SIMULTANEOUS RATIONAL PERIODIC POINTS OF DEGREE-2 RATIONAL MAPS

by
BURCU BARSAKÇI

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfilment of
the requirements for the degree of Master of Science

Sabancı University
June 2021

SIMULTANEOUS RATIONAL PERIODIC POINTS OF DEGREE-2 RATIONAL MAPS

Approved by:

Date of Approval: June 29, 2021

ABSTRACT

SIMULTANEOUS RATIONAL PERIODIC POINTS OF DEGREE-2 RATIONAL MAPS

BURCU BARSAKÇI

Mathematics, Master Thesis, JUNE 2021

Thesis Supervisor: Assoc. Prof. Mohammad Sadek

Keywords: dynamical systems, periodic point, degree-2 rational map, quadratic polynomial, orbit

In this thesis, we study the dynamical behaviour of degree two rational maps of the form $kx + b/x$ and quadratic polynomials with rational coefficients. Assuming standard conjectures on the period length of rational periodic points of these maps, we give the possible size of the intersection of their finite orbits. In particular, we give a necessary and sufficient condition for two rational points to be periodic for infinitely many such maps.

ÖZET

İKİNCİ DERECEDEN RASYONEL FONKSİYONLARIN ORTAK RASYONEL PERİYODİK NOKTALARI

BURCU BARSAKÇI

Matematik, Yüksek Lisans Tezi, HAZİRAN 2021

Tez Danışmanı: Assoc. Prof. Mohammad Sadek

Anahtar Kelimeler: dinamik sistemler, periyodik nokta, ikinci dereceden rasyonel fonksiyon, ikinci dereceden polinom fonksiyon, yörünge

Bu yüksek lisans tezinde, rasyonel katsayılı $kx + b/x$ formundaki ikinci dereceden rasyonel fonksiyonların ve rasyonel katsayılı ikinci dereceden polinom fonksiyonların dinamik davranışını inceliyoruz. Bu fonksiyonların rasyonel periyodik noktalarının periyot uzunluğu hakkındaki standart sanıları varsayarak, bu fonksiyonların sonlu yörüngelerinin kesişim kümelerine dair olası eleman sayılarını veriyoruz. Özellikle, iki tane rasyonel noktanın, sonsuz sayıda belirtilen formdaki fonksiyonun periyodik noktası olabilmesi için gerek ve yeter koşulu veriyoruz.

ACKNOWLEDGEMENTS

First of all, I would like to thank Mohammad Sadek. He has always been calm, kind and supportive. He is an exceptional instructor and a great advisor. I have learned a considerable amount from him. 'Thank you, Hocam!'.

In my six-year journey in mathematics, I have met many wonderful people at Bilgi and Sabancı universities. I would like to thank every single instructor for their efforts and all my friends for their support. It has been the most profound experience of my life.

I am grateful to my family for their love and care with a special thank you to my human diary: my sister. 'If you weren't there for me, this math journey wouldn't have even begun. I am so lucky to have you!'.

I was born and lived as a free woman with rights in Turkey. I owe this to Mustafa Kemal Atatürk, who changed the destiny of a nation with his faith. No matter how much I thank him, it won't be enough.

When work is a pleasure, life is a joy. When work is a duty, life is slavery!
Maxim Gorky

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Introduction

A (discrete) dynamical system consists of a set S and a function $\phi : S \to S$ mapping the set S to itself. This map permits iteration

$$\phi^n = \phi \circ \phi \circ \ldots \circ \phi = n^{th} \text{ iterate of } \phi.$$

For a given point $P \in S$, the orbit of P is the set

$$\mathcal{O}_\phi(P) = \mathcal{O}(P) = \{\phi^n(P) : n \geqslant 0\}.$$

One of the main goals of dynamics is to classify the points P in the set S according to their orbits $\mathcal{O}_\phi(P)$ [13]. Depending on the cardinality of their orbit under a map, we classify points in the set S. If the cardinality of $\mathcal{O}_\phi(P)$ is infinite, P is called a wandering point, and if the cardinality of $\mathcal{O}_\phi(P)$ is finite, P is called a preperiodic point. Let $PrePer(\phi, S)$ denote the set of preperiodic points of ϕ in S. Then $PrePer(\phi, S)$ is the set of points in S having finite orbit under the map ϕ. We also have the following equivalent definition:

$$PrePer(\phi, S) = \{\alpha \in S : \phi^{m+n}(\alpha) = \phi^m(\alpha) \text{ for some } n \geqslant 1, m \geqslant 0\}.$$

For a finite set S, we have $PrePer(\phi, S) = S$. If we take S to be an abelian group under multiplication and ϕ to be the map defined by $\phi(\alpha) = \alpha^d$ where d is an integer greater than 1, we see that $PrePer(\phi, S)$ is the torsion subgroup of S [13].

A point $P \in S$ is periodic if there exists an integer $n > 0$ such that $\phi^n(P) = P$. In this case we say that, P has period n. Here, if n is the smallest such integer, we say that P has exact (primitive) period n. Let $Per(\phi, S)$ denote the set of periodic points of ϕ in S. Clearly, $Per(\phi, S)$ is a subset of $PrePer(\phi, S)$.

In our study, we will take the set S to be the rational field $\mathbb{Q}$ and the map ϕ to be a degree-2 rational map of the form

$$\beta(z) = z^2 + c \ , \ \phi_{k,b}(z) = kz + \frac{b}{z},$$

where $k, b, c \in \mathbb{Q}$ with k, b are nonzero. We will focus on the rational periodic points of these maps. By Northcott [11], we know that these maps can have only finitely many rational periodic points.

In terms of dynamical behaviour linear conjugation is very important as linearly conjugate maps have the same dynamical behaviour. The reason why we focus on these degree-2 rational maps is that a quadratic polynomial over $\mathbb{Q}$ is linearly conjugate to $\beta(z) = z^2 + c$ for some $c \in \mathbb{Q}$, and a degree 2 rational map ϕ over $\mathbb{Q}$ has an automorphism group isomorphic to the cyclic group of order 2 if and only if the map ϕ is linearly conjugate over $\mathbb{Q}$ to some map of the form $\phi_{k,b}(z)$ with $k \in \mathbb{Q} \setminus \{0, -1/2\}$ and $b \in \mathbb{Q}^*$ [7].

We have a complete classification of periodic points of $\beta(z)$ with period length 1,2 and 3 [12, 16]. The polynomial $\beta(z)$ cannot have a rational periodic point of period 4 [10] or period 5 [5]. Under the assumption of Birch-Swinnerton- Dyer conjecture, Stoll proved that the polynomial $\beta(z)$ cannot have a rational periodic point of period 6 [15]. For details about the Birch-Swinnerton- Dyer conjecture see [2]. Poonen has conjectured that $\beta(z)$ cannot have a rational periodic point of period $\geqslant 6$ [12]. So in this thesis, we suppose that possible period lengths of rational points of $\beta(z)$ are $1, 2,$ or 3.

Following Manes's study in [7], we know that $\phi_{k,b}(z)$ can have rational points of period 1,2 or 4. It was proved $\phi_{k,b}(z)$ has no rational point of primitive period 3 [7]. She has also conjectured that if $\phi(z) \in \mathbb{Q}(z)$ is a degree-2 rational map with $Aut(\phi) \cong C_2$ then ϕ has no rational point of exact period greater than 4 [7]. Therefore, in this study, we suppose that the period length of a rational periodic point of $\phi_{k,b}$ is either $1, 2,$ or 4.

Chapter zero contains a brief information about elliptic curves and hyperelliptic curves as we use the theory to find rational points on these curves. All the definitions, and important facts related to periodic points of $\beta(z)$ and $\phi_{k,b}(z)$ can be found in the first chapter. To find the periodic points of $\phi_{k,b}(z)$ we reproduce some dynatomic polynomials of the map $\phi_{k,b}(z)$. All the computations related to the proof of the fact that the number of rational periodic points of $\phi_{k,b}$ with period length four is at most 4 are included in the second chapter.

In the third chapter, we parametrize all the maps $\beta(z)$ and $\phi_{k,b}$ such that they share the same periodic point p. Besides this, we find all the triples (k, b, c) satisfying

$$|Orb_{\beta}(p) \cap Orb_{\phi_{k,b}}(p)| \geqslant 2,$$

for some periodic point $p \in \mathbb{Q}$. Moreover, we prove that the cardinality of the intersection of $Orb_{\beta}(p)$ and $Orb_{\phi_{k,b}}(p)$ cannot be more than 2. We also show that we can find infinitely many $\phi_{k,b}$'s sharing same periodic point p, whereas we can find at most three $\beta(z)$ sharing same periodic point p.

Parametrization of all $\phi_{k,b}$'s with respect to their periodic points with period length $1, 2,$ and 4, and the list of all the maps ϕ_{k_1,b_1} and ϕ_{k_2,b_2} having simultaneous periodic points is included in the fourth chapter. For the maps ϕ_{k_1,b_1} and ϕ_{k_2,b_2} we find all possible four tuples (k_1, b_1, k_2, b_2) such that

$$|Orb_{\phi_{k_1,b_1}}(p) \cap Orb_{\phi_{k_2,b_2}}(p)| \geqslant 2,$$

for some periodic point $p \in \mathbb{Q}$. In particular, we show that the cardinality of the intersection is four implies that $\phi_{k_1,b_1} = \pm\phi_{k_2,b_2}$.

In the last chapter, we share the following analogue of Baker and De Marco's result in [1] which states that for fixed $c_1, c_2 \in \mathbb{C}$, the set of $t \in \mathbb{C}$ such that both c_1 and c_2 are preperiodic for $z^d + t$ is infinite if and only if $c_1^d = c_2^d$ where d is greater than one. Let

$$\phi_{t_1,t_2}(z) = t_1 \cdot z + \frac{t_2}{z}.$$

We prove that $a, b \in \mathbb{Q}$ are periodic points of $\phi_{t_1,t_2}(z)$ for infinitely many $(t_1, t_2) \in \mathbb{Q} \times \mathbb{Q}$ with $t_1 \cdot t_2 \neq 0$ if and only if $a^2 = b^2$.

In chapters three, four and five, we produce new results.

0. Elliptic Curves

Let K be a perfect field and $\overline{K}$ be the algebraic closure of K. An elliptic curve has an equation of the form

$$E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where $a_1, a_2, \ldots, a_6 \in \overline{K}$. The point $O = [0,1,0]$ on E is called the point at infinity. We usually write this equation in non-homogeneous coordinates $x = X/Z$ and $y = Y/Z$, as follows

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

keeping in mind that E is described by the latter equation together with the point at infinity. Let us define

$$b_2 := a_1^2 + 4a_2,$$

$$b_4 := a_1a_3 + 2a_4,$$

$$b_6 := a_3^2 + 4a_6,$$

$$b_8 := a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2,$$

$$\Delta := -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

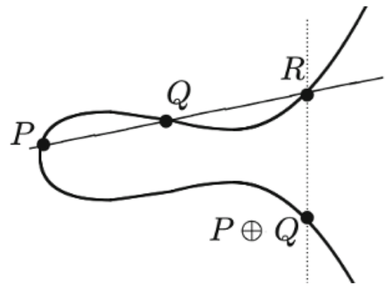
The quantity Δ is called the discriminant of the Weierstrass equation for E. Note that E defines an elliptic curve if and only if the discriminant is nonzero.

We define a composition law $\oplus$ on E, so that this composition law turns E into a group where the identity is the point at infinity. The rule for the composition law is the following:

Let P, $Q \in E$ be distinct points. We join P and Q using a line L. There will be a third intersection point R of the line L with E due to Bézout Theorem. The reflection of R about the x-axis is $P \oplus Q$, see figure 1.

Note that for $P = Q$, we take L to be the tangent of E at P.

Figure 1 Composition law on E



Let K be a number field. Then $E(K)$ denotes the set of the points $(x,y) \in K^2$ on E including the point at infinity. Note that, $E(K)$ is a subgroup of E. The following theorem gives the group structure of $E(K)$:

Theorem 0.0.1 (Mordell-Weil)**.** *([14], Chapter* 8*, §*6*, Theorem* 6.7*) The group $E(K)$ is finitely generated.*

Hence, in particular, we have

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \times \mathbb{Z}^r,$$

where $E(\mathbb{Q})_{tors}$ is the torsion subgroup which is finite and the rank r of $E(\mathbb{Q})$ is a nonnegative integer. By Mazur, we have the following theorem which describes all possibilities for the torsion subgroup of $E(\mathbb{Q})$:

Theorem 0.0.2 (Mazur, [8], [9])**.** *Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E(\mathbb{Q})_{tors}$ is isomorphic to one of the following groups:*

$$\mathbb{Z}/N\mathbb{Z} \text{ with } 1 \leqslant N \leqslant 10 \text{ or } N = 12,$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} \text{ with } 1 \leqslant N \leqslant 4.$$

Let K be a field. Now, we consider a curve C defined by an equation of the form

$$v^2 = au^4 + bu^3 + cu^2 + du + e,$$

where $a,b,c,d,e \in K$ with a is nonzero. Suppose we have a point $P \in C(K)$. This curve is an elliptic curve as long as $au^4 + bu^3 + cu^2 + du + e$ doesn't have a repeated factor. Then the following theorem shows that this curve can be transformed to a Weierstrass equation.

Theorem 0.0.3. *([17], Chapter 2, §5.3, Theorem 2.17) Let K be a field which is not of characteristic 2. Consider the equation*

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2$$

where $a,b,c,d,q \in K$. Let

$$x = \frac{2q(v+q)+du}{u^2}, \ y = \frac{4q^2(v+q)+2q(du+cu^2)-(d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q, \ ,a_2 = c-(d^2/4q^2), \ a_3 = 2qb, \ a_4 = -4q^2a, \ a_6 = a_2a_4 \ .$$

Then

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

For the inverse transformation we take

$$u = \frac{2q(x+c)-(d^2/2q)}{y}, \ v = -q + \frac{u(ux-d)}{2q}.$$

There is an implementation for Theorem 0.0.3 in the software Magma [3] that we are going to use in this thesis.

Let K be an algebraically closed field. Let g be a positive integer. Let $h(x), f(x) \in K[x]$ such that $\deg f = 2g+1$ and $\deg h \leqslant g$. Suppose that f is monic. The curve C given by the equation

$$C : y^2 + h(x)y = f(x)$$

is called a hyperelliptic curve of genus g if it is nonsingular for all $x, y \in K$ [17]. When $g = 1$, we obtain an elliptic curve in generalized Weierstrass form. For a curve of genus greater than one, we have the following theorem.

Theorem 0.0.4 (Faltings, [4])**.** *Let K be a number field. A curve of genus $g > 1$ over K has only finitely many rational points.*

1. Rational Maps of Degree 2

Throughout this chapter K will denote a field and $\overline{K}$ will denote the algebraic closure of K. A rational map $\phi(z) \in \overline{K}(z)$ is a quotient of polynomials

$$\phi(z) = \frac{F(z)}{G(z)} = \frac{a_0 + a_1 z + \ldots + a_d z^d}{b_0 + b_1 z + \ldots + b_d z^d},$$

with no common factors. The degree of the map is defined to be

$$\deg \phi = \max\{\deg F, \deg G\},$$

[13]. The affine n-space over K is the set of n-tuples

$$\mathbb{A}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \overline{K}\}.$$

The projective n-space over K is

$$\mathbb{P}^n = \mathbb{A}^{n+1} \setminus \{(0,0,\ldots,0)\} / \sim$$

where the equivalence relation $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$ if there exists $\lambda \in \overline{K}^*$ such that $x_i = \lambda \cdot y_i$ for all i.

Let F be homogeneous polynomial in $K[X_0, X_1, \ldots, X_n]$, and $V = V(F)$ be the zero set of F. $\overline{K}(V)$ is the set of elements of the form $\frac{f_1}{f_2}$ where the following conditions satisfied

- f_1, f_2 are homogeneous polynomials of the same degree,
- f_2 does not vanish identically on V,
- $\frac{f_1}{g_1} \sim \frac{f_2}{g_2}$ if $f_1 \cdot g_2 - f_2 \cdot g_1$ vanishes on V.

Let F_1, F_2 be homogeneous polynomials over $K[X_0, X_1, \ldots, X_n]$. Let $V_1 = V(F_1)$, $V_2 = V(F_2) \subset \mathbb{P}^n$ be zero sets of these polynomials. A rational map ϕ from V_1 to V_2

is of the form $\phi = (f_0, f_1, \ldots, f_n)$ with $f_0, f_1, \ldots, f_n \in \overline{K}(V_1)$, and it is defined by

$$\phi(P) = (f_0(P), f_1(P), \ldots, f_n(P)).$$

The rational map ϕ is said to be regular at $P \in V_1$ if there is a $g \in \overline{K}(V_1)$ such that gf_i is defined at P for all $i = 0, 1, \ldots, n$, and there exist some i such that $gf_i(P)$ is nonzero. A morphism is a rational map that is regular at every point. If V_1 is a smooth curve, then ϕ is a morphism ([14], Chapter 2, §2, Proposition 2.1). So if $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map defined over K, then ϕ is also a morphism.

A (discrete) dynamical system consists of a set S and a function $\phi : S \to S$ mapping the set S to itself. This map permits iteration

$$\phi^n = \phi \circ \phi \circ \ldots \circ \phi \ (n^{th} \text{ iterate of } \phi).$$

Orbit of a point P in the set S under a mapping $\phi : S \to S$ is the set which consists of the images of the point P under iterations ϕ^n of the mapping ϕ for all $n \geqslant 0$. We will denote it by $Orb_\phi(P)$.

One of the main goals of dynamical systems is to classify the points P in the set S according to their orbits $\mathcal{O}_\phi(P)$ [13]. A point P which has a finite orbit under a map is called a preperiodic point. In this thesis, we take the set S to be the rational field $\mathbb{Q}$, we focus on rational periodic points of some degree 2 rational maps. Now, let us give an equivalent definition of a preperiodic point besides definition of a periodic point:

A point $P \in S$ is preperiodic if there exist integers $n > m \geqslant 0$ such that $\phi^n(P) = \phi^m(P)$. A point $P \in S$ is periodic if there exists an integer $n > 0$ such that $\phi^n(P) = P$. In this case we say that, P has period n. Here, if n is the smallest such integer, we say that P has exact (primitive) period n.

When we consider the iterations of rational maps, we are not interested in the dynamical behaviour of linear maps as it is a trivial task. Clearly, any iteration of a linear map is again a linear map.

There is a great deal of work related to dynamical behaviour of quadratic polynomials in which researchers focused on the quadratic polynomial of the form $\beta(z) = z^2 + c$. Why they didn't focus on quadratic polynomials of the form

$$\alpha(z) = a_2 z^2 + a_1 z + a_0,$$

instead they worked on dynamical behaviour of $\beta(z) = z^2 + c$? Because any quadratic polynomial is linearly conjugate to $\beta(z)$ for some c.

