REGULARITY OF EDGE IDEALS AND THEIR POWERS

by

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"No two things have been combined better than knowledge and patience."

Prophet Muhammad

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Abstract

In this thesis, we study the Castelnuovo-Mumford regularity of edge ideals associated to graphs. We first give a detailed proof of celebrated theorem of Fröberg which characterizes edge ideals with linear resolution. We also collect recent results on different bounds and exact values of regularity of edge ideals associated to different classes of graphs. In the last part, we study some bounds and exact values of regularity of powers of edge ideals.

KENAR İDEALLERİN DÜZENLİLİĞİ VE ONLARIN KUVVETLERİ

Elshani Kamberi

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Tez Danışmanı: Yrd. Doç. Dr. Ayesha Asloob Qureshi

Anahtar Kelimeler: Kenar idealler, Castelnuovo-Mumford düzenliliği, graflar, eşleştirme, basit kompleksler

Özet

Bu tezde graflarla ilişkili kenar ideallerinin Castelnuovo-Memford düzenliliği üzerine çalıştık. Önce, Frönberg'in kenar ideallerini doğrusal çözünürlükle karakterize eden meşhur teoreminin detaylı bir ispatını vereceğiz. Ayrıca, farklı graf sınıflarıyla ilişkili kenar ideallerinin düzenliliğinin farklı sınırları ve kesin değerleri hakkındaki son sonuçları topluyoruz. Son bölümde, kenar ideallerin kuvvetlerinin düzenliliğinin bazı sınırlarını ve kesin değerlerini inceliyoruz.

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Introduction

Monomial ideals are one of the main algebraic tools that connects commutative algebra with combinatorics. Following the work of Richard Stanley, in the late 1970's a new and exciting trend started in commutative algebra, namely, the combinatorial study of squarefree monomial ideals. One can associate a simplicial complex with a squarefree monomial ideal and vice versa. In particular, every squarefree monomial ideal generated in degree 2 can be naturally associated with a finite simple graph. This translation allows us to describe the algebraic and homological properties of such ideals in terms of combinatorial data of graphs. In this work, we focus on the results obtained by several authors in last two decades on the Castelnuovo-Mumford regularity (or simply, regularity) of edge ideals. Regularity of ideals, modules or sheafs is an important tool to understand their complexity. The interpretation of regularity of modules in commutative algebra is given in terms of minimal graded free resolutions of modules.

Let G be a simple finite graph on $[n]$ and $K = [x_1, \ldots, x_n]$ be a polynomial ring over a field K. Then $I = (x_i x_j : \{i, j\} \in E(G))$ is called the edge ideal of G. In fact, every squarefree monomial ideal generated in degree 2 can be interpreted as an edge ideal. More generally, squarefree monomial ideals can be interpreted as edge ideals of *hypergraphs*. Restricting to the class of edge ideals, the first question that arises is for which classes of graph regularity of an edge ideal, denoted by $\text{reg}(I(G))$, is minimal possible, that is $\text{reg}(I(G)) = 2$. It is equivalent to say that when an edge ideal admits a linear resolution. The second question is that when an edge ideal does not have a linear resolution, then what are the natural bounds for it's regularity and if these bounds can be improved for restricted classes of graphs. Powers of edge ideals are also of particular interest and it is rapidly growing topic in combinatorial study of powers of squarefree monomial ideals. In [48] and [3], authors gave a detailed survey on regularity of edge ideals and their powers which provides a guideline for our work.

A breakdown of the contents of this thesis is given as follows: In Chapter 1, we give combinatorial and algebraic notation and definitions which will be used in later chapters. Chapter 2 is dedicated to the study of regularity of edge ideals of different classes of graphs. We divided Chapter 2 into 3 sections. In the first section, we give basic properties of edge ideals and certain inductive results that will be used in other subsections. In the second section, our main goal is to give a detailed proof of Fröberg's Theorem [20] that states that an edge ideal of a graph

G has linear resolution if and only if complement graph of G is chordal. Other than the original proof of Fröberg in [20], there are other proofs with different techniques from several authors, (for example, [26], [39]). However, we will present the proof given in [48]. In the last section, we have tried to collect most important and recent results about the upper and lower bounds for regularity of some special classes of edge ideals with proof. The most natural lower and upper bounds of regularity of $I(G)$ are the maximum size of an induced matching in G (see Theorem 2.3.2) and the minimum size of a maximum matching in G (see Theorem 2.3.5), respectively. In particular, in Theorem 2.4.9, Theorem 2.4.18, Theorem 2.4.20, Theorem 2.4.23 and Theorem 2.4.24 we list some recent results which gives different classes of graphs, for which the regularity for $I(G)$ is $\nu(G)+1$ where $\nu(G)$ is the maximum size of an induced matching in G. For each class of graphs, different techniques are used to obtain $\operatorname{reg} I(G) = \nu(G)+1$ and we have collected all these proof together here.

In Chapter 3, we focus on the powers of edge ideals. One of the main reason of the interest in the study of regularity of powers of edge ideals is due to well knows Theorem of Herzog, Cutkosk and Trung [10] and Kodiyalam [35], that states that if I is a graded ideal of a standard graded K-algebra, then the regularity of I^s is asymptotically a linear function of s. In simple words, there exists constants a and b such that $\operatorname{reg}(I^s) = as + b$ when s is large enough. To find the smallest number s_0 when this equality holds, is a hard problem. In recent years, it has been the topic of several papers, [2], [31], [6], [1], [30] etc, where authors tried to classify such s_0 or give some bounds for this function. In Chapter 3, we list some prominent results in this direction.

Chapter 1

Preliminaries

1.1 Combinatorial Preliminaries

First we recall some basic definitions and notions related to graph theory. All graphs considered in this work will be finite simple graphs, unless stated otherwise.

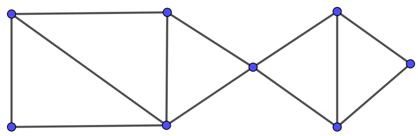
Let G be a simple finite graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Two vertices in G are called *adjacent* if they are connected by an edge. A subgraph H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called *induced* subgraph if for all $x, y \in V(H)$, we have $\{x, y\} \in E(H)$ whenever $\{x, y\} \in E(G)$. For any $x \in V(G)$, we denote by $G \setminus x$ the induced subgraph of G on $V(G) \setminus \{x\}$.

The *neighborhood* of a vertex x in G is denoted by $N_G(x)$ and it is the set of all vertices of G that are adjacent with x, that is $N_G(x) = \{y \ : \ \{x, y\} \in E(G)\}$. Morever we set $N_G[x] = N_G(x) \cup \{x\}$. The degree of a vertex $x \in V(G)$ is denoted by $\deg(x)$ and is equal to $|N_G(x)|$. A vertex of a graph is called a *leaf* if it has degree one. The complement graph of G is denoted by G^c and it is the graph with the same vertex set as of G and $e \in E(G^c)$ if and only if $e \notin E(G)$. A graph is called *complete* if every pair of its vertices is connected by an edge. A complete graph on n vertices is denoted by K_n. A complete graph is also called a *clique*.

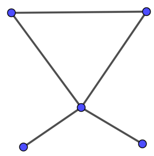
A walk in G is a sequence of vertices $x_0, x_1, \ldots, x_n$ such that $\{x_{i-1}, x_i\} \in E(G)$, for all $i = 1, \ldots, n$. A walk is called a path if $x_i \neq x_j$, for all $0 \leq i < j \leq n$. The length of a path on $n + 1$ vertices is set to be n. The distance between two vertices x and y is denoted by $d(x, y)$ and is defined to be the number of edges in a shortest path connecting them. A cycle in G is a sequence of distinct vertices $x_1, \ldots, x_n$ such that $\{x_1, x_n\} \in E(G)$ and $\{x_{i-1}, x_i\} \in E(G)$ for all $i = 2, \ldots, n$. A cycle on n vertices has length n and is denoted by C_n. If the complement of a graph is a cycle then the graph itself is called *anticycle*.

Two vertices $x, y \in V(G)$ are said to be connected if there is a path in G that starts with x and end with y. A graph G is called *connected* if all pairs of its vertices are connected, otherwise it is called *disconnected*.

A graph is called a *forest* if it does not contain any cycle as a subgraph. A connected forest is called a *tree*. A chord in a cycle is the edge $x_i x_k$ where $k \neq i+1, i-1$ for any $i = 1, \ldots, t$. A graph G is called *chordal* if every cycle of length $n \geq 4$ in G has a chord. A graph G is said to be co-chordal if G^c is chordal.



A chordal graph



A co-chordal graph

Figure 1.1:

The *intersection graph* of a finite family of non-empty sets is obtained by representing each set by a vertex, and two vertices are connected by an edge if and only if the corresponding sets intersect. We have following characterization of chordal graphs in terms of intersections graph of family of subtree of a tree.

Theorem 1.1.1. *[17] A graph is chordal if and only if it is the intersection graph of subtrees of a tree.*

Let H and K be two graphs with disjoint set of vertices. Then the union of H and K is the graph $G = H \cup K$ with vertex set $V(G) = V(H) \cup V(K)$ and edge set $E(G) = E(H) \cup E(K)$ together with edges obtained by connecting the vertices of H with all the vertices of K. The disjoint union of H and K is a graph G that has two connected components, namely, H and K.

A graph G is called a *bipartite graph* if its vertex set can be partitioned into two disjoint sets X and Y such that $\{x, y\} \in E(G)$ only if $x \in X$ and $y \in Y$. It is a well known fact that a graph is bipartite if and only if it does not contain cycles of odd length. A bipartite graph is called *complete* if $\{x, y\} \in E(G)$ for all $x \in X$ and $y \in Y$. A complete bipartite graph with $|X| = n$ and $|Y| = m$ is denoted by $K_{n,m}$.

A *matching* in a graph G is a set of pairwise disjoint edges. An *induced* matching is a matching such that the induced graph on its vertex set contains the edges only from the matching itself. A *maximal* matching is a matching of G which is not contained in any other matching of G. The size of a matching is the number of edges in it. We denote by $\nu(G)$ the maximum size of an induced matching of G and by $\beta(G)$ the minimum size of a maximal matching of G.

A vertex cover of G is a subset of $V(G)$ such that it intersects every edge of G. A vertex cover is said to be minimal if none of its proper subsets is a vertex cover.

A subset of vertices of a graph is called *independent* if it does not contain any adjacent vertices. If G can be partitioned into a clique and an independent set of vertices, then it is called a *split graph*. In other words a split graph is a chordal graph with a chordal complement.

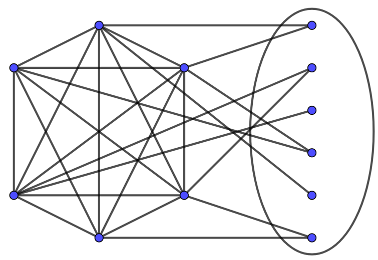


Figure 1.2: A split graph

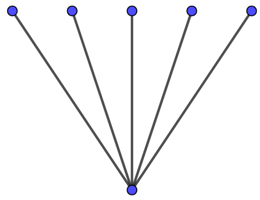
Let $H_1, H_2, \ldots, H_n$ be induced subgraphs of G. We say that $H_1, H_2, \ldots, H_n$ covers the edges of G if $E(G) = \cup_{i=1}^{n} E(H_i)$ as a union or a disjoint union. The splitting cover number of G is the minimum number of induced split subgraphs to cover the edges of G.

