A STUDY ON BINOMIAL EDGE IDEALS

by

BÜŞRA ATAR

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"My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously."

Emmy Noether

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Büşra Atar

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Thesis Supervisor: Asst. Prof. Dr. Ayesha Asloob Qureshi

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Abstract

In this thesis, we study the binomial edge ideals associated with graphs. We review their reduced Gröbner basis and minimal primary decompositions. Moreover, we also give an overview of recent results related to the depth, betti numbers, and regularity of binomial edge ideals arising from different classes of simple graphs.

BİNOM KENAR İDEALLER ÜZERİNE BİR ÇALIŞMA

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Anahtar Kelimeler: Binom kenar idealleri, çizgeler, Gröbner bazı, birincil ayrışma, derinlik, Betti sayıları, Castelnuovo-Mumford düzenliliği.

Özet

Bu tezde basit çizgelerle ilişkin binom kenar ideallerini çalıştık. Binom kenar ideallerinin indirgenmiş Gröbner bazları ve minimal birincil ayrışmaları üzerine çalışmalarını inceledik. Ayrıca, çeşitli basit çizgeler sınıfları ile ilişkin binom kenar ideallerinin derinlikleri, Betti sayıları ve düzenliliği hakkındaki en son çalışmalar ve sonuçlar genel olarak değerlendirildi.

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Contents

Introduction

The study of the ideal of t-minors of a $m \times n$ matrix X of indeterminates is a classical subject in commutative algebra and algebraic geometry. For any integer t, the ideal generated by all t-minors of X are well understood, for example, see [3]. Motivated by the application in algebraic statistics, the ideals generated by an arbitrary set of 2-minors were studied by Herzog, Hibi et al. in [12]. To define a combinatorial structure on these ideals, in [12], the authors defined the concept of binomial edge ideals of simple graphs and showed that how ideals generated by 2-minors can be used in hypothesis testing and to study the connectedness of contingency tables. In this thesis, we review several article related to the algebraic and homological properties of these ideals. Let G be a simple graph on $[n] = \{1, \dots, n\}$. The binomial edge ideal of G, denoted by J_G, is generated by all binomials $f_{ij} = x_i y_j - x_j y_i$ such that $\{i, j\} \in E(G)$. With this identification, the main aim is to describe algebraic and homological properties of J_G in terms of the information provided by G.

In the first chapter, we give certain combinatorial and algebraic terminologies and notations which are used in the following chapters. We divide the second chapter into four sections. In Section 2.1, we recall the reduced Gröbner basis of binomial edge ideals as described in [12]. The authors introduced the admissible paths in graphs to describe the binomials in the reduced Gröbner basis of binomial edge ideals. This description leads to an important consequence: given any graph G, the associated binomial edge ideal is a radical ideal, and hence it is the intersection of its minimal prime ideals. In Section 2.2, we study the minimal primes of binomial edge ideals which can be described by using the cut points of the graphs. In Section 2.3, we mention certain important results on the depth and Betti numbers of binomial edge ideals given in [7], [18]. Afterwards, the Cohen-Macaulay property for the binomial edge ideals is given for some classes of simple graphs in Section 2.4.

In Chapter 3, we discuss the Castelnuovo-Mumford regularity (or simply, the regularity), of binomial edge ideals. In last couple of years, several articles have been published where authors have tried to give tight upper and lower bounds for the regularity of J_G. We briefly survey these articles in Section 3.1. Matsuda and Murai, in [19] proved that if G is a connected graph then $l + 1 \leq \operatorname{reg}(J_G) \leq n$ where l is the length of the longest induced path of G. Moreover, it was shown in [8] that $\operatorname{reg}(J_G) = l + 1$ if G is a closed connected graph. Afterwards, in [16] Kiani and Madani emphasized that $\operatorname{reg}(J_G) \leq n - 1$ if G is not a path graph. Later, in [26] they improved this inequality for the closed graphs by proving that $\operatorname{reg}(J_G) \leq c(G)$ where $c(G)$ is the number of maximal cliques of G. In [27], they formulated the conjecture that $\operatorname{reg}(J_G) \leq c(G)$, for any graph G. This conjecture was first proved for the generalized block and chordal graphs, see [15], [8], [25]. Recently, a complete proof has been given for the above conjecture in [24] by Kiani, Madani, and Malayeri with a new terminology of compatible maps. In Section 3.2, we discuss those classes of binomial edge ideals for which the exact value of their regularity is known.

Chapter 1

Preliminaries

1.1 Combinatorial Preliminaries

1.1.1 Graph Theory

Let G be a graph on the vertex set $V(G)$ and the edge set $E(G)$. A graph G is said to be simple if G has no loops and no multiple edges. In the following text, all the graphs are simple and finite. Two distinct vertices u, v of G are called **adjacent** if $e = \{u, v\} \in E(G)$, and e is called **incident** with u and v. The degree of v in G, denoted by $\deg_G(v)$ or simply $\deg(v)$, is the number of vertices adjacent to v. The **complement** of a graph G, denoted by G^c, is the graph with $V(G) = V(G^c)$ such that two distinct vertices are adjacent in G^c if and only if they are not adjacent in G. For any $v \in V(G)$, the **neigborhood** of v is the set $N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$. The vertex $v \in V(G)$ is said to be **isolated** if $\deg(v) = 0$. A vertex $v \in V(G)$ is called **pendant vertex** if $\deg(v) = 1$. A pendant vertex is also called a leaf.
For any $e \in E(G)$, we denote $G \setminus e$ the graph obtained by removing e from G. Moreover, the induced graph of G on the vertex set $V(G) \setminus T$ is denoted by $G \setminus T$. Let u, v be non-adjacent vertices in G. Then G_e denotes the graph with vertex set $V(G)$, and

$$E(G_e) = E(G) \cup \{\{x, y\} \mid x, y \in N_G(u) \text{ or } x, y \in N_G(v)\}.$$

A simple graph in which all vertices are adjacent to each other is called a **complete graph**. The complete graph on n vertices is denoted by K_n. A graph G is called **bipartite graph** if its vertex set $V(G)$ can be partitioned into two disjoint sets X and Y such that $\{x, y\} \in E(G)$ if $x \in X$ and $y \in Y$. It is a well known fact that a graph is bipartite if and only if it does not contain cycles of odd length. A bipartite graph is called **complete** if $\{x, y\} \in E(G)$ for all $x \in X$ and $y \in Y$. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. Then the complete bipartite graph $K_{1,3}$ is called a **claw**, and a graph does not contain a claw as an induced subgraph is called **claw-free**.

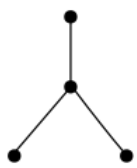


Figure 1.1.1: A claw

Two distinct vertices $u, v \in V(G)$ are said to be **connected** if there exists a path between u and v in G. The graph G is said to be connected if every pair vertices of G is connected. We denote by P_n the **path** graph on n vertices.

Definition 1.1.1. Let G be a graph on $[n] = \{1, \ldots, n\}$. Then G is called l-**vertex-connected** for some $l < n$ if for every subset $S \subseteq V(G)$ such that $|S| < l$, the induced graph $G \setminus S$ is connected. The **vertex-connectivity** of G, denoted by $\kappa(G)$, is defined as the maximum integer l such that G is l-vertex-connected.

A graph G is called **chordal** if every cycle of G of length bigger than 3 has a chord. A graph G is called **cochordal** if G^c is chordal. The **cochordal cover number** of a graph G is the minimum number of cochordal subgraphs $H_1, \ldots, H_s$ in G such that $\bigcup_{i=1}^{s} E(H_i) = E(G)$.

Lemma 1.1.2. *[7, Lemma 3.3] Let G be a bipartite graph on the vertex set $\{x_1 \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ with the edge set $\{x_i, y_j\}_{1 \leq i \leq j \leq n}$. Then G^c is a chordal graph.*

Definition 1.1.3. A graph G is called **closed** with respect to a given labeling of the vertices if G satisfies the following condition: For each $\{i, j\}$ and $\{k, l\}$ in $E(G)$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$. For example, P_n is a closed graph with the natural labelling of vertices.

A graph G is called an **interval graph** if for each $v \in V(G)$ there exists a closed interval $I_v = [l_v, r_v] \subset \mathbb{R}$ such that two distinct vertices u and v of G are adjacent if and only if $I_u \cap I_v \neq \emptyset$. Moreover, an interval graph G is a **proper interval graph** if there exists a family $\{I_v\}_{v \in V(G)}$ such that $I_v \nsubseteq I_u$ for all $u, v \in V(G)$. It is known from [4, Theorem 2.3] that every closed graph is isomorphic to a proper interval graph.

The **induced matching** of G is an induced subgraph of G in which every pair of edges is disjoint. We denote by indmatch(G), the maximal number of edges of H

where H is an induced matching.

Let G and H be two graphs on $[m]$ and $[n]$, respectively. We denote by $G * H$, the **join product** (or join) of two graphs G and H, that is the graph with vertex set $[m] \cup [n]$, and the edge set $E(G) \cup E(H) \cup \{\{v, w\} : v \in [m], w \in [n]\}$.

Now, we recall several combinatorial results which will be useful in the following chapters.

Proposition 1.1.4. *If G is closed, then G is chordal and has no claw.*

Proof. On the contrary, assume that G is not chordal. This implies that there is a cycle C of length bigger than 3 without a chord. Let $\{i, j\}$ and $\{i, k\}$ be the edges of C such that $i < j$, $i < k$ and $\{j, k\} \notin E(G)$. This contradicts the closedness of G.
Next, we show that G has no claw. Since G is closed, then any induced subgraph of G is also closed. Now, suppose that G has a subgraph H with three edges $e_1 = \{i, j\}$, $e_2 = \{i, k\}$ and $e_3 = \{i, l\}$. Then $i \neq \min\{i, j, k, l\}$, otherwise H is not closed. If $j < i$, then $k > i$ and $l > i$ so that $\{j, k\} \notin E(G)$ and $\{j, l\} \notin E(G)$ respectively. In this case, $\{l, k\}$ must be an edge of G because G is closed. Therefore, H is not a claw. □

Corollary 1.1.5. *A bipartite graph is closed if and only if it is a path.*

Proof. Let G be a closed bipartite graph. Recall that there is no odd cycle in a bipartite graph. Moreover, a tree which is not a path, contains a claw. Then the Proposition 1.1.4 shows that G must be a path.
Let G be a path of length n. We can label the vertices of G such that $E(G) = \{1, 2\}, \{2, 3\}, \ldots, \{n, n + 1\}$. This shows that G is closed. □

To a graph G, we associate the directed graph G^* with $V(G^*) = V(G)$ such that (i, j) is an arrow in G^* with $i < j$ if and only if $\{i, j\} \in E(G)$.

Proposition 1.1.6. *Let G be a graph on $[n]$. Then G is closed if and only if for any two vertices $i \neq j$ of the associated directed graph G^*, all paths of shortest length from i to j are directed.*

Proof. Suppose that G is closed, then one can find a labeling so that G^* is closed. Let $i, j \in E(G)$ with $i \neq j$, and P be a path of shortest length from i to j. Suppose that P is not directed. Hence, there is a subpath r, s, t such that either (r, s) and (t, s) or (s, r) and (s, t) are arrows of G^*. For both cases, we may suppose that $r < t$. Then (r, t) is an arrow in G^*, since G^* is closed. Then r, t is a shorter path from i to j which yields a contradiction.
To converse, we assume that all paths from i to j are directed in G^*. Let (i, j) and

(i,k) be two arrows where $j < k$. Thus, $\{j,i\},\{i,k\}$ is a non-directed path from j to k. Hence, there must be a shortest path in G^*. This implies that (j,k) is an arrow of G^*. Likewise, if (i,k) and (j,k) are arrows in G^* then (i,j) is an arrow of G^*. As a result, G is closed. □

A graph is isomorphic to the graph displayed in the Figure 1.1.2 is called a net graph and the graph is isomorphic to the graph displayed in the Figure 1.1.3. There

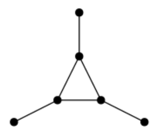


Figure 1.1.2: A net

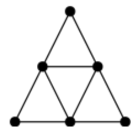


Figure 1.1.3: A tent

is an equivalent definition of closed graph independent of vertex labelling given in the following theorem.

Theorem 1.1.7. *[13, Theorem 7.10] A graph G is closed if and only if G is chordal, claw-free, net-free, and tent-free.*

1.1.2 Simplicial Complexes

Let Δ be a non-empty collection of subsets of $[n] = \{1,\dots,n\}$. Then Δ is called a **simplicial complex** if for each $F \in \Delta$, we have $F' \in \Delta$ whenever $F' \subset F$. The elements of Δ are called the **faces** of Δ. The maximal faces of Δ with respect to inclusion are called the **facets**. Note that every graph can be seen as a simplicial complex.

Let G be a simple graph on $[n]$. A **clique** of G is a subset $F \subset [n]$ with the property that each 2-element subset of F is an edge of G, that is, F is a complete component of G. The set of all cliques in G forms a simplicial complex $\Delta(G)$, called the **clique complex of** G. A vertex $v \in V(G)$ is called a **free (simplicial vertex)** of G if v belongs to exactly one maximal clique of Δ. In other words, v is called a free vertex of G, if the induced subgraph of G on the vertex set $N_G(v)$ is a complete graph. We denote the number of maximal cliques of G by $c(G)$. If a vertex v is not free then it is called an **internal** vertex. The number of internal vertices in G is denoted by $iv(G)$.

Definition 1.1.8. *Let G be a simple graph on $[n]$ and e be a cut edge of G so that the endpoints of e are free vertices in $G \setminus e$. Then e is said to be* ***free cut edge*** *of G. We set $\mathcal{R}(G) = G \setminus \{e_1, \dots, e_t\}$ where $\{e_1, \dots, e_t\}$ consists of all free cut edges of G. In particular, $\mathcal{R}(G) = G$ whenever G has not any free vertices.*

Furthermore, to any $v \in V(G)$, we associate a graph G_v with $V(G_v) = V(G)$ and the edge set $E(G_v) = E(G) \cup \{\{u, w\} \mid u, w \in N_G(v)\}$. Clearly if v is a free vertex, then $G_v = G$.
A vertex v in G is called a **cut point** of G if G has less connected components than $G_{[n]\setminus v}$. Let $T \subset V(G)$. If each $i \in T$ is a cut point of the graph $G_{([n]\setminus T)\cup\{i\}}$ then T is said to have the *cut point property* for G. We set

$$\mathcal{C}(G) := \{\emptyset\} \cup \{T \subseteq [n] \mid T \text{ has cut point property for } G\}$$

A maximal subgraph without a cut point in G is called a **block** of G. The graph G is called **block graph** if every block is a complete graph.
Let G be a connected chordal graph such that every $F_i, F_j, F_k \in \Delta(G)$, if $F_i \cap F_j \cap F_k \neq \emptyset$ then $F_i \cap F_j = F_i \cap F_k = F_j \cap F_k$. In this case, we say that G is **generalized block graph**. Then all block graphs are generalized block graphs. If $G \setminus e$ have more components than G for some $e \in E(G)$, then e is called a **cut edge** of G. A facet F of Δ is called a **leaf**, if there exists a facet E of Δ with $F \neq E$ such that $H \cap F \subset E \cap F$ for all facets H with $H \neq F$. The facet E is called a **branch** of F. The simplicial complex Δ is called a **quasi-forest**, if the facets of Δ can be ordered $F_1, \dots, F_m$ such that F_i is a leaf of $< F_1, \dots, F_i >$ for all $i > 1$. This type of order of facets of Δ is called a **leaf order**. Moreover, a connected graph (viewed as a simplicial complex) which is a quasi-forest is called a **quasi-tree**.

Theorem 1.1.9. *A graph G is chordal if and only if $\Delta(G)$ is quasi-forest.*

Due to the powerful similarity and relationship between simplicial complexes and graphs we have another equivalent definition for the closed graphs as follows.

Theorem 1.1.10. *Let G be a graph on $[n]$. Then the following are equivalent:*

(i) G is closed.

(ii) There exists a labelling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subseteq [n]$.

In addition, if the above equivalent holds and the facets $F_1, \dots, F_r$ of $\Delta(G)$ are labelled such that $\min(F_1) < \ldots < \min(F_r)$*, then $F_1, \dots, F_r$ is a leaf order of $\Delta(G)$.*

Proof. To prove this theorem we assume that G is connected. For disconnected graphs, we may look at their connected components.
$(i) \Rightarrow (ii)$: First, we consider a set $F = \{j \mid \{j,n\} \in E(G)\} \cup \{n\}$ and let $k := \min\{j \mid j \in F\}$. We want to have that $F = [k,n] \subseteq [n]$. For any $i \in F$ with $k \leq i < n$ we have the edges $\{i,n\}$ and $\{k,n\}$ in G. If $k = i$, we are done. If $k < i$, since G is closed, then $\{k,i\} \in E(G)$. Let $i,j \in F$ be two distinct vertices, then $\{i,n\},\{j,n\} \in E(G)$. Due to closedness of G, we have $\{i,j\} \in E(G)$. Then $\{i,i+1\} \in E(G)$ for all $i \in F$ with $i \neq n$.
Secondly, we show that F is a facet of $\Delta(G)$. Let $i,j \in F$ such that $i < j < n$. Then $\{i,n\},\{j,n\} \in E(G)$. In addition, since G is closed we have $\{i,j\} \in E(G)$. Hence F is a facet of $\Delta(G)$. If $j \notin F$, then $\{j,n\} \notin E(G)$. Thus, F is a maximal face in $\Delta(G)$.

