CONSTRUCTION OF EVIDENTLY POSITIVE SERIES AND AN ALTERNATIVE CONSTRUCTION FOR A FAMILY OF PARTITION GENERATING FUNCTIONS DUE TO KANADE AND RUSSELL

by

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Abstract

Construction of generating functions for partitions, especially construction evidently positive series as generating functions for partitions is a quite interesting problem. Recently, Kurşungöz has been constructed evidently positive series as generating functions for the partitions satisfying the difference conditions imposed by Capparelli's identities and Göllnitz-Gordon identities, for the partitions satisfying certain difference conditions in six conjectures by Kanade and Russell and the partitions satisfying the multiplicity condition in Schur's partition theorem. In this thesis, we give an alternative construction for a family of partition generating functions due to Kanade and Russell. In our alternative construction, we use ordinary partitions instead of jagged partitions.

We also present new generating functions which are evidently positive series for partitions due to Kanade and Russell. To obtain those generating functions, we first construct an evidently positive series for a key infinite product. In that construction, a series of combinatorial moves is used to decompose an arbitrary partition into a base partition together with some auxiliary partitions that bijectively record the moves.

BARİZ POZİTİF KATSAYILI SERİLERİN İNŞASI VE KANADE VE RUSSELL'IN BİR PARÇALANIŞ ÜRETEÇ FONKSİYON AİLESİ İÇİN FARKLI BİR İNŞA

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Anahtar Kelimeler: tam sayı parçalanışı, parçalanış üreteç fonksiyonu, bariz pozitif üreteç fonksiyonlar, Rogers-Ramanujan tipi parçalanış özdeşlikleri.

Özet

Parçalanışlar için üreteç fonksiyonlar inşa etmek, özellikle parçalanışların üreteç fonksiyonları olarak bariz pozitif seriler inşa etmek oldukça ilginç bir problemdir. Son zamanlarda, Kurşungöz, Capparelli özdeşlikleri ve Göllnitz-Gordon özdeşlikleri tarafından sunulan fark koşullarını sağlayan parçalanışlar için, Kanade ve Russell sanılarının altısında sunulan belirli fark koşullarını sağlayan parçalanışlar için ve Schur'un parçalanış teoremindeki tekrar koşulunu sağlayan parçalanışlar için üreteç fonksiyonlar olarak bariz pozitif seriler inşa etmiştir. Bu tezde, Kanade ve Russell'a ait bir parçalanış üreteç fonksiyon ailesi için alternatif bir inşa sunduk. Alternatif inşamızda, tırtıklı parçalanışlar yerine sıradan parçalanışları kullandık.

Kanade ve Russell'a ait parçalanışlar için bariz pozitif seriler olan yeni üreteç fonksiyonlar da sunduk. Bu üreteç fonksiyonları elde edebilmek için, öncelikle bir anahtar sonsuz çarpım için bariz pozitif bir seri inşa ettik. Bu inşada, rastgele bir parçalanışı, bir taban parçalanışla beraber, hareketleri birebir olarak kaydeden bazı yardımcı parçalanışlara ayrıştırmak için bir dizi kombinatoryal hareket kullanılmaktadır.

to my beloved father

…The sun was shining; while he was washing his hands, he stared at me and said: "Every new morning is a new beginning."…

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CHAPTER 1

Introduction

"The primary source (Urquell) of all of mathematics are the integers."
Hermann Minkowski

The theory of integer partitions is a fascinating branch of number theory and the history of it dates back to Middle Ages; however, the first striking discoveries were made by Euler. He proved many significant partition theorems. In this sense, we can say that Euler laid the foundations of the theory. Although Euler was the first person to make deep discoveries, Leibniz appeared to be the first person to consider the partitioning of integers into sums. The theory of partitions had been studied and discussed by many other famous mathematicians, like Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur, and Sylvester [2], [14].

After Euler's introduction of integer partitions, and his partition identity [6], a milestone in the theory is the Rogers-Ramanujan identities [11]. Although more than a hundred years have passed since the discovery of the identities, they are still the subject of active research. For a history of Rogers-Ramanujan identities, we invite the reader to take a glance at the book of Sills [13].

We can roughly say that a *partition* of a positive integer n is a way of writing n as a sum of positive integers. For instance, in how many ways can 3 be partitioned into positive integers? The answer is 3. We can take 3 itself as the first partition or we can write 3 as the sum $1 + 2$ or we can write it as a sum of three 1's. We note that the order does not matter, i.e. if two partitions differ only in order, then they are the same partitions for us [4]. It is very amazing that the theory starting with such a simple definition has many applications in a wide variety of fields, like physics and statistics.

For more details on the subject and its history, the curious reader is invited to read the book of Andrews [2] and the book of Andrews and Eriksson [4].

A main branch of research in partitions deals with partition identities, discovering and proving them, constructing generating functions for partitions. Generally, discovering a partition identity is harder than proving it. Lately, we are witnessed the opposite. For instance, some conjectures presented by Kanade and Russell [7] are still waiting to be proven. Some of the conjectures in [7] are proved by Bringmann, Jennings-Shaffer and Mahlburg [5] and by Rosengren [12].

In this thesis, we focus on constructing generating functions for partitions with certain difference conditions and multiplicity conditions.

In Chapter 2, we state some notations, basic definitions and well-known identities concerning partitions.

In Chapter 3, we give alternative proofs for construction of analytic sum-sides to some partition identities of Rogers-Ramanujan type, where these analytic sum-sides are provided by Kanade and Russell in [7] by using jagged partitions, 2-staircases and the q-series identities due to Euler. We use ordinary partitions instead of jagged partitions in the proofs we present.

In 2019, Kurşungöz constructed Andrews-Gordon type evidently positive series as generating functions for the partitions satisfying the difference conditions imposed by Capparelli's identities and Göllnitz-Gordon identities. Below, we present the theorem in which an evidently positive series is constructed for the partitions satisfying the difference conditions imposed by the first Capparelli Identity [8].

Theorem 1.1 (cf. the first Capparelli Identity). *For $n, m \in \mathbb{N}$, let $cp_1(n,m)$ denote the number of partitions of n into m parts such that all parts are at least 2, the difference is at least 2 at distance 1 and it is at least 4 unless the successive parts add up to a multiple of 3. Then,*

$$\sum_{m,n\geq 0} cp_1(n,m)q^n x^m = \sum_{n_1,n_2\geq 0} \frac{q^{2n_1^2+6n_1n_2+n_2^2}x^{n_1+2n_2}}{(q;q)_{n_1}(q^3;q^3)_{n_2}}.$$

In 2019, Kurşungöz also constructed evidently positive series as generating functions for the partitions satisfying certain difference conditions in six conjectures by Kanade and Russell and the partitions satisfying the multiplicity condition in Schur's partition theorem, respectively in [9] and [10]. In the construction, a series of combinatorial moves is used to decompose an arbitrary partition into a base partition together with some auxiliary partitions that bijectively record the moves.

In Chapter 4, we construct an evidently positive series for a key infinite product, namely

$$H(t;q) = \prod_{n=1}^{\infty}(1 + tq^n + t^2q^{2n}).$$

by using the construction in [8], [9] and [10] with a different approach. Then, we use that series to obtain evidently positive series as the generating functions of partitions studied in Chapter 3.

CHAPTER 2

Preliminaries

We state some notations, basic definitions and well-known identities concerning partitions in this chapter.

Definition 2.1 [2] A *partition* of a positive integer n is a finite non-decreasing sequence of positive integers $\lambda = \lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \ldots + \lambda_k = n$. The λ_i's are called the *parts* of the partition and the *weight* $|\lambda|$ of the partition λ is defined to be $|\lambda| = n$. We consider the empty sequence as the only partition of zero.

Let $n = 4$. The partitions of 4 are:

$$4, \quad 1+3, \quad 2+2, \quad 1+1+2, \quad 1+1+1+1.$$

We use the standard notations throughout the thesis:

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^n)$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = (1-a)(1-aq)(1-aq^2)(1-aq^3)\ldots.$$

Definition 2.2 [2] The *generating function* $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series $f(q) = \sum_{n\geq 0} a_n q^n$.

For the absolute convergence of a generating function, we take $|q| < 1$, [3]. Throughout the thesis, we will have some generating functions of the form $A(t;q;a)$. Here, we also take $|t|, |q| < 1$ in the same manner.

The generating function $P(q) = \sum_{n\geq 0} p(n)q^n$ counts all partitions of all non-negative integers n, where $p(n)$ denotes the number of all partitions of n and the exponent of q is the number being partitioned. We also have a nice infinite product representation for $P(q)$ as follows:

$$P(q) = \sum_{n\geq 0} p(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^k} = \frac{1}{(q;q)_\infty}.$$

Definition 2.3 A *partition identity* is an identity which stipulates that "The partitions of n satisfying condition A are equinumerous with the partitions of n satisfying condition B".

We recall the famous partition identity due to Euler: The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Example 2.1 Let $n = 8$. The partitions of 8 into distinct parts are:

$$8, \quad 1+7, \quad 2+6, \quad 3+5, \quad 1+2+5, \quad 1+3+4.$$

That is, there are six partitions of 8 into distinct parts. The partitions of 8 into odd parts are:

$$1+7, \quad 3+5, \quad 1+1+1+5, \quad 1+1+3+3, \quad 1+1+1+1+1+3,$$
$$1+1+1+1+1+1+1+1.$$

So that there are also six partitions of 8 into odd parts.

Generating functions play a very important role in the proofs of partition identities. The proof of Euler's identity using generating functions is as follows:

$$\begin{aligned}
\sum_{n\geq 0} \mathcal{D}(n)q^n &= (1+q)(1+q^2)(1+q^3)\dots \\
&= \frac{(1-q^2)}{(1-q)} . \frac{(1-q^4)}{(1-q^2)} . \frac{(1-q^6)}{(1-q^3)} \dots \\
&= \frac{1}{(1-q)} . \frac{1}{(1-q^3)} . \frac{1}{(1-q^5)} \dots \\
&= \sum_{n\geq 0} \mathcal{O}(n)q^n,
\end{aligned}$$

where $\mathcal{D}(n)$ denotes the number of partitions of n into distinct parts and $\mathcal{O}(n)$ denotes the number of partitions of n into odd parts.

Many partition identities in the literature have the following form: "The partitions of integer n with some difference conditions are equinumerous with the partitions of n with some congruence conditions, for all n".

We recall the famous Rogers-Ramanujan identities [11]:

(1) For any positive integer n, the partitions of n in which the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts congruent to 1 or 4 (modulo 5).

This partition identity can be written in generating function form as follows:

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv 1,4 \pmod 5}} \frac{1}{1-q^k} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

(2) For any positive integer n, the partitions of n in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of n into parts 2 or 3 (modulo 5).