Theorem 1.1.2. *[8, Theorem 4] Let G be a chordal graph. Then, the split cover number of G is equal to $\beta(G)$.*

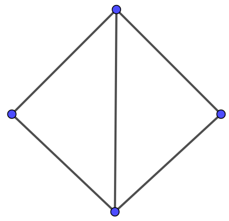
Let G be a graph and let a, b, c and d be four distinct vertices of G such that $\{a, b\}$ and $\{c, d\}$ are edges in G. The edges $\{a, b\}$ and $\{c, d\}$ form a *gap* in G if there does not exists any edge in G with one end point in $\{a, b\}$ and other end point in $\{c, d\}$. A graph is called *gap-free* if it does not have any pair of edges that form a gap. In other words, G is a gap-free graph if it does not contain two vertex-disjoint edges as an induced subgraph. Note that a graph G is gap-free if and only if G^c does not contain C_4 as an induced subgraph.

A graph isomophic to $K_{1,3}$ is called a *claw*. More generally, a graph isomorphic to $K_{1,n}$ is called n-claw. The unique vertex with degree n in $K_{1,n}$ is called the *root*. A graph which does not contain a claw as an induced subgraph is called *claw-free*. Similarly, a graph which does not contain an n-claw as an induced subgraph is called an *n-claw-free*.

A graph G is called *diamond* if it is isomorphic to the graph with vertex set $\{x, y, z, w\}$ and edge set $\{xy, yz, xz, xw, zw\}$. A graph which does not contain a diamond as an induced subgraph is called *diamond-free*.



5-claw



Diamond

A graph G is called a *cricket* if it is isomorphic to the graph with vertex set $\{x_1, x_2, x_3, x_4, x_5\}$ and edge set $\{x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_3x_4\}$. The co-chordal graph in Figure 1.1 is a cricket. A graph that does not contain a cricket as an induced subgraph is called *cricket-free*. Note that a claw-free graph is also cricket-free.

1.1.1 Simplicial Complexes

Let Δ be a collection of subsets of $[n] = \{1, 2, 3, \ldots, n\}$. The collection Δ is called a *simplicial complex* if for any $F \in \Delta$ all subsets of F also belong to Δ. Each element of Δ is called a *face* of Δ. The dimension of a face F is $|F| - 1$. Thus, an edge of Δ is a face of dimension 1 and a vertex of Δ is a face of dimension 0. A maximal face is called *facet* of Δ. The set $\mathcal{F}(\Delta)$ represents the set of all facets in Δ. If $F(\Delta) = \{F_1, \ldots, F_r\}$, then we write $\Delta = \langle F_1, \ldots, F_r \rangle$.The dimension of a simplicial complex, denoted by $\dim(\Delta)$, is $\max\{|F| - 1 : F \in \Delta\}$. A simplicial complex is called *pure* if all of its facets have same dimension. A *nonface* of a simplicial complex is a subset $F \subseteq [n]$ such that $F \notin \Delta$. We denote by $\mathcal{N}(\Delta)$ the set of all minimal nonfaces of Δ. The simplicial complex Δ is said to be *connected* if for any two facets F and T, there exists a sequence of facets $F = F_0, F_1, \ldots, F_{k-1}, F_k = T$ such that $F_i \cap F_{i+1} \neq \emptyset$.

For any $W \subseteq V$, we denote by Δ_W the simplicial complex obtained by restricting Δ to W and is given by $\Delta_W = \{F \in \Delta : F \subseteq W\}$. The Alexander dual of Δ, denoted by $\Delta^\vee$ is the given by

$$\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}$$

Example 1.1.3. Let $\Delta = \langle \{1, 2, 3\}, \{1, 3, 5\}, \{3, 4\}, \{2, 4\} \rangle$, then

$$\mathcal{N}(\Delta) = \{\{1, 4\}, \{4, 5\}, \{2, 5\}, \{2, 3, 4\}\}.$$

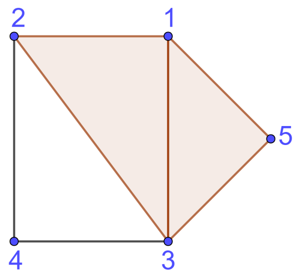


Figure 1.3: The geometrical realization of Δ

Also we can draw Δ geometrically as shown in Firgure 1.3:

Let G be a graph. Then a subset of vertices in which no pair of vertices is adjacent in G is called an *independent set* in G. The independence complex of G is the simplicial complex $\Delta(G)$ whose facets are the independent sets in G.

Let Δ be a simplicial complex and $A \in \Delta$. Then the deletion and the link of A is defined to be simplicial complexes respectively as follows:

$$del_{\Delta}(A) = \{B \in \Delta : A \nsubseteq B\},$$

$$link_{\Delta}(A) = \{F \in \Delta : F \cap A = \emptyset, A \cup F \in \Delta\}.$$

Definition 1.1.4. *A simplicial complex Δ is vertex decomposable if either:*

1. *Δ is a simplex, or*
2. *there exists a vertex $v \in \Delta$ such that both $del_{\Delta}(v)$ and $link_{\Delta}(v)$ are vertex decomposable and all facets of $del_{\Delta}(x)$ are facets of Δ.*

The vertex v with the property in above definition is called the shedding vertex of Δ. We say that a graph G is vertex decomposable if $\Delta(G)$ is vertex decomposable. We say that G is pure vertex decomposable if $\Delta(G)$ is pure and vertex decomposable. Now we give the following two lemmas which gives inductive method to determine vertex decomposable graphs.

Lemma 1.1.5. *[52, Lemma 20] Let H_1 and H_2 be two graphs such that $V(H_1) \cap V(H_2) = \emptyset$ and $G = H_1 \cup H_2$. Then H_1 and H_2 are vertex decomposable if and only if G is vertex decomposable.*

Lemma 1.1.6. *[13, Lemma 4.2] Let G be a graph, and suppose that $x, y \in V(G)$ are two vertices such that $\{x\} \cup N_G(x) \subseteq \{y\} \cup N_G(y)$. If $G \setminus y$ and $G \setminus (\{y\} \cup N_G(y))$ are both vertex decomposable, then G is vertex decomposable.*

Proof. Let $x, y \in V(G)$ be two vertices with the property in the statement. Firstly note that $del_{\Delta}(y) = \Delta(G \setminus \{y\})$ and $link_{\Delta}(y) = \Delta(G \setminus (\{y\} \cup N_G(y)))$, and they both are vertex decomposable by the assumption. Then to verify Definition 1.1.4, we have to check if every facet of $del_{\Delta}(y) = \Delta(G \setminus \{y\})$ is a facet of $\Delta(G)$. Let F be a facet of $del_{\Delta}(y)$ and suppose that $F \cup \{y\}$ is a facet of $\Delta(G)$. Note that $x \in F$ because $N_G(x) \subseteq N_G(y)$. But, x and y are adjacent since $x \in N_G(y)$, and hence x and y cannot be both elements of $F \cup \{y\}$. So, we have a contradiction. Hence F is a facet of $\Delta(G)$, and we are done. □

Definition 1.1.7. *We say that a simplicial complex Δ is shellable if its facets can be orederd, say $F_1, \ldots, F_k$, such that for $1 \leq i < j \leq k$, there exist some $x \in F_j \setminus F_i$ and some $t \in \{1, \ldots, j-1\}$ with $F_j \setminus F_t = \{x\}$. If Δ is also pure then we call it pure shellable.*

So we have the following useful theorem.

Theorem 1.1.8. *[5, Theorem 11.3] If Δ is a vertex decomposable simplicial complex, then Δ is also shellable.*

1.2 Algebraic Preliminaries

We recall some fundamental notions for commutative rings. In the following text R is a commutative Noetherian ring, that is, every ideal in commutative ring R is finitely generated. The reasons for this assumption is that we will use these definitions for standard graded algebras over fields, which are particular case of commutative Noetherian rings.

For a ring R, the spectrum of R is set of all prime ideals in R and is denoted by $\mathrm{Spec}(R)$. The set of minimal elements in $\mathrm{Spec}(R)$ is denoted by $\mathrm{Min}(R)$ whereas the set of associated primes of R is denoted by $\mathrm{Ass}(R)$. For any ideal I of R, by $\mathrm{Min}(I)$ and $\mathrm{Ass}(I)$, we mean $\mathrm{Min}(R/I)$ and $\mathrm{Ass}(R/I)$ respectively. Recall that a prime ideal P is in $\mathrm{Ass}(I)$ if P is the annihilator of some element $x \in R/I$, that is, $P = I : x$.

Let $P \in \mathrm{Spec}(R)$, then height of P is $\mathrm{height}(P) = \max\{n : P_0 \subset P_1 \subset \ldots \subset p_n = P\}$. Then the Krull dimension of R, or simply, the dimension of R is defined to be as follows:

$$\dim R = \max\{\mathrm{height}(P) : P \text{ is a prime ideal in} R\}.$$

Note that, we have $\mathrm{height}(P) = \dim R_P$. We say that an ideal $I \subseteq S$ is *unmixed* if $\mathrm{Min}(I) = \mathrm{Ass}(I)$.

An element $x \in R$ is called *regular* if for all $y \in R$ with $y \neq 0$, we have $xy \neq 0$. In addition, let R be a local ring with the unique maximal ideal $\mathfrak{m}$. Then a sequence of elements of $x_1, \dots, x_n$ in $\mathfrak{m}$ is called a *regular sequence* if x_i is regular on $R/(x_1, \dots, x_{i-1})$, for all $i = 1, \dots, n$. The maximal length of such a sequence if called *depth* of R. A ring local ring R is called Cohen-Macaulay if $\dim R = \operatorname{depth} R$.

Let K be a field and $S = k[x_1, \dots, x_n]$ be a polynomial ringin n indeterminates. The product $\mathbf{x}_\mathfrak{a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is called a monomial where $\mathfrak{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. We denote by $\operatorname{Mon}(S)$ the set of monomials in S. A polynomial f in S is a unique K-linear combination of elements in $\operatorname{Mon}(S)$ as follow:

$$f = \sum_{v \in Mon(S)} b_v v \quad \text{where} \quad b_v \in K.$$

The support of a polynomial f in S, denoted by *supp*(f), is

$$\mathit{supp}(f) = \{v \in Mon(S) : b_v \neq 0\}.$$

An ideal I is called a *monomial ideal* in S if it is generated by monomials. A monomial ideal is called *squarefree* if it is generated by squarefree monomials. We denote the unique minimal set of generators of a monomial ideal I by $G(I)$.

Remark 1.2.1. It is well known that the usual ideal operations applied on monomial ideal again results in monomial ideals. In particular, If I and J are monomial ideals, then we have

1. $\mathcal{G}(I + J) \subset \mathcal{G}(I) \cup \mathcal{G}(J)$,
2. $\mathcal{G}(IJ) \subset \mathcal{G}(I)\mathcal{G}(J)$,
3. $\mathcal{G}(I \cap J) = \{lcm(u, v) : u \in \mathcal{G}(I), v \in \mathcal{G}(J)\}$,
4. $I : J = \bigcap_{u \in \mathcal{G}(J)} I : (v)$, where $\mathcal{G}(I : (v)) = \{u/\gcd(u, v) \ : \ u \in \mathcal{G}(I)\}$.

