem we present that that there is an optimal solution with at most two fractional resources.

Here, we call a resource as *"fractional"* if its production level is strictly less than its capacity. Namely, a resource j is fractional if its production level is positive but not equal to its capacity, i.e. $0 < x_t^j < C^j$.

Theorem 3.2. *There exists an optimal solution for CLS-PC that uses at most two fractional resources at any period.*

Proof. Let X_t be the total amount that will be produced in period t: $\sum_m x_t^m$. As it is done in Absi et al. (2013), we can break down the problem using Bender's decomposition approach into a master problem and a set of T independent subproblems:

$$\begin{aligned}
&(3.3a) && \min \quad \sum_{t=1}^{T} \gamma_t(X_t) + \sum_{t=1}^{T} h_t s_t && \\
&(3.3b) && \text{s.t.} \quad s_{t-1} + X_t = d_t + s_t && t = 1, \dots, T \\
&(3.3c) && X_t \leq \bar{C}_t && t = 1, \dots, T \\
&(3.3d) && s_0 = 0 && \\
&(3.3e) && X_t, s_t \geq 0 && t = 1, \dots, T
\end{aligned}$$

where $\bar{C}_t$ is the maximum amount that can be produced in period t, and $\gamma_t(X_t)$ is the minimum cost for producing X_t units in period t which is given by the optimal value of the following subproblem for period t:

$$\begin{aligned}
&(3.4a) && \min \quad \sum_{m=1}^{M} (f_t^m y_t^m + p_t^m x_t^m) && \\
&(3.4b) && \text{s.t.} \quad \sum_{m=1}^{M} x_t^m = X_t && \\
&(3.4c) && \sum_{m=1}^{M} (e_t^m - \bar{e}_t^m) x_t^m \leq 0 && \\
&(3.4d) && x_t^m \leq C_t^m y_t^m && m = 1, \dots, M \\
&(3.4e) && x_t^m \geq 0, \; y_t^m \in \{0, 1\} && m = 1, \dots, M
\end{aligned}$$

It is important to realize that the master problem portrays the lot sizing problem (production planning problem) whereas the subproblems represent the allocation problems. As a matter of fact, the subproblem for period t is always feasible. This is because the problem is a single period allocation problem where the total production quantity is less than or equal to the actual capacity of period t, i.e. $X_t \leq \bar{C}_t$. Note that, we assume that $\bar{C}_t$'s are known in advance and determined based on the capacities of the resources and emission restrictions.

First, we examine the no setup cost case, i.e. y_t^m are removed from the problem. In this case, the subproblem reduces to a linear program with M decision variables, $M+2$ constraints and M non-negativity constraints. By LP theory, in any basic solution there must be M active linearly independent constraints. Yet, if the emission constraint is tight, then $M-2$ of the capacity and non-negativity constraints should be tight, indicating that at most two resources might be fractional. Conversely, if emission constraint is not active, then $M-1$ of the capacity and non-negativity constraints should be tight implicating that at most one resource might be fractional.

Second, we analyze the case with setup costs. Let $(\hat{y}, \hat{x})$ be a feasible solution and define the subset of resources in period t, i.e. $\hat{M}_t = \{m : \bar{y}_t^m = 1\}$. Observe the subproblem for period t where only the resources in the set $\hat{M}_t$ can be used, i.e. $y_t^m = \hat{y}_t^m$ reduces into a LP with $|\hat{M}_t|$ decision variables where in an optimal solution there exists at most two fractional resources. Consequently, $(\hat{y}, \hat{x})$ can be transformed into a solution with a less than or equal to cost solution with the desired property. □

By the proof of Theorem 3.2, the minimum production cost $\gamma_t(X_t)$ for a given quantity X_t is equal to the minimum of a set of linear cost functions. Hence, at any period the total production cost function is a piecewise linear function. This suggests that if one can construct the total production cost function effectively, then the problem can be solved in polynomial time since Koca, Yaman & Aktürk (2014) (or Ou (2017)) showed that when the number of breakpoints of the cost function is fixed, the lot sizing problem with piecewise concave production cost functions is polynomially solvable. In the following chapters, we demonstrate the construction of the total production cost functions, and study the special cases of CLS-PC.

4. CLS-PC WITH TWO MODES (C2LS-PC)

In this chapter, we consider a special case of the CLS-PC problem, called capacitated two-mode lot sizing problem with periodic carbon emission constraints (C2LS-PC). This special case enables us to see the total production cost function structure of each period. It is important to note that due to property Property 3.1, when both of the resources are regular the problem becomes infeasible. As a result, in the following two subsections we will analyze two cases: i) one of the resources is green the other one is regular, ii) both of the resources are green.

Before moving on to analyze the two cases of C2LS-PC, it is important to briefly mention the work of Hong et al. (2016). The authors investigated the two resource lot sizing problem with carbon emission constraints. Similar to our problem C2LS-PC, they analyzed the single item problem with two resources, one green and one regular resource. They entitled the two resource problem as "dual-mode" production planning problem.

In their version, there is no capacity limit for the production. Moreover, different from our problem there is no average carbon emission constraint. For each period the total carbon emission generated by production activities are limited by μ. In C2LS-PC the periodic carbon emission constraint forces that we start production using the green resource then include the standard resource to increase the production amount while minimizing the cost.

Due to the difference in emission relation, the production function of Hong et al. (2016) differs from our problem. In Hong et al. (2016) first production interval is via using the regular resource whereas in our problem (C2LS-PC) we start production using the green resource. (The detailed analysis of production function of the C2LS-PC can be found in the next sub section, Section 4.1.) For the dual mode production planning problem, green and regular resources are used together only when the emission constraint is binding. This case is not true for C2LS-PC problem, for each period our periodic carbon emission constraint (3.2d) ensures that the total production amount never violates the limiting average carbon emission level.

Hong et al. (2016) extend the problem by including the carbon-cap and trade emission scheme. They also propose a polynomial dynamic programming algorithm using multi-level decomposition approach with the structural properties of the problem. Phouratsamay & Cheng (2019) study the special instance of the same problem with inventory bounds and fixed periodic carbon emission constraints. They introduce a polynomial time dynamic programming algorithm to solve this problem. Since they use the mathematical model of Hong et al., their problem also differs from our problem C2LS-PC.

4.1 One Green - One Regular Resource Case

We assume that there exist two resources satisfying $e^1 < \bar{e} < e^2$ condition. Our aim is to develop the total production cost function while analyzing the conceivable relations between the unit production costs of the resources (p^1, p^2). Since the cost and emission parameters of the resources are time-independent, the relations observed here will be true for each period. Hence, we ignore the time index of the parameters in the following.

Case 1: $p^1 \geq p^2$

First, assume that there were no setup costs and no emission constraints (periodic carbon emission constraint (3.2d) did not exist). Then, to satisfy demand of a given period, resource 2 would be used until full capacity and if resource 2 capacity is not enough resource 1 would be used additionally to satisfy the demand. If this happens, total production capacity for any period would be $C^1 + C^2$.

Second, when there are emission constraints and the setup costs are zero, i.e. $f^1 = f^2 = 0$, the cheapest solution is equal to produce both resources proportional to their emission levels since there exists one green and one regular resource available and the periodic carbon emission must be satisfied. To demonstrate, if $e^1 = 1$, $e^2 = 4$ and $\bar{e} = 3$ the policy would be to produce one unit in resource 1 for every two units produced in resource 2. Whereas if $e^1 = 1$, $e^2 = 5$ and $\bar{e} = 3$ the policy would be to produce one unit in resource 1 for every one unit of resource 2.

As a general rule, we use ratios to determine the production amounts. We find these ratios by assuming that in the optimal solution emission constraint is tight. We denote r_m as proportional production amount of resource m. For the CLS-PC with two resources we have to solve the following equality.

$$e^1 r_1 + e^2 r_2 = \bar{e}(r_1 + r_2) \quad (4.1)$$

Since we assume that $e^1 < \bar{e} < e^2$, this equation has infinitely many solutions. However, the ratio of r_1, r_2 is critical and will be same in all of the solutions. To solve (4.1), we follow the resulting steps:

$$\bar{e}r_1 - e^1 r_1 = e^2 r_2 - \bar{e}r_2 \Rightarrow r_1(\bar{e} - e^1) = r_2(e^2 - \bar{e}) \quad (4.2)$$

$$r_1 = e^2 - \bar{e},\ r_2 = \bar{e} - e^1 \quad (4.3)$$

As a result we set r_1, r_2 as shown in (4.3). We use these ratio values to solve the no setup cost case. Knowing that quantity x (such that $x = x^1 + x^2$, $x^1 \leq C^1$, $x^2 \leq C^2$) should be produced using these resources, we calculate the production quantity of each resource in (4.4).

$$x^1 = x\frac{r_1}{r_1 + r_2},\ x^2 = x\frac{r_2}{r_1 + r_2} \quad (4.4)$$

The maximum amount of production using both resources is b_1, which acknowledges that one of the resources is fully utilized.

$$b_1 = r_1 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} + r_2 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = (r_1 + r_2)\min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} \quad (4.5)$$

b_1 explained in (4.5) represents the end point of the first segment of the total production cost function. The slope of this function is $\frac{p^1 r_1 + p^2 r_2}{r_1 + r_2}$ and equal to the unit production cost of using both resources together.

After the end of the first line segment we create a second line segment which has an ending point b_2. If resource 1 is fully used in production at the first segment i.e. $\min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = \{\frac{C^1}{r_1}\}$ and $b_1 = (r_1 + r_2)\{\frac{C^1}{r_1}\}$. Due to the emission constraint, regular resource (resource 2) cannot be used alone in production. Meaning that the total production cost function will have a single segment and $b_2 = b_1$. On the other hand, if resource 2 is fully used in production, there is available capacity for resource 1 implicating this resource can be used until its capacity is reached. In this case, total production cost function will have two segments and the slope of the second segment will be equal to the unit production cost of resource 2, p^2. The maximum amount of production will be equal to: $b_2 = b_1 + C^1 - r_1 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = (r_1 + r_2)\min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} + C^1 - r_1 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = C^1 + r_2 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}$.

