Nonlinear Vibration Analysis of Uniform and Functionally Graded Beams with Spectral Chebyshev Technique and Harmonic Balance Method

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ABSTRACT

In this paper, nonlinear forced vibrations of uniform and functionally graded Euler-Bernoulli Beams with large deformation are studied. Spectral and temporal boundary value problems of beam vibrations do not always have closed-form analytical solutions. As a result, many approximate methods are used to obtain the solution by discretizing the spatial problem. Spectral Chebyshev Technique (SCT) utilizes the Chebyshev Polynomials for spatial discretization and applies Galerkin's method to obtain boundary conditions and spatially discretized equations of motions. Boundary conditions are imposed using basis recombination into the problem and as a result of this, the solution can be obtained to any linear boundary condition without the need for re-derivation. System matrices are generated with the SCT, and natural frequencies and mode shapes are obtained by eigenvalue problem solution. Harmonic Balance Method (HBM) is used to solve nonlinear equation of motion in frequency domain, with large deformation nonlinearity. As a result, a generic method is constructed to solve nonlinear vibrations of uniform and functionally graded beams in frequency domain, subjected to different boundary conditions.

Keywords: Nonlinear Vibration, Functionally Graded Beam, Forced Response, Harmonic Balance Method

INTRODUCTION

Nonlinear vibration analysis of beams plays an important role in the design of many engineering structures, especially those experience dynamic loads such as airplane wings, wind turbines and jet engine blades, electronic boards, etc. Many failures in these structures can be predicted through nonlinear vibration analysis of beams. To prevent failures, without constructing costly analysis models, it is possible to change the design of the structure (e.g., geometry, material property, etc.). Nonetheless, the complexity of the nonlinear problem and the deficiency in the literature increases the need for research in nonlinear beam vibrations.

In large deformation nonlinearity, the deformations higher than the thickness of the beam results in stretching force which induces nonlinear behavior. This nonlinearity is reflected in the differential equations of motion. The resulting nonlinear differential equations can be solved by time domain or frequency domain methods. For instance, Chakrapani *et al.* [1] and Swain *et al.* [2] studied the force vibration of composite beams in time domain. Similarly, Liao-Liang Ke *et al.* [3] worked on free vibration of geometrically nonlinear composite beams. However, frequency domain methods are computationally very efficient compared to time domain methods to obtain the steady state response. Harmonic Balance Method (HBM) and

Describing Function Method (DFM) are the most common frequency domain methods used to obtain the steady state response of nonlinear systems. For example, H. Youzera *et al.* [4] implemented HBM in the solution of forced vibration of symmetric laminated composite beams.

In this paper, HBM is employed to convert the nonlinear differential equations of motion into nonlinear algebraic equations. The resulting set of nonlinear algebraic equations are solved by using Newton's method with arc-length continuation. Several case studies are performed on uniform and functionally graded beams. Simply supported, fixed-fixed and fixed-pinned boundary conditions are considered in the case studies. Deflections at the middle point of the beam are presented as a function of frequency for different mechanical properties and different external excitations.

THEORY

In the SCT, Chebyshev polynomials are used to spatially discretize the beam. Transverse displacement function of a beam can be expressed by Chebyshev series expansion as

$$y(x) = \sum_{k=0}^{\infty} \alpha_k T_k(x). \tag{1}$$

Where $T_k(x)$ is Chebyshev polynomials of the first kind which can be given as follows [5]

$$T_k(x) = \cos\left(k \cos^{-1}(x)\right) \quad for \;\; k = 0, 1, 2 \cdots. \tag{2}$$

The displacement function of the beam can be represented by sampled points at certain increments for numerical calculations. If the sampling point number is selected the same as the number of Chebyshev polynomials, there occurs a one-to-one mapping between the sampled points and Chebyshev coefficients α_k [5]. For *N* number of Chebyshev polynomials, *N* number of Gauss-Lobatto points are used for sampling spatial domain, which are defined as

$$p_k = \cos\left(\frac{(k-1)\pi}{N-1}\right). \tag{3}$$

The relation between sampled displacement function of the beam and the Chebyshev expansion coefficients can be written as

$$\mathbf{a} = \mathbf{\Gamma}_F \mathbf{y} \tag{4}$$

where $\mathbf{\Gamma}_F$ is an *N* x *N* forward transformation matrix. Additionally, backward transformation matrix is defined which is the inverse of forward transformation matrix.