This partition identity can be written in generating function form as follows:

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{\substack{k\geq 1 \\ k\equiv 2,3 \pmod 5}} \frac{1}{1-q^k} == \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

The following well-known identities are proved by Euler and we shall frequently use them in the proofs of theorems in the next chapters:

$$\frac{1}{(x;q)_\infty} = \sum_{n\geq 0} \frac{x^n}{(q;q)_n} \tag{2.1}$$

$$(-x;q)_\infty = \sum_{n\geq 0} \frac{x^n q^{n(n-1)/2}}{(q;q)_n} \tag{2.2}$$

CHAPTER 3

An Alternative Construction for a Family of Partition Generating Functions due to Kanade and Russell

In this chapter, we give alternative proofs for construction of analytic sum-sides to some partition identities of Rogers-Ramanujan type, where these analytic sum-sides are provided by Kanade and Russell in [7] by using jagged partitions, 2-staircases and the q-series identities (2.1) and (2.2) due to Euler.

We use ordinary partitions instead of jagged partitions [7] in the proofs we present.

Theorem 3.1 *[7] Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) Odd parts do not repeat.

(c) For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, *we have* $|\lambda_i - \lambda_{i+2}| \geq 4$ *if* λ_{i+1} *is even and appears more than once.*

(d) $2 + 2$ *is not allowed as a sub-partition.*

For $n, m \in \mathbb{N}$, *let* $kr_1(n, m)$ *denote the number of partitions of* n *into* m *parts such that the partitions satisfy the conditions (a), (b), (c) and (d). Then,*

$$\sum_{m,n\geq 0} kr_1(n,m)q^n t^m = \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+i+6j+3k^2+6k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k}. \tag{3.1}$$

Before we present the proof of the theorem, we give some examples, a definition and prove a proposition.

Example 3.2 Let $n = 12$. There are ten partitions of 12 satisfying the conditions (a), (b), (c) and (d):

$$12, \quad 1+11, \quad 2+10, \quad 3+9, \quad 4+8, \quad 1+3+8,$$
$$5+7, \quad 1+4+7, \quad 6+6, \quad 2+4+6.$$

The partitions $2+2+8$, $1+1+10$, $4+4+4$ of 12 do not satisfy all conditions.

Definition 3.1 Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition counted by $kr_1(n,m)$. If there exist repeating even parts $(2k)+(2k)$ in λ, then we rewrite those parts as consecutive odd parts $(2k-1)+(2k+1)$. We call the partition formed after this transformation a "*seed partition*".

Example 3.3 Let $\lambda = 3+5+8+12+12+16+18+24+24$. Then, the corresponding seed partition to λ is given by $\overline{\lambda} = 3+5+8+11+13+16+18+23+25$.

In some cases, a partition can be the same as its corresponding seed partition. We also note that a seed partition does not contain any repeating parts due to its construction.

Example 3.4 Let $\lambda = 3+5+8+17+19$. Then, the corresponding seed partition to λ is given by $\overline{\lambda} = 3+5+8+17+19$. We observe that $\lambda = \overline{\lambda}$.

Proposition 3.2 *The ordinary partitions in which* 0 *may appear as a part is generated by*

$$A(t;q;a) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+(a-1)t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t)},$$

where the exponent of t *keeps track of the number of parts and the exponent of* a *keeps track of the number of non-zero even parts that appear an even number of times.*

Proof: We will keep track of number of parts by the exponent of t, and keep track of the number of non-zero even parts that appear an even number of times by the exponent of a. Since we don't put any restrictions on parts, we will use all parts in all possible ways when we write the generating function. Let

$$\begin{aligned} A(t;q;a) =& (1+tq^0+t^2q^0+t^3q^0+t^4q^0+t^5q^0+\ldots). \\ & (1+tq+t^2q^2+t^3q^3+t^4q^4+t^5q^5+\ldots). \\ & (1+tq^2+at^2q^4+t^3q^6+at^4q^8+t^5q^{10}+\ldots). \\ & (1+tq^3+t^2q^6+t^3q^9+t^4q^{12}+t^5q^{15}+\ldots). \\ & (1+tq^4+at^2q^8+t^3q^{12}+at^4q^{16}+t^5q^{20}+\ldots). \\ & (1+tq^5+t^2q^{10}+t^3q^{15}+t^4q^{20}+t^5q^{25}+\ldots). \\ & \vdots \end{aligned}$$

The first infinite sum is for 0's, the second infinite sum is for 1's, etc. For example, the term at^4q^8 means that we use four 2's, and since 2 appears an even number of times in this case, we have a factor of a.

Using geometric series and elementary factorization identities [1], we obtain:

$$A(t;q;a) = \frac{1}{(1-t)}.\frac{1}{(1-tq)}.\frac{(1+tq^2+(a-1)t^2q^4)}{(1-t^2q^4)}.$$
$$.\frac{1}{(1-tq^3)}.\frac{(1+tq^4+(a-1)t^2q^8)}{(1-t^2q^8)}\cdots$$
$$= \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+(a-1)t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t)}.$$

□

Proof of Theorem 3.1: Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition counted by $kr_1(n,m)$ and $\overline{\lambda} = \overline{\lambda_1} + \ldots + \overline{\lambda_m}$ be the corresponding seed partition. We define $\beta = 1+3+5+\ldots+(2m-1)$ as the "base partition" with m consecutive odd parts. Let $\mu = \mu_1 + \ldots + \mu_m$, where $\mu_i = \overline{\lambda_i} - \beta_i$ for all $i = 1, \ldots, m$. Whenever the seed partition $\overline{\lambda}$ has a streak of consecutive odd parts, the number of parts in that streak must be even and that streak will give rise to a group of the same repeating even parts in μ. Assume that $\overline{\lambda} = 3+5+7+10$. Then, by the definition of the seed partition, we expect that $\lambda = 4+4+7+10$ or $\lambda = 3+6+6+10$. But, in both cases, the condition (c) is violated. From this example, we see that the number of consecutive odd parts in a streak of $\overline{\lambda}$ must be even.

The generating function for ordinary partitions where we keep track of the number of non-zero even parts that appear an even number of times is given in Proposition 3.2. We recall that the exponent of a keeps track of the number of non-zero even parts that appear an even number of times, and from a seed partition we can generate

$$2^{\text{the number of non-zero even parts appearing an even number of times in } \mu}$$

partitions satisfying the conditions, together with the seed partition itself. Therefore, we take $a = 2$ in $A(t;q;a)$ and we get the following infinite product:

$$A(t;q;2) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t)}.$$

After some straightforward calculations, we obtain:

$$\prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)} = \prod_{n=1}^{\infty} \frac{(1-t^3q^{6n})}{(1-tq^{2n-1})(1-tq^{2n})(1-t^2q^{4n})(1-t)}$$
$$= \frac{(t^3q^6;q^6)_\infty}{(t;q)_\infty (t^2q^4;q^4)_\infty}$$
$$= \left(\sum_{i\geq 0} \frac{t^i}{(q;q)_i}\right)\left(\sum_{j\geq 0} \frac{t^{2j}q^{4j}}{(q^4;q^4)_j}\right)\left(\sum_{k\geq 0} \frac{(-1)^k t^{3k} q^{6k+3k(k-1)}}{(q^6;q^6)_k}\right)$$
$$= \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{4j+3k^2+3k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \tag{3.2}$$

For the third identity, we use the identities (2.1) and (2.2). The remaining work is adding the weight of base partition to the exponent of q in (3.2). Consider the base partition $\beta = 1+3+5+\ldots+(2m-1)$. The partition β has m parts and its weight is m^2. Let $m = i+2j+3k$. The weight of the base partition β becomes $(i+2j+3k)^2$. Therefore, the generating function of partitions satisfying the conditions (a), (b), (c) and (d) is

$$\sum_{m,n\geq 0} kr_1(n,m)q^n t^m = \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{4j+3k^2+3k} q^{(i+2j+3k)^2}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}$$
$$= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+i+6j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}.$$

We note that after bringing back the base partition β, we can substitute $t=1$ in (3.1). The sum (3.1) is exactly the sum (3.2) which is constructed by Kanade and Russell in [7](Where they used x for t). □

Example 3.5 Let $\lambda = 3+5+8+11+13+19+21+23+25$. In this case, the seed partition is $\overline{\lambda} = 3+5+8+11+13+19+21+23+25$ and the base partition is $\beta = 1+3+5+7+9+11+13+15+17$. Then, we get $\mu = 2+2+3+4+4+8+8+8+8$. The sum $2+2$ corresponds to $3+5$ in $\overline{\lambda}$ and it corresponds to $1+3$ in β. The sum $4+4$ corresponds to $11+13$ in $\overline{\lambda}$ and it corresponds to $7+9$ in β. The sum $8+8+8+8$ correspond to $19+21+23+25$ in $\overline{\lambda}$ and it corresponds to $11+13+15+17$ in β. Moreover, the number of non-zero even parts that appear an even number of times in μ determines the number of partitions that can be generated from the seed partition $\overline{\lambda}$. In this example, we have three non-zero even parts that appear an even number of times in μ, namely 2, 4 and 8. So, from the seed partition $\overline{\lambda}$, we can generate $2^3 = 8$ new partitions (together with $\overline{\lambda}$) counted by $kr_1(n,m)$, where $n=128$, $m=9$. Let us list all of these partitions:

$$3+5+8+11+13+19+21+23+25 = \overline{\lambda}$$

$$4+4+8+11+13+19+21+23+25$$
$$3+5+8+12+12+19+21+23+25$$
$$4+4+8+12+12+19+21+23+25$$
$$3+5+8+11+13+20+20+24+24$$
$$4+4+8+11+13+20+20+24+24$$
$$3+5+8+12+12+20+20+24+24$$
$$4+4+8+12+12+20+20+24+24.$$

Theorem 3.3 *[7] Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) Odd parts do not repeat.

(c) For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.

(d') 1 is not allowed to appear as a part.

For $n, m \in \mathbb{N}$, let $kr_2(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d'). Then,

$$\sum_{m,n\geq 0} kr_2(n,m)q^n t^m = \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \tag{3.3}$$

Before the proof of the theorem, we give a definition and prove a proposition.

Definition 3.2 Let λ be a partition counted by $kr_2(n,m)$. If there exist repeating even parts $(2k)+(2k)$ in λ, then we rewrite those parts as consecutive odd parts $(2k-1)+(2k+1)$. We call the partition formed after this transformation an "*almost-seed partition*".

If the partition λ has the sub-partition $2+2$, the partition which is constructed by the method in Definition 3.2 is a partition satisfying all above conditions $(a), (b), (c)$ except the last condition (d'). After the transformation $2+2$ becomes $1+3$ which is ruled out. That is why we use the term "almost-seed" in our definition.

Example 3.6 Let $\lambda = 2+2+8+12+12+16+18+24+24$. Then, the corresponding almost-seed partition to λ is given by $\overline{\lambda} = 1+3+8+11+13+16+18+23+25$.

