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On some normability conditions

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Various normability conditions of locally convex spaces (including Vogt interpolation classes DN_φ and Ω_φ as well as quasi- and asymptotic normability) are investigated. In particular, it is shown that on the class of Schwartz spaces the property of asymptotic normability coincides with the property GS, which is a natural generalization of Gelfand–Shilov countable normability (cf. [9, 25], where the metrizable case was treated). It is observed also that there are certain natural duality relationships among some of normability conditions.

1 Introduction

The class of quasi-normable locally convex spaces, introduced by Grothendieck [11], was studied intensively in the last two decades (see e.g., [3–7, 18, 27]). In particular, there are various concrete characterizations of quasi-normability for Köthe spaces and their generalizations. Meise and Vogt [18] found that the class of quasi-normable Fréchet spaces coincides with the union of classes Ω_φ (see, [28, 29, 31]) with φ running through the set Φ of all strictly increasing functions from $\mathbb{R}_+$ to itself. The class of asymptotically normable Fréchet spaces was introduced by Terzioğlu and Vogt in [24, 25] as a natural counterpart to the class of quasi-normable spaces; they showed also that asymptotically normable Fréchet spaces are those which admit a DN_φ-condition with $\varphi \in \Phi$.

Natural dual relationships among above and some related invariant classes of locally convex spaces are investigated here. Some of the previous results about normability conditions are extended to non-metrizable locally convex spaces. One of them (Theorem 2.14) tells that a Schwartz space is asymptotically normable if and only if it satisfies the condition GS, which is a natural generalization of Gelfand–Shilov's countable normability. The metrizable case was treated by in [9, 25]; we simplify the proof, in particular, we avoid the closed graph theorem considerations (cf. [9, proof of Proposition 5, iii) $\Rightarrow$ i)]).

Finally, a concrete characterization of quasi-normability is considered for locally convex spaces with unconditional bases in terms of the condition (G) (see, [4, Proposition 3.9]; cf. [7]).

2 Interpolational and normability properties

We use the standard terminology of theory of locally convex spaces as in [15, 17]. On a locally convex space E we always consider a *basic system of seminorms* $\{|\cdot|_p,\ p \in P\}$, which means that the corresponding system of unit balls

$$U_p := \left\{x \in E : |x|_p \le 1\right\}, \quad p \in P,$$

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forms a base of neighborhoods of $0 \in E$. The set P is a directed set with the natural partial order

$$p \le q \iff |x|_p \le |x|_q\,, \ x \in E\,.$$

On the strong dual E' we have the system of polar norms

$$|x'|_p^* := \sup\{|x'(x)| : x \in E,\ |x|_p \le 1\}\,, \quad x' \in E'\,, \quad p \in P\,.$$

For $p \in P$ the Banach space E_p is a completion of the space $E/ker\,|\cdot|_p$, considered with the corresponding quotient norm, and $I_{q,p} : E_q \to E_p$ is the natural linking map if $q \ge p$.

We denote by $\mathcal{B}(E)$ the locally convex bornology of E consisting of all absolutely convex bounded sets in E. For each $M \in \mathcal{B}(E)$ we consider its gauge norm $|x|_M$ and its polar norm $|x'|_M^* := \sup\{|x'(x)| : x \in M\}$ on E'.

By $\lambda(A)$ we denote the Köthe space defined by the matrix $A = (a_{i,p})$ with $0 \le a_{i,p} \le a_{i,p+1}$ for all $(i,p) \in \mathbb{N}^2$.

Now we discuss the direct generalization of two important classes, introduced by Vogt for Fréchet spaces ([28, 29, 31]). Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing function.

Definition 2.1 A locally convex space E *satisfies the property* DN_φ $(E \in (DN_\varphi))$ if there is $p \in P$ and a map $\rho \in P^P$ such that for each $q \in P$ there is $C = C(q) > 0$ such that

$$|x|_q \le C\,\varphi(t)\,|x|_p + \frac{1}{t}\,|x|_{\rho(q)}\,, \quad x \in E\,, \quad t > 0\,. \tag{2.1}$$

Definition 2.2 A locally convex space E *satisfies the property* Ω_φ $(E \in (\Omega_\varphi))$ if there is a map $\sigma \in P^P$ such that for any $p \in P$ and any $r \in P$ there is $C = C(p,r) > 0$ such that

$$|x'|^*_{\sigma(p)} \le C\,\varphi(t)\,|x'|^*_r + \frac{1}{t}\,|x'|^*_p\,, \quad x' \in E'\,, \quad t > 0\,. \tag{2.2}$$

This condition can be written equivalently in terms of neighborhoods as

$$U_{\sigma(p)} \subset C\,\varphi(t) U_r + \frac{1}{t}\,U_p\,, \quad t > 0\,. \tag{2.3}$$

We say that E *satisfies the property* $\widehat{\Omega}_\varphi$ $\big($and write $E \in \big(\widehat{\Omega}_\varphi\big)\big)$ if the constant in Eq. (2.2) can be taken in the form $C = B(p) \cdot D(r)$.

The classes (DN_φ) and (Ω_φ) have been proved to be of a great importance in the study of linear topological structure of locally convex spaces (see, e.g., [10, 14, 17, 18, 24–26, 28, 29, 31]). Both properties, DN_φ and Ω_φ, are linear topological invariants; the first is inherited by subspaces, while the latter by quotient spaces ([18], see also [31]).

Remark 2.3 (Cf. [28, 10]) By iterating Definitions 2.1, 2.2 one can easily derive $\Omega_\varphi = \Omega_\psi$, $DN_\varphi = DN_\psi$ if $\psi(t) = \varphi(t^\alpha)$ for any $\alpha > 0$.