Now, let $H \neq F$ be a facet of $\Delta(G)$ with $H \cap F = \emptyset$. Then there exists $l = \max\{j \mid j \in H \cap F\}$. We show that $H \cap F = [k,l]$. In the case that $k = l$, we are done. Assume that $k < l$. Then there exists t with $k \leq t < l$. Let $s \in H \setminus F$. Then $s,t < l$ and $\{s,l\}$, $\{t,l\}$ are edges of G. Therefore, $\{s,t\}$ is not edge in G as G is closed. It follows that $t \in H$. Our claim implies that H is the facet which is a branch of F, that is, F is a leaf. Let $H \cap F = [k,l]$ such that H is a branch of F and let G_l be the restriction of G to $[l]$. By induction on the cardinality of the vertex set of G, all facets of $\Delta(G_l)$ are intervals. We take a facet F' in $\Delta(G)$. If $F = F'$, then F' is an interval. Otherwise, $F' \in \Delta(G_l)$, then it is again an interval.

$(ii) \Rightarrow (i)$: Let $\{i,j\}$ and $\{k,l\}$ be edges of G with $i < j$ and $k < l$. In the case that $i = k$, the edges $\{i,j\}$ and $\{i,l\}$ fall into the maximal clique which is an interval containing i. Consequently, $\{j,l\} \in E(G)$. If $j = l$, then $\{i,j\}$ and $\{k,j\}$ are in the same interval containing j, therefore, $\{i,k\} \in E(G)$.

Lastly, we conclude that the order of facets in $\Delta(G)$ with respect to their minimal elements yields a leaf order, since F_{i-1} has maximal intersection with F_i for all i in this order. □

1.2 Algebraic Preliminaries

Now, we recall some definitions and facts related to commutative algebra.

1.2.1 Gröbner basis

Let $K[x]$ be the polynomial ring over a field K. Given two polynomials $f(x), g(x) \in K[x]$ with $g(x) \neq 0$, there exist uniquely determined polynomials $q(x)$ and $r(x)$ in $K[x]$ such that

$$f(x) = q(x)g(x) + r(x) \quad \text{and} \quad \deg r(x) < \deg g(x).$$

The polynomial $r(x)$ is the remainder of $f(x)$ with respect to $g(x)$. There is a generalization of division algorithm in $K[x]$ to the polynomial ring in several variables. Let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring. A monomial in S is a product of variables. The set of all monomial in S is denoted by $\mathrm{Mon}(S)$. Let $u = x_1^{a_1} ... x_n^{a_n}$ be a monomial. Then, we set $x^{\mathbf{a}} = u$, with $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. A total order on $\mathrm{Mon}(S)$ is called a **monomial order** if it satisfies the following:

(i) $1 < x^{\mathbf{a}}$ for all $x^{\mathbf{a}} \in \mathrm{Mon}(S)$,

(ii) If $x^{\mathbf{a}} > x^{\mathbf{b}}$ then $x^{\mathbf{a}}x^{\mathbf{c}} > x^{\mathbf{b}}x^{\mathbf{c}}$ for all monomials $x^{\mathbf{a}}, x^{\mathbf{b}}, x^{\mathbf{c}} \in \mathrm{Mon}(S)$.

The lexicographic order on $\mathrm{Mon}(S)$ is defined as follows:
Lexicographic Order: $x^a > x^b$ if either $\sum_{i=0}^n a_i > \sum_{i=0}^n b_i$ or $\sum_{i=0}^n a_i = \sum_{i=0}^n b_i$ and the leftmost non-zero component $a - b$ is positive.

A polynomial f in S can be written as $f = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} x^{\mathbf{a}}$ such that $c_{\mathbf{a}} \in K$ and all but finitely many $c_{\mathbf{a}} = 0$. We set $\mathrm{supp}(f) := \{x^{\mathbf{a}} : c_{\mathbf{a}} \neq 0\}$, and call this finite set of monomials the **support** of f. Let $<$ be a monomial order on S and let $f \in S$ with $f \neq 0$. The initial monomial of f, denoted by $\mathrm{in}_<(f)$, is the largest monomial $u \in \mathrm{supp}(f)$ with respect to $<$. If c is the coefficient of $\mathrm{in}_<(f)$ in f then $c.\mathrm{in}_<(f)$ is called the **leading term** of f.

Let $I \subset S$ be a non-zero ideal. The **initial ideal** of I with respect to $<$ is

$$\mathrm{in}_<(I) = \big(\mathrm{in}_<(f) : f \neq 0 \in I\big).$$

Definition 1.2.1. Fix a monomial order $<$ on the polynomial ring S. Let $I \subset S$ be a non-zero ideal. Then a **Gröbner basis** of I with respect to $<$ is a finite set $\{g_1, \ldots, g_s\}$ of non-zero polynomials belonging to I such that

$$\{\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)\}$$

is a system of generators of $\mathrm{in}_<(I)$.

For more details on Gröbner bases in commutative algebra see the book of Ene and Herzog, [6]. The following theorem gives the generalization of division algorithm.

Theorem 1.2.2. *[6, Theorem 2.11] Let f and $g_1, \ldots, g_m$ be polynomials in S with $g_i \neq 0$. Given a monomial order $<$, there exist polynomials $r, q_1, \ldots, q_m$ in S with*

$$f = q_1 g_1 + q_2 g_2 + \cdots + q_m g_m + r \tag{1.2.1}$$

such that the following conditions are satisfied:

(i) No element of $\operatorname{supp}(r)$ *is contained in the ideal* $\big(\operatorname{in}_<(g_1), \ldots, \operatorname{in}_<(g_m)\big)$*;*

(ii) $\operatorname{in}_<(f) \geq \operatorname{in}_<(q_i g_i)$ *for all i.*

The right-hand side of the 1.2.1 satisfying the conditions (i) and (ii) is called a **standard expression** of f, and r is called the **remainder** of f with respect to $g_1, \ldots, g_m$. The polynomial f may have **different standard expressions** and **different remainders** with respect to $g_1, \ldots, g_m$ as the following example demonstrates.

Example 1.2.3. Let $f = x_1 x_2 + x_2^2$, $g_1 = x_1 + x_2$ and $g_2 = x_1$. Let $<$ be the lexicographic order. Then

$$f = x_2 g_1 \quad \text{as well as} \quad f = x_2 g_2 + x_2^2.$$

We say that f **reduces to** 0 with respect to $g_1, \ldots, g_m$ if f has a remainder which is zero in a standard expression.

Proposition 1.2.4. *[6, Proposition 2.12] Let $<$ be a monomial order on S. Assume that the polynomials $g_1, \ldots, g_m$ form a Gröbner basis of the ideal I. Then each polynomial $f \in S$ has a* ***unique remainder*** *with respect to $g_1, \ldots, g_m$.*

Gröbner basis provides a solution for the ideal membership problem in a polynomial ring with multiple variables.

Corollary 1.2.5. *Let $I = (g_1, \ldots, g_m) \subseteq S$ be an ideal and assume that for some monomial order, $\{g_1, \ldots, g_m\}$ is a Gröbner basis of I. Then a polynomial $f \in S$ belongs to I if and only if f reduces to 0 with respect to $g_1, \ldots, g_m$.*

Let f and g be non-zero polynomials of S. Let c_f be the coefficient of $\operatorname{in}_<(f)$ in f and c_g that of $\operatorname{in}_<(g)$ in g. Then the polynomial

$$\mathsf{S}(f, g) = \frac{\operatorname{lcm}(\operatorname{in}_<(f), \operatorname{in}_<(g))}{c_f \operatorname{in}_<(f)} f - \frac{\operatorname{lcm}(\operatorname{in}_<(f), \operatorname{in}_<(g))}{c_g \operatorname{in}_<(g)} g$$

is called the **S-polynomial** of f and g.

Theorem 1.2.6. *Let $<$ be a monomial order on S, and let $I = (g_1, \ldots, g_m)$ be an ideal in S with $g_i \neq 0$ for all i. Then the following conditions are equivalent:*

(a) $g_1, \ldots, g_m$ *is a Gröbner basis of* I *with respect to* $<$,

(b) $\mathsf{S}(g_i, g_j)$ *reduces to zero with respect to* $g_1, \ldots, g_m$ *for all* $i < j$.

An important consequence of Theorem 1.2.6 is given in the following.

Proposition 1.2.7. *Let* $<$ *be a monomial order on* S*, and let* $f, g \in S$ *be two polynomials such that* $\mathrm{in}_<(f)$ *and* $\mathrm{in}_<(g)$ *are relatively prime. Then* $\mathsf{S}(f, g)$ *reduces to* 0 *with respect to* f, g.

The **Buchberger's Algorithm** provides a method to compute a Gröbner basis of an ideal $I \subset S$. Let G be a finite system of generators of I.

Step 1: For each pair of distinct elements of G we compute the S-polynomial and its remainder with respect to G.

Step 2: If all S-polynomials reduce to 0, then the algorithm ends and G is a Gröbner basis of I. Otherwise we add one of the non-zero remainders to our system of generators, call this new system of generators again G and go back to Step 1.

Example 1.2.8. Let $S = K[x_1, \ldots, x_7]$ and $f = x_1x_4 - x_2x_3$, $g = x_4x_7 - x_5x_6$ and $I = (f, g)$. With respect to the lexicographic order $<$ we get $\mathrm{in}_<(f) = x_1x_4$, $\mathrm{in}_<(g) = x_4x_7$. Set $G = \{f, g\}$. Then $\mathsf{S}(f, g) = x_7f - x_1g = x_1x_5x_6 - x_2x_3x_7 = h$. Since $\mathrm{in}_<(h) \notin \mathrm{in}_<(I)$, we set $G' = \{f, g, h\}$ as a new candidate for a Gröbner basis. Note that $\mathsf{S}(f, h) = x_2x_3g$ reduces to zero, $\mathrm{in}_<(g)$ and $\mathrm{in}_<(h)$ are relatively prime, and as $\mathsf{S}(g, h)$ reduces to 0 with respect to g, h. Thus G' forms a Gröbner basis of I with respect to the lexicographic order.

1.2.2 Minimal free resolutions and homological invariants

First, we introduce $\mathbb{N}$-grading on the polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K. Let $\deg(x_i) = 1$ for each $1 \leq i \leq n$. Let $x^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n} \in \mathrm{Mon}(S)$. Then $\deg(x^{\mathbf{a}}) = \sum_{i=1}^{n} a_i$. We denote by S_i the K-vector space generated by all monomials of degree i. We assign 0 an arbitrary degree, and $0 \in S_i$ for each i. In particular, $S_0 = K$. Then S has a direct sum decomposition $S = \bigoplus_{i\in\mathbb{N}} S_i$ as a K-vector space and it satisfies the following: $S_iS_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$. In this setting, we say that S is **standard graded** or $\mathbb{N}$-**graded**. Each polynomial $p \in S$ can be written uniquely as a finite sum of non-zero $p_i \in S_i$, that is $p = \sum_{i=1}^{n} p_i$ for some n, where p_i is called the homogeneous component of p in degree i. A polynomial $p \in S$ is called **homogeneous** if $p \in S_i$ for some i, and we have $\deg(p) = i$.
An S-module M is said to be $\mathbb{N}$-graded if it can be written as a direct sum of homogeneous components $M = \bigoplus_{i\in\mathbb{N}} M_i$ as a K-vector space and $S_iM_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{N}$. The K- vector space M_i is called homogeneous component of M

for all i. Let $M = \oplus_{i\in\mathbb{N}} M_i$ and $N = \oplus_{i\in\mathbb{N}} N_i$ be graded S-modules. Then S-module homomorphism $f : M \to N$ is called *graded* if $f(M_i) \subseteq N_i$. An ideal in S is called graded if it is $\mathbb{N}$-graded S-module. The quotient ring $R = S/I$ inherits the grading from S by $R_i = S_i/I_i$ for every $i \in \mathbb{N}$. For any S-module M and $d \in \mathbb{N}$, the $\mathbb{N}$-graded S-module $M(-d)$ is obtained with shifting M by d degrees, that is, $M(-d)_i = M(i-d)$, for all i. Next, we define $\mathbb{N}^n$-grading on $S = K[x_1, \ldots, x_n]$. The **multidegree** of a monomial $x^{\mathbf{a}}$, denoted by $\mathrm{mdeg}(x^{\mathbf{a}})$, is $\mathbf{a}$. Then $\mathrm{mdeg}(x_i)$ is the i-th standard vector of the canonical basis of $\mathbb{N}^n$. In particular, $\mathrm{mdeg}(x_1) = (1, 0, \ldots, 0)$. Then for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ there exists unique monomial of degree $\mathbf{a}$ in $\mathrm{Mon}(S)$. In this case, we say that S is $\mathbb{N}^n$-graded. The direct sum decomposition of $\mathbb{N}^n$-graded S is

$$S = \bigoplus_{m\in Mon(S)} S_m$$

where each S_m is a K-vector space generated by the monomial m. Moreover, we have $S_m S_n = S_{mn}$ for every two monomials m, n. The $\mathbb{N}$-grading and $\mathbb{N}^n$-grading on S are also referred as $\mathbb{Z}$-grading and $\mathbb{Z}^n$-grading, respectively.

A sequence of S-module homomorphisms

$$\mathcal{F} : \ldots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is called a **complex over** S if $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. The i-th homology of $\mathcal{F}$ is

$$H_i(\mathcal{F}) = \ker(d_i)/\mathrm{im}(d_{i+1}).$$

If $H_i(\mathcal{F}) = 0$ for all i then $\mathcal{F}$ is exact. In particular, if $F_i = 0$ for all $i < 0$ and the $H_i(\mathcal{F}) = 0$ for all $i \geq 0$ then $\mathcal{F}$ is called **acyclic**. The complex $\mathcal{F}$ is said to be **graded** if the modules F_i are graded and each d_i is graded.

Let M be a finitely generated S-module, and $\mathfrak{m} = (x_1, \ldots, x_n)$ be the unique graded maximal ideal of S. Next, we give the definition of minimal $\mathbb{N}$-graded free resolution of M over S.

Definition 1.2.9. An exact complex $\mathcal{F}$ of $\mathbb{N}$-graded free modules over S

$$\mathcal{F} : \quad \cdots \longrightarrow F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} F_0$$

with $F_i = \bigoplus_{j\in\mathbb{N}} S(-j)^{\beta_{i,j}}$, is the **minimal $\mathbb{N}$-graded free resolution**, or simply a graded free resolution, of M over S if

1. each f_i is a graded S-module homomorphism;

2. $M \cong \mathrm{Coker}(f_1)$;

3. $f_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$, for all $i \geq 1$.

Here, $\beta_{i,j}$ is called the (i,j)-th graded Betti numbers of M. A Betti number $\beta_{i,i+j} \neq 0$ is called **extremal** if $\beta_{k,k+l} = 0$ for all pairs $(k,l) \neq (i,j)$ such that $k \geq i$ and $l \geq j$. Let R be a Noetherian graded local ring. The R-module M has **linear relations** if M has generators of degree d and $\beta_{1,j}(M) = 0$ for all $j \neq d+1$. Every finitely generated graded S-module admits a minimal graded free resolution over S which is unique up to isomorphism ([22, Theorem 7.5]). The Hilbert's syzygy theorem shows that every finitely generated S-module has a *finite* minimal graded free resolution (see [22, Theorem 15.2]). Moreover, the length of a minimal free resolution can be at most the number of variables in the polynomial ring. The length of the minimal graded free resolution is called **projective dimension** of M and is denoted by $\operatorname{proj}\dim(M)$, and is given by

$$\operatorname{proj}\dim M = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

The Castelnuovo-Mumford regularity,(or simply, the regularity), is defined to be as follows:

$$\operatorname{reg}(M) = \max\{j : \beta_{i,i+j} \neq 0\} = \max\{j - i : \operatorname{Tor}_i(M,K)_j \neq 0\}$$

Definition 1.2.10. Let M be a graded S-module. Then M is said to have a d-linear resolution if its graded minimal free resolution is in the following form:

$$0 \longrightarrow S(-d-p)^{\beta_p} \longrightarrow \cdots \longrightarrow S(-d-1)^{\beta_1} \longrightarrow S(-d)^{\beta_0} \longrightarrow N \longrightarrow 0.$$

Furthermore, M is said to have **linear quotients** if its minimal set of generators can be ordered as $m_1, \ldots, m_s$ such that $(m_1, \ldots, m_i) : m_{i+1}$, for all $i = 1, \ldots, s-1$ is generated in degree one.

Proposition 1.2.11. *[13, Proposition 2.11] Let M be a finitely generated S-module. If M has linear quotients then M has a linear resolution.*

An element a in a ring R is called **regular** if $ab \neq 0$ for every non-zero elements $b \in R$. Since the polynomial ring S is an integral domain every element in $\mathfrak{m}$ is regular. Let M be a finitely generated S-module. A sequence of elements $f_1, \ldots, f_r$ in S is called an M-regular sequence if f_i is regular on $M/(f_1, \ldots, f_{i-1})M$ for all $i = 1, \ldots, r$.

The maximal length of such a sequence is called **depth** of M. Equivalently, for an arbitrary finitely generated S-module M we have

$$\operatorname{depth}(M) = \min\{\operatorname{Ext}^i_S(K,M) \neq 0\}.$$

Let S be a Noetherian graded local ring with the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, and M be a finitely generated graded S-module. If $\dim(M) = \operatorname{depth}(M)$, M is said to be **Cohen-Macaulay**.

The following classical results will be used in the subsequent chapters.

