RATIONAL PREPERIODIC POINTS OF CUBIC POST-CRITICALLY FINITE POLYNOMIALS

by

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Abstract

All cubic post-critically finite polynomials defined over $\mathbb{Q}$ are classified up to $\overline{\mathbb{Q}}$-conjugacy, [3]. There are 15 such classes. For each representative of these classes we find all rational preperiodic points by using an algorithm of B. Hutz [8]. We find that the number of rational preperiodic points of a cubic post-critically finite polynomial defined over $\mathbb{Q}$ cannot exceed 6.

ÜÇÜNCÜ DERECEDEN KRİTİK-ÖTESİ SONLU POLİNOMLARIN RASYONEL PREPERİYODİK NOKTALARI

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Özet

Bütün $\mathbb{Q}$ üzerinde tanımlı üçüncü dereceden kritik ötesi sonlu polinomların $\overline{\mathbb{Q}}$-eşleniklerine göre sınıflandırılması [3]'da yapılmıştır. Toplam 15 eşlenik sınıfı vardır. B. Hutz'ın [8] bir algoritmasını kullanarak, bu eşlenik sınıflarından seçilmiş bir temsilcinin bütün rasyonel preperiyodik noktalarını bulacağız. Bir $\mathbb{Q}$ üzerinde tanımlı üçüncü dereceden kritik ötesi sonlu polinomun en fazla 6 rasyonel preperiyodik noktası olabileceğini söyleyeceğiz.

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Contents

Introduction

In this thesis, we are investigating discrete dynamical systems attached to a rational map. A discrete dynamical system consists of a set S and a self-map ϕ on S where iteration is permitted. We define the n-th iteration of ϕ as the composition of ϕ with itself n times. In arithmetic dynamics we are concerned about the behaviour of points under some iterations of a given self-map. Here, we refer to the definition of periodic and preperiodic points: a point α is a preperiodic point under ϕ if $\phi^{n+m}(\alpha) = \phi^m(\alpha)$ for some $n \geq 1$, $m \geq 0$. If $m = 0$ then we say that α is a periodic point. Since n is a positive integer then one may ask the following question: for given integers $n \geq 1$ and $d \geq 1$ is there any map of degree d which has periodic point of exact period n under ϕ?

In arithmetic dynamics the sets maybe chosen to be a field K and ϕ is chosen to be in $K(x)$. The same question above is valid for this field and rational map. Furthermore, finiteness of preperiodic points occupies an important place in arithmetic dynamics. If we restrict our attention to the rational maps defined over a number field K, by Northcott's Theorem [18] there are only finitely many rational preperiodic points under these maps. Since the set of preperiodic points is finite, we may investigate the existence of a uniform bound on the size of this set that depends on the degree of ϕ. In 1994, P. Morton and J.Silverman [17] conjectured that the number of K-rational preperiodic points of a rational map defined over a number field K is bounded by some constant that depends on the degree of this rational map. Until now, we have limited knowledge about this conjecture. In 1995, Poonen [19] states that a quadratic polynomial defined over $\mathbb{Q}$ can have at most 9 rational preperiodic points under the assumption that a quadratic polynomial cannot have a periodic point of exact period greater than 4. Until now it is proven that a quadratic polynomial defined over $\mathbb{Q}$ cannot have a periodic point of exact period 4, 5 and 6 in [15], [7] and [22] respectively, whereas the case $n = 6$ is proven under the assumption that Birch-Swinnerton Dyer conjecture is true.

A polynomial is post-critically finite if each critical point is a preperiodic point. In [13], the authors find out that there are 12 $\overline{\mathbb{Q}}$-conjugacy classes of quadratic post-critically finite rational maps defined over $\mathbb{Q}$. Moreover, they managed to find that a quadratic post-critically finite rational map can have at most 6 rational preperiodic points.

In this thesis we will be focusing on cubic post-critically finite polynomials defined over $\mathbb{Q}$. There are 15 $\overline{\mathbb{Q}}$-conjugacy classes of cubic post-critically finite polynomials which are classified and listed here [3]. Our main work is based on summarizing the classification results and then finding the rational preperiodic points of these polynomials using an algorithm of B. Hutz, see [8]. This algorithm was first introduced by J. Silverman [20] for local fields with non-archimeden absolute value.

Later on, B. Hutz gave a version of this algorithm for number fields.

In chapter 1 we will give some preliminary definitions. In chapter 2 we will show how to classify cubic unicritical and bicritical post-critically finite polynomials by finding some archimedean and non-archimedean bounds on the coefficients of these polynomials. According to these bounds only finitely many cubic polynomials are obtained. Here, authors of [3] use Sage to determine whether such polynomial with determined coefficients is post-critically finite or not. For cubic bicritical post-critically finite polynomials with rational critical points there are 11 conjugacy classes out of 126 candidates and for cubic bicritical post-critically finite polynomials with irrational critical points there are 2 conjugacy classes out of 23828 candidates. There are only 2 conjugacy classes of cubic unicritical post-critically finite polynomials.

In chapter 3, we will introduce Hutz' algorithm with an example to show how one can apply this algorithm. In fact this example is a representative of one of the conjugacy classes determined in chapter 2.

In chapter 4, we will give all calculations and some results deduced from these calculations. We found that a cubic post-critically finite polynomial defined over $\mathbb{Q}$ can have at most 6 rational preperiodic points. Furthermore, the exact period of a periodic points of a cubic post-critically finite polynomial can be at most 3. Moreover, a preperiodic point of such polynomial is of type (m, n) where $m \in \{1, 2\}$ and $n \in \{1, 2, 3\}$ but $(m, n) \neq (2, 2)$ or $(3, 3)$.

Chapter 1

Preliminaries

A *(discrete) dynamical system* consists of a set S and a self-map $\phi : S \to S$. We denote the dynamical system as $(S,\ \phi)$. If the set S is known, we basically denote it by ϕ. We define the n-th iteration of this map to be $\phi^n = \phi \circ \phi \circ ... \circ \phi$, i.e, composition of ϕ with itself n-times. Conventionally, ϕ^0 is the identity map on S and $\phi^1 = \phi$.

Let $\alpha \in S$, the (forward) *orbit of* α is the set

$$\mathcal{O}_\phi(\alpha) = \mathcal{O}(\alpha) = \{\phi^n(\alpha) \mid n \in \mathbb{Z}_{\geq 0}\}$$

The (backward) *orbit of* α is the set

$$\mathcal{O}_\phi^{-1}(\alpha) = \mathcal{O}^{-1}(\alpha) = \{\phi^{-n}(\alpha) \mid n \in \mathbb{Z}_{\geq 0}\}$$

Throughout this thesis, we will be only interested in forward orbits. Therefore, to fix the notation, we will only use *orbit* to indicate forward orbit.

In arithmetic dynamics, one of the main objects to study is orbits and finiteness property of these orbits. Asking when an orbit is finite is very natural.

If $\phi^n(\alpha) = \alpha$ for some $n \geq 1$ then the point α is a point of period n under ϕ. If this n is the smallest integer then the exact period of α is n. If $\phi(\alpha) = \alpha$, i.e., α is a point of period 1 under ϕ, then we say α to be a fixed point under ϕ. If $\phi^m(\alpha) = \phi^{m+n}(\alpha)$ for some $n \geq 1,\ m \geq 0$ then the point α is preperiodic under ϕ, we often say α is a preperiodic point under ϕ of type (m, n). Notice that every periodic point is a preperiodic point by setting $m = 0$ where $\phi^n(\alpha) = \phi^0(\alpha) = \alpha$. We denote sets of periodic and preperiodic points of ϕ respectively as follows

$$Per_S(\phi) = \{\alpha \in S \mid \phi^n(\alpha) = \alpha \ for\ some\ n \geq 1\}$$

$$PrePer_S(\phi) = \{\alpha \in S \mid \phi^m(\alpha) = \phi^{m+n}(\alpha) \ for\ some\ n \geq 1,\ m \geq 0\}.$$

The orbit of a preperiodic (or periodic) point under a map is finite. If the orbit of a point α under a map ϕ is infinite then we say that α is a wandering point of ϕ.

Notation: If the set S is known then we denote above sets as $Per(\phi)$ and $PrePer(\phi)$.

For a finite set S and a map $\phi : S \to S$, one can easily see that every point of S is preperiodic under ϕ.

Example 1.0.1. *[20, Example 0.1]*

Now, we may ask how big $Per_S(\phi)$ and $PrePer_S(\phi)$ are.

If we take $S = \mathbb{F}_p$ where p is a prime number and a polynomial $\phi(x) = x^p \in \mathbb{F}_p[x]$ then

$$Per_{\mathbb{F}_p}(x^p) = \mathbb{F}_p$$

Moreover, the exact period of any point $x \in \mathbb{F}_p$ under ϕ is 1. In fact, it is the Fermat's Little Theorem in a dynamical context.

Example 1.0.2. *Let $\phi(x) = x^2 - 3/4 \in \mathbb{Q}[x]$. The periodic points under ϕ can be found using the following Sage code [21]:*

```
[language= Python]
P.<x,y> = ProjectiveSpace(QQ,1)
f = DynamicalSystem([x^2-3/4*y^2,y^2])
f.rational_periodic_points()
```

This Sage code is based on an efficient algorithm that finds periodic and preperiodic points under a given rational map which we will introduce in Chapter 3. The output is the set $\{(-1/2 : 1), (3/2 : 1), (1 : 0)\}$ where $(1 : 0)$ is the point at infinity. Now, we may find periods of these points.

$$\phi(-1/2) = (-1/2)^2 - 3/4 = -2/4 = -1/2$$

Then $-1/2$ is a fixed point under ϕ.

$$\phi(3/2) = (3/2)^2 - 3/4 = 6/4 = 3/2$$

Then $3/2$ is a fixed point under ϕ.

These are only periodic points under ϕ. Now, if we change last phrase of the code for the same polynomial to be

```
f.rational_preperiodic_points()
```

The output is the set $\{(-1/2 : 1), (3/2 : 1), (-3/2 : 1), (1/2 : 1), (1 : 0)\}$ where $(1 : 0)$ is the point at infinity. There is an intersection with this set and the set of periodic points above. It is because every periodic point under a given polynomial is a preperiodic point under the same polynomial. We may call these fixed points to be preperiodic points of type $(0, 1)$. Now we can calculate types of these preperiodic points calculated above.

As we explained before, the points $(-1/2 : 1)$ and $(3/2 : 1)$ are both preperiodic points of type $(0, 1)$, i.e, fixed points under ϕ.

$$\phi(-3/2) = (-3/2)^2 - 3/4 = 6/4 = 3/2$$
$$\phi(3/2) = (3/2)^2 - 3/4 = 6/4 = 3/2$$

Since $\phi \circ \phi(-3/2) = \phi(-3/2)$ *then the point* $-3/2$ *is a preperiodic point under* ϕ *of type* $(1,1)$*.*

$$\phi(1/2) = (1/2)^2 - 3/4 = -2/4 = -1/2$$
$$\phi(-1/2) = (-1/2)^2 - 3/4 = -2/4 = -1/2$$

Since $\phi \circ \phi(1/2) = \phi(1/2)$ *then the point* $1/2$ *is a preperiodic point under* ϕ *of type* $(1,1)$*.*

For instance, the orbit of the point $x = 2$ *under* $\phi(x) = x^2 - 3/4 \in \mathbb{Q}[x]$ *is infinite since the cardinality of its orbit* $\mathcal{O}_f(2) = \{2, 13/4, 157/6, ...\}$ *is going to infinity as the ratio between every two consecutive term is greater than* 2*.*

In arithmetic dynamics, one of the main questions is to classify orbits of polynomials. Let K be a field with zero characteristic and $\phi(x) \in K[x]$ be a polynomial of degree d. When $d = 1$, a linear polynomial $\phi(x) = ax + b$ where $a, b \in K$ with $a \neq 0$ has periodic points of every period n given by the formula

$$x = \frac{-b\sum_{k=0}^{n-1}(a^k)}{a^n - 1}$$

where $a \neq \pm 1$.

It can be proven by induction on n. Let assume that this formula is true for $n-1$, i.e, there is a periodic point $\alpha \in K$ under $\phi(x) \in K[x]$ given by the formula

$$\alpha = \frac{-b\sum_{k=0}^{n-2}(a^k)}{a^{n-1} - 1}$$

It is obtained by equalizing $\phi^{n-1}(x) = a^{n-1}x + a^{n-2}b + ... + ab + b = x$. Hence if we apply ϕ once again to ϕ^{n-1} we obtain

$$\phi(\phi^{n-1}(x)) = a(a^{n-1}x + a^{n-2}b + ... + ab + b) + b = a^n x + a^{n-1}b + a^{n-2}b + ... + a^2 b + ab + b$$

Again, $\phi^n(x) = x$ gives us the desired result.

When $a = 1$, $\phi^n(x) = x + nb$. If we search for its periodic points, we obtain that $x + nb = x$, then $nb = 0$. Since $n \geq 1$ then $b = 0$. It means that whenever a polynomial $\phi(x) = x + b$ has a periodic point then $b = 0$. Hence the only polynomial which has periodic points of the form $\phi(x) = x + b$ is $\phi(x) = x$.

When $a = -1$,

$$\phi^n(x) = \begin{cases} x, & if\ n\ is\ even \\ -x + b, & if\ n\ is\ odd \end{cases}$$

Therefore, for even n, every point is periodic under ϕ. For odd case periodic points are points of the form $x = b/2$.

In fact, If $x \in K$ is a preperiodic point of type (m, n) where $n \geq 1$ and $m \geq 0$ under $\phi(x) = ax + b$ then

$$\phi^{m+n}(x) = \phi^m(x)$$

If we expand both sides

$$a^{m+n}x + a^{m+n-1}b + ... + ab + b = a^m x + a^{m-1}b + ... + ab + b$$

If we subtract same terms on both sides

$$a^{m+n}x + a^{m+n-1}b + ... + a^m b = a^m x$$

Since $a \neq 0$ we obtain

$$a^n x + a^{n-1}b + ... + b = x$$

Hence, x is a point of period n under ϕ. We already found periodic points of a linear polynomial.

In what follows we will be interested in polynomials with $d \geq 2$. Later on, in Chapter 2 we will be considering polynomials defined over a number field K.

Definition 1.0.3. *A number field is a finite extension of degree d of the field of rational numbers $\mathbb{Q}$. The degree d of K is the degree of the extension $d = [K : \mathbb{Q}]$.*

1.1 Conjugacy Classes

A polynomial of fixed degree enjoying a certain property can be classified into finitely many conjugacy classes. It would be more practical to study a certain dynamical property of these polynomials by discussing only representatives of these classes.

Let K be a field. Let $M_{m,n}(K)$ be the set of $m \times n$ matrices with entries in the field K. We denote their entries as $a_{i,j}$ where $1 \leq i \leq m$, $1 \leq j \leq n$ and (i, j) indicates the position of $a_{i,j}$. If the dimension of a matrix is $n \times n$ then we denote the set as $M_n(K)$, [4].

The set $M_{m,n}(K)$ forms a group under addition but does not form a group under multiplication since it is not closed under multiplication. However, $M_n(K)$ forms a (not generally commutative) ring under matrix addition and matrix multiplication. The additive identity is the $n \times n$ zero matrix.

$$0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

The multiplicative identity is the $n \times n$ matrix I where its entries are 0 if $i \neq j$ and 1 if $i = j$.

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Now, we recall the *determinant* of a matrix defined by $det : M_n(K) \to K$ defined by $A \longmapsto det(A)$ satisfying the following properties.

Proposition 1.1.1. *Let $A, B \in M_n(K)$ be matrices of dimension $n \times n$.*

- $det(AB) = det(A)det(B)$.
- $det(I) = 1$.
- *The matrix A is invertible if and only if $det(A) \neq 0$.*

The *General Linear Group* is defined as the set of invertible matrices, i.e,

$$GL_n(K) = \{A \in M_n(K) \mid det(A) \neq 0\}.$$

The *Special Linear Group* is defined as the set of invertible matrices which have determinant 1, i.e,

$$SL_n(K) = \{A \in M_n(K) \mid det(A) = 1\}.$$

The aforementioned sets are groups under the matrix multiplication. Moreover $SL_n(K)$ is a subgroup of $GL_n(K)$; $SL_n(K) \leq GL_n(K)$.

The *projective general linear group* and *projective special linear group* are defined as the quotient groups $GL_n(K)/\{\lambda I \mid \lambda \in K\} := PGL_n(K)$ and $SL_n(K)/\{\lambda I \mid \lambda \in K\} := PSL_n(K)$, respectively. Throughout this thesis, we will be only using $PGL_n(K)$.

A *Möbius transformation* from K to itself is given by

$$z \longmapsto \frac{az + b}{cz + d}$$

with $a, b, c, d \in K$ and $ad - bc \neq 0$. This transformation in fact defines an automorphism of $\mathbb{P}^1(K)$. Moreover, it corresponds to an invertible 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

since $ad - bc \neq 0$. A composition of two Möbius transformations corresponds to multiplication of their corresponding matrices. If two matrices A, B give the same transformation then $A = \lambda B$ where $\lambda \in \mathbb{C}$ and vice versa. Moreover, $Aut(\mathbb{P}^1(K)) = PGL_2(K)$ [20, Page 10].

Definition 1.1.2. *[6] Let G be a group and S be a set. A group action of G on S is a map $G \times S \to S$ defined as $(g, s) \longmapsto g \cdot s$ for all $g \in G$ and $s \in S$ with the following properties:*

1. *$g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$, for all $g_1, g_2 \in G$ and $s \in S$.*

2. *$1 \cdot s = s$, for all $s \in S$.*

Let G be a group acting on itself. The *conjugation* action is defined as follows; $g \cdot a = gag^{-1}$ for all $g, a \in G$. We say that two elements $a, b \in G$ are *conjugate* if there exists an element $g \in G$ such that $b = gag^{-1}$. Hence, this is an equivalence relation since it is reflexive $b = gbg^{-1}$ for $g = 1$, symmetric since if $b = gag^{-1}$ then $g^{-1}b(g^{-1})^{-1} = a$ and transitive since if $a = g_1 b g_1^{-1}$ and $b = g_2 c g_2^{-1}$ then $a = (g_2 g_1) c g_2 g_1^{-1}$. Equivalence classes of this relation are called *conjugacy classes* denoted by $C(a) = \{b \in G \mid b = gag^{-1} \; for \; all \; g \in G\}$.

Definition 1.1.3. *Let $f, g \in K(x)$ be two rational maps. f and g are linearly conjugate over $\overline{K}$ if there exists $T(x) = ax + b \in PGL_2(\overline{K})$ such that $f^T(x) = T \circ f \circ T^{-1}(x) = g(x)$.*

Example 1.1.4. *Let $f(x) = 3x^3 + 2 \in \mathbb{Q}[x]$ and $T(x) = x/2 \in PGL_2(\mathbb{Q})$ with the inverse $T^{-1}(x) = 2x$. then $g(x) = f^T(x) = T \circ f \circ T^{-1}(x) = T \circ f(2x) = T(2^3 \cdot 3 \cdot x^3 + 2) = 2^2 \cdot 3 \cdot x^3 + 1$. Hence, $f(x) = 3x^3 + 2$ and $g(x) = 12x^3 + 1$ are linearly conjugates.*

The Multiplier

Assume K is an algebraically closed field. Let $\alpha \in K$ and $\phi \in K(x)$ be a rational map then the multiplier of ϕ at α is the first derivative

$$\lambda_\alpha(\phi) = \phi'(\alpha).$$

In what follows, we are going to show that the multiplier of a rational map ϕ defined over $\mathbb{C}$ at a fixed point is invariant under linear conjugacy.