In the following text we will consider two different gradings on S. For this, we first recall the following: The $\mathbb{Z}$-grading on S is defined as follows: For a monomial $\mathbf{x}_\mathfrak{a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, the degree of $\mathbf{x}_\mathfrak{a}$ is $\deg(\mathbf{x}_\mathfrak{a}) = \sum_{i=1}^n a_i$. Then $S = \oplus_{i \in \mathbb{Z}} S_i$, where $S_0 = K$ and each S_i is K-vector space generated by monomials of degree i. Note that 0 is assigned an arbitrary degree. One can see that $S = \oplus_{i \in \mathbb{N}} S_i$ since there are no elements in S of negative degree. Each S_i is called the i-th graded component of S and a polynomial is called *homogeneous* of degree i if every elements in its support is of degree i. A S-module M is called $\mathbb{Z}$-graded if it admits a decomposition $M = \oplus_{i \in \mathbb{Z}} M_i$ as $\mathbb{Z}$-module and $S_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. If M and N are two $\mathbb{Z}$-graded S-modules then a S-module homomorphism $\phi : M \to N$ is a graded S-module homomorphism of degree i is $\deg(\phi(m)) = i + \deg(m)$ for all $m \notin \mathrm{Ker}(\phi)$. Moreover, for any S-module M and $d \in Z$, the $\mathbb{Z}$-graded S-module $M(-d)$ is obtained by shifting M by d degrees, that is , $M(-d)_i = M_{i-d}$, for all i. In particular, note that if I is a graded ideal of S, that is, I is generated by homogenous elements, then S/I naturally inherits $\mathbb{Z}$-grading structure.

The $\mathbb{Z}^n$-grading on S is defined as follows: For a monomial $\mathbf{x}_{\mathfrak{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, the multidegree of $\mathbf{x}_{\mathfrak{a}}$ is set to be $\mathfrak{a} = (a_1, \ldots, a_n)$. This shows that every monomial has a unique multidegree. Then S is $\mathbb{Z}^n$-graded with $S = \oplus_{\mathfrak{a} \in \mathbb{Z}^n} S_{\mathfrak{a}}$, where each $S_{\mathfrak{a}}$ is K-vector space generated by monomial of degree $\mathfrak{a}$. In particular, $S_{\mathfrak{a}} = 0$ is $\mathfrak{a} \notin \mathbb{N}^n$. One can define the analouge of all definitions in previous paragraph in case of $\mathbb{Z}^n$-grading as well.

1.2.1 Minimal free resolutions and homological invariants

Let M be a finitely generated $\mathbb{Z}$-graded S-module and

$$0 \to \oplus_j S(-j)^{\beta_{p,j}} \xrightarrow{\phi_p} \cdots \to \oplus_j S(-j)^{\beta_{1,j}} \xrightarrow{\phi_1} \oplus_j S(-j)^{\beta_{0,j}} \xrightarrow{\phi_0} M \to 0$$

be the minimal $\mathbb{Z}$-graded free resolution of M. The numbers $\beta_{i,j}(M) = \beta_{i,j}$ are uniquely determined by M and are called (i,j)-th graded Betti numbers of M. The integer i is called the homological degree of $F_i = \oplus_j S(-j)^{\beta_{i,j}}$ and j is called its internal degree. More precisely, we have

$$\beta_{i,j}(M) = \dim_K \mathrm{Tor}_i(M, K)_j$$

The length of the minimal graded free resolution is a homological invariant and is called *projective dimension* of M and is denoted by $\mathrm{proj}\dim(M)$. Then we have

$$\mathrm{proj}\dim M = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

The Castelnuovo-Mumford regularity,(or simply, the regularity), is defined to be as follows:

$$\mathrm{reg}(M) = \max\{j - i : \beta_{i,j} \neq 0\} = \max\{j - i : \mathrm{Tor}_i(M, K)_j \neq 0\}$$

When we consider $\mathbb{Z}^n$ grading on M and S, then the minimal $\mathbb{Z}$-graded free resolution takes the following form

$$F: \quad \cdots \to \bigoplus_{\mathfrak{a} \in \mathbb{Z}^n} S(-\mathfrak{a})^{\beta_{i,\mathfrak{a}}} \to\longrightarrow \cdots \to \bigoplus_{\mathfrak{a} \in \mathbb{Z}^n} S(-\mathfrak{a})^{\beta_{i,\mathfrak{a}}} \to \bigoplus_{\mathfrak{a} \in \mathbb{Z}^n} S(-\mathfrak{a})^{\beta_{i,\mathfrak{a}}} \to N \longrightarrow 0$$

The above form of F is called the minimal $\mathbb{Z}^n$-graded free resolution of M. The numbers $\beta_{i,\mathfrak{a}}$ are called the multigraded Betti numbers of M. Note that $\beta_{i,j} = \sum_{|\mathfrak{a}|=j} \beta_{i,\mathfrak{a}}$.

In the following text, by writing, minimal graded free resolution, we will mean the $\mathbb{Z}$-graded minimal free resolution.

Definition 1.2.2. *Let N be a graded S-module. We say that N has a d-linear resolution if its graded minimal free resolution is of the form*

$$0 \longrightarrow S(-d-p)^{\beta_p} \longrightarrow \cdots \longrightarrow S(-d-1)^{\beta_1} \longrightarrow S(-d)^{\beta_0} \longrightarrow N \longrightarrow 0.$$

Remark 1.2.3. *Let $S = k[x, x_1, \ldots, x_n]$. Then the monomial ideal $I = (xx_1, \ldots, xx_n)$ has a 2-linear resolution.*

1.2.2 Some important results related to regularity

Below, we will list some of the important well-known properties and facts related to regularity. Let I be a graded ideal of S. First note that $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. The following lemma is an immediate consequence of the definition of regularity.

Lemma 1.2.4. *A homogeneous ideal I generated in degree d has a linear resolution if and only if $\operatorname{reg}(I) = d$.*

Proof. ($\Rightarrow$) If I has a linear resolution, then we have $\beta_{i,i+j}(I) = 0$ for all $j \neq d$. Then,

$$\operatorname{reg}(I) = \max\{j - i : \beta_{i,j}(I) \neq 0\} = d.$$

($\Leftarrow$) Now, assume that $\operatorname{reg}(I) = \max\{j - i : \beta_{i,j} \neq 0\} = d$. Since I is generated in degree d, we have $\beta_{0,j} = 0$ for all $j \neq d$. Then $\beta_{1,j} \geq d$ for all j. This implies that $\beta_{1,j} = d$. Continuing in this way, we see that I has a d-linear resolution. $\square$

If a graded ideal I in S is generated by a homogenous regular sequence in same degree then we know the regularity for any power of I.

Lemma 1.2.5. *[6, Lemma 4.4] Let $x_1, x_2, \ldots, x_r$ be a regular sequence of graded elements in S with $\deg x_i = d$ for all $i = 1, \ldots, r$. let $I = (x_1, \ldots, x_r)$. Then*

$$\operatorname{reg}(I^s) = ds + (d-1)(r-1),$$

for all $s \geq 1$.

The following well known lemma is one of the main tool to relate the regularity of modules in a short exact sequence and its proof is obtained by the induced long exact sequence of Tor.

Lemma 1.2.6. *Let*

$$0 \to M \to N \to P \to 0.$$

be a short exact sequence of finitely generated graded S-modules, with graded homomorphisms of degree 0. Then

$$\operatorname{reg} N \leq \{\operatorname{reg} M, \operatorname{reg} P\},$$

$$\operatorname{reg} M \leq \{\operatorname{reg} N, \operatorname{reg} P + 1\},$$

$$\operatorname{reg} p \leq \{\operatorname{reg} M - 1, \operatorname{reg} N\}.$$

The following result is due to [32].

Theorem 1.2.7. *Let $I_1, \ldots, I_n$ be squarefree monomial ideals in S. Then*

$$\operatorname{reg}(S/\sum_{i=1}^{n} I_i) \leq \sum_{i=1}^{n} \operatorname{reg}(S/I_i).$$

Let I be a square free monomial ideal with primary decomposition as follows

$$I = \langle x_{1,1}, x_{1,2}, \ldots, x_{1,t_1}\rangle \cap \langle x_{2,1}, x_{2,2}, \ldots, x_{2,t_2}\rangle \cap \cdots \cap \langle x_{k,1}, x_{k,1}, \cdots, x_{k,t_k}\rangle,$$

Then the *Alexander dual* of I is denoted by $I^\vee$ and is defined to be

$$I^\vee = \langle x_{1,1}x_{1,2}\cdots x_{1,t_1}, x_{2,1}x_{2,2}\cdots x_{2,t_2}, \ldots, x_{k,1}x_{k,1}\cdots x_{k,t_k}\rangle$$

Then we have the following theorem making the relation between the projective dimension of $I^\vee$ and the regularity of I.

Theorem 1.2.8. *[45, Terai] Let I be an square-free monomial ideal. Then* $\operatorname{proj\,dim}(I^\vee) = \operatorname{reg}(R/I)$.

1.2.3 Stanley-Reisner Ideals

Let Δ be a simplicial complex on $[n]$ and $S = K[x_1, \ldots, x_n]$ be the polynomial ring over the field K. For each $F \in \mathcal{F}(\Delta)$ we set

$$x_F = \prod_{i \in F} x_i \in \operatorname{Mon}(S).$$

The *Stanley-Reisner ideal* of a simplicial complex Δ is denoted by I_Δ and it is defined to be the ideal generated by x_F such that $F \notin \Delta$. It is easy to see that I_Δ is generated by minimal non-faces of Δ, that is,

$$I_\Delta = \langle x_F : F \in \mathcal{N}(\Delta)\rangle.$$

The quotient ring $K[\Delta] = R/I_\Delta$ is called *Stanley-Reisner ring* and Krull dimension of $K[\Delta]$ is equal to $\dim \Delta - 1$. The facet ideal of Δ is denoted by $I(\Delta)$ and it is defined to be:

$$I(\Delta) = \langle x_{F_1}, \ldots, x_{F_k} \rangle, \text{ where } F_i \in \mathcal{F}(\Delta) \text{ for all } i = 1, \ldots, k.$$

To each facet $F \in \Delta$ we associate a monomial prime ideal P_F as follows:

$$P_F = (x_i : i \in F).$$

Example 1.2.9. Consider the simplicial complex in Example (1.1.3), then we have:

$$I_\Delta = (x_1x_2, x_4x_5, x_2x_5, x_2x_3x_4),$$

and

$$I(\Delta) = (x_1x_2x_3, x_1x_3x_5, x_3x_4, x_2x_4).$$

Chapter 2

Regularity of Edge Ideals

In this chapter we introduce the edge ideal of graph and discuss their regularity. The regularity of edge ideals has been a topic of dozens of articles in past two decades. We collect the main results on this topic. In particular, we discuss Fröberg's theorem which characterize the edge ideals with linear resolution, that is, with regularity equal to 2. Moreover, we also discuss the well known upper and lower bounds of regularity of edge ideals.