Lemma 1.2.12. *Let S be a standard graded polynomial ring and A, B and C be finitely generated graded S-modules. If the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact where f, g are graded homomorphisms of degree zero, then*

(i) $\operatorname{depth}(A) \geq \min\{\operatorname{depth}_S(B), \operatorname{depth}(C) + 1\}$,

(ii) $\operatorname{depth}(A) = \operatorname{depth}(C) + 1$ *if* $\operatorname{depth}(B) > \operatorname{depth}(C)$,

(iii) $\operatorname{depth}(A) = \operatorname{depth}(B)$ *if* $\operatorname{depth}(B) < \operatorname{depth}(C)$.

Theorem 1.2.13. ***(Auslander-Buchsbaum)*** *Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring and M a finitely generated R-module with finite projective dimension. Then:*

$$\operatorname{depth}(M) + \operatorname{proj\,dim}(M) = \operatorname{depth}(R).$$

1.2.3 Edge ideals and binomial edge ideals

Let G be a graph on $[n]$. The **edge ideal** $I(G)$ of G is generated by the monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. The edge ideals of graphs have been studied extensively over the last three decades by many algebraist. Every squarefree quadratic monomial ideal can be viewed as an edge ideal of a suitable graph. This helps to translate many combinatorial problems in graph theory in the language of monomial ideal and vice versa.

In [12], authors defined binomial edge ideal associated to simple graphs. To each $\{i, j\} \in E(G)$ with $i < j$, we associate a binomial $f_{ij} = x_i y_j - x_j y_i$. The binomial edge ideal J_G of G is generated by all f_{ij} with $\{i, j\} \in E(G)$. This generator set of J_G is a subset of the 2-minors of the following $(2 \times n)$-matrix

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{bmatrix}.$$

Example 1.2.14. Let G be the following graph,

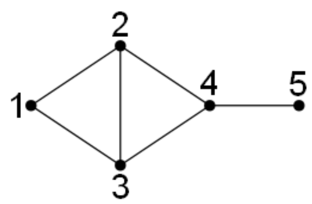


Figure 1.2.1

Then the binomial edge ideal of G is

$$J_G = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3, x_4y_5 - x_5y_4).$$

We will state some well-known results about the edge ideals and the binomial edge ideals that will be used in the following chapters.

Theorem 1.2.15. *[9] Let G be a graph. Then the edge ideal $I(G)$ has a linear resolution if and only if G^c is a chordal graph.*

In particular, a complete bipartite graph has a linear resolution, since its complement is chordal. Thanks to Woodroofe (see [28]), the next result gives an upper bound for the regularity of the edge ideal of a graph.

Theorem 1.2.16. *[28, Theorem 1] For any graph G, we have $reg(S/I(G)) \leq \mathrm{cochord}(G)$.*

Proposition 1.2.17. *[8, Proposition 3.5] Let G be a connected closed graph on $[n]$ and $I(H) = \mathrm{in}_<(J_G)$. Then $\mathrm{indmatch}(H) = l$ where l is the length of the longest induced path of G.*

Theorem 1.2.18. *[28] If G is a weakly chordal graph on $[n]$, then*

$$\mathrm{reg}(K[x_1, \ldots, x_n]/I(G)) = \mathrm{indmatch}(G).$$

Lemma 1.2.19. *[8] Let G be a connected closed graph on $[n]$. Then the bipartite graph $I(H) = \mathrm{in}_<(J_G)$ is weakly chordal.*

Lemma 1.2.20. *[21, Lemma 4.8] Let G be a graph and let $v \in V(G)$be a non-simplicial vertex, then*

$$J_G = (J_{G \setminus v} + (x_v, y_v)) \cap (J_{G_v}).$$

Theorem 1.2.21. *[20, Theorem 3.7] Let G be a simple graph on $[n]$ and $e = \{i, j\} \in E(G)$. Then*

$$J_{G \setminus e} : f_e = J_{(G \setminus e)_e} + I_{G,e}$$

where

$$I_{G,e} = (g_{\pi,t} \mid \pi : i, i_1, \ldots, i_s, j \text{ is a path with ends } i \text{ and } j \text{ in } G \text{ and } 0 \leq t \leq s)$$

such that $g_{\pi,0} = x_{i_1} \ldots x_{i_s}$ and $g_{\pi,t} = y_{i_1} \ldots y_{i_t} x_{i_{t+1}} \ldots x_{i_s}$ for $1 \leq t \leq s$.

For simplicity, let $f_{ij} = f_e$ for $e = \{i, j\}$. The following exact sequence

$$0 \longrightarrow S/(J_{G \setminus e} : f_e)(-2) \xrightarrow{f_e} S/(J_{G \setminus e}) \longrightarrow S/(J_G) \longrightarrow 0 \tag{1.2.2}$$

together with Theorem 1.2.21 provides an inductive technique to estimate the regularity of J_G as shown in the following:

Proposition 1.2.22. *Let G be a simple graph on $[n]$ and $e \in E(G)$. Then we have* *(i)* $\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus e}), \operatorname{reg}(J_{G\setminus e} : f_e) + 1\}$,

(ii) $\operatorname{reg}(J_{G\setminus e}) \leq \max\{\operatorname{reg}(J_G), \operatorname{reg}(J_{G\setminus e} : f_e) + 2\}$,

(iii) $\operatorname{reg}(J_{G\setminus e} : f_e) + 2 \leq \max\{\operatorname{reg}(J_{G\setminus e}), \operatorname{reg}(J_G) + 1\}$.

If e is chosen carefully, then the sequence 1.2.2 has even better consequences in ne, as show in [15, Proposition 3.7] and [20, Theorem 3.4] we formulate these two results in the following.

Proposition 1.2.23. *Let G be a simple graph on $[n]$ and e be a cut edge of G. Then*

(i) $\beta_{i,j}(J_G) \leq \beta_{i,j}(J_{G\setminus e}) + \beta_{i-1,j-2}(J_{(G\setminus e)_e})$, *for every* $i, j \geq 1$.

(ii) $\operatorname{proj\,dim}(J_G) \leq \max\{\operatorname{proj\,dim}(J_{G\setminus e}), \operatorname{proj\,dim}(J_{(G\setminus e)_e}) + 1\}$,

(iii) $\operatorname{reg}(J_G) \leq \max\{\operatorname{reg}(J_{G\setminus e}), \operatorname{reg}(J_{(G\setminus e)_e}) + 1\}$.

Also, we consider the following short exact sequence to state the next result.

$$0 \longrightarrow S/(J_{G\setminus e})(-2) \xrightarrow{f_e} S/(J_{G\setminus e}) \longrightarrow S/(J_G) \longrightarrow 0 \tag{1.2.3}$$

Proposition 1.2.24. *[15, Proposition 3.9] Let G be a simple graph on $[n]$ and e be a free cut edge of G. Then*

(i) $\beta_{i,j}(J_G) = \beta_{i,j}(J_{G\setminus e}) + \beta_{i-1,j-2}J_{(G\setminus e)}$, *for every* $i, j \geq 1$.

(ii) $\operatorname{proj\,dim}(J_G) = \operatorname{proj\,dim}(J_{G\setminus e}) + 1$,

(iii) $\operatorname{reg}(J_G) = \operatorname{reg}(J_{G\setminus e}) + 1$.

After applying the Proposition 1.2.24 repeatedly, we reach $\operatorname{reg}(J_G) = \operatorname{reg}(J_{\mathcal{R}(G)}) + q - 1$, if the number of free cut edges in G is $q-1$. Then we get the following result.

Corollary 1.2.25. *[15, Corollary 3.10] Let G be a connected graph on the vertex set $[n]$ with the connected components $R_1, \ldots, R_q$ of $\mathcal{R}(G)$. Therefore,* $\operatorname{reg}(J_G) = \sum_{i=1}^{q} \operatorname{reg}(J_{R_i})$.

The following result is the direct consequence of the fact that if H is a subgraph of G, then $J_H \subset J_G$

Proposition 1.2.26. *[27, Proposition 8] Let G be graph and H be a subgraph of G. Then*

(i) $\beta_{i,j}(J_H) \leq \beta_{i,j}(J_G)$, *for all* i, j;

(ii) $\operatorname{reg}(J_H) \leq \operatorname{reg}(J_G)$;

(iii) $\operatorname{proj\,dim}(J_H) \leq \operatorname{proj\,dim}(J_G)$.

Theorem 1.2.27. *[1, Theorem 3.19,3.20] Let G be a connected graph on $[n]$ and assume that G is not complete. If G is l-vertex-connected, then* $\operatorname{proj\,dim}(S/J_G) \geq n + l - 2$ *and* $\operatorname{depth}(S/J_G) \leq n - l + 2$ *where* $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Chapter 2

Binomial Edge Ideals of Graphs

In this chapter, we first discuss the Gröbner basis of a binomial edge ideal as described in [12]. Later we discuss the some algebraic and homological properties of binomial edge ideals.

2.1 Gröbner basis of binomial edge ideals

The authors introduced the binomial edge ideals associated to simple graphs in the paper [12]. In this paper, they characterized all the graphs G such that generators of J_G form the reduced Gröbner basis of J_G. Let G be a graph on $[n]$. Moreover, let $<$ be the lexicographic order induced by a total order $x_1 > \ldots > x_n > y_1 > \ldots > y_n$ on $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Theorem 2.1.1. *[12, Theorem 1.1] The following conditions are equivalent:*

(a) The generators f_{ij} of J_G form a quadratic Gröbner basis;

(b) G is closed.

Proof. **(a)⇒ (b):** On the contrary, assume that G is not closed. Then there exist some edges $\{i, j\}, \{i, k\} \in E(G)$ such that $i < j < k$, but $\{j, k\} \notin E(G)$. Since J_G forms a Gröbner basis, we must have $\mathsf{S}(f_{ik}, f_{ij}) = y_i f_{jk} \in J_G$ by Corollary 1.2.5. It follows that $\text{in}_<(y_i f_{jk}) = x_j y_i y_k \in \text{in}_<(J_G)$. However, $x_j y_i \notin \text{in}_<(J_G)$ since $\text{in}_<(f_{ij}) = x_i y_j \in \text{in}_<(J_G)$. Hence, $x_j y_k \in \text{in}_<(J_G)$ which implies that $\{j, k\} \in E(G)$.
(b)⇒ (a): Suppose that G is closed. Let $\{i, j\}, \{k, l\} \in E(G)$ with $i < j, k < l$. We want to show that each S-pair reduces to zero.

1. If $i \neq k$ and $j \neq l$ then $\{j, l\}, \{i, k\} \notin E(G)$. In this case, $\text{in}_<(f_{ij}) = x_i y_j$ and $\text{in}_<(f_{kl}) = x_k y_l$ are relatively prime. Then $\mathsf{S}(f_{ij}, f_{kl})$ reduces to zero by Proposition 1.2.7.

2. If $i = k$, then $\{j, l\} \in E(G)$. Therefore, $\mathsf{S}(f_{ij}, f_{kl}) = \mathsf{S}(f_{ij}, f_{il}) = y_i f_{lj}$ reduces to zero. Similarly, if we assume that $j = l$ then $\mathsf{S}(f_{ij}, f_{il}) = x_j f_{ik}$ reduces to zero.

2.1.1 The reduced Gröbner basis of a binomial edge ideal

To compute the reduced Gröbner basis of a binomial edge ideal of any graph G, the admissible path is defined as follows in [12].

Definition 2.1.2. Let G be a graph on $[n]$, and let i, j be two vertices of G such that $i < j$. A path $i = i_0, i_1, \dots, i_r = j$ from i to j is called **admissible**, if

1. $i_k \neq i_l$ if $k \neq l$;
2. for each $k = 1, \dots, r-1$ one has either $i_k < i$ or $i_k > j$;
3. for any proper subset $\{j_1, \dots, j_s\}$ of $\{i_1, \dots, i_{r-1}\}$, the sequence $i, j_1, \dots, j_s, j$ is not a path.

For an admissible path $\pi : i = i_0, i_1, \dots, i_r = j$ from i to j with $i < j$, we associate the monomial

$$u_\pi = \Big(\prod_{i_k > j} x_{i_k}\Big)\Big(\prod_{i_l < i} y_{i_l}\Big).$$

Theorem 2.1.3. *[12, Theorem 2.1] Let G be a graph on $[n]$, and $<$ be a lexicographic order on S. Then the set of binomials*

$$\mathcal{G} = \bigcup_{i<j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G with respect to $<$.

Proof. First, we show that $\mathcal{G} \subset J_G$. We prove this by applying induction on the length of the admissible path r. Let $\pi : i = i_0, i_1 \dots, i_r = j$. If $r = 1$, then $\pi : i, j$ and $u_\pi = 1$. Consequently, $u_\pi f_{ij} \in J_G$. Now, assume that $r > 1$, $P = \{i_k \mid i_k < i\}$, and $Q = \{i_l \mid i_l > j\}$. If $P \neq \emptyset$, we can take $i_{k_0} := \max(P)$. Therefore, the paths $\pi_1 : i_{k_0}, i_{k_0 - 1}, \dots, i_0 = i$ and $\pi_2 : i_{k_0}, i_{k_0+1}, \dots, i_r = j$ are admissible. The induction hypothesis ensures that each of $u_{\pi_1} f_{i_{k_0} i}$ and $u_{\pi_2} f_{i_{k_0} j}$ belongs to J_G, and moreover, $\mathsf{S}(u_{\pi_1} f_{i_{k_0} i}, u_{\pi_2} f_{i_{k_0} j}) = u_\pi f_{ij}$ belongs to J_G. If $Q \neq \emptyset$, then we take $i_{l_0} := \min(Q)$. A similar argument as in the case that $P \neq \emptyset$, gives the desired conclusion.
Secondly, we prove that $\mathcal{G}$ forms a Gröbner basis of J_G. Then we need to show that all $\mathsf{S}-$pairs $\mathsf{S}(u_\pi f_{ij}, u_\sigma f_{kl})$ reduce to zero. We consider the following cases.

1. If $i = k$ and $j = l$, then $\mathsf{S}(u_\pi f_{ij}, u_\sigma f_{kl}) = 0$.

2. If $\{i,j\}\cap\{k,l\}=\emptyset$, then $\text{in}_<(f_{ij})$ and $\text{in}_<(f_{kl})$ are relatively prime. Consequently, their S-polynomial reduces to zero by Proposition 1.2.7.

3. If $i=l$ or $k=j$, we may assume that $i=k$ and $j\neq l$. The latter case that $i\neq k$ and $j=l$ can be shown in a similar way. We take the paths $\pi : i = i_0, i_1, \dots, i_r = j$ and $\sigma : i = i'_0, i'_1, \dots, i'_s = l$. Then there exist two indices a and b such that $i_a = i'_b$ and $\{i_{a+1},\dots,i_r\}\cap\{i'_{b+1},\dots,i'_s\}=\emptyset$. Now, we consider the path $\tau : j = i_r, j_{r-1}, \dots, i_{a+1}, i_a = i'_b, i'_{b+1}, \dots, i'_s = l$ from i to l. Let us write $\tau : j = j_0, \dots, j_t = l$ to simplify the notation. We can take $j_{t(1)} = \min\{j_c \mid j_c > j, c = 1, \dots, t\}$, and $j_{t(m)} = \min\{j_c \mid j_c > j, c = t(i)+1, \dots t\}$ for all $i = 2, \dots, t-1$. This yields an admissible path

$$\tau_c : j_{t(c-1)}, j_{t(c-1)+1}, \dots, j_{t(c)-1}, j_{t(c)}.$$

Hence, we get the standard expression

$$\mathsf{S}(u_\pi f_{ij}, u_\sigma f_{kl}) = \sum_{c=1}^{q} v_{\tau_c} u_{\tau_c} f_{j_{t(c-1)} j_{t(c)}}$$

that reduces to zero, (see [12, Theorem 2.1] for the detailed computations).

In the last step, it is enough to show that $\mathcal{G}$ is reduced. Let $u_\pi f_{ij}$ and $u_\sigma f_{kl}$ be two distinct elements in $\mathcal{G}$ such that $i<j$ and $k<l$. Let $\pi : i = i_0, i_1 \dots, i_r = j$ and $\sigma : k = k_0, \dots, k_s = l$. On the contrary, suppose that $u_\pi x_i y_j$ divides $u_\sigma x_k y_l$ or $u_\sigma x_l y_k$. Therefore, $\{i_0, i_1 \dots, i_r\}$ is a proper subset of $\{k_0, \dots, k_s\}$ as $x_i y_j$ can divide neither of $x_k y_l$ and $x_l y_k$.

1. If $i=k$ and $j=l$, then $\{i_0, i_1 \dots, i_{r-1}\} \subsetneqq \{k_0, \dots, k_s\}$ and $k, i_1 \dots, i_{r-1}, l$ is an admissible path. This contradicts the third property of Definition 2.1.2 because σ is an admissible path.

2. If $i=k$ and $j\neq l$, then y_j divides u_σ since either $u_\pi x_i y_j \mid u_\sigma x_k y_l$ or $u_\pi x_i y_j \mid u_\sigma x_l y_k$. Thus, $j<k$ and this yields a contradiction because $k=i<j$.

3. If $\{i,j\}\cap\{k,l\}=\emptyset$, then $x_i y_j$ divides u_σ since either $u_\pi x_i y_j \mid u_\sigma x_k y_l$ or $u_\pi x_i y_j \mid u_\sigma x_l y_k$. Thus, $i>l$ and $j<k$, that is, $j<k<l<i$ which contradicts $i<j$.