Proposition 1.1.5. *[20] Let $\phi \in \mathbb{C}(x)$ be a rational map and $\alpha \in \mathbb{C}$ with $\phi(\alpha) = \alpha$. Let $g \in PGL_2(\mathbb{C})$ and β be a point in the pre-image of α under g. It implies that $\phi^g(\beta) = g^{-1} \circ \phi \circ g(\beta) = \beta$ then*

$$\phi'(\alpha) = (\phi^g)'(\beta).$$

Proof. Its proof is straightforward since we just apply the chain rule

$$(\phi^g)'(x) = (g^{-1} \circ \phi \circ g)'(x) = (g^{-1})'(\phi(g(x)))\phi'(g(x))g'(x)$$

Then if we evaluate this equation at $x = \beta$,

$$\begin{aligned} (\phi^g)'(\beta) &= (g^{-1})'(\phi(\alpha))\phi'(\alpha)g'(\beta) && (1) \\ &= (g^{-1})'(\alpha)\phi'(\alpha)g'(\beta) && (2) \\ &= (g^{-1})'(g(\beta))\phi'(\alpha)g'(\beta) && (3) \\ &= \phi'(\alpha) && (4) \end{aligned}$$

(1) Since β is in the pre-image of α under g.
(2) Since α is a fixed point of ϕ.
(3) Since α is an image of β under g.
(4) As we can see that the terms in the latter equation except $\phi'(\alpha)$ is the derivative of the composition $(g^{-1} \circ g)'(z)$ and it is 1. $\square$

The multiplier of ϕ at its fixed point α, $\lambda_\alpha(\phi)$, is well-defined within a $PGL_2(\mathbb{C})$ linear conjugacy classes of ϕ. Next question to consider is whether the multiplier of ϕ^n at α is well-defined under linear conjugacy?

If α is a periodic point of ϕ with the exact period n, then α is a fixed point of the map ϕ^n. We define the multiplier of ϕ at α when $\phi^n(\alpha) = \alpha$ to be

$$\lambda_\alpha(\phi) = (\phi^n)'(\alpha)$$

This can be shown using the chain rule again to $(\phi^n)'(\alpha)$.

$$(\phi^n)'(\alpha) = \phi'(\alpha) \cdot \phi'(\phi(\alpha)) \cdot \phi'(\phi^2(\alpha))...\phi'(\phi^{n-1}(\alpha))$$

Remark 1.1.6. *In other words, as in the previous proposition, the multiplier of a map at a periodic point is invariant under the linear conjugation by $T(x) = ax + b \in PGL_2(K)$.*

An observation to make is that if the absolute value of $\lambda_\alpha(\phi)$ is less than 1, then a small neighborhood of α will get smaller every time we touch to α. Otherwise, it will get bigger.

Now, we classify multipliers over $\mathbb{C}$.

Definition 1.1.7. *[20] Let α be a point of exact period n for a rational function $\phi \in \mathbb{C}(x)$ and $\lambda_\alpha(\phi) = (\phi^n)'(\alpha)$. Then*

1. *α is called superattracting if $\lambda_\alpha(\phi) = 0$*
2. *α is called attracting if $|\lambda_\alpha(\phi)| < 1$*
3. *α is called neutral if $|\lambda_\alpha(\phi)| = 1$*
4. *α is called repelling if $|\lambda_\alpha(\phi)| > 1$*

Remark 1.1.8. *If a critical point α of a rational map ϕ is a periodic point then the point α is superattracting since the derivative of ϕ^n at α will be 0. Furthermore, since a rational map of degree d has at most $2d-2$ critical points then it has at most $2d-2$ superattracting points.*

1.2 Good Reduction

As in $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, we can study the dynamics of a map on local fields such as $\mathbb{Q}_p$. We may obtain some fruitful results on dynamics on $\mathbb{Q}$ by using some properties of the dynamics on local fields, i.e, we can make use of local information to obtain insight about global question.

Firstly, we will define archimedean and non-archimedean absolute values.

Definition 1.2.1. *[11] Let K be a field. An absolute value on K is a map*

$$\eta : K \to \mathbb{R}$$

satisfying the following properties:

1. $\eta(\alpha) \geq 0$ *for all* $\alpha \in K$,
2. $\eta(\alpha) = 0$ *if and only if* $\alpha = 0$,
3. $\eta(\alpha\beta) = \eta(\alpha)\eta(\beta)$ *for all* $\alpha, \beta \in K$,
4. $\eta(\alpha + \beta) \leq \eta(\alpha) + \eta(\beta)$ *for all* $\alpha, \beta \in K$.

Notation: For convenience, we will denote the map η as $| \, . \, |$ which is a much more conventional notation.

The pair $(K, | \, . \, |_K)$ consisting of a field K and an absolute value on this field is called a *valued field*.

Definition 1.2.2. *[11] Let K be a valued field with an absolute value $| \, . \, |$. If for all $\alpha, \beta \in K$ the absolute value satisfies*

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$$

then we say the absolute value is nonarchimedean or ultrametric. Hence, the valuation on K

$$v : K^* \to \mathbb{R}$$

where $v(\alpha) = -\log|\alpha|$ is an homomorphism and v satisfies

$$v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}.$$

Definition 1.2.3. *[14] Now we define the p-adic field $\mathbb{Q}_p$ with the absolute value $| \, . \, |_p$ where p is a prime number. Let $x \in \mathbb{Q}$. Absolute value of x is*

1. $|x|_p = 0$ *if* $x = 0$,
2. $x = p^e \frac{a}{b}$ *where* $gcd(a, p) = gcd(b, p) = 1$ *if* $x \neq 0$. *The valuation is* $|x|_p = \frac{1}{p^e}$.

This absolute value $| \, . \, |_p$ satisfies properties $(1) - (4)$ of definition 2, plus the ultrametric property. Therefore, it is a nonarchimedean absolute value.

This construction is for finite prime numbers. For $p = \infty$ we define the absolute value of x as follows

$$|x|_\infty = \max\{x, -x\}$$

Let $| \, . \, |_{p_1}$ and $| \, . \, |_{p_2}$ be two absolute values on $\mathbb{Q}_{p_1}$ and $\mathbb{Q}_{p2}$, respectively. They are said to be equivalent if there exist $k \in \mathbb{Z}_{>0}$ such that $|x|_{p_1} = |x|_{p_2}^k$ for all $x \in \mathbb{Q}$. The following theorem of Ostrowski classifies the absolute values on $\mathbb{Q}$ up to equivalence.

Theorem 1.2.4. *[11, Theorem 1] Every absolute value on $\mathbb{Q}$ is equivalent to an absolute value $| \,.\, |_p$ for $p \neq \infty$ or $| \,.\, |_\infty$.*

As we mentioned before, we may obtain some information by reducing a map modulo a prime p. This way, a standard technique in Arithmetic Geometry is to reduce the map modulo several primes, then try to obtain some information about the map over $\mathbb{Q}$. Hence, now we introduce reduction of maps modulo a prime p.

Definition 1.2.5. *Let $\phi \in K(x)$ be a rational map. We say that ϕ has a good reduction modulo $\mathfrak{p}$ where $\mathfrak{p}$ is a maximal ideal of the ring of integers $\mathbf{O}_K$ of K. Define $\overline{\phi} \in \mathbf{O}_K/\mathfrak{p}$ as the reduced map of ϕ. If $\deg\phi = \deg\overline{\phi}$ where $\overline{\phi}$ is the reduction map of ϕ modulo $\mathfrak{p}$. Otherwise we say that ϕ has a bad reduction.*

Example 1.2.6. *Let $\phi(x) = 2x^3 + 3x + 3 \in \mathbb{Q}[x]$. Then ϕ has a bad reduction modulo $p = 2$ since $\phi(x) = 2x^3 + 3x + 3 \equiv x + 1 \mod 2$ and for $\overline{\phi}(x) = x + 1$ which is the reduced map modulo 2, $\deg\overline{\phi} = 1 \neq 3 = \deg\phi$. However, for $p = 3$, ϕ has a good reduction modulo 3 since $\phi(x) = 2x^3 + 3x + 3 \equiv 2x^3 \mod 3$ and for $\overline{\phi}(x) = 2x^3$ which is the reduced map of ϕ modulo 3, $\deg\overline{\phi} = 3 = \deg\phi$.*

1.3 Dynatomic Polynomials

There are several methods to find periodic points of a polynomial one of them is searching roots of n-th *dynatomic polynomials* as one may link periodic points of a polynomial to the roots of a relevant n-th dynatomic polynomials. Firstly, we will define the Möbius function μ.

$$\mu(n) = \begin{cases} 1 & if\ n = p_1^{\epsilon_1}...p_k^{\epsilon_k}\ where\ \epsilon_i = 1\ for\ all\ i = 1, ..., k\ and\ k\ is\ even \\ -1 & if\ n = p_1^{\epsilon_1}...p_k^{\epsilon_k}\ where\ \epsilon_i = 1\ for\ all\ i = 1, ..., k\ and\ k\ is\ odd \\ 0 & if\ n = p_1^{\epsilon_1}...p_k^{\epsilon_k}\ at\ least\ one\ \epsilon_i \geq 2 \end{cases}$$

Here are some small values of μ: $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$...

The n-th dynatomic polynomial of a polynomial $f \in K[x]$ is defined to be

$$\Phi_{n,f}(x) = \prod_{d|n} (f^d(x) - x)^{\mu(\frac{n}{d})}$$

The formula may suggest that $\Phi_{n,f}(x)$ to be a rational function in $K(x)$ due to possible negative values of the Möbius function μ, but we will show that $\Phi_{n,f}(x) \in K[x]$, a polynomial in $K[x]$.

Theorem 1.3.1. *[16, Theorem 2.4(a)] Let K be a field with char$K = 0$ or p and $f(x) \in K[x]$ be a polynomial. If char$K = 0$ or char$K = p \nmid n$ then*

$$f^n(x) - x = \prod_{d|n} \Phi_{d,f}(x)$$

Proof. We know that

$$\Phi_{n,f}(x) = \prod_{d|n} (f^d(x) - x)^{\mu(\frac{n}{d})}$$

By Möbius Inversion Formula

$$f^n(x) - x = \prod_{d|n} \Phi_{d,f}(x)$$

which is the desired result. □

Theorem 1.3.2. *[16, Theorem 2.5] Let $n \in \mathbb{N}$ and K be a field. Then the n-th dynatomic polynomial $\Phi_{n,f}(x)$ of a polynomial $f(x) \in K[x]$ is a polynomial defined over K.*

Proof. Let $g(x) = a_k x^k + ... + a_1 x + a_0 \in \mathbb{Q}(a_k, ..., a_0)[x]$ be a generic polynomial of degree m defined over $\mathbb{Q}$. Then by Theorem 1.3.1

$$g^n(x) - x = \prod_{d|n} \Phi_{d,g}(x)$$

where $\Phi_{d,g}(x) \in \mathbb{Q}(a_k, ..., a_0)[x]$ for all $d \mid n$. Also, we know that by definition

$$\Phi_{d,g}(x) = \prod_{e|d} (g^e(x) - x)^{\mu(\frac{d}{e})}.$$

By the definition of the Möbius function it is a rational function whose coefficients are in $\mathbb{Z}[a_k, ..., a_0]$, i.e, the smallest ring where the coefficients of $\Phi_{d,g}(x)$ live in. By the factorization of $\Phi_{d,g}(x)$, the leading coefficient is a power of a_k. By taking the leading coefficient a_k as a common factor we may assume that

$$\Phi_{d,g}(x) \in \mathbb{Z}[1/a_k, a_k, ..., a_0, x]$$

So, the leading coefficient is still a power of a_k. Now, we will show that $\Phi_{d,f}(x) \in K[x]$. Let $f(x) = b_k x^k + ... + b_1 x + b_0$ be a polynomial with coefficients in K and $b_k \neq 0$. Then the map

$$h : \mathbb{Z}[1/a_k, a_k, ..., a_0] \to K$$

defined by $a_i \longmapsto b_i$ for $0 \leq i \leq k$ is in fact a homomorphism. Therefore, by sending $x \longmapsto x$ the homomorphism h can be extended to a homomorphism

$$H : \mathbb{Z}[1/a_k, a_k, ..., a_0, x] \to K[x]$$

Using the homomorphism H we see that $g(x) \longmapsto f(x)$ and for all d, $g^d(x) \to f^d(x)$. Hence

$$\Phi_{d,g}(x) \longmapsto \Phi_{d,f}(x)$$

It means that $\Phi_{d,f}(x) \in K[x]$. Hence it is a polynomial. □

As the degree of the n-th dynatomic polynomial depends on n and $\deg f$ then we may compute it by the following formula:

$$\deg \Phi_{n,f} = \sum_{d|n} \mu(\frac{n}{d})(\deg f)^d$$

This formula is coming from the definition of the n-th dynatomic polynomial by applying Möbius Inversion formula to

$$(\deg f)^n = \sum_{d|n} \mu(\frac{n}{d}) \deg \Phi_{d,f}.$$

Example 1.3.3. *Take a polynomial $f(x) = x^2 - 2 \in \mathbb{Q}[x]$. Then*

$$\Phi_{1,f}(x) = f(x) - x = x^2 - 2 - x$$

$$\Phi_{2,f}(x) = (f(x) - x)^{-1}(f^2(x) - x) = x^4 - 2x^3 - 4x^2 + 5x + 4$$

$$\Phi_{3,f}(x) = (f(x)-x)^{-1}(f^3(x)-x) = x^8 - 4x^7 - 4x^6 + 26x^5 + 3x^4 - 54x^3 - 3x^2 + 35x + 10$$

To prove the following proposition, we need to prove a lemma and a corollary.

Lemma 1.3.4. *[16, Lemma 2.1] Let $f(x) \in K[x]$ be a polynomial and $\alpha \in K$.*

1. *If α is a point of period n under f then α is a point of period $n \cdot m$ under f for $m \geq 1$.*
2. *If α is a point of period n under f then α is a point of exact period d under f for d a unique divisor of n.*
3. *If α is a point of exact period n under f then $f^i(\alpha)$ for $0 \leq i \leq n-1$ are points of exact period n under f and are all distinct. In addition, orbits of every point of exact period n under f are all disjoint and are of size n.*
4. *Each $\beta \in \mathcal{O}_f(\alpha)$ generates the same field over K.*

Proof. 1. Suppose that α is a point of period n under f. It means that $f^n(x) = x$. So, for all $m \geq 1$

$$f^{n \cdot m}(\alpha) = f^n(f^n(...(f^n(\alpha)))...) = \alpha$$

. Hence, α is a point of period $n \cdot m$ under f.

2. Suppose that α is a point of exact period n under f. If n is a prime number then α is a fixed point or a point of exact period n under f. If n is a composite number, $n = a \cdot b$. If α is a point of period a then we are done, respectively b. If not then $a = x_1 \cdot x_2$ (respectively $b = y_1 \cdot y_2$). If α is a point of exact period x_1 then we are done, respectively for x_2 and $b = y_1 \cdot y_2$. We continue in the same manner to obtain a minimal period for α, say k where $f^l(\alpha) \neq \alpha$ for $0 < l < k$.

3. Suppose that α is a point of exact period n under f. Then $f^n(f^i(\alpha)) = f^i(f^n(\alpha)) = f^i(\alpha)$. So $f^i(\alpha)$ are points of exact period n under f for $0 \leq i < n$ since n is exact for α. Now, we will show that the points $f^i(\alpha)$ for $0 \leq i < n$ are all distinct. Assume that these points are not distinct, i.e, $f^i(\alpha) = f^j(\alpha)$ for $i \neq j$. So, either $i < j$ or $i > j$. Without loss of generality, assume that $i < j$. We have $f^i(\alpha) = f^n(f^i(\alpha)) = f^n(f^j(\alpha)) = f^j(\alpha)$. Since $i < j$ then the point $f^j(\alpha)$ is not a point of exact period n because $n + i < n + j$.

   The points $f^i(\alpha)$ are all distinct for $0 \leq i < n$. Hence, the orbits $\mathcal{O}_f(f^i(\alpha))$ are all distinct since otherwise two points can be the same and it implies that one of them is not a point of exact period n under f.

4. Let $f^i(\alpha) \in \mathcal{O}_f(\alpha)$ for $i \in \mathbb{Z}^+$. Define $\alpha_i = f^i(\alpha)$ as a polynomial in α over K and define $\alpha = f^{n-i}(\alpha_i)$ as a polynomial in α_i over K. So, α_i can be written as a linear combination of powers of α and α can be written as a linear combination of powers of α_i. Hence this means that they generate the same field.

Corollary 1.3.5. *[16, Corollary 2] Let $f(x) = x^d + a_{d-1}x^{d-1} + ... + a_1x + a_0$ be a generic polynomial of degree d defined over $\mathbb{Q}$ where coefficients of f are independent indeterminates over $\mathbb{Q}$. Then the polynomials $f^n(x)-x$ and $\Phi_{n,f}(x)$ have no multiple roots over $\mathbb{Q}(a_{d-1}, ..., a_0)$.*

Proof. The discriminant $\text{disc}(f^n(x)-x)$ is a polynomial in indeterminates $a_{d-1}, ...a_0$ with integer coefficients. If the discriminant vanishes when $a_i = 0$ for all $0 \leq i \leq d-1$, then it means that $f(x) = x^d$ then $f^n(x) - x = x^{d^n} - x$ and we know that this has not multiple roots over $\mathbb{Q}$. So, $\text{disc}(f^n(x) - x) \neq 0$ and $\text{disc}\, \Phi_{n,f}(x) \neq 0$. Hence, $f^n(x) - x$ and $\Phi_{n,f}(x)$ have no multiple roots over $\mathbb{Q}(a_{d-1}, ..., a_0)$. ☐

Proposition 1.3.6. *[12, Proposition 2.8] Let F be a field extension of K and let $\alpha \in F$.*

1. *If $f^n(\alpha) = \alpha$ then $\Phi_{n,f}(\alpha) = 0$.*
2. *If $\Phi_{n,f}(\alpha) = 0$ then $f^n(\alpha) = \alpha$.*
3. *If $\Phi_{n,f}(\alpha) = 0$ then $\Phi_{n,f}(f(\alpha)) = 0$.*
4. *Suppose that $\text{disc}\, \Phi_{n,f} \neq 0$ then for all α such that $\Phi_{n,f}(\alpha) = 0$ we have $f^n(\alpha) = \alpha$.*

Proof.

1. This is clear from the definition of $\Phi_{n,f}$.
2. By Theorem 1.3.1, the factorization $f^n(x)-x = \prod_{d|n} \Phi_{d,f}(x)$ gives the desired result since $\Phi_{n,f}$ always appears in the factorization.
3. Let $g(x) = a_kx^k + ... + a_0$ be a generic polynomial of degree k defined over $\mathbb{Q}$. Since $g(x)$ is generic then by Corollary 1.3.5 , $\Phi_{n,g}(x)$ has no multiple roots. So, roots of $\Phi_{n,g}(x)$ are periodic points of exact period n under g over $\mathbb{Q}(a_k, ..., a_0)$. We know that by Lemma 1.3.4 g permutes roots of $\Phi_{n,g}(x)$. So every root of $\Phi_{n,g}(x)$ is a root of $\Phi_{n,g}(g(x))$. Hence, $\Phi_{n,g}(x)|\Phi_{n,g}(g(x))$. Then by the homomorphism

$$h : \mathbb{Z}[a_k, ...a_0, x] \to K[x]$$

defined by $a_i \longmapsto b_i$ for $0 \leq i \leq k$ and $x \longmapsto x$ where b_i's are coefficients of a polynomial $f(x) = b_kx^k + ... + b_0 \in K[x]$. We see that $g(x) \longmapsto f(x)$ and $\Phi_{n,g}(x) \longmapsto \Phi_{n,f}(x)$. Hence, $\Phi_{n,g}(g(x)) \longmapsto \Phi_{n,f}(f(x))$. Therefore, since $\Phi_{n,g}(x)|\Phi_{n,g}(g(x))$ and h is a homomorphism then $\Phi_{n,f}(x)|\Phi_{n,f}(f(x))$.
4. This can be proven by using [16, Theorem 2.4(c)].