Derivative and integral of the any function and vector constructed by Chebyshev polynomials can be obtained as

$$\mathbf{y}^{(n)} = \mathbf{Q}_n \mathbf{y} \tag{5}$$

$$\int_{l_1}^{l_2} y(x)dx \; = \; \mathbf{v}^T \mathbf{a}. \tag{6}$$

Here $\mathbf{Q}_n$ is the derivative matrix with respect to order n. $\mathbf{v}$ is the definite integral vector. Derivation of the $\mathbf{Q}_n$ and $\mathbf{v}$ according to Spectral Chebyshev Method, are given in reference [6].

Equation of motion of a Bernoulli beam is written as

$$\frac{\partial^2}{\partial x^2}\left[E(x)I\,\frac{\partial^2 y(x,t)}{\partial x^2}\right] + \; \rho A\,\frac{\partial^2 y(x,t)}{\partial t^2} = f(x,t). \tag{7}$$

Boundary conditions for this equation can be written in a generic way such as

$$\boldsymbol{\beta}_{ij3}y'''' + \boldsymbol{\beta}_{ij2}\, y''' + \boldsymbol{\beta}_{ij1}y'' + \boldsymbol{\beta}_{ij0}y = \boldsymbol{\alpha}_{ij}(t) \tag{8}$$

Here β's are the constants of the spatial part of the boundary condition, whereas α's are the constants of the temporal part. Both can be written in vector form. The i and j indices corresponds to the boundary location (0 and L, i=1,2) and the number of the boundary condition (j=1,2). When boundary conditions change, the derivation of equation does not change, only these matrices change. An important step in imposing boundary conditions is expressing **y** by using projection matrices [6] as

$$\mathbf{y} = \mathbf{Pz} + \mathbf{R\alpha} \tag{9}$$

This technique makes the problem solvable for **z** which only satisfies homogenous boundary conditions, where **y** satisfies all the boundary conditions [6]. Calculation of **P** and **R** matrices are given in reference [6]. In order to obtain the approximate solution, Galerkin's Method is applied. Imposing projection matrices into Eq.8, the residual from approximation is written as

$$\boldsymbol{\phi} = \mathrm{m}(\mathbf{P}\ddot{\mathbf{z}} + \mathbf{R}\ddot{\boldsymbol{\alpha}}) - \mathbf{Q_4}(\mathbf{Pz} + \mathbf{R\alpha}) - \mathbf{f} \tag{10}$$

Where $\mathrm{m} = \left(\frac{\rho A}{EI}\right)$ and **f** is normalized by dividing it with EI. To minimize this residual, the inner product of weighted residuals must vanish, i.e.

$$\int_{l_1}^{l_2} \theta(x)\phi(x)dx = \boldsymbol{\theta}^T\mathbf{V}\boldsymbol{\phi} = 0 \tag{11}$$

The inner product of any two functions $\theta(x)$ and $\phi(x)$ can be constructed by Chebyshev polynomials by using the inner product matrix (**V**) which is described in Appendix.

Including derivative matrices, boundary projection matrices and applying Galerkin's Method, Eq. (7) can be written as

$$\left(\frac{\rho A}{EI}\right)\mathbf{P}^{\mathrm{T}}\mathbf{VP}\ddot{\mathrm{z}} + \mathbf{P}^{\mathrm{T}}\mathbf{VQ_4P} = \mathbf{P}^{\mathrm{T}}\mathbf{Vf} - \left(\frac{\rho A}{EI}\right)\mathbf{P}^{\mathrm{T}}\mathbf{VR}\ddot{\boldsymbol{\alpha}} - \mathbf{P}^{\mathrm{T}}\mathbf{VQ_4R\alpha} \tag{12}$$