Corollary 2.1.4. *[12, 2.2] J_G is a radical ideal.*

The proof of Corollary 2.1.4 follows from the general fact that each graded ideal $I \subseteq S$ is radical with the property that $\text{in}_<(I)$ is a squarefree monomial ideal for some fixed monomial order $<$, see [11, Corollary 1.2.5].

Proof. Let $\mathcal{G} = \bigcup_{i<j}\{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$ be the reduced Gröbner basis of J_G. Since u_π and f_{ij} do not contain any common variable, it follows that the reduced Gröbner basis is squarefree. If $\{g_1, \dots, g_s\}$ is a Gröbner basis then $\mathrm{in}_<(J_G) = (\mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_s))$. Therefore, $\mathrm{in}_<(J_G)$ is also squarefree, and hence radical by [11, Corollary 1.2.5]. Let $f^k \in J_G$ for some k. We show that $f \in J_G$ by induction on k. It is clear that $\mathrm{in}_<(f^k) = \mathrm{in}_<(f)^k \in \mathrm{in}_<(J_G)$. Hence $\mathrm{in}_<(f) \in \mathrm{in}_<(J_G)$ which implies that there exists $g \in J_G$ such that $\mathrm{in}_<(f) = \mathrm{in}_<(g)$. Then there is some $a \in K$ with $\mathrm{in}_<(f-ag) < \mathrm{in}_<(f)$ which gives that $(f-ag)^k \in J_G$. As $\mathrm{in}_<(f-ag) < \mathrm{in}_<(f)$, by induction hypothesis we obtain $f - ag \in J_G$. Thus, $f \in J_G$. □

2.2 Minimal prime ideals of binomial edge ideals

It is a well-known fact that every proper ideal I in S has a primary decomposition because S is a commutative Noetherian ring. In this section, we discuss the minimal primary decompositions of binomial edge ideals.

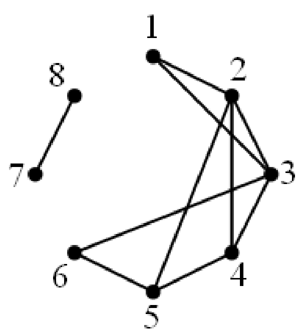
Let G be a graph on $[n]$. By [5, Theorem 7.1], the primary ideals in the primary decomposition of J_G must be either monomial or binomial ideals. Before giving the primary decomposition of J_G, we recall the following notation from [12].

For each $S \subset [n]$, we set a prime ideal $P_S(G)$ in the following way. Let $T = [n]\backslash S$, and let $G_1, \dots G_{t(S)}$ be the connected component of G_T which is the induced subgraph of G on T. We denote by $\tilde{G}_i$ the complete graph with $V(\tilde{G}_i) = V(G_i)$. Then we set

$$P_S(G) = \Big(\bigcup_{i\in S}\{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{t(S)}}\Big).$$

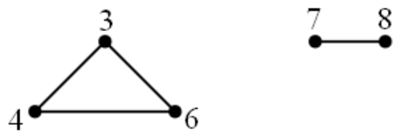
In fact, each $J_{\tilde{G}_i}$ is the ideal of generic $2 \times n_j$ matrix with $n_j = |V(G_j)|$. Since $\tilde{G}_i$ is a complete graph then $J_{\tilde{G}_i}$ is a prime ideal for every $1 \le i \le t(S)$. Moreover, $\bigcup_{i\in S}\big(\{x_i, y_i\}\big)$ is also a prime ideal since it is generated by variables. Hence, $P_S(G)$ is a prime ideal.

Example 2.2.1.



Let G be a graph displayed on the left hand side. Let $S = \{1, 2, 5\} \subset [8]$. Then
$T = [8] \setminus S = \{3, 4, 6, 7, 8\}$.

Therefore, $\tilde{G_1}$ and $\tilde{G_2}$ are displayed on the right hand side.



Hence, the minimal prime ideal of J_G is

$$P_S(G) = \big(\{x_1, y_1, x_2, y_2, x_5, y_5\}, f_{34}, f_{36}, f_{46}, f_{78}\big).$$

Lemma 2.2.2. *[12, Lemma 3.1] With the above notation,* height $P_S(G) = |S| + n - t(S)$.

Proof. We have the following equalities:

$$\begin{aligned}
\text{height } P_S(G) &= \text{height}\left(\bigcup_{i\in S}\{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{t(S)}}\right) \\
&= \text{height}\left(\bigcup_{i\in S}\{x_i, y_i\}\right) + \sum_{i=1}^{t(S)} \text{height } J_{\tilde{G}_i} \\
&= 2|S| + \sum_{i=1}^{t(S)}(n_i - 1) \qquad \text{where } n_i = |V(G_i)| \\
&= |S| + \left(|S| + \sum_{i=1}^{t(S)} n_i\right) - t(S) \\
&= |S| + n - t(S).
\end{aligned}$$

□

Theorem 2.2.3. *[12, Theorem 3.2] Let G be a graph on $[n]$. Then $J_G = \bigcap_{S\subset[n]} P_S(G)$.*

Proof. By Corollary 2.1.4, J_G is a radical ideal. It is a well-known that a radical ideal is the intersection of its minimal prime ideals.

- We show that $x_i \in P$ if and only if $y_i \in P$. We may assume that G is connected because if $G_1, \ldots, G_r$ are the connected components of G then each minimal prime ideal P of J_G is of the form $P_1 + \ldots + P_r$ where each P_i is a minimal prime ideal of J_{G_i}. Hence, if P_i is in the desired form for all i then so does P. We consider the set $T = \{x_i \mid i \in [n], x_i \in P, y_i \notin P\}$.
  First, we observe that $T \neq \{x_1, \ldots, x_n\}$, otherwise, we would have $J_G \subset J_{\tilde{G}} \subsetneq (x_1, \ldots, x_n) \subset P$ and P would not be a minimal prime ideal of J_G.
  We will show that $T = \emptyset$ by contradiction. Assume that there exists $\{i, j\} \in E(G)$ such that $x_i \in T$ but $x_j \notin T$. Since $f_{ij} = x_iy_j - x_jy_i \in J_G \subset P$ and $x_i \in P$, we obtain $x_iy_j \in P$. Now, we have that $f_{ij} = x_iy_j - x_jy_i \in J_G \subset P$ and $x_i \in P$. It follows that $x_jy_i \in P$, and then $x_j \in P$ as $y_i \notin P$. Thus, $x_j \in P$, and this gives us a contradiction. Hence there exists a subset $S \subset [n]$ such that $P = \big(\bigcup_{i\in S}\{x_i, y_i\}, P'\big)$ where P' does not contain any variable.

- Let G' be the graph $G \setminus S$. Next, we will show that $P' = (J_{\tilde{H}_1}, \dots J_{\tilde{H}_c})$ by assuming that $H_1, \dots, H_c$ are the connected components of G'. We claim that if $\{i, j\}$ with $i < j$ is an edge of $\tilde{H}_k$ for some k, then $f_{ij} \in P'$. Let $i = i_0, i_1, \dots, i_r = j$ be a path in H_k from i to j. Now, by induction on r we will show that $f_{ij} \in P'$. The base case is trivial for $r = 1$. Suppose that $r > 1$. Then by induction hypothesis, it follows that $f_{i_1 j} \in P'$, and $f_{ii_1} \in P'$. Hence we get $x_j f_{ii_1} + x_i f_{i_1 j} = x_{i_1} f_{ij} \in P'$. Since P' is prime, we have $x_{i_1} \in P'$ or $f_{ij} \in P'$. We conclude that $f_{ij} \in P'$ because P' has no variable. Hence, P' is a minimal prime ideal containing $J_{G'}$.

Thus we conclude that each minimal prime ideal containing J_G is of the form $P_S(G)$ for all $S \subseteq [n]$ by induction. □

Corollary 2.2.4. *[12, Corollary 3.3] Let G be a graph on $[n]$. Then*

$$\dim(S/J_G) = \max\{n - |S| + t(S) : S \subseteq [n]\}.$$

In particular, $\dim(S/J_G) \geq n + t$ *where t is the number of connected components of G.*

The Theorem 2.2.3 gives us a primary decomposition of a binomial edge ideal J_G. However, the primary decompositions need not to be minimal. One can observe that the minimal primes of J_G correspond to the several cut point property sets of G as shown in the following theorem.

Theorem 2.2.5. *[13, Theorem 7.20] Let G be a connected graph on $[n]$, and $V \subset [n]$. Then $P_V(G)$ is a minimal prime ideal of J_G if and only if $V = \emptyset$, or $V \neq \emptyset$ and $c(V \setminus \{i\}) < c(V)$ for all $i \in V$. In other words, this is the case, if and only if each $i \in V$ is a cut point of the graph $G_{([n]\setminus V)\cup\{i\}}$.*

The above theorem has a following consequence.

Corollary 2.2.6. *Let G be a graph on $[n]$. Then J_G is a prime ideal if and only if each component of G is a complete graph.*

Proof. Assume that $G_1, \dots, G_r$ are the connected components of G and J_G is a prime ideal. Then $P_\emptyset(G) = (J_{\tilde{G}_1}, \dots, J_{\tilde{G}_r})$ is a minimal prime ideal of J_G, and therefore, $J_G = (J_{\tilde{G}_1}, \dots, J_{\tilde{G}_r})$.
Let $G_1, \dots, G_r$ are the connected components of G. Then we have $J_G = (J_{G_1}, \dots, J_{G_r})$. Since each component of G is complete, J_{G_i} contains all 2-minors of $2 \times k$ matrix for some k where $i = 1, \dots, r$. Hence, J_{G_i} is a prime ideal for all i. □

2.3 Depth and Betti numbers of the binomial edge ideals

In this section, we give certain results about the depth and Betti numbers of binomial edge ideals.

Lemma 2.3.1. *[19, Lemma 2.1] Let G be a simple graph on $[n]$, and $T \subset [n]$. Then for any $\mathbf{a} \in \mathbb{N}^n$ with $\operatorname{supp}(\mathbf{a}) \subset T$, we have that*

$$\beta_{i,\mathbf{a}}(J_G) = \beta_{i,\mathbf{a}}(J_{G_T}) \quad \textit{for each } i.$$

Proof. Let $\operatorname{proj}\dim(J_G) = p$. We consider the $\mathbb{N}^n$-graded free resolution $\mathcal{F}$ of S/J_G,

$$\mathcal{F} : 0 \to \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S^{\beta_{p,\mathbf{a}}(J_G)}(-\mathbf{a}) \to \cdots \to \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S^{\beta_{0,\mathbf{a}}(J_G)}(-\mathbf{a}) \xrightarrow{\phi} S.$$

Now, we look at the subcomplex of F

$$\mathcal{F}' : 0 \to \bigoplus_{\substack{\mathbf{a}\in\mathbb{N}^n\\ \operatorname{supp}(\mathbf{a})\subset T}} S^{\beta_{p,\mathbf{a}}(J_G)}(-\mathbf{a}) \to \cdots \to \bigoplus_{\substack{\mathbf{a}\in\mathbb{N}^n\\ \operatorname{supp}(\mathbf{a})\subset T}} S^{\beta_{0,\mathbf{a}}(J_G)}(-\mathbf{a}) \xrightarrow{\phi} S.$$

Hence, we have trivially

$$\mathcal{F}' : 0 \to \bigoplus_{\substack{\mathbf{a}\in\mathbb{N}^n\\ \operatorname{supp}(\mathbf{a})\subset T}} S^{\beta_{p,\mathbf{a}}(J_G)}(-\mathbf{a}) \to \cdots \to \bigoplus_{\substack{\mathbf{a}\in\mathbb{N}^n\\ \operatorname{supp}(\mathbf{a})\subset T}} S^{\beta_{0,\mathbf{a}}(J_G)}(-\mathbf{a}) \xrightarrow{\phi} S \xrightarrow{\varphi} S/J_{G_T} \to J_{G_T} \to 0.$$

Whence, $\ker\varphi \cong \operatorname{coker}\phi' = S/J_{G_T}$. Next, we want to show that $\mathcal{F}'$ is an exact left complex. It is needed to show that each multigraded component $\mathcal{F}'_{\mathbf{a}}$ is exact where $\mathbf{a} \in \mathbb{N}^n$ and $\operatorname{supp}(\mathbf{a}) \subset T$. Let $\mathbf{a} \in \mathbb{N}^n$ such that $\operatorname{supp}(\mathbf{a}) \subset T$. For all $\mathbf{b} \in \mathbb{N}^n$, $S(-\mathbf{b})_{\mathbf{a}} = S(\mathbf{a} - \mathbf{b})$ is not zero if and only if $\mathbf{a} - \mathbf{b}$ is not a negative integer. Then $\mathcal{F}_{\mathbf{a}} = \mathcal{F}'_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{N}^n$. As $\mathcal{F}$ is a minimal free resolution then $\mathcal{F}_{\mathbf{a}}$ is exact for all $\mathbf{a} \in \mathbb{N}^n$, so is $\mathcal{F}'_{\mathbf{a}}$. □

Theorem 2.3.2. *[7, Theorem 1.1] Let G be a chordal graph on $[n]$ with the property that any two distinct maximal cliques intersect in at most one vertex. Then* $\operatorname{depth} S/J_G = n + t$, *where t is the number of connected components of G. Moreover, the following conditions are equivalent:*

(i) J_G is unmixed.

(ii) J_G is Cohen–Macaulay.

(iii) Each vertex of G is the intersection of at most two maximal cliques.

Let A be a simple graph on $[n]$ and $v \in V(A)$. Then the cone of v on A which is denoted by $v * A$ is the graph on the vertex set $V(v * A) = V(A) \sqcup \{v\}$ and the edge set $E(v * A) = E(A) \sqcup \{\{v, w\} \mid w \in E(A)\}$. Now, assume that A is not a complete graph, and we put

$$G = v * A, \quad S_A = K[x_i y_i \mid i \in V(G)], \quad \text{and } S = S_G[x_v, y_v].$$

If we take $G = v * A$, then $G_v = K_{n+1}$, $G_v \setminus v = K_n$, and $G \setminus v = A$. This implies the equality

$$(x_v, y_v) + J_{G \setminus v} + J_{G_v} = (x_v, y_v) + J_{K_n}.$$

Hence we obtain a short exact sequence by Lemma 1.2.20

$$0 \to \frac{S}{J_G} \to \frac{S}{(x_v, y_v) + J_A} \oplus \frac{S}{K_{n+1}} \to \frac{S}{(x_v, y_v) + J_{K_n}} \to 0,$$

Remark 2.3.3. If G is a complete graph on $[n]$, then the Theorem 2.3.2 implies that $\operatorname{depth} S/J_G = n + 1$ and S/J_G is Cohen-Macaulay. By Theorem 1.2.27, $\operatorname{proj\,dim}_S(S/J_G) \geq n - 2 + \kappa(G) \geq n - 1$ if G is connected, but not a complete graph. Hence $\operatorname{proj\,dim}_S(S/J_G) \geq n - 1$ for any connected graph G, and then from the Auslander-Buchsbaum formula (see 1.2.13) we have that $\operatorname{depth}(S/J_G) \leq n + 1$.

Next, we examine the depth of join of two graphs. Let G and H be graphs on $[m]$ and $[n]$, respectively, where $m, n > 1$. Suppose that either of G and H is not complete. We denote by $\Gamma = G * H$ the join of G and H on the vertex set $[m] \sqcup [n]$ and on the edge set $E(\Gamma) = E(G) \cup E(H) \cup \big\{\{i, j\} \mid i \in [m], j \in [n]\big\}$. The Theorem 2.2.3 and [12, Corollary 3.9] yields the following decomposition for the binomial edge ideal of Γ

$$J_\Gamma = P_\emptyset(\Gamma) \cap \big((x_i, y_i \mid i \in [m]) + J_H\big) \cap \big((x_j, y_j \mid j \in [n]) + J_G\big).$$

If we set

$$Q_1 = \big((x_i, y_i \mid i \in [n]) + J_G\big), \quad Q_2 = \big((x_i, y_i \mid i \in [m]) + J_H\big), \quad \text{and} \quad Q_3 = P_\emptyset(\Gamma) \cap Q_2,$$

then we have $Q_2 + P_\emptyset(G) = (x_i, y_i \mid i \in [m]) + J_{K_n}$ and $Q_1 + Q_3 = (x_i, y_i \mid i \in [n]) + J_{K_m}$. This gives the following exact sequence

(2.3.1) $$0 \to S/Q_3 \to S/P_\emptyset(\Gamma) \oplus S/Q_2 \to S/(Q_2 + P_\emptyset(G)) \to 0,$$

(2.3.2) $$0 \to S/J_\Gamma \to S/Q_1 \oplus S/Q_3 \to S/(Q_1 + Q_3) \to 0.$$

Let $S_G = K[x_i, y_i \mid i \in [m]]$ and $S_H = K[x_i, y_i \mid i \in [n]]$. Then

$$\operatorname{depth}_S(S/Q_1) = \operatorname{depth}_{S_G}(S_G/J_G), \quad \operatorname{depth}_S\big(S/(Q_2 + P_\emptyset(\Gamma))\big) = n + 1,$$

$$\operatorname{depth}_S\big(S/(Q_1+Q_3)\big)=m+1, \quad \text{and} \quad \operatorname{depth}_S(S/Q_2)=\operatorname{depth}_{S_H}(S_H/J_H).$$

Next, the Lemma1.2.12 and the exact sequences 2.3.1 and 2.3.2 yields the inequality

$$\operatorname{depth}_S(S/Q_3)\geq \min\{\operatorname{depth}_{S_H}(S_H/J_H), n+2\},$$

which gives

$$\operatorname{depth}_S(S/J_G)\geq \min\{\operatorname{depth}_{S_G}(S_G/J_G), \operatorname{depth}_{S_H}(S_H/J_H), m+2, n+2\}.$$

This inequality is improved in [18] as following.