Definition 2.4 A locally convex space E *satisfies the strict* Ω_φ *property* $(E \in (s\Omega_\varphi))$ if there exist a map $\sigma \in P^P$ and a set $M \in \mathcal{B}(E)$ such that for any $p \in P$ there is a constant $C = C(p) > 0$ such that

$$U_{\sigma(p)} \subset C\,\varphi(t) M + \frac{1}{t}\,U_p\,, \quad t > 0\,, \tag{2.4}$$

(due to the assumption that U_p is a base of neighborhoods of zero in E, we can even assume, without loss of generality, that $C(p) = 1$).

By Grothendieck ([11], p. 107), a locally convex space E is said to be *quasi-normable* if there is a map $\sigma \in P^P$ such that for each $p \in P$ and $\varepsilon > 0$ there exists $M \in \mathcal{B}(E)$ such that $U_{\sigma(p)} \subset M + \varepsilon U_p$. We say that E is *strictly quasi-normable* if there is $M \in \mathcal{B}(E)$ and a map $\sigma \in P^P$ such that for each $p \in P$ and $\delta > 0$ there exists $\Delta > 0$ such that

$$U_{\sigma(p)} \subset \Delta\, M + \delta U_p\,. \tag{2.5}$$

We say that E is *perfectly quasi-normable* ($E \in (pQN)$) if the constant Δ in Eq. (2.5) can be taken in the form $\Delta = C(p)D(\delta)$. It is well-known that for Fréchet spaces all these properties coincide (see [11, 13]). In general, all inclusions $(QN) \subset (sQN) \subset (pQN)$ are proper; in particular, the space ω^* is quasi-normable, but fails to be strictly quasi-normable. For the dual space E', besides strict quasi-normability, we consider the stronger property $\overline{sQN}$, demanding in Eq. (2.5) the equicontinuity of M (if E is quasi-barrelled these properties of E' coincide). Meise and Vogt [18] proved that a Fréchet space E is quasi-normable if and only if it belongs to the class (Ω_φ) for some φ.

Theorem 2.5 *Suppose a metrizable locally convex space E satisfies both properties Ω_φ and QN. Then $E \in (s\Omega_\psi)$ if $\psi(t)/\varphi(t) \to \infty$ as $t \to \infty$.*

Proof. Since E is metrizable, we may take $P = \mathbb{N}$. From $E \in (QN)$ we derive

$$U_{\pi(r)} \subset M(r,\varepsilon) + \varepsilon U_r\,, \quad \varepsilon > 0\,, \quad r \in \mathbb{N}\,, \tag{2.6}$$

with some $\pi \in \mathbb{N}^{\mathbb{N}}$ and $M(r,\varepsilon) \in \mathcal{B}(E)$.

On the other hand, from $E \in (\Omega_\varphi)$ we derive, without loss of generality, that the relation Eq. (2.3) holds with the constant depending only on r. Otherwise, since in the metrizable case the conditions Ω_φ and $\widehat{\Omega}_\varphi$ are obviously equivalent, the constant C can be chosen in the form $C = B(p) \cdot D(r)$, $B(p) \geq 1$, and we obtain this property by changing $\sigma \in \mathbb{N}^{\mathbb{N}}$ to $\widetilde{\sigma} \in \mathbb{N}^{\mathbb{N}}$ so that $B(p)\, U_{\widetilde{\sigma}(p)} \subset U_{\sigma(p)}$, $p \in \mathbb{N}$. Thus, taking in this relation $\pi(r)$ instead r, we get

$$U_{\sigma(p)} \subset C(\pi(r))\varphi(t)U_{\pi(r)} + \frac{1}{t}\, U_p\,, \quad t > 0\,, \quad p\,, r \in \mathbb{N}\,. \tag{2.7}$$

Now we choose $\tau_r \uparrow \infty$ so that $\varphi(t)C(\pi(r)) \leq \psi(t)$ for $t \geq \tau_r$, and define

$$\varepsilon_r := \left(C(\pi(r))\,\varphi(\tau_{r+1})\,\tau_{r+1}\right)^{-1}, \quad M_r := \left(M(r,\varepsilon_r) \cap 2\,U_r\right), \quad r \in \mathbb{N}\,.$$

Puting the estimate (2.6) considered with $\varepsilon = \varepsilon_r$ into Eq. (2.7), we obtain, due to our choice of τ_r and ε_r,

$$U_{\sigma(p)} \subset \psi(t)\, M_r + \frac{2}{t}\, U_p\,, \quad \tau_r \leq t \leq \tau_{r+1}\,, \quad r \geq p\,, \quad p \in \mathbb{N}\,. \tag{2.8}$$

Now we construct the set M as a convex hull of the union of the sets M_r, $r \in \mathbb{N}$. It is easy to see that $M \in \mathcal{B}(E)$. From Eq. (2.8) we obtain

$$\frac{1}{2}\, U_{\sigma(p)} \subset \psi(t)\, M + \frac{1}{t}\, U_p\,, \quad t \geq \tau_p\,, \quad p \in \mathbb{N}\,.$$

Finally, choosing $\rho \in \mathbb{N}^{\mathbb{N}}$ so that $U_{\rho(p)} \subset \left(\frac{1}{2}\, U_{\sigma(p)}\right) \cap \left(\frac{1}{\tau_p}\, U_p\right)$, we get

$$U_{\rho(p)} \subset \psi(t)\, M + \frac{1}{t}\, U_p\,, \quad t > 0\,, \quad p \in \mathbb{N}\,, \tag{2.9}$$

that is $E \in (s\Omega_\psi)$. □

Remark 2.6 Without assumption on quasinormability the statement may fail. Namely, Bonet and Dierolf [8] gave an example of a non-quasinormable metrizable (non-complete) locally convex space F, which is a dense linear subspace of a quasi-normable Fréchet space E. Since, by [18], E belongs to some class (Ω_φ), its dense linear subspace F also belongs to this class. But F cannot satisfy any $s\Omega_\varphi$, since otherwise it would be quasinormable.