In Eq. (12), mass and stiffness matrices, and the forcing vector can be obtained as follows

$$\mathbf{M} = \left(\frac{\rho A}{EI}\right)\mathbf{P}^{\mathrm{T}}\mathbf{VP}, \qquad \mathbf{K} = \mathbf{P}^{\mathrm{T}}\mathbf{VQ_4P}, \qquad \mathbf{f}^* = \mathbf{P}^{\mathrm{T}}\mathbf{Vf} \tag{13}$$

Since for basic boundary conditions $\alpha, \dot{\alpha}$ *and* $\ddot{\alpha}$ are 0, f term is simplified to the form given in Eq. (13).

When a mechanical property varies along the beam's longitudinal direction, the related inner product matrix changes. As a study, Young's modulus is considered to be varying along the beam's length such as

$$\mathrm{E(x)} = (E_2 - E_1)\mathrm{x/L} + E_1. \tag{14}$$

With respect to this variation, the E value in the equation of motion is not a scalar anymore but a function. By using chain rule, applying Galerkin's method and imposing boundary projection matrices to the equation of motion, the following equation is obtained:

$$I[\mathbf{P}^T[\mathbf{V}_E\mathbf{Q_4} + 2\mathbf{Q_1}\mathbf{V}_E\mathbf{Q_3} + \mathbf{Q_2}\mathbf{V}_E\mathbf{Q_2}]\,\mathbf{Pz}] + (\rho A)\mathbf{P}^T\mathbf{VP}\ddot{\mathbf{z}} = \mathbf{P}^T\mathbf{Vf} \tag{15}$$

Here, $\mathbf{V_E}$ is the inner product matrix with respect to varying Young's Modulus. Calculation of $\mathbf{V}_E$, $\mathbf{V}'_{\mathbf{E}}$ and $\mathbf{V}''_{\mathbf{E}}$ is given in Appendix.

With respect to Eq. (15), new mass and stiffness matrices and the forcing vector of the system becomes as follows

$$\mathbf{M} = \left(\frac{\rho A}{I}\right)\mathbf{P}^T\mathbf{V}\mathbf{P},\ K = \mathbf{P}^T[\mathbf{V}_E\mathbf{Q_4} + 2\mathbf{Q_1}\mathbf{V}_E\mathbf{Q_3} + \mathbf{Q_2}\mathbf{V}_E\mathbf{Q_2}]\mathbf{P},\ \mathbf{f}^* = \left(\frac{1}{I}\right)\mathbf{P}^T\mathbf{V}\mathbf{f} \tag{16}$$

Application of Harmonic Balance Method

The equation of motion of a Euler-Bernoulli Beam with geometric nonlinearity is given as

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = \left[\frac{EA}{2L}\int_{l_1}^{l_2}\left(\frac{\partial y}{\partial x}\right)^2 dx\right]\frac{\partial^2 y}{\partial x^2} + f(x,t). \tag{17}$$

Here the term $\left[\frac{EA}{2L}\int_{l_1}^{l_2}\left(\frac{\partial y}{\partial x}\right)^2 dx\right]\frac{\partial^2 y}{\partial x^2}$ comes due to the stretching effect occurring along the beam. This nonlinear phenomenon is studied by many researchers with different solution techniques [7,8,9,10].

The nonlinear term in Eq. (17) can be written by using Chebyshev Technique as follows

$$\left[\frac{EA}{2L}\int_{l_1}^{l_2}\left(\frac{\partial y}{\partial x}\right)^2 dx\right]\frac{\partial^2 y}{\partial x^2} = \frac{EA}{2L}\mathbf{v}^T\Gamma_{\boldsymbol{F}}(\mathbf{Q_1}\mathbf{Pz})^2\mathbf{Q_2}\mathbf{Pz} \tag{18}$$