The projective linear group of degree 2 over K is the quotient of the general linear group (the group of 2×2 nonsingular matrices) by the group of matrices of the form $k\cdot I$ where $k\in K$ is nonzero and I denotes the 2×2 identity matrix. We denote the projective linear group of degree 2 over K by $PGL_2(K)$.

Definition 1.0.1. *Let ϕ and ψ be two rational maps. These maps are linearly conjugate if there is some $f\in PGL_2(\overline{K})$ such that $f^{-1}\phi f=\psi$. They are linearly conjugate over K if there is some $f\in PGL_2(K)$ such that $f^{-1}\phi f=\psi$.*

Linearly conjugate maps have the same dynamical behaviour as

$$\alpha^n(P)=\alpha^m(P) \text{ if and only if } \beta^n(f^{-1}(P))=\beta^m(f^{-1}(P)),$$

where $\beta=f^{-1}\alpha f$. Because,

$$\begin{aligned}
\beta^n(f^{-1}(P)) &= (f^{-1}\alpha f)^n(f^{-1}(P))\\
&= f^{-1}\alpha^n f(f^{-1}(P))\\
&= f^{-1}\alpha^n(P)\\
&= f^{-1}\alpha^m(P)\\
&= f^{-1}\alpha^m f(f^{-1}(P))\\
&= (f^{-1}\alpha f)^m(f^{-1}(P))\\
&= \beta^m(f^{-1}(P)).
\end{aligned}$$

where $n>m\geqslant 0$.

Let $a_0,a_1,a_2\in\mathbb{Q}$ with $a_2\neq 0$. Now, we will show that all quadratic polynomials of the form

$$\alpha(z)=a_2z^2+a_1z+a_0$$

are linearly conjugate over $\mathbb{Q}$ to

$$\beta(z)=z^2+c$$

for some $c\in\mathbb{Q}$. Let $f(z)=dz+e$. Then, we have $f^{-1}(z)=(z-e)/d$. Now,

$$\begin{aligned}
f^{-1}\alpha f(z) &= f^{-1}\alpha(dz+e)\\
&= f^{-1}(a_2(dz+e)^2+a_1(dz+e)+a_0)\\
&= f^{-1}(a_2d^2z^2+(2a_2de+a_1d)z+(a_2e^2+a_1e+a_0))\\
&= \frac{(a_2d^2z^2+(2a_2de+a_1d)z+(a_2e^2+a_1e+a_0-e)}{d}.
\end{aligned}$$

The equality $f^{-1}\alpha f(z) = \beta(z)$ yields the following

$$d = \frac{1}{a_2}, \quad e = -\frac{a_1}{2a_2}, \quad c = \frac{-a_1^2 + 4a_0a_2 + 2a_1}{4}.$$

Hence, in terms of dynamical behaviour, $\beta(z) = z^2 + c$ is the representative of all quadratic maps over $\mathbb{Q}[z]$. We have a complete classification of periodic points of $\beta(z)$ with period length $1, 2$ and 3, see Theorem 1 in [12] and Theorem 1, 3 in [16].

Theorem 1.0.2. *[12, 16] Let $\beta(z) = z^2 + c$ where $c \in \mathbb{Q}$.*

a. $\beta(z)$ has a rational fixed point if and only if $c = 1/4 - \rho^2$ for some $\rho \in \mathbb{Q}$. In this case, there are exactly two, $1/2 + \rho$ and $1/2 - \rho$, unless $\rho = 0$, in which case they coincide.

b. $\beta(z)$ has a rational point of period 2 if and only if $c = -3/4 - \sigma^2$ for some $\sigma \in \mathbb{Q}^*$. In this case, there are exactly two, $-1/2 + \sigma$ and $-1/2 - \sigma$.

c. $\beta(z)$ has a rational point of period 3 if and only if

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2}$$

for some $\tau \in \mathbb{Q}$, $\tau \neq -1, 0$. In this case, there are exactly three,

$$x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)}$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)}$$

and these are cyclically permuted by $\beta(z)$.

Proof. In part $(a), (b), (c)$, taking c as above and using the corresponding periodic points we get one implication. Let p be a rational periodic point of $\beta(z)$ with period length 1. Then, we get

$$c = p - p^2.$$

On the other hand, roots of the equality

$$z^2 + p - p^2 = z,$$

are p and $1-p$. Hence $\beta(z)$ has a rational fixed point if and only if $c=p-p^2$ for some $p\in\mathbb{Q}$. In this case, rational fixed points are p and $1-p$. If we substitute

$$p=\frac{1}{2}+\rho,$$

this completes the proof for part (a).

Let p be a rational periodic point of $\beta(z)$ with exact period length 2. So we want p to be a root of the polynomial

$$\beta^2(z)-z=z^4+2cz^2-z+c^2+c.$$

If we factor this polynomial, we get

$$(z^2+c-z)(z^2+z+c+1).$$

Since we don't want to have $\beta(p)=p$, we only want p to be the root of

$$(z^2+z+c+1).$$

Hence, $c=-1-p-p^2$. Now, other root of the polynomial $(z^2+z-p-p^2)$ is $-1-p$. So $\beta(z)$ has a rational periodic point of exact period 2 if and only if $c=-1-p-p^2$ for some $p\in\mathbb{Q}$. In this case, rational period 2 points of the map are p and $-1-p$. If we substitute

$$p=-\frac{1}{2}+\sigma,$$

this completes the proof for part (b).

Let $\zeta\in\mathbb{Q}$ be an exact period 3 point $\beta(z)$. Assume that $\omega:=\beta(\zeta)=\zeta^2+c$ is not equal to ζ, that is ζ is not a rational fixed point of $\beta(z)$. Since we have $c=\omega-\zeta^2$, we get

$$\beta(z)=z^2+\omega-\zeta^2.$$

Now,

$$\begin{aligned}
\zeta &= \beta^3(\zeta)\\
&= \beta^2(\omega)\\
&= \beta(\omega^2+\omega-\zeta^2)\\
&= \omega^4+2\omega^3+\omega^2-2\omega^2\zeta^2-2\omega\zeta^2+\zeta^4+\omega-\zeta^2. \qquad (1.1)
\end{aligned}$$

Now, if we rearrange Equality (1.1) and divide by $\omega-\zeta$, we get an equivalent form of

$$(\omega+\zeta)^3+(2-2\zeta)(\omega+\zeta)^2+(1-2\zeta)(\omega+\zeta)+1=0.$$

Let $\tau=(\omega+\zeta)$. Using this in the previous equality, we get

$$\zeta=\frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)},$$

where $\tau\in\mathbb{Q}\setminus\{-1,0\}$. This is nothing but the formula of x_1. So, we get the following formulas

$$x_2=\beta(\zeta)=\omega=\tau-x_1,$$

$$c=\omega-\zeta^2=x_2-x_1^2,$$

$$x_3=\beta(x_2).$$

This completes the proof of part c. □

The polynomial $\beta(z)$ cannot have a rational periodic point of period 4 [10] or period 5 [5]. Under the assumption of Birch-Swinnerton- Dyer conjecture, it was proved that the polynomial $\beta(z)$ cannot have a rational periodic point of period 6 [15]. Finally, Poonen has conjectured that $\beta(z)$ cannot have a rational periodic point of period $\geqslant 6$ [12]. So in this thesis, we suppose that possible period lengths of rational points of $\beta(z)$ are $1,2,$ or 3.

Let $\phi:\mathbb{P}^1\to\mathbb{P}^1$ be a morphism defined over the rational field $\mathbb{Q}$. Automorphism group of a rational map ϕ is defined as follows:

$$Aut(\phi)=\{f\in PGL_2(\overline{K})|\ \phi^f=\phi\}$$

where $\phi^f=f^{-1}\phi f$ [7].

Lemma 1.0.3. *([7], Lemma 1) If ϕ is a rational map of degree 2 over $\mathbb{Q}$ then $Aut(\phi)$ is isomorphic to the cyclic group of order 2 if and only if ϕ is linearly conjugate over $\mathbb{Q}$ to some map of the form*

$$\phi_{k,b}(z)=kz+\frac{b}{z}$$

with $k\in\mathbb{Q}\setminus\{0,-1/2\}$ and $b\in\mathbb{Q}^*$.

Furthermore, $\phi_{k,b}$ and $\phi_{k',b'}$ are linearly conjugate over $\mathbb{Q}$ if and only if $k=k'$ and b/b' is a nonzero square in $\mathbb{Q}$.

By the previous lemma, we see that determining the dynamical behaviour of the map

$$\phi_{k,b}(z)=kz+\frac{b}{z}$$

is equivalent to determine the dynamical behaviour of all degree-2 rational maps over $\mathbb{Q}$ with automorphism group isomorphic to the cyclic group of order 2. This is why we focus on the rational periodic points of the map $\phi_{k,b}$.
By Manes, we have the following theorem and conjecture (see Proposition $1,2$, Theorem $3,4$ and Conjecture 1 in [7]):

Theorem 1.0.4. *[7] Let $\phi_{k,b}(z) = kz + \frac{b}{z}$ where $k \in \mathbb{Q}^*$ and $b \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.*

i. If $b \equiv (1-k)$ modulo squares, then $\phi_{k,b}(z)$ has two finite rational fixed points; otherwise, $\phi_{k,b}(z)$ has no finite rational fixed points.

ii. $\phi_{k,b}(z)$ has a rational point of primitive period 2 if and only if $b \equiv -(1+k)$ modulo squares.

iii. There is a one parameter family of such maps

$$\phi_m(z) = \frac{2mz}{m^2-1} - \frac{m}{z(m^4-1)}$$

where $m \in \mathbb{Q} \setminus \{0,1,-1\}$, with a rational point of primitive period 4. In this case $\phi_{k,b}(z)$ has exactly four points of primitive period 4.

iv. Let $\phi_{k,b}(z)$ be with $k,b \in \mathbb{Q}^*$. Then $\phi_{k,b}(z)$ has no rational point of primitive period 3.

Conjecture 1.0.5. *[7] If $\phi(z) \in \mathbb{Q}(z)$ is a degree-2 rational map with $Aut(\phi) \cong C_2$ then ϕ has no rational point of exact period greater than 4.*

In this thesis, we suppose that the period length of a rational periodic point of $\phi_{k,b}$ is either $1,2,$ or 4.

2. Dynatomic Polynomials of $\phi_{k,b}$

Using the definition of periodic point is one way to find periodic points of a rational map. Rather than using this way, we will use an easier and systematic way in which we deal with a special type of polynomials the so called 'dynatomic polynomials'.

Definition 2.0.1. *[13] Let $\phi(z) \in K(z)$ be a rational function of degree d. We write*

$$\phi^n = [F_n(x,y), G_n(x,y)]$$

where $F_n, G_n \in K[x,y]$ are homogeneous polynomials of degree d^n for any $n \geqslant 0$. The n- period polynomial of ϕ is the polynomial

$$\Phi_{\phi,n}(x,y) = y \cdot F_n(x,y) - x \cdot G_n(x,y).$$

The n^{th} dynatomic polynomial of ϕ is the polynomial

$$\Phi^*_{\phi,n}(x,y) = \prod_{k|n} (y \cdot F_k(x,y) - x \cdot G_k(x,y))^{\mu(n/k)} = \prod_{k|n} (\Phi_{\phi,k}(x,y))^{\mu(n/k)}$$

where μ is the Möbius function which is defined as $\mu(1) = 1$, $\mu(n) = (-1)^l$ if $n = p_1 p_2 \dots p_l$ with p_i, $i \in \{1, 2, \dots, l\}$ distinct primes and $\mu(n) = 0$ if n is not square free.
If ϕ is fixed, we write Φ_n and Φ^*_n. If $\phi(z) \in K[z]$ is a polynomial, then we dehomogenize $[x,y] = [z,1]$ and write $\Phi_n(z)$ and $\Phi^*_n(z)$.

Proposition 2.0.2. *Let $\phi(z) \in K(z)$ be a rational function of degree d. Let $\phi^n, \Phi_{\phi,n}(x,y), \Phi^*_{\phi,n}(x,y)$ be as in Definition 2.0.1. Then we have the following:*

i. $\Phi_{\phi,n}(P) = 0$ *if and only if* $\phi^n(P) = P$.

ii. *Suppose k and n are positive integers such that k divides n. If $\phi^k(P) = P$, then $\phi^n(P) = P$.*

iii. *If P is a root of $\Phi^*_{\phi,n}(x,y)$ then $\phi^n(P) = P$.*

Proof. Part (i): We first assume that $P=[1,0]$. Now, P is a root of $\Phi_{\phi,n}(x,y)$ if and only if

$$\Phi_{\phi,n}(P)=-G_n(P)=0.$$

This equality holds if and only if $G_n(P)=0$ and

$$\phi^n(P)=[F_n(P),0]=[1,0]=P.$$

Now, assume that $P=[a,1]$. P is a root of $\Phi_{\phi,n}(x,y)$ if and only if

$$\Phi_{\phi,n}(P)=F_n(P)-a\cdot G_n(P)=0.$$

This can happen if and only if

$$\phi^n(P)=[F_n(P),G_n(P)]=[a\cdot G_n(P),G_n(P)]=[a,1]=P.$$

Part (ii): Obvious.
Part(iii): Follows from part (i) and the definition of n-th dynatomic polynomial. □

By definition, every exact period n-point is a root of $\Phi^*_{\phi,n}(z)$. But it is important to notice that $\Phi^*_{\phi,n}(z)$ can have roots whose periods divide n and strictly smaller than n.

Now, for the convenience of the reader I am going to reproduce the dynatomic polynomials of $\phi_{k,b}$. First let us share some examples of dyanatomic polynomials for quadratic polynomial $\beta(z)$. As expected, this is straightforward:

$$\Phi^*_{\beta,1}(z)=z^2-z+c,$$

$$\Phi^*_{\beta,2}(z)=z^2+z+c+1,$$

$$\Phi^*_{\beta,3}(z)=z^6+z^5+(3c+1)z^4+(2c+1)z^3+(3c^2+3c+1)z^2+(c^2+2c+1)z+c^3+2c^2+c+1.$$

Now, we will find dynatomic polynomials $\Phi^*_{\phi,1}(z)$, $\Phi^*_{\phi,2}(z)$, $\Phi^*_{\phi,3}(z)$ and $\Phi^*_{\phi,4}(z)$ for

$$\phi_{k,b}(z)=\frac{kz^2+b}{z},$$

where $k,b\in\mathbb{Q}$ and k,b are nonzero. Since our $\phi=\phi_{k,b}$ is fixed, we will simplify the notation as in the Definition 2.0.1. To be able to find $\Phi^*_1(z)$, we will find $\phi(x,y)$, $\Phi_1(x,y)$ and $\Phi^*_1(x,y)$, respectively.

$$\phi(x,y)=[kx^2+by^2,xy],$$

$$\begin{aligned} \Phi_1(x,y) &= y\cdot F_1(x,y)-x\cdot G_1(x,y) \\ &= y(kx^2+by^2)-x(xy) \\ &= (k-1)x^2y+by^3. \end{aligned} \tag{2.1}$$

Since $\Phi_1^*(x,y)=\Phi_1(x,y)$, we get

$$\Phi_1^*(z)=(k-1)z^2+b.$$

To find $\Phi_2^*(z)$, we will find $\phi^2(z)$, $\phi^2(x,y)$, $\Phi_2(x,y)$ and $\Phi_2^*(x,y)$, respectively.

$$\phi^2(z)=\frac{k^3z^4+(2k^2b+b)z^2+kb^2}{kz^3+bz},$$

$$\phi^2(x,y)=[k^3x^4+(2k^2b+b)x^2y^2+kb^2y^4,kx^3y+bxy^3],$$

$$\begin{aligned} \Phi_2(x,y) &= y\cdot F_2(x,y)-x\cdot G_2(x,y) \\ &= y(k^3x^4+(2k^2b+b)x^2y^2+kb^2y^4)-x(kx^3y+bxy^3) \\ &= (k^3-k)x^4y+2k^2bx^2y^3+kb^2y^5. \end{aligned} \tag{2.2}$$

Using Equalities (2.1) and (2.2), we get

$$\begin{aligned} \Phi_2^*(x,y) &= \prod_{k|2}(\Phi_k(x,y))^{\mu(2/k)} \\ &= \Phi_1^{\mu(2/1)}(x,y)\cdot\Phi_2^{\mu(2/2)}(x,y) \\ &= \frac{\Phi_2(x,y)}{\Phi_1(x,y)} \\ &= \frac{(k^3-k)x^4y+2k^2bx^2y^3+kb^2y^5}{(k-1)x^2y+by^3} \\ &= (k^2+k)x^2+kby^2. \end{aligned}$$

Now, if we dehomogenize $\Phi_2^*(x,y)$, we get

$$\Phi_2^*(z)=(k^2+k)z^2+kb. \tag{2.3}$$

To find $\Phi_3^*(z)$, we will find $\phi^3(z)$, $\phi^3(x,y)$, $\Phi_3(x,y)$ and $\Phi_3^*(x,y)$, respectively.

$\phi^3(z) = \frac{k^7z^8+(4bk^6+2bk^4+bk^2)z^6+(6b^2k^5+4b^2k^3+3b^2k)z^4+(b^3+2b^3k^2+4b^3k^4)z^2+b^4k^3}{k^4z^7+(bk+3bk^3)z^5+(b^2+3b^2k^2)z^3+b^3kz}$,

$$\phi^3(x,y) = [(b^4k^3y^8 + b^3y^6x^2 + 2b^3k^2y^6x^2 + 4b^3k^4y^6x^2 + 3b^2ky^4x^4 + 4b^2k^3y^4x^4 + 6b^2k^5y^4x^4 + bk^2y^2x^6 + 2bk^4y^2x^6 + 4bk^6y^2x^6 + k^7x^8), (b^3kxy^7 + b^2x^3y^5 + 3b^2k^2x^3y^5 + bkx^5y^3 + 3bk^3x^5y^3 + k^4x^7y)],$$

$$\Phi_3(x,y) = y \cdot F_3(x,y) - x \cdot G_3(x,y),$$

(2.4) $$\Phi_3(x,y) = -k^4x^8y + k^7x^8y - bkx^6y^3 + bk^2x^6y^3 - 3bk^3x^6y^3 + 2bk^4x^6y^3 + 4bk^6x^6y^3 - b^2x^4y^5 + 3b^2kx^4y^5 - 3b^2k^2x^4y^5 + 4b^2k^3x^4y^5 + 6b^2k^5x^4y^5 + b^3x^2y^7 - b^3kx^2y^7 + 2b^3k^2x^2y^7 + 4b^3k^4x^2y^7 + b^4k^3y^9.$$

Now,

$$\begin{aligned} \Phi_3^*(x,y) &= \prod_{k|3}(\Phi_k(x,y))^{\mu(3/k)} \\ &= \Phi_1^{\mu(3/1)}(x,y) \cdot \Phi_3^{\mu(3/3)}(x,y) \\ &= \frac{\Phi_3(x,y)}{\Phi_1(x,y)}. \end{aligned}$$

Using Equalities (2.1) and (2.4), we get

$$\Phi_3^*(x,y) = (k^4 + k^5 + k^6)x^6 + (bk + 3bk^3 + 2bk^4 + 3bk^5)x^4y^2 + (b^2 - b^2k + 2b^2k^2 + b^2k^3 + 3b^2k^4)x^2y^4 + b^3k^3y.$$

Now, if we dehomogenize $\Phi_3^*(x,y)$, then we get

$$\Phi_3^*(z) = (k^4 + k^5 + k^6)z^6 + (bk + 3bk^3 + 2bk^4 + 3bk^5)z^4 + (b^2 - b^2k + 2b^2k^2 + b^2k^3 + 3b^2k^4)z^2 + b^3k^3.$$

Using similar process with the help of Mathematica, we also get

(2.5)
$$\Phi_4^*(z) = (k^{10} + k^{12})z^{12} + (bk^5 + 3bk^7 + 8bk^9 + 6bk^{11})z^{10} + (b^2k^2 + 6b^2k^4 + 13b^2k^6 + 22b^2k^8 + 15b^2k^{10})z^8 + (3b^3k + 10b^3k^3 + 18b^3k^5 + 28b^3k^7 + 20b^3k^9)z^6 + (2b^4 + 5b^4k^2 + 9b^4k^4 + 17b^4k^6 + 15b^4k^8)z^4 + (b^5k + b^5k^3 + 4b^5k^5 + 6b^5k^7)z^2 + b^6k^6.$$

In ([7], Theorem 4), it was proved that $\Phi_4^*(z)$ can have at most 4 rational root. In particular,

$$\phi_{k,b}(z) = kz + \frac{b}{z},$$

cannot have more than four points of exact period 4. We will reproduce this result.