Proposition 3.4 *The ordinary partitions in which 0 may appear as a part, and if it appears as a part, it must appear an even number of times is generated by*

$$B(t;q;a) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+(a-1)t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} . \frac{1}{(1-t^2)},$$

where the exponent of t keeps track of the number of parts and the exponent of a keeps track of the number of non-zero even parts that appear an even number of times.

Proof: We will keep track of number of parts by the exponent of t, and keep track of number of non-zero even parts that appear an even number of times by the exponent of a. Since we don't put any restrictions on parts, we will use all parts in all possible ways when we write the generating function. Let

$$\begin{aligned} B(t;q;a) =& (1+t^2q^0+t^4q^0+t^6q^0+t^8q^0+\ldots). \\ & (1+tq+t^2q^2+t^3q^3+t^4q^4+t^5q^5+\ldots). \\ & (1+tq^2+at^2q^4+t^3q^6+at^4q^8+t^5q^{10}+\ldots). \\ & (1+tq^3+t^2q^6+t^3q^9+t^4q^{12}+t^5q^{15}+\ldots). \\ & (1+tq^4+at^2q^8+t^3q^{12}+at^4q^{16}+t^5q^{20}+\ldots). \\ & (1+tq^5+t^2q^{10}+t^3q^{15}+t^4q^{20}+t^5q^{25}+\ldots). \\ & \vdots \end{aligned}$$

The first infinite sum is for 0's, the second infinite sum is for 1's, etc. For example, the term at^4q^8 means that we use four 2's, and since 2 appears an even number of times in this case, we have a factor of a.

Using geometric series and elementary factorization identities, we obtain:

$$\begin{aligned} B(t;q;a) =& \frac{1}{(1-t^2)} . \frac{1}{(1-tq)} . \frac{(1+tq^2+(a-1)t^2q^4)}{(1-t^2q^4)} . \\ & . \frac{1}{(1-tq^3)} . \frac{(1+tq^4+(a-1)t^2q^8)}{(1-t^2q^8)} \cdots \\ =& \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+(a-1)t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} . \frac{1}{(1-t^2)}. \end{aligned}$$

□

Proof of Theorem 3.3: We follow the idea in the proof of Theorem 3.1. Let $\lambda = \lambda_1 + \ldots + \lambda_m$ be a partition counted by $kr_2(n,m)$ and $\overline{\lambda} = \overline{\lambda_1} + \ldots + \overline{\lambda_m}$ be the corresponding almost-seed partition. We define $\beta = 1+3+5+\ldots+(2m-1)$ as the "base partition" with m consecutive odd parts. Let $\mu = \mu_1 + \ldots + \mu_m$, where $\mu_i = \overline{\lambda_i} - \beta_i$ for all $i = 1,\ldots,m$. Whenever the almost-seed partition $\overline{\lambda}$ has a streak of consecutive odd parts, this streak will give rise to a group of the same repeating even parts in μ. We also note that if the partition μ consists of a streak of 0's at the beginning, then the number of 0's in this streak has to be an even number. This fact follows from the difference condition (c). If an almost-seed partition starts with $1+3$, then during the generation of all partitions satisfying the conditions, we change $1+3$ as $2+2$ due to the condition (d'). We also observe that if an almost seed partition starts with a streak of consecutive odd parts, then this streak must have an even number of parts. To the contrary, assume that $\overline{\lambda}$ starts with an odd number of consecutive parts, set $\overline{\lambda} = 1+3+5+10$. By definition of the almost seed partition, the corresponding partition λ to $\overline{\lambda}$ can be $2+2+5+10$ or $1+4+4+10$. The partition $2+2+5+10$ violates the condition (c), the partition $1+4+4+10$ violates the conditions (c) and (d'). This also explains why the number of 0's in a streak of μ must be even, if μ starts with 0's.

The generating function for ordinary partitions where we keep track of the number of non-zero even parts that appear an even number of times and if 0 appears as a part then it must appear an even number of times is given in Proposition 3.4.

We recall that the exponent of a keeps track of the number of non-zero even parts that appear an even number of times, and from an almost-seed partition we can generate

$$2^{\text{the number of non-zero even parts appearing an even number of times in } \mu}$$

partitions satisfying the conditions. Therefore, we take $a = 2$ in $B(t;q;a)$ and we get the following infinite product:

$$B(t;q;2) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} . \frac{1}{(1-t^2)}.$$

After some straightforward calculations, we obtain:

$$\prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} . \frac{1}{(1-t^2)} = \prod_{n=1}^{\infty} \frac{(1-t^3q^{6n})}{(1-tq^{2n-1})(1-tq^{2n})(1-t^2q^{4n})(1-t^2)}$$
$$= \frac{(t^3q^6;q^6)_\infty}{(tq;q)_\infty (t^2;q^4)_\infty}$$
$$= \left(\sum_{i\geq 0} \frac{t^i q^i}{(q;q)_i}\right)\left(\sum_{j\geq 0} \frac{t^{2j}}{(q^4;q^4)_j}\right)\left(\sum_{k\geq 0} \frac{(-1)^k t^{3k} q^{6k+3k(k-1)}}{(q^6;q^6)_k}\right)$$
$$= \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{i+3k^2+3k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \tag{3.4}$$

For the third identity, we use the identities (2.1) and (2.2). The remaining work is adding the weight of the base partition to the exponent of q in (3.4). Consider the base partition $\beta = 1+3+5+\ldots+(2m-1)$. The partition β has m parts and its weight is m^2. Let $m = i+2j+3k$. Then the weight of the base partition β is $(i+2j+3k)^2$. Therefore, the generating function of partitions satisfying the conditions (a), (b), (c)

and (d') is

$$\sum_{m,n\geq 0} kr_2(n,m)q^n t^m = \sum_{i,j,k\geq 0} \frac{(-1)^k t^{i+2j+3k} q^{i+3k^2+3k} q^{(i+2j+3k)^2}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k}$$
$$= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k}.$$

We note that after bringing back the base partition β, we can substitute $t = 1$ in (3.3). The sum (3.3) is exactly the sum (3.4) which is constructed by Kanade and Russell in [7](Where they used x for t). □

Example 3.7 Let $\lambda = 2+2+6+12+12+16+18+24+24$. In this case, the almost-seed partition is $\overline{\lambda} = 1+3+6+11+13+16+18+23+25$ and we use the base partition $\beta = 1+3+5+7+9+11+13+15+17$. We have $\mu = 0+0+1+4+4+5+5+8+8$. Observe that μ has two non-zero even parts that appear an even number of times, namely 4 and 8. Therefore, we can generate $2^2 = 4$ partitions from the almost-seed partition $\overline{\lambda}$ counted by $kr_2(n,m)$, where $n = 116$ and $m = 9$. Let us list all of these partitions:

$$2+2+6+11+13+16+18+23+25$$
$$2+2+6+12+12+16+18+23+25$$
$$2+2+6+11+13+16+18+24+24$$
$$2+2+6+12+12+16+18+24+24.$$

Example 3.8 Let $\lambda = 3+5+10+10+17+26+26$. The almost-seed partition is $\overline{\lambda} = 3+5+9+11+17+25+27$ and we use the base partition $\beta = 1+3+5+7+9+11+13$. We have $\mu = 2+2+4+4+8+14+14$. Observe that μ has three non-zero even parts that appear an even number of times, namely 2, 4 and 14. Therefore, we can generate $2^3 = 8$ partitions from the almost-seed partition $\overline{\lambda}$ counted by $kr_2(n,m)$, where $n = 97$ and $m = 7$. Let us list all of these partitions:

$$3+5+9+11+17+25+27$$
$$4+4+9+11+17+25+27$$
$$3+5+10+10+17+25+27$$
$$4+4+10+10+17+25+27$$
$$3+5+9+11+17+26+26$$
$$4+4+9+11+17+26+26$$
$$3+5+10+10+17+26+26$$
$$4+4+10+10+17+26+26.$$

Theorem 3.5 *[7] Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) *Odd parts do not repeat.*

(c) *For a contiguous sub-partition* $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$*, we have* $|\lambda_i - \lambda_{i+2}| \geq 4$ *if* λ_{i+1} *is even and appears more than once.*

(d'') $1, 2$ *and* 3 *are not allowed to appear as parts.*

For $n, m \in \mathbb{N}$*, let* $kr_3(n, m)$ *denote the number of partitions of* n *into* m *parts such that the partitions satisfy the conditions (a), (b), (c) and (d"). Then,*

$$\sum_{m,n\geq 0} kr_3(n,m)q^n t^m = \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+4i+6j+3k^2+12k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}. \tag{3.5}$$

Proof: Let λ be a partition counted by $kr_2(n, m)$. Then we can find a partition λ' counted by $kr_3(n+2m, m)$ iterating each part in λ by 2. Conversely, if λ' is a partition counted by $kr_3(n, m)$, we can find a partition λ counted by $kr_2(n-2m, m)$ subtracting 2 from the each part in λ'. Therefore, there is a 1-1 correspondence between the partitions satisfying the conditions (a), (b), (c), (d') and the partitions satisfying the conditions (a), (b), (c), (d''). Now, by using the generating function in Theorem 3.3, we can provide a generating function for the partitions satisfying the conditions (a), (b), (c), (d''). For this purpose, we let $t^m \mapsto t^m q^{2m}$ to get that the required generating function is

$$\begin{aligned}
\sum_{m,n\geq 0} kr_3(n,m)q^n t^m &= \sum_{i,j,k\geq 0} (-1)^k \frac{(tq^2)^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k} \\
&= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{2i+4j+6k} q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k} \\
&= \sum_{i,j,k\geq 0} (-1)^k \frac{t^{i+2j+3k} q^{(i+2j+3k)(i+2j+3k-1)+4i+6j+3k^2+12k}}{(q;q)_i (q^4;q^4)_j (q^6;q^6)_k}.
\end{aligned}$$

We note that after bringing back the base partition β, we can substitute $t = 1$ in (3.5). The sum (3.5) is exactly the sum (3.6) which is constructed by Kanade and Russell in [7](Where they used x for t). □

In [7], Kanade and Russell presented the following identities as conjectures:

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+i+6j+3k^2+6k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = \frac{1}{(q,q^4,q^6,q^8,q^{11};q^{12})_\infty} \tag{3.6}$$

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+2i+2j+3k^2+6k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = \frac{(q^6;q^{12})_\infty}{(q^2,q^3,q^4,q^8,q^9,q^{10};q^{12})_\infty} = \frac{(q^6;q^{12})_\infty}{(q^2,q^3,q^4;q^6)_\infty} \tag{3.7}$$

$$\sum_{i,j,k\geq 0} \frac{(-1)^k q^{(i+2j+3k)(i+2j+3k-1)+4i+6j+3k^2+12k}}{(q;q)_i(q^4;q^4)_j(q^6;q^6)_k} = \frac{1}{(q^4,q^5,q^6,q^7,q^8;q^{12})_\infty}. \tag{3.8}$$

In 2020, Bringmann, Jennings-Shaffer and Mahlburg gave the proofs of conjectured identities (3.6), (3.7) and (3.8) [5].