Corollary 1.3.7. *Let K be a field, F/K is an extension field of K and $\alpha \in F$. Let $f(x) \in K[x]$ be a polynomial. If* $\operatorname{disc} \Phi_{n,f} \neq 0$ *then $f^n(\alpha) = \alpha$ where n is the period of α if and only if $\Phi_{n,f}(\alpha) = 0$ in F.*

Proof. Suppose that $\alpha \in F$ is a point of period n of the polynomial $f(x) \in K[x]$. Then by Proposition 1.3.6(1), $\Phi_{n,f}(\alpha) = 0$.
Suppose that $\Phi_{n,f}(x)$ has a root in F, say $\alpha \in F$. Since $\operatorname{disc} \Phi_{n,f} \neq 0$ then by Proposition 1.3.6(4) α is a point of period n of $f(x)$. □

In what follows, we show that given two linearly conjugate polynomials in $K[x]$ their n-th dynatomic polynomials are the same up to a scalar. First, we will briefly recall *Kronecker Delta Function.*

Let $i, j \in \mathbb{Z}$. Then the Kronecker Delta Function is defined as follows:

$$\delta_{i,j} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Proposition 1.3.8. *[12, Proposition 2.10] Let $f, g \in K[x]$ be two polynomials with $f = l \circ g \circ l^{-1}$ where $l(x) = ax + b \in K[x]$ is a linear polynomial. Then*

$$\Phi_{n,g}(l(x)) = a^{\delta_{1n}} \Phi_{n,f}(x)$$

where δ is the Kronecker delta function.

Proof. Suppose that $f(x) = l \circ g \circ l^{-1}(x)$. So, for all $d \in \mathbb{Z}_{>0}$, $f^d(x) = l \circ g^d \circ l^{-1}(x)$ Then $f^d \circ l(x) = l \circ g^d(x)$. It can be written as $a(g^d(x)) + b = f^d(l(x))$. With a simple manipulation, we find that $a(g^d(x) - x) = f^d(l(x)) - l(x)$. Therefore, the n-th dynatomic polynomial of $g(l(x))$ can be calculated as follows:

$$\begin{aligned} \Phi_{n,g}(l(x)) &= \prod_{d|n} (g^d(l(x)) - l(x))^{\mu(\frac{n}{d})} \\ &= \prod_{d|n} (a(f^d(x) - x))^{\mu(\frac{n}{d})} \\ &= \prod_{d|n} a^{\mu(\frac{n}{d})} (f^d(x) - x)^{\mu(\frac{n}{d})} \\ &= a^{\sum_{d|n} \mu(\frac{n}{d})} \prod_{d|n} (f^d(x) - x)^{\mu(\frac{n}{d})} \\ &= a^{\delta_{1n}} \Phi_{n,f}(x) \end{aligned}$$

□

In fact, if two polynomials are linearly conjugate over K then the corresponding dynatomic polynomials split in the same way.

Corollary 1.3.9. *[12, Corollary 2.11] Let $f, g \in K[x]$ be two polynomials as in the Proposition 1.3.8 . Let F be an extension of K, n be a positive integer and*

$$\Phi_{n,g} = \alpha \cdot A_1^{t_1} ... A_r^{t_r}$$

be a factorization of $\Phi_{n,g}$ *where* $\alpha \in F$, $A_1 ... A_r$ *are pairwise non-associate irreducible polynomials in* $F[x]$ *and* $t_i \in \mathbb{Z}^+$ *for all* $i = 1, ..., r$. *Then*

$$\Phi_{n,f} = \beta \cdot B_1^{t_1} ... B_s^{t_s}$$

where $\beta \in F$, *irreducible, pairwise and nonassociate factors* $B_1, ..., B_s \in F[x]$, $s = r$, $\deg A_i = \deg B_i$ *for all* $i = 1, ..., s$.

Proof. Suppose that $f, g \in K[x]$ are linearly conjugate. Let $l(x) = ax + b \in K[x]$ such that $f(x) = l \circ g \circ l^{-1}(x)$. As in the Proposition 1.3.8, we see that $\Phi_{n,g}(l(x)) = a^{\delta_{1n}} \Phi_{n,f}(x)$. On the other hand, the left hand side equals to $aA_1^{t_1}(l(x))...A_r^{t_r}(l(x))$. Hence

$$a^{1-\delta_{1n}} A_1^{t_1}(l(x))...A_r^{t_r}(l(x)) = \Phi_{n,f}(x).$$

Set $\beta = a^{1-\delta_{1n}}$ and $A_i(l(x)) = B_i(x)$ for all i. Hence $s = r$ comes immediately. □

So, the linear conjugation of f and g preserves the splitting behaviour of the corresponding dynatomic polynomials.

Corollary 1.3.10. *[12, Corollary 2.12] Let* $f, g \in K[x]$ *be two linearly conjugate polynomials over* K *and* n *be a positive integer. Then*

1. $\Phi_{n,g}$ *has a root in* K *if and only if* $\Phi_{n,f}$ *has a root in* K.
2. $\operatorname{disc} \Phi_{n,g} = 0$ *if and only if* $\operatorname{disc} \Phi_{n,f} = 0$.

Proof.
1. We know that $f, g \in K[x]$ are linearly conjugate, i.e, $g = l \circ f \circ l^{-1}$ for $l(x) = ax+b \in K[x]$. Then by Proposition 1.3.8 $\Phi_{n,g}(l(x)) = a^{\delta_{1n}} \Phi_{n,f}(x)$. If we consider them over K, not necessarily on an extension of K, then by Corollary 1.3.9 they factorize in the same way as $\beta = a^{1-\delta_{1n}}$ and $A_i(l(x)) = B_i(x)$ for all i. Hence, any root α of $\Phi_{n,g}$ will be a root of one of A_i. Since $l(x)$ is a linear polynomial, it only translates the roots of $\Phi_{n,g}$. Hence, $l(\alpha)$ will be a root of $\Phi_{n,g}(l(x))$. Therefore, it is a root of $\Phi_{n,f}$. We can show other direction in the same manner by letting $f = l \circ g \circ l^{-1}$ since the conjugation action is symmetric.

2. By the invariance property of discriminant, we know that

$$\operatorname{disc} \Phi_{n,g}(l(x)) = \operatorname{disc} \Phi_{n,g}(ax + b) = a^{k(k-1)} \operatorname{disc} \Phi_{n,g}(x).$$

Therefore if $\operatorname{disc} \Phi_{n,g}(x) = 0$ then $\operatorname{disc} \Phi_{n,g}(l(x)) = 0$. Since $\Phi_{n,g}(l(x)) = a^{\delta_{1n}} \Phi_{n,f}(x)$, then $\operatorname{disc} \Phi_{n,f}(x) = 0$. We can show other direction in the same manner by letting $f = l \circ g \circ l^{-1}$.

Chapter 2

Cubic Post-Critically Finite Polynomials

In this chapter we will show how to classify cubic post-critically finite polynomials defined over $\mathbb{Q}$. All of these works in this chapter is based on the classification results in [3]. Firstly, we will introduce *bicritical* and *unicritical* polynomials. We will see that a bicritical polynomial with two rational critical points is linearly conjugate to a special type of polynomials, the so called *Belyi maps*. We will be interested in a special example of Belyi maps $B_{d,k}(x)$ where d is the degree and $0 \leq k < d-1$. Further results can be found in [2]. After that we will focus on cubic polynomials defined over $\mathbb{Q}$. By finding bounds on coefficients of cubic post-critically finite polynomials and using Sage we will find explicit representatives of conjugacy classes of cubic post-critically finite polynomials defined over $\mathbb{Q}$. In this chapter K is a number field.

Since the first derivative of a cubic polynomial in $K[x]$ is a quadratic polynomial in $K[x]$ then we may classify cubic polynomials according to the zeros of the first derivative as follows:

1. $f(x) \in K[x]$ has two distinct K-rational critical points, namely $\alpha_1, \alpha_2 \in K$. These polynomials are called *bicritical* polynomials with two K-rational critical points.

2. $f(x) \in K[x]$ has two distinct irrational critical points, namely $\alpha_1, \alpha_2 \notin K$ with the quadratic extensions $K(\alpha_1) = K(\alpha_2)$. These polynomials are called *bicritical* polynomials with two irrational critical points.

3. $f(x) \in K[x]$ has a double critical point $\alpha \in K$, i.e., $(x-\alpha)^2 \mid f'(x)$. These polynomials are called *unicritical* polynomials.

2.1 Unicritical Polynomials

Firstly, due to their simplicity, we will begin with describing unicritical polynomials. Our first theorem is valid for polynomials of every degree d. We will see that unicritical polynomials of any degree d can be transformed into a polynomial of special form which is much more easier to study.

Theorem 2.1.1. *Let $f(x) \in K[x]$ be a polynomial of degree d. If $f(x)$ is an unicritical polynomial then there exists $l(x) \in \bar{K}[x]$ such that $f^l(x) = x^d \in K[x]$ or $f^l(x) = ax^d + 1 \in K[x]$ where $a \in K$ is unique.*

Proof. Let $f(x) = a_d x^d + ... + a_1 x + a_0 \in K[x]$ be an unicritical polynomial of degree d with derivative $f'(x) = \alpha(x - \beta)^{d-1}$. If we send x to $x + \beta$, we obtain $f'(x) = \alpha x^{d-1}$. Taking indefinite integral gives $g(x) = \lambda x^d + \gamma$ where $\lambda = \frac{\alpha}{d}$. The transformation does not change the field of definition, i.e., $g(x) \in K[x]$ because $\beta \in K$. So, $g(x) = \lambda x^d + \gamma \in K[x]$. Now we have two cases:

Case i) If $\gamma = 0$ then $f(x) = \lambda x^d$ where λ is a non-zero K-rational coefficient. We can get rid of λ by conjugating $f(x)$ with $l(x) = \lambda^{\frac{1}{(d-1)}} x$. Then

$$l \circ f \circ l^{-1}(x) = l \circ f\left(\frac{x}{\lambda^{1/(d-1)}}\right) = l\left(\lambda(\frac{x}{\lambda^{1/(d-1)}})^d\right) = \lambda^{\frac{1}{d-1} - \frac{1}{d-1}} x^d = x^d$$

Hence $f^l(x) = x^d$.
Case ii) If $\gamma \neq 0$ then we can conjugate $f(x)$ by $l(x) = \frac{x}{\gamma}$. Then

$$l \circ f \circ l^{-1}(x) = l \circ f(\gamma x) = l(\lambda \gamma^d x^d + \gamma) = \lambda \gamma^{d-1} x^d + 1$$

$\lambda\gamma^{d-1} \in K$ and non-zero since $\lambda, \gamma \in K$ and non-zero. If we set $a = \lambda\gamma^{d-1}$ we obtain the desired form. Now we have to show that a is unique.

The conjugation map $l(x) = \frac{x}{\gamma}$ is a map fixing 0 and satisfying $f^l(0) = 1$. Let us define $k(x) = cx + e \in \overline{K}[x]$. It must have same properties that $l(x)$ has. So, If $k(0) = 0$ then $e = 0$, and we have

$$k \circ f \circ k^{-1}(x) = k \circ f(\frac{x}{c}) = k(\lambda \frac{x^d}{c^d} + \gamma) = \lambda \frac{x^d}{c^{d-1}} + c\gamma$$

So If $f^k(0) = 1$ then $c\gamma = 1$ implies that $c = \frac{1}{\gamma}$. Hence $k(x) = l(x)$ and a is unique. □

As a result if $f(x) \in K[x]$ is an unicritical polynomial then it is linearly conjugate to either x^d or $ax^d + 1$. We need to show they are post-critically finite polynomials. Let us give the definition of a post-critically finite and post-critically bounded polynomial.

Definition 2.1.2. *A polynomial f is post-critically finite if every critical point of f is a preperiodic point under f, i.e., every critical point has a finite orbit. A polynomial f is post-critically bounded with respect to an absolute value $|\,.\,|$ if every element in the orbit of each critical point is bounded with respect to $|\,.\,|$.*

The critical point of $f(x) = x^d$ is 0 and notice that $f(0) = 0$. So 0 is a fixed point of f. It means that the critical point of f is a point of period 1 under f. Hence $f(x) = x^d$ is post-critically finite polynomial for every degree d.

The critical point of $f(x) = ax^d + 1$ is 0. Now, we prove that 0 is a preperiodic point under f if the leading coefficient of f is bounded by 2 with respect to the archimedean absolute value. Firstly, we will find some bounds on $a \in K^\times$ then we investigate the case where $a \in \mathbb{Q}^\times$. Later we will see that in fact $a \in \mathbb{Z}^\times$.

Proposition 2.1.3. *[5, Corollary 8] Let $f(x) \in K[x]$ be an unicritical polynomial of degree d linearly conjugate to $ax^d + 1$ where $a \in K^\times$. Then $|a| \leq 2$ if $f(x)$ is post-critically finite.*

Proof. We prove it by contradiction. We pick $\alpha \in K$. First, suppose that $|a| > 2$ and $|\alpha| \geq 1$. Then

$$|f(\alpha)| = |a\alpha^d + 1| > |a| \cdot |\alpha^d| - 1 > 2|\alpha| - 1 > 2|\alpha| - |\alpha| = |\alpha|.$$

It means that α is a wandering point. We know that the critical point of f is 0 and $f(0) = 1$. So f cannot be post-critically finite. Hence $|a| \leq 2$. □

Now, we can consider $f(x) = ax^d + 1 \in \mathbb{Q}$, i.e., $a \in \mathbb{Q}^\times$.

Theorem 2.1.4. *[3, Theorem 2.3] Let $f(x) \in \mathbb{Q}[x]$ be an unicritical polynomial of degree d linearly conjugate to $ax^d + 1 \in \mathbb{Q}[x]$. f is post-critically finite if and only if $a \in \{-2, -1\}$. In particular, If d is odd then f is post-critically finite if and only if $a = -1$.*

Proof. Firstly, suppose that $|a|_p > 1$ for some prime p. In the meantime, if $|x|_p \geq 1$ then x is a wandering point if there exists $n \geq 0$ such that $|f^n(x)|_p \geq 1$. We have

$$|f(x)|_p = |ax^d + 1|_p = |ax^d|_p > |x|_p$$

Notice that the critical point 0 of f is a wandering point since $f(0) = 1$ and $|a|_p > 1$. So, f is not post-critically finite. It means that whenever $|a|_p > 1$, f is not post-critically finite. Therefore $|a|_p \leq 1$ for every prime p. It means that a is an integer. By Proposition 2.1.3 we know that $|a| \leq 2$. Hence $a \in \{-2, -1, 1, 2\}$.

Now, we will find a bound on the size of wandering points of f. Let $\alpha \in \mathbb{Q}$ with $|\alpha| > 2$ then

$$|f(\alpha)| = |a\alpha^d + 1| > 2^{d-1}|\alpha| - 1 > |\alpha|$$

So, α is a wandering point. Hence preperiodic points of f are bounded by 2, i.e., $|\alpha| \leq 2$.

Now, we have to examine the possible values of a.

If $a = 1$ then $f(x) = x^d + 1$. The critical point of f is 0. We have $f(0) = 1$, $f \circ f(0) = f(1) = 1 + 1 = 2$, $f^3(0) = f(2) = 2^d + 1$. Hence 0 is clearly a wandering point. Hence, f is not post-critically finite.

If $a = -1$ then $f(x) = -x^d + 1$. The critical point of f is 0. We have $f(0) = 1$, $f \circ f(0) = f(1) = -1 + 1 = 0$. Hence 0 is a point of period 2 under f. It means that f is post-critically finite.

If $a = 2$ then $f(x) = 2x^d + 1$. The critical point of f is 0. We have $f(0) = 1$, $f \circ f(0) = f(1) = 2+1 = 3$, $f^3(0) = f(3) = 2 \cdot 3^d + 1$. Hence 0 is clearly a wandering point. Hence f is not post-critically finite.

If $a = -2$ then $f(x) = -2x^d + 1$. The critical point of f is 0. We have $f(0) = 1$, $f \circ f(0) = f(1) = -1$, $f^3(0) = f(-1) = -2 \cdot (-1)^d + 1$. In that case, If d is even then $f^3(0) = f(-1) = -1$ meaning that 0 is a preperiodic point of type $(2, 1)$ since $f^3(0) = f^2(0)$. Therefore f is post-critically finite. If d is odd then $f^3(0) = f(-1) = 2 + 1 = 3$ and $f^4(0) = f(3) = -2 \cdot 3^d + 1$. So, 0 is a wandering point. Hence f is not post-critically finite.

The only cases where f is post-critically finite are when $a \in \{-1, -2\}$. An unicritical polynomial defined over $\mathbb{Q}$ of any degree d is post-critically finite if it is linearly conjugate to one of the following polynomials:

- x^d
- $-x^d + 1$
- $-2x^d + 1$

2.2 Bicritical Polynomials

As we defined before, cubic polynomials with two critical points are called bicritical polynomials. There are two types of these polynomials. One is polynomials with two rational critical points and the other one is polynomials with two irrational critical points.

Let $f(x) \in K[x]$ be a cubic polynomial with two K-rational critical points $\alpha_1, \alpha_2 \in K$. We can move these critical points α_1, α_2 to 0 and 1 respectively by conjugating $f(x)$ with $l(x) = \frac{x-\alpha_1}{\alpha_2-\alpha_1} \in K[x]$ without changing the field of definition. Thus the conjugate polynomial $g(x) = f^l(x)$ has critical points 0 and 1. It can be generalized to degree d polynomials with two rational critical points.

When we move the critical points to 0 and 1 then the derivative of a polynomial of degree d with critical points 0 and 1 is the following

$$B'_{d,k}(x) = cx^{d-k-1}(x-1)^k$$

where $1 \leq k < d-1$ and $c \in K$. Since the degree of $B'_{d,k}(x)$ is $d-1$ then the degree of the term x and the term $(x-1)$ must sum up to $d-1$. Expanding $B'_{d,k}(x)$ and integrating term by term and requiring the critical points to be fixed give us

$$B_{d,k}(x) = \left(\frac{1}{k!}\prod_{j=0}^{k}(d-j)\right)x^{d-k}\sum_{i=0}^{k}\frac{(-1)^i}{(d-k+i)}\binom{k}{i}x^i$$

We are interested in the case where the degree $d = 3$. So $k = 1$, then

$$B_{3,1}(x) = -2x^3 + 3x^2$$

Now, we will prove that every bicritical polynomial of degree $d \geq 3$ over a number field K with two K-rational critical points is linearly conjugate to a scalar times $B_{d,k}(x)$ plus a constant.

Proposition 2.2.1. *Let $f(x) \in K[x]$ be a bicritical polynomial with two K-rational critical points of degree $d \geq 3$. Then there exists $T(x) = ax + b \in PGL_2(K)$ such that $f^T(x) = \alpha B_{d,k}(x) + \beta$.*

Proof. Let $\alpha_1, \alpha_2 \in K$ be two critical points of a bicritical polynomial $f(x) \in K[x]$ of degree $d \geq 3$. As we defined before the transformation $T(x) = \frac{x-\alpha_1}{\alpha_2-\alpha_1}$ moves α_1 and α_2 to 0 and 1 respectively. Fix a positive integer m such that the exponent of α_1 in $f'(x)$ is $d-m-1$ and the exponent of α_2 in $f'(x)$ is m. Then the polynomial $f^T(x) = g(x)$ has critical points at 0 and 1 with exponents $d-m-1$ and m respectively in $g'(x)$. Therefore

$$g'(x) = ax^{d-m-1}(x-1)^m = \alpha B'_{d,k}(x)$$

for some $\alpha \in \overline{K}^{\times}$. Integrating term by term results

$$g(x) = \alpha B_{d,k}(x) + \beta$$

Since $T(x) \in PGL_2(K)$ and $f(x) \in K[x]$ then $g(x) \in K[x]$. Hence $\alpha, \beta \in K$. □

We classified every bicritical polynomial of degree $d \geq 3$ with two K-rational critical points up to linear conjugacy. Now, we will classify every bicritical polynomial of odd degree with two irrational critical points up to linear conjugacy. We force d to be odd because if the first derivative of a polynomial has an irrational root then the conjugate of this irrational root is also a root of the derivative. So, every irrational critical point comes with its conjugate. Hence $d-1$ must be even then d must be odd.

Let $\alpha \in \mathbf{O}_K^{\times}$ where $\mathbf{O}_K$ is the ring of integer of K and $d \geq 3$ be odd. Since the first derivative of a polynomial in $K[x]$ with irrational critical points will have $(x^2-\alpha)$ as a factor, we define $P_{d,\alpha} \in K[x]$ to be a polynomial satisfying the following properties: $P'_{d,\alpha} = (x^2-\alpha)^{d-1/2}$ and $P_{d,\alpha}(0) = 0$, i.e., $P_{d,\alpha}$ is fixing 0.