Theorem 2.0.3. *Let*

$$\phi_{k,b}(z) = kz + \frac{b}{z},$$

where $k, b \in \mathbb{Q}^*$. The map $\phi_{k,b}(z)$ cannot have more than four points of exact period 4.

Proof. If we factorize $\Phi_4^*(z)$, we get

$$\Phi_4^*(z) = \Psi_4(z) \cdot \Lambda_4(z) \tag{2.6}$$

where
$\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$,
$\Lambda_4(z) = b^4k^5 + b^3z^2 + b^3k^2z^2 + 2b^3k^4z^2 + 4b^3k^6z^2 + b^2kz^4 + 3b^2k^3z^4 + 4b^2k^5z^4 + 6b^2k^7z^4 + bk^4z^6 + 2bk^6z^6 + 4bk^8z^6 + k^9z^8$.
We claim that $\Lambda_4(z)$ does not have a rational root. Suppose that $\Lambda_4(z) = 0$ for some $z \in \mathbb{Q}$. First let us substitute $z^2 = t$ in $\Lambda_4(z) = 0$, then we get

$$b^4k^5 + (1+k^2+2k^4+4k^6)tb^3 + (k+3k^3+4k^5+6k^7)t^2b^2 + (k^4+2k^6+4k^8)t^3b + k^9t^4 = 0.$$

If we change coordinates by using $t = t_1b$, and divide this equality by b^4, this yields

$$k^5 + (1+k^2+2k^4+4k^6)t_1 + (k+3k^3+4k^5+6k^7)t_1^2 + (k^4+2k^6+4k^8)t_1^3 + k^9t_1^4 = 0.$$

Since $k \in \mathbb{Q}^*$, $a = k^{-1}$ exists. Multiplying last equality by a^9 gives us

$$a^4 + (a^9+a^7+2a^5+4a^3)t_1 + (a^8+3a^6+4a^4+6a^2)t_1^2 + (a^5+2a^3+4a)t_1^3 + t_1^4 = 0.$$

Now, if we change coordinates by using $t_1 = t_2a$ and multiply the equality by a^{-4}, we finally get

$$1 + (a^6+a^4+2a^2+4)t_2 + (a^6+3a^4+4a^2+6)t_2^2 + (a^4+2a^2+4)t_2^3 + t_2^4 = 0. \tag{2.7}$$

Note that the fact that $\Lambda_4(z)$ has a rational root implies that Equation (2.7) has a rational root. But we will see that this doesn't happen. If we solve this equality in t_2, solutions are

$$s_1 = \tfrac{1}{4}(-4-2a^2-a^4)+\tfrac{1}{4}a^2\sqrt{4+a^4}-\frac{\sqrt{8a^2+4a^4+2a^6+a^8-\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{2\sqrt{2}},$$

$$s_2 = \tfrac{1}{4}(-4-2a^2-a^4)+\tfrac{1}{4}a^2\sqrt{4+a^4}+\frac{\sqrt{8a^2+4a^4+2a^6+a^8-\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{2\sqrt{2}},$$

$$s_3 = \tfrac{1}{4}(-4-2a^2-a^4)-\tfrac{1}{4}a^2\sqrt{4+a^4}-\frac{\sqrt{8a^2+4a^4+2a^6+a^8+\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{2\sqrt{2}},$$

$$s_4 = \tfrac{1}{4}(-4-2a^2-a^4)-\tfrac{1}{4}a^2\sqrt{4+a^4}+\frac{\sqrt{8a^2+4a^4+2a^6+a^8+\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{2\sqrt{2}}.$$

First let us consider s_1. To be able to have $s_1 \in \mathbb{Q}$, we need to have

$$\frac{1}{4}a^2\sqrt{4+a^4}-\frac{\sqrt{8a^2+4a^4+2a^6+a^8-\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{2\sqrt{2}} \in \mathbb{Q}.$$

Let $\sqrt{4+a^4}=q$, and $\frac{\sqrt{8a^2+4a^4+2a^6+a^8-\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{\sqrt{2}}=B$. Suppose that

$$\frac{1}{4}a^2q-\frac{B}{2}=C \tag{2.8}$$

for some $C \in \mathbb{Q}$. It turns out that q and B are both rational or both irrational. Suppose $q \in \mathbb{Q}$ and $B \notin \mathbb{Q}$. Using $q \in \mathbb{Q}$ in Equality (2.8) implies that $B \in \mathbb{Q}$, a contradiction. Similarly, $q \notin \mathbb{Q}$ and $B \in \mathbb{Q}$ also leads a contradiction. Now, we will show that having q and B both irrational is also not possible, so that our only case is q and B are both rational. Suppose $q \notin \mathbb{Q}$. Note that, we have $q^2 \in \mathbb{Q}$. If we rearrange Equality (2.8) in terms of q, we get

$$\frac{1}{4}a^2q-C=\frac{1}{2}\sqrt{\frac{1}{2}\left(q^2\cdot(a^4+2a^2)-\frac{a^2\cdot q^2\cdot(a^4+2a^2)}{q}\right)}. \tag{2.9}$$

If we take square of the both sides of the Equality (2.9), and rearrange, we get

$$\frac{1}{16}a^4q^2+C^2-\frac{1}{8}q^2(a^4+2a^2)=q\cdot(\frac{1}{2}a^2C-\frac{1}{8}a^2(a^4+2a^2)).$$

Since $C,a,q^2 \in \mathbb{Q}$, we get $q \in \mathbb{Q}$, a contradiction.

Now, since in Equality (2.8) we want to have $C \in \mathbb{Q}$, we must have $\sqrt{4+a^4}=q$ for some $q \in \mathbb{Q}^+$. If we take $a^2=s$ and rearrange this equality then we get

$$(\frac{q}{2})^2-(\frac{s}{2})^2=1.$$

So we have, $q = \frac{2(m^2+1)}{m^2-1}$ and $s = \frac{4m}{m^2-1}$ for some $m \in \mathbb{Q} \setminus \{0,-1,1\}$. We also want to have,

$$\frac{\sqrt{8a^2+4a^4+2a^6+a^8-\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{\sqrt{2}} = B \tag{2.10}$$

for some $B \in \mathbb{Q}$.
If we use $\sqrt{4+a^4} = q$ and $a^2 = s$, and rearrange Equality (2.10) we get,

$$(8s+4s^2+2s^3+s^4)(\frac{q-s}{q}) = 2B^2.$$

Now, if we write q and s in terms of m, this yields

$$\frac{16}{(m-1)^2(m+1)^4} \cdot (m^2+2m-1)(m^2+1)m = B^2$$

At this point, we have to investigate whether the hyperelliptic curve defined by

$$y^2 = (m^2+2m-1)(m^2+1)m \tag{2.11}$$

has rational points except for $m \in \{0,-1,1\}$, or not. LMFDB [6] tells us this hyperelliptic curve does not have any rational points except for $m = 0,-1,1$. Note that for s_2, we have the same result. If we consider s_3 and s_4 by taking

$$\sqrt{4+a^4} = q \text{ and } \frac{\sqrt{8a^2+4a^4+2a^6+a^8+\frac{(8a^4+4a^6+2a^8+a^{10})}{\sqrt{4+a^4}}}}{\sqrt{2}} = B,$$

we see that q and B must be both rational as before. If we parametrize $\sqrt{4+a^4} = q$ taking $a^2 = s$ as before, and express q and s in terms of m, we get

$$\frac{16}{(m-1)^4(m+1)^2} \cdot (m^2+2m-1)(m^2+1)m = B^2.$$

But this gives the same hyperelliptic curve defined by Equation (2.11). Therefore, in the factorization of $\Phi_4^*(z)$ in Equation (2.6), $\Lambda_4(z)$ does not give any rational root, and this implies that to investigate period 4 points of $\phi_{k,b}(z)$, it is enough to concentrate on the roots of the factor $\Psi_4(z)$ of the dynatomic polynomial $\Phi_4^*(z)$. $\square$

3. Simultaneous Rational Periodic Points of $\beta(z)$ and $\phi_{k,b}(z)$

Let $\beta(z) = z^2 + c$ and $\phi_{k,b}(z) = kz + b/z$ where $k, b, c \in \mathbb{Q}$ with k, b are nonzero. In this chapter, we find families (k, b, c) for some parameter p which is periodic point for $\beta(z)$ and $\phi_{k,b}(z)$. Let us share our results in the following table:

Table 3.1 Simultaneous Rational Periodic Points of $\phi_{k,b}(z)$ and $\beta(z)$

$\phi_{k,b}(z)$	Periodic Points	PL	$\beta(z)$	Periodic Points	PL
$\frac{q+p}{p}z - \frac{qp}{z}$	$p, -p$	1	$z^2 + p - p^2$	$p, 1-p$	1
$\frac{q-p}{p}z - \frac{qp}{z}$	$p, -p$	2	$z^2 + p - p^2$	$p, 1-p$	1
$\frac{2q}{q^2-1}z - \frac{p^2(q^2+1)}{q(q^2-1)z}$	$p, p/q, -p, -p/q$	4	$z^2 + p - p^2$	$p, 1-p$	1
$\frac{q+p}{p}z - \frac{qp}{z}$	$p, -p$	1	$z^2 - (p^2+p+1)$	$p, -p-1$	2
$\frac{q-p}{p}z - \frac{qp}{z}$	$p, -p$	2	$z^2 - (p^2+p+1)$	$p, -p-1$	2
$\frac{2q}{q^2-1}z - \frac{p^2(q^2+1)}{q(q^2-1)z}$	$p, p/q, -p, -p/q$	4	$z^2 - (p^2+p+1)$	$p, -p-1$	2
$\frac{q+p_\tau}{p_\tau}z - \frac{qp_\tau}{z}$	$p_\tau, -p_\tau$	1	$z^2 + c_\tau$	p_τ	3
$\frac{q-p_\tau}{p_\tau}z - \frac{qp_\tau}{z}$	$p_\tau, -p_\tau$	2	$z^2 + c_\tau$	p_τ	3
$\frac{2q}{q^2-1}z - \frac{p_\tau^2(q^2+1)}{q(q^2-1)z}$	$p_\tau, p_\tau/q, -p_\tau, -p_\tau/q$	4	$z^2 + c_\tau$	p_τ	3

where:

$$c_\tau = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2},$$

$$p_\tau \in \{\frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}, \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)}, -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)}\},$$

and PL is abbreviation of period length for the periodic points. This table arises from the following theorem:

Theorem 3.0.1. *Let $\beta(z) = z^2 + c$ and $\phi_{k,b}(z) = \frac{kz^2+b}{z}$ where $k, b, c \in \mathbb{Q}$ with k, b are nonzero. Let m denote the period length of a periodic point of $\beta(z)$. Let n denote the period length of a periodic point of $\phi_{k,b}(z)$. Suppose $m \in \{1, 2, 3\}$ and $n \in \{1, 2, 4\}$. Let $p \in \mathbb{Q}^*$ be simultaneous periodic point of $\beta(z)$ and $\phi_{k,b}(z)$. Then we have three* *families of one parameter maps in which k and b are in one of the following forms:*

$$(k,b) = (\frac{q+p}{p}, -qp) \text{ for } q \in \mathbb{Q} \setminus \{0, -p\}:$$

In this case, $p, -p$ are the only rational fixed points of $\phi_{k,b}$.

$$(k,b) = (\frac{q-p}{p}, -qp) \text{ for } q \in \mathbb{Q} \setminus \{0, p\}:$$

In this case, $(p, -p)$ is the only 2-cycle of $\phi_{k,b}$.

$$(k,b) = (\frac{2q}{(q^2-1)}, -\frac{(q^2+1)}{q \cdot (q^2-1)} p^2) \text{ for } q \in \mathbb{Q} \setminus \{0, -1, +1\}:$$

In this case, $(p, \frac{p}{q}, -p, -\frac{p}{q})$ is the only 4-cycle of $\phi_{k,b}$.

Once we prove the following nine propositions, this theorem will immediately follow.

Proposition 3.0.2. *Intersection of rational fixed points of $\beta(z)$ and $\phi_{k,b}(z)$ is nonempty if and only if k, b, and c are of the form:*

$$k = \frac{q+p}{p}, \tag{3.1}$$

$$b = -qp, \tag{3.2}$$

$$c = p - p^2, \tag{3.3}$$

for some $p, q \in \mathbb{Q}$ such that $p \neq 0$, and $q \notin \{0, -p\}$ where p is a rational fixed point of $\beta(z)$ and $\phi_{k,b}(z)$.

Proof. If we take k, b, c as in the given form, we easily see that p is a rational fixed point of $\beta(z)$ and $\phi_{k,b}(z)$. Now, suppose $\beta(z)$ has a rational fixed point. We know that $\beta(z)$ has a rational fixed point if and only if $c = 1/4 - \rho^2$ for some $\rho \in \mathbb{Q}$. In this case, there are exactly two, $1/2 + \rho$ and $1/2 - \rho$, unless $\rho = 0$, in which case they coincide [12].

Case 1 : Suppose $1/2 + \rho$ is a rational fixed point of $\phi_{k,b}(z)$. So we want to have

$$\Phi_1^*(1/2 + \rho) = 0$$

that is,

$$(k-1)(1/2+\rho)^2+b=0.$$

Such a ρ satisfies the following equation,

$$(k-1)/4+b+(k-1)\rho+(k-1)\rho^2=0.$$

where $\rho \neq -1/2$. In this case, ρ has the following form,

$$\rho=\frac{1-k\pm 2\sqrt{b-bk}}{2(-1+k)}, \tag{3.4}$$

whenever $k \neq 1$. Since ρ is rational, we must have

$$\sqrt{b-bk}=q \tag{3.5}$$

for some $q \in \mathbb{Q}^+$. Now using Equation (3.4) and (3.5) we see that our fixed point is of the form,

$$\frac{1}{2}+\rho=\frac{\pm q}{k-1}. \tag{3.6}$$

From Equation (3.6), we get

$$k=\frac{\pm 2q+2\rho+1}{2\rho+1}. \tag{3.7}$$

If we take squares of the both sides of the Equality (3.5), and if we use Equality (3.7), we get

$$b=\frac{\mp q\cdot(1+2\rho)}{2}.$$

Hence, if $\beta(z)$ and $\phi_{k,b}(z)$ have the same rational fixed point $\frac{1}{2}+\rho$, then k, b, and c are of the form

$$k=\frac{2q+2\rho+1}{2\rho+1},$$

$$b=\frac{-q\cdot(1+2\rho)}{2},$$

$$c=\frac{1}{4}-\rho^2,$$

for some $\rho, q \in \mathbb{Q}$ such that $\rho \neq -\frac{1}{2}$, and $q \neq 0$.

Case 2: Suppose $1/2-\rho$ is a fixed rational point of $\phi_{k,b}(z)$. So we want to have

$$\Phi_1^*(1/2-\rho)=0$$

that is,

$$(k-1)(1/2-\rho)^2+b=0.$$

Such a ρ satisfies the following equation,

$$(k-1)/4+b-(k-1)\rho+(k-1)\rho^2=0.$$

where $\rho \neq 1/2$. In this case, ρ has the following form,

$$\rho=\frac{-1+k\pm 2\sqrt{b-bk}}{2(-1+k)},$$

whenever $k \neq 1$. Since ρ is rational, we must have $\sqrt{b-bk}=q$ for some $q\in\mathbb{Q}$. If we proceed as before, we see that if $1/2-\rho$ is a rational fixed point of $\beta(z)$ and $\phi_{k,b}(z)$, then k, b, and c are of the form

$$k=\frac{2q-2\rho+1}{-2\rho+1},$$

$$b=\frac{-q\cdot(1-2\rho)}{2},$$

$$c=\frac{1}{4}-\rho^2,$$

for some $\rho,q\in\mathbb{Q}$ such that $\rho\neq\frac{1}{2}$ and $q\neq 0$. Replacing ρ by $-\rho$ for $k,b,$ and c, we get the same parametrization as in case 1. Substituting $\rho=p-1/2$ in case 1 completes our proof. $\square$

Example 3.0.3. *Let us take $p=3/2$ and $q=1$ in Proposition 3.0.2. By Equalities (3.1), (3.2), and (3.3) we get,*

$$\beta(z)=z^2-\frac{3}{4},$$

$$\phi_{k,b}(z)=\frac{5z}{3}-\frac{3}{2z}.$$

Hence, $\frac{3}{2}$ is a rational fixed point of $\beta(z)$ and $\phi_{k,b}(z)$.

Proposition 3.0.4. *Intersection of rational fixed points of $\beta(z)$ and rational period 2 points of $\phi_{k,b}(z)$ is nonempty if and only if k, b, and c are of the form:*

$$k=\frac{q-p}{p}, \tag{3.8}$$

(3.9)
$$b = -qp,$$

(3.10)
$$c = p - p^2,$$

for some $p, q \in \mathbb{Q}$ such that $p \neq 0$, and $q \notin \{0, p\}$ where p is a rational fixed point of $\beta(z)$ and a rational exact period 2 point of $\phi_{k,b}(z)$.