The case where setup costs are zero, using both of the resources together instead of using the green resource (resource 1) alone is cheaper. Yet, if the setup costs are positive the relation between setup cost of resource 1 and resource 2 must be considered. Considering resource 2 (regular resource) cannot be used alone, resource 1 (green resource) must be used alone until the setup cost of resource 2 is justified. This will add another segment *before* the first segment of the total production cost function (b_1) explained above. The endpoint of this segment can be calculated using the following equation:

$$f^1 + p^1 x = f^1 + p^1 x \frac{r_1}{r_1 + r_2} + f^2 + p^2 x \frac{r_2}{r_1 + r_2} \tag{4.6}$$

$$x = \frac{f^2(r_1 + r_2)}{(p^1 - p^2) r_2} \tag{4.7}$$

The endpoint of the first segment where the resources have positive setup costs is equal to $\frac{f^2(r_1+r_2)}{(p^1-p^2)r_2}$, as calculated in (4.7). And the slope of this segment is equal to the unit production cost of resource 1, p^1.

As a result, the total production cost function for any period will include at most three segments and three breakpoints as displayed in Figure 4.1. Considering the total production cost function is concave in $[0,b_1]$, the point defined in (4.7) will not be considered as a breakpoint. For this reason the total production cost function will have two breakpoints represented as b_1 and b_2.

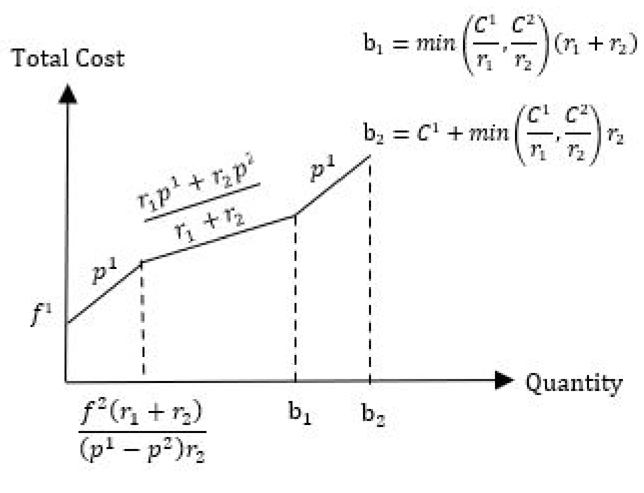

Figure 4.1 Total production cost function for one green- one regular resource case where $p^1 \geq p^2$

Case 2: $p^1 < p^2$

When $p^1 < p^2$, cheapest resource is the green resource (resource 1). Resource 1 will be utilized until its capacity is reached. Consequently, the first breakpoint of the total production cost function will be $b_1 = C^1$ and the slope will be equal to the unit production cost of resource 1, p^1. Next, we can use resource 2 in production till the capacity and carbon emission constraints are satisfied. Once more, we will use proportional production quantities to find the maximum amount we can produce using resource 2 only: $r_2 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\}$. Correspondingly, the end point of the second segment (b_2) will be equal to the summation of the fully utilized resource 1 and the maximum production amount using resource 2, as calculated in (4.8). And the slope of the second line segment will be the unit production cost of resource 2, p^2.

$$b_2 = b^1 + r_2 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} = C^1 + r_2 \min\{\frac{C^1}{r_1}, \frac{C^2}{r_2}\} \tag{4.8}$$

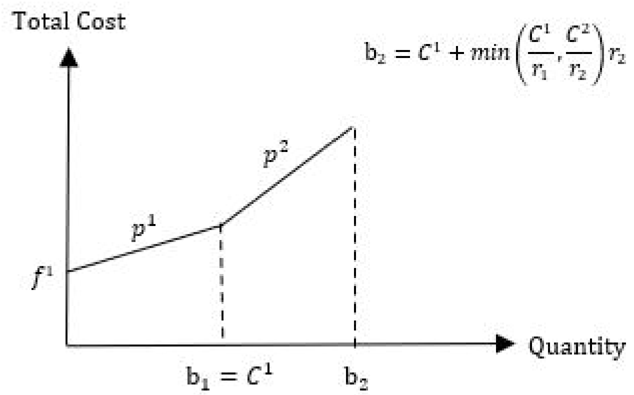

Figure 4.2 Total production cost function for one green- one regular resource case where $p^1 < p^2$

As shown in Figure 4.2, the total production cost function will have two breakpoints represented as b_1 and b_2.

4.2 Two Green Resource Case

The case when we have both green resources reduces to the capacitated MMLS and the periodic carbon emission constraints turn out to be irrelevant. Since we examine green-green resource pairs in the next chapters and the complexity of MMLS is open for discussion, we briefly analyze the possible cases.

Case 1: $f^1 \geq f^2$ and $p^1 \geq p^2$

In this case, compared to resource 1, resource 2 is cheaper to use in production. In other words resource 2 dominates resource 1. Therefore, we use resource 2 until its capacity is reached and continue production with resource 1. As a result, we obtain the total production cost function displayed in Figure 4.3. The total production cost function will include two breakpoints, $b_1 = C^2$, $b_2 = C^1 + C^2$ with slopes p^2,p^1 respectively.

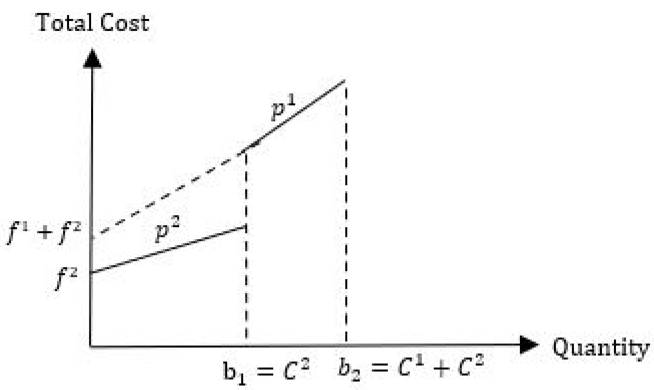

Figure 4.3 Total production cost function for two green resource case where $f^1 \geq f^2$ and $p^1 \geq p^2$

Case 2: $f^1 \geq f^2$ and $p^1 < p^2$

In this case, resource 1 has smaller unit production cost but larger setup cost. Since we aim to minimize total cost of production we need to solve the trade-off between using resource 1 and resource 2 together, and resource 1 alone. We need to solve the following inequality (4.9) to determine the trade-off point x:

(4.9) $$f^2 + p^2 x \leq f^1 + p^1 x \Rightarrow x = \frac{f^1 - f^2}{p^2 - p^1}$$

First, we will begin production with resource 2, till the trade-off point x is reached. Second, we will continue production by using resource 1 till its capacity is fulfilled, $b_1 = C^1$. Third, we will use resource 2 until its capacity is fully used, $b_2 = C^2$. Identical to Case 1 of one green-one regular resource total production cost function, the function is concave in the interval [0,b_1], point defined in (4.9) will not be considered as a breakpoint. As a result the total production cost function will have two breakpoints (b_1, b_2) as depicted in Figure 4.4.

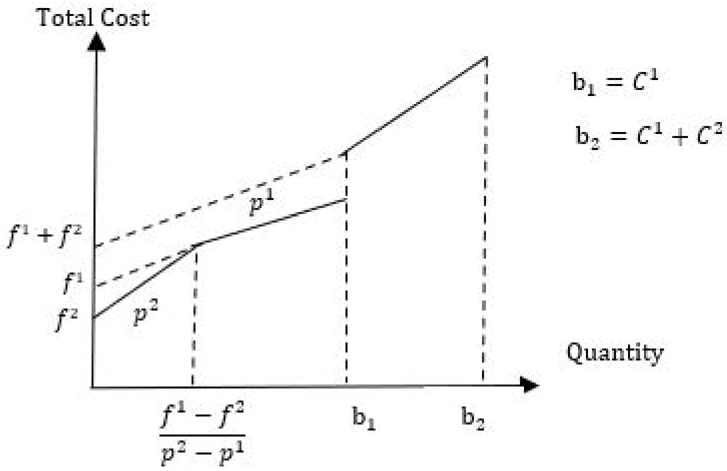

Figure 4.4 Total production cost function for two green resource case where $f^1 \geq f^2$ and $p^1 < p^2$

Case 3: $f^1 < f^2$ and $p^1 \geq p^2$

Case 3 is similar to Case 2. Again, we need to solve a trade-off to find the quantity x but in this case the parameters are switched. In this case, resource 2 has smaller unit production cost whereas resource 1 has smaller setup cost but larger unit production cost. The calculations are demonstrated in (4.10).

$$(4.10) \qquad f^1 + p^1 x \leq f^2 + p^2 x \Rightarrow x = \frac{f^2 - f^1}{p^1 - p^2}$$

Production will start with using resource 1 only until the trade-off quantity x is reached. Then, resource 2 will be used until its capacity is fully utilized. The production will continue with using resource 1 till its capacity is fulfilled. First breakpoint will be $b_1 = C^2$ and the second breakpoint will be $b_2 = C^1 + C^2$. Identical to the previous case, the total production cost function is concave in the interval [0,b_1], point defined in (4.10) will not be considered as a breakpoint. Figure 4.5 shows that the total production cost function will have two breakpoints with slopes

p^2, p^1 respectively.

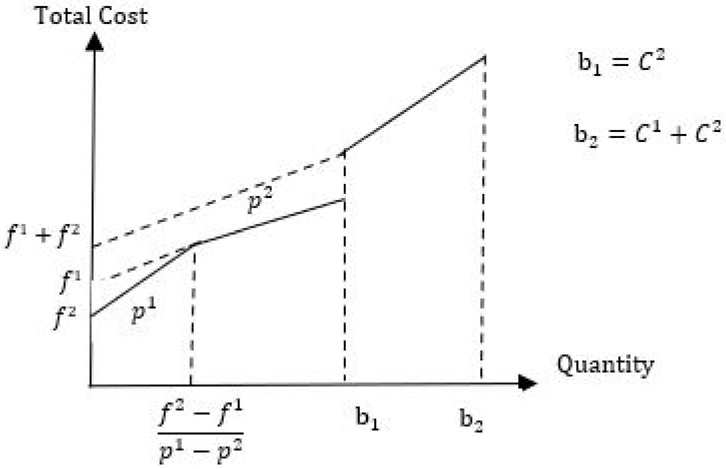

Figure 4.5 Total production cost function for two green resource case where $f^1 < f^2$ and $p^1 \geq p^2$

Case 4: $f^2 \geq f^1$ and $p^2 \geq p^1$

In this case, resource 1 dominates resource 2. In fact, Case 4 is the symmetric of Case 1. Therefore, this case will also have two breakpoints, where $b_1 = C^1, b_2 = C^1 + C^2$ with slopes p^1, p^2 respectively. The total production cost function of this case can be seen in figure Figure 4.6.