Theorem 2.3.4. *[18, Theorem 4.1] Let G and H be two connected graphs on $[m]$ and $[n]$ respectively, and let $\Gamma=G*H$ be their join product. Then*

$$\operatorname{depth}_S(S/J_\Gamma)=\min\{\operatorname{depth}_{S_G}(S_G/J_G), \operatorname{depth}_{S_H}(S_H/J_H)\}.$$

Moreoever, when G and H are disconnected graphs then we have the following.

Theorem 2.3.5. *[18, Theorem 4.4] Let G and H be two disconnected graphs on $[m]$ and $[n]$ respectively, and let $\Gamma=G*H$ be their join product. Suppose that $n\geq m$, then*

$$\operatorname{depth}_S(S/J_\Gamma)=\min\big\{\operatorname{depth}_{S_G}(S_G/J_G), \operatorname{depth}_{S_H}(S_H/J_H), m+2\big\}$$

2.4 Cohen-Macaulayness of the binomial edge ideals

Now, we discuss the Cohen-Macaulayness of the binomial edge ideals. First, we recall the characterisation of bipartite graphs whose edge ideals are Cohen-Macaulay, see [10].

Theorem 2.4.1. *Let G be a finite bipartite graph on the vertex set $W\cup W'$ where $W=\{x_1,\dots,x_n\}$ and $W'=\{y_1,\dots,y_n\}$. Then $I(G)$ is Cohen-Macaulay if and only if the following conditions are satisfied:*

1. *$\{x_i,y_i\}$ are edges for $i=1,\dots,n$;*
2. *if $\{x_i,y_j\}$ is an edge, then $i\leq j$;*
3. *if $\{x_i,y_j\}$ and $\{x_j,y_k\}$ are edges, then $\{x_i,y_k\}$ is an edge.*

In [12], the Cohen-Macaulay property for the binomial edge ideals of bipartite graphs is discussed. We simply say that G is Cohen-Macaulay if its binomial edge ideal J_G is Cohen-Macaulay. A graph G is Cohen-Macaulay if and only if each connected component of G is Cohen-Macaulay. Hence, we can examine the connected graphs.

Proposition 2.4.2. *[12, Proposition 1.6] Let G be a connected graph on $[n]$ which is closed with respect to a given labelling. Suppose that G satisfies the condition that $\{i, j+1\} \in E(G)$ with $i < j$ and $\{j, k+1\} \in E(G)$ with $j < k$ implies that $\{i, k+1\} \in E(G)$. Then S/J_G is Cohen-Macaulay.*

Proof. We recall that

$$\dim(S/I) \geq \operatorname{depth}(S/I) \geq \operatorname{depth}(S/\operatorname{in}_<(I)).$$

We show that $S/\operatorname{in}_<(J_G)$ is Cohen-Macaulay because it implies that S/J_G is Cohen-Macaulay. By Theorem 2.1.1, $\operatorname{in}_<(J_G)$ is a monomial ideal generated by $x_i y_j$ with $\{i, j\} \in E(G)$ and $i < j$. We consider the automorphism

$$\begin{aligned} \varphi : \quad & S \longrightarrow S \\ & x_i \mapsto x_i; \\ & y_j \mapsto y_{j-1} \quad \text{for } j > 1; \\ & y_1 \mapsto y_n. \end{aligned}$$

The map φ associate $\operatorname{in}_<(J_G)$ to the ideal I generated by all monomials $x_i y_j$ such that $\{i, j+1\} \in E(G)$. Note that I is an edge ideal of a bipartite graph H on the vertex set $\{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}\}$ with $\{x_i, x_j\} \in E(H)$ if and only if $\{i, j+1\} \in E(G)$. We put $S' = K[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}]$ such that $I \subseteq S'$. Due to the automorphism φ, $S/\operatorname{in}_<(J_G)$ is Cohen-Macaulay if and only if S'/I is Cohen-Macaulay.
Now, we are going to show that J_G is Cohen-Macaulay by satisfying the conditions in Definition 2.4.1. For the first condition, it follows from the Proposition 1.1.6 that $\{i, i+1\}$ is an edge of G for all i. Second condition is clearly satisfied by the structure of the graph. The third condition is provided due to our assumption. □

By Corollary 2.2.4, we have the following:

Corollary 2.4.3. *[12, Corollary 3.4] Let G be a simple graph on $[n]$ with t connected components. If S/J_G is Cohen-Macaulay, then $\dim(S/J_G) = n + t$.*

Moreover, the Proposition 2.2.3 gives the following result.

Corollary 2.4.4. *[12, Corollary 3.7] Let G be a cycle of length n. Then the following are equivalent:*

(1) $n = 3$,

(2) J_G is a prime ideal,

(3) J_G is unmixed,

(4) S/J_G is Cohen-Macaulay.

Proof. From Corollary 2.2.6, we get (1) $\Leftrightarrow$ (2).
(2) $\Rightarrow$ (4) : Owing to Corollary 2.2.6, if J_G is a prime ideal, then each of the components of G is a complete graph. Therefore, the binomial edge ideal of each component is the ideal of 2-minors of a $2 \times k$ matrix for some k. Therefore, J_G is Cohen-Macaulay.
(4) $\Rightarrow$ (3) : Every Cohen-Macaulay graph without isolated vertices is unmixed.
(3) $\Rightarrow$ (2) : We know that $P_\emptyset(G)$ is a minimal prime of J_G and $\dim(S/P_\emptyset(G)) = n+1$. Let $S \subset [n]$ be a non-empty subset. We consider the a counterclockwise labelling on the edges of the cycle. Assume that $S = \bigcup_{i=1}^{r}[a_i, b_i]$ where $0 = a_1 - 1 < b_1 < a_2 - 1 < b_2 < \ldots < b_r < n$. Then $t(S)$, the number of connected components of G_S, is equal to r. Also, by Corollary 2.2.4 we have $\dim(S/P_\emptyset(G)) = n - |S| + t(S)) = n - \sum_{i=1}^{r}(b_i - a_i) - r + r \leq n$. If J_G is unmixed then J_G has only one minimal prime ideal that is $P_\emptyset(G)$. Thus, J_G is reduced and hence prime. □

In [7], closed graphs with Cohen-Macaulay binomial edge ideals are completely classified. We state this result in the following theorem.

Theorem 2.4.5. *[7, Theorem 3.1] Let G be a connected graph on $[n]$ which is closed with respect to the given labelling. Then the following conditions are equivalent:*

(i) J_G is unmixed;

(ii) J_G is Cohen-Macaulay;

(iii) $\text{in}_<(J_G)$ is Cohen-Macaulay;

(iv) G satisfies the condition that whenever $\{i, j+1\}$ with $i < j$ and $\{j, k+1\}$ with $j < k$ are edges of G, then $\{i, k+1\}$ is an edge of G.

(v) there exist integers $1 = a_1 < a_2 < \ldots < a_{r+1} = n$, and a leaf order of the facets $F_1, \ldots, F_r$ of $\Delta(G)$ such that $F_i = [a_i, a_{i+1}]$ for all $i = 1, \ldots, r$.

Proof. We may also assume that G is not connected and then consider the connected components of G. Let G be a simple graph on $[n]$. We denote $t_S(G)$, the number of connected component in $G_{[n]\setminus S}$. $(i) \Rightarrow (v)$: By Theorem 1.1.10, the simplicial complex of G, $\Delta(G)$, has facets $F_1, \ldots F_r$ which are intervals in $[n]$. We order the intervals $F_i = [a_i, b_i]$ where $1 = a_1 < a_2 < \ldots < a_r \leq b_r = n$. From $1 = a_1 < a_2 < \ldots < a_r \leq b_r = n$, it follows that $a_i \leq b_{i-1}$ for each i. Otherwise, we would have $a_2 \in F_1$, that is we could not have a leaf order.
Let $W = [a_r, b_{r-1}]$. Then V has a cut point property. The Theorem 2.2.5 follows that $P_W(G)$ is a minimal prime ideal of J_G. Hence, $c(W) = 2$.
Hence, we have height $P_W(G) = n + |W| - c(W) = n + (b_{r-1} - a_r + 1) - 2 =$

$n + b_{r-1} - a_r - 1$.
We recall that if I is unmixed then all minimal prime ideals of I have the same height. Take $P_\emptyset(G) \in \mathrm{Min}(R/I)$. Then height $P_W(G) =$ height $P_\emptyset(G) = n-1$ as G is connected. It follows that $n+b_{r-1}-a_r-1 = n-1$, and then $a_r = b_{r-1}$. Let $r = 1$. Then $\Delta(G) = \{\emptyset, [a_1, b_1]\}$. Now, we consider the graph H so that $F_1, \dots F_{r-1}$ are the facets of $\Delta(H)$. We take $P_W(H)$ as a minimal prime ideal of J_H. Whence $b_{r-1} \notin W$. Then $c_H(W) = 2 = c_G(W)$. So, height $P_W(H) =$ height $P_W(G)$. Therefore, J_H is also an unmixed ideal.
By Proposition 2.4.2 we get that $(iv) \Rightarrow (iii)$. In addition, the implications $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are satisfied for all ideals.
Thus, it is enough to show that $(v) \Rightarrow (iv)$. Let i, j, k be in $V(G)$ such that $i < j < k$ and $\{i, j+1\}, \{j, k+1\} \in E(G)$. Therefore, i and $j+1$ have to be in the same facet, say F_i. Since $j+1 \in F_i$, we obtain $j \in F_i$ by Theorem 1.1.10. This implies that $k+1 \in F_i$. Thus, $\{i, k+1\} \in E(G)$ as desired. □

Proposition 2.4.6. *[7, Proposition 3.2] Let G be a closed graph with Cohen-Macaulay binomial edge ideal, and let $<$ be the lexicographic order on S. Then $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$ for all i and j.*

Proof. First, we suppose that G is connected. Let M be a graded S-module. The Betti polynomial of M is $B_M(s,t) = \sum_{i,j} \beta_{ij}(M) s^i t^j$.
By Theorem 2.4.5, $\mathrm{in}_<(J_G)$ is Cohen-Macaulay. The same theorem implies that $[n] = \bigcup_{k=1}^r [a_k, a_{k+1}]$ with $1 = a_1 < a_2 < \ldots < a_{r+1} = n$ such that $G_k = G_{[a_k, a_{k+1}]}$ is a complete graph. Moreover, G_k is a facet of clique complex $\Delta(G)$ for all k because G_k is a complete graph. Therefore, $\mathrm{in}_<(J_G)$ is minimally generated by the set of monomials $\bigcup_{k=1}^r M_k$ where $M_k = \{x_i y_j \mid a_k \le i < j \le a_{k+1}\}$ for all k.

$$\begin{aligned} \mathrm{in}_<(J_G) =& \Big(\bigcup_{k=1}^r \mathrm{in}_<(J_{G_k}) \Big) \\ =& \Big(\{x_i y_j \mid a_k \le i < j \le a_{k+1}\} \Big). \end{aligned}$$

For all distinct i, j, M_i and M_j are disjoint. Hence, $\mathrm{Tor}_k(S/(M_i), S/(M_j) = 0$ for all $i \neq j$ and all $k > 0$. It follows

$$B_{S/\mathrm{in}_<(J_G)}(s,t) = \prod_{i=1}^r B_{S/\mathrm{in}_<(M_i)}(s,t).$$

Now, we have that $\mathrm{Tor}_k(S/(M_i), S/(M_j) = 0$ for all k, and $\mathrm{in}_<(J_{G_i}) = (M_i)$ for all i. Owing to [2, Proposition 3.3] we have that

$$\mathrm{Tor}(S/J_{G_i}, S/J_{G_j}) \le \mathrm{Tor}\left(S/\mathrm{in}_<(J_{G_i}),\ S/\mathrm{in}_<(J_{G_j})\right) = 0.$$

That is, $\mathrm{Tor}(S/J_{G_i}, S/J_{G_j}) = 0$. Hence

$$B_{S/J_G}(s,t) = \prod_{i=1}^{r} B_{S/J_{G_i}}(s,t).$$

Next, we assume that G is a connected graph. Therefore, $\mathrm{in}_<(J_G)$ is an edge ideal of a complete bipartite graph. Then by Theorem 1.1.2, $\mathrm{in}_<(J_G)$ has a linear resolution. It is well-known that $\beta_{ij}(J_G) \leq \beta_{ij}(\mathrm{in}_<(J_G))$ (see Corollary 3.3.3 [11]). Moreover, the Betti numbers of an ideal which has a linear resolution are specified according to Hilbert function of the ideal (see Proposition 2.11 [13]). Since $\mathrm{Hilb}_{S/J_G}(t) = \mathrm{Hilb}_{S/\mathrm{in}_<(J_G)}(t)$ (by Proposition 2.6 in [13]), then $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$.
Lastly, we suppose that G is not connected. Let $G_1, \ldots, G_c$ be the connected components of G. Therefore, $\mathrm{in}_<(J_{G_i})$ and $\mathrm{in}_<(J_{G_j})$ consist of distinct monomials, and then we proceed the similar way as in the case of connected graphs. □

Chapter 3

Regularity of Binomial Edge Ideals

In this chapter, we study the results related to the regularity of binomial edge ideals. First, we state some upper and lower bounds for the regularity of binomial edge ideals.

3.1 Upper and lower bounds for the regularity

The bounds for the regularity of binomial edge ideal were studied in [19], however these bounds are not sharp. In [19], the authors proved the following:

Theorem 3.1.1. *[19, Theorem 1.1] Let G be a graph on $[n]$ and let l be the length of the longest induced path of G. Then*

$$l + 1 \leq \operatorname{reg}(J_G) \leq n.$$

Before stating the proof of this theorem we first recall the following notions related to monomials and monomial ideals. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Next, we define $\mathbb{N}^{2n}$-grading on S such that $\deg x_i = e_i$ and $\deg y_i = e_i + n$. We denote the multidegree $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{N}^{2n}$ and the monomial $x^{\mathbf{a}} y^{\mathbf{b}} = x_1^{a_1} \ldots x_n^{a_n} y_1^{b_1} \ldots y_n^{b_n}$. We observe that the monomials of S are $\mathbb{N}^{2n}$-graded, but the binomials are not $\mathbb{N}^{2n}$-graded in S. Let M be an $\mathbb{N}^{2n}$-graded monomial ideal. Then the *Poincaré series* of M is

$$P(M, t) = \sum_{k=0}^{2n} \sum_{(\mathbf{a},\mathbf{b}) \in \mathbb{N}^{2n}} \beta_{k, x^{\mathbf{a}} y^{\mathbf{b}}}(M) x^{\mathbf{a}} y^{\mathbf{b}} t^k.$$

Lemma 3.1.2. *[19, Lemma 3.1] Let I be an ideal generated by $m_1, \ldots, m_r$ in* $\operatorname{Mon}(S)$*. Then we have a coefficient-wise inequality*

$$P(S/I, t) \leq 1 + \sum_{m_j \notin (m_1, \ldots, m_{j-1})} P\Big(S/\big((m_1, \ldots, m_{j-1}) : m_j\big), t\Big) m_j t$$

Proof. We consider a short exact sequence

$$0 \longrightarrow S/\big((m_1,\ldots,m_{j-1}):m_j\big) \xrightarrow{m_j} S/(m_1,\ldots,m_{j-1}) \longrightarrow S/(m_1,\ldots,m_j) \longrightarrow 0$$

where m_j is the multiplication map. To make m_j a map of degree 0, we shift $S/\big((m_1,\ldots,m_{j-1}):m_j\big)$ by the degree of m_j, say $\mathbf{c} := \deg(m_j)$. Hence, we get the graded exact sequence

$$0 \longrightarrow S/\big((m_1,\ldots,m_{j-1}):m_j\big)(-\mathbf{c}) \xrightarrow{m_j} S/(m_1,\ldots,m_{j-1}) \longrightarrow S/(m_1,\ldots,m_j) \longrightarrow 0.$$

We suppose that a multigraded free resolution F_j of $S/(m_1,\ldots,m_j)$ and that a multigraded free resolution G_j of $S/\big((m_1,\ldots,m_j):m_{j+1}\big)$ are known. Then one can compute the multigraded free resolution of F_{j+1} owing to mapping cones. (see [22, Construction 27.3]) □

Let G be a simple graph on $[n]$. A path $\pi : s = p_1,\ldots,p_l = f$ of G is called **weakly admissible**, w-admissible in short, if it satisfies the second property of the Definition 2.1.2: one has either $p_k < s$ or $p_k > f$ for $k = 1,2,\ldots,l-1$ whenever $s < f$. Let $\pi : s = p_1,\ldots,p_l = f$ be a w-admissible path of G. We set a monomial

$$m_\pi = \Big(\prod_{p_k<s} y_k\Big)\Big(\prod_{p_k>f} x_k\Big)x_s y_t.$$

Also, we set $\mathcal{P}(G) := \{\text{w-admissible paths of } G\}$, and fix the lexicographic order $<$ on S such that $x_1 > \ldots > x_n > y_1 > \ldots > y_n$. Thanks to Herzog, Hibi, Hreinsdóttir [12] and Ohtani [21], we have the next result.

Lemma 3.1.3. *[19, Lemma 3.2]* $\mathrm{in}_<(J_G) = (m_\pi \mid \pi \in \mathcal{P}(G))$.