Proof. Suppose rational fixed points $1/2+\rho$ and $1/2-\rho$ of $\beta(z)$ are roots of $\Phi_2^*(z) = (k^2+k)z^2+kb$.
Case 1: Suppose $1/2+\rho$ is a root of $\Phi_2^*(z) = (k^2+k)z^2+kb$ where $\rho \neq -1/2$. This means,

(3.11)
$$(k^2+k)(\frac{1}{2}+\rho)^2 + kb = 0.$$

If we solve Equation (3.11) in ρ we get

$$\rho = \frac{-(1+k) \pm 2\sqrt{-b(1+k)}}{2(1+k)},$$

whenever $k \neq -1$. Since $\rho \in \mathbb{Q}$, we must have $\sqrt{-b(1+k)} = q$ for some $q \in \mathbb{Q}^+$. This yields

(3.12)
$$\frac{1}{2} + \rho = \frac{\pm q}{1+k}.$$

If we rearrange Equality (3.12), we get

$$k = \frac{\pm 2q - 2\rho - 1}{2\rho + 1}.$$

Using k with $\sqrt{-b(1+k)} = q$, we also have

$$b = \frac{\mp q \cdot (1+2\rho)}{2}.$$

Hence, if $\frac{1}{2}+\rho$ is a rational fixed point of $\beta(z)$ and a period 2 point of $\phi_{k,b}(z)$, then k, b, and c are of the form

$$k = \frac{2q - 2\rho - 1}{2\rho + 1},$$

$$b = \frac{-q\cdot(1+2\rho)}{2},$$

$$c = \frac{1}{4} - \rho^2,$$

for some $\rho, q \in \mathbb{Q}$ such that $\rho \neq -\frac{1}{2}$ and $q \neq 0$. Using these k and b, we get

$$\phi_{k,b}(\frac{1}{2}+\rho) = -(\frac{1}{2}+\rho) \neq \frac{1}{2}+\rho,$$

as $\rho \neq -1/2$. Therefore, $\frac{1}{2}+\rho$ is a fixed point of $\beta(z)$ and an exact period 2 point of $\phi_{k,b}(z)$ for k, b, and c as above. Substituting $\rho = p - 1/2$ gives the result.

Case 2: Suppose $1/2-\rho$ is a root of $\Phi_2^*(z) = (k^2+k)z^2 + kb$ where $\rho \neq 1/2$. If we proceed, as in previous case, we see that if $\frac{1}{2}-\rho$ is a rational fixed point of $\beta(z)$ and a rational period 2 point of $\phi_{k,b}(z)$, then k, b, and c are of the form

$$k = \frac{2q+2\rho-1}{1-2\rho},$$

$$b = \frac{-q\cdot(1-2\rho)}{2},$$

$$c = \frac{1}{4} - \rho^2,$$

for some $\rho, q \in \mathbb{Q}$ such that $\rho \neq \frac{1}{2}$ and $q \neq 0$. Clearly, this parametrization is equivalent to previous one. $\square$

Example 3.0.5. *Let us take $p = 3$ and $q = \frac{1}{2}$ in Proposition 3.0.4. By Equalities (3.8), (3.9), and (3.10) we get,*

$$\beta(z) = z^2 - 6,$$

$$\phi_{k,b}(z) = -\frac{5z}{6} - \frac{3}{2z}.$$

Hence, 3 is a rational fixed point of $\beta(z)$ and a rational exact period 2 point of $\phi_{k,b}(z)$.

Proposition 3.0.6. *Intersection of rational fixed points of $\beta(z)$ and rational period 4 points of $\phi_{k,b}(z)$ is nonempty if and only if k, b, and c are of the form:*

$$k = \frac{2m}{m^2-1}, \tag{3.13}$$

$$b = -\frac{p^2 \cdot (m^2+1)}{m(m^2-1)}, \tag{3.14}$$

$$c = p - p^2, \tag{3.15}$$

where $p \in \mathbb{Q}^*$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. In this case p is a rational fixed point of $\beta(z)$ and a rational exact period 4 point of $\phi_{k,b}(z)$.

Proof. Suppose rational fixed points $1/2+\rho$ and $1/2-\rho$ of $\beta(z)$ are roots of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$.

Case 1: Suppose $1/2+\rho$ is a root of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$ where $\rho \neq -1/2$. This means,

$$b^2k + 2b(1/2+\rho)^2 + 2bk^2(1/2+\rho)^2 + k(1/2+\rho)^4 + k^3(1/2+\rho)^4 = 0. \tag{3.16}$$

If we solve Equation (3.16) in ρ we get

$$\rho_1 = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 - 4\sqrt{b^2 + b^2k^2}}{k+k^3}}\right),$$

$$\rho_2 = \frac{1}{2}\left(-1 - \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 + 4\sqrt{b^2 + b^2k^2}}{k+k^3}}\right),$$

$$\rho_3 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 - 4\sqrt{b^2 + b^2k^2}}{k+k^3}}\right),$$

$$\rho_4 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 + 4\sqrt{b^2 + b^2k^2}}{k+k^3}}\right).$$

If we rearrange these solutions, without loss of generality we get our possible solutions in ρ:

$$\rho_1 = \frac{1}{2}\left(-1 - \sqrt{\frac{-4b \cdot (1 + k^2 + \sqrt{1+k^2})}{k \cdot (1+k^2)}}\right), \tag{3.17}$$

$$\rho_2 = \frac{1}{2}\left(-1 - \sqrt{\frac{-4b\cdot(1+k^2-\sqrt{1+k^2})}{k\cdot(1+k^2)}}\right), \tag{3.18}$$

$$\rho_3 = \frac{1}{2}\left(-1 + \sqrt{\frac{-4b\cdot(1+k^2+\sqrt{1+k^2})}{k\cdot(1+k^2)}}\right), \tag{3.19}$$

$$\rho_4 = \frac{1}{2}\left(-1 + \sqrt{\frac{-4b\cdot(1+k^2-\sqrt{1+k^2})}{k\cdot(1+k^2)}}\right). \tag{3.20}$$

First let us consider ρ_1. Let

$$s = \sqrt{\frac{-4b\cdot(1+k^2+\sqrt{1+k^2})}{k\cdot(1+k^2)}}, \tag{3.21}$$

where $s \neq 0$, and

$$q = \sqrt{1+k^2}. \tag{3.22}$$

Since $\rho_1 \in \mathbb{Q}$, we need to have $s \in \mathbb{Q}$. But then, this yields $q \in \mathbb{Q}$ by Equality (3.21). Now using Equality (3.22), we have

$$q = \frac{m^2+1}{m^2-1}, \tag{3.23}$$

$$k = \frac{2m}{m^2-1}, \tag{3.24}$$

for some $m \in \mathbb{Q}\setminus\{0,\pm 1\}$. Now, using Equations (3.23) and (3.24) in Equation (3.21), we get

$$b = -\frac{s^2\cdot(m^2+1)}{4m(m^2-1)}.$$

By Equations (3.17) and (3.21) we also have,

$$\frac{1}{2} + \rho_1 = -\frac{s}{2}.$$

Therefore, if $-\frac{s}{2}$ is fixed point of $\beta(z)$ and period 4 point of $\phi_{k,b}(z)$, then k, b and c

are of the form

$$k = \frac{2m}{m^2-1},$$

$$b = -\frac{s^2\cdot(m^2+1)}{4m(m^2-1)},$$

$$c = -\left(\frac{s^2+2s}{4}\right),$$

where $s \in \mathbb{Q}^*$ and $m \in \mathbb{Q}\setminus\{0,\pm 1\}$. Using these k and b, we will show that $-s/2$ is a rational exact period 4 point for $\phi_{k,b}$. Let us make some calculations:

$$\phi_{k,b}(-\frac{s}{2}) = -\frac{s}{2m},$$

$$\phi^2_{k,b}(-\frac{s}{2}) = \frac{s}{2},$$

$$\phi^3_{k,b}(-\frac{s}{2}) = \frac{s}{2m},$$

$$\phi^4_{k,b}(-\frac{s}{2}) = -\frac{s}{2}.$$

Note that $-s/2 \neq -s/2m$ as $m \neq 1$ and $-s/2 \neq s/2$ as $s \neq 0$. Therefore $-s/2$ is a rational exact period 4 point for $\phi_{k,b}$. As an equivalent result, if p is a rational fixed point of $\beta(z)$ and a rational exact period 4 point of $\phi_{k,b}(z)$ then k, b, and c are of the form

$$k = \frac{2m}{m^2-1},$$

$$b = -\frac{p^2\cdot(m^2+1)}{m(m^2-1)},$$

$$c = p - p^2,$$

where $p \in \mathbb{Q}^*$ and $m \in \mathbb{Q}\setminus\{0,\pm 1\}$. If we consider Equation (3.19) we get exactly the same parametrization. Let us investigate what Equation (3.18) gives us:
Let

$$t = \sqrt{\frac{-4b\cdot(1+k^2-\sqrt{1+k^2})}{k\cdot(1+k^2)}},$$

where $t \neq 0$, and

$$q = \sqrt{1+k^2}.$$

Since $\rho_2 \in \mathbb{Q}$, we need to have $t \in \mathbb{Q}$. But then, this yields $q \in \mathbb{Q}$ by Equation (3.18). Note that, we have the same parametrizations for q and k as in Equations (3.23) and (3.24). So we get,

$$b = -\frac{t^2 \cdot m(m^2+1)}{4(m^2-1)}$$

where $t \in \mathbb{Q}^*$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. If we proceed as before, we see that if t is fixed point of $\beta(z)$ and period 4 point of $\phi_{k,b}(z)$ then k, b, and c are of the form

$$k = \frac{2m}{m^2-1},$$

$$b = -\frac{t^2 \cdot m(m^2+1)}{(m^2-1)},$$

$$c = t - t^2,$$

where $t \in \mathbb{Q}^*$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. But in this parametrization, replacing m by $-1/m$ gives the same parametrization as before.

Case 2: Suppose $1/2 - \rho$ is a root of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$ where $\rho \neq 1/2$. This means,

$$b^2k + 2b(1/2-\rho)^2 + 2bk^2(1/2-\rho)^2 + k(1/2-\rho)^4 + k^3(1/2-\rho)^4 = 0. \tag{3.25}$$

If we solve Equation (3.25) in ρ we get

$$\rho_1 = \frac{1}{2}\left(1 - \sqrt{\frac{-4b \cdot (1+k^2+\sqrt{1+k^2})}{k \cdot (1+k^2)}}\right),$$

$$\rho_2 = \frac{1}{2}\left(1 - \sqrt{\frac{-4b \cdot (1+k^2-\sqrt{1+k^2})}{k \cdot (1+k^2)}}\right),$$

$$\rho_3 = \frac{1}{2}\left(1 + \sqrt{\frac{-4b \cdot (1+k^2+\sqrt{1+k^2})}{k \cdot (1+k^2)}}\right),$$

$$\rho_4 = \frac{1}{2}\left(1 + \sqrt{\frac{-4b \cdot (1+k^2-\sqrt{1+k^2})}{k \cdot (1+k^2)}}\right).$$

Proceeding as in Case 1, we get the same parametrization. $\square$

Example 3.0.7. *Let us take $m = 2$ and $p = 2$ in Proposition 3.0.6. By Equalities (3.13), (3.14), and (3.15) we get,*

$$\beta(z) = z^2 - 2,$$

$$\phi_{k,b}(z) = \frac{4z}{3} - \frac{10}{3z}.$$

Hence, $p = 2$ is a rational fixed point of $\beta(z)$ and a rational exact period 4 point of $\phi_{k,b}(z)$.

Proposition 3.0.8. *Intersection of rational period 2 points of $\beta(z)$ and rational fixed points of $\phi_{k,b}(z)$ is nonempty if and only k, b, and c are of the form:*

$$k = \frac{q+p}{p}, \tag{3.26}$$

$$b = -qp, \tag{3.27}$$

$$c = -(p^2+p+1), \tag{3.28}$$

for some for some $p \in \mathbb{Q} \setminus \{0, -1/2\}$ and $q \in \mathbb{Q} \setminus \{0, -p\}$ where p is a rational period 2 point of $\beta(z)$ and a rational fixed point of $\phi_{k,b}(z)$.

Proof. Suppose $\beta(z)$ has a rational point of period 2. We know that $\beta(z)$ has a rational point of period 2 if and only if $c = -3/4 - \sigma^2$ for some $\sigma \in \mathbb{Q}^*$. In this case, there are exactly two, $-1/2 + \sigma$ and $-1/2 - \sigma$. [12]

Case 1 : Suppose $-1/2 + \sigma$ is a rational fixed point of $\phi_{k,b}(z)$ where $\sigma \neq 1/2$. We want to have

$$\Phi_1^*(-1/2 + \sigma) = 0$$

that is,

$$(k-1)(-1/2+\sigma)^2 + b = 0.$$

Such a ρ satisfies the following equation,

$$(k-1)/4 + b - (k-1)\sigma + (k-1)\sigma^2 = 0,$$

where $\sigma \neq 1/2$. In this case, σ has the following form,

$$\sigma = \frac{-1+k \pm 2\sqrt{b-bk}}{2(-1+k)},$$

whenever $k \neq 1$. Since σ is rational, we must have

$$\sqrt{b-bk} = q,$$

for some $q \in \mathbb{Q}$. As a result, if $-\frac{1}{2}+\sigma$ is a rational period 2 point of $\beta(z)$ and a rational fixed point of $\phi_{k,b}(z)$, then k, b and c are of the form,

$$k = \frac{2q+2\sigma-1}{2\sigma-1},$$

$$b = \frac{-q\cdot(-1+2\sigma)}{2},$$

$$c = -\frac{3}{4} - \sigma^2,$$

for some $\sigma, q \in \mathbb{Q}$ such that $\sigma \in \mathbb{Q} \setminus \{0, 1/2\}$ and $q \neq 0$. Now, it is enough to substitute $\sigma = p + 1/2$.

Case 2 : Suppose $-1/2-\sigma$ is a rational fixed point of $\phi_{k,b}(z)$ where $\sigma \neq 0$. In this case, σ has the following form,

$$\sigma = \frac{1-k \pm 2\sqrt{b-bk}}{2(-1+k)},$$

whenever $k \neq 1$. Hence, if $-\frac{1}{2}-\sigma$ is a rational period 2 point of $\beta(z)$ and a rational fixed point of $\phi_{k,b}(z)$, k, b and c are of the form,

$$k = \frac{2q+2\sigma+1}{2\sigma+1},$$

$$b = \frac{-q\cdot(1+2\sigma)}{2},$$

$$c = -\frac{3}{4} - \sigma^2,$$

for some $\sigma, q \in \mathbb{Q}$ such that $\sigma \in \mathbb{Q} \setminus \{0, -1/2\}$ and $q \neq 0$. Substituting $-\sigma$ in place of σ, and $-q$ in place of q, we get the same parametrization as in previous case. □

Example 3.0.9. *Let us take $p = 1/2$ and $q = 1$ in Proposition 3.0.8. By Equalities*

(3.26), (3.27), and (3.28) we get,

$$\beta(z) = z^2 - \frac{7}{4},$$

$$\phi_{k,b}(z) = 3z - \frac{1}{2z}.$$

Hence, $\frac{1}{2}$ is a rational period 2 point of $\beta(z)$ and a rational fixed point of $\phi_{k,b}(z)$.

Proposition 3.0.10. *Intersection of rational period 2 points of $\beta(z)$ and $\phi_{k,b}(z)$ is nonempty if and only if k, b, and c are of the form:*

$$k = \frac{q-p}{p}, \tag{3.29}$$

$$b = -qp, \tag{3.30}$$

$$c = -(p^2+p+1), \tag{3.31}$$

for some $p \in \mathbb{Q} \setminus \{0, -1/2\}$ and $q \in \mathbb{Q} \setminus \{0, p\}$ where p is a rational period 2 point of $\beta(z)$ and $\phi_{k,b}(z)$.

Proof. Suppose rational period 2 points $-1/2+\sigma$ and $-1/2-\sigma$ of $\beta(z)$ are roots of $\Phi_2^*(z) = (k^2+k)z^2 + kb$ where $\sigma \neq 0$.
Case 1: Suppose $-1/2+\sigma$ is a root of $\Phi_2^*(z) = (k^2+k)z^2+kb$ where $\sigma \in \mathbb{Q} \setminus \{0, 1/2\}$. This means,

$$(k^2+k)(-\frac{1}{2}+\sigma)^2 + kb = 0. \tag{3.32}$$

If we solve Equation (3.32) in σ, then we get

$$\sigma = \frac{1+k \pm 2\sqrt{-b(1+k)}}{2(1+k)},$$

whenever $k \neq -1$. Since $\sigma \in \mathbb{Q}$, we must have $\sqrt{-b(1+k)} = q$ for some $q \in \mathbb{Q}^+$. This yields

$$-\frac{1}{2} + \sigma = \frac{\pm q}{1+k}. \tag{3.33}$$

If we rearrange Equality (3.33), then we get

$$k = \frac{\pm 2q - 2\sigma + 1}{2\sigma - 1}.$$

Using $\sqrt{-b(1+k)} = q$ and Equality (3.33), we also have

$$b = \frac{\mp q \cdot (-1 + 2\sigma)}{2}.$$

Hence, if $-\frac{1}{2} + \sigma$ is a rational period 2 point of $\beta(z)$ and $\phi_{k,b}(z)$, then k, b, and c are of the form

$$k = \frac{2q - 2\sigma + 1}{2\sigma - 1},$$
$$b = \frac{-q \cdot (-1 + 2\sigma)}{2},$$
$$c = -\frac{3}{4} - \sigma^2,$$

for some $\sigma, q \in \mathbb{Q}$ such that $\sigma \in \mathbb{Q} \setminus \{0, \frac{1}{2}\}$ and $q \neq 0$. Note that, we have

$$\phi_{k,b}(-\frac{1}{2} + \sigma) = -(-\frac{1}{2} + \sigma) \neq -\frac{1}{2} + \sigma,$$

as $\sigma \neq 1/2$. We also have,

$$\beta(-\frac{1}{2} + \sigma) = -\frac{1}{2} - \sigma \neq -\frac{1}{2} + \sigma,$$

as $\sigma \neq 0$. Therefore, $-\frac{1}{2} + \sigma$ is a rational exact period 2 point of $\beta(z)$ and $\phi_{k,b}(z)$ for k, b, and c as above. Substituting $\sigma = p + 1/2$ completes this case.

Case 2: Suppose $-1/2 - \sigma$ is a root of $\Phi_2^*(z) = (k^2 + k)z^2 + kb$ where $\sigma \in \mathbb{Q} \setminus \{0, -1/2\}$. This means,

$$(k^2 + k)(-\frac{1}{2} - \sigma)^2 + kb = 0. \tag{3.34}$$

If we solve Equation (3.34) in σ, then we get

$$\sigma = \frac{-(1+k) \pm 2\sqrt{-b(1+k)}}{2(1+k)},$$

whenever $k \neq -1$. So, if $-\frac{1}{2} - \sigma$ is a rational period 2 point of $\beta(z)$ and $\phi_{k,b}(z)$, then

k, b, and c are of the form

$$k = \frac{2q-2\sigma-1}{2\sigma+1},$$

$$b = \frac{-q\cdot(1+2\sigma)}{2},$$

$$c = -\frac{3}{4} - \sigma^2,$$

for some $\sigma, q \in \mathbb{Q}$ such that $\sigma \in \mathbb{Q}\setminus\{0, -\frac{1}{2}\}$ and $q \neq 0$. As before, this parametrization is equivalent to previous one. $\square$

Example 3.0.11. *Let us take $p = 1$ and $q = -1$ in Proposition 3.0.10. By Equalities (3.29), (3.30), and (3.31) we get,*

$$\beta(z) = z^2 - 3,$$

$$\phi_{k,b}(z) = -2z + \frac{1}{z}.$$

In this case, 1 is a rational exact period 2 point of $\beta(z)$ and $\phi_{k,b}(z)$.