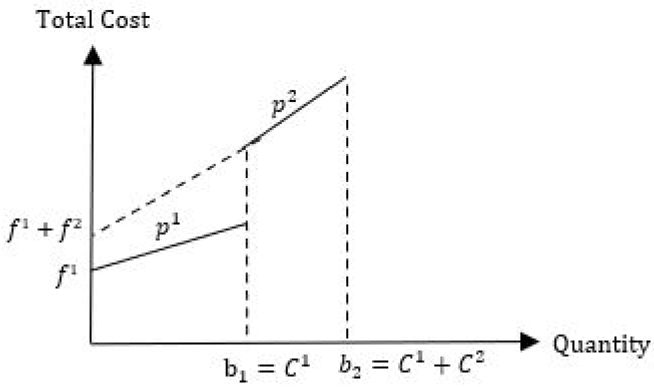

Figure 4.6 Total production cost function for two green resource case where $f^2 \geq f^1$ and $p^2 \geq p^1$

It is important to realize that in all cases, the total production cost function has two breakpoints.

Remark 1. *Note that the breakpoints of the total production cost function do not depend on the cost parameters in any case. However, the case that will apply depends on the relation between the cost parameters of the resources, and the breakpoints are different at different cases. This is the reason of our assumption for stationary cost parameters of the resources.*

Remark 2. *Since the number of breakpoints is 2 at each case, this special case of the problem can be solved in $O(T^6)$ time by using the dynamic programming algorithm of Koca et al. (2014).*

5. CLS-PC WITH NO SETUP COSTS (CLS-PC0)

In this chapter, we will discuss CLS-PC with no setup costs (CLS-PC0). The case with no setup costs $f_t^m = 0$ for all m and t, reduces CLS-PC (3.2) into a linear program. The reduced problem can be solved via interior point method of Karmakar (1984) in polynomial time. Yet, we include this special case in our analysis since the algorithm discussed in this section allows us to reformulate the problem as a minimum cost network flow problem with piecewise linear cost functions. The reformulated problem can be solved by using special algorithms of Orlin (1988) and Pinto & Shamir (1994).

CLS-PC with no setup costs and no carbon emission constraints reduces into an MMLS without setup costs. In this case, the resources are fully utilized in non-decreasing order of their unit production costs. In other words, since the objective is to minimize total production cost, a resource with smaller unit production cost will be used until full capacity. The resulting total production cost function will have the following breakpoints: $0, C^1, C^1 + C^2, ..., C^1 + C^2 + ... + C^M$ and the following slopes: $p^1, p^2, ..., p^M$, assuming that the resources are listed in non-decreasing unit production costs order. In Figure 5.1, we demonstrate an example, where there are four resources with unit production costs $p^1 < p^2 < p^3 < p^4$. The breakpoints of the total production cost function are given by breakpoints: $0, C^1, C^1 + C^2, C^1 + C^2 + C^3, C^1 + C^2 + C^3 + C^4$. Although this method is not applicable to CLS-PC0 (due to carbon emission constraints), we will discuss in the following how it is applicable to the artificial resources that will be created after a preprocessing step.

By Theorem 3.2, we know that to produce a given quantity there exists a subset of resources where at most two resources are fractional. Furthermore, since in CLS-PC0 there are no setup costs (fixed costs), the same subset of resources and the same subset of fractional resources would deliver the smallest production cost while satisfying capacity limits. Thus, we introduce a crucial property of the breakpoints of the total production cost function: *there will be at most one fractional resource at any breakpoint*. In order to construct the total production cost function for a period, we determine different resource combinations as the *artificial resources* and

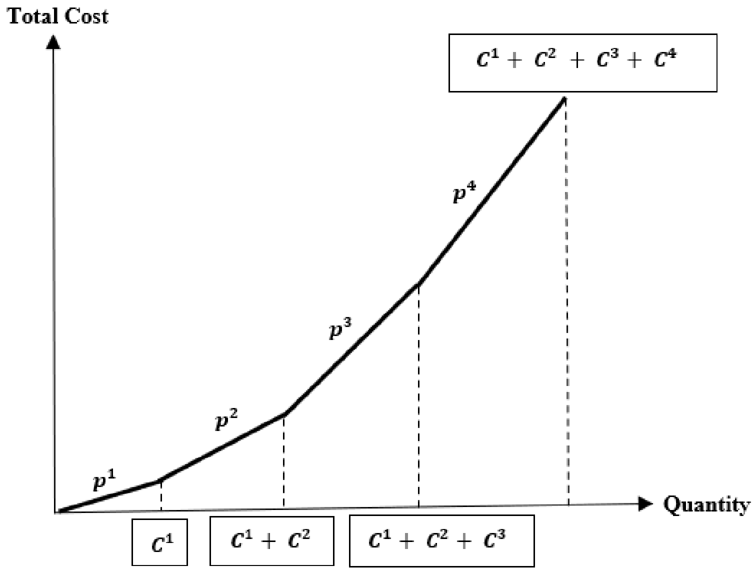

Figure 5.1 CLS-PC with no setup costs and no carbon emission constraints

regard the unit production cost of each artificial resource for a period.

In the remaining of this chapter, to formulate the breakpoints of the total production cost function of a period, we will define two possible subsets of resources where $\bar{S}^0$ is the subset of resources that will produce in full capacity and satisfy the carbon emission constraints, and $\bar{S}^1$ is the subset of resources with exactly one fractional resource. Then, we will determine their average unit production cost ($\bar{p}$) and total quantity (q). Note that $\bar{e}^j$ notation, defined by $\bar{e}^j = e^j - \bar{e}$ is used to represent the ratio values as demonstrated in Equation 4.3 for the two resource case. This notation is used to show the type of resource and determine the production amounts while satisfying the carbon emission constraints.

We define the subset of resources with no fractional resources, $\bar{S}^0$ in (5.1a). For each element in this subset, we calculate the average unit production cost in (5.1b) and total production quantity in (5.1c). Correspondingly, we define the subset of resources with exactly one fractional resource where resource i represents the fractional resource, $\bar{S}^1$ in (5.2a). For each element in this subset, we calculate the average unit production cost in (5.2b) and total production quantity in (5.2c).

(5.1a) $$\bar{S}^0 = \{S \subseteq \{1,...,M\} : \sum_{j \in S} \bar{e}^j C^j \leq 0\}$$

(5.1b) $$\bar{p}_S = \frac{\sum_{j \in S} p^j C^j}{\sum_{j \in S} C^j} \qquad \forall S \in \bar{S}^0$$

(5.1c) $$q_S = \sum_{j \in S} C^j \qquad \forall S \in \bar{S}^0$$

(5.2a)

$$\bar{S}^1 = \{(S,i) : 0 < x^{(S,i)} := \frac{-\sum_{j\in S\setminus\{i\}} \bar{e}^j C^j}{\bar{e}^i} < C^i, i \in S, S \subseteq \{1,...,M\}\}$$

(5.2b)

$$\bar{p}_{(S,i)} = \frac{\sum_{j\in S\setminus\{i\}} \bar{p}^j C^j + p^i x^{(S,i)}}{\sum_{j\in S\setminus\{i\}} \bar{C}^j + x^{(S,i)}} \qquad \forall (S,i) \in \bar{S}^1$$

(5.2c)

$$q_{(S,i)} = \sum_{j\in S\setminus\{i\}} C^j + x^{(S,i)} \qquad \forall (S,i) \in \bar{S}^1$$

It is important to realize that $\bar{p}$ values represent the average unit production cost at the potential breakpoints of the total production cost function. Given that the total production cost function is piecewise linear convex and there are no setup costs, the unit (average) production costs are non-decreasing in the segments of the function. We apply this property to construct the total production cost function for any period.

Let $\bar{S} = \bar{S}^0 \cup \bar{S}^1$. We create an ordered list L of the elements in $\bar{S}$. The elements in $\bar{S}$ are ordered in non-decreasing average unit production costs $\bar{p}$, and for the elements with the identical $\bar{p}$ value, the following rule will be applied: select the element with the smallest $|\bar{S}|$ and largest q value first in this order. Henceforth, we develop the the algorithm given in Algorithm 1 to calculate the total production cost function for any period.

Algorithm 1 Algorithm to determine the total production cost function for CLS-PC0

Input: L, $\bar{S}$, $\bar{p}$, q

Output: The breakpoints b of the total production cost function for CLS-PC0

```
Let b_0 = 0 and i = 1.
while L ≠ ∅ do
    Select the first element, say π in L
    Let b_i = q_π
    Remove all elements in L s.t. q_S ≤ b_i or q_{S,i} ≤ b_i
    Increase i by one
```

In Algorithm 1, we establish the breakpoints of the total production cost function. Moreover, by analyzing the difference between the total costs of two consecutive breakpoints, the unit production costs (slopes) of the breakpoints can be calculated.

In order to determine the time complexity of this algorithm, we need to acknowledge that by Property 3.1, all elements in the set $\bar{S}$ should have at least one green resource. The size of the set $\bar{S}^0$ is $O(2^M - 2^{M_r})$ and $\bar{S}^1$ is $O(M(2^M - 2^{M_r}))$. Hence, the size of the set $\bar{S}$ is $O((M+1)(2^M - 2^{M_r}))$. For an arbitrary M value, the size of $\bar{S}$ is exponential. However, when the number of resources M is fixed, then the complexity of determining all these sets will also be fixed. The main part of Algorithm 1 is constructing the list L, in which each element of set $\bar{S}$ is considered once and sorted. As a result, the time complexity of this algorithm is fixed, $O(|\bar{S}|\log|\bar{S}|)$.