2.1 Edge Ideals

Let G be a simple graph with the vertex set $V(G) = \{x_1, \dots, x_n\}$ and the edge set $E(G)$. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over field K. Here, for the sake of simplicity, we are going to denote the vertices of G and variables of S by $x_i's$. The edge ideal $I(G)$ associated to the graph G is the square-free monomial ideal defined by

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E(G)).$$

Let $I(G) = \bigcap_{k=1}^{r} P_k$ be the minimal primary decomposition of $I(G)$, where $P_k = (x_{k_1}, x_{k_2}, \dots, x_{k_m})$. Note that the set of generators of each $P_k = (x_{k_1}, x_{k_2}, \dots, x_{k_m})$ gives a minimal vertex cover of G. This can be easily seen because for any $x_i x_j \in I(G)$, we have $x_i x_j \in P_k$ if and only if either x_i or x_j is in P_k. This shows that the generators of $I(G)^\vee$ correspond to the minimal vertex covers of G. The Alexander dual $I(G)^\vee$ of $I(G)$ is called the vertex cover ideal of G and is denoted by $J(G)$.

Example 2.1.1. Consider the graph of $G = C_7$ in Figure 2.1. Then

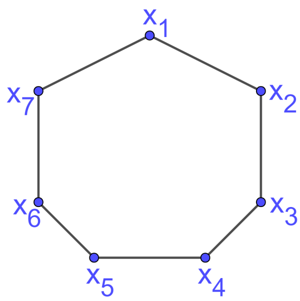


Figure 2.1

$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_1).$$

The minimal vertex covers of G are: $\{x_1, x_3, x_5, x_7\}$, $\{x_1, x_3, x_5, x_6\}$, $\{x_1, x_2, x_4, x_6\}$, $\{x_1, x_3, x_4, x_6\}$, $\{x_2, x_4, x_6, x_7\}$, $\{x_2, x_3, x_5, x_7\}$, $\{x_2, x_4, x_5, x_7\}$. So, the minimal primary decomposition of $I(G)$ is:

$$I(G) = \langle x_1, x_3, x_5, x_7\rangle \cap \langle x_1, x_3, x_5, x_6\rangle \cap \langle x_1, x_2, x_4, x_6\rangle \cap \langle x_1, x_3, x_4, x_6\rangle \cap \langle x_2, x_4, x_6, x_7\rangle$$

$$\cap \langle x_2, x_3, x_5, x_7\rangle \cap \langle x_2, x_4, x_5, x_7\rangle$$

Hence, the Alexander dual of $I(G)$ is

$$I(G)^{\vee} = \langle x_1x_3x_5x_7, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_6, x_2x_4x_6x_7, x_2x_3x_5x_7, x_2x_4x_5x_7\rangle,$$

Lemma 2.1.2. *Let G be a simple graph and H be any induced subgraph of G. Then,*

$$\operatorname{reg}(H) \leq \operatorname{reg}(G).$$

The characteristic of base field plays crucial role in minimal graded free resolution of an edge ideal because it affects the Betti numbers. In [46], Hibi and Terai showed that 3rd and 4th Betti number of a Stanley Reisner ring are independent of $\operatorname{char}(K)$ and this result was later improved by [33] where Katzman showed that 5th and 6th Betti number also have the same property. He also showed that if a simplicial complex has less than 11 vertices, then the Betti numbers of it's associated Stanley-Reisner ring are independent of $\operatorname{char}(K)$. From this, we see that if a graph has less than 11 vertices, then Betti numbers of $I(G)$ do not depend on $\operatorname{char}(K)$. Katzman, showed that there are exactly 4 non-isomorphic graphs on 11 vertices whose Betti number do not agree for characteristic 2 and 0. Moreover, in [11], it is shown that Betti numbers of an edge ideal of chordal graph are independent of $\operatorname{char}(K)$. Throughout this work we will have $\operatorname{char}(K) = 0$.

We begin our discussion on the regularity of $I(G)$ be giving the following exact sequence. Let I be a graded ideal and f be an element of degree d in S. Then the following sequences are exact:

$$0 \to S/(I:f)(-d) \xrightarrow{\cdot f} S/I \to S/(I+f) \to 0, \tag{2.1}$$

Then in the view of Lemma 1.2.6, we obtain the following

Lemma 2.1.3. *Let $I \subseteq S$ be a monomial ideal, and let t be a monomial of degree d. Then*

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(I:t)+d, \operatorname{reg}(I,t)\}.$$

Moreover, if x is a variable appearing in I then

$$\operatorname{reg}(I) \in \{\operatorname{reg}(I:x)+d, \operatorname{reg}(I,x)\}.$$

The inclusion mentioned in above lemma is due to [41]. Note that if x is a variable that does not divide any generator of monomial ideal I then we have $\operatorname{reg}(I,x) = \operatorname{reg} I$. Together with this, when we apply the above lemma to the case of edge ideal, then we obtain very useful consequences. If x is an isolated vertex to G then we can drop x from G and compute the regularity of $I(G \setminus x)$ instead. This helps us to reduce the discussion to the case when G does not have any isolated vertices. Moreover, $\operatorname{reg}(I(G):x) = \operatorname{reg} I(G \setminus N_G[x])$ and $\operatorname{reg}(I(G),x) = \operatorname{reg} I(G \setminus x)$ for all $x \in G$(when we say $x \in G$ it means $x \in V(G)$). Hence, Lemma 2.1.3 can be translated as:

Lemma 2.1.4. *Let $x \in V(G)$. Then*

$$\operatorname{reg}(I(G)) \in \{\operatorname{reg} I(G \setminus N_G[x]) + 1, \operatorname{reg} I(G \setminus x)\}.$$

Lemma 1.2.7 can be stated as follow in the case of edge ideals.

Corollary 2.1.5. *If G is a simple graph and $G_1, \ldots, G_n$ subgraphs of G such that $E(G) = \cup_{i=1}^{n} E(G_i)$ is a disjoint union, then*

$$reg(S/I(G)) \leq \sum_{i=1}^{n} regS/I(G_i).$$

An important consequence of Theorem [34] together with use of the Künneth Formula from algebraic topology is the following:

Corollary 2.1.6. *Let G be the same as in the previous corollary and the union of all $E(G_i)$ be a disjoint union. Then*

$$\text{reg}(S/I(G)) = \sum_{i=1}^{n}(S/I(G_i)).$$

In view of above corollary, one see that to understand the regularity of $I(G)$, it is enough to understand the regularity of the edge ideal of each of the connected components of G.

2.2 Edge ideals with linear resolutions

The first question in studying of regularity of edge ideal is that for which classes of graphs $\text{reg}(I(G) = 2$. It is equivalent to say that for which classes of graphs, $I(G)$ admits a 2-linear resolution (See Lemma 1.2.4). This question is answered by well celebrated result of Fröberg (See, [20]) that states $I(G)$ has a linear resolution if and only if G^c is a chordal graph. In addition to the original proof in [20], one can find other proofs of this result with different techniques, for example, in [26], [39]. We are going to present the proof of Fröberg's theorem by techniques presented in [48]. For this, we first introduce the notion of Betti splitting. Let I be a monomial ideal with set of generator $\mathcal{G}(I) = \{\alpha_1, \ldots, \alpha_n\}$. Then, we partition $\mathcal{G}(I)$ into two sets:

$$\mathcal{G}(I) = \mathcal{G}(J) \cup \mathcal{G}(N),$$

setting $J = \langle \alpha_1, \ldots, \alpha_m \rangle$ and $N = \langle \alpha_{m+1}, \ldots, \alpha_n \rangle$. Note that we have $I = J + N$. So, we can give the following definition.

Definition 2.2.1. *$I = J + N$ is called a Betti splitting if for all i and j, we have:*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(N) + \beta_{i-1,j}(J \cap N) \tag{2.2}$$

We may have a numerous different Betti splitting. So, now we give an important Betti splitting which is useful in the proof of our main theorem.

Theorem 2.2.2. *[19, Theorem 2.3] Let I be a monomial ideal in R and let J and N be two monomial ideals such that $\mathcal{G}(I) = \mathcal{G}(J) \cup \mathcal{G}(N)$. Suppose that for all homological degrees i and for all internal degrees j, $\beta_{i,j}(J \cap N) > 0$ implies $\beta_{i,j}(J) = \beta_{i,j}(N) = 0$. Then, $I = J + N$ is a Betti splitting.*

Proof. The following short sequence is exact:

$$0 \to J \cap N \to J \oplus N \to I \to 0. \tag{2.3}$$

Then, (2.3) induces the following long exact sequence of homologies:

$$\cdots \to Tor_i(K, J \cap N)_j \to Tor_i(K, J)_j \oplus Tor_i(K, N)_j \to Tor_i(K, J + N)_j \to$$

$$\to Tor_{i-1}(K, J\cap N)_j \to Tor_{i-1}(K, J)_j \oplus Tor_{i-1}(K, N)_j \to Tor_{i-1}(K, J+N)_j \to \cdots . \tag{2.4}$$

Then, recall that $\beta_{i,j}(I) = \dim_R Tor_i(K, I)_j$, and fix i and j. In addition we break our proof into two cases:

Case 1: Suppose that $\beta_{i,j}(J \cap N) = 0$. Then we seperate this case into two subcases:

(1) By hypothesis, if $\beta_{i-1,j}(J \cap N) \neq 0$ then $\beta_{i-1,j}(I) = \beta_{i-1,j}(N) = 0$, which implies that $Tor_{i-1}(K, J)_j = Tor_{i-1}(K, N)_j = 0$. So, from 2.4 then we get the following short exact sequence

$$0 \to Tor_i(K, J)_j \oplus Tor_i(K, N)_j \to Tor_i(K, I = J + N)_j \to Tor_{i-1}(K, J \cap N)_j \to 0. \tag{2.5}$$

Now, recall that the dimension is additive in exact sequences. Then, from (2.5) we get

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(N) + \beta_{i-1,j}(J \cap N)$$

for any i and j, which implies $I = J + N$ is a Betti splitting.

(2) If instead we have $\beta_{i-1,j}(J \cap N) = 0$, then from (2.5) we get the following exact sequence

$$0 \to Tor_i(K, J)_j \oplus Tor_i(K, N)_j \xrightarrow{\phi} Tor_i(K, I = J + N)_j \to 0,$$

which implies ϕ is bijective, and from the properties of dimension we get again

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(N) + \beta_{i-1,j}(J \cap N).$$

Case 2 Suppose $\beta_{i,j}(J\cap N) \neq 0$ and from the hypothesis we get $\beta_{i,j}(J) = \beta_{i,j}(N) = 0$. So, from (2.4) we get the following exact sequence

$$0 \to Tor_i(K, I)_j \to Tor_{i-1}(K, J \cap N)_j \to Tor_{i-1}(K, J)_j \oplus Tor_{i-1}(K, N)_j \to \cdots . \tag{2.6}$$

Then we seperate again it in two subcases

(1) If $\beta_{i-1,j}(J \cap N) \neq 0$, then by our hypothesis we get $\beta_{i-1,j}(J) = \beta_{i-1,j}(N) = 0$ which implies $Tor_{i-1}(K, J)_j = Tor_{i-1}(K, N)_j = 0$. So, we get the following exact sequence

$$0 \to Tor_i(K, I)_j \to Tor_{i-1}(K, J \cap N)_j \to 0.$$

Hence $\beta_{i,j}(I) = \beta_{i-1,j}(J \cap N)$, and we are done.

(2) If $\beta_{i-1,j}(J \cap N) = 0$, then $Tor_i(K, I)j = 0$, which implies $\beta_{i,j}(I) = 0$. Hence, from (2.6), (2.2) holds and we are done. □

Following from here we give the following corollary.