One should be careful about the definition of the admissible paths. In the case that we consider the w-admissible paths, the generators of $\mathrm{in}_<(J_G)$ in Lemma 3.1.3 need not to be minimal. The next properties play an important role to reach the main statement mentioned in Theorem 3.1.1.

Lemma 3.1.4. *[19, Lemma 3.3] Let $\pi : s = p_1,\ldots,p_l = f$ be a w-admissible path, and $1 \le k \le l-1$. Then*

(i) *If $p_k < s$, then there exists an $n > k$ with $\pi' : p_k, p_{k+1},\ldots,p_n$ is a w-admissible path of G, and $m_{\pi'} \mid x_{p_k} m_\pi$.*

(ii) *If $p_k > f$, then there exists an $n < k$ with $\pi' : p_n, p_{n+1},\ldots,p_k$ is a w-admissible path of G, and $m_{\pi'} \mid y_{p_k} m_\pi$.*

Proof. (i) Let $n > k$ be the smallest integer providing that $p_k < p_n \le f$. Then the path $\pi' : p_k,\ldots,p_n$ implies that $m_{\pi'} = y_{p_k}\ldots y_{p_n} x_s y_f$. Since m_π contains $m_{\pi'}$ as a multiple, the desired conclusion is obtained.

(ii) Now, we take the biggest integer providing $p_k > p_n \geq s$. Similarly, the path π' gives the desired conclusion.

A path satisfying the conditions (i),(ii) in the Lemma 3.1.4, is said to be a **wedge** of π at p_k. Let $r = |\mathcal{P}(G)|$. We order the elements of $\mathcal{P}(G)$ as $\pi_1, \ldots, \pi_r$ such that if $i < j$ then length of π_i is less than the length of π_j. We denote simply the monomials $m_k = m_{\pi_k}$ for $k = 1, \ldots, r$. Then the Lemma 3.1.3 yields that $\text{in}_<(J_G) = (m_1, \ldots, m_r)$.

Lemma 3.1.5. *[19, Lemma 3.4] Fix $1 < j < r$. Let $\pi_j : s = p_0, \ldots, p_l = f \in \mathcal{P}(G)$. Then for any $p_k \in \{p_1, \ldots, p_{l-1}\}$, one has $x_k \in ((m_1, \ldots, m_{j-1}) : m_j)$ whenever $k < s$ and $y_k \in ((m_1, \ldots, m_{j-1}) : m_j)$ whenever $k > t$.*

For an element $m \in \text{Mon}(S)$, we put

$$\text{mult}(m) = \{k \in [n] \mid x_k y_k \text{ divides } m\}.$$

We observe that if m is a squarefree monomial then we have $\deg(m) \leq n + |mult(m)|$. Thus, after the next result we can conclude the Theorem 3.1.1.

Proposition 3.1.6. *[19, Proposition 3.5] For any monomial $m \in S$ and a positive integer z, one has*

$$\beta_{z,m}(S/\text{in}_<(J_G)) = 0 \quad \textit{if} \quad |\text{mult}(m)| \geq z.$$

In particular, $\text{reg}(\text{in}_<(J_G)) \leq n$.

Proof. First, we give some definitions and notations to prove this proposition. We set $M = \{m_1, \ldots, m_r\}$ be a set of monomials. Any subset $F = \{m_{i_1}, \ldots, m_{i_k}\} \subset M$ with $i_1 < \ldots < i_k$ is called a **Lyubeznik subset** of M if for any $j = 1, \ldots, k$ and for any monomial m_l with $l < i_j$ does not divide the $\text{lcm}(m_{i_j}, m_{i_{j+1}} \ldots, m_{i_k})$. In a path $\pi : s = p_0, p_1, \ldots, p_l = f$ of G, s and f are called **ends** of π, and $p_1, \ldots, p_{l-1}$ are called **inner vertices**. We show the statement due to claims in what follows.

Claim 1: Let $F = \{m_{i_1}, \ldots, m_{i_k}\}$ be a Lyubeznik subset of M with $i_1 < \ldots < i_k$. Then

(a) $\text{mult}(\text{lcm}(F))$ contains no inner vertices of P_{i_1}.

(b) if $\text{mult}(\text{lcm}(F))$ contains no inner vertices of P_{i_j} for $j = 2, 3, \ldots, k$ then $|\text{mult}(\text{lcm}(F))| \leq k - 1$.

Claim 2: Let $F = \{m_{i_1}, \ldots, m_{i_k}\}$ be a Lyubeznik subset of M with $i_1 < \ldots < i_k$, and $m \in \text{Mon}(S)$. Let z be a positive integer. Assume

(a) $\beta_{z,m}(S/\big((m_1 \ldots, m_{i_1-1}) : m_{i_1}, \ldots m_{i_k}\big) \neq 0$, (*b*) $\operatorname{mult}(m.\operatorname{lcm}(F))$ contains no inner vertices of P_{i_δ} for $\delta = 2, 3, \dots, k$.

Then there exists a Lyubeznik subset $\tilde{F} = \{m_{j_1}, \dots, m_{j_l}\}$ of M with $j_1 < \dots < j_l$, and a monomial $\tilde{m}$ such that

(a') $\beta_{z-1,m}(S/\big((m_1 \dots, m_{j_1-1}) : m_{j_1}, \dots m_{j_l}\big) \neq 0$,

(b') $\operatorname{mult}(\tilde{m}.\operatorname{lcm}(\tilde{F}))$ contains no inner vertices of P_{j_δ} for $\delta = 2, 3, \dots, l$,

(c') $|\operatorname{mult}(\tilde{m}.\operatorname{lcm}(\tilde{F})) - |\tilde{F}|)| = |\operatorname{mult}(m.\operatorname{lcm}(F))| - |F| - 1$.

To begin, we demonstrate that these claims entail the desired conclusion. Let $m \in \operatorname{Mon}(S)$ such that $\beta_{z,m}(S/\operatorname{in}_<(J_G)) \neq 0$ with a positive integer z. We find a Lyubeznik subset F such that

$$|\operatorname{mult}(m)| = |\operatorname{mult}(\operatorname{lcm}(F))| - |F| + z \tag{3.1.1}$$

and, hereby F satisfies the assumption of Claim 1(b).

We recall that $\operatorname{in}_<(J_G) = (m_1, \dots, m_r)$ is known. Using the Lemma 3.1.2, one can find a Lyubeznik 1-element subset $\{m_j\}$ such that $\beta_{z-1,u/m_j}(S/((m_1, \dots, m_{j-1}) : m_j)) \neq 0$. We can see that $F = \{m_j\}$ and $u = m_j$ when $z = 1$ satisfies the expression 3.1.1. Moreover, the assumptions of Claim 2(a) and Claim 2(b) are satisfied for $F = \{m_j\}$ and $u = u/m_j$ when $z > 1$. After applying Claim 2 repeatedly, we can find a Lyubeznik subset $F = \{m_{i_1}, \dots, m_{i_k}\}$ and a monomial m with

- $\beta_{0,m}(S/\big((m_1 \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k}\big) \neq 0$,
- $\operatorname{mult}(m.\operatorname{lcm}(F)) - |F| = |\operatorname{mult}(u)| - z$.

The former implies that $m = x^0 y^0$ where $0 = (0, \dots, 0)$ and the latter one shows that F satisfies the expression 3.1.1 Next, we are going to prove Claims 1,2.

Proof of Claim 1. (a) On the contrary, assume that there is an inner vertex of P_{i_1} in $\operatorname{mult}(\operatorname{lcm}(F))$. Then we can find an inner vertex v. Let P_j be a wedge of P_{i_1} at v. By the Lemma 3.1.4 we obtain that $j < i_1$ and $m_j \mid \operatorname{lcm}(m_{i_1}, \dots, m_{i_k})$. This gives a contradiction with the definition of Lyubeznik subset.
(b) Let $s_1, t_1, s_2, t_2, \dots, s_k, t_k$ be end points of $P_{i_1}, \dots, P_{i_k}$ with $s_j < t_j$ for every j. By the part (a), there is no inner vertices of P_{i_j} for every j. Therefore,

$$|\operatorname{mult}(\operatorname{lcm}(F))| \leq |\operatorname{mult}(x_{s_1} y_{t_1} x_{s_2} y_{t_2} \dots x_{s_k} y_{t_k})|,$$

and since $s_j < t_j$ for every j, we get that
$|\operatorname{mult}(\operatorname{lcm}(F))| \leq |\operatorname{mult}(x_{s_1} y_{t_1} x_{s_2} y_{t_2} \dots x_{s_k} y_{t_k})| \leq k - 1$.

Proof of Claim 2. *Case* 1. If there exists an inner vertex $v \in \operatorname{mult}(m.\operatorname{lcm}(F))$ of P_{i_1}, then either $x_v \mid m_{i_1}$ or $y_v \mid m_{i_1}$. Suppose now that $x_v \mid m_{i_1}$, then y_v does not divide $\operatorname{lcm}(F)$ by the Claim 1(i). Therefore, $y_v \mid m$. We have that $y_v \in (m_1, \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k}$ by the Lemma 3.1.5 and hence $\beta_{z,m}(S/\big((m_1 \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k}\big) \neq 0$ if and only if $\beta_{z-1,m}(S/\big((m_1 \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k}\big) \neq 0$. Therefore, if we take $\tilde{F} = F$ and $\tilde{m} = m/y_v$ yields the desired conclusions as in $(a'), (b')$ and (c').

Case 2. Assume that $\operatorname{mult}(m.\operatorname{lcm}(F))$ have no inner vertex v of P_{i_1}, and

$$\bar{m}_j = \frac{m_j}{\gcd(m_j, m_{i_1} \dots m_{i_k})}$$

for every $j = 1, 2, \dots, i_1 - 1$. Then we get that

$$(\bar{m}_1, \dots \bar{m}_{i_1-1}) = (m_1 \dots, m_{i_1-1}) : m_{i_1} \dots m_{i_k}.$$

Thus, there is an $1 \leq i_0 < i_1$ such that $\bar{m}_{i_0} \notin (\bar{m}_1, \dots, \bar{m}_{i_0-1})$, and

$$\beta_{z-1,m/\bar{m}_{i_0}}\big(S/(\bar{m}_1, \dots, \bar{m}_{i_0-1}) : \bar{m}_{i_0}\big) \neq 0 \tag{3.1.2}$$

by the Lemma 3.1.2 and the assumption (a) in Claim 2. Let $\tilde{m} = m/\bar{m}_{i_0}$ and $\tilde{F} = \{m_{i_0}, \dots, m_{i_k}\}$. Since we have

$$(\bar{m}_1, \dots \bar{m}_{i_0-1}) : \bar{m}_{i_0} = (m_1 \dots, m_{i_0-1}) : m_{i_0} \dots m_{i_k},$$

and we have the expression 3.1.2, and that $m.\operatorname{lcm}(F) = \tilde{m}\operatorname{lcm}(\tilde{F})$, therefore, Lyubeznik subset $\tilde{F}$ and the monomial $\tilde{m}$ satisfy $(a'), (b')$ and (c').

□

Now, we prove the Theorem 3.1.1.

Proof of the Theorem 3.1.1. Let P be a longest induced path of G with length l. Note that P admits $l+1$ vertices. Then J_G consists of l generators of degree 2. The Lemma 2.3.1 gives the result that $\beta_{i,j}(J_G) \geq l+1$ because $\beta_{i,j}(J_G) \geq \beta_{i,j}(J_{G_P})$. Moreover, we proved that the regularity has an upper bound n, see Proposition 3.1.6.

□

An upper bound for the regularity of the binomial edge ideals of the closed graphs is given in [26] which is sharper than the one given in Theorem 3.1.1. We denote by $c(G)$ the number of maximal cliques of G.

Theorem 3.1.7. *[26, Theorem 3.2] Let G be a closed graph. Then* $\operatorname{reg}(J_G) \leq c(G) + 1$.

Proof. First, we notice that $\text{in}_<(J_G) = (x_i y_j \mid i < j,\ \{i,j\} \in E(G))$ can be seen as the edge ideal of a bipartite graph. We denote this bipartite graph by $\text{in}_<(G)$. Then we obtain $\text{in}_<(J_G) = I(\text{in}_<(G))$. Next, the [11, Theorem 3.3.4] implies that $\text{reg}(J_G) \leq \text{reg}(\text{in}_<(J_G)) = \text{reg}(I(\text{in}_<(G)))$. Then we need to demonstrate that $\text{reg}(I(\text{in}_<(G))) \leq c(G) + 1$. By Theorem 1.2.16, we get that $\text{reg}(I(\text{in}_<(G))) \leq \text{cochord}(\text{in}_<(G)) + 1$. Hence, it is needed to show that $\text{cochord}(\text{in}_<(G)) \leq c(G)$.
Let C be a maximal clique of G. Then $\text{in}_<(C)$ is a complete induced subgraph of $\text{in}_<(G)$. Then from Theorem 3.2.1, we know that $\text{in}_<(C)$ has a linear resolution. It follows from the Theorem 1.2.15 that the complementary of $\text{in}_<(C)$ is a chordal graph, that is, $\text{in}_<(C)$ is co-chordal. On the other hand, union of all maximal cliques of G, denoted by $\bigcup_{i=1}^{c(G)} C_i$, covers all edges of G. Thus, $\bigcup_{i=1}^{c(G)} \text{in}_<(C_i)$ covers all edges of $\text{in}_<(G)$. Then we conclude that $\text{cochord}(\text{in}_<(G)) \leq c(G)$. □

The proof of Theorem 3.1.7 is based on the known results about the regularity of the edge ideals. The upper bound given in Theorem 3.1.7 was later improved in [14] and an exact value of the regularity for the binomial edge ideal of closed graph was obtained.

Theorem 3.1.8. *[8, Theorem 2.2] Let G be a closed graph on $[n]$ and $G_1, \ldots, G_r$ be the connected components of G. Then*

$$\text{reg}(J_G) = l_1 + \ldots + l_r - r$$

where l_i is the length of the longest induced path of G_i for all $1 \leq i \leq r$.

Proof. Since S_i is a polynomial ring with variables coming from the vertex sets of G_i, then $S/J_G \cong \otimes_{i=1}^{r}(S_i/J_{G_i})$. Therefore, we obtain $\text{reg}(S/J_G) = \sum_{i=1}^{r}(S/J_{G_i})$. It follows that $\text{reg}(J_G) = \sum_{i=1}^{r} \text{reg}(J_{G_i}) - r$. Due to this equality, it is enough to show that every connected closed graph G satisfies the statement. Take a connected closed graph G. Let l be the length of the longest induced path of G. By Theorem 3.1.1, we get that $\text{reg}(J_G) \geq l + 1$. By [11, Theorem 3.3.4], we additionally have that $\text{reg}(J_G) \leq \text{reg}(\text{in}_<(J_G)) = \text{indmatch}(\text{in}_<(G))$. It follows from [8, Proposition 2.5] that $\text{indmatch}(\text{in}_<(G)) = l$, then we obtain the desired equality. □

In [8], the upper bound for the regularity of binomial edge ideal is examined for some special classes of chordal graphs.

Theorem 3.1.9. *[8, Theorem 2.9] Let G be a chordal graph on $[n]$ with the property that any two distinct maximal cliques intersect in at most one vertex. Then* $\text{reg}(J_G) \leq c(G) + 1$.

Proof. We may assume that G is connected similar to Theorem 3.1.8. We denote $c := c(G)$. If $c = 1$, then G is a complete graph. If $c > 1$, then we can consider the clique complex $\Delta(G)$ of G. Let $F_1,\ldots,F_c$ be a leaf order in $\Delta(G)$. As F_c is a leaf, there exists a unique vertex $i \in F_c$ such that $F_r \cap F_j = \{i\}$ for some j. We consider the facets $F_{k_1},\ldots,F_{k_s}$ that intersect F_r at i. We write a decomposition $J_G = P_1 \cap P_2$ such that $P_1 = \bigcap_{S\in\mathcal{P}([n]),\, i\notin S} P_S(G)$ and $P_2 = \bigcap_{S\in\mathcal{P}([n]),\, i\in S} P_S(G)$ where $\mathcal{P}([n])$ is the power set of $[n]$. This primary decomposition yields the sequence

$$0 \to S/J_G \to S/P_1 \oplus S/P_2 \to S/(P_1+P_2) \to 0 \tag{3.1.3}$$

Let G' be the graph by replacing the facets $F_{k_1},\ldots,F_{k_s},F_c$ by the clique on the vertex set $\left(\bigcup_{j=1}^{s} F_{k_j}\right) \cup F_r$. Thus, we notice that $P_1 = J(G')$ and G' is a connected chordal graph in which any two distinct cliques intersect in at most one vertex. On the other hand, the number of cliques of G' is smaller than that of G. Therefore, our inductive hypothesis implies that $\operatorname{reg}(S/J_{G'}) < c$.
Let G'' be the restriction of G on the vertex set $[n]\setminus\{i\}$ such that $P_S(G) = (x_i,y_i) + P_{S\setminus\{i\}}$ for all $S \in \mathcal{P}(G)$ and $i \in S$. Then we have that $P_2 = (x_i,y_i) + J_{G''}$. Therefore, we get that $S/P_2 \cong S_i/J_{G''}$ where $S_i = S/(x_i,y_i)$. Thus, G'' is the graph on $n-1$ vertices and $s+1$ components. Here, $s+1 > 2$. Similarly, there are at most c cliques by our assumption. Then $\operatorname{reg}(S/P_2) \leq c$. As a consequence, $\operatorname{reg}(S/P_1 \oplus S/P_2) \leq c$. We note that $P_1+P_2 = J'_G + (x_i,y_i) + J_{G''} = (x_i,y_i) + J_{G'\setminus i}$, therefore, $S/(P_1+P_2) \cong S_i/J_H$ with the graph H on $[n]\setminus\{i\}$ such that the number of cliques in H is less than r. As a consequence, we have that $\operatorname{reg}(S/(P_1+P_2)) \leq c$. The [22, Corollary 18.7] implies that $\operatorname{reg}(S/J_G) \leq \max\{\operatorname{reg}(S/P_1 \oplus S/P_2), \operatorname{reg}(S/P_1 \oplus S/P_2)+1\}$ which yields $\operatorname{reg} S/J_G \leq c$. □

Madani and Kiani gave a similar upper bound for the regularity of the generalized block graphs in [15] which is similar to the one given in Theorem 3.1.9.