After obtaining the total production cost function, we can reduce CLS-PC0 into a minimum cost network flow problem with piecewise linear convex costs. The reduced problem can be solved directly by utilizing the special purpose algorithm of Pinto & Shamir (1994). Alternatively, after converting the network we can define a dummy node to represent production (node 0) and connect this node with a separate arc to node t which represents period t for each segment of the total production cost function and then apply the algorithm of Orlin (1988).

In the following, we will give an example to demonstrate how the algorithm works.

Example 1. Consider the case where we have $M = 4$, $M_g = 2$, $M_r = 2$, $\bar{e} = 3$, $e^4 = \left[1,2,4,5\right]$, $p^4 = \left[10,5,15,5\right]$, $C^4 = \left[100,150,150,150\right]$.

First, we determine the set $\bar{S}^0$. For all subsets of M, we check (5.1a) and for each subset satisfying this condition calculate the average unit production cost ($\bar{p}$) and the total quantity values (q) using equations (5.1b) and Equation 5.1c. In Table 5.1, we show for each element in $\bar{S}^0$ their corresponding average unit production cost and total quantity values. We have $\bar{S}^0 = \{\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{1,2,4\}\}$.

Table 5.1 For each element in $\bar{S}^0$ average unit production cost ($\bar{p}$) and total quantity (q) values

π	$\bar{p}$	q
{1}	10	100
{2}	5	150
{1,2}	7	250
{1,3}	13	250
{2,3}	10	300
{1,2,3}	10	400
{1,2,4}	6.25	400

Second, we repeat the same analysis to find the elements in $\bar{S}^1$, to find the elements we use (5.2a) and solve (5.2b) and (5.2c). The results of these calculations give us

Table 5.2. We determine $\bar{S}^1 = \{(\{1,4\},4),(\{2,4\},4),(\{1,3,4\},4),(\{1,2,3,4\},4)\}$.

Table 5.2 For each element in $\bar{S}^1$ average unit production cost ($\bar{p}$) and total quantity (q) values

π	$\bar{p}$	q
({1,4}, 4)	7.5	200
({2,4}, 4)	5	225
({1,3,4}, 4)	12.272	275
({1,2,3,4}, 4)	9	500

Third, we create the ordered list L in non-decreasing order of average unit production costs for elements in $\bar{S} = \bar{S}^0 \cup \bar{S}^1$.

We get: $L = \{\{2\},\{2,4\},\{1,2,4\},\{1,2\},\{1,4\},\{1,2,3,4\},\{1\},\{2,3\},\{1,2,3\},\{1,3,4\},\{1,3\}\}$. Note that, as a tie breaker we selected the smallest $|\bar{S}|$. Moreover, to make the list L, easier to follow the fractional resource in the elements of $\bar{S}^1$ are not represented in the following.

Last, we start the algorithm.

- ***Step 0:*** Initialization
  $L = \{\{2\},\{2,4\},\{1,2,4\},\{1,2\},\{1,4\},\{1,2,3,4\},\{1\},\{2,3\},\{1,2,3\},\{1,3,4\},\{1,3\}\}$
  Let $b_0 = 0$, $i = 1$

- ***Step 1:*** $L \neq \emptyset$, enter **while**, set $i = 1$
  $\pi = \{2\}$, $b_1 = 150$ $q_1 = q_{\{2\}} = 150$, $\bar{p}_{\{2\}} = 5$

π	{2}	{2,4}	{1,2,4}	{1,2}	{1,4}	{1,2,3,4}	{1}	{2,3}	{1,2,3}	{1,3,4}	{1,3}
$\bar{p}$	5	5	6.25	7	7.5	9	10	10	10	12.272	13
q	150	225	400	250	200	500	100	300	400	275	250

  Remove all elements with $q \leq 150$ from L and increase i
  Updated list $L = \{\{2,4\},\{1,2,4\},\{1,2\},\{1,4\},\{1,2,3,4\},\{2,3\},\{1,2,3\},\{1,3,4\},\{1,3\}\}$

- ***Step 2:*** $L \neq \emptyset$, enter **while**, set $i = 2$
  $\pi = \{2,4\}$, $b_{\{2,4\}} = 225$, $q_2 = q_{\{2,4\}} = 225$, $\bar{p}_{\{2,4\}} = 5$

π	{2,4}	{1,2,4}	{1,2}	{1,4}	{1,2,3,4}	{2,3}	{1,2,3}	{1,3,4}	{1,3}
$\bar{p}$	5	6.25	7	7.5	9	10	10	12.272	13
q	225	400	250	200	500	300	400	275	250

  Remove all elements with $q \leq 225$ from L and increase i
  Updated list $L = \{\{1,2,4\},\{1,2\},\{1,2,3,4\},\{2,3\},\{1,2,3\},\{1,3,4\},\{1,3\}\}$

- ***Step 3:*** $L \neq \emptyset$, enter **while**, set $i = 3$
  $\pi = \{1,2,4\}$, $b_3 = 400$, $q_3 = q_{\{1,2,4\}} = 400$, $\bar{p}_{\{1,2,4\}} = 6.25$

π	{1,2,4}	{1,2}	{1,2,3,4}	{2,3}	{1,2,3}	{1,3,4}	{1,3}
$\bar{p}$	6.25	7	9	10	10	12.272	13
q	400	250	500	300	400	275	250

  Remove all elements with $q \leq 400$ from L and increase i
  Updated list $L = \{\{1,2,3,4\}\}$

- ***Step 4:*** $L \neq \emptyset$, enter **while**, set $i = 4$
  $\pi = \{1,2,3,4\}$, $b_4 = 500$, $q_4 = q_{\{1,2,3,4\}} = 500$, $\bar{p}_{\{1,2,3,4\}} = 9$

π	{1,2,3,4}
$\bar{p}$	9
q	500

  Remove all elements with $q \leq 225$ from $L \Rightarrow L = \emptyset$
  STOP!

As a result,we determine the breakpoints of the total production cost function of this example as: $b = \left[0, 150, 225, 400, 500\right]$. We also have the information of which resources are used and the average production cost for each breakpoint. Figure 5.2 shows the resulting total production cost function.

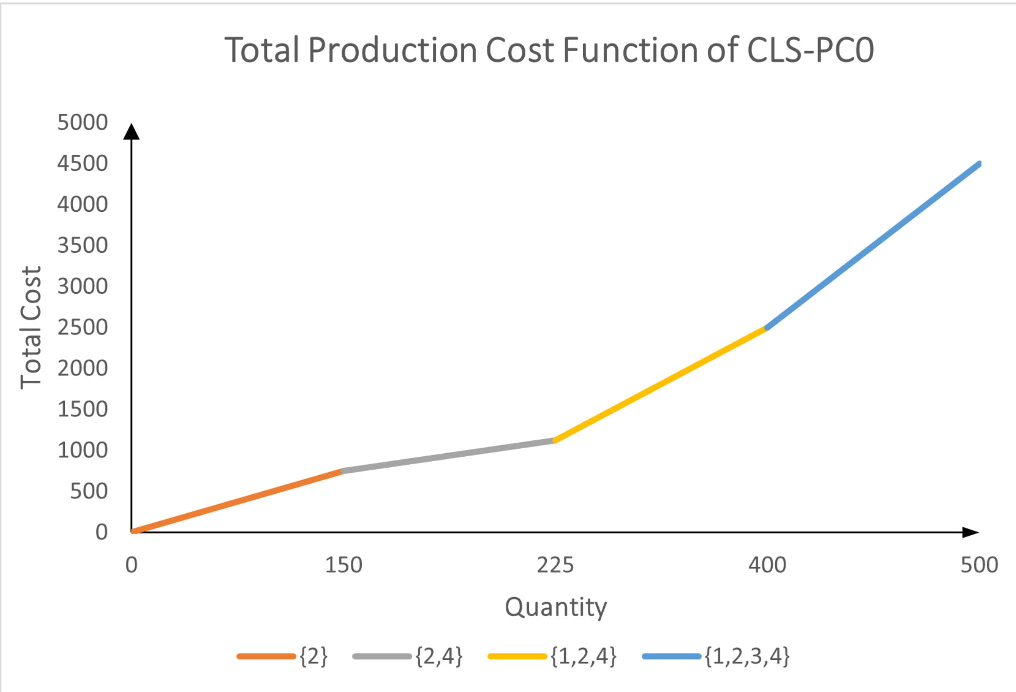

Figure 5.2 Total Production Cost Function of CLS-PC0 Example

6. CLS-PC TOTAL PRODUCTION COST FUNCTION STRUCTURE

In this chapter, we will examine the total production cost function structure of CLS-PC when the setup costs are positive. In the previous chapter, we discussed the total production cost function of CLS-PC0. The findings of this study suggest that the CLS-PC0 has at most one fractional resource as a breakpoint. However, for the positive setup cost case the single fractional resource property of the breakpoints might not hold since there might be additional breakpoints of the total production cost function with exactly two fractional resources because of the intersections of production cost functions of different resource combinations.

To demonstrate, in Figure 4.1 the first breakpoint occurs at point $\frac{f^2(r_1+r_2)}{(p^1-p^2)r_2}$ (calculated in (4.7)) as a result of the intersection of the production cost function of resource 1 (green resource) with the production cost function of using both resource together (resource 1 and 2 - green and regular resource). Furthermore, as demonstrated in Figure 4.1, when the production cost functions of two different resource combinations intersect, the slope of the total production cost decreases (because we select the breakpoint option with minimum cost). Consequently, the total production cost function is concave between the breakpoints. For this reason, these new breakpoints do not have to be considered as "breakpoints" in the sense of the dynamic programming algorithm of Koca et al. (2014).