Corollary 2.2.3. *[19, Corollary 2.7] Let $I \subseteq R = K[x_1, \ldots x_n]$ be a monomial ideal. Fix a variable x_i, and set*

$$J = \langle m \in \mathcal{G}(I) | x_i | m \rangle \text{ and } N = \langle m \in \mathcal{G}(I) | x_i \nmid m \rangle.$$

If $\beta_{i,j}(J \cap N) > 0$ implies $\beta_{i,j}(J) = 0$ for all i and j, then $I = J + N$ is a Betti splitting.

Proof. Firstly, we note that not all the multigraded Betti numbers of J and $J \cap N$ are zero at the i^{th} position since all elements of these two ideals are divisible by x_i. Also, as x_i does not divide any of elements of N, all multigraded Betti numbers of N are zero at i^{th} position. Therefore, for all i and for all j, $\beta_{i,j}(J \cap N) > 0$ implies $\beta_{i,j}(N) = 0$. Hence, as $\beta_{i,j}(J \cap N) > 0$ implies $\beta_{i,j}(J) = 0$ is from the hypothesis, using Theorem 2.2.2 we get $I = J + N$ as a Betti splitting. □

Note that such a Betti splitting with fixed x_i is called an x_i-partition. From the previous two results we get an important Theorem by Francisco, Ha and Van Tuyl. This theorem is of plays a vital role in the proof of Fröberg's theorem.

Corollary 2.2.4. *(Francisco-Há-Van Tuyl's Theorem)[19, Corollary 2.7] Let $I \subseteq R = K[x_1, \ldots x_n]$ be a monomial ideal. Fix a variable x_i, and set*

$$J = \langle m \in \mathcal{G}(I) : x_i | m \rangle \text{ and } N = \langle m \in \mathcal{G}(I) : x_i \nmid m \rangle.$$

If J has a linear resolution, then $I = J + N$ is a Betti splitting.

Proof. Assume that J has a linear resolution, which implies that it is generated by elements with the same degree(say in degree d). Then, let $a \in J$ with $\deg(a) \geq d$. In addition, to have a in $J \cap N$ by the construction of N we should have $\deg(a) > d$. So, $J \cap N$ is generated in degree greater than J. Therefore, as the shiftings j cannot be decreasing we have $\beta_{i,j}(J \cap N) > 0$ implies $\beta_{i,j}(J) = 0$. Hence, using Corollary 2.2.3 we are done. □

Also we can state the following corollary.

Corollary 2.2.5. *[48, Corollary 2.18] Suppose that $G \setminus \{x\}$ is not the graph of isolated vertices. Let $N_G(x) = \{x_1, \ldots, x_m\}$. Then*

$$I(G) = (xx_1, \ldots, xx_m) + I(G \setminus \{x\}) \tag{2.7}$$

is a Betti splitting.

Proof. Note that (2.7) is an x-partition. Then, considering Remark 1.2.3 and using Corollary 2.2.4 we finish the proof. □

Example 2.2.6. Consider the edge ideal obtained from the graph in Figure 2.2. Then

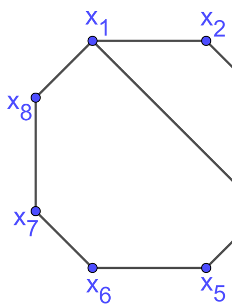


Figure 2.2

$$I = I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_8x_1, x_1x_4).$$

Set $J = (x_1x_2, x_1x_8, x_1x_4)$ and $N = (x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8)$. All generators of J are divisible by x_1, and no generator of N is divisible by x_1. Also note that

$$J \cap N = (x_1x_7x_8, x_1x_4x_5, x_1x_3x_4, x_1x_2x_3, x_1x_5x_6x_8, x_1x_4x_6x_7, x_1x_2x_6x_7, x_1x_2x_5x_6).$$

Then, we get the following graded minimal free resolutions:

$$0 \to S(-7)^4 \to S(-6)^{17} \to S(-4)^3 \oplus S(-5)^{24} \to S(-3)^{12} \oplus S(-4)^{10} \to S(-2)^9 \to I \to 0,$$

$$0 \to S(-4) \to S(-3)^3 \to S(-2)^3 \to J \to 0,$$

$$0 \to S(-6)^3 \to S(-5)^9 \to S(-4)^6 \oplus S(-3)^5 \to S(-2)^6 \to N \to 0.$$

and

$$0 \to S(-7)^4 \to S(-6)^{14} \to S(-4)^2 \oplus S(-5)^{15} \to S(-3)^4 \oplus S(-4)^4 \to J \cap N \to 0.$$

So, J has a linear resolution. Hence, from Corollary 2.7, $I = J + K$ is a Betti splitting. Indeed, if we consider the Betti numbers from the resolutions we see that the equality in the definition of Betti splitting is satisfied.

Fix $x \in V(G)$ and let $N_G(x) = \{x_1, \ldots, x_m\}$. Then we introduce the following two notations:

$$G_i := G \setminus (N_G(x) \cup N_G(x_i)) \tag{2.8}$$

and $G_{(x)}$ is the graph with edge set:

$$\{\{y, z\} \in E(G) : \{y, z\} \cap N_G(x) \neq \emptyset, y \neq x, z \neq x\}. \tag{2.9}$$

With this notation, we give the following Lemma.

Lemma 2.2.7. *[22, Corollary 4.3] Suppose that $G \setminus \{x\}$ is not the graph of isolated vertices. Let $N_G(x) = \{x_1, \ldots, x_m\}$. Then*

$$(xx_1, \ldots, xx_m) \cap I(G \setminus \{x\}) = xI(G_{(x)}) + xx_1 I(G_1) + \cdots + xx_m I(G_m).$$

Proof. Let $I = (xx_1, \ldots, xx_m)$ and $J = I(G \setminus \{x\})$. Then

$$I \cap J = \langle \mathrm{lcm}(u, v) : u \in (xx_1, \ldots, xx_m), v \in \mathcal{G}(G \setminus \{x\}) \rangle.$$

Therefore $\mathcal{G}(I \cap J)$ is disjoint union of $\{xx_ix_j : x_ix_j \in E(G)\}$ and $\{xx_iy_j : x_iy_j \in E(G)\}$ and $\{xx_iy_jy_k : y_jy_k \in E(G)$ but $x_iy_j, x_iy_k \notin E(G),\}$, where $y_i \in V(G) \setminus \{x, x_1, \ldots, x_m\}$ for $i \in \{1, \ldots, m\}$. Therefore,

$$I \cap J = xI(G_{(x)}) + xx_1 I(G_1) + \cdots + xx_m I(G_m).$$

□

Before we prove the main theorem in this section, we give the following characterization for G^c to be chordal. Note that when we write $G^c_{(x)}$ we mean $(G_{(x)})^c$.

Lemma 2.2.8. *[48, Lemma 2.23] Let G be a graph with $x \in V(G)$ and $G \setminus x$ is not the graph of isolated vertices. Then the followings are equivalent:*

1. *G^c is chordal*

2. *(a)* $(G \setminus x)^c$ *is chordal.*

   (b) $G^c_{(x)}$ *is chordal.*

   (c) G_i *has no edges. (See (2.8))*

Proof. Let $N_G(x) = \{x_1, x_2, \ldots, x_m\}$.

(i)⇒(ii) Suppose that G^c is chordal. We know that $(G \setminus x)^c = G^c \setminus x$. Since being chordal is preserved by induced graphs of a chordal graph, we conclude that $(G \setminus x)^c$ is chordal. We are done with (a).

Now we prove part (c). Suppose that there exists an i such that G_i has an edge, say $\{u, v\}$. Then, $\{x, x_1\}$ and $\{u, v\}$ belongs to G, but $\{x, u\}$,$\{x, v\}$,$\{x_i, u\}$ and $\{x_i, v\}$ do not. So, we have a four cycle belonging to G^c induced by x, u, x_1 and v. So we have a contradiction with the fact that G^c is chordal. This completes the proof of part (c)

Now we prove part (b). Let $v_1, \ldots, v_t$ be a minimal cycle in $G^c_{(x)}$. Our aim is to show that $t = 3$. Firstly note that:

$$V(G^c_{(x)}) = V(G_{(x)}) = N_G(x) \cup \{y : \{y, x_i\} \in E(G), y \notin N_G(x) \text{ and } x_i \in N_G(x)\}. \tag{2.10}$$

Set $A = \{y : \{y, x_i\} \in E(G), y \notin N_G(x) \text{ and } x_i \in N_G(x)\}$. Then we seperate our proof into four possibilities:

- If $\{v_1, \ldots, v_t\} \subseteq N_G(x)$, none of these vertices is in A. As for all $i = 1, \ldots, t$, v_i is a neighbor of x, then from the definition of $G_{(x)}$, any edge in $G_{(x)}$ is an edge in G. So, any cycle in ${G^c}_{(x)}$ is a cycle in G^c. Hence, the induced subgraph on $\{v_1, \ldots, v_t\}$ in G^c is again a cycle. As G^c is chordal, we conclude $t = 3$.

- If only one v_i belongs to A, then the same explanation is satisfied. So, again the induced subgraph on $\{v_1, \ldots, v_t\}$ in G^c is a cycle. Hence, $t = 3$.

- If exactly two of $v_1, \ldots, v_t$ is in A, say v_i and v_j, then $\{v_i, v_j\} \notin E(G_{(x)})$, and hence $\{v_i, v_j\} \in E(G^c_{(x)})$. By the minimality of the cycle $v_1, \ldots, v_t$, we must have $\{v_i, v_j\}$ as one of the edges in the cycle. Also, in G^c, x is not adjacent with any of elements in the cycle but v_i and v_j. Therefore, $v_1, \ldots, v_i, x, v_j, \ldots, v_t$ is a cycle in G^c. Hence, $t = 3$.

- Let three or more vertices of the cycle $v_1, \ldots, v_t$ belong to A, then, none of them is adjacent with each other in $G_{(x)}$, in other words, they form a clique in $(G_{(x)})^c$ of size ≥ 3. Hence $t = 3$, because $v_1 \ldots v_t$ is minimal cycle and contains a clique.

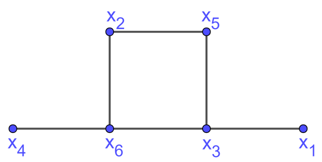
(ii)⇒(i) Let G^c has a cycle with length ≥ 4. As $(G_{(x)})^c$ is chordal, we must have that cycle passing through x. Let $x, v_1, v_2, \ldots, v_t$ with $t \geq 3$ be this cycle. Then we separate our proof into two cases:

- When $t = 3$. Then x, v_1, v_2, v_3 is a cycle of length 4 in G^c. Then $\{x, v_2\}$, $\{v_1, v_3\} \in E(G)$. Also we note that $v_2 \in N_G(x)$, but $v_1, v_3 \notin N_G(x)$. Because of this and as $G \setminus (N_G(x) \cup N_G(v_2))$ has no edge(from part(c)), one of v_1 or v_3 should belong to $N_G(v_2)$, because otherwise $\{v_1, v_3\}$ would be an edge of $G \setminus (N_G(x) \cup N_G(v_2))$. So, either $\{v_1, v_2\}$ or $\{v_2, v_3\}$ should be an edge of G. But, this is in contradiction with the fact that both $\{v_1, v_2\}, \{v_2, v_3\} \in E(G^c)$. Hence, G^c is chordal.