Theorem 3.1.10. *[15] Let G be a generalized block graph on $[n]$. Then* $\operatorname{reg}(J_G) \leq c(G)+1$.

The authors proved in which case the regularity of a binomial edge ideal is exactly less than n in [16]. In the case of path graphs of length n we get $\operatorname{reg}(J_G) = n$ as mentioned in [13]. To begin, we state a useful result for the rest.

Lemma 3.1.11. *[16, Lemma 3.1] Let G be a graph on $[n]$ and v be a simplicial vertex of G such that* $\deg(v) \geq 2$, *and e be an edge incident with v. Then*

$$\operatorname{reg}(J_{G\setminus e} : f_e) \leq n-2.$$

Proof. Let $d \geq 2$ be the degree of v and $v_1,\ldots,v_d$ be the neighbours of v. We denote the corresponding edges by $e_1,\ldots,e_d$. Assume without loss of generality that $e := e_d$. Since v is a simplicial complex, there is always an edge between all pairs of $v,v_1,\ldots,v_d$. Therefore, one can walk the path v,v_i,v_d for all $i = 1,\ldots,d-1$.

We define the ideal $I_{G,e}$ where $e = \{i, j\} \in E(G)$ in the Theorem 1.2.21 due to the paths from i to j. Now, in our case, one can see that $I_{G,e}$ is minimally generated by $(x_i, y_i \mid i = 1, \ldots, d-1)$. The paths from v to v_d in the graph $G \setminus e$ must pass at v_i for some $i = 1, \ldots, d-1$. Hence each monomial coming from these paths must be divided by either x_i or y_i for some $i = 1, \ldots, d-1$. Hence, we get

$$J_{G\setminus e} : f_e = J_{(G\setminus e)_e} + I_{G,e},$$

by Theorem 1.2.21. Moreover, one can realize that the generators of the binomial ideal $J_{(G\setminus e)_e}$ are contained by $I_{G,e}$. Let $H := (G \setminus e)_e$. Then we observe that v is an isolated vertex in $H_{[n]\setminus\{v_1,\ldots,v_{d-1}\}}$ which implies that

$$J_{G\setminus e} : f_e = J_{H_{[n]\setminus\{v_1,\ldots,v_{d-1}\}}} + (x_i, y_i \mid i = 1, \ldots, d-1).$$

Thus, $\operatorname{reg}(J_{G\setminus e} : f_e) = \operatorname{reg}(J_{H_{[n]\setminus\{v_1,\ldots,v_{d-1}\}}})$. The Theorem 3.1.1 follows that $\operatorname{reg}(J_{H_{[n]\setminus\{v_1,\ldots,v_{d-1}\}}}) \leq n-2$ and this yields the desired conclusion. □

Theorem 3.1.12. *[16, Theorem 3.2] For any graph $G \neq P_n$ with n vertices, we have* $\operatorname{reg}(J_G) \leq n-1$.

First, we define $\alpha_G(v) = \binom{\deg(v)}{2} - |E(G_{N(v)})|$ where $v \in V(G)$ and $\alpha(G) = \min\{\alpha_G(v) \mid v \in V(G)\}$. For instance, we get the values $\alpha_G(1) = \alpha_G(5) = 0$, $\alpha_G(2) = \alpha_G(3) = 1$, and $\alpha_G(4) = 2$; and hence $\alpha(G) = 0$ for the graph displayed in the Figure 3.1.1. Note that if there exists any simplicial vertex in a graph G then $\alpha(G) = 0$.

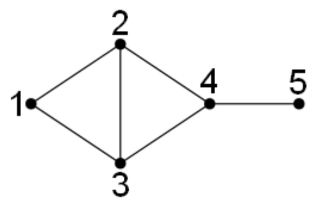


Figure 3.1.1

Proof. We show the statement in two cases depending on whether the graph G contains a simplicial vertex.

Case 1. Assume that G contains a simplicial vertex, that is, $\alpha(G) = 0$. Now, we want to prove $\operatorname{reg}(J_G) \leq n-1$ by induction on n. In the case that $n = 2$, G is the graph of two isolated vertices which implies that $J_G = 0$.

Next, let $n > 2$ and G be a graph on $[n]$ such that G has a simplicial vertex and $G \neq P_n$.

(i) Assume that there is a simplicial vertex v which is pendant in G, that is $\deg(v) = 1$. Let w be the neighbour of v such that $e = \{v, w\} \in E(G)$. Since v is pendant, it must be an isolated vertex in $G \setminus e$. It follows that $\mathrm{reg}(J_{G\setminus e}) = \mathrm{reg}(J_{(G\setminus e)_{[n]\setminus v}})$. By Theorem 3.1.1, we have $\mathrm{reg}(J_{G\setminus e}) \leq n - 1$. Also, by Theorem 1.2.21, we have $\mathrm{reg}(J_{G\setminus e} : f_e) = \mathrm{reg}(J_{(G\setminus e)_e})$. Since $(G \setminus e)_e$ does not contain any edge passing at v then $\mathrm{reg}((G \setminus e)_e)$ does not depend on v. Hence our inductive hypothesis implies that $\mathrm{reg}(J_{(G\setminus e)_e} \leq n - 2$ with the simplicial vertex w in $(G \setminus e)_e$. So, $\mathrm{reg}(J_{G\setminus e} : f_e) + 1 \leq n - 1$. Due to the Proposition 1.2.22 we have $\mathrm{reg}(J_G) \leq n - 1$.

(ii) Assume that the degrees of all simplicial vertices are greater than one. Let v be a simplicial vertex of G such that $N_G(v) = \{v_1, \ldots, v_t\}$ and $e_1 = \{v, v_1\}, \ldots, e_t = \{v, v_t\}$ for every $t \geq 2$. By the Proposition 1.2.22(i) and the Lemma 3.1.11, we get that $\mathrm{reg}(J_G) \leq \max\{\mathrm{reg}(J_{G\setminus e_1}), n - 1\}$. Likewise, after applying once again the Proposition 1.2.22(i) and the Lemma 3.1.11 on $G \setminus e_1$ one can see that $\mathrm{reg}(J_G) \leq \max\{\mathrm{reg}(J_{G\setminus\{e_1,e_2\}}), n - 1\}$. We proceed this, and at the end, we get $\mathrm{reg}(J_G) \leq \max\{\mathrm{reg}(J_{G\setminus\{e_1,\ldots,e_{t-1}\}})$. We consider that $G \setminus \{e_1, \ldots, e_{t-1}\}$ is a graph on $[n]$, where v is a pendant vertex. Thus, the first case implies that $\mathrm{reg}(J_G) \leq \mathrm{reg}(J_{G\setminus\{e_1,\ldots,e_{t-1}\}} \leq n - 1)$ as desired.

Case 2. Now, we look at the case that G has no simplicial vertex. Assume that G is not a path and $\mathrm{reg}(J_G) \geq n$. We could suppose that G is the graph satisfying these assumptions on the least number of vertices, say n. Since G has no simplicial vertex, then $\alpha(G) \geq 1$. Hence one can find a vertex $v \in V(G)$ such that $\alpha(G) = \alpha_G(v)$. Then there exist two vertices v_1, v_2 of G with $\{v, v_1\}$ and $\{v, v_2\}$ are edges in G, but $\{v_1, v_2\} \notin E(G)$. If $\{v_1, v_2\} \notin E(G)$, then $\alpha(G) = \alpha_G(v)$ would be zero which is a contradiction. Let us denote $e = \{v_1, v_2\}$. Then by Proposition 1.2.22(ii), we have

$$\mathrm{reg}(J_G) \leq \max\{\mathrm{reg}(J_{G\cup e}), \mathrm{reg}(J_G : f_e) + 2\}. \tag{3.1.4}$$

Note that $\alpha_{G\cup e}(v) = \alpha_G(v) - 1$ and it follows that $\alpha(G \cup e) \leq \alpha(G - 1)$. We also observe that $G\cup e$ is a graph which has a simplicial vertex v on n vertices. Therefore, from the Case 1, we get that $\mathrm{reg}(J_{G\cup e}) \leq n - 1$. As G has the vertices v, v_1, v_2 that form a cycle in G, G cannot be a path graph. We obtain $\mathrm{reg}(J_G) \leq \mathrm{reg}(J_{G\cup e}) \leq n-1$ if we have $\max\{\mathrm{reg}(J_{G\cup e}), \mathrm{reg}(J_G : f_e) + 2\} = \mathrm{reg}(J_{G\cup e})$. Then it follows from the Theorem 1.2.21 that $J_G : f_e = J_{G_e} + I_{G\cup e}$. Then $I_{G\cup e} = (x_v, y_v) + I_{(G\setminus v)\cup e}$, and as a consequence $J_G : f_e = J_{G_e} + (x_v, y_v) + I_{(G\setminus v)\cup e}$. □

In [25], an upper bound for the regularity of the binomial edge ideals of chordal graphs is given which is an improvement of results given in Theorem 3.1.9.

Theorem 3.1.13. *[25, Theorem 3.5] Let G be a chordal graph. Then* $\mathrm{reg}(J_G) \leq c(G) + 1$.

Very recently in the paper [24], the authors proved the conjecture that $\operatorname{reg}(J_G) \leq c(G)+1$ for every graph G. They introduced an invariant which is called compatible map from the set of all graphs to the set of all integers satisfying certain properties. For the next definition, we denote $\widehat{G} = G \setminus Is(G)$ where $Is(G)$ is the set of isolated vertices of G.

Definition 3.1.14. *[24, Definition 2.1]*Let $\mathcal{G}$ be the set of all graphs. We call a map $\varphi : \mathcal{G} \to \mathbb{N}$, compatible, if it satisfies the following conditions:

(a) $\varphi(\widehat{G}) \leq \varphi(G)$, for every $G \in \mathcal{G}$;

(b) if $G = \dot{\cup}_{i=1}^{t} K_{n_i}$, where $n_i \geq 2$ for every $1 \leq i \leq t$, then $\varphi(G) \geq t$;

(c) if $G \neq \dot{\cup}_{i=1}^{t} K_{n_i}$, then there exists $v \in V(G)$ such that

 (1) $\varphi(G - v) \leq \varphi(G)$, and

 (2) $\varphi(G_v) < \varphi(G)$

Theorem 3.1.15. *[24, Theorem 2.3] Let G be a graph on $[n]$ and φ be a compatible map. Then*

$$\operatorname{reg}(S/J_G) \leq \varphi(G).$$

To demonstrate this theorem, we provide the next result.

Lemma 3.1.16. *[17, Lemma 3.4] Let G be a graph and v be a internal vertex of G. Then*

$$\max\{iv(G_v), iv(G \setminus v), iv(G_v \setminus v)\} < iv(G).$$

Proof. We have $iv(G) \geq 1$ because v is an internal vertex of G. If $iv(G_v) = 0$, we get $iv(G) > iv(G_v)$ and $iv(G_v) = iv(G_v \setminus v)$. Now, we assume that $iv(G_v) = k > 0$. As v is an internal vertex of G, then $iv(G) \geq k+1 > iv(G_v)$. Since $iv(G_v) = iv(G_v \setminus v)$, we have $iv(G) > iv(G_v \setminus v)$. Let w be a free vertex of G. Therefore, w is either free vertex or isolated vertex in $G \setminus v$. Thus, $iv(G) > iv(G \setminus v)$. □

Now, we go back the Theorem 3.1.15.

Proof of Theorem 3.1.15. We apply an induction on $iv(G)$. If $iv(G) = 0$, then G can be written as a disjoint union of the complete graphs. consider $\widehat{G} = \bigcup_{i=1}^{t} K_{n_i}$ where $n \geq 2$ for every $1 \leq i \leq t$. It is well-known that $\operatorname{reg}(S/J_G) = \operatorname{reg}(\widehat{S}/J_{\widehat{G}})$ with $\widehat{S} = \mathbb{K}[x_i, y_i \mid i \in [n] \setminus \mathcal{I}_S(G)]$. Then it follows from [26, Theorem 2.1] that $\operatorname{reg}(\widehat{S}/J_{\widehat{G}}) = t$. By Definition 3.1.14 we have $t \leq \varphi(\widehat{G}) \leq \varphi(G)$ as desired.
Now, we suppose that $iv(G) > 0$. Let $v \in [n]$ be any vertex satisfying the condition (c) in the Definition 3.1.14. $P_1 = \bigcap_{S \in \mathcal{P}([n]),\, i \notin S} P_S(G)$ and $P_2 = \bigcap_{S \in \mathcal{P}([n]),\, i \in S} P_S(G)$ where $\mathcal{P}([n])$ is the power set of $[n]$. Therefore, $P_1 = J_{G_v}$ and $P_2 = (x_v, y_v) + J_{G-v}$ which

follows $P_1 + P_2 = (x_v, y_v) + J_{G_v - v}$ similarly to the proof of the Theorem 3.1.9. In addition, we have the following short exact sequence

$$0 \to S/J_G \to S/J_{G_v} \oplus S_v/J_{G-v} \to S_v/J_{G_v-v} \to 0$$

which is induced by the short exact sequence 3.1.3 and $S_v = K[x_i, y_i \mid i \in [n] \setminus v]$. Next, by the [22, Corollary 18.7], we have that

$$\operatorname{reg}(S/J_G) \leq \max\{\operatorname{reg}(S/J_{G_v} \oplus S_v/J_{G-v}), \operatorname{reg}(S_v/J_{G_v-v}) + 1\}. \tag{3.1.5}$$

Owing to the short exact sequence $0 \to S/J_{G_v} \to S/J_{G_v} \oplus S_v/J_{G-v} \to S_v/J_{G_v-v} \to 0$ and by the [22, Corollary 18.7], we have another inequality

$$\operatorname{reg}(S/J_{G_v} \oplus S_v/J_{G-v}) \leq \max\{\operatorname{reg}(S/J_{G_v}), \operatorname{reg}(S_v/J_{G-v})\}$$

Hence the inequality 3.1.5 becomes

$$\operatorname{reg}(S/J_G) \leq \max\{\operatorname{reg}(S/J_{G_v}), \operatorname{reg}(S_v/J_{G-v}), \operatorname{reg}(S_v/J_{G_v-v}) + 1\}. \tag{3.1.6}$$

Now, the Lemma 3.1.16 and our inductive hypothesis yield

$$\operatorname{reg}(S/J_{G_v}) \leq \varphi(G_v) \leq \varphi(G), \tag{3.1.7}$$

and

$$\operatorname{reg}(S_v/J_{G-v}) \leq \varphi(G - v) \leq \varphi(G). \tag{3.1.8}$$

As $G_v - v$ is induced by the graph G_v, then by [27, Proposition 8], we get

$$\operatorname{reg}(S_v/J_{G_v-v}) \leq \operatorname{reg}(S/J_{G_v}).$$

Therefore the inequality 3.1.7 implies that $\operatorname{reg}(S_v/J_{G_v-v}) \leq \varphi(G)$. Thus, by 3.1.6 we get $\operatorname{reg}(S/J_G) \leq \varphi(G)$ as desired. □

The authors introduced another terminology as follows.

Definition 3.1.17. *[24, Definition 2.4]*Let G be a graph and $\mathcal{H} \subseteq E(G)$ with the property that no two elements of $\mathcal{H}$ belong to the same clique of G. Then, we call the set $\mathcal{H}$, a **clique disjoint edge set** in G. Moreover, we put

$$\eta(G) := \max\{|\mathcal{H}| : \mathcal{H} \text{ is a clique disjoint edge set in } G\}.$$

Theorem 3.1.18. *[24, Theorem 2.5] The map $\eta : \mathcal{G} \to \mathbb{N}$ is compatible.*

Proof. To prove this theorem we show that η satisfies the conditions in the Definition 3.1.14. Since a clique must contain an edge, we have $\eta(\widehat{G}) = \eta(G)$. Secondly, we assume that $G = \dot{\cup}_{i=1}^{t} K_{n_i}$ with $n_i \geq 2$ for every $1 \leq i \leq t$. Then there are t disjoint cliques and it follows that $\eta(G) = t$.
Now, we assume that $G \neq \dot{\cup}_{i=1}^{t} K_{n_i}$. We recall that $N_G(v)$ is complete graph for any free vertex $v \in V(G)$. Therefore, one can find an internal vertex v of G. By definition, v belongs to at least two facets. This implies that $\eta(G - v) \leq \eta(G)$ because each clique disjoint edge set of $G - v$ is also a clique disjoint edge set in G. Lastly, we need to show that $\eta(G_v) < \eta(G)$.
Let $\mathcal{H} = \{e_1, \ldots, e_{\eta(G_v)}\}$ be a clique disjoint edge set of G_v with the same v such that $\eta(G_v) = |\mathcal{H}|$.
Case 1. Suppose that $v \in \bigcup\limits_{e_i \in \mathcal{H}} e_i$, without loss of generality that $v \in e_1$ as well as $v \notin e_j$ for any $2 \leq j \leq \eta(G_v)$.
On the contrary, assume that $v \in e_j$ for some j, $2 \leq j \leq \eta(G_v)$. Since v is free, then e_1 and e_j are contained by a clique. This contradicts the definition of $\mathcal{H}$.