Proposition 3.0.12. *Intersection of rational period 2 points of $\beta(x)$ and rational period 4 points of $\phi_{k,b}(x)$ is nonempty if and only if k, b, and c are of the form:*

$$k = \frac{2m}{m^2-1}, \tag{3.35}$$

$$b = -\frac{p^2\cdot(m^2+1)}{m(m^2-1)}, \tag{3.36}$$

$$c = -(p^2+p+1), \tag{3.37}$$

where $p \in \mathbb{Q}\setminus\{0, -1/2\}$ and $m \in \mathbb{Q}\setminus\{0, \pm 1\}$. In this case, p is a rational exact period 2 point of $\beta(z)$ and a rational exact period 4 point $\phi_{k,b}(z)$.

Proof. Suppose rational period 2 points $-1/2+\sigma$ and $-1/2-\sigma$ of $\beta(z)$ are roots of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$ where $\sigma \neq 0$.

<u>Case 1</u>: Suppose $-1/2+\sigma$ is a root of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$ where $\sigma \in \mathbb{Q}\setminus\{0, 1/2\}$. This means,

$$b^2k + 2b(-1/2+\sigma)^2 + 2bk^2(-1/2+\sigma)^2 + k(-1/2+\sigma)^4 + k^3(-1/2+\sigma)^4 = 0. \tag{3.38}$$

If we solve Equation (3.38) in σ we get

$$\sigma_1 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 - 4\sqrt{b^2 + b^2k^2}}{k + k^3}}\right),$$

$$\sigma_2 = \frac{1}{2}\left(1 - \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 + 4\sqrt{b^2 + b^2k^2}}{k + k^3}}\right),$$

$$\sigma_3 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 - 4\sqrt{b^2 + b^2k^2}}{k + k^3}}\right),$$

$$\sigma_4 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{-4b - k - 4bk^2 - k^3 + 4\sqrt{b^2 + b^2k^2}}{k + k^3}}\right).$$

If we rearrange these solutions, without loss of generality we get our possible solutions in σ:

$$\sigma_1 = \frac{1}{2}\left(1 - \sqrt{\frac{-4b \cdot (1 + k^2 + \sqrt{1 + k^2})}{k \cdot (1 + k^2)}}\right), \tag{3.39}$$

$$\sigma_2 = \frac{1}{2}\left(1 - \sqrt{\frac{-4b \cdot (1 + k^2 - \sqrt{1 + k^2})}{k \cdot (1 + k^2)}}\right), \tag{3.40}$$

$$\sigma_3 = \frac{1}{2}\left(1 + \sqrt{\frac{-4b \cdot (1 + k^2 + \sqrt{1 + k^2})}{k \cdot (1 + k^2)}}\right), \tag{3.41}$$

$$\sigma_4 = \frac{1}{2}\left(1 + \sqrt{\frac{-4b \cdot (1 + k^2 - \sqrt{1 + k^2})}{k \cdot (1 + k^2)}}\right). \tag{3.42}$$

First let us consider σ_1. Let

$$s=\sqrt{\frac{-4b\cdot(1+k^2+\sqrt{1+k^2})}{k\cdot(1+k^2)}}, \tag{3.43}$$

where $s\neq 0,1$, and

$$q=\sqrt{1+k^2}. \tag{3.44}$$

Since $\sigma_1\in\mathbb{Q}$, we need to have $s\in\mathbb{Q}$. But then, this yields $q\in\mathbb{Q}$ by Equality (3.43). Now using Equality (3.44), we have

$$q=\frac{m^2+1}{m^2-1}, \tag{3.45}$$

$$k=\frac{2m}{m^2-1}, \tag{3.46}$$

for some $m\in\mathbb{Q}\setminus\{0,\pm1\}$. Now, using Equations (3.45) and (3.46) in Equation (3.43), we get

$$b=-\frac{s^2\cdot(m^2+1)}{4m(m^2-1)}.$$

By Equations (3.39) and (3.43) we also have,

$$-\frac{1}{2}+\sigma_1=-\frac{s}{2}.$$

Therefore, if $-\frac{s}{2}$ is a rational period 2 point of $\beta(z)$ and a rational period 4 point of $\phi_{k,b}(z)$, then k, b, and c are of the form

$$k=\frac{2m}{m^2-1},$$

$$b=-\frac{s^2\cdot(m^2+1)}{4m(m^2-1)},$$

$$c=-\frac{(s^2-2s+4)}{4},$$

where $s\in\mathbb{Q}\setminus\{0,1\}$ and $m\in\mathbb{Q}\setminus\{0,\pm1\}$. Since we have exactly the same parametrization for k and b as in Proposition 3.0.6, we can easily conclude that $-s/2$ is an exact period 4 point for $\phi_{k,b}(z)$. As an equivalent result, if p is a rational exact period 2 point of $\beta(z)$ and a rational exact period 4 point of $\phi_{k,b}(z)$, then k,

b, and c are of the form

$$k = \frac{2m}{m^2-1},$$

$$b = -\frac{p^2 \cdot (m^2+1)}{m(m^2-1)},$$

$$c = -(p^2+p+1),$$

where $p \in \mathbb{Q} \setminus \{0, -1/2\}$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. We also have,

$$\beta(p) = -p-1 \neq p,$$

as $p \neq -\frac{1}{2}$. Hence p is a rational exact period 2 point of $\beta(z)$. Note that if we consider Equation (3.41) we get exactly the same parametrization. Now let us consider Equation (3.40). Let

$$t = \sqrt{\frac{-4b \cdot (1+k^2-\sqrt{1+k^2})}{k \cdot (1+k^2)}},$$

where $t \neq 0, 1$, and

$$q = \sqrt{1+k^2}.$$

Since $\sigma_2 \in \mathbb{Q}$, we need to have $t \in \mathbb{Q}$. But then, this yields $q \in \mathbb{Q}$ by Equation (3.40). Note that, we have the same parametrizations for q and k as in Equations (3.45) and (3.46). So we get,

$$b = -\frac{t^2 \cdot m(m^2+1)}{4(m^2-1)}$$

where $t \in \mathbb{Q} \setminus \{0, 1\}$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. If we proceed as before, we see that if t is a rational period 2 point of $\beta(z)$ and a rational period 4 point of $\phi_{k,b}(z)$ then k, b, and c are of the form

$$k = \frac{2m}{m^2-1},$$

$$b = -\frac{t^2 \cdot m(m^2+1)}{(m^2-1)},$$

$$c = -(t^2+t+1),$$

where $t \in \mathbb{Q} \setminus \{0, -1/2\}$ and $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. Replacing m by $-1/m$ gives the previous parametrization. Note that if we consider Equation (3.42), then we get the same parametrization as well.

Case 2: This case gives the same parametrizations as in previous case. □

Example 3.0.13. *Let us take $m=3$ and $p=-1$ in Proposition 3.0.12. By Equalities (3.35), (3.36), and (3.37) we get,*

$$\beta(z) = z^2 - 1,$$

$$\phi_{k,b}(z) = \frac{3z}{4} - \frac{5}{12z}.$$

In this case, $p=-1$ is a rational exact period 2 point of $\beta(z)$ and a rational exact period 4 point of $\phi_{k,b}(z)$.

Proposition 3.0.14. *Let*

$$x_1 = \frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3-\tau-1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)},$$

be period 3 points of $\beta(z)$ where $\tau \in \mathbb{Q}$, $\tau \neq -1,0$. Then, if x_i is rational fixed point of $\phi_{k,b}(z)$, then k, b, and c are of the form

$$b = q \cdot x_i^2, \tag{3.47}$$

$$k = 1 - q, \tag{3.48}$$

$$c = -\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2}, \tag{3.49}$$

where $q \in \mathbb{Q}\setminus\{0,1\}$ and $\tau \in \mathbb{Q}\setminus\{-1,0\}$ with $i = 1,2,3$.

Proof. Suppose $\beta(z)$ has a rational point of period 3. By Poonen [12], $\beta(z)$ has a rational point of period 3 if and only if

$$c = -\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2},$$

for some $\tau \in \mathbb{Q}$, $\tau \neq -1, 0$. In this case, there are exactly three,

$$x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)},$$

and these are cyclically permuted by $\beta(z)$.

<u>Case 1</u> : Suppose x_1 is a rational fixed point of $\phi_{k,b}(z)$. So we want x_1 to be the root of the dynatomic polynomial $\Phi_1^*(z) = (k-1)z^2 + b$, that is, we want to have

$$\pm\sqrt{\frac{b}{1-k}} = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)},$$

where $k \neq 0, 1$, and $b \neq 0$. If we rearrange this equality we get

$$\frac{b}{1-k} = \frac{(\tau^3 + 2\tau^2 + \tau + 1)^2}{4\tau^2(\tau+1)^2}.$$

Therefore, b and k are of the form

$$b = q \cdot \frac{(\tau^3 + 2\tau^2 + \tau + 1)^2}{4\tau^2(\tau+1)^2},$$

$$k = 1 - q,$$

for some $q \in \mathbb{Q} \setminus \{0, 1\}$ and $\tau \in \mathbb{Q} \setminus \{-1, 0\}$.

<u>Case 2</u>: Suppose x_2 is a rational fixed point of $\phi_{k,b}(z)$. Proceeding as in previous case, we get

$$b = q \cdot \frac{(\tau^3 - \tau - 1)^2}{4\tau^2(\tau+1)^2},$$

$$k = 1 - q,$$

for some $q \in \mathbb{Q} \setminus \{0, 1\}$ and $\tau \in \mathbb{Q} \setminus \{-1, 0\}$.

<u>Case 3</u>: Suppose x_3 is a rational fixed point of $\phi_{k,b}(z)$. Proceeding as before, we get

$$b = q \cdot \frac{(\tau^3+2\tau^2+3\tau+1)^2}{4\tau^2(\tau+1)^2},$$

$$k = 1-q,$$

for some $q \in \mathbb{Q} \setminus \{0,1\}$ and $\tau \in \mathbb{Q} \setminus \{-1,0\}$.

Setting $x_i = p$ and replacing q by $-q/p$ gives the result for Theorem 3.0.1. $\square$

Example 3.0.15. *Let us take $\tau = 1$ and $q = 16$. Then $x_1 = 5/4$, $x_2 = -1/4$, and $x_3 = -7/4$. By Proposition 3.0.14; x_1, x_2 and x_3 are rational period 3 points of $\beta(z) = z^2 - 29/16$.*

Moreover, $x_1 = \frac{5}{4}$ is a rational fixed point of

$$\phi_{k,b}(z) = -15z + \frac{25}{z},$$

$x_2 = -\frac{1}{4}$ is a rational fixed point of

$$\phi_{k,b}(z) = -15z + \frac{1}{z},$$

$x_3 = -\frac{7}{4}$ is a rational fixed point of

$$\phi_{k,b}(z) = -15z + \frac{49}{z}.$$

Proposition 3.0.16. *Let*

$$x_1 = \frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3-\tau-1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)},$$

be period 3 points of $\beta(z)$ where $\tau \in \mathbb{Q}$, $\tau \neq -1, 0$. Then, if x_i is a rational period 2 point of $\phi_{k,b}(z)$, then k, b, and c are of the form

$$b = -q \cdot x_i^2,$$

$$k = q - 1,$$

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2},$$

where $q \in \mathbb{Q} \setminus \{0,1\}$ *and* $\tau \in \mathbb{Q} \setminus \{-1,0\}$.

Proof. Suppose rational period 3 points x_1, x_2 and x_3 of $\beta(z)$ are roots of $\Phi_2^*(z) = (k^2+k)z^2 + kb$.

<u>Case 1</u>: Suppose x_1 is a root of $\Phi_2^*(z) = (k^2+k)z^2 + kb$. So we want to have

$$\pm\sqrt{\frac{-b}{k+1}} = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}.$$

where $k \neq 0, -1$, and $b \neq 0$. If we rearrange this equality we get

$$\frac{b}{k+1} = \frac{-(\tau^3 + 2\tau^2 + \tau + 1)^2}{4\tau^2(\tau+1)^2}.$$

Therefore, k and b are of the form

$$b = -q \cdot \frac{(\tau^3 + 2\tau^2 + \tau + 1)^2}{4\tau^2(\tau+1)^2},$$

$$k = q - 1,$$

for some $q \in \mathbb{Q} \setminus \{0,1\}$ and $\tau \in \mathbb{Q} \setminus \{-1,0\}$. Note that, in this case

$$\phi_{k,b}\left(\frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}\right) = -\frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)} \neq \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)},$$

as $(\tau^3 + 2\tau^2 + \tau + 1) \neq 0$ for any $\tau \in \mathbb{Q}$. Therefore x_1 is a rational exact period 2 point of $\phi_{k,b}(z)$.

<u>Case 2</u>: Suppose x_2 is a root of $\Phi_2^*(z) = (k^2+k)z^2 + kb$. Proceeding as in previous case, k and b are of the form

$$b = -q \cdot \frac{(\tau^3 - \tau - 1)^2}{4\tau^2(\tau+1)^2},$$

$$k = q - 1,$$

for some $q \in \mathbb{Q} \setminus \{0,1\}$ and $\tau \in \mathbb{Q} \setminus \{-1,0\}$. Note that, in this case

$$\phi_{k,b}\left(\frac{\tau^3-\tau-1}{2\tau(\tau+1)}\right) = -\frac{\tau^3-\tau-1}{2\tau(\tau+1)} \neq \frac{\tau^3-\tau-1}{2\tau(\tau+1)},$$

as $(\tau^3-\tau-1) \neq 0$ for any $\tau \in \mathbb{Q}$. Therefore x_2 is a rational exact period 2 point of $\phi_{k,b}(z)$.

Case 3: Suppose x_3 is a root of $\Phi_2^*(z) = (k^2+k)z^2+kb$. We get

$$b = -q \cdot \frac{(\tau^3+2\tau^2+3\tau+1)^2}{4\tau^2(\tau+1)^2},$$

$$k = q-1,$$

for some $q \in \mathbb{Q} \setminus \{0,1\}$ and $\tau \in \mathbb{Q} \setminus \{-1,0\}$. In this case,

$$\phi_{k,b}\left(-\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)}\right) = \frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)} \neq -\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)},$$

as $(\tau^3+2\tau^2+3\tau+1) \neq 0$ for any $\tau \in \mathbb{Q}$. Therefore x_3 is a rational exact period 2 point of $\phi_{k,b}(z)$.

Setting $x_i = p$ and replacing q by q/p gives the result for Theorem 3.0.1. □

Example 3.0.17. *Let us take $\tau = 1/2$ and $q = 9$. Then $x_1 = 17/12$, $x_2 = -11/12$, and $x_3 = -25/12$. By Proposition 3.0.16; x_1, x_2 and x_3 are rational period 3 points of $\beta(z) = z^2 - 421/144$.*

Moreover, $x_1 = \frac{17}{12}$ is a rational exact period 2 point of

$$\phi_{k,b}(z) = 8z - \frac{289}{16z},$$

$x_2 = -\frac{11}{12}$ is a rational exact period 2 point of

$$\phi_{k,b}(z) = 8z - \frac{121}{16z},$$

$x_3 = -\frac{25}{12}$ is a rational exact period 2 point of

$$\phi_{k,b}(z) = 8z - \frac{625}{16z}.$$

Proposition 3.0.18. *Let*

$$x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)},$$

be period 3 *points of* $\beta(z)$ *where* $\tau \in \mathbb{Q}$, $\tau \neq -1, 0$. *Then, if* x_i *is a period* 4 *point of* $\phi_{k,b}(z)$*, then* k *and* b *are of the form,*

$$k = \frac{2m}{m^2 - 1},$$

$$b = -x_i^2 \cdot \frac{(m^2+1)}{m(m^2-1)},$$

where $m \in \mathbb{Q} \setminus \{0, \pm 1\}$.

Proof. Suppose rational period 3 points x_1, x_2 and x_3 of $\beta(z)$ are roots of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$ where b and k are nonzero. First of all, let us find the roots of $\Psi_4(z)$. The roots of $\Psi_4(z)$ are of the form,

$$z_1 = -\sqrt{-\frac{b}{k+k^3} - \frac{bk^2}{k+k^3} - \frac{\sqrt{b^2(1+k^2)}}{k+k^3}},$$

$$z_2 = \sqrt{-\frac{b}{k+k^3} - \frac{bk^2}{k+k^3} - \frac{\sqrt{b^2(1+k^2)}}{k+k^3}},$$

$$z_3 = -\sqrt{-\frac{b}{k+k^3} - \frac{bk^2}{k+k^3} + \frac{\sqrt{b^2(1+k^2)}}{k+k^3}},$$

$$z_4 = \sqrt{-\frac{b}{k+k^3} - \frac{bk^2}{k+k^3} + \frac{\sqrt{b^2(1+k^2)}}{k+k^3}}.$$

For $b > 0$, we have the following general structure,

$$z^*_{1,2} = \pm\sqrt{\frac{-b}{k}\left(1 + \frac{1}{\sqrt{1+k^2}}\right)},$$

$$z^*_{3,4} = \pm\sqrt{\frac{-b}{k}\left(1 - \frac{1}{\sqrt{1+k^2}}\right)}.$$

Since for $b<0$, we also have the same structure; for a nonzero b, the roots of $\Psi_4(z)$ have the above general structure.
Now, suppose $x_1 = z^*_{1,2}$. Then we get,

$$x_1 = \pm\sqrt{\frac{-b}{k}\left(1+\frac{1}{\sqrt{1+k^2}}\right)}.$$

Taking square of both sides of this equality yields,

$$x_1^2 = \frac{-b}{k}\left(1+\frac{1}{\sqrt{1+k^2}}\right). \tag{3.50}$$

Since $x_1 \in \mathbb{Q}$, we must have

$$\sqrt{1+k^2} = q, \tag{3.51}$$

for some $q \in \mathbb{Q}^*$. Therefore, we get

$$q = \frac{m^2+1}{m^2-1}, \tag{3.52}$$

$$k = \frac{2m}{m^2-1}, \tag{3.53}$$

for some $m \in \mathbb{Q}\setminus\{0,\pm 1\}$. If we use, Equalities (3.51), (3.52), and (3.53) in Equality (3.50), we get

$$b = -x_1^2 \cdot \frac{(m^2+1)}{m(m^2-1)}.$$

Note that taking k and b as above we get,

$$\phi_{k,b}(x_1) = \frac{x_1}{m},$$

$$\phi^2_{k,b}(x_1) = -x_1,$$

$$\phi^3_{k,b}(x_1) = -\frac{x_1}{m},$$

$$\phi^4_{k,b}(x_1) = x_1.$$

Since $x_1 \neq 0$, and $m \in \mathbb{Q}\setminus\{0,\pm 1\}$, x_1 is a rational exact period 4 point of $\phi_{k,b}(z)$.