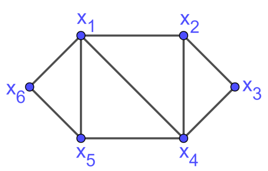
- When $t \geq 4$ we have $x, v_1, v_2, \ldots, v_t$ a cycle in G^c. So, obviously $\{x, v_2\}, \{x, v_3\}, \ldots \{x, v_{t-1}\} \in E(G)$, which implies $\{v_2, \ldots, v_{t-1}\} \subseteq N_G(x)$, and $\{x, v_1\}, \{x, v_t\} \in E(G^c)$, which implies $v_1, v_t \notin N_G(x)$. Also, $\{v_1, v_{t-1}\}, \{v_t, v_2\} \notin E(G^c)$, so they belong to $E(G)$, and we conclude that $v_1, v_2 \in A$, where A is from (2.10). Then, as $v_1, v_t \notin N_G(x)$ we have $\{v_1, v_t\} \notin E(G_{(x)})$, which implies $\{v_1, v_t\} \in E(G^c_{(x)})$. In addition, all $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{t-1}, v_t\}$ belongs to $E(G^c_{(x)})$. Hence, we have the minimal cycle $v_1, v_2, \ldots, v_t$ in $E((G_{(x)})^c)$ where $t \geq 4$, which contradicts part (b).

This completes the proof. □

Example 2.2.9. *Consider the following graph G and its complement G^c.Note that G^c is chordal. Then, choose $x_2 \in V(G)$. From the notation in (2.8), as $N_G(x_2) =$*



The graph G



The graph G^c

$\{x_5, x_6\}$, $N_G(x_5) = \{x_3\}$, and $N_G(x_6) = \{x_4, x_3\}$, we have: G_5 is the graph of the isolated vertices x_1, x_4 and x_2, and G_6 is the graph of isolated vertices x_1 and x_2. Also, $G_{(x_2)}$ is the path x_4, x_6, x_3, x_5, and its complement is the path x_3, x_4, x_5, x_6, which obviously is chordal.

Finally, we give a proof of Fröberg's Theorem.

Theorem 2.2.10. *(Fröberg's Theorem)[20] Let G be a graph. Then $I(G)$ has a linear resolution if and only if G^c is a chordal graph.*

Proof. We break our proof into two cases:

Case 1: When $V(G) = \{x, x_1, x_2, \ldots, x_n\}$ and $E(G) = \{\{x, x_i\} : 1 \leq i \leq n\}$. Then, we have $I(G) = (xx_1, \ldots, xx_n)$, which from Remark 1.2.3 has a linear resolution, and as G is an n-claw we have G^c as a complete graph of vertices $V(G^c) = \{x_1, \ldots, x_n\}$, which obviously is chordal.

Case 2: We use induction on the number of vertices of G. In the basis step we show for $|V(G)| = 1, 2$, and 3. Statement holds trivially if $|V(G)| = 1$. Let $|V(G)| = 2$, then we have $I(G) = (xy)$, and it has a linear resolution, and also G^c is chordal as it consist of only two isolated vertices. If G is edgeless then G^c is the graph containing one edge, assertion is clear. Let $|V(G)| = 3$, then if we have one isolated vertex, which means we have a path, we are done from the first case with $n = 1$. If G is a cycle of length 3, that it is the graph C_3, then C_3^c consist of three isolated vertices and it is chordal. Moreover, we have $I(G) = (xy, xz, yz)$, which has a $2-$linear resolution. Hence, the theorem holds for any case where $|V(G)| \leq 3$.

In addition, we continue with the inductive step, where we assume that our theorem holds for any graph with $|V(G)| \leq n-1$. Now we prove it when $|V(G)| = n \geq 4$. Then there exists a vertex x such that $G \setminus \{x\}$ is not a graph of isolated vertices, because otherwise we would have Case 1. Then, from Corollary 2.2.5 we have a Betti splitting $I(G) = J + N$, where $J = (xx_1, \ldots, xx_n)$ and $N = I(G \setminus x)$. So, we have

$$\beta_{i,j}(I(G)) = \beta_{i,j}((xx_1, \ldots, xx_n)) + \beta_{i,j}(G \setminus x) + \beta_{i-1,j}(J \cap N), \tag{2.11}$$

where from Lemma 2.2.7 we have

$$J \cap N = xI(G_{(x)}) + xx_1 I(G_1) + \cdots + xx_n I(G_n).$$

($\Rightarrow$) Suppose that $I(G)$ has a linear resolution. Then this implies that J and N has a linear resolution because we have a Betti splitting, and clearly $J \cap N$ has it too. By induction $G_{(x)}^c$ is chordal. In addition, because of $J \cap N$ has a linear resolution, it is generated in the same degrees. So, it cannot have degree three or four generators, hence $J \cap N = xI(G_{(x)})$. Hence, G_i has no edges for any i. Finally, as $J \cap N = xI(G_{(x)})$ has a linear resolution, then $I(G_{(x)})$ also has. So, by induction $G_{(x)}^c$ is chordal. Therefore, applying Lemma 2.2.8 we conclude G^c is chordal.

($\Leftarrow$) Now, suppose G^c is chordal. From Remark 1.2.3, $(xx_1, \ldots, xx_n)$ always has a linear resolution. By Lemma 2.2.8 we have all G_i have no edges, so $J \cap N = xI(G_{(x)})$, where $G^c_{(x)}$ is chordal. Then, by induction we get that $J \cap N$ has a linear resolution. Finally, from the Betti splitting formula (2.11) we conclude that $I(G)$ has a linear resolution. $\square$

Example 2.2.11. *Consider the graph G in Example 2.2.9. Then, G^c is chordal, and*

$$I(G) = (x_1x_3, x_3x_5, x_5x_2, x_2x_6, x_6x_3, x_4x_6).$$

The graded minimal free resolution of $I(G)$ is as follows:

$$0 \to S(-4)^3 \to S(-3)^8 \to S(-2)^6 \to I(G) \to 0.$$

which is a 2-linear resolution and $\operatorname{reg}(I(G)) = 2$.

2.3 Lower and upper bounds

In this part where we show results about the bounds of the regularity of any edge Ideal of a simple graph. We will see that the role of the induced matching number and the matching number of the graph is significant. So, before we start let us give the following example.

Example 2.3.1. *Consider the graph in Figure 2.3.*

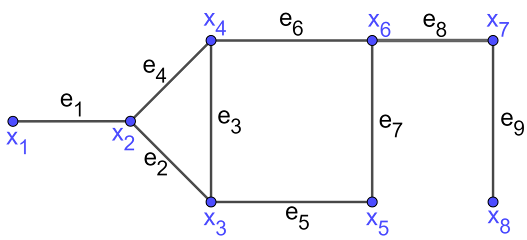


Figure 2.3

Then $A = \{e_4, e_5, e_9\}$ is a maximal matching and it is of minimum size. Also $B = \{e_1, e_7\}$ is an induced matching which is of maximal size. So

$$\nu(G) = |B| = 2, \text{ and } \beta(G) = |A| = 3.$$

Theorem 2.3.2. *[33, Lemma 2.2] Let G be a simple graph . Then,*

$$\operatorname{reg}(I(G)) \geq \nu(G) + 1.$$

Proof. Let $\nu(G) = k$ and $A = \{e_1, \ldots, e_k\}$ be the set of an induced matching of maximum size in G. Then, let H be the subgraph of G which has $V(H) = A$. Note that $E(H)$ consists of disjoint edges. Therefore, by Corollary 2.1.6 we have $\operatorname{reg}(S/I(H)) = k$. So, $\operatorname{reg}(I(H)) = k+1$. Hence, by Lemma 2.1.2 we are done. □

As we are done with the lower bound of the regularity, we continue with the upper bound of any edge ideal of a simple graph. Here we first give the proof of a stronger result of $\operatorname{reg}(I(G))$ which is bounded by co-chord(G), then we conclude the weaker one which depends on $\beta(G)$.

Definition 2.3.3. *The* co-chordal number *of G is denoted by co-chord(G) and is defined to be the least number of co-chordal subgraphs of G which cover the edges of G.*

Theorem 2.3.4. *[53, Lemma 1] Let G be a simple graph. Then*

$$\operatorname{reg}(I(G)) \leq \textit{co-chord}(G) + 1$$

Proof. Let co-chord$(G) = m$ and $H_1, \ldots, H_m$ be a co-chordal cover of G. As H_i^c are chordal for all i, from Fröberg's Theorem we have for each $i = 1, \ldots, m$, $\operatorname{reg}(H_i) = 2$. This implies that $\operatorname{reg}(S/I(H_i)) = 1$. Note that $E(G) = \cup_{i=1}^{m} E(H_i)$. So, using Corollary 2.1.5 we get $\operatorname{reg}(S/I(G)) \leq m$, which implies $\operatorname{reg}(I(G)) \leq m + 1$. □

Now, we give the following theorem which give us another upper bound even if it is weaker than the previous one. Note that the reason why we give this weaker result is that in general the computation of $\beta(G)$ is much easier than the computation of co-chord(G).

Theorem 2.3.5. *[23, Theorem 6.7],[53, Theorem 11] Let G be a simple graph. Then,*

$$\operatorname{reg}(I(G)) \leq \beta(G) + 1.$$

Proof. Let $\{e_1, \ldots, e_m\}$ be a maximal matching of minimal size in G. Also, let H_i be the induced subgraph of G such that $E(H_i)$ consists $e_1, \ldots, e_m$ and all edges in G adjacent to each e_i for $i = 1, \ldots, m$. Then, $H_1, \ldots, H_m$ forms a co-chordal cover of G. Therefore, co-chord$(G) \leq \beta(G)$. So, from Theorem 2.3.4 we are done. □

Example 2.3.6. *Consider the graph in Figure 2.3. Then*

$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_2x_4, x_3x_5, x_5x_6, x_4x_6, x_6x_7, x_7x_8),$$

and its graded minimal free graded resolution is

$$0 \to S(-7)^3 \to S(-6)^{13} \to S(-4)^5 \oplus S(-5)^{18} \to S(-3)^{13} \oplus S(-4)^8 \to S(-2)^9 \to I(G) \to 0.$$

So, from Theorem 2.3.2 and Theorem 2.3.5 we have

$$\operatorname{reg}(I(G)) = 4 \textit{ or } \operatorname{reg}(I(G)) = 3.$$

From the graded minimal free resolution of $I(G)$ *we conclude that* $\operatorname{reg}(I(G)) = 3$.

Now consider a graph G in which $V(G)$ can be partitioned into sets $I_0, I_1, \ldots, I_k$ where I_0 is an independent set and C_i is a clique for each $i = 1, \ldots, k$. Also let H_i be subgraphs of G which have those edges which are incident to at least one vertex of I_i. Then H_i is partitioned to I_i as the clique and $V(G \setminus V(I_i))$ as an independent set, thus it is a split graph. So, $H_1, \ldots, H_n$ is a split graph covering. Therefore, we get the following theorem.