By using the method that we use for constructing generating functions (3.1), (3.3) and (3.5), one can construct hundreds of generating functions. By the constructing of hundreds of generating functions, we mean that we may change the conditions concerning unallowed initial parts and we can construct generating functions for those partitions. The key point is that the generating functions (3.1), (3.3) and (3.5) have a nice infinite product representations as in (3.6), (3.7)and (3.8) (When we take $t=1$).

CHAPTER 4

Construction of Evidently Positive Series for a Key Infinite Product

In this chapter, we construct evidently positive series as the generating functions of partitions stated in Theorem 3.1, Theorem 3.3 and Theorem 3.5.

In the proof of Theorem 3.1, we showed that

$$A(t;q;2) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} . \frac{1}{(1-t)},$$

where $A(t;q;a)$ is the generating function of partitions in which 0 may appear as a part, the exponent of t keeps track of the number of parts, the exponent of q keeps track of the number being partitioned and the exponent of a keeps track of the number of non-zero even parts that appear an even number of times. The infinite product

$$H_0(t;q) = \prod_{n=1}^{\infty} (1+tq^{2n}+t^2q^{4n}) \tag{4.1}$$

which appears as the numerator of $A(t;q;2)$ is the generating function of partitions with parts occurring at most twice and all parts are even. We highlight that the exponent of t keeps track of the number of parts. We try to represent $H_0(t;q)$ as a series with evidently positive coefficients. In this way, we can write an evidently positive series as the generating function of partitions in Theorem 3.1. More precisely, assume that

$$H_0(t;q) = \sum_{n\geq 0} h_n q^n t^n,$$

where $h_n \geq 0$ for all n. Then, we get

$$
\begin{aligned}
A(t;q;2) &= \frac{H_0(t;q)}{\prod_{n=1}^{\infty}(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t)} \\
&= \left(\sum_{n\geq 0} h_n q^n t^n\right).\frac{1}{\prod_{n=1}^{\infty}(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t)} \\
&= \left(\sum_{n\geq 0} h_n q^n t^n\right).\frac{1}{(tq;q^2)_\infty(t^2q^4;q^4)_\infty}.\frac{1}{(1-t)} \\
&= \left(\sum_{n\geq 0} h_n q^n t^n\right)\cdot\left(\sum_{i\geq 0}\frac{(tq)^i}{(q^2;q^2)_i}\right)\cdot\left(\sum_{j\geq 0}\frac{(t^2q^4)^j}{(q^4;q^4)_j}\right)\cdot\left(\sum_{k\geq 0} t^k\right) \\
&= \sum_{n,i,j,k\geq 0}\frac{h_n q^{n+i+4j} t^{n+i+2j+k}}{(q^2;q^2)_i(q^4;q^4)_j}.
\end{aligned}
$$

The latter sum is an evidently positive series. After adding the weight of the base partition β to this series, we obtain an evidently positive series as the generating function of partitions in Theorem 3.1. Moreover, we can also construct evidently positive series as the generating functions of partitions in Theorem 3.3 and Theorem 3.5 by using the same positive series representation of $H_0(t;q)$. Consider

$$
H(t;q) = \prod_{n=1}^{\infty}(1+tq^n+t^2q^{2n}).
$$

We observe that $H_0(t;q) = H(t;q^2)$. Therefore, it is sufficient to represent $H(t;q)$ as an evidently positive series. We also observe that $H(t;q)$ is the generating function of partitions with parts occurring at most twice.

Theorem 4.1 *For $n,m\in\mathbb{N}$, let $h(n,m)$ denote the number of partitions of n into m parts such that each part appears at most twice. Then,*

$$
\sum_{m,n\geq 0} h(n,m)q^n t^m = \sum_{\substack{n_1,n_2\geq 0\\ n_1=n_{11}+n_{12}\\ \beta}} \frac{q^{|\beta|}t^{2n_2+n_1}}{(q;q)_{n_{12}}(q^3;q^3)_{n_2}} \tag{4.2}
$$

where β is the base partition with n_2 pairs, n_{11} immobile singletons, n_{12} moveable singletons.

Proof: We will show that each partition λ counted by $h(n,m)$ corresponds to a triple of partitions (β,μ,θ) by a series of backward moves, where

- β is the base partition into $m = 2n_2+n_1$ parts, has n_2 pairs, n_{11} immobile singletons, n_{12} moveable singletons, where $n_1 = n_{11}+n_{12}$ (pairs, immobile singletons and moveable singletons will be defined shortly), and minimum possible weight,
- μ is a partition into n_2 multiples of 3 (0 is allowed as a part),

- θ is a partition into $n_1 = n_{11} + n_{12}$ parts (0 is allowed as a part), where n_{11} will be determined once we perform all backward moves on the pairs and the number of 0's in $\theta \geq n_{11}$.

- $|\lambda| = |\beta| + |\mu| + |\theta|$.

Conversely, we will obtain a unique λ from a given triple of partitions (β, μ, θ) by a series of forward moves. We show that the sequences of backward and forward moves are inverses of each other. Therefore, we will obtain

$$\sum_{m,n \geq 0} h(n,m) q^n t^m = \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 = n_{11} + n_{12} \\ \beta, \mu, \theta}} q^{|\beta|+|\mu|+|\theta|} t^{2n_2+n_1}$$

We now define pairs, singletons, the backward moves on pairs and singletons. Let λ be a partition counted by $h(n,m)$. The *pairs* in λ are defined as a pair of two repeating parts or a pair of two consecutive parts. We pair the leftmost parts first, and continue recursively for yet unbound parts. We show the pairs in brackets for convenience. Once we determine all pairs, the remaining parts will be defined as *singletons*. The singletons will be divided into two categories; *immobile singletons* and *moveable singletons*. It is not possible to distinguish a singleton as an immobile or a moveable singleton in the first place. Once we perform all possible backward moves on the pairs (the backward moves on the pairs will be defined shortly), the singletons which exist between the pairs will be immobile singletons and no backward or forward moves will be possible on those singletons. The singletons which appear just after a streak containing pairs and immobile singletons will be moveable singletons. The details will be explained in due course. A pair will not be moved through other pairs in the backward or forward direction, but it can be moved through singletons. We indicate the pair or singleton being moved in boldface, here and elsewhere. The backward moves on pairs are defined as follows:

<u>case Ia</u>

$$(\text{parts} \leq k-3),\ [\mathbf{k}, \mathbf{k+1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \leq k-3),\ [\mathbf{k-1}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

<u>case Ib</u>

$$(\text{parts} \leq k-4),\ k-2,\ [\mathbf{k}, \mathbf{k+1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-4),\ k-2,\ [\mathbf{k-1}, \mathbf{k-1}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \le k-4),\ [\mathbf{k-2}, \mathbf{k-1}],\ k-1,\ (\text{parts} \ge k+1)$$

case IIa

$$(\text{parts} \le k-4),\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-4),\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \ge k+1)$$

case IIb

$$(\text{parts} \le k-4),\ k-2,\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-4),\ k-2,\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \le k-4),\ [\mathbf{k-2}, \mathbf{k-2}],\ k-1,\ (\text{parts} \ge k+1)$$

case IIc

$$(\text{parts} \le k-5),\ k-3,\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-5),\ k-3,\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \ge k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \le k-5),\ [\mathbf{k-3}, \mathbf{k-2}],\ k-1,\ (\text{parts} \ge k+1)$$

case IIIa

$$\underbrace{\ldots\ldots\ldots\ldots\ldots,\ [k-3, k-2]}_{\text{a streak of pairs which can not be moved further back}},\ k-2,\ [\mathbf{k}, \mathbf{k+1}],\ [k+2, k+3],\ (\text{parts} \ge k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-2],\ k-2,\ [\mathbf{k-1},\mathbf{k-1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-2],\ \underbrace{[k-2,k-1]}_{\substack{\text{can not be moved}\\ \text{further back}}},\ k-1,\ [\mathbf{k+2},\mathbf{k+3}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-2],\ [k-2,k-1],\ \underbrace{k-1}_{\substack{\text{an immobile}\\ \text{singleton}}},\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

The pair $[k+1,k+1]$ can not be moved further back and the part $k-1$ is an immobile singleton. If we try to move the pair $[k+1,k+1]$ back once more, we get:

$$\ldots\ldots\ldots\ldots,[k-3,k-2],\ [k-2,k-1],\ k-1,\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-2],\ [k-2,k-1],\ k-1,\ [\mathbf{k-1},\mathbf{k}],\ (\text{parts} \geq k+3)$$

As we see, the part $k-1$ appears thrice in this case, but appearance of a part more than twice is not allowed.

case IIIb

$$\underbrace{\ldots\ldots\ldots\ldots\ldots,\ [k-3,k-3]}_{\substack{\text{a streak of pairs which can not be}\\ \text{moved further back}}},\ k-2,\ [\mathbf{k},\mathbf{k+1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-3],\ k-2,\ [\mathbf{k-1},\mathbf{k-1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-3],\ \underbrace{[k-2,k-1]}_{\substack{\text{can not be moved}\\ \text{further back}}},\ k-1,\ [\mathbf{k+2},\mathbf{k+3}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots\ldots,[k-3,k-3],\ [k-2,k-1],\ \underbrace{k-1}_{\substack{\text{an immobile}\\ \text{singleton}}},\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

The pair $[k+1,k+1]$ can not be moved further back and the part $k-1$ is an immobile singleton as in case **IIIa**.

case IIIc

$$\underbrace{\ldots\ldots\ldots\ldots,[k-4,k-3]}_{\substack{\text{a streak of pairs which can not be}\\ \text{moved further back}}},\ k-2,\ [\mathbf{k},\mathbf{k+1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots,[k-4,k-3],\ k-2,\ [\mathbf{k-1},\mathbf{k-1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots,[k-4,k-3],\ \underbrace{[k-2,k-1]}_{\substack{\text{can not be moved}\\ \text{further back}}},\ k-1,\ [\mathbf{k+2},\mathbf{k+3}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$\ldots\ldots\ldots,[k-4,k-3],\ [k-2,k-1],\ \underbrace{k-1}_{\substack{\text{an immobile}\\ \text{singleton}}},\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

The pair $[k+1,k+1]$ can not be moved further back and the part $k-1$ is an immobile singleton as in the cases **IIIa** and **IIIb**.