Nonetheless, in order to apply Koca et al. (2014)'s dynamic programming algorithm and accurately compute the total production cost function we need to determine these additional breakpoints and the slopes of these segments. With this intention, we benefit from the definitions made in the previous chapter, i.e. $\bar{S}, \bar{S}^0$ and $\bar{S}^1$. In contrast to our previous methodology, since the total production cost function is neither convex nor concave, considering the points in the plane is not enough to build the total production cost function. Therefore, it is necessary to determine a line segment that represents the production cost of each resource combination. For each resource in $\bar{S}$, we need to determine the total cost, unit price, and the additional quantity that can be produced with this combination, as well as the minimum quantity that can be produced. Observe that this information is sufficient to plot the total production cost function for this combination. Next, we introduce a systematic approach to do this.

To formulate the total production cost function for each resource combination in $\bar{S}$, we will define $Q(\bar{S})$ representing the maximum quantity of production using all resources, and $F(\bar{S})$ representing the total cost of using maximum quantity of production using all resources.

For each resource combination in the subset of no fractional resources, $S \in \bar{S}^0$, we calculate the maximum quantity of production using all resources in (6.1a) and the cost of maximum quantity of production in (6.1b).

$$Q(S) = \sum_{j \in S} C^j \tag{6.1a}$$

$$F(S) = \sum_{j \in S} \left(f^j + p^j C^j \right) \tag{6.1b}$$

For this resource combination, it is possible to increase production, if a new resource $i \notin S$ is added to the combination S under the assumption that the carbon emission constraint is still satisfied with this new resource combination. For each $S \in \bar{S}^0$, the following three scenarios should be analyzed:

1.1 A new resource $i \notin S$ might be included in the combination, and this new combination can be used until either i is fully utilized or the carbon emission constraint (3.2d) becomes tight.

1.2 If the emission constraint is tight, then two new resources $i, j \notin S$ might be included in the combination, and this new combination can be used until either i or j is fully utilized.

1.3 If the emission constraint is tight, then a new resource $i \notin S$ might be included in the combination, and production level of a resource $j \in S$ that was fully used in the combination might be decreased while the production level of i is increased until either i is fully used or j's production level reaches to zero.

Likewise for each resource combination in the subset of resources with exactly one fractional resources where resource i is the fractional resource, $(S, i) \in \bar{S}^1$, we compute the maximum quantity of production using all resources in (6.2a) and the cost of maximum quantity of production in (6.2b).

$$Q(S,i) = \sum_{j \in S} C^j + x^{(S,i)} \quad \text{(6.2a)}$$

$$F(S,i) = \sum_{j \in S} f^j + \sum_{j \in S \setminus \{i\}} p^j C^j + p^i x^{(S,i)} \quad \text{(6.2b)}$$

Notice that for this case the carbon emission constraint (3.2d) should be tight. And, if we want to produce more, this is possible either by using resource i along with a new resource $j \notin S$, or along with a resource $j \in S$ that was fully utilized (in the reverse direction). Hence for each $(S,i) \in \bar{S}^1$, the following three scenarios should be considered:

2.1 A new resource $j \notin S$ might be included in the combination, and this new combination can be used until either i or j is fully utilized. Remark: i and j should be from different resource groups (green vs regular) since the emission constraint is tight.

2.2 Production level of a resource $j \in S$ that was fully used in the combination might be decreased while the production level of i is increased until either i is fully used or j's production level reaches to zero.

2.3 A new resource $j \notin S$ might be included in the combination, and the production level of j might be increased while the production level of i is decreased until either i's production level reaches zero or j is fully utilized.

In Table 6.1 and Table 6.2 we show the data necessary to determine the line segment that represents the production cost of each resource combination in $S \in \bar{S}^0$ and $(S,i) \in \bar{S}^1$. Both of the tables provide an overview of the condition of the possible scenario, the starting point which represents the coordinate - quantity and cost of the starting point in the two-dimensional plane, the slope which is the unit production cost for the new combination, and the additional quantity that can be produced with the new combination (assume that division by zero results ∞). In other words, additional quantity is the length of the line segment in the first axis. Since the end points of the line segments can be effortlessly calculated, we do not include the costs and the quantity values of the ending points.

We give the scenarios' necessary conditions in Table 6.1 and Table 6.2 under the column of "Condition". It is important to highlight that we only study the cases where producing more units has a positive cost with that combination since the reverse corresponds to another combination, and it will be also considered as another element in the set $\bar{S}$. Conversely, for the scenarios where the production level of

Table 6.1 Possible scenarios for $S \in \bar{S}^0$

Condition	**Starting point**	**Slope**	**Additional quantity**
$i \notin S$, $\bar{e}^i \leq 0$ if $\sum_{j\in S} \bar{e}^j C^j = 0$	$(Q(S), F(S) + f^i)$	p^i	$\min\left\{\frac{-\sum_{j\in S}\bar{e}^j C^j}{\max\{\bar{e}^i,0\}}, C^i\right\}$
$i, j \notin S$, $\bar{e}^i\bar{e}^j < 0$, $\sum_{k\in S} \bar{e}^k C^k = 0$	$(Q(S), F(S) + f^i + f^j)$	$\frac{p^i\|\bar{e}^j\|+p^j\|\bar{e}^i\|}{\|\bar{e}^i\|+\|\bar{e}^j\|}$	$\min\left\{\frac{C^i}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}(\|\bar{e}^j\|+\|\bar{e}^i\|)$
$i \notin S$, $j \in S$, $\sum_{k\in S} \bar{e}^k C^k = 0$, $\bar{e}^i\bar{e}^j > 0$, $\|\bar{e}^j\| > \|\bar{e}^i\|$, $p^i\|\bar{e}^j\| > p^j\|\bar{e}^i\|$	$(Q(S), F(S) + f^i)$	$\frac{p^i\|\bar{e}^j\|-p^j\|\bar{e}^i\|}{\|\bar{e}^j\|-\|\bar{e}^i\|}$	$\min\left\{\frac{C^i}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}(\|\bar{e}^j\|-\|\bar{e}^i\|)$

Table 6.2 Possible scenarios for $(S, i) \in \bar{S}^1$

Condition	**Starting point**	**Slope**	**Additional quantity**
$j \notin S$, $\bar{e}^i\bar{e}^j < 0$	$(Q(S,i), F(S,i) + f^j)$	$\frac{p^i\|\bar{e}^j\|+p^j\|\bar{e}^i\|}{\|\bar{e}^i\|+\|\bar{e}^j\|}$	$\min\left\{\frac{C^i - x^{(S,i)}}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}(\|\bar{e}^i\|+\|\bar{e}^j\|)$
$j \in S$, $\|\bar{e}^j\| > \|\bar{e}^i\|$, $p^i\|\bar{e}^j\| > p^j\|\bar{e}^i\|$	$(Q(S,i), F(S,i))$	$\frac{p^i\|\bar{e}^j\|-p^j\|\bar{e}^i\|}{\|\bar{e}^j\|-\|\bar{e}^i\|}$	$\min\left\{\frac{C^i - x^{(S,i)}}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}(\|\bar{e}^j\|-\|\bar{e}^i\|)$
$j \notin S$, $\|\bar{e}^i\| > \|\bar{e}^j\|$, $p^j\|\bar{e}^i\| > p^i\|\bar{e}^j\|$	$(Q(S,i), F(S,i) + f^j)$	$\frac{-p^i\|\bar{e}^j\|+p^j\|\bar{e}^i\|}{\|\bar{e}^i\|-\|\bar{e}^j\|}$	$\min\left\{\frac{x^{(S,i)}}{\|\bar{e}^j\|}, \frac{C^j}{\|\bar{e}^i\|}\right\}(\|\bar{e}^i\|-\|\bar{e}^j\|)$

a resource is reduced, it is possible for a resource in the endpoint to be eliminated (its production reduces to zero or production level reduced to zero). To calculate the cost of end point correctly, one has to subtract the setup cost of that resource from the total cost. Nevertheless, we do not have to perform this analysis, because after all production cost functions are determined, we take the lower envelope of the functions.

In order to assess the time complexity of this systematic approach of formulating the total production cost function of each resource combination in $\bar{S}$, we need to study $O(M)$ line segments. Then, we need to determine the production cost functions for each potential breakpoint in the set $\bar{S}$ which can be identified by taking the lower envelope of these line segments. To find the lower envelope of these line segments we can use the algorithm of Shamos & Hoey (1976) which will take $O\left(M|\bar{S}|logM|\bar{S}|\right)$ time. For fixed M, this step will be in constant time.

Once the total production cost function is determined together with its breakpoints and slopes, one can solve the problem CLS-PC using the dynamic programming algorithm of Koca et al. (2014). This solution method will take $O\left(T^{2M'+2}\right)$ time where M' represents the number of breakpoints of the total production cost function.

In the following, we will show how the CLS-PC total production cost function is constructed using the same example we used in Chapter 5. Since now we are in the CLS-PC setting, we add setup costs to the problem formulation.

Example 1. Consider the case where we have $M = 4$, $M_g = 2$, $M_r = 2$, $\bar{e} = 3$, $e^4 = \left[1,2,4,5\right]$, $p^4 = \left[10,5,15,5\right]$, $C^4 = \left[100,150,150,150\right]$, $f^4 = \left[200,400,500,100\right]$.

First, we determine the maximum quantity ($Q(S)$) of production and the total cost of using maximum quantity of production using all resources ($F(S)$) for $S \in \bar{S}^0$ where $\bar{S}^0 = \{\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{1,2,4\}\}$. We use (6.1a) and (6.1b). The results of these calculations are shown in Table 6.3.

Table 6.3 $Q(S)$ and $F(S)$ values for each $\bar{S}^0$

S	$Q(S)$	$F(S)$
{1}	100	1200
{2}	150	1150
{1,2}	250	2350
{1,3}	250	3950
{2,3}	300	3900
{1,2,3}	400	5100
{1,2,4}	400	3200

Second, we repeat the calculations of $Q(S,i)$ and $F(S,i)$ for $(S,i) \in \bar{S}^1$ where $\bar{S}^1 = \{(\{1,4\},4),(\{2,4\},4),(\{1,3,4\},4),(\{1,2,3,4\},4)\}$ and i represents the fractional resource. We use (6.2a) and (6.2b). In Table 6.4 we show the results of these calculations.