So $P_{d,\alpha}(x)$ is a bicritical polynomial with irrational critical points $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ each with exponent $\frac{d-1}{2}$. Expanding $P'_{d,\alpha}(x)$ and integrating term by term and requiring 0 to be a fixed point give us the following polynomial

$$P_{d,\alpha}(x) = \sum_{j=0}^{\frac{d-1}{2}} (-\alpha)^{\frac{d-1}{2}-j} \binom{\frac{d-1}{2}}{j} \frac{x^{2j+1}}{2j+1}$$

For $d = 3$ we have $P_{3,\alpha}(x) = \frac{x^3}{3} - \alpha x$.

Proposition 2.2.2. *[3, Proposition 3.4] Let $f(x) \in K[x]$ be a bicritical polynomial of degree $d \geq 3$ with two irrational critical points $\alpha_1, \alpha_2 \notin K$. Then there exists $T(x) = ax + b \in PGL_2(K)$ such that $f^T(x) = aP_{d,\alpha}(x) + b$ where $a, b \in K$ and $\alpha \in \mathbf{O}_K^{\times}/\mathbf{O}_K^2$.*

Proof. Suppose that $\alpha_1, \alpha_2 \notin K$ are critical points of $f(x) \in K[x]$. It means that these critical points live in an extension of K, namely $K(\alpha_1)$ and $K(\alpha_2)$. In fact, $K(\alpha_1) = K(\alpha_2)$ since the polynomial $f'(x)$ must have an irreducible quadratic factor $h(x) \in K[x]$, i.e., $f'(x) = \beta(h(x))^s$. So d must be odd since $f'(x)$ is of even degree $d-1$. We know that $f'(\alpha_1) = f'(\alpha_2) = 0$ and α_1 and α_2 are conjugate. So, there exists $y_1, y_2 \in K$ with $y_2 \neq 0$ and $\alpha \in \mathbf{O}_K^{\times}/\mathbf{O}_K^2$, $\alpha_1 = y_1 + y_2\sqrt{\alpha}$ and $\alpha_2 = y_1 - y_2\sqrt{\alpha}$.

Now, define $T(x) = \frac{x-y_1}{y_2} \in PGL_2(K)$. Notice that $T(\alpha_1) = \sqrt{\alpha}$ and $T(\alpha_2) = -\sqrt{\alpha}$. $T(x)$ moves critical points to the pure irrational parts of these points. The polynomial $f^T(x) = g(x) \in K[x]$ since $T(x), f(x) \in K[x]$. Necessarily $g'(x) \in K[x]$, so $\sqrt{\alpha}$ and $-\sqrt{\alpha}$ are roots of $g'(x)$. Then $(x^2-\alpha) \mid g'(x)$. We know that $\deg g'(x) = d-1$ and $f'(x) = \beta(h(x))^s$ where $h(x) \in K[x]$ is an irreducible quadratic polynomial then $g'(x) = a(x^2-\alpha)^{d-1/2}$ where $a \in K$. Expanding and integrating term by term give us $g(x) = aP_{d,\alpha}(x) + b \in K[x]$, the desired result. □

2.3 Bounds for Coefficients of Cubic Post-critically Finite Polynomials

In [10], Ingram gives us an effective method to find bound on coefficients of cubic post-critically finite polynomials. In what follows we will introduce this bound briefly, then we will focus on some results related to this method. In fact, this method will give us bounds on coefficients of cubic post-critically finite polynomials defined over $\mathbb{Q}$. By using Sage [21] we can find all representatives of conjugacy classes of each cubic post-critically finite polynomials. Moreover, there are only finitely many conjugacy classes.

Proposition 2.3.1. *[10, Corollary 2] Choose $d \geq 2$ and $D \geq 1$. Then the number of conjugacy classes of post-critically finite polynomials of degree d which has coefficients of algebraic degree at most D is finite and the set of representatives of these conjugacy classes can be computed effectively.*

Now, we introduce a symbol for $x \in \mathbb{R}$.

$$(x)_v = \begin{cases} x & if\ v\ is\ archimedean \\ 1 & if\ v\ is\ non-archimedean \end{cases}$$

We give a v-adic bound depends on coefficients and the degree d of $f(x) = b_d x^d + ... + b_1 x + b_0 \in K[x]$.

$$C_{f,v} = (2d)_v \max_{1 \leq i < d} \left\{ 1, \left| \frac{b_i}{b_d} \right|_v^{\frac{1}{d-i}}, |b_d|_v^{\frac{-1}{d-1}} \right\}$$

This value $C_{f,v}$ can be computed effectively for polynomials defined over a number field. In the following lemma, we show that $C_{f,v}$ is a v-adic bound for preperiodic points of a given polynomial.

Lemma 2.3.2. *[3, Lemma 4.1] Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $d \geq 2$, $\alpha \in \mathbb{Q}$. If there exists $n \in \mathbb{N}$ and an absolute value v on $\mathbb{Q}$ such that*

$$|f^n(\alpha)|_v > C_{f,v},$$

then α is a wandering point of f, i.e., α has infinite orbit under f.

Proof. We know that α has an infinite orbit $\mathcal{O}_f(\alpha)$ under f if and only if $f^n(\alpha)$ has an infinite orbit $\mathcal{O}_f(f^n(\alpha))$ under f since $f^n(\alpha) \in \mathcal{O}_f(\alpha)$ for all $n \in \mathbb{N}$. So, we can replace $f^n(\alpha)$ by α in the lemma. Assume that $|\alpha|_v > C_{f,v}$ for an absolute value on $\mathbb{Q}$. We will show that if α has infinite orbit under f then $|f(\alpha)|_v > |\alpha|_v$ when $|\alpha|_v > C_{f,v}$. We have two cases:

1. If v is archimedean then the first inequality is $|\alpha|_v > 2d\left|\frac{b_i}{b_d}\right|_v^{\frac{1}{d-i}}$ for all $0 \leq i < d$.

   Then
   $$|b_d^{\frac{1}{d-i}}\alpha|_v > 2d|b_i|_v^{\frac{1}{d-i}}.$$
   Taking $(d-i)^{th}$ power of both sides, we have
   $$|b_d\alpha^{d-i}|_v > (2d)^{d-i}|b_i|_v.$$
   Multiplying each side by $|\alpha^i|_v$ we obtain
   $$|b_d\alpha^d|_v > (2d)^{d-i}|b_i\alpha^i|_v.$$
   Taking the inequality for all $0 \leq i < d$ we obtain the following
   $$|b_d\alpha^d|_v > \max_{0\leq i<d}\{(2d)^{d-i}|b_i\alpha^i|_v\}.$$
   Finally, due to the following inequality
   $$|f(\alpha)|_v = \left|\sum_{i=0}^{d} b_i\alpha^i\right|_v \geq |b_d\alpha^d|_v - d\max_{0\leq i<d}|b_i\alpha^i|_v > \frac{1}{2}|b_d\alpha^d|_v.$$
   We obtain the desired result
   $$|f(\alpha)|_v > \frac{1}{2}|b_d\alpha^d|_v > |\alpha|_v.$$
   Hence, α has infinite orbit under f.

   The second inequality is $|\alpha|_v > 2d|b_d|_v^{\frac{-1}{d-1}}$. We multiply both sides with $|b_d|_v^{\frac{1}{d-1}}$, we obtain
   $$|\alpha|_v|b_d|_v^{\frac{1}{d-1}} > 2d.$$
   Take $(d-1)^{th}$ power of both sides
   $$|\alpha|_v^{d-1}|b_d|_v > (2d)^{d-1}.$$
   Multiply both sides by $|\alpha|_v$
   $$|b_d\alpha^d|_v > (2d)^{d-1}|\alpha|_v > |\alpha|_v.$$
   Since
   $$|f(\alpha)|_v = \left|\sum_{i=0}^{d} b_i\alpha^i\right|_v > |b_d\alpha^d|_v > |\alpha|_v.$$
   We found that $|f(\alpha)|_v > |\alpha|_v$. By assumption α has infinite orbit under f.

2. If v is non-archimedean then the first inequality is $|\alpha|_v > \left|\frac{b_i}{b_d}\right|_v^{\frac{1}{d-i}}$ then

$$|b_d^{\frac{1}{d-i}}\alpha|_v > |b_i|_v^{\frac{1}{d-i}}$$

for all $0 \leq i < d$. Then take $(d-i)^{th}$ power of both sides

$$|b_d\alpha^{d-i}|_v > |b_i|_v.$$

Then multiply both sides with α^i

$$|b_d\alpha^d|_v > |b_i\alpha^i|_v$$

for all $0 \leq i < d$. Also we have

$$|\alpha|_v > C_{f,v} \geq |b_d|_v^{\frac{-1}{d-1}}.$$

Therefore,

$$|f(\alpha)|_v = \left|\sum_{i=0}^{d} b_i\alpha^i\right|_v = |b_d\alpha^d|_v > |\alpha|_v.$$

In both cases $|f(\alpha)|_v > |\alpha|_v$. Hence α has infinite orbit under f, i.e., α is a wandering point of f. □

2.3.1 Cubic PCF Polynomials With Rational Critical Points

We first begin with finding bounds on the coefficients of bicritical cubic polynomials with rational critical points. Lemma 2.3.2 gives us an efficient bound for cubic post-critically finite polynomials which we introduces above. We apply this bound to the polynomial $B_{3,1}(x)$ which we found in the previous section.

Lemma 2.3.3. *[3, Lemma 4.2] Let $f(x) \in \mathbb{Q}[x]$ be a bicritical polynomial of degree 3 with rational critical points. Then $f(x) = aB_{3,1} + c = a(-2x^3 + 3x^2) + c \in \mathbb{Q}[x]$. If there exists $n \in \mathbb{N}$ and an absolute value $v \in M_{\mathbb{Q}}$ such that*

$$|f^n(\alpha)|_v > C_{f,v} = (6)_v max\left\{1, \left|\frac{3}{2}\right|_v, \left|\frac{1}{2a}\right|_v^{\frac{1}{2}}, \left|\frac{c}{2a}\right|_v^{\frac{1}{3}}\right\}$$

then α has infinite orbit under f, i.e., α is a wandering point of f.

Proof. Proof of this lemma can be obtained easily by applying Lemma 2.3.2 to the coefficients of $f(x) = a(-2x^3 + 3x^2) + c$. □

Remark 2.3.4. *An observation to make is that the critical points of $f(x) = a(-2x^3 + 3x^2) + c \in \mathbb{Q}[x]$ are 0 and 1. If f is post-critically finite polynomial then 0 and 1 are preperiodic, i.e., these points have finite orbits under f and bounded by $C_{f,v}$. So,*

$$|f(1)|_v = |a + c|_v \leq C_{f,v},$$

$$|f(0)|_v = |c|_v \leq C_{f,v}.$$

As a result for any absolute value $v \in M_{\mathbb{Q}}$ if f is post-critically finite then $\max\{|c|_v, |a + c|_v\} \leq C_{f_v}$. *However, particularly if v is non-archimedean then* $|a+c|_v \leq \max\{|c|_v, |a|_v\}$. *Hence,* $\max\{|c|_v, |a|_v\} \leq C_{f,v}$.

Now, we will give an archimedean bound on coefficients of $f(x) = a(-2x^3 + 3x^2) + c \in \mathbb{Q}[x]$

Lemma 2.3.5. *[3, Lemma 4.4] If $f(x) = a(-2x^3+3x^2)+c \in \mathbb{Q}[x]$ is post-critically finite then $|a| < 4$.*

Proof. Suppose $f(x) = a(-2x^3 + 3x^2) + c$ is post-critically finite and $|a| \geq 4$ and $|\alpha| \geq \max\{|c|, 2\}$. So

$$|f(\alpha)| = |a(-2\alpha^3 + 3\alpha^2) + c| = |a\alpha^2(-2\alpha + 3) + c|$$

We know that $|\alpha| \geq \max\{|c|, 2\}$ then

$$|a\alpha^2(-2\alpha + 3) + c| \geq |a\alpha^2(-2\alpha + 3)| - |c| \geq |\alpha|(|a\alpha(-2\alpha + 3)| - 1) > |\alpha|$$

because we have $|a\alpha(-2\alpha + 3)| \geq 1$. Therefore $|f(\alpha)| > |\alpha|$ means that α is a wandering point of f. If $|c| \geq 2$ then $|f(0)| = |c| \geq 2$. So 0 is a wandering point of f. If $|c| < 2$ then $|f(1)| = |a + c| \geq |a| - |c| > 2$. So 1 is a wandering point of f. Hence it is a contradiction. □

This proof can be repeated for any post-critically finite polynomial of degree d of the form $a(-(d-1)\alpha^d + d\alpha^{d-1}) + c \in \mathbb{Q}[x]$. Hence $|a| < 4$ for post-critically finite polynomials of these forms.

Notice that the bound $C_{f,v}$ can be rewritten by excluding $\left|\frac{c}{2a}\right|_v^{\frac{1}{3}}$.

Lemma 2.3.6. *[3, Lemma 4.5] If $f(x) = a(-2x^3+3x^2)+c \in \mathbb{Q}[x]$ is post-critically finite then for a non-archimedean $v \in M_{\mathbb{Q}}$ we have*

$$C_{f,v} = \max\left\{1, \left|\frac{3}{2}\right|_v, \left|\frac{1}{2a}\right|_v^{\frac{1}{2}}\right\}.$$

Proof. Suppose $v \in M_{\mathbb{Q}}$ is non-archimedean. Suppose first $C_{f,v} = \left|\frac{c}{2a}\right|_v^{\frac{1}{3}} > \left|\frac{1}{2a}\right|_v^{\frac{1}{2}}$. We take 6^{th} power of both sides,

$$\left|\frac{c}{2a}\right|_v^2 > \left|\frac{1}{2a}\right|_v^3.$$

After some manipulations we have,

$$|c|_v^2 > \left|\frac{1}{2a}\right|_v.$$

Since we assume f is post-critically finite then by Remark 2.3.4, we must have $|c|_v \leq C_{f,v} = \left|\frac{c}{2a}\right|_v^{\frac{1}{3}}$. So this implies that $|c|_v^2 \leq \left|\frac{1}{2a}\right|_v$. It is a contradiction. □

The following lemma holds for any number field K and non-archimedean $v \in M_K$.

Lemma 2.3.7. *[3, Lemma 4.6] If $f(x) = a(-2x^3+3x^2)+c \in \mathbb{Q}[x]$ is post-critically finite, $p > 2$ is a prime and $|\cdot|_p$ is a p-adic absolute value then a is a p-adic integer and $|c|_p^2 \le |a|_p^{-1}$.*

Proof. By Lemma 2.3.6 we know that for a p-adic absolute value $|\cdot|_p$ for an odd prime p we have

$$C_{f,p} = \max\left\{1, \left|\frac{3}{2}\right|_p, \left|\frac{1}{2a}\right|_p^{\frac{1}{2}}\right\}.$$

Since p is odd then $\left|\frac{3}{2}\right|_p = |3|_p$ and $\left|\frac{1}{2a}\right|_p^{\frac{1}{2}} = \left|\frac{1}{a}\right|_p^{\frac{1}{2}} = |a|_p^{\frac{-1}{2}}$. So it turns out that

$$C_{f,p} = \max\{1, |3|_p, |a|_p^{\frac{-1}{2}}\}.$$

If $p = 3$ then $|3|_3 = \frac{1}{3} < 1$. If $p \neq 3$ then $|3|_p = 1$. We can exclude $|3|_p$ from the list.

$$C_{f,p} = \max\{1, |a|_p^{\frac{-1}{2}}\}$$

There are two cases:

- Case i: If $C_{f,p} = 1$ then $1 \ge |a|_p^{\frac{-1}{2}}$. We take $(-2)^{nd}$ power of both sides. It gives us $|a|_p \ge 1$. We assume that f is post-critically finite so $|a|_p, |c|_p \le C_{f,p} = 1$. Therefore $|a|_p = 1$ and $|a|_p^{-1} = 1$. Moreover $|c|_p^2 \le 1 = |a|_p^{-1}$ which is the desired result for the Case i.
- Case ii: If $C_{f,p} = |a|_p^{\frac{-1}{2}}$ then $|a|_p^{\frac{-1}{2}} > 1$. We take $(-2)^{nd}$ power of both sides. It gives us $|a|_p < 1$. We assume that f is post-critically finite. So $|c|_p \le C_{f,p} = |a|_p^{\frac{-1}{2}}$. We take 2^{nd} power of both sides again. It gives us $|c|_p^2 \le |a|_p^{-1}$ which is the desired result for the case ii.

Lemma 2.3.7 works when p is an odd prime. Now, we will find 2-adic bounds on the coefficients of $f(x) = a(-2x^3 + 3x^2) + c$.

Lemma 2.3.8. *[3, Lemma 4.7] If $f(x) = a(-2x^3+3x^2)+c \in \mathbb{Q}[x]$ is post-critically finite then $|2a|_2 \le 1$ and $|2c|_2 \le 1$. Moreover $2a \in \mathbb{Z}$.*

Proof. We know that $|a|_p \le 1$ for odd primes p. So $|2a|_p \le 1$ for odd primes p. If we prove that $|2a|_2 \le 1$ then we will deduce immediately that $2a \in \mathbb{Z}$. So,

$$C_{f,2} = \max\left\{\left|\frac{3}{2}\right|_2, \left|\frac{1}{2a}\right|_2^{\frac{1}{2}}\right\}.$$

Here we have $\left|\frac{3}{2}\right|_2 = 2 > 1$. It turns out that

$$C_{f,2} = \max\left\{2, \left|\frac{1}{2a}\right|_2^{\frac{1}{2}}\right\}.$$

We have two cases:

- Case i: If $C_{f,2} = 2$ then $|a|_2 \leq 2$ and $|c|_2 \leq 2$ since f is post-critically finite. Thus we divide both two inequalities by $|2|_2 = \frac{1}{2}$. So $|2a|_2 \leq 1$ and $|2c|_2 \leq 1$. Hence $2a \in \mathbb{Z}$.

- Case ii: If $C_{f,2} = \left|\frac{1}{2a}\right|_2^{\frac{1}{2}}$ then $\left|\frac{1}{2a}\right|_2^{\frac{1}{2}} > 2$. We take 2^{nd} power of both sides $\left|\frac{1}{2a}\right|_2 > 4$. We multiply both sides by $|2a|_2$ and $\frac{1}{4}$ then we obtain $|2a|_2 < \frac{1}{4} < 1$. Since $|a| < 4$ and $2a \in \mathbb{Z}$ then

$$a \in \left\{\frac{-7}{2}, \frac{-6}{2}, ..., \frac{6}{2}, \frac{7}{2}\right\} - \{0\}.$$

  However, $|2a|_2 = |\pm 7|_2 = 1$, $|2a|_2 = |\pm 6|_2 = \frac{1}{2}$,...,$|2a|_2 = |\pm 2|_2 = \frac{1}{2}$, $|2a|_2 = |\pm 1|_2 = 1$. On the other hand we found that $|2a|_2 > \frac{1}{2}$. Hence we cannot have $C_{f,2} = \left|\frac{1}{2a}\right|_2^{\frac{1}{2}}$. Thus the only possibility is $C_{f,2} = 2$ when f is post-critically finite.

If f is post-critically finite then $|2a|_2 \leq 1$ and $|2c|_2 \leq 1$ and $2a \in \mathbb{Z}$.