In the cases **IIIa**, **IIIb** and **IIIc**, the streak of pairs at the very beginning, where the pairs are tightly packed, i.e. no backward moves are possible to perform on any of them can not be described more precisely. Because, we have two kinds of pairs; a pair of two repeating parts and a pair of two consecutive parts. Therefore, there may exist three different pairs after a fixed pair. For example, if we consider the pair $[k,k+1]$, there are three possibilities for the pair just coming after $[k,k+1]$: $[k+1,k+2]$, $[k+2,k+2]$ and $[k+2,k+3]$.

We highlight that an immobile singleton may exist between two pairs such that the first pair is a pair of consecutive parts and the second pair is a pair of repeating parts.

We observe that any backward move on a pair decreases the total weight by 3. Once we determine all pairs and singletons in λ, we start with the smallest pair and we perform $\frac{\mu_1}{3}$ backward moves on it until it becomes $[1,1]$, $[1,2]$ or $[2,2]$, thus determining μ_1, the smallest part of μ. If the smallest pair is already $[1,1]$, $[1,2]$ or $[2,2]$, then we set $\mu_1 = 0$. Once the smallest pair is stowed as $[1,1]$, $[1,2]$ or $[2,2]$, we continue with the next smallest pair. We perform $\frac{\mu_2}{3}$ backward moves on the second smallest pair, thus determining μ_2 and so on. When the smallest two pairs are moved back as much as possible, we may have the following partitions at the beginning of the base partition:

$[1,1],[2,2]$	$[1,2],2,[4,4]$	$[2,2],[3,3]$
$[1,1],[2,3]$	$[1,2],[2,3]$	$[2,2],[3,4]$
$[1,1],[3,3]$	$[1,2],[3,3]$	$[2,2],[4,4]$
	$[1,2],[3,4]$	

We now show that any backward move on the smaller pair allows a backward move on the immediately succeeding pair. We may assume that there exist no other parts between the smaller pair and the succeeding pair. Because, if there exist some singletons between two pairs, then we can move the larger pair in the backward direction so that there exist no singletons between the pairs. Here, we also assume that there is no immobile singleton appearing between the pairs during the backward moves. The case when an immobile singleton exists between two pairs is already considered in the cases **V** and **VI**. It is obvious that one backward move on the smaller pair allows one backward move on the larger pair in the cases **V** and **VI**. We now consider several cases:

case i

$$(\text{parts} \le k-2),\ [\mathbf{k},\mathbf{k+1}],\ [k+1,k+2],\ (\text{parts} \ge k+2)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1,k-1],\ [\mathbf{k+1},\mathbf{k+2}],\ (\text{parts} \ge k+2)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1,k-1],\ [\mathbf{k},\mathbf{k}],\ (\text{parts} \ge k+2)$$

case ii

$$(\text{parts} \le k-2),\ [\mathbf{k},\mathbf{k+1}],\ [k+2,k+2],\ (\text{parts} \ge k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1,k-1],\ [\mathbf{k+2},\mathbf{k+2}],\ (\text{parts} \ge k+3)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1,k-1],\ [\mathbf{k},\mathbf{k+1}],\ (\text{parts} \ge k+3)$$

case iii

$$(\text{parts} \le k-2),\ [\mathbf{k}, \mathbf{k+1}],\ [k+2, k+3],\ (\text{parts} \ge k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1, k-1],\ [\mathbf{k+2}, \mathbf{k+3}],\ (\text{parts} \ge k+3)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1, k-1],\ [\mathbf{k+1}, \mathbf{k+1}],\ (\text{parts} \ge k+3)$$

case iv

$$(\text{parts} \le k-2),\ [\mathbf{k}, \mathbf{k}],\ [k+1, k+1],\ (\text{parts} \ge k+2)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-2, k-1],\ [\mathbf{k+1}, \mathbf{k+1}],\ (\text{parts} \ge k+2)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$ or $k-3$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-2, k-1],\ [\mathbf{k-1}, \mathbf{k}],\ (\text{parts} \ge k+2)$$

case v

$$(\text{parts} \le k-2),\ [\mathbf{k}, \mathbf{k}],\ [k+1, k+2],\ (\text{parts} \ge k+2)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-2, k-1],\ [\mathbf{k+1}, \mathbf{k+2}],\ (\text{parts} \ge k+2)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$ or $k-3$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1, k-1],\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \ge k+2)$$

case vi

$$(\text{parts} \le k-2),\ [\mathbf{k}, \mathbf{k}],\ [k+2, k+2],\ (\text{parts} \ge k+3)$$

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-2, k-1],\ [\mathbf{k+2}, \mathbf{k+2}],\ (\text{parts} \ge k+3)$$

Here, there is a potential regrouping for determining pairs if there is a $k-2$ or $k-3$.

$$\Big\downarrow \text{one backward move}$$

$$(\text{parts} \le k-2),\ [k-1, k-1],\ [\mathbf{k}, \mathbf{k+1}],\ (\text{parts} \ge k+3)$$

In all cases **i**, **ii**, **iii**, **iv**, **v** and **vi**, one backward move on the smaller pair allows a backward move on the larger pair. Thus, it is obvious that

$$\mu_1 \le \mu_2 \le \ldots \le \mu_{n_2}.$$

Once we perform all possible backward moves on the pairs, we will have pairs which are fixed in its places, i.e. they can not be moved further back and some immobile singletons may exist between the pairs. More precisely, once we perform all possible backward moves on the pairs, the intermediate partition looks like

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{n_2-1 \text{ pairs which can not be moved further back} \\ \text{and } n_{11} \text{ immobile singletons}}},\ [k, k],\ \text{moveable singletons} \ge k+1$$

or

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{n_2-1 \text{ pairs which can not be moved further back} \\ \text{and } n_{11} \text{ immobile singletons}}},\ [k, k+1],\ \text{moveable singletons} \ge k+1$$

To make the n_{12} moveable singletons $k+1, k+3, k+5, \ldots, k+2n_{12}-1$, we move the singletons one by one in backward direction. We subtract $\theta_{n_{11}+1}$ from the smallest moveable singleton, $\theta_{n_{11}+2}$ from the next smallest moveable singleton and so on. Since we have n_{11} immobile singletons, the backward moves of those singletons will correspond to n_{11} 0's in the partition θ. It is clear that any backward move on a moveable singleton allows a backward move on the immediately succeeding moveable singleton. It is immediate that

$$\theta_1 = \theta_2 = \ldots = \theta_{n_{11}} = 0 \le \theta_{n_{11}+1} \le \theta_{n_{11}+2} \le \ldots \le \theta_{n_{11}+n_{12}} = \theta_{n_1}.$$

We have now produced a triple of partitions (β, μ, θ) from the given partition λ counted by $h(n, m)$.

Conversely, given a triple of partitions (β, μ, θ), we will produce a unique partition λ counted by $h(n, m)$ via forward moves.

Let (β, μ, θ) be a triple of partitions such that β is the base partition into $m = 2n_2 + n_1$ parts, has n_2 pairs, n_{11} immobile singletons, n_{12} moveable singletons, where $n_1 = n_{11} + n_{12}$ and minimum possible weight, μ is a partition into n_2 multiples of 3 (0 is allowed as a part), θ is a partition into $n_1 = n_{11} + n_{12}$ parts (0 is allowed as a part), where the number of 0's in $\theta \geq n_{11}$. We first add the ith largest part of θ to the ith largest singleton in β for $i = 1, 2, \ldots, n_{12}$ in this order. These are the forward moves on the moveable singletons. Once we perform all forward moves on the moveable singletons, we perform $\frac{1}{3}$.(the ith largest part of μ) forward moves on the ith largest pair in β for $i = 1, 2, \ldots, n_2$, in this order. As we showed for the backward moves on the pairs , one can easily show that one forward move on the larger pair allows one forward move of the preceding smaller pair. Any forward move on a pair will increase the total weight by 3. The forward moves on the pairs are defined as follows:

case I'a

$$(\text{parts} \leq k-3),\ [\mathbf{k-1}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one forward move}$$

$$(\text{parts} \leq k-3),\ [\mathbf{k}, \mathbf{k+1}],\ (\text{parts} \geq k+1)$$

Here, there is a potential regrouping for determining pairs if there is a $k+1$ or $k+2$.

case I'b

$$(\text{parts} \leq k-4),\ [\mathbf{k-2}, \mathbf{k-1}],\ k-1,\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \leq k-4),\ k-2,\ [\mathbf{k-1}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one forward move}$$

$$(\text{parts} \leq k-4),\ k-2,\ [\mathbf{k}, \mathbf{k+1}],\ (\text{parts} \geq k+1)$$

Here, there is a potential regrouping for determining pairs if there is a $k+1$ or $k+2$.

case II'a

$$(\text{parts} \leq k-4),\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one forward move}$$

$$(\text{parts} \leq k-4),\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \geq k+1)$$

Here again, there is a potential regrouping for determining pairs if there is a $k+1$.

case II'b

$$(\text{parts} \leq k-4),\ [\mathbf{k-2}, \mathbf{k-2}],\ k-1,\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \leq k-4),\ k-2,\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one forward move}$$

$$(\text{parts} \leq k-4),\ k-2,\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \geq k+1)$$

Here again, there is a potential regrouping for determining pairs if there is a $k+1$.

case II'c

$$(\text{parts} \leq k-5),\ [\mathbf{k-3}, \mathbf{k-2}],\ k-1,\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$(\text{parts} \leq k-5),\ k-3,\ [\mathbf{k-2}, \mathbf{k-1}],\ (\text{parts} \geq k+1)$$

$$\Big\downarrow \text{one forward move}$$

$$(\text{parts} \leq k-5),\ k-3,\ [\mathbf{k}, \mathbf{k}],\ (\text{parts} \geq k+1)$$

Here again, there is a potential regrouping for determining pairs if there is a $k+1$.

case III'a

$$\underbrace{\ldots\ldots\ldots\ldots,\ [k-3, k-2], [k-2, k-1]}_{\text{a streak of pairs which are tightly packed}},\ k-1,\ [\mathbf{k+1}, \mathbf{k+1}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$\ldots\ldots\ldots\ldots,\ [k-3, k-2], [k-2, k-1],\ k-1,\ [\mathbf{k+2}, \mathbf{k+3}],\ (\text{parts} \geq k+3)$$

Here, there is a potential regrouping of the pairs if there is a $k+3$ or $k+4$. We now perform a forward move on the preceding smaller pair. We first regroup the pairs and we perform a forward move on the new pair:

$$\ldots\ldots\ldots\ldots,\ [k-3, k-2],\ [k-2,\ \underbrace{k-1], k-1},\ [k+2, k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots\ldots,\ [k-3, k-2],\ k-2,\ [\mathbf{k-1}, \mathbf{k-1}],\ [k+2, k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$\ldots\ldots\ldots\ldots,\ [k-3, k-2],\ k-2,\ [\mathbf{k}, \mathbf{k+1}],\ [k+2, k+3],\ (\text{parts} \geq k+3)$$

case III'b

$$\underbrace{\ldots\ldots\ldots\ldots, [k-3,k-3],[k-2,k-1]}_{\text{a streak of pairs which are tightly packed}},\ k-1,\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$\ldots\ldots\ldots\ldots, [k-3,k-3],[k-2,k-1],\ k-1,\ [\mathbf{k+2},\mathbf{k+3}],\ (\text{parts} \geq k+3)$$