Corollary 2.7 *Let a Fréchet space E have the property Ω_φ. Then E belongs to the class $(s\Omega_\psi)$, if $\psi(t)/\varphi(t) \to \infty$ as $t \to \infty$.*

Proof. Since in this case, by [18], $E \in (QN)$ is a consequence of $E \in (\Omega_\varphi)$, we can apply Theorem 2.5. □

Under an additional restriction we can prove this fact with $\psi = \varphi$.

Proposition 2.8 *Suppose that a function φ is such that*

$$\lim_{t\to\infty} \varphi(t^\alpha)/\varphi(t) \;=\; 0 \tag{2.10}$$

for some $\alpha > 0$. Then a Fréchet space E satisfies Ω_φ if and only if $E \in (s\Omega_\varphi)$.

In the proof of Theorem 2.5 we have basically followed Vogt ([28, Lemma 1.4] and Meise-Vogt [17, Lemma 29.16]), where the case $\varphi(t) = t$, very important for applications, is considered (the property Ω_φ in that case is called also D_2 or $\overline{\Omega}$). Those considerations relate to so-called "dead-end spaces", which are useful in studying of bases in certain spaces, especially in finite centers of Hilbert scales and in some spaces of analytic functions (see, e.g., [1, 2, 19–21, 32, 33, 36]).

A locally convex space E is said to be *asymptotically normable* (cf. [25]) (we write $E \in (AN)$) if there is $p \in P$ and a map $\rho \in P^P$ such that for each $q \in P$ the q-topology coincides with the p-topology on the unit ball $U_{\rho(q)}$, which means that for each $\delta > 0$ there is $\Delta = \Delta(q, \delta) > 0$ such that

$$|x|_q \;\le\; \Delta\,|x|_p + \delta\,|x|_{\rho(q)}\,, \qquad x \in E\,. \tag{2.11}$$

We say that E is *perfectly asymptotically normable* ($E \in (pAN)$) if the constant Δ in Eq. (2.11) can be taken in the form $C(q) \cdot D(\delta)$. For non-metrizable locally convex spaces these two properties may not coincide.

Terzioğlu and Vogt [24] introduced also the following property, which is a weakening of AN: a locally convex space E is *locally normable* ($E \in (LN)$) if there is $p \in P$ such that on each $M \in \mathcal{B}(E)$ the p-topology coincides with the topology of E.

We say that a locally convex space E is *strictly locally normable* ($E \in (sLN)$) if there is $p \in P$ such that for each $M \in \mathcal{B}(E)$ there is $L \in \mathcal{B}(E)$ such that the p-topology coincides with the L-topology on M, or

$$|x|_L \;\le\; \Delta\,|x|_p + \delta\,|x|_M\,, \quad x \in E\,, \tag{2.12}$$

for arbitrary $\delta > 0$ and $\Delta = \Delta(M, \delta) > 0$.

Proposition 2.9 *A locally convex space E is perfectly asymptotically normable if and only if $E \in (DN_\varphi)$ for some function φ.*

This is an easy extension of Terzioğlu-Vogt's result [25] for Fréchet spaces.

The property sLN is intermediate between pAN and LN: the inclusion $(sLN) \subset (LN)$ is obvious, while $(pAN) \subset (sLN)$ will be proved in the next section (Lemma 3.6). For Montel spaces the relations among some of above properties are more transparent.

Theorem 2.10 *Let E be a Montel locally convex space. Then*
(a) *$E \in (LN)$ if and only if it admits a continuous norm;*
(b) *$E \in (sLN)$ if and only if E admits a continuous norm and is co-Schwartz;*
(c) *E is quasi-normable if and only if it is Schwartz;*
(d) *$E \in (sQN)$ if and only if it is Schwartz and admits a total bounded set.*

P r o o f. Notice first that (c) is due to Grothendieck (see, e.g., [13, Section 10.7]).

(a) Only the if-part needs a proof. Since E is Montel, any $M \in \mathcal{B}(E)$ considered with the topology $\tau(E)$ induced from E is a compact topological space. Suppose that there is $p \in P$ such that $|\cdot|_p$ is a norm. Then the p-topology is a Hausdorff topology on M coarser than $\tau(E)$, so these topologies must coincide.

(b) Let $E \in (sLN)$. Since E is Montel, any $M \in \mathcal{B}(E)$ is precompact in E, but by the definition of sLN there are $p \in P$ and $L \in \mathcal{B}(E)$ such that the topology $\mathcal{T}_L$, as considered on M, coincides with the topology defined by $|\cdot|_p$. Hence M is also precompact with respect to L, which means that E is co-Schwartz. Suppose now that E is co-Schwartz. Then, putting the norm $|\cdot|_L$ instead of $|\cdot|_q$ everywhere in the proof of (a), we obtain, by the same token, that $E \in (sLN)$.

(d) Let $E \in (sQN)$. Then, due to (c), E is Schwartz, and it is easy to see that the set M from Eq. (2.5) is total. Suppose that E is Schwartz and admits a total set $M \in \mathcal{B}(E)$. Then for any $p \in P$ there is $q \in P$ such that for each δ there is a finite set $T \subset U_q$ such that

$$U_q \;\subset\; T + \delta/2\,U_p\,.$$

Applying the totality of M we get

$$T \subset \bigcup_{n=1}^{\infty} n\,M + \delta/2\,U_p\,.$$

Since T is finite, we can find $\Delta \in \mathbb{N}$ so that $T \subset \Delta M + \delta/2\,U_p$. Hence the relation (2.5) holds, and $E \in (sQN)$. □

The following notion is a natural extension of the concept of *countable normability* (Gelfand–Shilov, see, e.g., [9]).