Theorem 2.3.7. *[53, Theorem 2] Let* G *be a simple graph in which* $V(G)$ *can be partitioned into sets* $I_0, I_1, \ldots, I_k$ *as mentioned previously. Then*

$$\operatorname{reg}(I(G)) \leq k + 1$$

Proof. Let $H_1, \ldots, H_k$ be a split graph covering as mentioned previously. Then it is a co-chordal covering of G. Hence by Theorem 2.3.4 we are done. □

Now we discuss the regularity of an interesting class of graphs which is Gap-free graphs. But, finding an exact value or an exact bound for the regularity of such graphs is not easy. So, here we consider the more special case by adding one more property to be satisfied which is that the graph should be also a claw-free graph. Before proving the theorem about the regularity we prove a proposition which will be used later. Recall that if $x, y \in V(G)$, then $d(x, y)$ denotes the length of the shortest path between x and y.

Proposition 2.3.8. *[12, Proposition 3.2] Let* G *be a gap-free graph. If* x *is a vertex of maximal degree we have* $d(x, y) \leq 2$ *for all vertices* $y \in V(G)$.

Proof. Assume that for all $y \in V(G)$ we have $d(x, y) > 2$. So, there exists $y \in V(G)$ such that $d(x, y) = 3$. Let $\deg(x) = k$ and $N_G(x) = \{x_1, \dots, x_k\}$. We may choose x_1 such that $\{x_1, z\}, \{z, y\} \in E(G)$. By the assumption we have that $\{x, x_i\}$ and $\{z, y\}$ do not form a gap for any $i \in \{1, \dots, k\}$. So, there must be an $e_i \in E(G)$ such that it has one endpoint in $\{x, x_i\}$ and one in $\{y, z\}$ for each $i \in \{1, \dots, k\}$. As we have $d(x, y) > 1$ and $d(x, z) > 1$, these e_i's cannot be incident to x. Also, $\{x_i, y\} \notin E(G)$ for all $i \in \{1, \dots, k\}$, as then $d(x, y) \leq 2$ which contradicts the assumption. So, $\{x_i, z\} \in E(G)$ for all $i \in \{1, \dots, k\}$. Hence, $\deg(z) > k$, which gives us a contradiction. □

Theorem 2.3.9. *[12, Theorem 3.4],[39, Theorem 1.2] Let G be a simple gap-free and claw-free graph. Then*

$$\operatorname{reg}(I(G)) \leq 3.$$

Proof. Here we use induction on the number of vertices of G. When $|V(G)|$ =2 or 3, we have nothing to prove, as in any case the graph is gap-free and claw-free and its regularity does not exceed 3. So we are done with the basis case. Assume that the theorem holds when $|V(G)| = n - 1$. Let x be of the highest degree. Note that by removing one of the vertices of G the graph remains claw-free and gap-free. Therefore, both $G \setminus x$ and $G \setminus N_G[x]$ are claw-free and gap-free. Using induction we conclude that $\operatorname{reg}(G \setminus x) \leq 3$. Now we prove that $(G \setminus N_G[x])^c$ is chordal.
Suppose that $v_1, \dots, v_m$ form a cycle in $(G \setminus N_G[x])^c$ with $m \geq 4$. Then, by Proposition 2.3.8 we have all vertices of G have distance two to x. So, there is $y \in G$ such that $\{x, y\}$ and $\{y, v_1\}$ are edges in G. Note that one of $\{y, v_2\}$ or $\{y, v_m\}$ should be in $E(G)$, because if not then $\{v_2, v_n\}$ and $\{x, y\}$ form a gap in G. So, without loss of generality $\{y, v_m\} \in E(G)$. Then we have $\{x, y, v_1, v_m\}$ is a claw in G, which contradicts the assumption.
In addition, as $(G \setminus N_G[x])^c$ is chordal by Fröberg's theorem we get $\operatorname{reg}(G \setminus N_G[x]) = 2$ and by Lemma 2.1.4 we have $\operatorname{reg}(G) \in \{\operatorname{reg}(G \setminus N_G[x]) + 1, \operatorname{reg}(G \setminus x)\}$. Hence, we are done. □

The same result is true for gap-free and cricket-free graphs. Note that a claw-free graph is also a cricket free graph. The proof of the following theorem is similar to that for gap-free and claw-free graphs.

Theorem 2.3.10. *[2, Theorem 3.4] Suppose G is a cricket-free and gap-free graph. Then*

$$\operatorname{reg}(I(G)) \leq 3.$$

We also have such a result which is generalized to gap-free and n-claw-free graphs.

Theorem 2.3.11. *[2, Theorem 3.5] Let G be a simple gap-free and n-claw-free graph. Then*

$$\operatorname{reg}(I(G)) \leq n.$$

Proof. As for $n = 3$ we are done by the previous theorem, we assume that $n \geq 4$. Again we use induction on $|V(G)|$. The basis case is obvious and exactly the same as the basis case in the previous theorem. Assume that the theorem holds for $|V(G)|-1$. So, from induction we have $\operatorname{reg}(G \setminus x) \leq n$. By Lemma 2.1.4 it is enough to show that $\operatorname{reg}(G \setminus N_G[x]) \leq n-1$, so it is enough to show that $G \setminus N_G[x]$ is $(n-1)$-claw free.

Let $x_1, x_2, \dots, x_m$ be an $(n-1)$-claw in $G \setminus N_G[x]$ with root at x_1. Then for any $y \in N_G(x)$ we have that it is connected with x_1 or all $x_2, \dots, x_n$ in G. Otherwise, we have $\{x, y\}$ and $\{x_1, x_i\}$ forming a gap in G, which would be a contradiction. If x_1 is connetcted to all elements of $N_G(x)$ then we have $\deg(x_1) > \deg(x)$, which contradicts the assumption in the statement of the theorem. Therefore, there is a $v \in N_G(x)$ such that it is not connected to x_1, but it is connected to all $x_2, \dots, x_m$. Also, knowing that $x \notin N_G(x_i)$ for any $i \in \{1, \dots, m\}$, we get $\{x, v, x_2, \dots, x_m\}$ as an n-claw in G with root at v. Hence, we have a contradiction. □

Example 2.3.12. *Let G be the graph in Figure 2.4. Note that it is a gap-free and 4-claw-free graph. Then,*

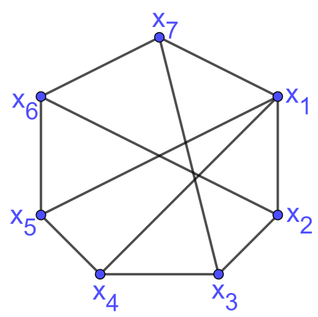


Figure 2.4

$$I(G) = (x_1x_2, x_2x_3, x_4x_3, x_4x_5, x_6x_5, x_6x_7, x_7x_1, x_1x_4, x_1x_5, x_3x_7, x_6x_2).$$

So, the graded minimal free resolution of $I(G)$ is as follow:

$$0 \to S(-7) \to S(-5)^4 \oplus S(-6)^3 \to S(-4)^{17} \oplus S(-5)^2 \to S(-3)^{23} \to S(-2)^{11} \to I(G) \to 0.$$

So, $reg(I(G)) = 3 \leq 4$. Note that $\{x_7, x_1\}$, $\{x_7, x_3\}$ and $\{x_7, x_6\}$ form a claw in G.

Now we state a result about regularity of edge ideals of planar graphs. A graph is called *planar* graph if it can be drawn in a way that no pair of edges cross. Even though the regularity of a planar graph can be as large as desired we see that the regularity of its complement is bounded.

Theorem 2.3.13. *[53, Proposition 19] Let G be a planar graph. Then*

$$\operatorname{reg}(I(G^c)) \leq 4.$$

The proof of this theorem is based on the arguments related to the "boxicity" of G.

2.4 Exact Values

It is difficult to find the exact value for the regularity of an edge ideal. That is why we do not know much about it. However, for some special cases we have good results. We have seen that the induced matching number was essential for the bounds of the regularity of edge Ideals. Now we will see that it plays a big role to find the exact value of regularity of edge ideals of some special classes of graphs.

Definition 2.4.1. *Let M be a finitely generated graded module in S. Then , M is called sequentially Cohen-Macaulay(written SCM) if there exists a finite filtration*

$$0 = M_o \subset M_1 \subset M_2 \subset \cdots \subset M_r = M,$$

of M by submodules M_i such that M_i/M_{i-1} is Cohen-Macaulay, and

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

Definition 2.4.2. *We say that a simple graph G is Sequentially Cohen-Macaulay if $S/I(G)$ is Sequentially Cohen-Macaulay.*

So, we have two general results. For any simplicial complex we should have the property of shellability in order to have S/I_Δ SCM.

Theorem 2.4.3. *[44] Let Δ be a simplicial complex, and suppose that S/I_Δ is the associated Stanley-Reisner ring. If Δ is shellable, then S/I_Δ is sequentially Cohen-Macaulay.*

And, we have the following SCM property of any simple graph G.

Theorem 2.4.4. *[49, Theorem 3.3] Let $x \in V(G)$ be any vertex of G and set $H = G \setminus (\{x\} \cup N_G(x))$. If G is SCM, then H is SCM.*

When G is a bipartite graph and it has the property of being SCM we have the following useful property.

Theorem 2.4.5. *[49, Theorem 3.9] Let G be a bipartite graph. If G is SCM, then there exists a vertex $x \in V(G)$ such that* $\deg(x) = 1$.

In addition, we can prove the following theorem in the case G is a bipartite graph.

Theorem 2.4.6. *[47, Theorem 2.10] Let G be a bipartite graph. Then the following are equivalent:*

1. *G is SCM,*

2. *G is shellable,*

3. *G is vertex decomposable.*

Proof. From Theorem 1.1.8 we have 3. $\Rightarrow$ 2. and from Theorem 2.4.3 we get 2. $\Rightarrow$ 1. So to finish the proof we have just to show 1. $\Rightarrow$ 3.

Here we use induction on $|V(G)| = n$. For $n = 2$ we have only one edge, so its graph is SCM, and also the complex is a simplex which means it is vertex decomposable. Hence, the theorem holds in this case. So, assume that we have $n \geq 2$. From Theorem 2.4.5 there exists a vertex x with degree 1 and let $N_G(x) = \{y\}$.

Set $G_x = G \setminus N_G[x]$ and $G_y = G \setminus N_G[y]$. Then from Theorem 2.4.4 we have both of G_x and G_y SCM. So by induction we get both of them vertex decomposable. Denote by H the graph G_x added the vertex x. Then, H is the same with the graph $G \setminus y$, since x and y are adjacent. So, from Lemma 1.1.5, vertex decomposability of G_x gives H vertex decomposable. Therefore, $G \setminus y$ and $G \setminus N_G[y]$ both are vertex decomposable. Note that here we get $N_G[x] \subseteq N_G[y]$. Hence, from Lemma 1.1.6 we are done. □

From now on in this section we will use the following definition of a vertex decomposable graph G which is equivalent with Definition 1.1.4.

Definition 2.4.7. *We say that a vertex x of G is a shedding vertex if for every independent set A in $G \setminus N_G[x]$, there is some vertex $v \in N_G(x)$ so that $A \cup v$ is independent. A graph G is called vertex decomposable if either it is an edgeless graph(a graph with no edges) or it has a shedding vertex x such that $G \setminus x$ and $G \setminus N_G[x]$ are both vertex-decomposable.*

We say that a vertex x in G is codominated if there is a vertex $y \in V(G) \setminus x$ such that $N_G[y] \subseteq N_G[x]$.

Then, we have the following lemma for C_5-free graph.