Note that setting $x_1 = p$ and replacing m by q gives the result for Theorem 3.0.1.

If we suppose $x_1 = z^*_{3,4}$, and apply the same process we get the same parametrization for k, and the following parametrization for b:

$$b = -x_1^2 \cdot \frac{m(m^2+1)}{(m^2-1)},$$

for some $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. Substituting $-1/m$ in place of m, we get the previous parametrization for b. $\square$

Example 3.0.19. *Let us take $\tau = -1/2$ and $m = 2$. Then $x_1 = -7/4$, $x_2 = 5/4$, and $x_3 = -1/4$. By Proposition 3.0.18; x_1, x_2 and x_3 are rational period 3 points of $\beta(z) = z^2 - 29/16$.*
Moreover, $x_1 = -\frac{7}{4}$ is a rational exact period 4 point of

$$\phi_{k,b}(z) = \frac{4z}{3} - \frac{245}{96z},$$

$x_2 = \frac{5}{4}$ is a rational exact period 4 point of

$$\phi_{k,b}(z) = \frac{4z}{3} - \frac{125}{96z},$$

$x_3 = -\frac{1}{4}$ is a rational exact period 4 point of

$$\phi_{k,b}(z) = \frac{4z}{3} - \frac{5}{96z}.$$

Now we will try to answer the following question: Can we find the triples (k,b,c) such that β and $\phi_{k,b}$ satisfy the following: There exists a rational periodic point p such that

$$|Orb_\beta(p) \cap Orb_{\phi_{k,b}}(p)| \geqslant 2.$$

With the following proposition we see that the answer is positive.

Proposition 3.0.20. *Let $\beta(z)$ and $\phi_{k,b}(z)$ as before. Suppose that possible period lengths of rational points of $\beta(z)$ are $1, 2,$ or 3, besides the period length of a rational periodic point of $\phi_{k,b}$ is either $1, 2,$ or 4. If there exists a rational periodic point s such that*

$$|Orb_\beta(s) \cap Orb_{\phi_{k,b}}(s)| \geqslant 2,$$

then the triples (k,b,c) are in one of the following forms:

$$(k,b,c) = \left(\frac{2s(s+1)}{2s+1}, -\frac{s(s+1)[s^2+(s+1)^2]}{2s+1}, -(s^2+s+1)\right)$$

$$(k,b,c)=\left(-\frac{2s(s+1)}{2s+1},\frac{s(s+1)[s^2+(s+1)^2]}{2s+1},-(s^2+s+1)\right)$$

where $s\in\mathbb{Q}\setminus\{0,-1/2,-1\}$, *and* $Orb_\beta(s)\cap Orb_{\phi_{k,b}}(s)=\{s,-s-1\}$,

$$(k,b,c)=\left(\frac{2m}{m^2-1},\ -x_1^2\cdot\frac{(m^2+1)}{m\cdot(m^2-1)},-\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2}\right)$$

where $m=\pm\frac{\tau^3+2\tau^2+\tau+1}{\tau^3-\tau-1}$, *and* $Orb_\beta(s)\cap Orb_{\phi_{k,b}}(s)=\{x_1(\tau),x_2(\tau)\}$ *with* $s\in\{x_1(\tau),x_2(\tau)\}$,

$$(k,b,c)=\left(\frac{2m}{m^2-1},\ -x_1^2\cdot\frac{(m^2+1)}{m\cdot(m^2-1)},-\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2}\right)$$

where $m=\pm\frac{\tau^3+2\tau^2+\tau+1}{\tau^3+2\tau^2+3\tau+1}$, *and* $Orb_\beta(s)\cap Orb_{\phi_{k,b}}(s)=\{x_1(\tau),x_3(\tau)\}$ *with* $s\in\{x_1(\tau),x_3(\tau)\}$,

$$(k,b,c)=\left(\frac{2m}{m^2-1},\ -x_2^2\cdot\frac{(m^2+1)}{m\cdot(m^2-1)},-\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2}\right)$$

where $m=\mp\frac{\tau^3-\tau-1}{\tau^3+2\tau^2+3\tau+1}$, *and* $Orb_\beta(s)\cap Orb_{\phi_{k,b}}(s)=\{x_2(\tau),x_3(\tau)\}$ *with* $s\in\{x_2(\tau),x_3(\tau)\}$, *and*

$$x_1(\tau)=\frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)}$$

$$x_2(\tau)=\frac{\tau^3-\tau-1}{2\tau(\tau+1)}$$

$$x_3(\tau)=-\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)}$$

for some $\tau\in\mathbb{Q}$, $\tau\neq-1,0$. *Moreover, cardinality of the set*

$$Orb_\beta(s)\cap Orb_{\phi_{k,b}}(s)$$

cannot be more than 2.

Proof. Case 1: Suppose s is a rational period 2 point of β and $\phi_{k,b}$. So we have the following cycles

$$(s,-s-1)\text{ and }(s,-s)$$

for β and $\phi_{k,b}$, respectively. But

$$-s-1 \neq -s.$$

In this case, the cardinality of the intersection is 1.

Case 2: Suppose s is a rational period 2 point of β and a rational period 4 point of $\phi_{k,b}$. So we have the following cycles

$$(s,-s-1) \text{ and } \left(s,\frac{s}{m},-s,-\frac{s}{m}\right)$$

for β and $\phi_{k,b}$, respectively. We have,

$$-s-1=\frac{s}{m} \text{ if and only if } m=-\frac{s}{s+1}, \text{ and}$$

$$-s-1=-\frac{s}{m} \text{ if and only if } m=\frac{s}{s+1}.$$

Hence,

$$|\{s,-s-1\}\cap\{s,\frac{s}{m},-s,-\frac{s}{m}\}|=2, \text{ with}$$

$$(k,b,c)=\left(\frac{2m}{m^2-1},-s^2\cdot\frac{(m^2+1)}{m\cdot(m^2-1)},-(s^2+s+1)\right),$$

where $m=-\frac{s}{s+1}$ or $m=\frac{s}{s+1}$. This yields the following triples:

$$(k,b,c)=\left(\frac{2s(s+1)}{2s+1},-\frac{s(s+1)[s^2+(s+1)^2]}{2s+1},-(s^2+s+1)\right),$$

$$(k,b,c)=\left(-\frac{2s(s+1)}{2s+1},\frac{s(s+1)[s^2+(s+1)^2]}{2s+1},-(s^2+s+1)\right).$$

where $s\in\mathbb{Q}\setminus\{0,-1/2,-1\}$.

Case 3: Suppose $\beta(z)$ has a rational point of period 3. Then, $\beta(z)$ has a rational point of period 3 if and only if

$$c=-\frac{\tau^6+2\tau^5+4\tau^4+8\tau^3+9\tau^2+4\tau+1}{4\tau^2(\tau+1)^2},$$

for some $\tau\in\mathbb{Q}$, $\tau\neq-1,0$. In this case, there are exactly three,

$$x_1=\frac{\tau^3+2\tau^2+\tau+1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)},$$

and these are cyclically permuted by $\beta(z)$ [12]. Suppose x_1 is a rational period 2 point of $\phi_{k,b}(z)$. Now consider, $\{x_1, x_2, x_3\} \cap \{x_1, -x_1\}$. Since $\tau \neq 0, -1$, we get $-x_1 \neq x_2$, and $-x_1 \neq x_3$. Considering x_2 and x_3 to be the period 2 points of $\phi_{k,b}$ give the same result: The cardinality of the intersection is 1, for this case.

Case 4: Suppose x_1 is a rational period 4 point of $\phi_{k,b}(z)$. Consider,

$$\{x_1, x_2, x_3\} \cap \{x_1, \frac{x_1}{m}, -x_1, -\frac{x_1}{m}\},$$

$$\text{for } k = \frac{2m}{m^2-1},\ b = -x_1^2 \cdot \frac{(m^2+1)}{m \cdot (m^2-1)}.$$

We know that the cases $-x_1 = x_2$ and $-x_1 = x_3$ are not possible. Note also that

$$x_2 = \frac{x_1}{m} \text{ and } x_3 = -\frac{x_1}{m};$$

cannot happen simultaneously. Therefore

$$|\{x_1, x_2, x_3\} \cap \{x_1, \frac{x_1}{m}, -x_1, -\frac{x_1}{m}\}| = 2,$$

if the following cases hold:

$$x_2 = \frac{x_1}{m} \text{ if and only if } m = \frac{\tau^3 + 2\tau^2 + \tau + 1}{\tau^3 - \tau - 1},$$

$$x_2 = -\frac{x_1}{m} \text{ if and only if } m = -\frac{\tau^3 + 2\tau^2 + \tau + 1}{\tau^3 - \tau - 1},$$

$$x_3 = -\frac{x_1}{m} \text{ if and only if } m = \frac{\tau^3 + 2\tau^2 + \tau + 1}{\tau^3 + 2\tau^2 + 3\tau + 1},$$

$$x_3 = \frac{x_1}{m} \text{ if and only if } m = -\frac{\tau^3 + 2\tau^2 + \tau + 1}{\tau^3 + 2\tau^2 + 3\tau + 1}.$$

Suppose x_2 is a rational period 4 point of $\phi_{k,b}(z)$. Consider,

$$\{x_1, x_2, x_3\} \cap \{x_2, \frac{x_2}{m}, -x_2, -\frac{x_2}{m}\},$$

$$\text{for } k = \frac{2m}{m^2-1},\ b = -x_2^2 \cdot \frac{(m^2+1)}{m \cdot (m^2-1)}.$$

We know that the cases $-x_2 = x_1$ and $-x_2 = x_3$ are not possible. Note also that

$$x_1 = \frac{x_2}{m} \text{ and } x_3 = -\frac{x_2}{m};$$

cannot happen simultaneously. Therefore

$$|\{x_1, x_2, x_3\} \cap \{x_2, \frac{x_2}{m}, -x_2, -\frac{x_2}{m}\}| = 2,$$

if the following cases hold:

$$x_1 = \frac{x_2}{m} \text{ if and only if } m = \frac{\tau^3 - \tau - 1}{\tau^3 + 2\tau^2 + \tau + 1},$$

$$x_1 = -\frac{x_2}{m} \text{ if and only if } m = -\frac{\tau^3 - \tau - 1}{\tau^3 + 2\tau^2 + \tau + 1},$$

$$x_3 = \frac{x_2}{m} \text{ if and only if } m = -\frac{\tau^3 - \tau - 1}{\tau^3 + 2\tau^2 + 3\tau + 1},$$

$$x_3 = -\frac{x_2}{m} \text{ if and only if } m = \frac{\tau^3 - \tau - 1}{\tau^3 + 2\tau^2 + 3\tau + 1}.$$

Suppose x_3 is a rational period 4 point of $\phi_{k,b}(z)$. Consider,

$$\{x_1, x_2, x_3\} \cap \{x_3, \frac{x_3}{m}, -x_3, -\frac{x_3}{m}\},$$

$$\text{for } k = \frac{2m}{m^2 - 1}, \ b = -x_3^2 \cdot \frac{(m^2 + 1)}{m \cdot (m^2 - 1)}.$$

We know that the cases $-x_3 = x_1$ and $-x_3 = x_2$ are not possible. Note also that

$$x_1 = \frac{x_3}{m} \text{ and } x_2 = -\frac{x_3}{m};$$

cannot happen simultaneously. Therefore

$$|\{x_1, x_2, x_3\} \cap \{x_3, \frac{x_3}{m}, -x_3, -\frac{x_3}{m}\}| = 2,$$

if the following cases hold:

$$x_1 = \frac{x_3}{m} \text{ if and only if } m = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{\tau^3 + 2\tau^2 + \tau + 1},$$

$$x_1 = -\frac{x_3}{m} \text{ if and only if } m = \frac{\tau^3 + 2\tau^2 + 3\tau + 1}{\tau^3 + 2\tau^2 + \tau + 1},$$

$$x_2 = \frac{x_3}{m} \text{ if and only if } m = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{\tau^3 - \tau - 1},$$

$$x_2 = -\frac{x_3}{m} \text{ if and only if } m = \frac{\tau^3+2\tau^2+3\tau+1}{\tau^3-\tau-1}.$$

Note that from these twelve m's we get only six $\phi_{k,b}$ s. More precisely, taking $m = x_i/x_j$ for

$$k = \frac{2m}{m^2-1},\ b = -x_i^2 \cdot \frac{(m^2+1)}{m\cdot(m^2-1)}, \text{ and}$$

taking $m = -x_j/x_i$ for

$$k = \frac{2m}{m^2-1},\ b = -x_j^2 \cdot \frac{(m^2+1)}{m\cdot(m^2-1)}.$$

give the same $\phi_{k,b}(z)$. □

Example 3.0.21. *For each triple in Proposition 3.0.20, taking following values respectively, we get the following maps and cycles:*

Take $s = 3$. Then, for

$$\beta(z) = z^2 - 13 \textit{ and } \phi_{k,b}(z) = \frac{24z}{7} - \frac{300}{7z},$$

we have the following cycles, respectively:

$$(3,-4) \textit{ and } (3,-4,-3,4)$$

For

$$\beta(z) = z^2 - 13 \textit{ and } \phi_{k,b}(z) = -\frac{24z}{7} + \frac{300}{7z},$$

we have the following cycles, respectively:

$$(3,-4) \textit{ and } (3,4,-3,-4)$$

Take $\tau = 2$. Then, for

$$\beta(z) = z^2 - \frac{301}{144} \textit{ and } \phi_{k,b}(z) = \frac{95z}{168} - \frac{18335}{24192z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12},\frac{5}{12},-\frac{23}{12}\right) \textit{ and } \left(\frac{19}{12},\frac{5}{12},-\frac{19}{12},-\frac{5}{12}\right).$$

For

$$\beta(z) = z^2 - \frac{301}{144} \textit{ and } \phi_{k,b}(z) = -\frac{95z}{168} + \frac{18335}{24192z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12}, \frac{5}{12}, -\frac{23}{12}\right) \text{ and } \left(\frac{19}{12}, -\frac{5}{12}, -\frac{19}{12}, \frac{5}{12}\right).$$

For

$$\beta(z) = z^2 - \frac{301}{144} \text{ and } \phi_{k,b}(z) = -\frac{437z}{84} + \frac{194465}{12096z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12}, \frac{5}{12}, -\frac{23}{12}\right) \text{ and } \left(\frac{19}{12}, \frac{23}{12}, -\frac{19}{12}, -\frac{23}{12}\right).$$

For

$$\beta(z) = z^2 - \frac{301}{144} \text{ and } \phi_{k,b}(z) = \frac{437z}{84} - \frac{194465}{12096z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12}, \frac{5}{12}, -\frac{23}{12}\right) \text{ and } \left(\frac{19}{12}, -\frac{23}{12}, -\frac{19}{12}, \frac{23}{12}\right).$$

For

$$\beta(z) = z^2 - \frac{301}{144} \text{ and } \phi_{k,b}(z) = \frac{115z}{252} - \frac{31855}{36288z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12}, \frac{5}{12}, -\frac{23}{12}\right) \text{ and } \left(\frac{5}{12}, -\frac{23}{12}, -\frac{5}{12}, \frac{23}{12}\right).$$

For

$$\beta(z) = z^2 - \frac{301}{144} \text{ and } \phi_{k,b}(z) = -\frac{115z}{252} + \frac{31855}{36288z},$$

we have the following cycles, respectively:

$$\left(\frac{19}{12}, \frac{5}{12}, -\frac{23}{12}\right) \text{ and } \left(\frac{5}{12}, \frac{23}{12}, -\frac{5}{12}, -\frac{23}{12}\right).$$

4. Simultaneous Rational Periodic Points of ϕ_{k_1,b_1} and ϕ_{k_2,b_2}

The aim of this chapter is to characterize rational periodic points of $\phi_{k,b}$ and to find families (k_1,b_1,k_2,b_2) for some parameters related to common rational periodic points of the maps $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$ where

$$\phi_{k_1,b_1}(z) = k_1 z + \frac{b_1}{z}, \phi_{k_2,b_2}(z) = k_2 z + \frac{b_2}{z} \text{ with } k_1,b_1,k_2,b_2 \in \mathbb{Q}^*.$$

Furthermore, we will find four tuples (k_1,b_1,k_2,b_2) such that ϕ_{k_1,b_1} and ϕ_{k_2,b_2} satisfy the following: There exists a rational periodic point p such that

$$|Orb_{\phi_{k_1,b_1}}(p) \cap Orb_{\phi_{k_2,b_2}}(p)| \geqslant 2.$$

The following theorem gives the characterization of rational periodic points of $\phi_{k,b}(z)$ for periods 1, 2, and 4.

Theorem 4.0.1. *Let $p \in \mathbb{Q}^*$ be a periodic point of $\phi_{k,b}(z)$. Suppose that possible period lengths of rational points of $\phi_{k,b}$ is either $1, 2,$ or 4. Then we have three families of one parameter maps in which (k,b) is in one of the following forms:*

$$(k,b) = ((1-s),(s \cdot p^2)) \text{ for some } s \in \mathbb{Q} \setminus \{0,1\}:$$

In this case, $p, -p$ are the only rational fixed points of $\phi_{k,b}$.

$$(k,b) = ((s-1),(-s \cdot p^2)) \text{ for some } s \in \mathbb{Q} \setminus \{0,1\}:$$

In this case, $(p,-p)$ is the only 2-cycle of $\phi_{k,b}$.

$$(k,b) = (\frac{2s}{(s^2-1)}, -\frac{(s^2+1)}{s \cdot (s^2-1)} p^2) \text{ for some } s \in \mathbb{Q} \setminus \{0,\pm 1\}:$$

In this case, $(p, \frac{p}{s}, -p, -\frac{p}{s})$ is the only 4-cycle of $\phi_{k,b}$. In particular, there are infinitely many $\phi_{k,b}(z)$ sharing the same rational periodic point p.

Proof. Let $p \in \mathbb{Q}^*$ be a rational fixed point of $\phi_{k,b}(z)$. We want p to be a root of the dynatomic polynomial $\Phi_1^*(z) = (k-1)z^2 + b$. So, we need to have

$$\pm\sqrt{\frac{b}{1-k}} = p,$$

where $k \in \mathbb{Q} \setminus \{0,1\}$, $b \in \mathbb{Q}^*$, that is,

$$\frac{b}{1-k} = p^2.$$

Hence, if p is a rational fixed point of $\phi_{k,b}(z)$, then k and b are of the form

$$b = s \cdot p^2,$$

$$k = 1 - s,$$

for some $s \in \mathbb{Q} \setminus \{0,1\}$.