Here, there is a potential regrouping of the pairs if there is a $k+3$ or $k+4$. We now perform a forward move on the preceding smaller pair. We first regroup the pairs and we perform a forward move on the new pair:

$$\ldots\ldots\ldots\ldots, [k-3,k-3],\ [k-2,\ \underbrace{k-1],k-1},\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots\ldots, [k-3,k-3],\ k-2,\ [\mathbf{k-1},\mathbf{k-1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$\ldots\ldots\ldots\ldots, [k-3,k-3],\ k-2,\ [\mathbf{k},\mathbf{k+1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

case III'c

$$\underbrace{\ldots\ldots\ldots\ldots, [k-4,k-3],[k-2,k-1]}_{\text{a streak of pairs which are tightly packed}},\ k-1,\ [\mathbf{k+1},\mathbf{k+1}],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$\ldots\ldots\ldots\ldots, [k-4,k-3],[k-2,k-1],\ k-1,\ [\mathbf{k+2},\mathbf{k+3}],\ (\text{parts} \geq k+3)$$

Here, there is a potential regrouping of the pairs if there is a $k+3$ or $k+4$. We now perform a forward move on the preceding smaller pair. We first regroup the pairs and we perform a forward move on the new pair:

$$\ldots\ldots\ldots\ldots, [k-4,k-3],\ [k-2,\ \underbrace{k-1],k-1},\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$\ldots\ldots\ldots\ldots, [k-4,k-3],\ k-2,\ [\mathbf{k-1},\mathbf{k-1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

$$\Big\downarrow \text{one forward move}$$

$$.................., [k-4,k-3],\ k-2,\ [\mathbf{k},\mathbf{k+1}],\ [k+2,k+3],\ (\text{parts} \geq k+3)$$

We notice that the corresponding cases for the backward and forward moves have switched inputs and outputs and we perform the backward moves and the forward moves in the exact reverse order. Therefore, λ's enumerated by $h(n,m)$ are in 1-1 correspondence with the triples (β,μ,θ).

It follows that

$$\sum_{m,n\geq 0} h(n,m)q^n t^m = \sum_{\substack{n_1,n_2\geq 0\\ n_1=n_{11}+n_{12}\\ \beta,\mu,\theta}} q^{|\beta|+|\mu|+|\theta|}t^{2n_2+n_1} = \sum_{\substack{n_1,n_2\geq 0\\ n_1=n_{11}+n_{12}\\ \beta}} \frac{q^{|\beta|}t^{2n_2+n_1}}{(q;q)_{n_{12}}(q^3;q^3)_{n_2}}.$$

□

Example 4.9 Let $\lambda = 1,4,4,5,6,6,9,10,11,12,12,14$. We first determine the pairs and the singletons:

$$\lambda = 1,[4,4],[5,6],6,[9,10],[11,12],12,14.$$

We have $n_2 = 4$ pairs and $n_1 = 4$ singletons. We start to perform backward moves on the pairs, and once all possible backward moves are performed on the pairs, we continue with the backward moves on the moveable singletons.

$$\lambda = 1,[\mathbf{4},\mathbf{4}],[5,6],6,[9,10],[11,12],12,14$$

$\big\downarrow$ one backward move

$$1,[\mathbf{2},\mathbf{3}],[5,6],6,[9,10],[11,12],12,14$$

$\big\downarrow$ regrouping the pairs

$$[1,2],3,[\mathbf{5},\mathbf{6}],6,[9,10],[11,12],12,14$$

$\big\downarrow$ one backward move

$$[1,2],3,[\mathbf{4},\mathbf{4}],6,[9,10],[11,12],12,14$$

$\big\downarrow$ regrouping the pairs

$$[1,2],[3,4],4,6,[\mathbf{9},\mathbf{10}],[11,12],12,14$$

$\big\downarrow$ one backward move

$$[1,2],[3,4],4,6,[\mathbf{8},\mathbf{8}],[11,12],12,14$$

$\big\downarrow$ one backward move

$$[1,2],[3,4],4,6,[\mathbf{6},\mathbf{7}],[11,12],12,14$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$[1,2],[3,4],4,[6,6],7,[\mathbf{11},\mathbf{12}],12,14$$

We observe that the pair $[6,6]$ can not be moved further back and 4 is an immobile singleton.

$$\Big\downarrow \text{one backward move}$$

$$[1,2],[3,4],4,[6,6],7,[\mathbf{10},\mathbf{10}],12,14$$

$$\Big\downarrow \text{one backward move}$$

$$[1,2],[3,4],4,[6,6],7,[\mathbf{8},\mathbf{9}],12,14$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$[1,2],[3,4],4,[6,6],[7,8],\mathbf{9},12,14$$

$$\Big\downarrow \text{one backward move}$$

$$[1,2],[3,4],4,[6,6],[7,8],8,\mathbf{12},14$$

$$\Big\downarrow \text{two backward moves}$$

$$[1,2],[3,4],4,[6,6],[7,8],8,10,\mathbf{14}$$

$$\Big\downarrow \text{two backward moves}$$

$$\beta=[1,2],[3,4],4,[6,6],[7,8],8,10,12$$

We have $\mu=3+3+6+6$ and $\theta=0+1+2+2$. We observe that

$$|\lambda|=94=|\beta|+|\mu|+|\theta|=71+18+5.$$

Example 4.10 Let $\beta=[2,2],[3,4],4,[6,6],[7,8],8,[10,10],11,13,15$ be a base partition with 5 pairs, 2 immobile singletons and 3 moveable singletons, $\mu=3+3+3+6+6$ and $\theta=0+0+2+3+5$. After incorporating parts of θ as forward moves on the moveable singletons, the intermediate partition becomes

$$[2,2],[3,4],4,[6,6],[7,8],8,[10,10],13,16,20.$$

We perform $\frac{\mu_5}{3}=2$ forward moves on the largest pair $[10,10]$:

$$[2,2],[3,4],4,[6,6],[7,8],8,[\mathbf{10},\mathbf{10}],13,16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[2,2],[3,4],4,[6,6],[7,8],8,[\mathbf{11},\mathbf{12}],13,16,20$$

$$\Big\downarrow \text{regrouping the pairs}$$

$$[2,2],[3,4],4,[6,6],[7,8],8,11,[\mathbf{12},\mathbf{13}],16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[2,2],[3,4],4,[6,6],[\mathbf{7},\mathbf{8}],8,11,[14,14],16,20$$

We now perform $\frac{\mu_4}{3} = 2$ forward moves on the second largest pair. We first need to regroup the pairs.

$$\Big\downarrow \text{regrouping of pairs}$$

$$[2,2],[3,4],4,[6,6],7,[\mathbf{8},\mathbf{8}],11,[14,14],16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[2,2],[3,4],4,[6,6],7,[\mathbf{9},\mathbf{10}],11,[14,14],16,20$$

$$\Big\downarrow \text{regrouping of pairs}$$

$$[2,2],[3,4],4,[6,6],7,9,[\mathbf{10},\mathbf{11}],[14,14],16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[2,2],[3,4],4,[\mathbf{6},\mathbf{6}],7,9,[12,12],[14,14],16,20$$

We now perform $\frac{\mu_3}{3} = 1$ forward move on the third largest pair. We first need to regroup the pairs.

$$\Big\downarrow \text{regrouping of pairs}$$

$$[2,2],[3,4],4,6,[\mathbf{6},\mathbf{7}],9,[12,12],[14,14],16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[2,2],[\mathbf{3},\mathbf{4}],4,6,[8,8],9,[12,12],[14,14],16,20$$

We now perform $\frac{\mu_2}{3} = 1$ forward move on the fourth largest pair. We first need to regroup the pairs.

$$\Big\downarrow \text{regrouping of pairs}$$

$$[2,2],3,[\mathbf{4},\mathbf{4}],6,[8,8],9,[12,12],[14,14],16,20$$

$$\Big\downarrow \text{one forward move}$$

$$[\mathbf{2}, \mathbf{2}], 3, [5, 6], 6, [8, 8], 9, [12, 12], [14, 14], 16, 20$$

We now perform $\frac{\mu_1}{3} = 1$ forward move on the smallest pair. We first need to regroup the pairs.

$$\Big\downarrow \text{regrouping of pairs}$$

$$2, [\mathbf{2}, \mathbf{3}], [5, 6], 6, [8, 8], 9, [12, 12], [14, 14], 16, 20$$

$$\Big\downarrow \text{one forward move}$$

$$2, [\mathbf{4}, \mathbf{4}], [5, 6], 6, [8, 8], 9, [12, 12], [14, 14], 16, 20$$

This final partition is λ. Its weight is $140 = |\lambda| = |\beta| + |\mu| + |\theta| = 109 + 21 + 10$ indeed.

The only shortcoming in the generating function (4.2) is that $|\beta|$, the weight of the base partition β, is not given explicitly. To give an explicit formula for $|\beta|$ with respect to the number of pairs and the number of singletons has not been possible. Because, for a fixed number of pairs and singletons, there exist different base partitions. But, we will be able to construct some polynomials as generating functions of the base partitions with a given number of pairs and singletons.

Lemma 4.2 *For $m_1, m_2, m_3 \in \mathbb{N}$ and $m \in \mathbb{Z}$, let $P(m_1, m_2, m_3, m+1; q)$ be the generating function of the base partitions β's defined in the proof of Theorem 4.1 with m_1 pairs of two repeating parts, m_2 pairs of two consecutive parts and m_3 blocks, where a block is a partition into five parts which have the form $[k-1, k], k, [k+2, k+2]$. Then,*

$$P(m_1, m_2, m_3, m+1; q) = P_0(m_1, m_2, m_3, m+1; q) + P_1(m_1, m_2, m_3, m+1; q) \qquad (4.3)$$

where $P_0(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m]$ and $P_1(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m+1]$. $P_0(m_1, m_2, m_3, m+1; q)$ and $P_1(m_1, m_2, m_3, m+1; q)$ satisfy the following functional equations:

$$\begin{aligned} P_0(m_1, m_2, m_3, m+1; q) = & q^{2m}\Big[P_0(m_1 - 1, m_2, m_3, m; q) \\ & + P_1(m_1 - 1, m_2, m_3, m-1; q) \\ & + P_0(m_1 - 1, m_2, m_3, m-1; q)\Big] \\ & + q^{5m-7}\Big[P_1(m_1, m_2, m_3 - 1, m-3; q) \\ & + P_0(m_1, m_2, m_3 - 1, m-3; q) \\ & + P_1(m_1, m_2, m_3 - 1, m-4; q)\Big] \end{aligned} \qquad (4.4)$$

$$
\begin{aligned}
P_1(m_1, m_2, m_3, m+1; q) =& q^{2m+1}\Big[P_1(m_1, m_2-1, m_3, m; q) \\
&+ P_0(m_1, m_2-1, m_3, m; q) \\
&+ P_1(m_1, m_2-1, m_3, m-1; q)\Big]
\end{aligned}
\tag{4.5}
$$

$$
\left.
\begin{aligned}
&P_{0/1}(m_1, m_2, m_3, m; q) &&= 0 && \textit{if } m < 0 \\
&P_{0/1}(m_1, m_2, m_3, 0; q) &&= 1 && \\
&P_{0/1}(0, 0, 0, m; q) &&= 0 && \textit{if } m \neq 1 \\
&P_0(0, 0, 0, 1; q) &&= 1 && \\
&P_1(0, 0, 0, 1; q) &&= 0 &&
\end{aligned}
\right\} \textit{initial conditions}
\tag{4.6}
$$

Moreover, $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients.