Proposition 2.3.9. *[3, Proposition 4.8] If $f(x) = a(-2x^3 + 3x^2) + c \in \mathbb{Q}[x]$ is post-critically finite then*

$$a \in \left\{\frac{k}{2} : k \in \mathbb{Z}^{\times}, -8 < k < 8\right\}$$

and

$$c \in \left\{\frac{l}{2} : l \in \mathbb{Z}, -4 \leq l \leq 4\right\}.$$

Proof. The set for a is the same set which we introduced in the proof of Lemma 2.3.8. Remember that $|a| < 4$ and $2a \in \mathbb{Z}$. So, $|a|_p = 1$ whenever $p > 7$. On the other hand, if $p = 3$ then $|a|_3 \geq \frac{1}{3}$ for all possible values for a. If $p = 5$ then $|a|_5 \geq \frac{1}{5}$ for all possible values for a. If $p = 7$ then $|a|_7 \geq \frac{1}{7}$. So $|a|_p^{-1} \leq p$ for $p \in \{3, 5, 7\}$. By Lemma 2.3.7 $|c|_p \leq 1$ in all cases. We know that $|2c|_p \leq 1$. Thus $|2c|_p \leq 1$ for all primes p and $2c \in \mathbb{Z}$. Now, suppose that $|\alpha| \geq |c| \geq \frac{5}{2}$ and $a \in \left\{\frac{k}{2} : k \in \mathbb{Z}^{\times}, -8 < k < 8\right\}$. We have the following inequalities:

$$\begin{aligned}
|f(\alpha)| &= |a(-2\alpha^3 + 3\alpha^2) + c| \\
&\geq |a\alpha^2(-2\alpha + 3)| - |c| \\
&\geq |a| \cdot |\alpha^2| \cdot | - 2\alpha + 3| - |c| \\
&\geq |a| \cdot |\alpha^2| \cdot | - 2\alpha + 3| - |\alpha| \\
&\geq |\alpha|(|a| \cdot |\alpha| \cdot | - 2\alpha + 3| - 1)
\end{aligned}$$

Here $|a| \geq \frac{1}{2}$ and $|\alpha| \geq \frac{5}{2}$ implies that $|a| \cdot |\alpha| \geq \frac{5}{4} > 1$ and $| - 2\alpha + 3| > 1$. So we have

$$|\alpha|(|a| \cdot |\alpha| \cdot | - 2\alpha + 3| - 1) > |\alpha|.$$

Hence $|f(\alpha)| > |\alpha|$ means that α is a wandering point of f. Since 0 is a critical point of f then 0 must be a wandering point because $f(0) = c$ and $|c| \geq \frac{5}{2}$. However, f is post-critically finite . It is a contradiction. Thus $|c| < \frac{5}{2}$.

Therefore

$$c \in \left\{ \frac{l}{2} : \ l \in \mathbb{Z}, -4 \leq l \leq 4 \right\}.$$

□

Theorem 2.3.10. *[3, Theorem 5.1] Let $f(x) \in \mathbb{Q}[x]$ be a bicritical post-critically finite polynomial of degree 3 with rational critical points. Then $f(x)$ is linearly conjugate to the polynomial $f_{(a,c)}(x) = a(-2x^3 + 3x^2) + c \in \mathbb{Q}[x]$ where*

$$(a,c) \in \left\{ \Big(-2, \frac{3}{2}\Big), \Big(\frac{-3}{2}, 1\Big), \Big(-1, \frac{1}{2}\Big), \Big(1, \frac{1}{2}\Big), \Big(\frac{-1}{2}, 0\Big), \Big(-1, 1\Big), \right.$$
$$\left. \Big(\frac{1}{2}, -1\Big), \Big(\frac{1}{2}, 1\Big), \Big(1, 0\Big), \Big(\frac{3}{2}, 0\Big), \Big(2, \frac{-1}{2}\Big) \right\}.$$

Proof. These values for (a,c) come from the sets $a \in \left\{ \frac{k}{2} : \ k \in \mathbb{Z}^{\times}, \ -8 < k < 8 \right\}$ and $c \in \left\{ \frac{l}{2} : \ l \in \mathbb{Z}, -4 \leq l \leq 4 \right\}$. So there are 126 possible value for (a,c) that makes $f_{(a,c)}(x)$ a post-critically finite polynomial. After checking all values with Sage [21], we see that only 11 of them is actually a post-critically finite polynomial. □

2.3.2 Cubic PCF Polynomials with Irrational Critical Points

We previously found in Proposition 2.2.2 that a bicritical polynomial of degree 3 with two irrational critical points is linearly conjugate to a polynomial $f(x) = a(\frac{x^3}{3} - \alpha x) + c \in \mathbb{Q}[x]$. In this section we will find some relations between the quantites a, α and c. Then using an algorithm which we will introduce later this section, we will give the possible triples (α, a, c) which makes $f(x)$ a post-critically finite polynomial.

Proposition 2.3.11. *[3, Proposition 4.9] If $f(x) = a(\frac{x^3}{3} - \alpha x) + c \in \mathbb{Q}[x]$ is post-critically finite then*

$$a\alpha \in \left\{ \pm\frac{3}{4}, \pm\frac{3}{2}, \pm\frac{9}{4}, \pm 3, \pm\frac{15}{4}, \pm\frac{9}{2}, \pm\frac{21}{4} \right\}.$$

Proof. In fact this list is coming from the set for the coefficient a for bicritical polynomials of degree 3 with rational critical points that are post-critically finite which we found in the previous section, Proposition 2.3.9 . By conjugating the polynomial $f(x) = a(\frac{x^3}{3} - \alpha x) + c \in \mathbb{Q}[x]$ with $l(x) = \frac{x-\sqrt{\alpha}}{-2\sqrt{\alpha}} \in \mathbb{Q}(\sqrt{\alpha})[x]$ we obtain

$$f^l(x) = \frac{-2a}{3} \cdot \alpha(-2x^3 + 3x^2) + \frac{a}{3} \cdot \alpha - \frac{1}{2}\left(\frac{c\sqrt{\alpha} - \alpha}{\alpha}\right).$$

Since the bound on a does not depend on the constant term c in the last section therefore if $f^l(x)$ is post-critically finite then

$$\frac{-2}{3}a\alpha \in \left\{\frac{k}{2} : k \in \mathbb{Z}^\times, -8 < k < 8\right\}.$$

Finally,

$$a\alpha \in \left\{\frac{3k}{4} : k \in \mathbb{Z}^\times, -8 < k < 8\right\}.$$

□

We have to find bounds on the constant term $c \in \mathbb{Q}$.

Lemma 2.3.12. *[3, Lemma 4.10] If $f(x) = a(\frac{x^3}{3} - \alpha x) + c \in \mathbb{Q}[x]$ is post-critically finite and $a\alpha \in \left\{\frac{3k}{4} : k \in \mathbb{Z}^\times, -8 < k < 8\right\}$ then $|c|^2 < 11|\alpha|$.*

Proof. We know that the critical points of $f(x)$ are $\sqrt{\alpha}$ and $-\sqrt{\alpha}$. So

$$f(\sqrt{\alpha}) = \frac{a\alpha\sqrt{\alpha}}{3} - a\alpha\sqrt{\alpha} + c = \frac{a\alpha\sqrt{\alpha} - 3a\alpha\sqrt{\alpha}}{3} + c = \frac{-2a\alpha\sqrt{\alpha}}{3} + c$$

and $f(-\sqrt{\alpha}) = \frac{2a\alpha\sqrt{\alpha}}{3} + c$. Suppose that $|c|^2 \geq 11|\alpha|$. Let $\alpha \in \mathbb{C}$ such that $|\alpha| > |c|$. Then

$$\begin{aligned}
|f(\alpha)| &= \left|\frac{a\alpha^3}{3} - a\alpha^2 + c\right| \\
&> \left|\frac{a\alpha^3}{3} - a\alpha^2\right| - |c| \\
&> \left|\frac{a\alpha^3}{3} - a\alpha^2\right| - |\alpha| \\
&= |\alpha|^2\left|\frac{a\alpha}{3} - a\right| - |\alpha| \\
&= |\alpha|(|\alpha| \cdot |a| \cdot \left|\frac{\alpha}{3} - 1\right| - 1) \\
&> |\alpha|
\end{aligned}$$

Thus $|f(\alpha)| > |\alpha| > |c|$ means that α is a wandering point of f. We know that $a \neq 0$. It implies that one of the critical points α_1 will satisfy $|f(\alpha_1)| > |c|$. It is a contradiction. Hence $|c|^2 < 11|\alpha|$. □

Before giving the algorithm which serves to find post-critically finite polynomials of degree 3 with two irrational critical points, we will find bounds on a and c.

Lemma 2.3.13. *[3, Lemma 4.11] If $f(x) = a(\frac{x^3}{3} - \alpha x) + c \in \mathbb{Q}[x]$ is post-critically bounded then*

- $|c\sqrt{a}|_2 \leq 2^3$,
- $|c\sqrt{a}|_3 \leq \frac{\sqrt{3}}{3}$,

- $|c\sqrt{a}|_p \leq 1$, *if* $p \geq 5$.

Proof. First we define $l(x) = \sqrt{\frac{a}{3}}x - \beta \in PGL_2(\bar{\mathbb{Q}})$ where $\beta^3 - (a\alpha+1)\beta + \sqrt{\frac{a}{3}}c = 0$ so that the polynomial $g(x) = f^l(x)$ is monic and 0 is a fixed point of $g(x)$. Then

$$g(x) = x^3 + 3\beta x^2 + (3\beta^2 - a\alpha)x$$

Critical points of g are $-\beta \pm \sqrt{\frac{a\alpha}{3}}$. By [1, Theorems 1.2 and 4.1] if g is post-critically bounded then for $p \geq 3$

$$\left| -\beta \pm \sqrt{\frac{a\alpha}{3}} \right|_p \leq 1$$

and

$$\left| -\beta \pm \sqrt{\frac{a\alpha}{3}} \right|_2 \leq 2.$$

Using the ultrametric triangle inequality

$$\left| -\beta + \sqrt{\frac{a\alpha}{3}} + \left(-\beta - \sqrt{\frac{a\alpha}{3}} \right) \right|_p = |-2\beta|_p \leq \max\left\{ \left| -\beta + \sqrt{\frac{a\alpha}{3}} \right|_p, \left| -\beta - \sqrt{\frac{a\alpha}{3}} \right|_p \right\} \leq 1.$$

For $p \geq 3$ we have $|\beta|_p \leq 1$.

Now, suppose a coefficient of the polynomial $h(x) = x^3 - (a\alpha + 1)x + \sqrt{\frac{a}{3}}c$ has negative valuation for a prime p. Say $a_3 = 1$, $a_2 = -(a\alpha + 1)$ and $a_1 = \sqrt{\frac{a}{3}}c$. So there exists an $i \in \{1, 2, 3\}$ such that $v_p(a_i) < 0$. A segment of the Newton polygon of $h(x)$ have positive slope λ. By Newton polygon algorithm it means that the valuation of at least one root of $h(x)$ equals $-\lambda$ which implies that the p-adic absolute value of this root is greater than 1. Every coefficient must lie in the p-adic unit disk. So we have $|c\sqrt{\frac{a}{3}}|_p \leq 1$. In particular $|c\sqrt{a}|_3 \leq \frac{1}{\sqrt{3}}$ and $|c\sqrt{a}|_p \leq 1$ if $p > 5$. We know that $|-\beta \pm \sqrt{\frac{a\alpha}{3}}|_2 \leq 2$ and $a\alpha \in \left\{ \frac{3k}{4} : k \in \mathbb{Z}^\times, -8 < k < 8 \right\}$. For all values of $a\alpha$ we have $|\sqrt{a\alpha}|_2 \leq 2$. So $|\sqrt{\frac{a\alpha}{3}}|_2 \leq 2$ since $|\frac{1}{\sqrt{3}}|_2 = 1$. By triangle inequality

$$\left| -\beta \pm \sqrt{\frac{a\alpha}{3}} \right|_2 \leq \max\left\{ |-\beta|_2, \left| \pm \sqrt{\frac{a\alpha}{3}} \right|_2 \right\}.$$

If $|\pm \sqrt{\frac{a\alpha}{3}}|_2$ is the maximum then $|-\beta|_2 \leq 2$. If $|-\beta|_2$ is the maximum then $|-\beta|_2 \leq 2$. Because otherwise $|-\beta \pm \sqrt{\frac{a\alpha}{3}}|_2$ would be greater than 2. Therefore again by Newton polygon algorithm at $p = 2$, the Newton polygon of $h(x)$ has a segment of slope at most 1. Since the polynomial $h(x)$ is cubic, it means that the constant term must satisfy $v_2(c\sqrt{\frac{a}{3}}) \geq -3$. Hence $|c\sqrt{a}|_2 \leq 2^3$. □

Currently, we have enough information about bounds on coefficients of a post-critically finite polynomial of degree 3 with irrational critical points. However, not all numbers satisfying these bounds are coefficients of a post-critically finite polynomial. We introduce an algorithm to determine which triple of the form (α, a, c) makes the polynomial $f_{(\alpha,a,c)}(x) = a(x^3/3 - \alpha x) + c \in \mathbb{Q}[x]$ post-critically finite.

1. If $f_{(\alpha,a,c)}(x)$ is post-critically finite then $a\alpha \in \left\{ \frac{3k}{4} : k \in \mathbb{Z}^\times, -8 < k < 8 \right\}$. We go over all possible values of $a\alpha$.

2. Let $\alpha \in \mathbb{Z}^\times/(\mathbb{Z}^\times)^2$. We look at the 2-adic absolute value for every value of $a\alpha$. We calculate $|a|_2$ by using the parity of α. We may avoid negative values of $a\alpha$ for now since the calculations are the same as for positive values.

   - For example, if $a\alpha = \frac{3}{2}$ then $|a\alpha|_2 = |a|_2|\alpha|_2 = |\frac{3}{2}|_2 = 2$. If $\alpha = 2k$ where k is odd and squarefree, then $|a|_2|2k|_2 = \frac{1}{2}|a|_2 = 2$ implies that $|a|_2 = 4$. If $\alpha = 2k+1$ then $|a|_2 = 2$.

   Repeat this calculation for every value of $a\alpha$. We find that $|a|_2 \in \{1, 2, 4, 8\}$ depending on the parity of α.

3. By Lemma 2.3.13, if $f_{(\alpha,a,c)}(x)$ is post-critically bounded then $|c\sqrt{a}|_2 \le 2^3$, $|c\sqrt{a}|_3 \le \frac{1}{\sqrt{3}}$ and $|c\sqrt{a}|_p \le 1$ if $p > 3$. Using values of $|a|_2$ we found in the second step, we can find $t < 4$ such that $|c|_2 \le 2^t$. We know values $|a\alpha|_p$ for $p \ge 3$ using the list for $a\alpha$.

   - For example, for $\alpha = pk$ where $\gcd(k,p) = 1$ and k is squarefree, we have $|a\alpha|_p = \frac{1}{p}|a|_p = 1$ then $|a|_p = p$. In that case $|c|_p \le 1$ since $|c\sqrt{a}|_p \le 1$ for $p \ge 3$. For $\alpha = pk + i$ where $0 < i < p$, $\gcd(k,p) = 1$ and k is squarefree, we have $|a|_p = 1$. Thus $|c|_p \le 1$.

   Therefore for all values of $a\alpha$ and primes $p \ge 3$ we have $|c|_p \le 1$.

4. We have two cases for α: $\alpha = ks$ or $\alpha = 2ks$ where k and s are squarefree odd integers and $gcd(k,s) = 1$ such that when $a = \frac{a_1}{a_2}$ where $\gcd(a_1,a_2) = 1$, we have $s \mid a_2$. Since $|c\sqrt{a}|_p \le 1$ for $p \ge 3$ then $c\sqrt{a}$ lies in the p-adic unit disk for $p \ge 3$. Thus $s \mid c$. Therefore, $c = \frac{sl}{2^t}$ for $l \in \mathbb{Z}$.

5. By Lemma 2.3.12, if $f_{(\alpha,a,c)}(x)$ is post-critically bounded at archimedean place then $|c|^2 < 11|\alpha|$. For $\alpha = ks$ we have
$$\frac{s^2l^2}{2^{2t}} < 11ks.$$
   For $\alpha = 2ks$ we have
$$\frac{s^2l^2}{2^{2t}} < 22ks.$$
   Therefore, $sl^2 < 2^{2t} \cdot 11k$ if α is odd or $sl^2 < 2^{2t+1} \cdot 11k$ if α is even.

6. Fixing each odd squarefree integer $s < 2^{2t} \cdot 11k$ or $s < 2^{2t+1} \cdot 11k$ we find all numbers l such that $sl^2 < 2^{2t} \cdot 11k$ or $sl^2 < 2^{2t+1} \cdot 11k$.

7. By finding triples of the form (k, s, l) we can find the triples of the form $(\alpha, a, c) = (ks, \frac{a\alpha}{ks}, \frac{sl}{2^t})$ or $(\alpha, a, c) = (2ks, \frac{a\alpha}{2ks}, \frac{sl}{2^t})$. For each a, c in these triples we check that $3 \mid ac$ to satisfy the condition $|c\sqrt{a}|_3 \le \frac{1}{\sqrt{3}}$. If $3 \mid ac$ then the triple (α, a, c) becomes a candidate.

There are 5957 triples which are outcomes of this algorithm and they correspond to 23828 possible post-critically finite polynomials. We used Sage [21] to test these triples whether they correspond to post-critically finite polynomials or not. This code gives only two triples which correspond to two polynomials. They are not linearly conjugate to post-critically finite polynomials with rational critical points.

Theorem 2.3.14. *[3, Theorem 5.2] Let $f(x) \in \mathbb{Q}[x]$ be a bicritical post-critically finite polynomial of degree 3. Assume that $f(x)$ is not linearly conjugate to a bicritical polynomial with rational critical points. Then there exists $l(x) = ax + b \in PGL_2(\bar{\mathbb{Q}})$ such that $f^l(x) = f_{(\alpha,a,c)}(x) = a(\frac{x^3}{3} - \alpha x) + c$ where $(\alpha, a, c) = (2, \frac{-3}{4}, 2)$ or $(\alpha, a, c) = (-7, \frac{-3}{28}, \frac{7}{2})$.*

Proof. If $c = 0$ then there exists $l(x) = \frac{a-\sqrt{\alpha}}{-x\sqrt{\alpha}} \in PGL_2(\bar{\mathbb{Q}})$ such that $f_{(\alpha,a,0)}(x)$ is conjugate to a polynomial of degree 3 with rational critical points via $l(x) = \frac{x-\sqrt{\alpha}}{-2\sqrt{\alpha}}$. In fact $f_{(\alpha,a,-c)}(x)$ is linearly conjugate to $f_{(\alpha,a,c)}(x)$ via $l(x) = -x$. So we can suppose that $c > 0$. Thus each triple (α, a, c) where α, a, c are all positive corresponds to four triples depending on the signs of α and a. Running the above algorithm and using Sage will give us the desired result. □

Chapter 3

An Algorithm to Find Preperiodic Points

In this chapter we will introduce an algorithm which finds all rational preperiodic points of a given map. The following proposition is introduced by B. Hutz in [8] over number fields. It is a special case of the [9, Theorem 1]. Also, the algorithm over local field can be found in [20, Theorem 2.21].

Let R be a discrete valuation ring, K its field of fractions, $\mathfrak{p}$ a uniformizer and k is the residue field with $char(k) = p$.

If a map f has good reduction modulo $\mathfrak{p}$ then $\overline{f^n(x)} = \overline{f^n}(\overline{x})$. Firstly, we need to show that when we reduce a periodic point α under f modulo some prime then the period of the reduced point $\bar{\alpha}$ must divide the period of α.

Proposition 3.0.1. *[20, Corollary 2.20] Let $f(x) \in K(x)$ be a rational map with good reduction. Then the reduced preperiodic points of $f(x)$ are preperiodic points of the reduced rational map $\overline{f(x)}$. Moreoever, the exact period n of a reduced periodic point divides the exact period m of the corresponding periodic point in K.*

Proof. Suppose that α is a point of exact period m, i.e., $f^m(\alpha) = \alpha$. We have $\overline{\alpha} = \overline{f^m(\alpha)} = \overline{f^m}(\overline{\alpha})$. It means that $\overline{\alpha}$ is a point of exact period n under the reduced map $\overline{f}$. Now, let n be the exact period of $\overline{\alpha}$. So $m = nq + r$ where $0 \leq r < n$. We have

$$\overline{\alpha} = \overline{f^m}(\overline{\alpha}) = \overline{f^{nq+r}}(\alpha) = \overline{f^r} \circ \overline{f^n} \circ ... \circ \overline{f^n}(\overline{\alpha}) = \overline{f^r}(\overline{\alpha})$$

Thus $\overline{f^r}(\overline{\alpha}) = \overline{\alpha}$ where $0 \leq r < n$ which contradicts the fact that n is the exact period of $\overline{\alpha}$. Hence n divides m. □

Now, we introduce the proposition and the algorithm of Hutz for rational maps defined over a number field K.