Let $p \in \mathbb{Q}^*$ be a rational period 2 point of $\phi_{k,b}(z)$. Now, we want p to be a root of the dynatomic polynomial

$$\Phi_2^*(z) = (k^2 + k)z^2 + kb.$$

So, we need to have

$$\pm\sqrt{\frac{-b}{1+k}} = p$$

where $k \in \mathbb{Q} \setminus \{0,-1\}$, $b \in \mathbb{Q}^*$, that is,

$$\frac{-b}{1+k} = p^2.$$

Hence, if p is a period 2 point of $\phi_{k,b}(z)$, then k and b are of the form

$$b = -s \cdot p^2,$$

$$k = s - 1,$$

for some $s \in \mathbb{Q} \setminus \{0,1\}$. In this case, we have

$$\phi_{k,b}(p) = -p.$$

Since $p \neq 0$, $p \neq -p$. This means p is a rational exact period 2 point of $\phi_{k,b}(z)$.

Let $p \in \mathbb{Q}^*$ be a rational period 4 point of $\phi_{k,b}(z)$. Now, we want p to be a root of $\Psi_4(z) = b^2k + 2bz^2 + 2bk^2z^2 + kz^4 + k^3z^4$. As we mentioned earlier, the roots of $\Psi_4(z)$ have the following general structure,

$$z^*_{1,2} = \pm\sqrt{\frac{-b}{k}\left(1+\frac{1}{\sqrt{1+k^2}}\right)},$$

$$z^*_{3,4} = \pm\sqrt{\frac{-b}{k}\left(1-\frac{1}{\sqrt{1+k^2}}\right)},$$

where b and k are nonzero. So, we have two cases, either $p = z^*_{1,2}$ or $p = z^*_{3,4}$. Taking $q = \sqrt{1+k^2}$, we have the same parametrizations for q and k as in Equations (3.52) and (3.53). Now if $p = z^*_{1,2}$, we have

$$p = \pm\sqrt{\frac{-b}{k}\left(1+\frac{1}{\sqrt{1+k^2}}\right)}.$$

If we take q and k as in Equations (3.52) and (3.53) in this equation, we get

$$p^2 = -b\frac{(m^2-1)m}{(m^2+1)},$$

where $m \in \mathbb{Q}\setminus\{0,\pm1\}$. If $p = z^*_{3,4}$, we have

$$p = \pm\sqrt{\frac{-b}{k}\left(1-\frac{1}{\sqrt{1+k^2}}\right)}.$$

If we take q and k as in Equations (3.52) and (3.53) in this equation, we get

$$p^2 = -b\frac{(m^2-1)}{m(m^2+1)},$$

where $m \in \mathbb{Q}\setminus\{0,\pm1\}$. Therefore, if p is a period 4 point of $\phi_{k,b}(z)$, then k and b have one of the following forms; either

$$k = \frac{2m}{m^2-1},$$

$$b^* = -p^2\cdot\frac{(m^2+1)}{m(m^2-1)},$$

or

$$k = \frac{2m}{m^2-1},$$

$$b^{**} = -p^2 \cdot \frac{m(m^2+1)}{(m^2-1)}, \tag{4.1}$$

where $m \in \mathbb{Q} \setminus \{0, \pm 1\}$. Replacing m by $-1/m$ in Equality (4.1), we get b^*. Note that using k and b^*, we get

$$\phi_{k,b}(p) = \frac{p}{m},$$
$$\phi_{k,b}^2(p) = -p,$$
$$\phi_{k,b}^3(p) = -\frac{p}{m},$$
$$\phi_{k,b}^4(p) = p.$$

Since $m \in \mathbb{Q} \setminus \{0, \pm 1\}$, we guarantee that p is a rational exact period 4 point of $\phi_{k,b}(z)$. $\square$

The following table in which we see families (k_1, b_1, k_2, b_2) such that the maps $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$ having a simultaneous rational periodic point p is an immediate corollary of Theorem 4.0.1 where

$$\phi_{k_1,b_1}(z) = k_1 z + \frac{b_1}{z}, \phi_{k_2,b_2}(z) = k_2 z + \frac{b_2}{z} \text{ with } k_1, b_1, k_2, b_2 \in \mathbb{Q}^*.$$

Table 4.1 Simultaneous Rational Periodic Points of ϕ_{k_1,b_1} and ϕ_{k_2,b_2}

(k_1, b_1)	Periodic Points	L	(k_2, b_2)	Periodic Points	L
$(1-s_1, s_1 \cdot p^2)$	$p, -p$	1	$(1-s_2, s_2 \cdot p^2)$	$p, -p$	1
$(s_1-1, -s_1 \cdot p^2)$	$p, -p$	2	$(s_2-1, -s_2 \cdot p^2)$	$p, -p$	2
$(\frac{2s_1}{s_1^2-1}, -\frac{p^2(s_1^2+1)}{s_1(s_1^2-1)})$	$p, \frac{p}{s_1}, -p, -\frac{p}{s_1}$	4	$(\frac{2s_2}{s_2^2-1}, -\frac{p^2(s_2^2+1)}{s_2(s_2^2-1)})$	$p, \frac{p}{s_2}, -p, -\frac{p}{s_2}$	4
$(1-s_1, s_1 \cdot p^2)$	$p, -p$	1	$(s_2-1, -s_2 \cdot p^2)$	$p, -p$	2
$(1-s_1, s_1 \cdot p^2)$	$p, -p$	1	$(\frac{2s_2}{s_2^2-1}, -\frac{p^2(s_2^2+1)}{s_2(s_2^2-1)})$	$p, \frac{p}{s_2}, -p, -\frac{p}{s_2}$	4
$(s_1-1, -s_1 \cdot p^2)$	$p, -p$	2	$(\frac{2s_2}{s_2^2-1}, -\frac{p^2(s_2^2+1)}{s_2(s_2^2-1)})$	$p, \frac{p}{s_2}, -p, -\frac{p}{s_2}$	4

Note that L denotes the period length of the periodic points in the table.

Remark 4.0.2. *Let p be a rational fixed point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Then, k_1, b_1, k_2 and b_2 are as in the table for some $s_1, s_2 \in \mathbb{Q} \setminus \{0, 1\}$. Note that $s_1 \neq s_2$ implies that $k_1 \neq k_2$ and $b_1 \neq b_2$.*

Example 4.0.3. *Let $p=\frac{4}{5}$ be a rational fixed point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Let us take $s_1=\frac{25}{8}$ and $s_2=50$. Then, we get*

$$\phi_{k_1,b_1}=-\frac{17z}{8}+\frac{2}{z},\ \phi_{k_2,b_2}=-49z+\frac{32}{z}\ \textit{with cyles}$$

$$(\frac{4}{5}),(-\frac{4}{5}).$$

Remark 4.0.4. *Let p be a rational fixed point of $\phi_{k_1,b_1}(z)$ and a rational exact period 2 point of $\phi_{k_2,b_2}(z)$. Then, k_1,b_1,k_2 and b_2 are as in the table for some $s_1,s_2\in\mathbb{Q}\setminus\{0,1\}$. In this case, $k_1\neq k_2$ or $b_1\neq b_2$. Suppose not, that is, suppose $k_1=k_2$ and $b_1=b_2$. Then, we get*

$$1-s_1=s_2-1, \tag{4.2}$$

$$s_1\cdot p^2=-s_2\cdot p^2. \tag{4.3}$$

Since $p\neq 0$, by Equation (4.3) we get $s_1=-s_2$. Using this in Equation (4.2) yields $2=0$, a contradiction.

Example 4.0.5. *Let $p=\frac{2}{3}$ be a rational fixed point of $\phi_{k_1,b_1}(z)$ and a rational exact period 2 point of $\phi_{k_2,b_2}(z)$. Let us take $s_1=\frac{9}{2}$ and $s_2=9$. Then, we get*

$$\phi_{k_1,b_1}=-\frac{7z}{2}+\frac{2}{z}\ \textit{with cycles}\ (\frac{2}{3}),(-\frac{2}{3}),$$

$$\phi_{k_2,b_2}=8z-\frac{4}{z}\ \textit{with the cyle}\ (\frac{2}{3},-\frac{2}{3}).$$

Remark 4.0.6. *Let p be a rational fixed point of $\phi_{k_1,b_1}(z)$ and a rational exact period 4 point of $\phi_{k_2,b_2}(z)$. Then, k_1,b_1,k_2 and b_2 are as in the table where $s_1\in\mathbb{Q}\setminus\{0,1\}$ and $s_2\in\mathbb{Q}\setminus\{0,\pm 1\}$. In this case, $k_1\neq k_2$ or $b_1\neq b_2$. Suppose on the contrary. Then,*

$$1-s_1=\frac{2s_2}{s_2^2-1},$$

$$s_1\cdot p^2=-p^2\cdot\frac{(s_2^2+1)}{s_2(s_2^2-1)}.$$

Using these equalities together yields

$$1-\frac{2s_2}{(s_2^2-1)}=-\frac{(s_2^2+1)}{s_2(s_2^2-1)}$$

and we get $s_2 = \pm 1$. This contradicts with the choice of s_2.

Example 4.0.7. *Let $p = \frac{3}{2}$ be a rational fixed point of $\phi_{k_1,b_1}(z)$ and a rational exact period 4 point of $\phi_{k_2,b_2}(z)$. Let us take $s_1 = 8$ and $s_2 = 2$. Then, we get*

$$\phi_{k_1,b_1} = -7z + \frac{18}{z} \text{ with cycles } (\frac{3}{2}), (-\frac{3}{2}),$$

$$\phi_{k_2,b_2} = \frac{4z}{3} - \frac{15}{8z} \text{ with the cycle } (\frac{3}{2}, \frac{3}{4}, -\frac{3}{2}, -\frac{3}{4}).$$

Remark 4.0.8. *Let p be a rational exact period 2 point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Then, k_1, b_1, k_2 and b_2 are as in the table for some $s_1, s_2 \in \mathbb{Q} \setminus \{0,1\}$. In this case, $s_1 \neq s_2$ implies that $k_1 \neq k_2$ and $b_1 \neq b_2$.*

Example 4.0.9. *Let $p = 3$ be a rational exact period 2 point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Let us take $s_1 = -1$ and $s_2 = 2$. Then, we get*

$$\phi_{k_1,b_1} = -2z + \frac{9}{z},\ \phi_{k_2,b_2} = z - \frac{18}{z} \text{ with the cycle}$$

$$(3, -3).$$

Remark 4.0.10. *Let p be a rational exact period 2 point of $\phi_{k_1,b_1}(z)$ and a rational exact period 4 point of $\phi_{k_2,b_2}(z)$. Then, k_1, b_1, k_2 and b_2 are as in the table where $s_1 \in \mathbb{Q} \setminus \{0,1\}$ and $s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$. In this case, $k_1 \neq k_2$ or $b_1 \neq b_2$. Suppose on the contrary. Then,*

$$s_1 - 1 = \frac{2s_2}{s_2^2 - 1},$$

$$-s_1 \cdot p^2 = -p^2 \cdot \frac{(s_2^2 + 1)}{s_2(s_2^2 - 1)}.$$

Using these equalities together yields

$$1 + \frac{2s_2}{s_2^2 - 1} = \frac{(s_2^2 + 1)}{s_2(s_2^2 - 1)}$$

and we get $s_2 = \pm 1$. This contradicts with the choice of s_2.

Example 4.0.11. *Let $p = -\frac{2}{5}$ be a rational exact period 2 point of $\phi_{k_1,b_1}(z)$ and a rational exact period 4 point of $\phi_{k_2,b_2}(z)$. Let us take $s_1 = \frac{25}{2}$ and $s_2 = \frac{1}{3}$. Then, we get*

$$\phi_{k_1,b_1} = \frac{23z}{2} - \frac{2}{z}, \text{ with the cycle } (-\frac{2}{5}, \frac{2}{5}),$$

$$\phi_{k_2,b_2} = -\frac{3z}{4} + \frac{3}{5z} \text{ with the cyle } (-\frac{2}{5}, -\frac{6}{5}, \frac{2}{5}, \frac{6}{5}).$$

Remark 4.0.12. *Let p be a rational exact period 4 point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Then, k_1, b_1, k_2 and b_2 are as in the table where $s_1, s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$. In this case, $s_1 \neq s_2$ implies that $k_1 \neq k_2$ or $b_1 \neq b_2$. Suppose $k_1 = k_2$ and $b_1 = b_2$. Then, we get*

$$\frac{s_1}{s_1^2 - 1} = \frac{s_2}{s_2^2 - 1}, \tag{4.4}$$

$$\frac{(s_1^2 + 1)}{s_1(s_1^2 - 1)} = \frac{(s_2^2 + 1)}{s_2(s_2^2 - 1)}. \tag{4.5}$$

By Equation (4.4), we get $s_1 = s_2$ or $s_1 s_2 = -1$. Now, let us use $s_1 s_2 = -1$ in Equation (4.5). This gives us $s_1 = \pm 1$, $s_2 = \pm 1$, but this is not the case by the choice of s_1, s_2.

Example 4.0.13. *Let $p = \frac{3}{5}$ be a rational exact period 4 point of $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$. Let us take $s_1 = 2$ and $s_2 = -2$. Then, we get*

$$\phi_{k_1,b_1} = \frac{4z}{3} - \frac{3}{10z} \text{ with the cycle } (\frac{3}{5}, \frac{3}{10}, -\frac{3}{5}, -\frac{3}{10}),$$

$$\phi_{k_2,b_2} = -\frac{4z}{3} + \frac{3}{10z} \text{ with the cyle } (\frac{3}{5}, -\frac{3}{10}, -\frac{3}{5}, \frac{3}{10}).$$

The following proposition gives conditions on the four tuples (k_1, b_1, k_2, b_2) such that there exists a rational periodic point p such that

$$|Orb_{\phi_{k_1,b_1}}(p) \cap Orb_{\phi_{k_2,b_2}}(p)| \geqslant 2.$$

Proposition 4.0.14. *Let*

$$\phi_{k_1,b_1}(z) = k_1 z + \frac{b_1}{z}, \ \ \phi_{k_2,b_2}(z) = k_2 z + \frac{b_2}{z}.$$

with $k_1, b_1, k_2, b_2 \in \mathbb{Q}^*$. Suppose that possible period lengths of ϕ_{k_1,b_1} and ϕ_{k_2,b_2} are either $1, 2,$ or 4. If there exists a rational periodic point p such that

$$|Orb_{\phi_{k_1,b_1}}(p) \cap Orb_{\phi_{k_2,b_2}}(p)| \geqslant 2,$$

then we have the following four tuples (k_1, b_1, k_2, b_2):

$$(k_1, b_1, k_2, b_2) = (s_1 - 1, -s_1 \cdot p^2, s_2 - 1, -s_2 \cdot p^2),$$

for some distinct $s_1, s_2 \in \mathbb{Q} \setminus \{0,1\}$ *where*

$$Orb(\phi_{k_1,b_1})(p) \cap Orb(\phi_{k_2,b_2})(p) = \{p, -p\}.$$

$$(k_1, b_1, k_2, b_2) = \left(s_1 - 1, -s_1 \cdot p^2, \frac{2s_2}{(s_2^2 - 1)}, -p^2 \cdot \frac{(s_2^2 + 1)}{s_2(s_2^2 - 1)}\right),$$

for some $s_1 \in \mathbb{Q} \setminus \{0,1\}$*, and* $s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$ *where*

$$Orb(\phi_{k_1,b_1})(p) \cap Orb(\phi_{k_2,b_2})(p) = \{p, -p\}.$$

$$(k_1, b_1, k_2, b_2) = \left(\frac{2s_1}{(s_1^2 - 1)}, -p^2 \cdot \frac{(s_1^2 + 1)}{s_1(s_1^2 - 1)}, \frac{2s_2}{(s_2^2 - 1)}, -p^2 \cdot \frac{(s_2^2 + 1)}{s_2(s_2^2 - 1)}\right),$$

where $s_1, s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$ *and*

$$Orb(\phi_{k_1,b_1})(p) \cap Orb(\phi_{k_2,b_2})(p) = \{p, -p\} \text{ for } s_1 \neq \pm s_2.$$

$$(k_1, b_1, k_2, b_2) = \left(\frac{2s_1}{(s_1^2 - 1)}, -p^2 \cdot \frac{(s_1^2 + 1)}{s_1(s_1^2 - 1)}, -\frac{2s_1}{(s_1^2 - 1)}, p^2 \cdot \frac{(s_1^2 + 1)}{s_1(s_1^2 - 1)}\right),$$

where $s_1 \in \mathbb{Q} \setminus \{0, \pm 1\}$ *and*

$$Orb(\phi_{k_1,b_1})(p) \cap Orb(\phi_{k_2,b_2})(p) = \{p, p/s_1, -p, -p/s_1\} \text{ for } s_1 = -s_2.$$

Moreover, $|Orb_{\phi_{k_1,b_1}}(p) \cap Orb_{\phi_{k_2,b_2}}(p)| = 4$ *implies that* $\phi_{k_1,b_1} = \pm \phi_{k_2,b_2}$.