Table 6.4 $Q(S,i)$ and $F(S,i)$ values for each $\bar{S}^1$

(S,i)	$Q(S,i)$	$F(S,i)$
({1,4}, 4)	200	1800
({2,4}, 4)	225	1625
({1,3,4}, 4)	275	4175
({1,2,3,4}, 4)	500	5700

Third, we check for each resource combination in $\bar{S} = \bar{S}^0 \cup \bar{S}^1$ whether there exists additional breakpoints and slopes of these segments or not. If there exists such resource combination we will compute the minimum quantity that will be produced, its total cost, the unit production cost and the maximum quantity that can be produced with this combination. The systematic way of doing this differs for $\bar{S}^0$ and $\bar{S}^1$.

For each resource combination in $\bar{S}^0$ we follow Table 6.1. We analyze the three scenarios in this table for each $S \in \bar{S}^0$. As a result, we create Table 6.5. The first column represents the case it is possible to use $\{1,2,3\}$ together. This means when we have {2,3} we can include resource {1} in production and produce 100 units more. The second column represents the case when we include {1,4} in production, this means we can use one green and one regular resource that is not in S to produce 200 units more. In Figure 6.1 we show the graph of total production cost function of $\{2,3\} \in \bar{S}^0$. Note that with Figure 6.1 we prove that the information obtained by this algorithm is sufficient to draw the production cost function for this resource combination.

Table 6.5 Possible scenarios for $S \in \bar{S}^0$

S	{2,3}	{2,3}
Condition on j	$i \notin S$, $\bar{e}^i \leq 0$ if $\sum_{j \in S} \bar{e}^j C^j = 0$	$i,j \notin S$, $\bar{e}^i \bar{e}^j < 0$, $\sum_{k \in S} \bar{e}^k C^k = 0$
Resources	{1}	{1,4}
Starting Point	(300, 4100)	(300, 4200)
Slope	10	7.5
Additional Quantity	100	200

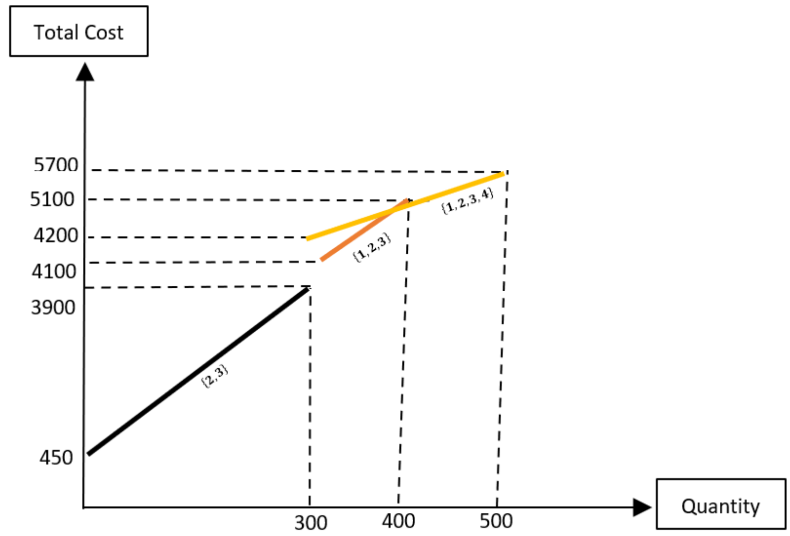

Figure 6.1 Possible cases for $S = \{2,3\}$

For each resource combination in $\bar{S}^1$ we follow Table 6.2. We analyze the three scenarios in this table for each $(S,i) \in \bar{S}^1$. As a result we obtain Table 6.6. In each column we represent a resource combination and the related information of the additional production quantity. For instance, we can add resource {2} into {1,4} resource pair and produce 150 additional units by using all three resources {1,2,4} together.

In the previous steps, we have analyzed for each resource combination the additional production quantities and formulated the total production cost function for each

Table 6.6 Possible scenarios for $(S, i) \in \bar{S}^1$ where i is the fractional period

(S,i)	({1,4}, 4)	({2,4}, 4)	({2,4}, 4)	({1,3,4}, 4)
Condition on j	$j \notin S,\ \bar{e}^i\bar{e}^j < 0$	$j \notin S,\ \bar{e}^i\bar{e}^j < 0$	$j \notin S,\ \lvert\bar{e}^i\rvert > \lvert\bar{e}^j\rvert,\ p^j\lvert\bar{e}^i\rvert > p^i\lvert\bar{e}^j\rvert$	$j \notin S,\ \bar{e}^i\bar{e}^j < 0$
(i,j) Pairs	{4,2}	{4,1}	{4,3}	{4,2}
Starting Point	(200, 2200)	(225, 1825)	(225, 2125)	(275, 4575)
Slope	5	5	25	5
Additional Quantity	150	150	75	225

potential breakpoint. We combine all of the line segments and draw the overall graph in Figure 6.2. This figure shows the total production cost function of all resource combinations with their potential breakpoints. These points are calculated in the previous steps (in Table 6.3, Table 6.4, Table 6.5, Table 6.6). The total production cost function is found by taking the lower envelope of these line segments using Shamos & Hoey (1976)'s algorithm. Figure 6.3 is the total production cost function obtained after taking the lower envelope of Figure 6.2. Since, the total production cost function is determined with its breakpoints and slopes, we can solve this problem using the dynamic programming algorithm of Koca et al. (2014).

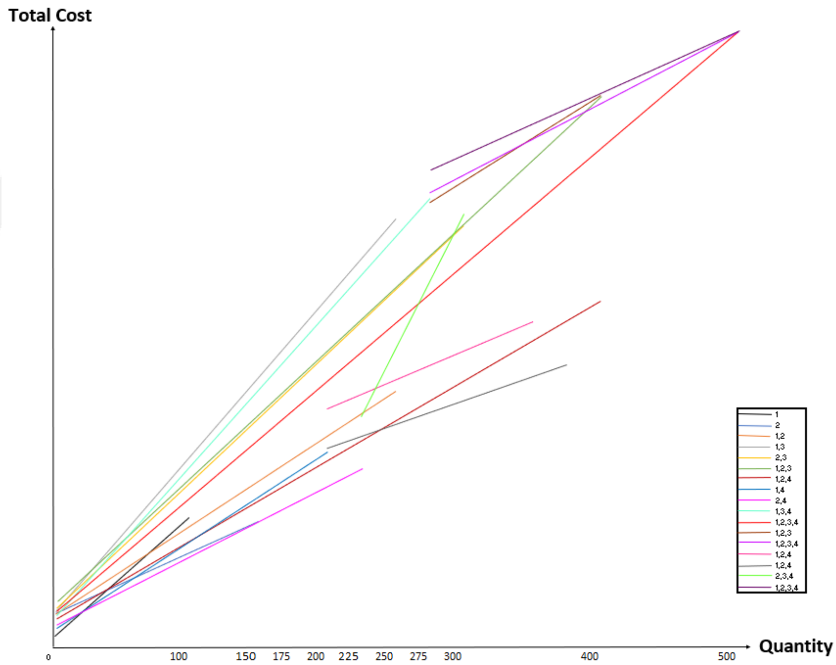

Figure 6.2 All Resource Combinations of CLS-PC Example with their Potential Breakpoints

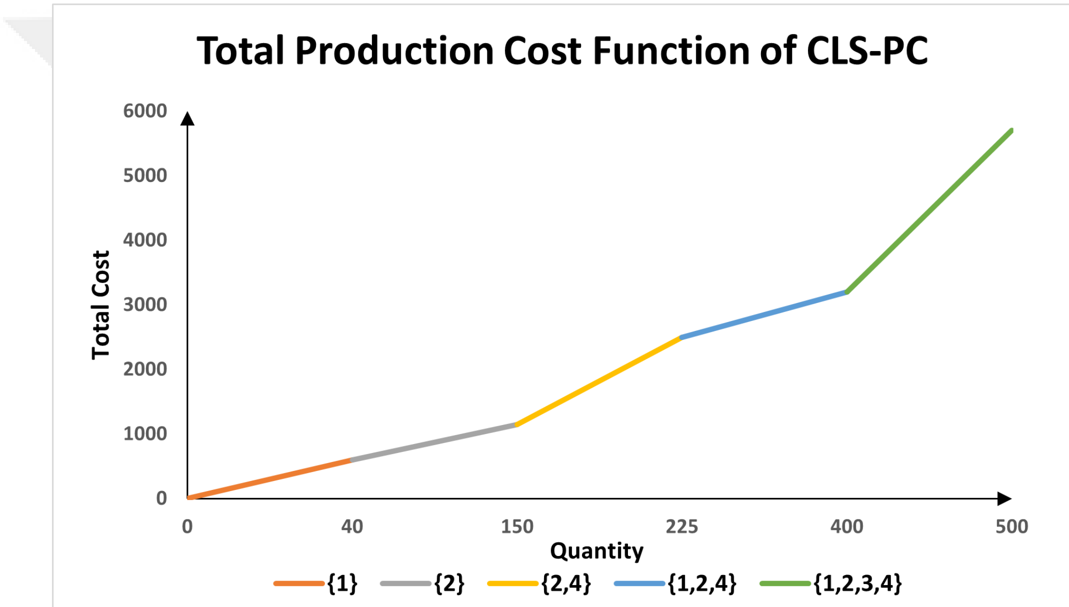

Figure 6.3 Total Production Cost Function of CLS-PC Example

7. CLS-PC WITH AT MOST TWO SETUPS AT EACH PERIOD (CLS2-PC)

In this chapter, we consider another special case of the CLS-PC problem where each period at most two of the resources can be used (CLS2-PC). This problem can occur in various situations; for example: in a production system that requires selection of some resources at each time period, in a system that can contract with a limited number of suppliers, or in a distribution system where there are restricted number of transportation modes. The main idea of this problem is to ensure that the system is more controllable while the number of resources utilized at any period is at a limited level. In fact, CLS-PC where at most $k \leq M$ of the resources are available for production would be a more general extension of this problem. However, since the complexity of this problem increases with k, the special case with at most two resources is discussed in the remaining of this chapter.