Definition 2.11 We say that a locally convex space E *satisfies the Gelfand–Shilov property* $(E \in (GS))$ if there is a basic system of norms $\{\|\cdot\|_k,\ k \in K\}$ defining the topology of E and such that for each $k \in K$ and every $m \in K$ provided $m \geq k$ the linking map $I_{m,k} : E_m \to E_k$ is injective, where E_k stands for a completion of $(E, \|\cdot\|_k)$.

Let $A = (a(t,p))_{t \in T,\, p \in P}$, where T is any set, P is a directed set, $0 \leq a(t,p) \leq a(t,q)$ for $p \leq q$. Lez $T_p := \{t \in T : a(t,p) > 0\}$, $p \in P$, and $T = \bigcup_{p \in P} T_p$. We use a term *weighted sup-space* for the space

$$\lambda_\infty(T, A) \ := \ \left\{u \in \mathbb{R}^T : |u|_p := \sup\{|u(t)|a(t,p) : t \in T\} < \infty,\ p \in P\right\} \tag{2.13}$$

endowed with the locally convex topology defined by the system of seminorms $\{|\cdot|_p,\ p \in P\}$; if $a(t,p)$ is positive for all $t,\ p$, the space (2.13) is called a *weighted sup-norm space* (see, e.g., [30]). Denote by $l_\infty\big(a^{(p)}\big)$ the Banach space consisting of all functions $u \in \mathbb{R}^{T_p}$ with a finite sup-norm, defined by the weight $a^{(p)} := a(t,p),\ t \in T_p$, and consider the map $R_p : \lambda_\infty(T, A) \to l_\infty\big(a^{(p)}\big)$ such that $R_p(u) := u|T_p$.

Definition 2.12 Let F be a subspace of the weighted sup-space defined by Eq. (2.13) and F_p be the closure of $R_p(F)$ in $l_\infty\big(a^{(p)}\big)$, $p \in P$. Then we say that a subspace F is *well-imbedded* into this space if, for any $p \leq q$, the conditions $u \in F_q$ and $u(t) = 0$, $t \in T_p$, imply that $u(t) \equiv 0$ on $t \in T_q$.

The following lemma is a generalization of Vogt's result [30]; we are following basically his proof with some simplifications in the part (ii) ⇒ (i).

Lemma 2.13 *Let E be a locally convex space and $\{|\cdot|_p,\ p \in P\}$ be any system of seminorms defining its topology. Then the following statements are equivalent:*

(i) *$E \in (GS)$;*

(ii) *There exists $p \in P$ and a mapping $\rho \in P^P$ such that for each $q \geq p$ any $|\cdot|_{\rho(q)}$-Cauchy sequence $\{x_k\} \subset E$ with $|x_k|_p \to 0$ converges to 0 also by $|\cdot|_q$;*

(iii) *E is isomorphic to a well-imbedded subspace of some weighted sup-space of the space defined by Eq. (2.13).*

P r o o f. (ii) ⇒ (i). Taking p and ρ as in (ii), we define for any $q \geq p$ a new norm by the relation

$$\|x\|_q \ := \ \inf\left\{\lim_{k\to\infty} |x_k|_{\rho(q)}\right\}, \quad x \in E\,, \tag{2.14}$$

where the infimum is over the set of all $\rho(q)$-Cauchy sequences (x_k) in E such that $|x_k - x|_q \to 0$. It is obvious that this norm satisfies the estimates

$$|x|_q \ \leq \ \|x\|_q \ \leq \ |x|_{\rho(q)}\,, \quad x \in E\,. \tag{2.15}$$

Hence the system of norms $\{\|\cdot\|_q\}_{q \in Q}$ with $Q := \{q \in P :\ p \leq q\}$ defines the original topology of E.

Denote by $\widetilde{E}_q$ the completion of the normed space $(E, \|\cdot\|_q)$, $q \in Q$, and by $J_{r,q} : \widetilde{E}_r \to \widetilde{E}_q$, $r \geq q$, the corresponding unique extension of the identity. In particular, since $\rho(p) = p$ is available in Eq. (2.11), we can assume that $\|\cdot\|_p = |\cdot|_p$ and $\widetilde{E}_p = E_p$. By the definition (2.14), we may identify $\widetilde{E}_q$ with the quotient space $E_{\rho(q)}/N_q$, where $N_q = ker I_{\rho(q),q}$. Let $\tau_q : E_{\rho(q)} \to E_{\rho(q)}/N_q$ be the canonical quotient map.

The condition (ii) means that for any q the linking map $I_{q,p}$ is injective as considered on the image $R_q := I_{\rho(q),q}\big(E_{\rho(q)}\big)$, $q \in P$. Therefore the map $\sigma_q : \widetilde{E}_q = E_{\rho(q)}/N_q \to R_q$, such that $I_{\rho(q),q} = \sigma_q \circ \tau_q$, is a bijection, which implies the injectivity of the map $J_{q,p} = I_{q,p} \circ \sigma_q$ for every $q \in Q$. Hence we deduce now the injectivity of the mapping $J_{r,q}$, $r \geq q$, from the obvious relation $J_{r,p} = J_{q,p} \circ J_{r,q}$. So, the new equivalent system of norms (2.14), $q \in Q$, on E is constructed, which complies with the property GS (Definition 2.11).