Proposition 3.0.2. *[8, Proposition 2] Let $f(x) \in K(x)$ be a rational map with good reduction at $\mathfrak{p}$. If $\alpha \in K$ is a point of exact period n and $\overline{\alpha} \in k$ is a point of exact period m_p under $\overline{f(x)}$ then*

$$n = m_p \text{ or } n = m_p \cdot r_p \cdot p^e$$

for some $e \geq 0$ and r_p is the multiplicative order of the multiplier $\overline{(f^{m_p})'(\alpha)}$ in k.

Here is the steps of the algorithm;

1. For some prime $\mathfrak{p}$ where f has good reduction, determine all periods of periodic points modulo $\mathfrak{p}$. Find the multiplicative order of the multiplier at every periodic point in the residue field k and p^e.

2. Take the gcd of all determined possible periods modulo every chosen prime.

3. There will be several possible periods n over K. Solve the equation $f^n(x)-x = 0$ for all n we found in Step 2. We will obtain all rational periodic points of f.

4. Forward images of preperiodic points are periodic points. To find a rational preperiodic point β, find rational solutions to the equation $f(\beta) = \alpha$ for every known rational preperiodic point α. Repeat this process for every rational preperiodic points. We will obtain all rational preperiodic points of f.

Since a polynomial defined over $\mathbb{Q}$ has bad reduction at finitely many primes then we will consider only a small number of good primes. These small primes will give us enough information to obtain global information.

Here is an example which shows how this algorithm applies to a polynomial.

Example 3.0.3. *Let $f(x) = 2x^3 - 3x^2 + 1 \in \mathbb{Q}[x]$. So f has a bad reduction at $p = 2$. All primes different from 2 are good primes according to our definition of good reduction. For now, choose two small primes $p = 3$ and $p = 5$. We will compute possible periods m_p and n_p for each $p = 3$ and $p = 5$.*

1. ***p=3**, $f(x) \equiv 2x^3 + 1 \mod 3$. We iterate f for all points in modulo 3 which are $0, 1, 2$.*

   - $f(0) \equiv 1 \mod 3$
   - $f(1) \equiv 0 \mod 3$

   So 0 and 1 are points of period 2 modulo 3.

   - $f(2) \equiv 2 \mod 3$

   So 2 is a fixed point modulo 3. Thus the possible periods modulo 3 are $m_3 = 1, 2$. To ease the calculations we may just take all possible multiplicative orders r_3 of the multipliers at each periodic point since the multiplicative group of the residue field is of order 2. So $r_3 \mid 2$. Finally all possible periods modulo 3 are $n_3 = 2^k 3^e$ where $0 \leq k \leq 2$.

2. ***p=5**, $f(x) \equiv 2x^3 - 3x^2 + 1 \mod 5$. We iterate f for each point in modulo 5 which are $0, 1, 2, 3, 4$.*

   - $f(0) \equiv 1 \mod 5$
   - $f(1) \equiv 0 \mod 5$

So 0 *and* 1 *are points of period* 2 *modulo* 5.

- $f(2) \equiv 0 \bmod 5$

We know that 0 *is a point of period* 2. *So* 2 *is a preperiodic point of type* $(1,2)$ *modulo* 5.

- $f(3) \equiv 3 \bmod 5$

So 3 *is a fixed point modulo* 5.

- $f(4) \equiv 1 \bmod 5$

We know that 1 *is a point of period* 2. *So* 4 *is a preperiodic point of type* $(1,2)$ *modulo* 5. *Thus* $m_5 = 1, 2$ *and* $r_5 \mid 4$. *Therefore* $n_5 = 2^k 5^e$ *where* $0 \leq k \leq 3$.

By computing the possible values for n_3 *and* n_5 *and using Proposition 3.0.2, we obtain* $n = 2^k$ *where* $0 \leq k \leq 2$. *So we need to check that if there is any rational periodic point of period* $1, 2$ *or* 4. *It can be done by finding rational roots of* 1^{st}, 2^{nd} *and* 4^{th} *dynatomic polynomials of* f. *After searching rational roots, It turns out that;*

- *The only* $\mathbb{Q}$*-rational root of* $\Phi_{1,f}(x)$ *is* $\frac{1}{2}$.
- $\mathbb{Q}$*-Rational roots of* $\Phi_{2,f}(x)$ *are* 0 *and* 1.
- $\Phi_{4,f}(x)$ *has no* $\mathbb{Q}$*-rational root.*

Hence, $\frac{1}{2}$ *is a fixed point of* f, 0 *and* 1 *are periodic points of exact period* 2 *under* f, *and* f *has no* $\mathbb{Q}$*-rational periodic point of period* 4.

Now, we have to determine all rational preperiodic points of f. *To do this assume that* $\alpha \in \mathbb{Q}$ *is a point. We will try to find rational solutions to the equation* $f(\alpha) - \beta = 0$ *for each known rational preperiodic points. For now the only known rational preperiodic points are periodic points* $\beta = 0, \frac{1}{2}, 1$. *After solving the equation, we repeat this process until no new rational preperiodic point appears.*

- $f(\alpha) = 0$ *has two rational solutions which are* $\alpha = \frac{-1}{2}$ *and* $\alpha = 1$. *We know that every periodic point of period* n *is a preperiodic points of type* $(0, n)$. *Hence* $\alpha = \frac{-1}{2}$ *is a possible rational preperiodic point. Since* $f(\frac{-1}{2}) = 0$ *and* 0 *is a point of exact period* 2 *then* $f^3(\frac{-1}{2}) = f(\frac{-1}{2})$ *and* $\frac{-1}{2}$ *is a preperiodic point of type* $(1, 2)$.
- $f(\alpha) = \frac{1}{2}$ *has one rational solution which is* $\frac{1}{2}$. *However* $\frac{1}{2}$ *is a fixed point.*
- $f(\alpha) = 1$ *has two rational solutions which are* 0 *and* $\frac{3}{2}$. *However* 0 *is a periodic point of exact period* 2. *So* $\frac{3}{2}$ *is a possible rational preperiodic point. Since* $f(\frac{3}{2}) = 1$ *and* 1 *is a point of exact period* 2 *then* $f^3(\frac{3}{2}) = f(\frac{3}{2})$ *and* $\frac{3}{2}$ *is a preperiodic point of type* $(1, 2)$.
- $f(\alpha) = -\frac{1}{2}$ *has no rational solution.*
- $f(\alpha) = \frac{3}{2}$ *has no rational solution.*

Finally, all rational preperiodic points of f *are* $\frac{-1}{2}$ *and* $\frac{3}{2}$. *Each of them are of type* $(1, 2)$.

Chapter 4

Preperiodic Points of Cubic PCF Polynomials

In this chapter, we will find all rational preperiodic points of all classified cubic post-critically finite polynomials defined over $\mathbb{Q}$, [3], also found in Chapter 2, Theorem 2.1.4, Theorem 2.3.10 and Theorem 2.3.14. We will apply Hutz' Algorithm (Proposition 3.0.2) to these polynomials. There are 15 $\overline{\mathbb{Q}}$-conjugacy classes of such polynomials. For 13 classes, our aim is to determine $\mathbb{Q}$-rational preperiodic points since these polynomials are bicritical cubic post-critically finite polynomials with one or two rational critical points. For remaining two classes our treatment will be quite different from the previous classes. These polynomials are bicritical cubic post-critically finite polynomials with two irrational critical points and these points are in a quadratic extension of $\mathbb{Q}$. Therefore, our aim for these polynomials is to determine $\mathbb{Q}(\sqrt{\alpha})$-rational preperiodic points where $\alpha = 2$ for one representative and $\alpha = -7$ for the representative of the other class.

4.1 Calculations

4.1.1 Cubic Unicritical PCF Polynomials

Some small good primes of $f(x) = x^3$ are 3 and 5.

- $f(x) \equiv x^3 \bmod 3$
- $f(x) \equiv x^3 \bmod 5$

In Table 4.1, we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 1$. We check whether the polynomial $f(x) = x^3$ has rational periodic points of exact period 1 or 2. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial by Proposition 1.3.6. Here we will find rational roots of first and second dynatomic polynomial of $f(x) = x^3$.

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
x^3	$p = 3$	$f(0) \equiv 0 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$, $0 \leq k \leq 1$
	$p = 5$	$f(0) \equiv 0 \bmod 5$ $f(1) \equiv 1 \bmod 5$ $f(2) \equiv 3 \bmod 5$ $f(3) \equiv 2 \bmod 5$ $f(4) \equiv 4 \bmod 5$	$m_5 = 1, 2$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$, $0 \leq k \leq 3$

Table 4.1: Polynomial 1

- $\Phi_{1,f}(x) = 0$ has three rational solutions which are $-1, 0, 1$.
- $\Phi_{2,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $-1, 0, 1$. Now, we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -1$ has only one rational solution which is -1. However -1 is a periodic point.
- $f(\alpha) = 0$ has only one rational solution which is 0. However 0 is a periodic point.
- $f(\alpha) = 1$ has only one rational solution which is 1. However 1 is a periodic point.

Since we couldn't find any new rational solution to the equations above then $f(x)$ has no rational preperiodic points other than these periodic points.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
x^3	$(-1, 1), (0, 1), (1, 1)$	——	3

Figure 4.1.1: Polynomial 1

Some small good primes of $f(x) = -x^3 + 1$ are 2 and 3.

- $f(x) \equiv x^3 + 1 \bmod 2$
- $f(x) \equiv -x^3 + 1 \bmod 3$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-x^3+1$	$p=2$	$f(0)\equiv 1 \bmod 2$ $f(1)\equiv 0 \bmod 2$	$m_2=2$	$r_2 \mid 1$	$n_2=2^k$
	$p=3$	$f(0)\equiv 1 \bmod 3$ $f(1)\equiv 0 \bmod 3$ $f(2)\equiv 2 \bmod 3$	$m_3=1,2$	$r_3 \mid 2$	$n_3=2^k\cdot 3^e$, $0\le k\le 2$

Table 4.2: Polynomial 2

In Table 4.2, we have found n_2 and n_3. Now, according to the algorithm, Proposition 3.0.2, the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \le k \le 2$. We check whether the polynomial $f(x) = -x^3 + 1$ has rational periodic points of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -x^3 + 1$.

- $\Phi_{1,f}(x) = 0$ has no rational solution.
- $\Phi_{2,f}(x) = 0$ has two rational solutions which are $0, 1$.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $0, 1$. Now, we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = 0$ has only one rational solution which is 0. However 0 is a periodic point.
- $f(\alpha) = 1$ has only one rational solution which is 1. However 1 is a periodic point.

Since we couldn't find any new rational solution to the equations above then $f(x)$ has no rational preperiodic points other than these periodic points. According to

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-x^3+1$	$(0,2),(1,2)$	——	2

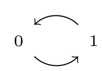

Figure 4.1.2: Polynomial 2

these calculations an immediate result can be stated as follows:

Theorem 4.1.1. *Let $f(x) \in \mathbb{Q}[x]$ be an unicritical polynomial of degree 3 linearly conjugate to either x^3 or $-x^3+1$. Then $f(x)$ can have at most 3 $\mathbb{Q}$-rational preperiodic points. In fact these preperiodic points are periodic.*

4.1.2 Cubic Bicritical PCF Polynomials with Two Rational Critical Points

Some small good primes of $f(x) = -2x^3 + 3x^2 + \frac{1}{2}$ are 3 and 5.

- $f(x) \equiv -2x^3 + 2 \bmod 3$
- $f(x) \equiv -2x^3 + 3x^2 + 3 \bmod 5$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-2x^3+3x^2+\frac{1}{2}$	$p=3$	$f(0) \equiv 2 \bmod 3$ $f(2) \equiv 1 \bmod 3$ $f(1) \equiv 0 \bmod 3$	$m_3 = 3$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \le k \le 1$
	$p=5$	$f(0) \equiv 3 \bmod 5$ $f(3) \equiv 1 \bmod 5$ $f(1) \equiv 4 \bmod 5$ $f(4) \equiv 3 \bmod 5$ $f(2) \equiv 4 \bmod 5$	$m_5 = 3$	$r_5 \mid 4$	$n_5 = 2^k \cdot 3^l \cdot 5^e$ $0 \le k \le 2$ $0 \le l \le 1$

Table 4.3: Polynomial 3

In Table 4.3, we have found n_3 and n_5. Now, according to the algorithm, Proposition 3.0.2, the possible periods over $\mathbb{Q}$ are given by $n = 2^k \cdot 3^l$ where $0 \le k \le 1$ and $0 \le l \le 1$. We check whether the polynomial $f(x) = -2x^3 + 3x^2 + \frac{1}{2}$ has rational periodic points of exact period $1, 2, 3$ or 6. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of f are rational periodic points under f. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -2x^3 + 3x^2 + \frac{1}{2}$.

- $\Phi_{1,f}(x) = 0$ has no rational solution.
- $\Phi_{2,f}(x) = 0$ has no rational solution.
- $\Phi_{3,f}(x) = 0$ has three rational solutions which are $1, \frac{1}{2}, \frac{3}{2}$.
- $\Phi_{6,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $1, \frac{1}{2}, \frac{3}{2}$. Now, we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = 1$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.

- $f(\alpha) = \frac{1}{2}$ has two rational solutions which are $0, \frac{3}{2}$. However $\frac{3}{2}$ is a periodic point. So 0 is a rational preperiodic point of type $(1,3)$ since $f^4(0) = f(0)$.

- $f(\alpha) = \frac{3}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However 1 is a periodic point. So $-\frac{1}{2}$ is a rational preperiodic point of type $(1,3)$ since $f^4(-\frac{1}{2}) = f(-\frac{1}{2})$.

- $f(\alpha) = \frac{-1}{2}$ has no rational root.

- $f(\alpha) = 0$ has no rational root.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-2x^3 + 3x^2 + \frac{1}{2}$	$(1,3), (\frac{1}{2},3), (\frac{3}{2},3)$	$(-\frac{1}{2},(1,3)), (0,(1,3))$	5

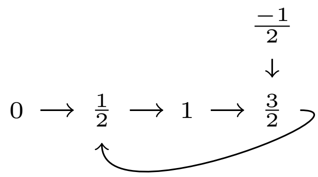

Figure 4.1.3: Polynomial 3

Some small good primes of $f(x) = -2x^3 + 3x^2$ are 3 and 5.

- $f(x) \equiv x^3 \bmod 3$

- $f(x) \equiv -2x^3 + 3x^2 \bmod 5$

In fact, rational preperiodic points of $f(x) = -2x^3 + 3x^2$ and more general form of this polynomial are found here [2]. We gave this result in this thesis for completeness. In Table 4.4 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-2x^3 + 3x^2$	$p = 3$	$f(0) \equiv 0 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 1$
	$p = 5$	$f(0) \equiv 0 \bmod 5$ $f(1) \equiv 1 \bmod 5$ $f(2) \equiv 1 \bmod 5$ $f(3) \equiv 3 \bmod 5$ $f(4) \equiv 0 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$

Table 4.4: Polynomial 4

the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 1$. We check whether the polynomial $f(x) = -2x^3 + 3x^2$ has rational periodic points of exact period 1 or 2. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first and second dynatomic polynomial of $f(x) = -2x^3 + 3x^2$.

- $\Phi_{1,f}(x) = 0$ has three rational solutions which are $0, \frac{1}{2}, 1$.
- $\Phi_{2,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $0, \frac{1}{2}, 1$. Now, we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = 0$ has two rational solutions which are $0, \frac{3}{2}$. However 0 is a periodic point. So $\frac{3}{2}$ is a rational preperiodic point of type $(1,1)$ since $f^2(\frac{3}{2}) = f(\frac{3}{2})$.
- $f(\alpha) = \frac{1}{2}$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.
- $f(\alpha) = 1$ has two rational solutions which are $-\frac{1}{2}, 1$. However 1 is a periodic point. So $-\frac{1}{2}$ is a rational preperiodic point of type $(1,1)$ since $f^2(-\frac{1}{2}) = f(-\frac{1}{2})$.
- $f(\alpha) = \frac{-1}{2}$ has no rational root.
- $f(\alpha) = \frac{3}{2}$ has no rational root.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-2x^3 + 3x^2$	$(0,1), (\frac{1}{2},1), (1,1)$	$(-\frac{1}{2},(1,1)), (\frac{3}{2},(1,1))$	5

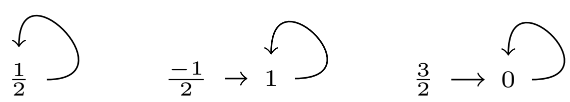

Figure 4.1.4: Polynomial 4

Some small good primes of $f(x) = -x^3 + \frac{3}{2}x^2 - 1$ are 3 and 5.

- $f(x) \equiv -x^3 - 1 \bmod 3$
- $f(x) \equiv -x^3 + 4x^2 - 1 \bmod 5$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-x^3 + \frac{3}{2}x^2 - 1$	$p = 3$	$f(0) \equiv 2 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 0 \bmod 3$	$m_3 = 1, 2$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 2$
	$p = 5$	$f(0) \equiv 4 \bmod 5$ $f(1) \equiv 2 \bmod 5$ $f(2) \equiv 2 \bmod 5$ $f(3) \equiv 3 \bmod 5$ $f(4) \equiv 4 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$

Table 4.5: Polynomial 5

In Table 4.5 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 2$. We check whether the polynomial $f(x) = -x^3 + \frac{3}{2}x^2 - 1$ has rational periodic points of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -x^3 + \frac{3}{2}x^2 - 1$.

- $\Phi_{1,f}(x) = 0$ has only one rational solution which is $-\frac{1}{2}$.
- $\Phi_{2,f}(x) = 0$ has two rational solutions which are $-1, \frac{3}{2}$.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $-1, -\frac{1}{2}, \frac{3}{2}$. Now, we will find rational preperiodic point. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -1$ has two rational solutions which are $0, \frac{3}{2}$. However, $\frac{3}{2}$ is a periodic points. So 0 is a rational preperiodic point of type $(1, 2)$ since $f^3(0) = f(0)$.
- $f(\alpha) = -\frac{1}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However, $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1, 1)$ since $f^2(1) = f(1)$.
- $f(\alpha) = \frac{3}{2}$ has only one rational solution which is -1. However, -1 is a periodic point.
- $f(\alpha) = 1$ has no rational solution.
- $f(\alpha) = \frac{3}{2}$ has a rational solution which is -1. However -1 is a periodic point.

Some small good primes of $f(x) = 2x^3 - 3x^2 + 1$ are 3 and 5.

- $f(x) \equiv 2x^3 + 1 \bmod 3$

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-x^3+\frac{3}{2}x^2-1$	$(-1,2),(-\frac{1}{2},1),(\frac{3}{2},2)$	$(0,(1,2)),(1,(1,1))$	5

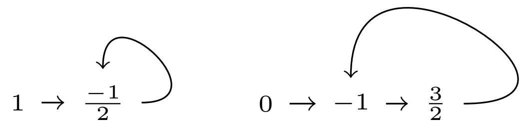

Figure 4.1.5: Polynomial 5

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$2x^3-3x^2+1$	$p=3$	$f(0)\equiv 1 \bmod 3$ $f(1)\equiv 0 \bmod 3$ $f(2)\equiv 2 \bmod 3$	$m_3=1,2$	$r_3 \mid 2$	$n_3=2^k\cdot 3^e$ $0\le k\le 2$
	$p=5$	$f(0)\equiv 1 \bmod 5$ $f(1)\equiv 0 \bmod 5$ $f(2)\equiv 0 \bmod 5$ $f(3)\equiv 3 \bmod 5$ $f(4)\equiv 2 \bmod 5$	$m_5=1,2$	$r_5 \mid 4$	$n_5=2^k\cdot 5^e$ $0\le k\le 3$

Table 4.6: Polynomial 6

- $f(x)\equiv 2x^3-3x^2+1 \bmod 5$

In Table 4.6 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n=2^k$ where $0\le k\le 2$. We check whether the polynomial $f(x)=2x^3-3x^2+1$ has rational periodic point of exact period $1,2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x)=2x^3-3x^2+1$.