Besides, we consider that $\mathcal{H} \setminus \{e_1\} \subseteq E(G)$. On the contrary, we suppose that $e_j = \{u_j, w_j\} \notin E(G)$ for some $2 \leq j \leq \eta(G)$. It follows that $\{v, u_j\}, \{v, w_j\} \in E(G)$, and the edges e_1, e_j are in the same clique of G_v. Then this yields a contradiction. Therefore, we can find the vertices $a, b \in V(G)$ such that $\{a, b\}$ is an edge of $N_G(v)$, however, $\{a, b\} \notin E(G)$ as v belongs to at least two maximal cliques. Hence, we obtain another disjoint clique set $\mathcal{H}' = \{e_2, \ldots, e_{\eta(G_v)}, \{v, a\}, \{v, b\}\}$ in G. Indeed, assume that either $\{v, a\}$ and e_j or $\{v, b\}$ and e_k belong to the same clique where $2 \leq j, k \leq \eta(G_v)$. Therefore, either e_1 and e_j or e_1 and e_k belong to the same clique, since $v \in e_1$. This also yields a contradiction. Furthermore, since $v \notin e_j$ for all $2 \leq j \leq \eta(G_v)$, we get $\{\{v, a\}, \{v, b\} \cap \{e_2, \ldots, e_{\eta(G_v)}\}\} = \emptyset$. Thus, $|\mathcal{H}'| = \eta(G_v) + 1 \geq \eta(G)$ as required.

Case 2. Next, we suppose that $v \notin \bigcup\limits_{e_i \in \mathcal{H}} e_i$. If $\mathcal{H} \nsubseteq E(G)$, which means that we find $j = 1, \ldots, \eta(G_v)$ such that $e_j = \{u_j, w_j\}$ is not an edge of G, then we can apply the similar method as given in the first case. One can obtain another disjoint clique set $\mathcal{H}' = (\mathcal{H} \setminus \{e_j\}) \cup \{\{v, u_j\}, \{v, w_j\}\}$ in G. Then $|\mathcal{H}'| = \eta(G_v) + 1 \leq \eta(G)$ as desired.

Accordingly, we may suppose that $\mathcal{H} \subseteq E(G)$. Then there exist $a, b \in N_G(v)$ such that $\{a, b\} \notin E(G)$, since v is not free in G. We note that if e_i and $\{v, a\}$ are not in the same clique for all $1 \leq i \leq \eta(G_v)$, then we set a clique disjoint edge set $\mathcal{H}_a = \mathcal{H} \cup \{\{v, a\}\}$ of G. Likewise, $\mathcal{H}_b = \mathcal{H} \cup \{\{v, b\}\}$ is also a clique disjoint edge set of G. Therefore, $|\mathcal{H}_a| = |\mathcal{H}_b| = \eta(G_v) + 1 \leq \eta(G)$. Otherwise, if e_i and $\{\{v, a\}\}$ are in the same clique as well as e_j and $\{\{v, b\}\}$ are in the same clique for some

$e_i, e_j \in \mathcal{H}$ then $e_i = e_j$. If $e_i \neq e_j$, then e_i and e_j also belong to the same clique of G_v which is a contradiction. Hence, we have that

$$\mathcal{H}'' = (\mathcal{H} \setminus \{e_i\}) \cup \{\{v,a\},\{v,b\}\}$$

is a clique disjoint edge set of G with the cardinality $\eta(G_v)+1$. Thus, we conclude that $\eta(G_v) < \eta(G)$.

The Theorem 3.1.15 and Theorem 3.1.18 enable us to find an upper bound for the regularity given by $\eta(G)$.

Corollary 3.1.19. *[24, Corollary 2.6] Let G be a graph on $[n]$. Then $\operatorname{reg}(S/J_G) \leq \eta(G)$.*

We can create a clique disjoint edge set by taking one edge from every maximal clique. We have clearly that $\eta(G) \leq c(G)$ for every graph G. The main result follows the Corollary 3.1.19.

Corollary 3.1.20. *[24, Corollary 2.7] Let G be a graph on $[n]$. Then*

$$\operatorname{reg}(S/J_G) \leq c(G).$$

The following conjecture is still not answered.

Conjecture 3.1.21. *[7] Let G be a graph on $[n]$. Then we have $\operatorname{reg}(J_G) = \operatorname{reg}(\operatorname{in}_<(J_G))$.*

3.2 Exact values for the regularity

In this section, we will give the exact values for the regularity of different graphs. The following theorem shows that the minimum value of the regularity that is 2 is obtained only in the case of complete graph.

Theorem 3.2.1. *[26, Theorem 2.1] Let G be a graph. Then the following are equivalent:*

(i) J_G has a linear resolution;

(ii) J_G has linear relations;

(iii) $\operatorname{in}_<(J_G)$ is generated in degree 2 and has linear quotients;

(iv) $\operatorname{in}_<(J_G)$ has a linear resolution;

(v) G is a complete graph.

Proof. By Proposition 1.2.11, we have $(iii) \Rightarrow (iv)$ and additionally, by [13, Theorem 2.19] $(iv) \Rightarrow (i)$. As well, we have trivially $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (v)$: Let $F = (e_{ij} \mid \{i,j\} \in E(G))$ be the graded S-module such that the homomorphism

$$\psi : F \longrightarrow J_G$$
$$e_{ij} \mapsto f_{ij}$$

is surjective. We put $\deg(e_{ij}) = \epsilon_i + \epsilon_j$ for each $\{i,j\} \in E(G)$. Therefore, ψ is a $\mathbb{Z}^n$-graded epimorphism, and $K := Ker\psi$ is also a $\mathbb{Z}^n$ graded S-module.
Now, assume on the contrary that G is not a complete graph. Hence, there is a path as an induced subgraph of G with the vertices $\{i,j,k\}$. Let $\{i,j\}$ and $\{j,k\}$ be in $E(G)$. We can suppose that $i < j < k$. We take an element $r = f_{ij}e_{jk} - f_{jk}e_{ij} \in F$ of degree 4, indeed, of multidegree $\epsilon_i + 2\epsilon_j + \epsilon_k$.

- **Claim:** $r = f_{ij}e_{jk} - f_{jk}e_{ij} \in F$ is of degree 4.
  If r can be reduced by elements of degree 3, then there must be another element e_{st} such that $s,t \in \{i,j,k\}$ and $s \neq t$. Since the path consists of two edges $\{i,j\}$ and $\{j,k\}$, then the degree 3 relation is a relation which involves e_{ij} and e_{jk}. However, f_{ij} and f_{jk} form a regular sequence, then there is no such a relation.

Since $\psi(f_{ij}e_{jk} - f_{jk}e_{ij}) = f_{ij}\psi(e_{jk}) - f_{jk}\psi(e_{ij}) = 0$, then $r \in K$. This implies that $\beta_{1,4}(J_G)$ is non-zero, that is, J_G does not have only linear relations.
$(v) \Rightarrow (iii)$: By Theorem 1.1.7, G is a closed graph. Since G is complete graph then $\text{in}_<(J_G) = (x_iy_j \mid \text{ for all } 1 \leq i < j \leq n)$. Due to lexicographic order, we order the monomials of $\text{in}_<(J_G)$:

$$x_1y_2 > x_1y_3 > \ldots > x_1y_n > x_2y_3 > \ldots > x_2y_n > \ldots x_{n-1}y_n.$$

For the simplicity let's denote by $u_1, \ldots, u_{\binom{n}{2}}$ such that $u_1 > \ldots > u_{\binom{n}{2}}$.

- **Claim:** $\text{in}_<(J_G)$ has linear quotients.
  We show that the colon ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by certain indeterminates for each i.
  We observe

  $$(u_1, \ldots, u_{i-1}) : u_i = \{u_j \gcd(u_j, u_i) \mid 1 \leq j \leq i-1\}.$$

  For each $1 \leq l \leq n-2$, the ideal

  $$(x_1y_2, \ldots, x_1y_n, x_2y_3, \ldots, x_2y_n, \ldots, x_ly_{l+1}, \ldots, x_ly_t) : x_{l+1}y_{l+2}$$

is generated by the set of $\{x_1, \ldots, x_l\}$, while for each $1 \leq l \leq n-2$ and $l \leq t \leq n-1$, the ideal

$$(x_1y_2, \ldots, x_1y_n, x_2y_3, \ldots, x_2y_n, \ldots, x_ly_{l+1}, \ldots, x_ly_t) : x_ly_{t+1}$$

is generated by $\{x_1, \ldots x_{l-1}, y_{l+1}, \ldots, y_t\}$ as desired in the claim.

We conclude that $(v) \Rightarrow (iii)$, and this completes the proof. □

Remark 3.2.2. *[26, Remark 3.3]*

(i) We indicate that any complete graph G has a linear resolution by Theorem 3.2.1. Then $\text{reg}(J_G) \leq c(G) + 1 = 2$ whenever G is complete by Corollary 3.1.20. Hence, if G is complete then $\text{reg}(S/J_G) + 1 = \text{reg}(J_G) = 2$.

(ii) Let P_n be a path graph. We recall that P_n is closed with respect to a labelling. And also, J_{P_n} is Cohen-Macaulay and is a complete intersection by [7, Corollary 1.2]. Then $in_<(J_{P_n})$ is the edge ideal of $n-1$ disjoint edges. Then it follows from Proposition 2.4.6 that $\text{reg}(J_{P_n}) = \text{reg}(in_<(J_{P_n})) = c(P_n) + 1 = n$.

(iii) As the graphs with $\text{reg}(J_G) = 2$ are characterized in (i), therefore, any closed graph consisting of exactly two maximal cliques has $\text{reg}(J_G) = 3$ by Corollary 3.1.20.

In [23], Rauf and Rinaldo looked at the binomial edge ideals of graphs, which are created by gluing two graphs together at their free vertices. Later, the extension of their argument is given in [14]. There is another gluing process to compute regularity of a binomial edge ideal considering its components in [14]. Let G be a graph and v be a *cut point* in G. Let $G_1, \ldots, G_k$ be the connected components of $G - \{v\}$. Let G'_i, the subgraph of G induced by $V(G_i) \cup \{v\}$. Then, $G'_1, \ldots, G'_k$ is called the **split** of G at v and we say that G is obtained by gluing $G_1, \ldots, G_k$ at v.

Theorem 3.2.3. *[14, Theorem 3.1] Let G_1 and G_2 be the split of a graph G at v. If v is a free vertex in both G_1 and G_2, then*

$$\text{reg}(S/J_G) = \text{reg}(S/J_{G_1}) + \text{reg}(S/J_{G_2}).$$

Corollary 3.2.4. *[14, Corollary 3.2] Let $G = G_1 \cup \cdots \cup G_k$ such that*

(i) for $i \neq j$, if $G_i \cap G_j \neq \emptyset$, then $G_i \cap G_j = \{v_{ij}\}$, for some vertex v_{ij} which is a free vertex in G_i as well as in G_j;

(ii) for pairwise distinct i, j, k we have $G_i \cap G_j \cap G_k = \emptyset$.

Then $reg(S/J_G) = \sum_{i=1}^{k} reg(S/J_{G_i})$.

In 3.1.10, the upper bound is given for the generalized block graph. We have the equality for block graphs as given in the next result.

Corollary 3.2.5. *[14, Corollary 3.3] Let G be a block graph such that every vertex is contained in at most two maximal cliques, then $reg(J_G) = c(G) + 1$.*

Proof. If $c(G) = 1$, then G is a complete graph and then $reg(J_G) = 2$, see Remark 3.2.2. Assume that $c(G) > 1$. Let v be a cut vertex of G and G_1, G_2 split at $\{v\}$. Therefore, $c(G) = c(G_1) + c(G_2)$. Then our inductive hypothesis yields the desired conclusion by using Corollary 3.2.4. □

We denote $\mathcal{L}(G)$ the sum of the lengths of longest induced paths of connected components of G. We see a classification of all chordal graphs for which $\mathcal{L}(G) = c(G)$ in the same paper. In addition, we call *strongly interval graph* the graphs G for which $\text{reg}(S/J_G) = \mathcal{L}(G) = c(G)$.

Recall that a graph G is said to be an **interval graph** if every vertex $v \in V(G)$ has an associated real closed interval $I_v = [a_v, b_v]$ in such a way that for all adjacent vertices $v, w \in V(G)$ their corresponding intervals have non-empty intersection.

Theorem 3.2.6. *[25, Theorem 4.2] Let G be a chordal graph. Then the following are equivalent:*

(a) $\mathcal{L}(G) = c(G)$.

(b) G is a strongly interval graph.

Corollary 3.2.7. *[25, Corollary 4.3] Let G be a chordal graph. Then the following are equivalent:*

(a) $\text{reg}(S/J_G) = \mathcal{L}(G) = c(G)$.

(b) G is a strongly interval graph.

Bibliography

[1] Arindam Banerjee and Luis Núñez Betancourt. Graph connectivity and binomial edge ideals. *Proc. Amer. Math. Soc.*, 145(2):487–499, 2017.

[2] Winfried Bruns and Aldo Conca. Gröbner bases, initial ideals and initial algebras. *arXiv:0308102 [math.AC]*, 2003.

[3] Winfried Bruns and Udo Vetter. *Determinantal rings*, volume 1327 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[4] Marilena Crupi and Giancarlo Rinaldo. Closed graphs are proper interval graphs. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.*, 22(3):37–44, 2014.

[5] David Eisenbud and Bernd Sturmfels. Binomial ideals. *Duke Math. J.*, 84(1):1–45, 1996.

[6] Viviana Ene and Jürgen Herzog. *Gröbner bases in commutative algebra*, volume 130 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[7] Viviana Ene, Jürgen Herzog, and Takayuki Hibi. Cohen-Macaulay binomial edge ideals. *Nagoya Math. J.*, 204:57–68, 2011.

[8] Viviana Ene and Andrei Zarojanu. On the regularity of binomial edge ideals. *Math. Nachr.*, 288(1):19–24, 2015.

[9] Ralf Fröberg. On Stanley-Reisner rings. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 57–70. PWN, Warsaw, 1990.

[10] Jürgen Herzog and Takayuki Hibi. Distributive lattices, bipartite graphs and Alexander duality. *J. Algebraic Combin.*, 22(3):289–302, 2005.

[11] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[12] Jürgen Herzog, Takayuki Hibi, Freyja Hreinsdóttir, Thomas Kahle, and Johannes Rauh. Binomial edge ideals and conditional independence statements. *Adv. in Appl. Math.*, 45(3):317–333, 2010.

[13] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi. *Binomial ideals*, volume 279 of *Graduate Texts in Mathematics*. Springer, Cham, 2018.

[14] A. V. Jayanthan, N. Narayanan, and B. V. Raghavendra Rao. Regularity of binomial edge ideals of certain block graphs. *Proc. Indian Acad. Sci. Math. Sci.*, 129(3):Paper No. 36, 10, 2019.

[15] Dariush Kiani and Sara Saeedi Madani. The regularity of binomial edge ideals of graphs. *AUT Journal of Mathematics and Computing*, 1:217–221, 2013.

[16] Dariush Kiani and Sara Saeedi Madani. The Castelnuovo-Mumford regularity of binomial edge ideals. *J. Combin. Theory Ser. A*, 139:80–86, 2016.

[17] Arvind Kumar. Regularity bound of generalized binomial edge ideal of graphs. *J. Algebra*, 546:357–369, 2020.

[18] Arvind Kumar and Rajib Sarkar. Depth and extremal Betti number of binomial edge ideals. *Math. Nachr.*, 293(9):1746–1761, 2020.

[19] Kazunori Matsuda and Satoshi Murai. Regularity bounds for binomial edge ideals. *J. Commut. Algebra*, 5(1):141–149, 2013.

[20] Fatemeh Mohammadi and Leila Sharifan. Hilbert function of binomial edge ideals. *Comm. Algebra*, 42(2):688–703, 2014.

[21] Masahiro Ohtani. Graphs and ideals generated by some 2-minors. *Comm. Algebra*, 39(3):905–917, 2011.

[22] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[23] Asia Rauf and Giancarlo Rinaldo. Construction of Cohen-Macaulay binomial edge ideals. *Comm. Algebra*, 42(1):238–252, 2014.

[24] Mohammad Rouzbahani Malayeri, Sara Saeedi Madani, and Dariush Kiani. A proof for a conjecture on the regularity of binomial edge ideals. *J. Combin. Theory Ser. A*, 180:105432, 9, 2021.

[25] Mohammad Rouzbahani Malayeri, Sara Saeedi Madani, and Dariush Kiani. Regularity of binomial edge ideals of chordal graphs. *Collect. Math.*, 72(2):411–422, 2021.

[26] Sara Saeedi Madani and Dariush Kiani. Binomial edge ideals of graphs. *Electron. J. Combin.*, 19(2):Paper 44, 6, 2012.

[27] Sara Saeedi Madani and Dariush Kiani. On the binomial edge ideal of a pair of graphs. *Electron. J. Combin.*, 20(1):Paper 48, 13, 2013.

[28] Russ Woodroofe. Matchings, coverings, and Castelnuovo-Mumford regularity. *J. Commut. Algebra*, 6(2):287–304, 2014.