(i) ⇒ (iii). Let $E \in (GS)$ and $\{\|\cdot\|_k,\ k \in K\}$, $\widetilde{E}_k$, $I_{m,k}$ be as in Definition 2.11; let $\|\cdot\|_k^*$ be the corresponding polar norms on the strong dual E'. Set $T = E' \setminus \{0\}$, $T_k = \{t \in T : |t|_k^* < \infty\}$, and $a(t,k) := 1/|t|_k^*$ if $t \in T_k$ and $a(t,k) := 0$ if $t \in T \setminus T_k$.

Consider the operator $S : E \to \lambda_\infty(T, A)$ defined by $(Sx)(t) := t(x)$, $t \in T$. Since

$$\|x\|_k \ = \ \sup\{|t(x)|\, a(t,k) : t \in T\}\,, \quad x \in E\,, \quad k \in K\,, \tag{2.16}$$

this operator is an isomorphism of E onto its image F. Let F_k be the spaces, derived from F as in Definition 2.12. Then, due to Eq. (2.16), the formula $S_k(x)(t) = t(x)$, $t \in T_k$, defines an isometric surjection $S_k : \widetilde{E}_k \to F_k$ for any k. Hence the natural map $R_{m,k} : F_m \to F_k$, defined as a restriction from T_m onto T_k, $k \leq m$, can be represented in the form $R_{m,k} = S_m^{-1} \circ I_{m,k} \circ S_k$. Since, by the assumption, $I_{m,k}$ is an injection, so is $R_{m,k}$, which means that the subspace F is well-imbedded into $\lambda_\infty(T, A)$.

(iii) ⇒ (ii). We obtain this inclusion, taking into account that, on the one hand, any well-imbedded subspace of a sup-space (2.13) satisfies obviously the condition (ii) with any p and $\rho(q) \equiv q$ and, on the other hand, the condition (ii) is invariant under isomorphisms. □

The next theorem generalizes Terzioğlu–Vogt's result ([24, 25]).

Theorem 2.14 *For a locally convex space E to be asymptotically normable it is necessary and, if E is Schwartz, sufficient that $E \in (GS)$.*

Proof. Suppose first that $E \in (AN)$ and $\{|\cdot|_q,\ q \in P\}$ is a basic system of seminorms in it. Then there is $p \in P$ and a non-decreasing map $\rho \in P^P$ such that the condition (2.11) holds. Hence, we get $E \in (GS)$ by Proposition 2.13, since the condition (ii) follows obviously from Eq. (2.11).

Suppose now that a Schwartz locally convex space E satisfies the property GS, so that it can be endowed with a basic system of norms from Definition 2.11. Choose $\rho \in K^K$ such that $U_{\rho(q)}$ is precompact in the q-topology for all $q \in K$. It is sufficient to show that, given $p \in K$, for any $q \geq p$ the q-topology coincides with the p-topology on the ball $U_{\rho(q)}$, which will provide $E \in (AN)$. Indeed, supposing the contrary, we get a sequence $\{x_j\} \subset U_{\rho(q)}$ which converges to 0 in the p-topology but $|x_j|_q \geq \delta > 0$ for all j. Then, using the precompactness of $U_{\rho(q)}$ in the q-topology, we find a subsequence $\{x_{j_n}\}$ which is q-Cauchy but does not converge to 0 in the q-topology. The sequence $\{x_{j_n}\}$ generates a non-zero element $z \in E_q$ such that $I_{q,p}(z) = 0$, which is in contradiction with Definition 2.11. □

For a given locally convex space E and a fixed $p \in P$ we consider a new system of seminorms defined by

$$\|x\|_q^{(p)} \ := \ \sup\big\{|t(x)| : |t|_q^* \leq 1,\ |t|_p^* < \infty\big\}\,, \quad q \in P\,, \quad q \geq p\,, \tag{2.17}$$

and denote by $\mathcal{T}_E^{(p)}$ the topology on E generated by it (note that this topology is Hausdorff if and only if $|\cdot|_p$ is a norm). The following two notions, introduced in [22, 23, 30], are defined here in a slightly different equivalent form.

Definition 2.15 E is called
(a) *locally closed* ($E \in (LC)$) if the topology $\mathcal{T}_E^{(p)}$ coincides with the original topology of E for some $p \in P$;
(b) *locally admissible* ($E \in (y)$) if the topology $\mathcal{T}_E^{(p)}$ is (E, E')-admissible for some $p \in P$.

Vogt proved in [30] that E is locally closed if and only if it is isomorphic to a subspace of some weighted sup-norm space. The following characterization of locally admissible spaces is a simple corollary from this result.

Proposition 2.16 *A locally convex space E is locally admissible if and only if it is weakly isomorphic to a subspace of some weighted sup-norm space* (2.13).

Now we compare the properties AN and LC.

Proposition 2.17 *If E is asymptotically normable then it is locally closed.*

P r o o f. Suppose $E \in (AN)$. Then there exist $p \in P$ and $\rho \in P^P$ such that the condition (2.11) holds for every $q \in P$ and each $\delta > 0$ with some $\Delta = \Delta(q, \delta)$. Rewriting this condition in equivalent form in terms polars of neighborhoods, we obtain

$$U_q^\circ \subset \Delta U_p^\circ + \delta U_{\rho(q)}^\circ \subset E_p' + \delta U_{\rho(q)}^\circ\,, \quad \delta > 0\,, \quad q \in P\,. \tag{2.18}$$

Take an arbitrary $t \in U_q^\circ$. Applying Eq. (2.18), we find $t_n \in E_p'$ such that

$$|t(x) - t_n(x)| \leq 1/n\,|x|_{\rho(q)}\,, \quad x \in E\,,$$

from where we derive also that $t_n \in (1 + 1/n)\,U_{\rho(q)}^\circ \subset 2\,U_{\rho(q)}^\circ$. From these considerations we conclude that

$$U_q^\circ \subset \overline{E_p' \cap 2\,U_{\rho(q)}^\circ}\,, \quad q \in P\,,$$

where the closure is considered in the weak topology $\sigma(E', E)$. Hence $|x|_q \leq 2\,\|x\|_{\rho(q)}^{(p)}$, $q \in P$, which means that the topology $\mathcal{T}_E^{(p)}$ coincides with the topology of E. So, $E \in (LC)$. □

Using previous considerations and results from [25, 30] we get that, for Schwartz spaces, the above properties have especially simple description.