- $\Phi_{1,f}(x)=0$ has only one rational solution which is $\frac{1}{2}$.
- $\Phi_{2,f}(x)=0$ has two rational solutions which are $0,1$.
- $\Phi_{4,f}(x)=0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $0,\frac{1}{2},1$. Now we will find rational preperiodic points. Let $\alpha\in\mathbb{Q}$.

- $f(\alpha)=0$ has two rational solutions which are $-\frac{1}{2},1$. However 1 is a periodic point. So $-\frac{1}{2}$ is a rational preperiodic point of type $(1,2)$ since $f^3(-\frac{1}{2})=f(-\frac{1}{2})$.

- $f(\alpha) = \frac{1}{2}$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.
- $f(\alpha) = 1$ has two rational solutions which are $0, \frac{3}{2}$. However, 0 is a periodic point. So $\frac{3}{2}$ is a rational preperiodic point of type $(1, 2)$ since $f^3(\frac{3}{2}) = f(\frac{3}{2})$.
- $f(\alpha) = \frac{-1}{2}$ has no rational solution.
- $f(\alpha) = \frac{3}{2}$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$2x^3 - 3x^2 + 1$	$(0, 2), (\frac{1}{2}, 1), (1, 2)$	$(-\frac{1}{2}, (1, 2)), (\frac{3}{2}, (1, 2))$	5

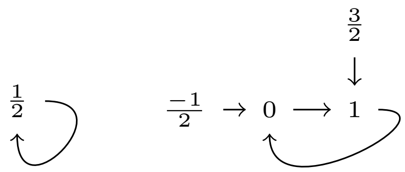

Figure 4.1.6: Polynomial 6

Some small good primes of $f(x) = 2x^3 - 3x^2 + \frac{1}{2}$ are 3 and 5.

- $f(x) \equiv 2x^3 + 2 \bmod 3$
- $f(x) \equiv 2x^3 - 3x^2 + 3 \bmod 5$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$2x^3 - 3x^2 + \frac{1}{2}$	$p = 3$	$f(0) \equiv 2 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 0 \bmod 3$	$m_3 = 1, 2$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 2$
	$p = 5$	$f(0) \equiv 3 \bmod 5$ $f(1) \equiv 2 \bmod 5$ $f(2) \equiv 2 \bmod 5$ $f(3) \equiv 0 \bmod 5$ $f(4) \equiv 3 \bmod 5$	$m_5 = 1, 2$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 3$

Table 4.7: Polynomial 7

In Table 4.7 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 2$. We check whether the polynomial $f(x) = 2x^3 - 3x^2 + \frac{1}{2}$ has rational periodic point of exact period 1, 2 or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic point under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = 2x^3 - 3x^2 + \frac{1}{2}$.

- $\Phi_{1,f}(x) = 0$ has only one rational solution which is $-\frac{1}{2}$.
- $\Phi_{2,f}(x) = 0$ has two rational solutions which are $0, \frac{1}{2}$.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $-\frac{1}{2}, 0, \frac{1}{2}$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -\frac{1}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1,1)$ since $f^2(1) = f(1)$.
- $f(\alpha) = 0$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.
- $f(\alpha) = \frac{1}{2}$ has two rational solutions which are $0, \frac{3}{2}$. However, 0 is a periodic point. So $\frac{3}{2}$ is a rational preperiodic point of type $(1,2)$ since $f^3(\frac{3}{2}) = f(\frac{3}{2})$.
- $f(\alpha) = 1$ has no rational solution.
- $f(\alpha) = \frac{3}{2}$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$2x^3 - 3x^2 + \frac{1}{2}$	$(-\frac{1}{2}, 1), (0, 2), (\frac{1}{2}, 2)$	$(1, (1, 1)), (\frac{3}{2}, (1, 2))$	5

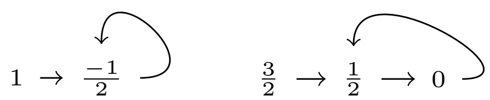

Figure 4.1.7: Polynomial 7

Some small good primes of $f(x) = x^3 - \frac{3}{2}x^2$ are 3 and 5.

- $f(x) \equiv x^3 \bmod 3$
- $f(x) \equiv x^3 + x^2 \bmod 5$

In Table 4.8 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 1$. We check whether the polynomial $f(x) = x^3 - \frac{3}{2}x^2$ has rational periodic point of exact period

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$x^3 - \frac{3}{2}x^2$	$p = 3$	$f(0) \equiv 0 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 1$
	$p = 5$	$f(0) \equiv 0 \bmod 5$ $f(1) \equiv 2 \bmod 5$ $f(2) \equiv 2 \bmod 5$ $f(3) \equiv 1 \bmod 5$ $f(4) \equiv 0 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$

Table 4.8: Polynomial 8

1 or 2. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first and second dynatomic polynomial of $f(x) = x^3 - \frac{3}{2}x^2$.

- $\Phi_{1,f}(x) = 0$ has three rational solutions which are $-\frac{1}{2}, 0, 2$.
- $\Phi_{2,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $-\frac{1}{2}, 0, 2$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -\frac{1}{2}$ has two rational solutions $-\frac{1}{2}, 1$. However $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1, 1)$ since $f^2(1) = f(1)$.
- $f(\alpha) = 0$ has two rational solutions $0, \frac{3}{2}$. However 0 is a periodic point. So $\frac{3}{2}$ is a rational preperiodic point of type $(1, 1)$ since $f^2(\frac{3}{2}) = f(\frac{3}{2})$.
- $f(\alpha) = 2$ has only one rational solution which is 2. However 2 is a periodic point.
- $f(\alpha) = 1$ has no rational solution.
- $f(\alpha) = \frac{3}{2}$ has no rational solution.

Some small good primes of $f(x) = -3x^3 + \frac{9}{2}x^2$ are 5 and 7.

- $f(x) \equiv 2x^3 + 2x^2 \bmod 5$
- $f(x) \equiv -3x^3 + x^2 \bmod 7$

In Table 4.9 we have found n_5 and n_7, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 2$. We check whether the polynomial $f(x) = -3x^3 + \frac{9}{2}x^2$ has rational periodic point

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$x^3 - \frac{3}{2}x^2$	$(-\frac{1}{2}, 1), (0, 1), (2, 1)$	$(1, (1, 1)), (\frac{3}{2}, (1, 1))$	5

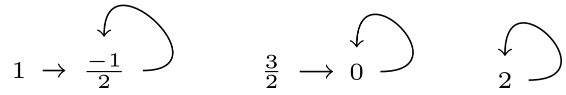

Figure 4.1.8: Polynomial 8

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-3x^3 + \frac{9}{2}x^2$	$p = 5$	$f(0) \equiv 0 \bmod 5$ $f(1) \equiv 4 \bmod 5$ $f(2) \equiv 4 \bmod 5$ $f(3) \equiv 2 \bmod 5$ $f(4) \equiv 0 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$
	$p = 7$	$f(0) \equiv 0 \bmod 7$ $f(1) \equiv 5 \bmod 7$ $f(2) \equiv 1 \bmod 7$ $f(3) \equiv 5 \bmod 7$ $f(4) \equiv 6 \bmod 7$ $f(5) \equiv 0 \bmod 7$ $f(6) \equiv 4 \bmod 7$	$m_7 = 1, 2$	$r_7 \mid 6$	$n_7 = 2^k \cdot 3^l \cdot 7^e$ $0 \leq k \leq 2$ $0 \leq l \leq 1$

Table 4.9: Polynomial 9

of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -3x^3 + \frac{9}{2}x^2$.

- $\Phi_{1,f}(x) = 0$ has only one rational solution which is 0.
- $\Phi_{2,f}(x) = 0$ has no rational solution.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, only rational periodic point of $f(x)$ is 0. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = 0$ has two rational solutions which are $0, \frac{3}{2}$. However 0 is a periodic point. So $\frac{3}{2}$ is a rational preperiodic point of type $(1, 1)$ since $f^2(\frac{3}{2}) = f(\frac{3}{2})$.
- $f(\alpha) = \frac{3}{2}$ has two rational solutions $-\frac{1}{2}, 1$. So $-\frac{1}{2}$ is a rational preperiodic point of type $(2, 1)$ since $f^3(-\frac{1}{2}) = f^2(-\frac{1}{2})$. So 1 is a rational preperiodic point of type $(2, 1)$ since $f^3(1) = f^2(1)$.

- $f(\alpha) = 1$ has no rational solution.
- $f(\alpha) = -\frac{1}{2}$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-3x^3 + \frac{9}{2}x^2$	$(0, 1)$	$(-\frac{1}{2}, (2, 1)), (1, (2, 1))(\frac{3}{2}, (1, 1))$	4

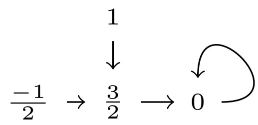

Figure 4.1.9: Polynomial 9

Some small good primes of $f(x) = -4x^3 + 6x^2 - \frac{1}{2}$ are 3 and 5.

- $f(x) \equiv -x^3 + 1 \bmod 3$
- $f(x) \equiv x^3 + x^2 + 2 \bmod 5$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-4x^3 + 6x^2 - \frac{1}{2}$	$p = 3$	$f(0) \equiv 1 \bmod 3$ $f(1) \equiv 0 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1, 2$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 2$
	$p = 5$	$f(0) \equiv 2 \bmod 5$ $f(1) \equiv 4 \bmod 5$ $f(2) \equiv 4 \bmod 5$ $f(3) \equiv 3 \bmod 5$ $f(4) \equiv 2 \bmod 5$	$m_5 = 1, 2$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 3$

Table 4.10: Polynomial 10

In Table 4.10 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 2$. We check whether the polynomial $f(x) = -4x^3 + 6x^2 - \frac{1}{2}$ has rational periodic point of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -4x^3 + 6x^2 - \frac{1}{2}$.

- $\Phi_{1,f}(x) = 0$ has only one rational solution which is $\frac{1}{2}$.
- $\Phi_{2,f}(x) = 0$ has two rational solutions which are $-\frac{1}{2}, \frac{3}{2}$.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -\frac{1}{2}$ has two rational solutions which are $0, \frac{3}{2}$. However $\frac{3}{2}$ is a periodic point. So 0 is a rational preperiodic point of type $(1, 2)$ since $f^3(0) = f(0)$.
- $f(\alpha) = \frac{1}{2}$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.
- $f(\alpha) = \frac{3}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1, 2)$ since $f^3(1) = f(1)$.
- $f(\alpha) = 0$ has no rational solution.
- $f(\alpha) = 1$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-4x^3+6x^2-\frac{1}{2}$	$(\frac{1}{2}, 1), (-\frac{1}{2}, 2), (\frac{3}{2}, 2)$	$(0, (1, 2)), (1, (1, 2))$	5

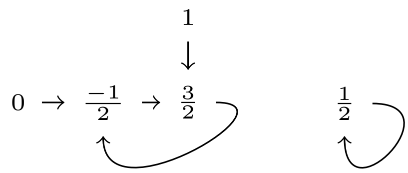

Figure 4.1.10: Polynomial 10

Some small good primes of $f(x) = 4x^3 - 6x^2 + \frac{3}{2}$ are 3 and 5.

- $f(x) \equiv x^3 \bmod 3$
- $f(x) \equiv -x^3 - x^2 - 1 \bmod 5$

In Table 4.11 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 1$. We check whether the polynomial $f(x) = 4x^3 - 6x^2 + \frac{3}{2}$ has rational periodic point of exact period 1 or 2. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first and second dynatomic polynomial of $f(x) = 4x^3 - 6x^2 + \frac{3}{2}$.

- $\Phi_{1,f}(x) = 0$ has three rational solutions which are $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$.
- $\Phi_{2,f}(x) = 0$ has no rational solution.

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$4x^3 - 6x^2 + \frac{3}{2}$	$p = 3$	$f(0) \equiv 0 \bmod 3$ $f(1) \equiv 1 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \leq k \leq 1$
	$p = 5$	$f(0) \equiv 4 \bmod 5$ $f(1) \equiv 2 \bmod 5$ $f(2) \equiv 2 \bmod 5$ $f(3) \equiv 3 \bmod 5$ $f(4) \equiv 4 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$

Table 4.11: Polynomial 11

Hence, rational periodic points of $f(x)$ are $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -\frac{1}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1, 1)$ since $f^2(1) = f(1)$.
- $f(\alpha) = \frac{1}{2}$ has only one rational solution which is $\frac{1}{2}$. However $\frac{1}{2}$ is a periodic point.
- $f(\alpha) = \frac{3}{2}$ has two rational solutions which are $0, \frac{3}{2}$. However $\frac{3}{2}$ is a periodic point. So 0 is a rational preperiodic point of type $(1, 1)$ since $f^2(0) = f(0)$.
- $f(\alpha) = 0$ has no rational solution.
- $f(\alpha) = 1$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$4x^3 - 6x^2 + \frac{3}{2}$	$(-\frac{1}{2}, 1), (\frac{1}{2}, 1), (\frac{3}{2}, 1)$	$(0, (1, 1)), (1, (1, 1))$	5

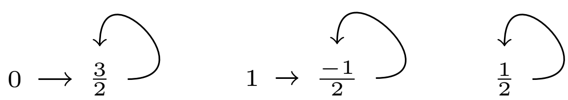

Figure 4.1.11: Polynomial 11

Some small good primes of $f(x) = 3x^3 - \frac{9}{2}x^2 + 1$ are 5 and 7.

- $f(x) \equiv 3x^3 + 3x^2 + 1 \bmod 5$
- $f(x) \equiv 3x^3 - x^2 + 1 \bmod 7$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$3x^3 - \frac{9}{2}x^2 + 1$	$p = 5$	$f(0) \equiv 1 \bmod 5$ $f(1) \equiv 2 \bmod 5$ $f(2) \equiv 2 \bmod 5$ $f(3) \equiv 4 \bmod 5$ $f(4) \equiv 1 \bmod 5$	$m_5 = 1$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \leq k \leq 2$
	$p = 7$	$f(0) \equiv 1 \bmod 7$ $f(1) \equiv 3 \bmod 7$ $f(2) \equiv 0 \bmod 7$ $f(3) \equiv 3 \bmod 7$ $f(4) \equiv 2 \bmod 7$ $f(5) \equiv 1 \bmod 7$ $f(6) \equiv 4 \bmod 7$	$m_7 = 1$	$r_7 \mid 6$	$n_7 = 2^k \cdot 3^l \cdot 7^e$ $0 \leq k \leq 1$ $0 \leq l \leq 1$

Table 4.12: Polynomial 12

In Table 4.12 we have found n_5 and n_7, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \leq k \leq 1$. We check whether the polynomial $f(x) = 3x^3 - \frac{9}{2}x^2 + 1$ has rational periodic point of exact period 1 or 2. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first and second dynatomic polynomial of $f(x) = 3x^3 - \frac{9}{2}x^2 + 1$.

- $\Phi_{1,f}(x) = 0$ has only one rational solution which is $-\frac{1}{2}$.
- $\Phi_{2,f}(x) = 0$ has no rational solution.

Hence, the only rational periodic point of $f(x)$ is $-\frac{1}{2}$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = -\frac{1}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However $-\frac{1}{2}$ is a periodic point. So 1 is a rational preperiodic point of type $(1, 1)$ since $f^2(1) = f(1)$.
- $f(\alpha) = 1$ has two rational solutions which are $0, \frac{3}{2}$. So 0 is a rational preperiodic point of type $(2, 1)$ since $f^3(0) = f^2(0)$. So $\frac{3}{2}$ is a rational preperiodic point of type $(2, 1)$ since $f^3(\frac{3}{2}) = f^2(\frac{3}{2})$.
- $f(\alpha) = 0$ has no rational solution.
- $f(\alpha) = \frac{3}{2}$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$3x^3 - \frac{9}{2}x^2 + 1$	$(-\frac{1}{2}, 1)$	$(0,(2,1)),(1,(1,1)),(\frac{3}{2},(2,1))$	4

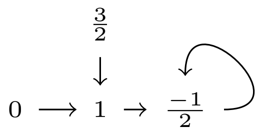

Figure 4.1.12: Polynomial 12

Some small good primes of $f(x) = -x^3 + \frac{3}{2}x^2 + 1$ are 3 and 5.

- $f(x) \equiv -x^3 + 1 \bmod 3$
- $f(x) \equiv -x^3 - x^2 + 1 \bmod 5$

Polynomial	Good Primes	Reduced Polynomial	m_p	r_p	n_p
$-x^3 + \frac{3}{2}x^2 + 1$	$p = 3$	$f(0) \equiv 1 \bmod 3$ $f(1) \equiv 0 \bmod 3$ $f(2) \equiv 2 \bmod 3$	$m_3 = 1, 2$	$r_3 \mid 2$	$n_3 = 2^k \cdot 3^e$ $0 \le k \le 2$
	$p = 5$	$f(0) \equiv 1 \bmod 5$ $f(1) \equiv 4 \bmod 5$ $f(2) \equiv 4 \bmod 5$ $f(3) \equiv 0 \bmod 5$ $f(4) \equiv 1 \bmod 5$	$m_5 = 2$	$r_5 \mid 4$	$n_5 = 2^k \cdot 5^e$ $0 \le k \le 3$

Table 4.13: Polynomial 13

In Table 4.13 we have found n_3 and n_5, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}$ are given by $n = 2^k$ where $0 \le k \le 2$. We check whether the polynomial $f(x) = -x^3 + \frac{3}{2}x^2 + 1$ has rational periodic point of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -x^3 + \frac{3}{2}x^2 + 1$.

- $\Phi_{1,f}(x) = 0$ has no rational solution.
- $\Phi_{2,f}(x) = 0$ has two rational solutions which are $1, \frac{3}{2}$.
- $\Phi_{4,f}(x) = 0$ has no rational solution.

Hence, rational periodic points of $f(x)$ are $1, \frac{3}{2}$. Now we will find rational preperiodic points. Let $\alpha \in \mathbb{Q}$.

- $f(\alpha) = 1$ has two rational solutions which are $0, \frac{3}{2}$. However $\frac{3}{2}$ is a periodic point. So 0 is a rational preperiodic point of type $(1, 2)$ since $f^3(0) = f(0)$.
- $f(\alpha) = \frac{3}{2}$ has two rational solutions which are $-\frac{1}{2}, 1$. However 1 is a periodic point. So $-\frac{1}{2}$ is a rational preperiodic point of type $(1, 2)$ since $f^3(\frac{-1}{2}) = f(\frac{-1}{2})$.
- $f(\alpha) = -\frac{1}{2}$ has no rational solution.
- $f(\alpha) = 0$ has no rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-x^3 + \frac{3}{2}x^2 + 1$	$(1, 2), (\frac{3}{2}, 2)$	$(0, (1, 2)), (-\frac{1}{2}, (1, 2))$	4

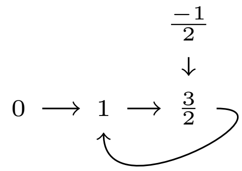

Figure 4.1.13: Polynomial 13

According to these calculations an immediate result can be stated as follows:

Theorem 4.1.2. *Let $f(x) \in \mathbb{Q}[x]$ be a bicritical post-critically finite polynomial of degree 3 with two rational critical points. Then $f(x)$ can have at most 5 $\mathbb{Q}$-rational preperiodic points.*

4.1.3 Cubic Bicritical PCF Polynomials with Two Irrational Critical Points

The next two polynomials are representatives of conjugacy classes of cubic bicritical post-critically finite polynomials with two irrational critical points. Whereas they are defined over $\mathbb{Q}$, their critical points are in a quadratic extension of $\mathbb{Q}$. As a result, we will search for periodic points in quadratic extensions. It means that we will find preperiodic points and solutions to dynatomic polynomials in these extensions.