For the beam with variation of its Young Modulus along length, nonlinear term can be defined as below, with respect to the system parameters defined at Eq. (16)

$$\left[\frac{EA}{2L}\int_{l_1}^{l_2}\left(\frac{\partial y}{\partial x}\right)^2 dx\right]\frac{\partial^2 y}{\partial x^2} = \frac{\mathbf{V}_\mathrm{E}A}{2L}\mathbf{v}^T\Gamma_{\boldsymbol{F}}(\mathbf{Q_1}\mathbf{Pz})^2\mathbf{Q_2}\mathbf{Pz} \tag{19}$$

Eq. (17) can be written in the following form

$$\left(\frac{\rho A}{EI}\right)\mathbf{P}^T\mathbf{V}\mathbf{P}\ddot{\mathbf{z}} + \mathbf{P}^T\mathbf{V}\mathbf{Q_4}\mathbf{P} = \frac{A}{2IL}\mathbf{v}^T\Gamma_{\boldsymbol{F}}(\mathbf{Q_1}\mathbf{Pz})^2\mathbf{Q_2}\mathbf{Pz} + \mathbf{P}^T\mathbf{V}\mathbf{f}. \tag{20}$$

A single harmonic solution is assumed as follows

$$\mathbf{z} = \begin{Bmatrix} z_{s1}\sin\theta\ +\ z_{c1}\cos\theta \\ \cdot \\ \cdot \\ z_{sn}\sin\theta\ +\ z_{cn}\cos\theta \end{Bmatrix}. \tag{21}$$

The $\mathbf{z}$ vector is placed into the Eq. (20) and coefficients of the similar terms are balanced to determine the unknowns. The nonlinear term in the equation can be concluded to form given below with the help of trigonometric relations.

$$\left(\frac{A}{2IL}\mathbf{v}^T\Gamma_{\boldsymbol{F}}(\mathbf{Q}_1\mathbf{Pz})^2\mathbf{Q}_2\mathbf{Pz}\right)(\mathbf{z}) = [\mathbf{V}_L]\begin{bmatrix} \psi_{s_1}\ sin\theta\ +\ \psi_{c_1}\ cos\theta \\ \cdot \\ \cdot \\ \psi_{s_n} sin\theta\ +\ \psi_{c_n}\ cos\theta \end{bmatrix} \tag{22}$$

Where $[\mathbf{V}_L] = \mathbf{Q_2}\mathbf{Pz}$ and ψ_{s_1} , ψ_{c_1} are the numerical expressions arising from Harmonic Balance Method.

With the addition of the Eq. (21), nonlinear equation of motion of can be converted into a set of nonlinear algebraic equations in frequency domain for sine cosine terms of the displacement as the unknowns. These nonlinear algebraic equations can be solved by using Newton’s method with arc-length continuation [11].

CASE STUDY

Solving the eigenvalue problem with the system matrices defined in Eq.13, the first 5 natural frequencies of a Euler-Bernoulli Beam subjected to pinned-pinned, fixed-fixed and fixed-pinned boundary conditions are given in Table 1. Additionally, exact solutions are calculated for the problem and given in the Table. In these results, material of the beam is aluminum with the following properties E=71 GPa, ρ=2770 kg/m^3, w=0.03 m (width), L=1 m and, h=0.01 m (thickness).

	Natural Frequencies (rad/s)								
	Pinned - Pinned			**Fixed - Fixed**			**Fixed - Pinned**		
	Exact Solution	SCT with 9 polynomials	SCT with 13 polynomials	Exact Solution	SCT with 9 polynomials	SCT with 13 polynomials	Exact Solution	SCT with 9 polynomials	SCT with 13 polynomials
1st	144.244	144.244	144.244	326.98	326.985	326.985	225.336	225.337	225.337
2nd	576.9769	576.994	576.977	901.347	901.647	901.348	730.239	730.309	730.236
3rd	1298.197	1298.677	1298.198	1767	1770.323	1767.003	1523.586	1525.876	1523.58
4th	2307.907	2452.392	2307.957	2920.96	3242.881	2921.322	2605.423	2811.163	2605.539
5th	3606.105	4012.955	3606.596	4363.407	5165.42	4365.938	3975.749	4601.434	3977.551

Table 1: First 5 Natural Frequencies of Uniform Beam Subjected to Different Boundary Conditions

The first five natural frequencies of the functionally graded beam are obtained with the previous parameters that are given, except E_1 and E_2, which are taken as 85.2 GPa and 28.4 GPa.