Proof. Case 1:: Suppose p is a rational period 2 point of ϕ_{k_1,b_1} and ϕ_{k_2,b_2}. Then we have the following cycle for ϕ_{k_1,b_1} and ϕ_{k_2,b_2}:

$$(p, -p).$$

In this case, we have

$$(k_1, b_1, k_2, b_2) = (s_1 - 1, -s_1 \cdot p^2, s_2 - 1, -s_2 \cdot p^2),$$

for some distinct $s_1, s_2 \in \mathbb{Q} \setminus \{0,1\}$. Hence,

$$|Orb(\phi_{k_1,b_1})(p) \cap Orb(\phi_{k_2,b_2})(p)| = 2.$$

Case 2: Suppose p is a rational period 2 point of ϕ_{k_1,b_1} and a rational period 4 point of ϕ_{k_2,b_2}. Then we have the following cycles for ϕ_{k_1,b_1} and ϕ_{k_2,b_2}, respectively:

$$(p, -p), (p, \frac{p}{s_2}, -p, -\frac{p}{s_2}).$$

In this case, we have

$$(k_1,b_1,k_2,b_2) = \left(s_1-1, -s_1\cdot p^2, \frac{2s_2}{(s_2^2-1)}, -p^2\cdot\frac{(s_2^2+1)}{s_2(s_2^2-1)}\right),$$

for some $s_1 \in \mathbb{Q}\setminus\{0,1\}$, and $s_2 \in \mathbb{Q}\setminus\{0,\pm1\}$. Hence,

$$|Orb(\phi_{k_1,b_1})(p)\cap Orb(\phi_{k_2,b_2})(p)| = 2.$$

Case 3: Suppose p is a rational period 4 point of ϕ_{k_1,b_1} and ϕ_{k_2,b_2}. Then for

$$(k_1,b_1) = (\frac{2s_1}{(s_1^2-1)}, -p^2\cdot\frac{(s_1^2+1)}{s_1(s_1^2-1)}), \text{ and}$$

$$(k_2,b_2) = (\frac{2s_2}{(s_2^2-1)}, -p^2\cdot\frac{(s_2^2+1)}{s_2(s_2^2-1)}),$$

where $s_1,s_2 \in \mathbb{Q}\setminus\{0,\pm1\}$, we have the following cycles for ϕ_{k_1,b_1} and ϕ_{k_2,b_2}, respectively:

$$(p, \frac{p}{s_1}, -p, -\frac{p}{s_1}), (p, \frac{p}{s_2}, -p, -\frac{p}{s_2}).$$

If $s_1 = \pm s_2$, then

$$|Orb(\phi_{k_1,b_1})(p)\cap Orb(\phi_{k_2,b_2})(p)| = 4.$$

Otherwise,

$$|Orb(\phi_{k_1,b_1})(p)\cap Orb(\phi_{k_2,b_2})(p)| = 2.$$

□

Example 4.0.15. *Using the four tuples in Proposition 4.0.14, respectively, we get the following maps and cycles: Fix* $p=\frac{3}{5}$
Take $s_1 = 25$, $s_2 = 50$. *Then* ϕ_{k_1,b_1} *and* ϕ_{k_2,b_2} *have the following cycle*

$$(\frac{3}{5}, -\frac{3}{5}),$$

where

$$\phi_{k_1,b_1}(z) = 24z - \frac{9}{z},\ \phi_{k_2,b_2}(z) = 49z - \frac{18}{z}.$$

Take $s_1 = 25$, $s_2 = 1/3$. *Then* ϕ_{k_1,b_1} *and* ϕ_{k_2,b_2} *have the following cycles, respectively*

$$(\frac{3}{5}, -\frac{3}{5}),\ (\frac{3}{5}, \frac{9}{5}, -\frac{3}{5}, -\frac{9}{5})$$

where

$$\phi_{k_1,b_1}(z) = 24z - \frac{9}{z},\ \phi_{k_2,b_2}(z) = -\frac{3z}{4} + \frac{27}{20z}.$$

Take $s_1 = -1/3$, $s_2 = 1/3$. Then ϕ_{k_1,b_1} and ϕ_{k_2,b_2} have the following cycles, respectively

$$(\frac{3}{5}, -\frac{9}{5}, -\frac{3}{5}, \frac{9}{5}),\ (\frac{3}{5}, \frac{9}{5}, -\frac{3}{5}, -\frac{9}{5})$$

where

$$\phi_{k_1,b_1}(z) = \frac{3z}{4} - \frac{27}{20z},\ \phi_{k_2,b_2}(z) = -\frac{3z}{4} + \frac{27}{20z}.$$

Take $s_1 = 2$, $s_2 = 1/3$. Then ϕ_{k_1,b_1} and ϕ_{k_2,b_2} have the following cycles, respectively

$$(\frac{3}{5}, \frac{3}{10}, -\frac{3}{5}, -\frac{3}{10}),\ (\frac{3}{5}, \frac{9}{5}, -\frac{3}{5}, -\frac{9}{5})$$

where

$$\phi_{k_1,b_1}(z) = \frac{4z}{3} - \frac{3}{10z},\ \phi_{k_2,b_2}(z) = -\frac{3z}{4} + \frac{27}{20z}.$$

Proposition 4.0.16. *Let*

$$S = \{\phi_{k_1,b_1}(z), \phi_{k_2,b_2}(z), \ldots, \phi_{k_m,b_m}(z), \beta_1(z), \beta_2(z), \ldots, \beta_n(z)\}$$

where

$$\phi_{k_i,b_i}(z) = k_i z + \frac{b_i}{z}$$

for all $i \in \{1,2,\ldots,m\}$, and

$$\beta_j(z) = z^2 + c_j$$

for all $j \in \{1,2,\ldots,n\}$. Suppose that possible period lengths of rational points of $\beta_j(z)$'s are $1,2,$ or 3, and the period length of rational periodic points of $\phi_{k_i,b_i}(z)$'s are either $1,2,$ or 4. Assume every element of S share the same rational periodic point. Then, m can be chosen arbitrarily large, but n is at most 3.

Proof. First part of the proposition is clear. So it is enough to show that n is at most 3. Let $q \in \mathbb{Q}$.

Suppose q is a rational fixed point of $\beta(z) = z^2 + c$. Then c is $\frac{1}{4} - \rho^2$ for some $\rho \in \mathbb{Q}$. In this case q is either $1/2 + \rho$ or $1/2 - \rho$ [12], so that ρ is either $q - 1/2$ or $-(q - 1/2)$. In each case, we get one c.

Suppose q is a rational exact period 2 point of $\beta(z) = z^2 + c$. Then c is $-\frac{3}{4} - \sigma^2$ for some $\sigma \in \mathbb{Q}^*$. In this case q is either $-1/2 + \sigma$ or $-1/2 - \sigma$ [12], so that σ is either $q + 1/2$ or $-(q + 1/2)$. But in each case, we get one c.

Suppose q is a rational exact period 3 point of $\beta(z) = z^2 + c$. Then,

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2},$$

for some $\tau \in \mathbb{Q} \setminus \{-1, 0\}$. In this case q is one the following x_i's:

$$x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)},$$

$$x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)},$$

$$x_3 = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)},$$

where $\tau \in \mathbb{Q} \setminus \{0, -1\}$ [12]. First suppose that

$$q = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau+1)}.$$

Then, we get

$$\tau^3 + (2 - 2q)\tau^2 + (1 - 2q)\tau + 1 = 0.$$

This means τ is a root of the following polynomial:

$$x^3 + (2 - 2q)x^2 + (1 - 2q)x + 1. \tag{4.6}$$

We want to see whether this polynomial can have a rational root other than τ, or not. We know that for other roots x_1, x_2 of this polynomial we have

$$x_1 + x_2 = 2q - 2 - \tau, \;\; x_1 \cdot x_2 = -1/\tau.$$

Therefore x_1 and x_2 satisfy the following quadratic polynomial:

$$x^2 + (\tau + 2 - 2q)x - \frac{1}{\tau}.$$

This polynomial has rational roots if and only if the discriminant

$$(\tau + 2 - 2q)^2 + \frac{4}{\tau},$$

is a square in $\mathbb{Q}$. Using q in terms of τ, this is if and only if

$$\frac{\tau^4 + 6\tau^3 + 7\tau^2 + 2\tau + 1}{\tau^2(\tau+1)^2}$$

is a square in $\mathbb{Q}$. Now, consider the elliptic curve defined by

$$\tau^4+6\tau^3+7\tau^2+2\tau+1=y^2.$$

There is an implementation for Theorem 0.0.3 [17] in the software Magma [3] that allows us to find rational points on this elliptic curve. Using Magma [3] one can check that $(0,1),(0,-1),(-1,1),(-1,-1)$, and the two points at infinity are the only rational points on the elliptic curve. Since $\tau \neq 0,-1$, the discriminant cannot be a square in $\mathbb{Q}$, so that from this case we get one c.

Suppose that

$$q=\frac{\tau^3-\tau-1}{2\tau(\tau+1)}.$$

Then, we get

$$\tau^3-2q\tau^2-(1+2q)\tau-1=0.$$

This means τ is a root of the following polynomial:

$$x^3-2qx^2-(1+2q)x-1. \tag{4.7}$$

Let x_1 and x_2 be other roots of this polynomial. They should satisfy the following quadratic polynomial:

$$x^2+(\tau-2q)x+\frac{1}{\tau}.$$

This polynomial has rational roots if and only if the discriminant

$$(\tau-2q)^2-\frac{4}{\tau},$$

is a square in $\mathbb{Q}$. Using q in terms of τ, this is if and only if

$$\frac{\tau^4-2\tau^3-5\tau^2-2\tau+1}{\tau^2(\tau+1)^2}$$

is a square in $\mathbb{Q}$. Consider the elliptic curve defined by

$$\tau^4-2\tau^3-5\tau^2-2\tau+1=y^2.$$

Similarly, using Magma [3] one can check that $(0,1),(0,-1),(-1,1),(-1,-1)$, and the two points at infinity are the only rational points on the elliptic curve. Hence from this case again we get one c.

Now, suppose that

$$q=-\frac{\tau^3+2\tau^2+3\tau+1}{2\tau(\tau+1)}.$$

Then, we get

$$\tau^3 + (2+2q)\tau^2 + (3+2q)\tau + 1 = 0.$$

This means τ is a root of the following polynomial:

$$x^3 + (2+2q)x^2 + (3+2q)x + 1. \tag{4.8}$$

Let x_1 and x_2 be other roots of this polynomial. They should satisfy the following quadratic polynomial:

$$x^2 + (\tau + 2 + 2q)x - \frac{1}{\tau}.$$

This polynomial has rational roots if and only if the discriminant

$$(\tau + 2 + 2q)^2 + \frac{4}{\tau},$$

is a square in $\mathbb{Q}$. Using q in terms of τ, this is if and only if

$$\frac{\tau^4 + 2\tau^3 + 7\tau^2 + 6\tau + 1}{\tau^2(\tau+1)^2}$$

is a square in $\mathbb{Q}$. Consider the elliptic curve defined by

$$\tau^4 + 2\tau^3 + 7\tau^2 + 6\tau + 1 = y^2.$$

Using Magma [3] one can check that $(0,1), (0,-1), (-1,1), (-1,-1)$, and the two points at infinity are the only rational points on the elliptic curve. Once again, we get one c. This completes our proof. $\square$

5. Further Investigation of Periodic Points of $\phi_{k,b}(z)$

In this chapter, we will prove a mild analogue of Baker and De Marco's following theorem:

Theorem 5.0.1. *[1] Let $d \geqslant 2$ be an integer. Fix $c_1, c_2 \in \mathbb{C}$. The set of $t \in \mathbb{C}$ such that both c_1 and c_2 are preperiodic for $z^d + t$ is infinite if and only if $c_1^d = c_2^d$.*

Main theorem of this chapter is as follows:

Theorem 5.0.2. *Suppose $a, b \in \mathbb{Q}$ are periodic points of*

$$\phi_{t_1,t_2}(z) = t_1 \cdot z + \frac{t_2}{z}$$

where $(t_1, t_2) \in (\mathbb{Q}^* \times \mathbb{Q}^*)$. Suppose that possible period lengths of $\phi_{t_1,t_2}(z)$ is either $1, 2,$ or 4. Then, there exists infinitely many rational pairs (t_1, t_2) such that a and b are both rational periodic points of $\phi_{t_1,t_2}(z)$ if and only if $a^2 = b^2$.

Before we start proving this theorem we need a further investigation of periodic points of the map $\phi_{k,b}(z)$

Proposition 5.0.3. *Let $\phi_{k,b}(z) = kz + \frac{b}{z}$ with $k, b \in \mathbb{Q}^*$. If $\phi_{k,b}$ has two distinct rational periodic points q_1, q_2 with period lengths 1, 2, respectively, then*

$$\phi_{k,b}(z) = \frac{-q_2^2 - q_1^2}{q_2^2 - q_1^2} z + \frac{2q_1^2 q_2^2}{(q_2^2 - q_1^2)z}.$$

Proof. Suppose q_1, q_2 are rational periodic points of $\phi_{k,b}(z)$ with exact periods 1 and 2, respectively. Then, by Theorem 4.0.1, we have

$$b = s_1 \cdot q_1^2 = -s_2 \cdot q_2^2, \text{ and} \tag{5.1}$$

$$k = 1 - s_1 = s_2 - 1, \tag{5.2}$$

for some $s_1, s_2 \in \mathbb{Q} \setminus \{0,1\}$. Using Equalities (5.1) and (5.2) together, we get

$$b = \frac{2q_1^2 q_2^2}{(q_2^2 - q_1^2)},$$

$$k = \frac{-q_2^2 - q_1^2}{q_2^2 - q_1^2}.$$

$\square$

The following proposition was proved by Manes ([7], Proposition 10). We reproduce the proof.

Proposition 5.0.4. *Let $\phi_{k,b}(z) = kz + \frac{b}{z}$ where $k, b \in \mathbb{Q}^*$. Then,*

- *$\phi_{k,b}(z)$ cannot have periodic points with exact period 1 and exact period 4.*
- *$\phi_{k,b}(z)$ cannot have periodic points with exact period 2 and exact period 4.*

Proof. Suppose p and r are rational periodic points of $\phi_{k,b}(z)$ with period length 1 and 4, respectively. Then,

$$k = (1 - s_1) = \frac{2s_2}{(s_2^2 - 1)},$$

$$b = (s_1 \cdot p^2)) = -\frac{(s_2^2 + 1)}{s_2 \cdot (s_2^2 - 1)} r^2,$$

for some $s_1 \in \mathbb{Q} \setminus \{0,1\}$ and $s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$. From these equalities, we get

$$\frac{s_2^2 - 2s_2 - 1}{s_2^2 - 1} \cdot p^2 = -\frac{s_2^2 + 1}{s_2 \cdot (s_2^2 - 1)} \cdot r^2.$$

This yields

$$(\frac{p}{r})^2 = -\frac{s_2(s_2^2 + 1)(s_2^2 - 2s_2 - 1)}{s_2^2(s_2^2 - 2s_2 - 1)^2}.$$

If we can show that there is no rational point on the hyperelliptic curve defined by the equation

$$y^2 = -x(x^2 + 1)(x^2 - 2x - 1) \tag{5.3}$$

where $x \in \mathbb{Q} \setminus \{0, \pm 1\}$, we are done for the first part. Equation (5.3) defines a genus 2 curve. LMFDB [6] tells us there isn't any rational point on the curve except for $x = 0, -1, 1$.

Suppose p and r are rational periodic points of $\phi_{k,b}(z)$ with period length 2 and 4, respectively. Then,

$$k = (s_1 - 1) = \frac{2s_2}{(s_2^2 - 1)},$$

$$b = -(s_1 \cdot p^2)) = -\frac{(s_2^2 + 1)}{s_2 \cdot (s_2^2 - 1)} r^2,$$

for some $s_1 \in \mathbb{Q} \setminus \{0, 1\}$ and $s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$. From these equalities, we get

$$\frac{s_2^2 + 2s_2 - 1}{s_2^2 - 1} \cdot p^2 = \frac{s_2^2 + 1}{s_2 \cdot (s_2^2 - 1)} \cdot r^2.$$

This yields

$$(\frac{p}{r})^2 = \frac{s_2(s_2^2 + 1)(s_2^2 + 2s_2 - 1)}{s_2^2(s_2^2 + 2s_2 - 1)^2}.$$

We will show that there is no rational point on the hyperelliptic curve defined by the equation

$$y^2 = x(x^2 + 1)(x^2 + 2x - 1) \tag{5.4}$$

where $x \in \mathbb{Q} \setminus \{0, \pm 1\}$. But if we replace x by $-x$ in Equation (5.3), we get the same curve. So again we don't have any rational point on the curve defined by Equation (5.4) except for $x = 0, \pm 1$. □

Now let us prove Theorem 5.0.2

Proof. We already know that supposing $a^2 = b^2$, we can find infinitely many rational pairs (t_1, t_2). Now, suppose that there exists infinitely many rational parameters (t_1, t_2) such that a and b are both rational periodic points of $\phi_{t_1,t_2}(z)$. By proposition 5.0.3 and 5.0.4, a and b cannot have distinct period lengths. Now, if a and b both have period length 1, or period length 2 then by theorem 4.0.1, we have $a = -b$, this yields $a^2 = b^2$. Suppose a and b both have period length 4, then by theorem 4.0.1

$$a, b \in \{p, \frac{p}{s}, -p, -\frac{p}{s}\},$$

for some $s \in \mathbb{Q} \setminus \{0, \pm 1\}$. If $a = p$ and $b = -p$, or $a = p/s$ and $b = -p/s$ we are done. From the cases $a = p$, $b = p/s$; $a = p$, $b = -p/s$; $a = -p$, $b = p/s$; and $a = -p$, $b = -p/s$, we get $s = \pm a/b$. But these give only two rational pairs (t_1, t_2), contradicting to our assumption. □

BIBLIOGRAPHY

[1] M. Baker and L. Demarco. Preperiodic points and unlikely intersections. *Duke Mathematical Journal*, 159(1), 2011. doi: 10.1215/00127094-1384773.

[2] B. Birch and H. Swinnerton-Dyer. Notes on elliptic curves II. *Journal für die reine und angewandte Mathematik*, 218:79–108, 1965.

[3] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24:235 – 265, 1997.

[4] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Inventiones Mathematicae*, 73(3):349–366, 1983. doi: 10.1007/bf01388432.

[5] E. V. Flynn, B. Poonen, and E. F. Schaefer. Cycles of quadratic polynomials and rational points on a genus-2 curve. *Duke Mathematical Journal*, 90(3), 1997. doi: 10.1215/s0012-7094-97-09011-6.

[6] T. LMFDB Collaboration. The L-functions and modular forms database. http://www.lmfdb.org, 2021.

[7] M. Manes. $\mathbb{Q}$-rational cycles for degree-2 rational maps having an automorphism. *Proceedings of the London Mathematical Society*, 96(3):669–696, 2007. doi: 10.1112/plms/pdm044.

[8] B. Mazur. Modular curves and the Eisenstein ideal. *Publications mathématiques de l'IHÉS*, 47(1):33–186, 1977. doi: 10.1007/bf02684339.

[9] B. Mazur and D. Goldfeld. Rational isogenies of prime degree. *Inventiones Mathematicae*, 44(2):129–162, 1978. doi: 10.1007/bf01390348.

[10] P. Morton. Arithmetic properties of periodic points of quadratic maps, ii. *Acta Arithmetica*, 87(2):89–102, 1998. doi: 10.4064/aa-87-2-89-102.

[11] D. G. Northcott. Periodic Points on an Algebraic Variety. *The Annals of Mathematics*, 51(1):167–177, 1950. doi: 10.2307/1969504.

[12] B. Poonen. The classification of rational preperiodic points of quadratic polynomials over $\mathbb{Q}$: a refined conjecture. *Mathematische Zeitschrift*, 228(1):11–29, 1998. doi: 10.1007/pl00004405.

[13] J. H. Silverman. *The Arithmetic of Dynamical Systems*. Springer, 2007.

[14] J. H. Silverman. *The Arithmetic of Elliptic Curves*. Springer, New York, 2009.

[15] M. Stoll. Rational 6-cycles under iteration of quadratic polynomials. *LMS Journal of Computation and Mathematics*, 11:367–380, 2008. doi: 10.1112/s1461157000000644.

[16] R. Walde and P. Russo. Rational Periodic Points of the Quadratic Function $\mathbb{Q}_c(x) = x^2 + c$. *The American Mathematical Monthly*, 101(4):318–331, 1994. doi: 10.1080/00029890.1994.11996950.

[17] L. C. Washington. *Elliptic curves: Number theory and cryptography (2nd ed.)*. Chapman-Hall/CRC, 2008.