In CLS2-PC we would like to find the minimum cost production plan satisfying the carbon emission constraints over a planning horizon of T periods where at most two resources are used each period. We have the same general formulation of CLS-PC given by (3.2a)-(3.2g) with the inclusion of the constraints $\sum_{m=1}^{M} y_t^m \leq 2$ for $t = 1, \ldots, T$.

The new limit on the number of resources available at any time means that all machine pairs can be taken into account in the preprocessing step. We can consider all machine pairs (m_1, m_2) where m_1 is a green resource and m_2 is a green or regular resource, and construct the production cost function for each machine pair as demonstrated in Chapter 4. The study in Chapter 4 confirm that CLS2-PC can be reduced to an instance of multi-mode lot sizing problem with piecewise linear production cost functions (MMLS-PL) where at most one resource can be used at any period in $O(M_g \max\{M_g, M_r\})$ time. This is possible because constructing the production cost function for any machine pair can be done in a fixed time. The number of machine pairs that should be considered is $\bar{M} = M_g \times M_r + M_g \times M_g$. Furthermore, as discussed in Chapter 4, the total production cost function have at most two breakpoints for each resource pair.

As far as we know, there exists no algorithm in the literature for solving multi-mode lot sizing problem with piecewise linear production cost functions (MMLS-PL). As a matter of fact, the MMLS-PL can be transformed into an instance of a single resource lot sizing problem with piecewise linear production cost functions which can be solved using the algorithms of Koca et al. (2014) or Ou (2017). In the previous chapter (Chapter 6), we explained that to apply Koca et al. (2014)'s algorithm we need to take the lower envelope of the piecewise linear cost functions of resources. In this chapter, alternative to construct a single piecewise linear cost function, we will modify the algorithm of Koca et al. (2014) to solve MMLS-PC directly in polynomial time when $\bar{M}$ is fixed.

Before starting to modify Koca et al. (2014)'s algorithm we need to make some definitions. We will use the following two widely used concepts in the lot sizing literature:

Definition 7.1. *An interval* $[k,l]$ *where* $1 \leq k \leq l \leq T$, $s_{k-1} = s_l = 0$ *and* $s_t > 0$ *for* $k \leq t < l$ *is called a* ***regeneration interval****.*

Definition 7.2. *A production period* t *is called a* ***fractional period*** *if the production amount* x_t^m *is not equal to any of the breakpoints of the production cost function of resource* m*.*

Henceforth, we assume that the number of resources $\bar{M}$ is fixed and address the *resource pairs* as *resources*. We denote the number of breakpoints of the production cost function of resource r as B_r. By Theorem 3.2 we have: $B_r \leq 2$ for all $r = 1,\ldots,\bar{M}$. We represent the vector of breakpoints b such that b_{ri} is the ith breakpoint of the production cost function of resource r for $i = 1,\ldots,B_r$, $r = 1,\ldots,\bar{M}$. For each resource $r = 1,\ldots,\bar{M}$ we assume $0 < b_1 < \ldots < b_{B_r}$ holds. To show the no production case, we create a dummy resource 0 with one breakpoint, $B_0 = 1$ and $b_{01} = 0$. We calculate the total number of breakpoints as $\bar{B} = \sum_{r=0}^{\bar{M}} B_r$ and assume that it is also fixed. We depict the unit production cost of resource r as c_{ri} if $b_{r,i-1} < x \leq b_{ri}$ for $i = 1,\ldots,B_r$ and $r = 1,\ldots,\bar{M}$, in words if the production amount x lies in the ith segment of the cost function.

There exists an optimal solution to MMLS-PL with zero ending inventory ($s_T = 0$) because for each resource the production cost function is monotonic. Moreover, there exists an optimal solution to MMLS-PC that is composed of a series of regeneration intervals that cover the planning horizon $[1,T]$. Hence for each regeneration interval of an optimal solution we have the following property.

Theorem 7.1. *There exists an optimal solution for MMLS-PL such that there exists at most one fractional period at each regeneration interval.*

Proof. Let $[k,l]$ be a regeneration interval in an optimal solution (x,s), and assume that there are more than one fractional periods in $[k,l]$. Let u and v be two consecutive fractional periods in this regeneration interval with $u < v$. Assume that resources r_1 and r_2 are used in periods u and v, respectively, and the quantities x_{ur_1} and x_{vr_2} lie in the i_1th and i_2th segments of their production cost functions, respectively, i.e. $b_{r_1,i_1-1} < x_{ur_1} < b_{r_1,i_1}$ and $b_{r_2,i_2-1} < x_{vr_2} < b_{r_2,i_2}$. Define $\Delta^1 = \min\{\min_{t=u}^{v-1} s_t, x_{ur_1} - b_{r_1,i_1-1}, b_{r_2,i_2} - x_v\} > 0$ and $\Delta^2 = \min\{b_{r_1,i_1} - x_{ur_1}, x_{vr_2} - b_{r_2,i_2-1}\} > 0$. Now consider the following two solutions (x^1,s^1) and (x^2,s^2) same with (x,s) except

$$x^1_{ur_1} = x_{ur_1} - \Delta^1,\ s^1_t = s_t - \Delta^1 \ \text{ for } t = u,\dots,v-1,\ x^1_{vr_2} = x_{vr_2} + \Delta^1$$

$$x^2_{ur_1} = x_{ur_1} + \Delta^2,\ s^2_t = s_t + \Delta^2 \ \text{ for } t = u,\dots,v-1,\ x^2_{vr_2} = x_{vr_2} - \Delta^2$$

Note that both solutions are feasible. Moreover, since (x,s) is optimal we should have

$$\Delta^1 \left(-c_{r_1 i_1} - \sum_{t=u}^{v-1} h_t + c_{r_2 i_2} \right) \geq 0 \text{ and } \Delta^2 \left(c_{r_1 i_1} + \sum_{t=u}^{v-1} h_t - c_{r_2 i_2} \right) \geq 0.$$

Since $\Delta^1, \Delta^2 > 0$, $c_{r_1 i_1} + \sum_{t=u}^{v-1} h_t - c_{r_2 i_2} = 0$ should hold. This means that both (x^1,s^1) and (x^2,s^2) are also optimal. Note that (x^2,s^2) is an optimal solution where the number of fractional periods in the regeneration interval $[k,l]$ is reduced by one since u or v is not a fractional period: i)if $\Delta^2 = b_{r_1,i_1} - x_{ur_1}$, then u is not a fractional period, ii) if $\Delta^2 = x_v - b_{r_2,i_2-1}$, then v is not a fractional period. This procedure can be applied until an optimal solution with the desired property is obtained, so the result follows. □

Remark 1. *These results are very similar to Theorems 1 and 2 of Koca et al. (2014) developed for the single resource lot sizing problem with piecewise concave production cost functions.*

This finding confirms our decision of modifying the algorithm of Koca et al. (2014) to solve MMLS-PC directly in polynomial time when $\bar{M}$ is fixed. Eventually, we use the notation of Koca et al. (2014) in our algorithm. The modified algorithm is divided into two main steps: first we determine the minimum cost for each regeneration interval. Second, we solve the resulting problem as a shortest path problem. In the following two sections, we discuss the details of these steps.

7.1 Minimum cost for a regeneration interval $[k,l]$

First, we identify the minimum cost for a regeneration interval $[k,l]$ $s_{k-1} = s_l = 0$ and $s_t > 0$ for $t = k,\dots,l-1$. We make the following definitions:

- $\tau_{ri} \in \mathbb{Z}_+$ for $r = 1,\dots,\bar{M}$, $i = 1,\dots,B_r$: is the number of times production occurs at the level of breakpoint b_{ri} before period t
- $\pi_{ri} \in \mathbb{Z}_+$ $r = 1,\dots,\bar{M}$, $i = 1,\dots,B_r$: is the number of times production occurs at the level of breakpoint b_{ri} after period t
- $d_{kl} = \sum_{t=k}^{l} d_t$: is the total demand for the time interval $[k,l]$
- $\rho_{kl}(\tau,\pi)$: is the quantity produced in a fractional period

Note that, for the case when $r = \bar{M}$ if τ_{ri} times b_{ri} units are produced between periods k and t for $r = 1,\dots,\bar{M}$, $i = 1,\dots,B_r$, and π_{ri} times b_{ri} units will be produced between periods $t+1$ and l for $r = 1,\dots,\bar{M}-1$, $i = 1,\dots,B_r$, and $i = 1,\dots,B_{\bar{M}}-1$. Then the number of times $b_{\bar{M}B_{\bar{M}}}$ units produced is calculated as $\pi_{\bar{M}B_{\bar{M}}}$ in equation (7.1).

$$\pi_{\bar{M}B_{\bar{M}}} := \left\lfloor \frac{d_{kl} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} - \sum_{r=1}^{\bar{M}-1} \sum_{i=1}^{B_r} \pi_{ri} b_{ri} - \sum_{i=1}^{B_{\bar{M}}-1} \pi_{\bar{M}i} b_{\bar{M}i}}{b_B} \right\rfloor \tag{7.1}$$

Thus, we describe $\pi_{\bar{M}}$ as a $B_{\bar{M}}$-vector while its last component is determined by the expression given in equation (7.1). Then in (7.1), we calculate the remaining production using τ and π which is equal to the quantity that will be produced in the fractional period.

$$\rho_{kl}(\tau,\pi) = d_{kl} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \tau_{ri} b_{ri} - \sum_{r=1}^{\bar{M}} \sum_{i=1}^{B_r} \pi_{ri} b_{ri} \tag{7.2}$$

It is important to highlight that in a single resource problem we need to determine the fractional period and the fractional production amount. When we have multiple resources, we need to include the decision of which resource will produce fractional quantity in our calculations.

Our aim is to find an optimal solution where at each regeneration interval there exists at most one fractional period. For this reason, we introduce functions, $F_{kl}(t,\tau)$ and

$G_{kl}(t,\tau,\pi)$ to depict the cases where the interval $[k,t]$ does not contain any fractional period, and contains a single fractional period, respectively.