	Natural Frequencies (rad/s)					
	Pinned - Pinned		**Fixed - Fixed**		**Fixed - Pinned**	
	SCT with 9 polynomials	SCT with 13 polynomials	SCT with 9 polynomials	SCT with 13 polynomials	SCT with 9 polynomials	SCT with 13 polynomials
1st	126.63	126.63	282.558	282.557	203.332	203.332
2nd	503.519	503.158	781.206	781.171	641.88	641.876
3rd	1133.394	1130.656	1543.647	1533.353	1339.53	1331.586
4th	2070.412	2008.442	2682.509	2536.243	2341.85	2272.009
5th	3672.316	3139.349	4767.871	3799.932	4423.264	3468.51

Table 2: First 5 Natural Frequencies of a Beam with Varying Young Modulus
Subjected to Different Boundary Conditions

As expected, the natural frequencies of the beam decreased, since overall stiffness of the beam is reduced due to the new variation of Young Modulus.

Frequency response of the beams defined above (uniform and functionally graded) is solved with geometric nonlinearity. A sinusoidal force is applied at the mid-point of the beam and the transverse deflection at the mid-point is obtained for different excitation forcing amplitudes. A viscous damping coefficient of 0.03 is considered for the whole system by assuming proportional damping. The response was given as normalized amplitude which is displacement divided by force.

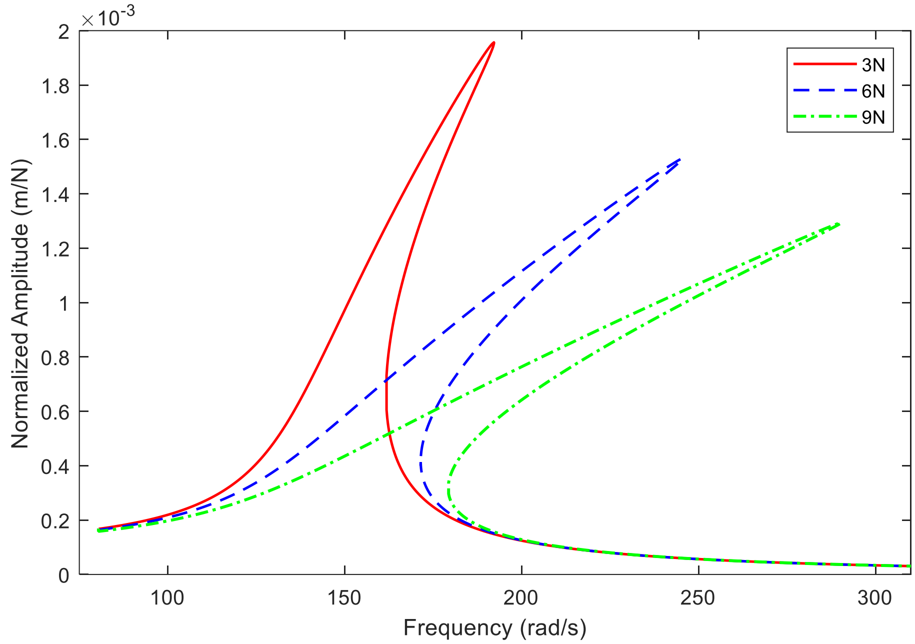

Figure 1: Mid-Point Normalized Deflection of the Uniform Beam with Pinned-Pinned Boundary Conditions