Theorem 2.18 *Let E be a Schwartz space. Then the following are equivalent:*
(1) *E is asymptotically normable;*
(2) *E has the Gelfand–Shilov property;*
(3) *E is locally closed;*
(4) *E is isomorphic to a subspace of some weighted sup-norm space.*
If E is also Fréchet space then this list may be extended by
(5) *E is locally admissible;*
(6) *E is isomorphic to a subspace of $l_\infty \widehat{\otimes}_\pi \lambda(A)$, where $\lambda(A)$ is a nuclear Köthe space.*

3 Duality

Here we study certain dual relationships between some of the properties we have already considered.

Theorem 3.1 *If a locally convex space E satisfies DN_φ then its strong dual E' satisfies $\widehat{\Omega}_\varphi$.*

P r o o f. Suppose that $E \in (DN_\varphi)$, then Eq. (2.1) is true. Without loss of generality we assume that $r = \rho(q) \geq q$, $q \in P$, hence the system of seminorms

$$\{|x|_r,\ r \in R := \rho(P)\}$$

defines the original topology on E.

We will use the special base $\mathcal{M}$ of bornology $\mathcal{B}(E)$ consisting of sets

$$M = M_\alpha := \{x \in E : |x|_M := \sup\{\alpha(r)|x|_r : r \in R\} \leq 1\}$$

with α running over the set of all functions $\alpha : R \to (0, 1]$. Given an arbitrary $M = M_\alpha \in \mathcal{M}$, we multiply the inequality Eq. (2.1) by $\frac{\alpha(\rho(q))}{C(q)}$ and take the supremum by q over P. Then we get the estimate (we assume that $C(q) \geq 1$)

$$\sup\left\{\frac{\alpha(\rho(q))}{C(q)}\,|x|_q : q \in P\right\} \leq \varphi(t)\,|x|_p + \frac{1}{t}\,|x|_M\,, \quad t > 0\,. \tag{3.1}$$

It is clear that the set $A = \left\{x \in E : |x|_q \le \frac{C(q)}{\alpha(\rho(q))},\ q \in P\right\}$ is bounded in E so there is $L = M_\beta \in \mathcal{M}$ such that $A \subset L$, hence Eq. (3.1) implies that

$$|x|_L \;\le\; \varphi(t)\,|x|_p + \frac{1}{t}\,|x|_M\,, \quad t \;>\; 0\,.$$

Finally we have that for every $M \in \mathcal{M}$ there is $L \in \mathcal{M}$ and an equicontinuous (hence bounded) set U_p° in E' such that

$$L^\circ \;\subset\; \varphi(t)\,U_p^\circ + \frac{1}{t}\,M^\circ\,, \quad t \;>\; 0\,.$$

Since $\mathcal{M}^\circ := \{M^\circ : M \in \mathcal{M}\}$ is a basis of neighborhoods of E', we have that the space E' satisfies the property $\widehat{\Omega}_\varphi$. □

For a given Köthe space $E = \lambda(A)$ and sequence of positive numbers $\gamma = (\gamma_i)$ we introduce the notation

$$\mathbf{B}(\gamma) \;:=\; \left\{x = (\xi_i) \in \lambda(A) : \sum_{i=1}^{\infty} |\xi_i|\,\gamma_i \le 1\right\}.$$

Theorem 3.2 *Let a Montel Köthe space $E = \lambda(A)$ admit a continuous norm and satisfy Ω_φ. If $\psi(t)/\varphi(t) \to \infty$ as $t \to \infty$, then the strong dual E' satisfies DN_ψ. If Eq.* (2.10) *holds for some $\alpha > 0$, then $E' \in (DN_\varphi)$.*

We need the following quite well-known fact (see, e.g., [4]); its proof is represented here for the sake of completeness.

Lemma 3.3 *Let $E = \lambda(A)$ be a Montel space admitting a continuous norm and Π be a family of all non-decreasing positive integer-valued sequences (p_i) tending to ∞. Then the collection*

$$C\,\{\mathbf{B}((a_{i,p_i}))\}\,, \quad (p_i) \in \Pi\,, \quad C \;>\; 0\,, \tag{3.2}$$

forms a base of bornology $\mathcal{B}(E)$.

Proof. Denote by Γ the set of all sequences $\gamma = (\gamma_i)$ which can be represented in the form

$$\gamma_i \;=\; \sup\,\{\alpha(p)a_{i,p} : p \in \mathbb{N}\}\,,$$

where $\alpha = \alpha(p)$ runs over the set of all non-increasing functions such that

$$\lim_{p\to\infty} \alpha(p)a_{i,p} \;=\; 0$$

for every $i \in \mathbb{N}$ and $\alpha(p) \le 1$. Then the collection

$$\{\mathbf{B}(\gamma),\ \gamma \in \Gamma\} \tag{3.3}$$

forms a base of $\mathcal{B}(E)$.