Let $F_{kl}(t,\tau)$ be the minimum cost for the interval $[k,t]$ during which τ_{ri} times b_{ri} units are produced at the corresponding resources and no fractional production is done between periods k and t given that $s_t > 0$ for $t = k,\ldots,l-1$.

For the first period k of the regeneration interval, for $r = 0,\ldots,\bar{M}$, $i = 1,\ldots,B_r$ where e_{ri} is a vector of the appropriate size (which has a component for each breakpoint of each resource) and only the ith component of the rth vector is one with the convention that e_{01} represents the zero vector (no production case) we have the following:

$$(7.3) \qquad F_{kl}(k,e_{ri}) = \begin{cases} c_{ri}b_{ri} + h_k(b_{ri} - d_k), & \text{if } b_{ri} > d_k \text{ and } k < l \\ c_{ri}b_{ri}, & \text{if } b_{ri} = d_k \text{ and } k = l \\ \infty, & \text{otherwise.} \end{cases}$$

For the remaining values of τ, where $\tau \neq e_{ri}$ for all $r \in \{0,\ldots,\bar{M}\}$, $i \in \{1,\ldots,B_r\}$ we set F_{kl} to be ∞. For the remaining intervals in the set for $t \in \{k+1,\ldots,l\}$ we have the recursive relations in (7.1) where τ_r is the vectors for $r = 0,\ldots,\bar{M}$ satisfying equation (7.1).

$$(7.4) \qquad F_{kl}(t,\tau) = \min_{\substack{r=0,\ldots,\bar{M},\\ i=1,\ldots,B_r:\\ \tau \geq_{ri}}} \{F_{kl}(t-1,\tau -_{ri}) + c_{ri}b_{ri}\} + h_t\left(\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} - d_{kt}\right)$$

(7.5)

$$\tau_r = \{\tau_r \in \mathbb{Z}_+^{B_r} : \sum_{r=0}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri} \leq t-k+1\} = \begin{cases} \sum_{r=0}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} > d_{kt}, & \text{when } t < l \\ \sum_{r=0}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} = d_{kt}, & \text{when} t = l \\ F_{kl}(t,\tau) = \infty, & \text{otherwise.} \end{cases}$$

To calculate the value of $F_{kl}(t,\tau)$, all possible production quantities for period t are acknowledged with the vectors e_{ri} for $r = 0,\ldots,\bar{M}$, $i = 1,\ldots,B_r$. For all possible values of τ and for increasing values of t we evaluate the above recursion (7.1). Since we assume $\bar{M}$ and $\bar{B}$ are fixed, for a given t and τ, F_{kl} can be computed in fixed time. The number of possible τ vectors is $O\left(T^{\bar{B}}\right)$ consequently for a regeneration interval $[k,l]$ F_{kl} can be evaluated in $O\left(T^{\bar{B}+1}\right)$ time .

Let $G_{kl}(t,\tau,\pi)$ be the minimum cost for the interval $[k,t]$ during which τ_{ri} times b_{ri} units are produced and a fractional production is done in the level of $\rho_{kl}(\tau,\pi)$ between periods k and t, given that π_{ri} times b_{ri} units are produced between periods $t+1$ and l, and $s_t > 0$ for $t = k,\ldots,l-1$.

We define $\bar{M}_\rho$ as the subset of resources such that $\rho_{kl}(\tau,\pi)$ is not equal to any of the breakpoints. For the first period k of the regeneration interval we calculate $G_{kl}(k,e_{01},\pi)$ in (7.1). In this computation, we take π satisfying $d_{kl} > \sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\pi_{ri}b_{ri}$ and $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\pi_{ri} \leq l-k$, and $\rho_{kl}(e_{01},\pi)$ such that $\bar{M}_\rho \neq \emptyset$. If not we allow G_{kl} to be ∞.

(7.6)

$$G_{kl}(k,e_{01},\pi) = \begin{cases} \min_{r\in\bar{M}_\rho}\{c_{ri(r,\rho_{kl})}\}\rho_{kl}(e_{01},\pi) + h_k(\rho_{kl}(e_0,\pi) - d_k), \text{if } \rho_{kl}(e_{01},\pi) > d_k \text{ and } k < l \\ \min_{r\in\bar{M}_\rho}\{c_{ri(r,\rho_{kl})}\}\rho_{kl}(_{01},\pi), \text{if } \rho_{kl}(e_{01},\pi) = d_k \text{ and } k = l \\ \infty, \text{otherwise.} \end{cases}$$

For the remaining $t \in \{k+1,\ldots,l\}$, the following recursive equations (7.7) will be used to compute G_{kl}.

(7.7)

$$G_{kl}(t,\tau,\pi) = \min\Bigg\{F_{kl}(t-1,\tau) + \min_{r\in\bar{M}_\rho}\{c_{ri(r,\rho)}\}\rho_{kl}(\tau,\pi),$$
$$\min_{\substack{r=0,\ldots,\bar{M},:\\ i=1,\ldots,B_r:\\ \tau\geq ri}}\{G_{kl}(t-1,\tau-e_{ri},\pi+e_{ri}) + c_{ri}b_{ri}\}\Bigg\} + h_t\left(\sum_{i=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} + \rho_{kl}(\tau,\pi) - d_{kt}\right)$$

We solve the recursive relation in (7.7) for $\tau_r \in \mathbb{Z}_+^{B_r}$ and $\pi_r \in \mathbb{Z}_+^{B_r}$ satisfying: $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri} \leq t-k$, $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\pi_{ri} \leq l-t$, $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} + \rho_{kl}(\tau,\pi) > d_{kt}$ when $t < l$, and $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} + \rho_{kl}(\tau,\pi) = d_{kt}$ when $t = l$, $\sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\tau_{ri}b_{ri} + \sum_{r=1}^{\bar{M}}\sum_{i=1}^{B_r}\pi_{ri}b_{ri} \leq d_{kl}$ and $\bar{M}_\rho \neq \emptyset$. Otherwise, we let $G_{kl}(t,\tau,\pi) = \infty$.

Throughout calculating the value of $G_{kl}(t,\tau,\pi)$, two possibilities are examined. First expression represents the case where the fractional production quantity is produced in period t and the previous productions occur in breakpoint levels. Second expression applies to the case where fractional production quantity is produced at a previous period, and production at period t occurs at a breakpoint level of a resource. Similar to the calculation of F_{kl}, for given t, τ, and π with fixed number of resources $\bar{M}$ and the total number of breakpoints $\bar{B}$ the G_{kl} can be evaluated in a fixed time. The number of possible τ vectors is $O\left(T^{\bar{B}}\right)$ and the number of possible π vectors is $O\left(T^{\bar{B}-1}\right)$. Therefore, for a regeneration interval $[k,l]$ G_{kl} can be evaluated in $O\left(T^{2\bar{B}}\right)$ time.

7.2 Minimum cost for the problem MMLS-PL

In order to calculate the minimum cost for the regeneration interval $[k,l]$, all potential F_{kl} and G_{kl} must be evaluated for all possible t, τ and π values. When this step is complete we continue by solving the following recursive relation (7.2):

$$\mu_{kl} = \min_{\substack{r=0,\dots,\bar{M}: \\ \tau_r \in \{0,\dots,T\}^{B_r}}} \{F_{kl}(l,\tau), G_{kl}(l,\tau_{,01})\}. \tag{7.8}$$

By solving the (7.2) we can obtain the minimum cost for a given regeneration interval in $O\left(T^{2\bar{B}}\right)$ time. In total we have $O\left(T^2\right)$ regeneration intervals. Altogether, for all regeneration intervals the minimum cost can be identified in $O\left(T^{2\bar{B}+2}\right)$ time.

Once for all $1 \le k \le l \le T$, μ_{kl} is determined, we can construct a directed graph $G = (N, A)$ where each node represents a period, $N = \{1, \dots, T+1\}$, and each arc $(k, l+1)$ represents a regeneration interval $[k,l]$, $A = \{(k, l+1) : 1 \le k \le l \le T\}$. We assume that cost of arc $(k, l+1)$ is given by μ_{kl}, i.e. $c_{k,l+1} = \mu_{kl}$. Subsequently, the shortest path problem from node 1 to node $T+1$ gives us the solution to the MMLS-PL. Knowing that shortest path problem can be solved in $O\left(T^2\right)$ time we can solve MMLS-PL in $O\left(T^{2\bar{B}+2}\right)$ time. Note that the dominant part of this solution algorithm is to construct the directed graph G.

As a result, we demonstrated that CLS2-PC can be solved in polynomial time, $O\left(T^{2\bar{B}+2}\right)$ when the cost and emission parameters are time-invariant and the number of resources is fixed using modified Koca et al. (2014)'s algorithm.

8. CONCLUSION

In this thesis, we investigated the single item capacitated multi-mode lot sizing problem with periodic carbon emission constraints. We studied several structural properties of the problem and proved that the resource capacities make the problem NP-Hard. In the aim of applying the existing algorithms developed for the lot sizing problem with piecewise concave production cost functions (Koca et al. (2014),Ou (2017)) we introduced algorithms to construct the piecewise linear total production cost function of the CLS-PC problem and its extensions. Moreover, we examined an extension of CLS-PC with at most two resources can be used at any period, and developed a polynomial time dynamic programming algorithm.

In Chapters 5 and 6, we show that when the number of resources are fixed, the complexity of algorithms introduced are also fixed. It is important to recognize that, an alternative approach can be an attempt to construct the total production cost functions by considering the possible amounts that can be produced in any period, which can lead to another exponential algorithm when the number of resources is not fixed. In further research, more efficient heuristics or approximation algorithms can be designed in the aim of constructing the total production cost function.

All things considered, to the best of our knowledge this is the first study along with Koca & Koksalan (2021) considering the multi-mode capacitated lot sizing problem with periodic carbon emission constraints. We analyzed CLS-PC and its extensions, while focusing on the complexities of the problems. Further research might explore different carbon emission constraints such as nonlinear, global or cumulative instead of periodic carbon emission constraint (the one considered in this thesis). It would be interesting to see the effects of carbon emission constraints and resource capacities via computational analysis.

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