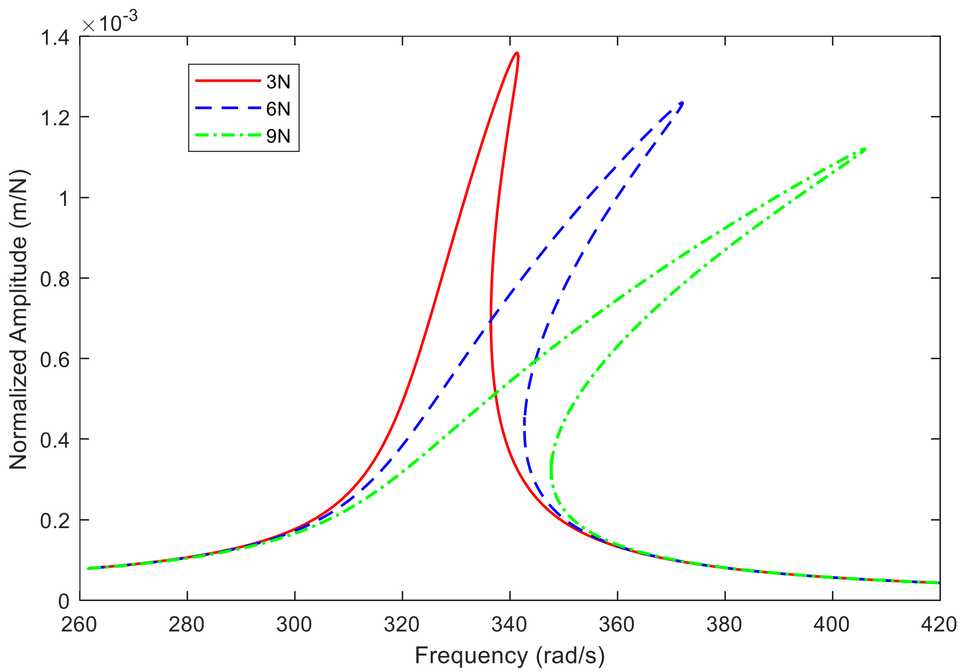

Figure 2: Mid-Point Normalized Deflection of the Uniform Beam with Fixed-Fixed Boundary Conditions

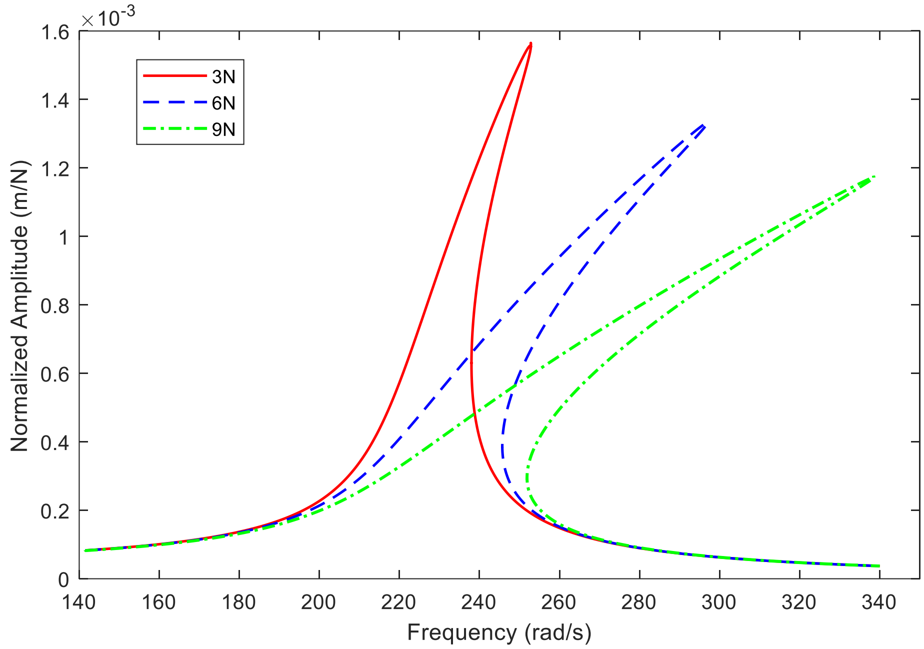

Figure 3: Mid-Point Normalized Deflection of the Uniform Beam with Fixed-Pinned Boundary Conditions

Frequency response plot for functionally graded beam (E_1 = 85.2 GPa,,E_2 =28.4 GPa.) with fixed-pinned boundary conditions is given below.