We check first that each set (3.2) is bounded. Indeed, if $(p_i) \in \Pi$ then for each $p \in \mathbb{N}$ there is i_0 such that $p \le p_i$ for $i \ge i_0$; therefore we get that

$$\mathbf{B}((a_{i,p_i})) \;\subset\; C\,U_p \quad \text{where} \quad C \;=\; \sup\left\{\frac{a_{i,p}}{a_{i,p_i}} : i < i_0\right\}.$$

Thus the set $\mathbf{B}(a_{i,p_i})$ is absorbed by any zero neighborhood, hence it is bounded.

Now we show that the collection (3.2) is a base of bornology $\mathcal{B}(E)$. Without loss of generality we assume that $a_{i,p} > 0$ for all $(i,p) \in \mathbb{N}^2$. Since E is Montel, each set (3.3) is precompact, hence for every $p \in \mathbb{N}$ we have

$$\frac{\gamma_i}{a_{i,p}} \;\longrightarrow\; \infty \quad \text{as} \quad i \;\longrightarrow\; \infty\,.$$

Hence there exists a strictly increasing sequence $k(p) \uparrow \infty$ such that

$$a_{i,p} \le \gamma_i\,, \quad i \ge k(p)\,.$$

Define a sequence $(q_i) \in \Pi$ as follows: $q_i := p$ if $k(p) \le i < k(p+1)$ for $p \in \mathbb{N}$ and $q_i := 1$ if $1 \le i \le k(1)$. Then, by the construction,

$$a_{i,q_i} \le \gamma_i\,, \quad i \ge k(1)\,.$$

Therefore for every $\gamma \in \Gamma$ there is a sequence $(q_i) \in \Pi$ and a constant $C > 0$ such that

$$a_{i,q_i} \le C\gamma_i\,, \quad i \in \mathbb{N}\,,$$

which means that the collection (3.2) forms a base of bornology $\mathcal{B}(E)$. □

Proof of Theorem 3.2. Due to Theorem 2.5, we can assume that there is $M \in \mathcal{B}(E)$ and a strictly increasing function $\rho \in \mathbf{N}^{\mathbf{N}}$ such that for every $p \in \mathbf{N}$ the inclusion (2.9) holds. Applying its dual form to the elements of the canonical basis e_i' we obtain

$$\frac{1}{a_{i,\rho(p)}} \le \psi(t)\,\frac{1}{a_{i,M}} + \frac{1}{ta_{i,p}}\,, \quad t > 0\,, \quad p \in \mathbf{N}\,,$$

where $a_{i,M} := |e_i|_M$. Hence we can find a map $\sigma \in \mathbf{N}^{\mathbf{N}}$ such that

$$\frac{1}{a_{i,q}} \le \psi(t)\frac{1}{a_{i,M}} + \frac{1}{ta_{i,\sigma(q)}}\,, \quad t > 0\,, \quad q \ge \rho(1)\,. \tag{3.4}$$

For an arbitrary $L \in \mathcal{B}(E)$, by Lemma 3.3, we can choose a sequence $(q_i) \in \Pi$ and a constant $D > 0$ so that

$$|x'|_L^* \le D \sup\left\{\frac{|\xi_i'|}{a_{i,q_i}} : i \in \mathbf{N}\right\}, \tag{3.5}$$

where $x' = (\xi_i')$. Letting $q = q_i$ in Eq. (3.4) we get

$$\frac{1}{a_{i,q_i}} \le \psi(t)\frac{1}{a_{i,M}} + \frac{1}{ta_{i,r_i}}\,, \quad i \in \mathbf{N}\,, \quad t > 0\,,$$

with $r_i = \sigma(q_i)$. Hence, taking into account Eq. (3.5), we obtain

$$|x'|_L^* \le D\,\psi(t)\,|x'|_M^* + |x'|_K^*\,, \quad x' \in E'\,, \quad t > 0\,, \tag{3.6}$$

with $K := D \cdot \mathbf{B}((a_{i,r_i})) \in \mathcal{B}(E)$. So, there is $M \in \mathcal{B}(E)$ such that for any $L \in \mathcal{B}(E)$ there is $K \in \mathcal{B}(E)$ and $D > 0$ such that Eq. (3.6) holds, which means that $E' \in (DN_\psi)$. If the condition (2.10) holds, we can take $\psi = \varphi$ here, due to Proposition 2.8. □

Since any Montel space is reflexive, the last theorem may be considered as a partial converse to Theorem 3.1. There is a natural dual relationship between the properties sLN and $\overline{sQN}$ (sQN, if E is quasi-barrelled).

Proposition 3.4 *For $E \in (sLN)$ it is necessary and sufficient that $E' \in \left(\overline{sQN}\right)$.*

Proof. Indeed, the relation (2.12) can be expressed equivalently in the form

$$L^\circ \subset \Delta U_p^\circ + \delta M^\circ\,,$$

preserving all parameters and quantifiers in the definition of sLN. But this just means that $E' \in \left(\overline{sQN}\right)$. □

Now we deal with the duality for quasi- and asymptotical normability.

Theorem 3.5 *If $E \in (pAN)$ then $E' \in \left(\overline{sQN}\right)$ and, consequently, E' belongs to the classes (sQN) and (QN).*

This fact follows from Proposition 3.4 and the next lemma.