Proof: We first prove the equation (4.3). Let β be a base partition counted by $P(m_1, m_2, m_3, m+1; q)$. Then, it has one of the following forms:

$$
\underbrace{\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1-1 \text{ pairs of two repeating parts} \\ m_2 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m, m], \underbrace{m+1, m+3, \ldots, m+2n_{12}-1}_{n_{12} \text{ moveable singletons}} \tag{$*$}
$$

or

$$
\underbrace{\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1 \text{ pairs of two repeating parts} \\ m_2-1 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m, m+1], \underbrace{m+1, m+3, \ldots, m+2n_{12}-1}_{n_{12} \text{ moveable singletons}} \tag{$**$}
$$

The partition in $(*)$ is a base partition counted by $P_0(m_1, m_2, m_3, m+1; q)$ and the partition in $(**)$ is a base partition counted by $P_1(m_1, m_2, m_3, m+1; q)$. It is clear that there is no base partition which is counted by both $P_0(m_1, m_2, m_3, m+1; q)$ and $P_1(m_1, m_2, m_3, m+1; q)$. Hence, (4.3) follows.

We now prove (4.4). Let β be a base partition counted by $P_0(m_1, m_2, m_3, m+1; q)$. β has the form in $(*)$. The largest pair $[m, m]$ may be contained in a block or not. We consider both cases separately. If the largest pair $[m, m]$ is not contained in a block and we delete it from β, the remaining partition may have one of the following forms:

$$
\underbrace{\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1-2 \text{ pairs of two repeating parts} \\ m_2 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m-1, m-1], \underbrace{m, m+2, \ldots, m+2n_{12}-2}_{n_{12} \text{ moveable singletons}} \tag{a}
$$

$$
\underbrace{\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1-1 \text{ pairs of two repeating parts} \\ m_2-1 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m-2, m-1], \underbrace{m-1, m+1, \ldots, m+2n_{12}-3}_{n_{12} \text{ moveable singletons}} \tag{b}
$$

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{m_1-2 \text{ pairs of two repeating parts}\\ m_2 \text{ pairs of two consecutive parts}\\ m_3 \text{ blocks}}}, [m-2,m-2], \underbrace{m-1, m+1, \ldots, m+2n_{12}-3}_{n_{12} \text{ moveable singletons}} \quad \text{(c)}$$

If the largest pair $[m,m]$ is contained in a block, then we have:

$$\underbrace{\ldots\ldots\ldots\ldots\ldots}_{\substack{m_1 \text{ pairs of}\\ \text{two repeating parts}\\ m_2 \text{ pairs of}\\ \text{two consecutive parts}\\ m_3-1 \text{ blocks}}}, \underbrace{[m-3,m-2], m-2, [m,m]}_{\text{a block}}, \underbrace{m+1, \ldots, m+2n_{12}-1}_{n_{12} \text{ moveable singletons}} \quad (***)$$

If we delete the block $[m-3,m-2], m-2, [m,m]$ in $(***)$, the remaining partition may have one of the following forms:

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{m_1 \text{ pairs of two repeating parts}\\ m_2-1 \text{ pairs of two consecutive parts}\\ m_3-1 \text{ blocks}}}, [m-4,m-3], \underbrace{m-3, m-1, \ldots, m+2n_{12}-5}_{n_{12} \text{ moveable singletons}} \quad \text{(a')}$$

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{m_1-1 \text{ pairs of two repeating parts}\\ m_2 \text{ pairs of two consecutive parts}\\ m_3-1 \text{ blocks}}}, [m-4,m-4], \underbrace{m-3, m-1, m+2n_{12}-5}_{n_{12} \text{ moveable singletons}} \quad \text{(b')}$$

$$\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{m_1 \text{ pairs of two repeating parts}\\ m_2-1 \text{ pairs of two consecutive parts}\\ m_3-1 \text{ blocks}}}, [m-5,m-4], \underbrace{m-4, m-2, \ldots, m+2n_{12}-6}_{n_{12} \text{ moveable singletons}} \quad \text{(c')}$$

The base partitions in (a), (b), (c), (a'), (b') and (c') are counted by $P_0(m_1-1, m_2, m_3, m; q)$, $P_1(m_1-1, m_2, m_3, m-1; q)$, $P_0(m_1-1, m_2, m_3, m-1; q)$, $P_1(m_1, m_2, m_3-1, m-3; q)$, $P_0(m_1, m_2, m_3-1, m-3; q)$ and $P_1(m_1, m_2, m_3-1, m-4; q)$, respectively. Therefore, we have:

$$\begin{aligned} P_0(m_1, m_2, m_3, m+1; q) = & \underbrace{q^{2m}}_{\substack{\text{for the deleted}\\ \text{pair } [m,m]}} \Big[P_0(m_1-1, m_2, m_3, m; q) \\ & + P_1(m_1-1, m_2, m_3, m-1; q) \\ & + P_0(m_1-1, m_2, m_3, m-1; q) \Big] \\ & + \underbrace{q^{5m-7}}_{\substack{\text{for the deleted block}\\ [m-3,m-2], m-2, [m,m]}} \Big[P_1(m_1, m_2, m_3-1, m-3; q) \\ & + P_0(m_1, m_2, m_3-1, m-3; q) \\ & + P_1(m_1, m_2, m_3-1, m-4; q) \Big] \end{aligned}$$

We now prove (4.5). Let β be a base partition counted by $P_1(m_1, m_2, m_3, m+1; q)$. β has the form in $(**)$. If we delete the largest pair $[m, m+1]$, the remaining partition may have one of the following forms:

$$\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1 \text{ pairs of two repeating parts} \\ m_2-2 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m-1, m], \underbrace{m, m+2, \ldots, m+2n_{12}-2}_{n_{12} \text{ moveable singletons}} \quad \text{(d)}$$

$$\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1-1 \text{ pairs of two repeating parts} \\ m_2-1 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m-1, m-1], \underbrace{m, m+2, \ldots, m+2n_{12}-2}_{n_{12} \text{ moveable singletons}} \quad \text{(e)}$$

$$\underbrace{\cdots\cdots\cdots\cdots\cdots\cdots}_{\substack{m_1 \text{ pairs of two repeating parts} \\ m_2-2 \text{ pairs of two consecutive parts} \\ m_3 \text{ blocks}}}, [m-2, m-1], \underbrace{m-1, m+1, \ldots, m+2n_{12}-3}_{n_{12} \text{ moveable singletons}} \quad \text{(f)}$$

The base partitions in (d), (e) and (f) are counted by $P_1(m_1, m_2-1, m_3, m; q)$, $P_0(m_1, m_2-1, m_3, m; q)$ and $P_1(m_1, m_2-1, m_3, m-1; q)$, respectively. Therefore, we have:

$$\begin{aligned} P_1(m_1, m_2, m_3, m+1; q) = \underbrace{q^{2m+1}}_{\substack{\text{for the deleted} \\ \text{pair } [m, m+1]}} \Big[& P_1(m_1, m_2-1, m_3, m; q) \\ & + P_0(m_1, m_2-1, m_3, m; q) \\ & + P_1(m_1, m_2-1, m_3, m-1; q) \Big] \end{aligned}$$

We note that in the polynomials $P(m_1, m_2, m_3, m+1; q)$, we ignore the weights of the moveable singletons, but if they exist, they have the forms in the streaks coloured blue. We recall that $P_0(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m]$ and $P_1(m_1, m_2, m_3, m+1; q)$ is the generating function of the base partitions in which the largest pair is $[m, m+1]$. $P_0(0, 0, 0, 1; q)$ is the generating function of the base partitions in which the largest pair is $[0, 0]$, but since we take $m_1, m_2, m_3 = 0$, we set $P_0(0, 0, 0, 1; q) = 1$ which counts the base partition $\beta = 1$ with only one moveable singleton. $P_1(0, 0, 0, 1; q)$ is the generating function of the base partitions in which the largest pair is $[0, 1]$, where $m_1, m_2, m_3 = 0$. Therefore, we set $P_1(0, 0, 0, 1; q) = 0$. The other initial conditions in (4.6) are obvious.

The polynomials $P_0(m_1, m_2, m_3, m+1; q)$ and $P_1(m_1, m_2, m_3, m+1; q)$ satisfy the functional equations (4.4) and (4.5) and the initial values of the polynomials are 0 or 1, (4.6). Therefore, $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients.

□

By Lemma 4.2, we have:

$$\sum_{\substack{n_1,n_2\geq 0\\ n_1=n_{11}+n_{12}\\ \beta}} \frac{q^{|\beta|}t^{2n_2+n_1}}{(q;q)_{n_{12}}(q^3;q^3)_{n_2}}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q)q^{mn_{12}+n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q;q)_{n_{12}}(q^3;q^3)_{m_1+m_2+2m_3}} \tag{4.7}$$

Combining (4.7) and Theorem 4.1, we have the following theorem:

Theorem 4.3 *For $n,m \in \mathbb{N}$, let $h(n,m)$ denote the number of partitions of n into m parts such that each part appears at most twice. Then,*

$$H(t;q) = \sum_{m,n\geq 0} h(n,m)q^n t^m$$
$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q)q^{mn_{12}+n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q;q)_{n_{12}}(q^3;q^3)_{m_1+m_2+2m_3}}.$$

where $P(m_1,m_2,m_3,m+1;q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 4.2.

Theorem 4.4 *Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) Odd parts do not repeat.

(c) For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.

(d) $2+2$ is not allowed as a sub-partition.