We will begin with $f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x + 2 \in \mathbb{Q}[x]$. Critical points of this polynomial are $\pm\sqrt{2}$. Hence we will find preperiodic points in the extension $\mathbb{Q}(\sqrt{2})$. The ring of integers of $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Z}[\sqrt{2}]$. Since $\mathbb{Z}[\sqrt{2}]$ is an Euclidean domain then it is a principal ideal domain hence it is a unique factorization domain. Therefore, every irreducible element is prime. For $f(x)$ we choose $\mathfrak{p}_1$ and $\mathfrak{p}_2$ to be the primes ideals generated by $p_1 = 3$ and $p_2 = 3\sqrt{2} + 1$, respectively. $f(x) - \frac{1}{4}x^3 + \frac{3}{2}x + 2$ has good reduction in $(\mathbb{Z}(\sqrt{2})/\mathfrak{p}_{\mathfrak{i}})[x]$ for each $\mathfrak{p}_{\mathfrak{i}}$. Notice that $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(3) = 9$. So the residue field $\mathbb{Z}[\sqrt{2}]/\mathfrak{p}_1$ has 9 elements. Similarly, $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(3\sqrt{2}+1) = 17$. So the residue field $\mathbb{Z}[\sqrt{2}]/\mathfrak{p}_2$ has 17 elements, meaning that $\mathbb{Z}[\sqrt{2}]/\mathfrak{p}_2 \simeq \mathbb{Z}/17\mathbb{Z}$. Therefore, we can consider $f(x)$ modulo 17.

- $f(x) \equiv -x^3 + 2 \bmod \mathfrak{p}_1$
- $f(x) \equiv 4x^3 + 10x^2 + 2 \bmod \mathfrak{p}_2$

Now, we will give all cycles of $f(x)$ in $\mathbb{Z}[\sqrt{2}]/\mathfrak{p}_1$ and $\mathbb{Z}[\sqrt{2}]/\mathfrak{p}_2$, respectively. In

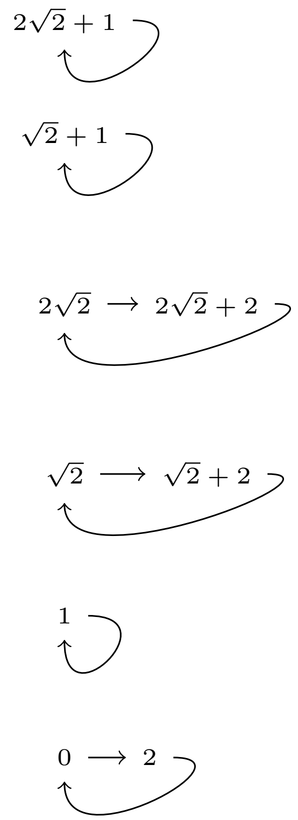

Figure 4.1.14: Polynomial 14, $p_1 = 3$

Figures 4.1.14 and 4.1.15 we have found $m_3 = 1, 2$, $r_3 \mid 6$ and $n_3 = 2^k \cdot 3^l \cdot 9^e$ where $0 \leq k \leq 2$ and $0 \leq l \leq 1$, and $m_{17} = 2$, $r_{17} \mid 16$ and $n_{17} = 2^k \cdot 17^e$ where $0 \leq k \leq 5$, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}(\sqrt{2})$ are given by $n = 2^k$ where $0 \leq k \leq 2$. We check whether the polynomial $f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x + 2$ has rational periodic point of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find rational roots of first, second and fourth dynatomic polynomial of $f(x) = -\frac{1}{4}x^3 + \frac{3}{2}x + 2$. Since $f(x)$ is a bicritical cubic post-critically finite polynomial

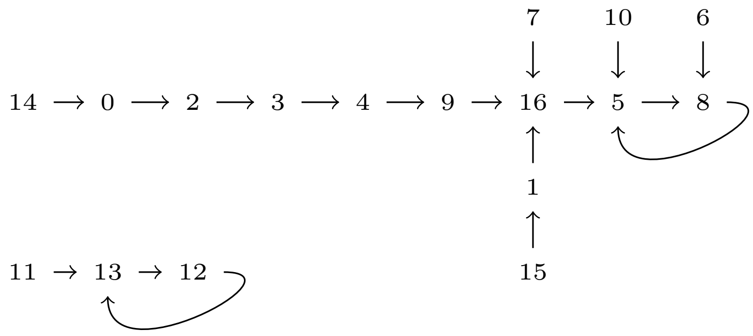

Figure 4.1.15: Polynomial 14, $p_2 = 17$

with two irrational critical points, then its critical points are preperiodic under $f(x)$. Therefore, we will search for $\mathbb{Q}(\sqrt{2})$-rational roots of the following dynatomic polynomials.

- $\Phi_{1,f}(x) = 0$ has no $\mathbb{Q}(\sqrt{2})$-rational solution.
- $\Phi_{2,f}(x) = 0$ has four $\mathbb{Q}(\sqrt{2})$-rational solutions which are $\pm 2\sqrt{2}, \pm\sqrt{2} + 2$.
- $\Phi_{4,f}(x) = 0$ has no $\mathbb{Q}(\sqrt{2})$-rational solution.

Hence, $\mathbb{Q}(\sqrt{2})$-rational periodic points of $f(x)$ are $\pm 2\sqrt{2}, \pm\sqrt{2}+2$. Now we will find $\mathbb{Q}(\sqrt{2})$-rational preperiodic points. Let $\alpha \in \mathbb{Q}(\sqrt{2})$.

- $f(\alpha) = 2\sqrt{2}$ has only one $\mathbb{Q}(\sqrt{2})$-rational solution which is $-\sqrt{2}+2$. However, $-\sqrt{2} + 2$ is a periodic point.
- $f(\alpha) = -2\sqrt{2}$ has only one $\mathbb{Q}(\sqrt{2})$-rational solution which is $\sqrt{2}+2$. However, $\sqrt{2} + 2$ is a periodic point
- $f(\alpha) = \sqrt{2}+2$ has two $\mathbb{Q}(\sqrt{2})$-rational solutions which are $\sqrt{2}, -2\sqrt{2}$. However $-2\sqrt{2}$ is a periodic point. So $\sqrt{2}$ is a $\mathbb{Q}(\sqrt{2})$-rational preperiodic point of type $(1, 2)$ since $f^3(\sqrt{2}) = f(\sqrt{2})$.
- $f(\alpha) = -\sqrt{2}+2$ has two $\mathbb{Q}(\sqrt{2})$-rational solutions which are $-\sqrt{2}, 2\sqrt{2}$. However $2\sqrt{2}$ is a periodic point. So $-\sqrt{2}$ is a $\mathbb{Q}(\sqrt{2})$-rational preperiodic point of type $(1, 2)$ since $f^3(-\sqrt{2}) = f(-\sqrt{2})$.
- $f(\alpha) = \sqrt{2}$ has no $\mathbb{Q}(\sqrt{2})$-rational solution.
- $f(\alpha) = -\sqrt{2}$ has no $\mathbb{Q}(\sqrt{2})$-rational solution.

We continue with $f(x) = -\frac{1}{28}x^3 - \frac{3}{4}x + \frac{7}{2} \in \mathbb{Q}[x]$. Critical points of this polynomial are $\pm\sqrt{-7}$. Hence we will find preperiodic points in the extension $\mathbb{Q}(\sqrt{-7})$. The

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-\frac{1}{4}x^3+\frac{3}{2}x+2$	$(\pm2\sqrt{2},2),(\pm\sqrt{2}+2,2)$	$(\pm\sqrt{2},(1,2))$	6

Figure 4.1.16: Polynomial 14

ring of integers of $\mathbb{Q}(\sqrt{-7})$ is $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. Since $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ is an Euclidean domain then it is a principal ideal domain hence it is a unique factorization domain. Therefore, every irreducible element is prime. For $f(x)$ we choose $\mathfrak{p}_1$ and $\mathfrak{p}_2$ to be the prime ideals generated by $p_1 = 3$ and $p_2 = 5$, respectively. $f(x) = -\frac{1}{28}x^3 - \frac{3}{4}x + \frac{7}{2}$ has good reduction in $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]/\mathfrak{p}_i$ for each $\mathfrak{p}_i$. Notice that, $N_{\mathbb{Q}(\sqrt{-7})/\mathbb{Q}}(3) = 9$. So the residue field $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]/\mathfrak{p}_1$ has 9 elements. Similarly $N_{\mathbb{Q}(\sqrt{-7})/\mathbb{Q}}(5) = 25$. So the residue field $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]/\mathfrak{p}_2$ has 25 elements.

- $f(x) \equiv -x^3 + 2 \bmod \mathfrak{p}_1$
- $f(x) \equiv -2x^3 - 3x + 1 \bmod \mathfrak{p}_2$

In Figures 4.1.17 and 4.1.18 we have found $m_3 = 1, 2$, $r_3 \mid 6$ and $n_3 = 2^k \cdot 3^l \cdot 9^e$ where $0 \le k \le 2$ and $0 \le l \le 1$, and $m_5 = 1, 4$, $r_5 \mid 20$ and $n_5 = 2^k \cdot 5^e$ where $0 \le k \le 4$, see Proposition 3.0.2. Now, according to the algorithm the possible periods over $\mathbb{Q}(\sqrt{-7})$ are given by $n = 2^k$ where $0 \le k \le 2$. We check whether the polynomial $f(x) = -\frac{1}{28}x^3 - \frac{3}{4}x + \frac{7}{2}$ has rational periodic point of exact period $1, 2$ or 4. To do this we use the fact that rational roots of the corresponding dynatomic polynomial of a polynomial are rational periodic points under this polynomial. Here we will find $\mathbb{Q}(\sqrt{-7})$-rational roots of first, second and fourth dynatomic polynomial of $f(x) = -\frac{1}{28}x^3 - \frac{3}{4}x + \frac{7}{2}$. Since $f(x)$ is a bicritical cubic post-critically finite polynomial with two irrational critical points, then its critical points are preperiodic under $f(x)$. Therefore we will search for $\mathbb{Q}(\sqrt{-7})$-rational roots of the following dynatomic polynomials.

- $\Phi_{1,f}(x) = 0$ has no $\mathbb{Q}(\sqrt{-7})$-rational solution.
- $\Phi_{2,f}(x) = 0$ has four $\mathbb{Q}(\sqrt{-7})$-rational solutions which are $\pm\sqrt{-7}, \pm\frac{\sqrt{-7}}{2} + \frac{7}{2}$.
- $\Phi_{4,f}(x) = 0$ has no $\mathbb{Q}(\sqrt{-7})$-rational solution.

Hence, $\mathbb{Q}(\sqrt{-7})$-rational periodic points of $f(x)$ are $\pm\sqrt{-7}, \pm\frac{\sqrt{-7}}{2} + \frac{7}{2}$. Now we will find $\mathbb{Q}(\sqrt{-7})$-rational preperiodic points. Let $\alpha \in \mathbb{Q}(\sqrt{-7})$.

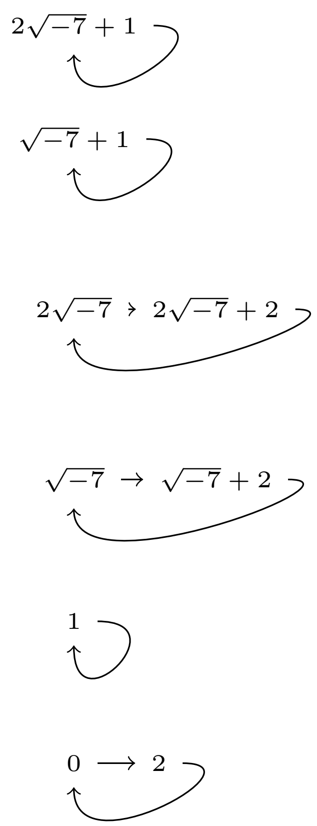

Figure 4.1.17: Polynomial 15, $p_1 = 3$

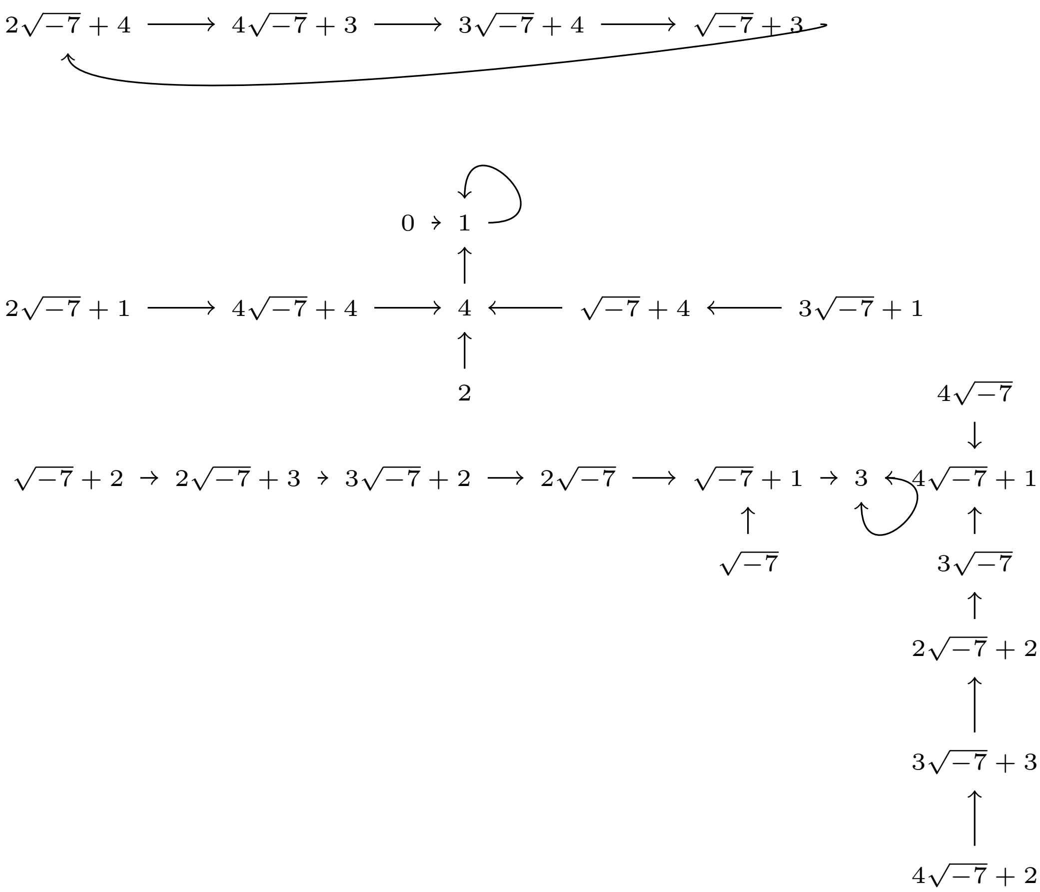

Figure 4.1.18: Polynomial 14, $p_2 = 5$

- $f(\alpha)=\sqrt{-7}$ has only one $\mathbb{Q}(\sqrt{-7})$-rational solution which is $\frac{-\sqrt{-7}}{2}+\frac{7}{2}$. However $\frac{-\sqrt{-7}}{2}+\frac{7}{2}$ is a periodic point.

- $f(\alpha)=-\sqrt{-7}$ has only one $\mathbb{Q}(\sqrt{-7})$-rational solution which is $\frac{\sqrt{-7}}{2}+\frac{7}{2}$. However $\frac{\sqrt{-7}}{2}+\frac{7}{2}$ is a periodic point.

- $f(\alpha)=\frac{-\sqrt{-7}}{2}+\frac{7}{2}$ has two $\mathbb{Q}(\sqrt{-7})$-rational solutions which are $\sqrt{-7}, -2\sqrt{-7}$. However $\sqrt{-7}$ is a periodic point. So $-2\sqrt{-7}$ is a $\mathbb{Q}(\sqrt{-7})$-rational preperiodic point of type $(1,2)$ since $f^3(-2\sqrt{-7})=f(-2\sqrt{-7})$.

- $f(\alpha)=\frac{\sqrt{-7}}{2}+\frac{7}{2}$ has two $\mathbb{Q}(\sqrt{-7})$-rational solutions which are $-\sqrt{-7}, 2\sqrt{-7}$. However $-\sqrt{-7}$ is a periodic point. So $2\sqrt{-7}$ is a $\mathbb{Q}(\sqrt{-7})$-rational preperiodic point of type $(1,2)$ since $f^3(2\sqrt{-7})=f(\sqrt{-7})$.

- $f(\alpha)=2\sqrt{-7}$ has no $\mathbb{Q}(\sqrt{-7})$-rational solution.

- $f(\alpha)=-2\sqrt{-7}$ has no $\mathbb{Q}(\sqrt{-7})$-rational solution.

Polynomial	(Periodic Point, Period)	(Preperiodic Point, Type)	Total
$-\frac{1}{28}x^3-\frac{3}{4}x+\frac{7}{2}$	$(\pm\sqrt{-7},2),(\pm\frac{\sqrt{-7}}{2}+\frac{7}{2},2)$	$(\pm 2\sqrt{-7},(1,2))$	6

Figure 4.1.19: Polynomial 15

According to these calculations an immediate result can be stated as follows:

Theorem 4.1.3. *Let $f(x)\in\mathbb{Q}[x]$ be a bicritical post-critically polynomial of degree 3 with two irrational critical points $a\pm b\sqrt{\alpha}\in\mathbb{Q}(\sqrt{\alpha})$. Then $f(x)$ can have at most 6 $\mathbb{Q}(\sqrt{\alpha})$-rational preperiodic points.*

4.2 Results

As we have seen from calculations, representatives of conjugacy classes of cubic post-critically finite polynomials can have at most 6 rational preperiodic points. We know that linear conjugation preserves the dynamical behaviour. If two polynomials $f,g\in K[x]$ are linearly conjugate by $T(x)=ax+b\in K[x]$, i.e., $f^T(x)=T\circ f\circ T^{-1}(x)=g(x)$ then $\mathcal{O}_f(\alpha)=\mathcal{O}_{g^T}(\alpha)=T(\mathcal{O}_g(\alpha))$. Since $T(x)$ is a linear polynomial then $|\mathcal{O}_f(\alpha)|=|\mathcal{O}_g(\alpha)|$. Moreover,

$$(T\circ f\circ T^{-1})^n(x)=T\circ f^n\circ T^{-1}(x)=g^n(x).$$

If $\alpha \in K$ is a periodic point of exact period n under f then $T \circ f^n \circ T^{-1}(\alpha) = g^n(\alpha)$. If we take a composition with $T(x)$ from right then

$$T \circ f^n(\alpha) = g^n \circ T(\alpha).$$

Since α is a periodic point under f, we have

$$T(\alpha) = g^n(a\alpha + b).$$

Therefore

$$a\alpha + b = g^n(a\alpha + b).$$

It means that $a\alpha + b$ is a periodic point of exact period n under g. Therefore the exact period of a periodic point α under f is equal to the exact period of the point $a\alpha + b$ under g. Hence we can state our result.

Theorem 4.2.1. *Let $f(x) \in \mathbb{Q}[x]$ be a post-critically finite polynomial of degree 3 then the number of rational preperiodic points cannot exceed 6.*

- An unicritical cubic PCF polynomial defined over $\mathbb{Q}$ with a rational critical point can have at most 3 rational preperiodic points. In fact these points are periodic.
- A bicritical cubic PCF polynomial defined over $\mathbb{Q}$ with two rational critical points can have at most 5 rational preperiodic points.
- A bicritical cubic PCF polynomial defined over $\mathbb{Q}$ with two irrational critical points have at most 6 preperiodic points in $\mathbb{Q}(\sqrt{\alpha})$ where $a \pm b\sqrt{\alpha}$, $a, b \in \mathbb{Q}$, are its critical points.

We have three immediate corollaries of this theorem which are clear from calculations in Chapter 4.

Corollary 4.2.2. *Let $f(x) \in \mathbb{Q}[x]$ be a post-critically finite polynomial of degree 3. Then the exact period of a periodic point under f cannot exceed 3.*

Corollary 4.2.3. *Let $f(x) \in \mathbb{Q}[x]$ be a post-critically finite polynomial of degree 3. If $f(x)$ has a periodic point of exact period 1 or 2 then $f(x)$ cannot have a periodic point of exact period 3. If $f(x)$ has a periodic point of exact period 3 then $f(x)$ cannot have a periodic point of exact period 1 or 2.*

Corollary 4.2.4. *Let $f(x) \in \mathbb{Q}[x]$ be a post-critically finite polynomial of degree 3 and α be a preperiodic point (not periodic) under f of type (m, n). Then $m \in \{1, 2, 3\}$ and $n \in \{1, 2\}$ where $(m, n) \neq (2, 2), (3, 3)$.*

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