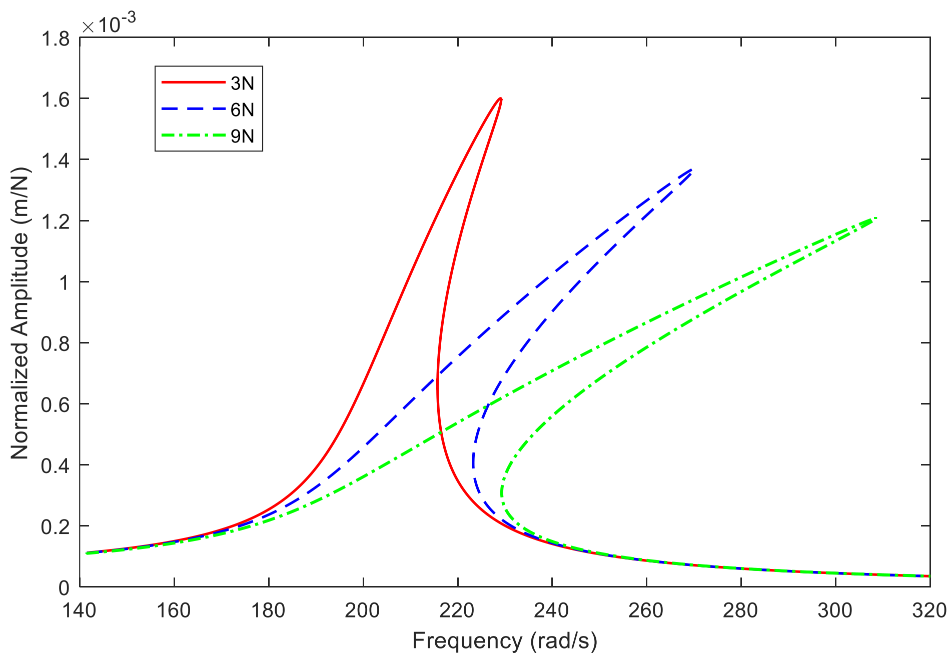

Figure 4: Mid-Point Deflection of the Functionally Graded Beam with Fixed-Pinned Boundary Conditions

Additionally, a frequency response plot for a varying Young Modulus beam with different scenarios is given in Fig. 5 (with 6N force applied).

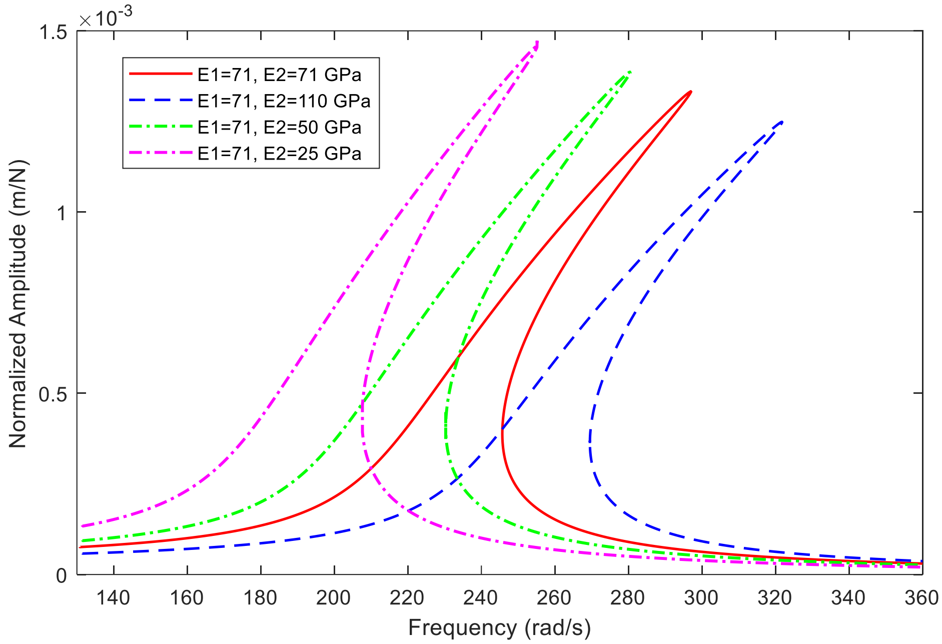

Figure 5: Mid-Point Deflection of the Functionally Graded Beam with Fixed-Pinned Boundary Conditions and Different Cases

As expected, if Young Modulus at the end of the beam decreases, the overall stiffness of the beam also decreases. Hence, the natural frequency of the beam becomes lower and the deflection of the beam increases.