For $n,m \in \mathbb{N}$, let $kr_1(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d). Then,

$$\sum_{m,n\geq 0} kr_1(n,m)q^n t^m$$
$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j,k\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+4j+(2m_1+2m_2+5m_3+n_{12}+i+2j+k)^2}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$
$$\times\, t^{2m_1+2m_2+5m_3+n_{12}+i+2j+k} \tag{4.8}$$

where $P(m_1,m_2,m_3,m+1;q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 4.2. Moreover, the generating function (4.8) is an evidently positive series.

Proof: We have:

$$A(t;q;2) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)} \qquad \text{(by Proposition 3.2)}$$

$$= \left(\prod_{n=1}^{\infty}(1+tq^{2n}+t^2q^{4n})\right) \prod_{n=1}^{\infty} \frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)}$$

$$= H_0(t;q) \prod_{n=1}^{\infty} \frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)} \qquad \text{(by (4.1))}$$

$$= H(t;q^2) \prod_{n=1}^{\infty} \frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})} \cdot \frac{1}{(1-t)}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}} \qquad \text{(by Theorem 4.3)}$$

$$\times \frac{1}{(tq;q^2)_\infty(t^2q^4;q^4)_\infty} \cdot \frac{1}{(1-t)}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}} \qquad \text{(by (2.1))}$$

$$\times \left(\sum_{i\geq 0}\frac{(tq)^i}{(q^2;q^2)_i}\right) \cdot \left(\sum_{j\geq 0}\frac{(t^2q^4)^j}{(q^4;q^4)_j}\right) \cdot \left(\sum_{k\geq 0} t^k\right)$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j,k\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+4j}t^{2m_1+2m_2+5m_3+n_{12}+i+2j+k}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \tag{4.9}$$

We now add the weight of the base partition β to the exponent of q in (4.9). The exponent of t keeps track of the number of parts in the partition. Therefore, we need to consider the base partition β with $2m_1+2m_2+5m_3+n_{12}+i+2j+k$ parts, namely $\beta = 1+3+5+\ldots+2(2m_1+2m_2+5m_3+n_{12}+i+2j+k)-1$. The weight of β is $(2m_1+2m_2+5m_3+n_{12}+i+2j+k)^2$. It follows that

$$\sum_{m,n\geq 0} kr_1(n,m)q^nt^m$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j,k\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+4j}t^{2m_1+2m_2+5m_3+n_{12}+i+2j+k}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$

$$\times q^{(2m_1+2m_2+5m_3+n_{12}+i+2j+k)^2}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j,k\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+4j+(2m_1+2m_2+5m_3+n_{12}+i+2j+k)^2}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$

$$\times t^{2m_1+2m_2+5m_3+n_{12}+i+2j+k}$$

This proves (4.8). The polynomials $P(m_1, m_2, m_3, m+1; q)$'s have evidently positive coefficients by Lemma 4.2. Therefore, the generating function (4.8) is an evidently positive series. $\square$

Theorem 4.5 *Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) Odd parts do not repeat.

(c) For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.

(d') 1 is not allowed to appear as a part.

For $n, m \in \mathbb{N}$, let $kr_2(n, m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d'). Then,

$$\sum_{m,n\geq 0} kr_2(n,m)q^n t^m$$
$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$
$$\times t^{2m_1+2m_2+5m_3+n_{12}+i+2j} \tag{4.10}$$

where $P(m_1, m_2, m_3, m+1; q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 4.2. Moreover, the generating function (4.10) is an evidently positive series.

Proof: We have:

$$B(t;q;2) = \prod_{n=1}^{\infty} \frac{(1+tq^{2n}+t^2q^{4n})}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t^2)} \qquad \text{(by Proposition 3.4)}$$
$$= \left(\prod_{n=1}^{\infty}(1+tq^{2n}+t^2q^{4n})\right)\prod_{n=1}^{\infty}\frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t^2)}$$
$$= H_0(t;q)\prod_{n=1}^{\infty}\frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t^2)} \qquad \text{(by (4.1))}$$
$$= H(t;q^2)\prod_{n=1}^{\infty}\frac{1}{(1-tq^{2n-1})(1-t^2q^{4n})}.\frac{1}{(1-t^2)}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}} \quad \text{(by Theorem 4.3)}$$

$$\times \frac{1}{(tq;q^2)_\infty(t^2;q^4)_\infty}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12}\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2}t^{2m_1+2m_2+5m_3+n_{12}}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}} \quad \text{(by (2.1))}$$

$$\times \left(\sum_{i\geq 0}\frac{(tq)^i}{(q^2;q^2)_i}\right)\cdot\left(\sum_{j\geq 0}\frac{(t^2)^j}{(q^4;q^4)_j}\right)$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i}t^{2m_1+2m_2+5m_3+n_{12}+i+2j}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \tag{4.11}$$

We now add the weight of the base partition β to the exponent of q in (4.11). The exponent of t keeps track of the number of parts in the partition. Therefore, we need to consider the base partition β with $2m_1 + 2m_2 + 5m_3 + n_{12} + i + 2j$ parts, namely $\beta = 1 + 3 + 5 + \ldots + 2(2m_1 + 2m_2 + 5m_3 + n_{12} + i + 2j) - 1$. The weight of β is $(2m_1 + 2m_2 + 5m_3 + n_{12} + i + 2j)^2$. It follows that

$$\sum_{m,n\geq 0} kr_2(n,m)q^n t^m$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i}t^{2m_1+2m_2+5m_3+n_{12}+i+2j}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$

$$\times q^{(2m_1+2m_2+5m_3+n_{12}+i+2j)^2}$$

$$= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j}$$

$$\times t^{2m_1+2m_2+5m_3+n_{12}+i+2j}$$

This proves (4.10). The polynomials $P(m_1, m_2, m_3, m + 1; q)$'s have evidently positive coefficients by Lemma 4.2. Therefore, the generating function (4.10) is an evidently positive series. □

Theorem 4.6 *Consider the partitions satisfying the following conditions:*

(a) No consecutive parts allowed.

(b) Odd parts do not repeat.

(c) For a contiguous sub-partition $\lambda_i + \lambda_{i+1} + \lambda_{i+2}$, we have $|\lambda_i - \lambda_{i+2}| \geq 4$ if λ_{i+1} is even and appears more than once.

(d″) 1, 2 and 3 are not allowed to appear as parts.

For $n, m \in \mathbb{N}$, let $kr_3(n,m)$ denote the number of partitions of n into m parts such that the partitions satisfy the conditions (a), (b), (c) and (d"). Then,

$$\begin{aligned}
&\sum_{m,n\geq 0} kr_3(n,m)q^n t^m \\
&= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \\
&\times q^{2mn_{12}+2n_{12}^2+3i+4j+4m_1+4m_2+10m_3+2n_{12}+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2} t^{2m_1+2m_2+5m_3+n_{12}+i+2j}
\end{aligned} \tag{4.12}$$

where $P(m_1,m_2,m_3,m+1;q)$'s are the polynomials of q with evidently positive coefficients constructed in Lemma 4.2. Moreover, the generating function (4.12) is an evidently positive series.

Proof: In the proof of Theorem 3.5, we showed that there is a 1-1 correspondence between the partitions enumerated by $kr_2(n,m)$ and the partitions enumerated by $kr_3(n,m)$. To construct an evidently positive series as the generating function of the partitions enumerated by $kr_3(n,m)$, we let $t^m \mapsto t^m q^{2m}$ in (4.10) and we get:

$$\begin{aligned}
&\sum_{m,n\geq 0} kr_3(n,m)q^n t^m \\
&= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)q^{2mn_{12}+2n_{12}^2+i+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2}}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \\
&\times (tq^2)^{2m_1+2m_2+5m_3+n_{12}+i+2j} \\
&= \sum_{\substack{m_1,m_2,m_3,\\ m,n_{12},i,j\geq 0}} \frac{P(m_1,m_2,m_3,m+1;q^2)}{(q^2;q^2)_{n_{12}}(q^6;q^6)_{m_1+m_2+2m_3}(q^2;q^2)_i(q^4;q^4)_j} \\
&\times q^{2mn_{12}+2n_{12}^2+3i+4j+4m_1+4m_2+10m_3+2n_{12}+(2m_1+2m_2+5m_3+n_{12}+i+2j)^2} t^{2m_1+2m_2+5m_3+n_{12}+i+2j}
\end{aligned}$$

This proves (4.12). The polynomials $P(m_1,m_2,m_3,m+1;q)$'s have evidently positive coefficients by Lemma 4.2. Therefore, the generating function (4.12) is an evidently positive series. □

CHAPTER 5

Conclusion and Remarks

In Chapter 3, we presented an alternative proof for the construction of generating functions (3.1), (3.3) and (3.5). Those generating functions are not new. They are also constructed by Kanade and Russell [7] by using jagged partitions, 2-staircases and the q-series identities due to Euler. In our alternative construction, we used ordinary partitions instead of jagged partitions. In this sense, our construction is more elementary.

In Chapter 4, we constructed evidently positive series as the generating functions of the partitions in Theorem 4.4, Theorem 4.5 and Theorem 4.6. The partitions described in Theorem 4.4, Theorem 4.5 and Theorem 4.6 are exactly the same partitions in Theorem 3.1, Theorem 3.3 and Theorem 3.5, respectively. In the construction of the generating functions (3.1) and (3.3), in both of them, the following infinite product appeared:

$$H_0(t;q) = \prod_{n=1}^{\infty}(1 + tq^{2n} + t^2q^{4n}).$$

We realized that $H_0(t;q)$ is the generating function of partitions with parts occurring at most twice and all parts are even. Then, we aimed to construct an evidently positive series for $H_0(t;q)$ and use that series for constructing new generating functions. We simplified the problem a little bit and instead of $H_0(t;q)$, we studied with

$$H(t;q) = \prod_{n=1}^{\infty}(1 + tq^{n} + t^2q^{2n}),$$

which is the generating function of partitions with parts occuring at most twice and it satisfies the equation $H_0(t;q) = H(t;q^2)$. We first constructed an evidently positive series for $H(t;q)$ in Theorem 4.3. Then, by using the relation between $H_0(t;q)$ and $H(t;q)$, we obtained an evidently positive series for $H_0(t;q)$. All these provided us to construct the generating functions (4.8), (4.10) and (4.12). Since we use the polynomials, namely $P(m_1, m_2, m_3, m+1; q)$'s defined in Lemma 4.2, the generating functions (4.8), (4.10) and (4.12) are evidently positive. We had to consider the polynomials for the construction of a formula for the weight of the base partition β. Because, for a fixed number of pairs and singletons, there exist different base partitions. It would be better if we were able to write a closed formula for the polynomial $P(m_1, m_2, m_3, m+1; q)$, but at least we know the functional equations concerning those polynomials. Better than that would be constructing a monomial of q as the generating function of the base partitions β's with a given number of pairs and singletons. It is a quite interesting question for us and we intend to study on it in the future.

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