Lemma 3.6 *If $E \in (pAN)$ then $E \in (sLN)$.*

P r o o f. Since $E \in (pAN)$, the constant Δ in Eq. (2.11) may be chosen in the form $\Delta = C(q)D(\delta)$. Without loss of generality we can assume that $\rho(q) \geq q,\ q \in P$, so that $\{U_r : r \in R := \rho(P)\}$ is a base of neighborhoods of 0. Given $M \in \mathcal{B}(E)$ we choose a function $\alpha(r)$ (we assume that $\alpha(r)C(\sigma(r)) \leq 1$ and $\alpha(q) \to 0$ as $q \to \infty$) so that

$$|x|_M \ \geq\ \sup\{\alpha(r)|x|_r : r \in R\}\,.$$

Then, multiplying the inequality by $\alpha(\rho(q))$ and taking the supremum by q, we obtain

$$|x|_L \ \leq\ D(\delta)\,|x|_p + \delta\,|x|_M\,, \quad t\ >\ 0\,, \quad x \in E\,,$$

with $L := \{x \in E : \alpha(\rho(q))|x|_q \leq 1,\ r \in R\} \in \mathcal{B}(E)$. Thus, $E \in (sLN)$. □

Remark 3.7 If E in Theorem 3.5 is metrizable then pAN can be substituted by AN.

4 Quasi-normability and (G)-condition

We denote by $\mathcal{N}$ the class of all Fréchet spaces E with an unconditional basis $\{e_i\}$ satisfying the following conditions: there is a fundamental non-decreasing sequence of seminorms $\{|\cdot|_p\}$ in E such that

N1: there is a strictly increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that the inequality $|e_i|_q \leq t\,|e_i|_p, t > 0, i \in J \subset \mathbb{N}$, implies the estimate $|x|_q \leq Ch(t)|x|_p$ for all $x \in E_J := \overline{\mathrm{span}}\{e_i\}_{i\in J}$ with some constant $C = C(p,q)$, $p \leq q$;

N2: there is a strictly increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$, converging to 0 as $t \to 0$, such that the inequality $|e_i|_p \leq t|e_i|_q$, $t > 0$, $i \in J \subset \mathbb{N}$, implies the estimate $|x|_p \leq Cg(t)|x|_q$ for all $x \in E_J := \overline{\mathrm{span}}\{e_i\}_{i\in J}$ with some constant $C = C(p,q)$, $p \leq q$.

It is worth noticing that N1 is related to the condition "$c\Omega$ without the restriction $k < t$" from [6].

The next statement is a generalization of Bierstedt–Meise–Summers' result (see [4, Theorem 3.4 and Proposition 3.9]) about characterization of quasi-normable Köthe spaces in terms of the condition (G), which is an improvement of the original Grothendieck's claim [12, p. 102] (see also [7]).

Theorem 4.1 *A space $E \in \mathcal{N}$ is quasi-normable if and only if*

(G) *for each $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for any $\varepsilon > 0$ there is $J \subset \mathbb{N}$ providing*

(a) *the induced topology on E_J coincides with the p-topology, and*

(b) $U_q \subset E_J + \varepsilon U_p$.

P r o o f. Let $E \in \mathcal{N}$, $\{e_i\}$ be a corresponding unconditional basis in E, and $\{e'_i\}$ be its biorthogonal system in E'. Without loss of generality we can assume that the fundamental system of norms $\{|\cdot|_p\}_{p\in\mathbb{N}}$ satisfies the condition

$$\left|\sum_{i\in J} e'_i(x)e_i\right|_p \ \leq\ |x|_p \tag{4.1}$$

for any $J \subset \mathbb{N}$.

The "if-part" is quite obvious, so we consider the "only if-part". Suppose that E is quasi-normable. Then, by Meise–Vogt [18], $(E \in \Omega_\varphi)$ for some strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Therefore, applying the condition (2.2) in a multiplicative form ([18, Theorem 7]) to the functionals e'_i, we get that for any p there is q such that for every r one can choose a constant $C > 0$ so that

$$|e'_i|^*_q \ \leq\ C\varphi\left(\frac{|e'_i|^*_p}{|e'_i|^*_q}\right)|e'_i|^*_r\,, \quad i \in \mathbb{N}\,.$$

Since, due to Eq. (4.1),

$$|e_i'|_p^* \;=\; \frac{1}{|e_i|_p}\,,$$

the above inequality can be written as

$$\frac{|e_i|_r}{|e_i|_q} \;\le\; C\,\varphi\Big(\frac{|e_i|_q}{|e_i|_p}\Big), \quad i \in \mathbb{N}\,. \tag{4.2}$$

For an arbitrary $\varepsilon > 0$ we find $\delta > 0$ such that

$$h(\delta) \;<\; \varepsilon\,. \tag{4.3}$$

Then we set $J := \{i : \delta\,|e_i|_q \le |e_i|_p\}$ and $E_J := \overline{\operatorname{span}\{e_i\}_{i\in J}}$. From Eq. (4.2) we get that

$$|e_i|_r \;\le\; C\,\varphi\Big(\frac{|e_i|_q}{|e_i|_p}\Big)|e_i|_q \;\le\; C\,\varphi\Big(\frac{1}{\delta}\Big)\delta^{-1}\,|e_i|_p$$

for each $i \in J$. Therefore, taking into account the condition N1, we obtain that

$$|x|_r \;\le\; h\Big(C\,\varphi\Big(\frac{1}{\delta}\Big)\delta^{-1}\Big)|x|_p$$

for all $x \in E_J$. So, the condition (a) is fulfilled for the chosen subspace E_J.

On the other hand, $|e_i|_p < \delta\,\|e_i|_q$ for $i \notin J$. Therefore, using the condition N2 and Eq. (4.3), we obtain that

$$|x|_p \;\le\; h(\delta)\,|x|_q \;\le\; \varepsilon\,|x|_q$$

holds for all $x \in E_{\mathbb{N}\setminus J}$, which means that the condition (b) is true, too. □

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