CONCLUSION

In this paper a generic method is proposed to solve nonlinear vibrations of uniform and functionally graded beams. The method generates a fast solution for the problem in the frequency domain. If one is after the frequency response of any point along the nonlinear beam subjected to different boundary conditions, the method yields efficient solutions that are not computationally expensive. As case studies, the frequency response of the mid-point deflections of uniform and functionally graded beams subjected to different basic boundary conditions are presented.

APPENDIX

Calculation of Inner Product Matrix

Values of any two functions f(x) and g(x) at *N* Gauss-Lobatto points are written as f_N and g_N. Product of interpolated functions has order of 2*N*.

$$\mathbf{f}_{2N}=\mathbf{S}_2\mathbf{f}_N \tag{A-1}$$

S2 is constructed as:

$$\mathbf{S}_{2N}=\boldsymbol{\Gamma}_{\mathrm{B}_{2N}}[\mathbf{I}_N\,;\,\mathbf{O}_N]\,\boldsymbol{\Gamma}_{\mathrm{F}_N} \tag{A-2}$$

Here $\mathbf{\Gamma}_{B_{2N}}$ is the $2N$ x $2N$ backward transformation matrix. $\mathbf{I}_N$ and $\mathbf{O}_N$ are the N x N dimensional identity and zero matrices. The inner product of f(x) and g(x) can be written as

$$\int_{l_1}^{l_2} \mathrm{f}(x)\mathrm{g}(x)dx = \mathbf{f}^T\mathbf{V}\,\mathbf{g} = \mathbf{f}_{2N}^{\boldsymbol{T}}\mathbf{v}_{d,2N}\,\boldsymbol{g}_{2N} \qquad (A-3)$$

Here $\mathbf{v}_{d,2N}$is a matrix whose diagonal has the elements of multiplication $\mathbf{v}_{2N}^{\boldsymbol{T}}\mathbf{\Gamma}_{F_{2N}}$. Then the inner product matrix is written as:

$$\mathbf{V} = \mathbf{S}_{2N}^{\boldsymbol{T}}\boldsymbol{v}_{d,2N}\,\mathbf{S}_2 \qquad (A-4)$$

When the differential equation has variable coefficients, a weighted inner product is defined with respect to a weighting function γ(x). In the problem given in case study, γ(x) is variation of the Young Modulus distribution, E(x). Since there is a weighting function, the inner product has order of $3N$. Consequently, the inner product and inner product matrix can be described as

$$\int_{l_1}^{l_2} f(x)\mathrm{g}(x)E(x)dx = \mathbf{f}_{3N}^{\boldsymbol{T}}\mathbf{V}_{\mathbf{E}}\,\mathbf{g}_{\boldsymbol{N}} \qquad (A-5)$$

$$\mathbf{V}_{\mathbf{E}} = \mathbf{S}_{3N}^{\boldsymbol{T}}\mathbf{v}_{d,3N}\,\mathbf{E}_{d,3N}\mathbf{S}_3 \qquad (A-6)$$

Where $\mathbf{v}_{d,3n}$ and $\mathbf{E}_{d,3N}$ are $3N$ x $3N$ matrices whose diagonals have the values of $\mathbf{f}_{3N}$ and $\mathbf{E}_{3N}$

The first and second derivative (with respect to x) of the E(x) are E′(x) and E″(x). While finding the inner product matrix as described above, if E′(x) is used then $V_E'(x)$ is obtained. Similarly, if E″(x) is used then $V_E''(x)$ is obtained.

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