Lemma 2.4.8. *[34, Lemma 2.3] Let G be a C_5-free graph. If x is a shedding vertex, then it is codominated.*

Proof. We will prove this by contradiction. Let x be a shedding vertex such that $N_G(x) = \{x_1, \ldots, x_m\}$ and suppose that there is no such $y \in V(G) \setminus \{x\}$ such that $N_G[y] \subseteq N_G[x]$. Then, there exists a vertex $z_i \in N_G(x_i) \cap (V(G) \setminus N_G[x])$ for all $i = 1, \ldots, m$. If we have for some $i \neq j$, where $i, j \in \{1, \ldots, m\}$, z_i adjacent to z_j then x, x_i, z_i, z_j would form a cycle C_5 as an induced subgraph of G, which would be a contradiction. So, $B = \{z_1, \ldots, z_m\}$ form an independent set in $G \setminus N_G[x]$. Let A be a maximal independent set which contains B. Then A is maximal independent set in $G \setminus \{x\}$ also. As we assumed that x is a shedding vertex, we have a contradiction. Hence, x is codominated. □

From Theorem 2.4.6 any SCM bipartite graph is vertex decomposable. Also, any bipartite graph has no odd cycle. So, in the following theorem (1) implies (2) and (3).

Theorem 2.4.9. *If G is a simple graph, then*

$$\operatorname{reg}(I(G)) = \nu(G) + 1$$

in the following cases:

1. *G is vertex decomposable graph and C_5-free;[34, Theorem 2.4]*
2. *G is a (C_4, C_5)-free vertex decomposable graph;[4, Theorem 24]*
3. *G is sequentially Cohen-Macaulay bipartite graph;[47]*

Proof. We will prove only the first statement. We know from Theorem 2.3.2 that for any graph, $\operatorname{reg}(I(G)) \geq \nu(G) + 1$.

(1) Let G be a vertex decomposable and C_5-free graph. From Theorem 1.2.8 we have just to show that $\operatorname{proj}\dim(I(G)^\vee) \leq \nu(G)$. Let $|V(G)| = n$. We use induction on the number of vertices n. If $n = 2$, then we have either a total disconnected graph or a single edge graph. If G is totally disconnected then $I(G)^\vee = 0$ and $\operatorname{proj}\dim(I(G)^\vee) = 0$. If G is a single edge graph then $I(G)^\vee = \langle x_1, x_2 \rangle$ and $\operatorname{proj}\dim(I(G)^\vee) = 1 \leq 1$. Note that in both cases the graph is obviously vertex decomposable and C_5-free graph.

So, let $n \geq 2$. From the definition 2.4.7 there exists a shedding vertex $x \in V(G)$ such that both $G \setminus \{x\}$ and $G \setminus N_G[x]$ are vertex decomposable. Note that both $G \setminus \{x\}$ and $G \setminus N_G[x]$ are C_5-free graphs. Now, let $N_G(x) = \{x_1, \ldots, x_m\}$. By induction, we have $\operatorname{proj}\dim(I(G \setminus \{x\})^\vee) \leq \nu(G \setminus \{x\})$ and $\operatorname{proj}\dim(I(G \setminus N_G[x])^\vee) \leq \nu(G \setminus N_G[x])$. So, from Lemma 2.1.4 we have:

$$\operatorname{proj}\dim(I(G)^\vee) \leq \max\{\nu(G \setminus \{x\}), \nu(G \setminus N_G[x]) + 1\}.$$

Obviously $\nu(G \setminus \{x\}) \leq \nu(G)$. So, we have to show that inequality holds for the other part. From Lemma 2.4.8, x is codominated, hence there exists a $y \in V(G \setminus \{x\})$ such that $N_G[y] \subset N_G[x]$. Therefore, it is no problem if we add $\{x, y\}$ to any of induced matching in $G \setminus N_G[x]$, which will give us an induced matching of G. So $\nu(G \setminus N_G[x]) + 1 \leq \nu(G)$. Hence, we are done. □

Example 2.4.10. *Let G be the graph in the following figure. Then,the edge ideal related to this graph is as follows:*

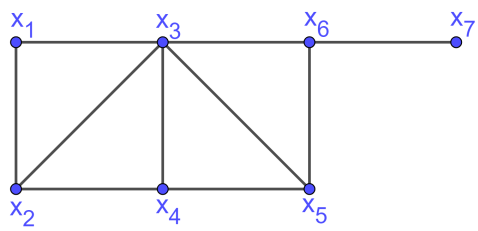


Figure 2.5

$$I(G) = (x_1x_2, x_1x_3, x_4x_3, x_2x_3, x_2x_4, x_3x_5, x_4x_5, x_3x_6, x_5x_6, x_6x_7).$$

So, the graded minimal free resolution of $I(G)$ is as follows:

$$0 \to S(-6) \oplus S(-7) \to S(-5)^5 \oplus S(-6)^5 \to S(-4)^{14} \oplus S(-5)^7 \to S(-3)^{19} \oplus S(-4)^3$$
$$\to S(-2)^{10} \to I(G) \to 0.$$

G is vertex decomposable graph, such that $\nu(G) = 2$ and $\operatorname{reg}(I(G)) = 3$.

Definition 2.4.11. *A graph G is said to be a weakly chordal graph if neither G nor G^c contains any C_n, for $n \geq 5$.*

Note that the graph in Figure 2.5 is a (weakly) chordal graph. Let G be a graph. Then, denote by G_L the graph whose vertices are edges of G, and $e_i, e_j \in E(G)$ are nonadjacent in G_L if and only if they form a gap in G. Note that any co-chordal subgraph of G maps to a clique in G_L. Also, if we denote by $\alpha(G_L)$ the size of the largest independent set in G_L, then $\nu(G) = \alpha(G_L)$.

It is well known that if G is weakly chordal graph then G_L is weakly chordal, see [9]. Further, from [25] we have that every weakly chordal graph is perfect.

Definition 2.4.12. *The clique cover of G_L is defined to be the partition of the vertices of G_L into cliques. The minimal clique number of G_L is denoted by $\theta(G_L)$ and is defined to be the minimum number of cliques needed to form a clique cover of G_L.*

So, if G is a weakly chordal graph, then $\nu(G) = \alpha(G_L) = \theta(G_L)$. We know that for any graph G, $\nu(G) \leq \text{co-chord}(G)$. Hence, if G is weakly chordal then $\theta(G_L) \leq \text{co-chord}(G)$.

Definition 2.4.13. *We say that an edge $\{x, y\}$ in G is a co-pair of the graph G, if x and y are end vertices of any path P_k, for $k \geq 4$, in G^c.*

The following two propositions are well known when G is a weakly chordal graph.

Proposition 2.4.14. *[43] If $e \in E(G)$ is a co-pair of G, then G is weakly chordal graph if and only if $G \setminus e$ is weakly chordal.*

Proposition 2.4.15. *[15] Let G be a weakly chordal graph which has a gap in it. Then, G contains co-pairs e and f such that they form a gap in G.*

Lemma 2.4.16. *[7, Lemma 6] Let G be a weakly chordal graph. If e is a co-pair of G, then $G_L \setminus e = (G \setminus e)_L$.*

Proof. Firstly, note that the vertices of $G_L \setminus e$ and $(G \setminus e)_L$ consist of edges of $G \setminus e$. If $\{x, y\} \in E(G)$ is not one of the edges that form a gap in G, then removing it from G never destroys the gap. If when we delete delete the edge $\{x, y\}$ a new gap is formed, then $\{x, y\}$ must be the middle edge of a path P_4 in G. So, x and y are the end vertices of a path P_4 in G^c. Hence, for any co-pair e in G, two edges form a gap in $G \setminus e$ if and only if they form a gap in G that does not include e. Thus, the edge set of $G_L \setminus e$ and $(G \setminus e)_L$ are identical. Hence, $(G \setminus e)_L = G_L \setminus e$. □

Observe that from the definition of $\theta(G_L)$ every member of a clique cover of G_L can be assumed to be maximal, which is there is no other clique of G_L which contains it. The same is possible from Definition 2.3.3 for members of a co-chordal cover of G.

Theorem 2.4.17. *[7, Theorem 7] Let G be a weakly chordal graph. Then, every maximal clique of G_L is the edge set of a maximal co-chordal subgraph of G.*

Proof. Here we use induction on $|E(G)|$. When G is an edgeless graph or a graph with only one edge in it, then obviously the theorem holds. Assume that for any weakly chordal graph with $|E(G)|$ up to $n - 1$, the statement is true. Let G be a weakly chordal graph such that $|E(G)| = n$. Now we have two cases:

Case 1: If G is gap-free, then G is co-chordal and G_L is itself a clique, so we are done.

Case 2: Suppose G has a gap. So, from Proposition 2.4.15 there are $e_1, e_2 \in E(G)$, both co-pair edges of G forming a gap in G. Let C be a maximal clique of G_L. Note that C can not contain both e_1 and e_2. So, without loss of generality choose $e_1 \notin C$. Hence, C is a maximal clique of $G_L \setminus e_1$. From Lemma 2.4.16 $G_L \setminus e_1 = (G \setminus e_1)_L$. Also, from Proposition 2.4.14, $G \setminus e_1$ is weakly chordal. Hence, by induction, C is the edge set of a maximal co-chordal subgraph of $G \setminus e_1$, and denote it by H. Note that H is also co-chordal in G. Now we claim that H is a maximal co-chordal subgraph of G. Suppose that it is not. So, there exists a co-chordal subgraph of G, call it H' such that $C \subset E(H')$. As every co-chordal subgraph of G maps to a clique of G_L, then $E(H')$ is a clique of G_L such that $C \subset E(H')$. So, the maximality of C gives us a contradiction. Hence, C is the edge set of a maximal co-chordal subraph of G. □

Then, we get the following theorem. Note that any chordal graph is weakly chordal. So, in this theorem (1) implies (2). Also, in [52, 18] we see that a chordal graph is vertex decomposable and in [47] we can see that a vertex decomposable graph G is SCM. So, as any chordal graph is C_5-free, we see that (2) of the following theorem is also a consequence of Theorem 2.4.9.

Theorem 2.4.18. *If G is a simple graph, then*

$$\operatorname{reg}(I(G)) = \nu(G) + 1$$

in the following cases:

1. *G is a weakly chordal graph;[53]*

2. *G is a chordal graph;[23]*

Proof. Here we prove only the first one.

From Theorem 2.4.17 we get co-chord$(G) \leq \nu(G)$. Hence, from Theorem 2.3.4 we get $\operatorname{reg}(I(G)) \leq \nu(G) + 1$. $\square$

Definition 2.4.19. *A graph G is said to be well covered if it has no isolated vertices and all maximal independent sets have the same cardinality. Moreover, if $|V(G)|$ is equal to two times the cardinality of maximal independent sets, then G is said to be very-well covered graph.*

Note that in [21] we can see that in a well covered graph we have $2\,\operatorname{height}(I(G)) \geq |V(G)|$. Also we know that the complement of any maximal independent set is a minimal vertex cover. So, if G is a well covered graph, then it is said to be a very-well covered graph if and only if $2\,\operatorname{height}(I(G)) = |V(G)|$. So, when G is a very-well covered graph we can assume that $|V(G)| = 2m$, for $m = \operatorname{height}(I(G))$. Note that any bipartite well covered graph is a very-well covered graph.

For very-well covered graphs and unmixed bipartite graphs, the exact value of regularity is known, as given in following:

Theorem 2.4.20. *If G is a simple graph, then*

$$\operatorname{reg}(I(G)) = \nu(G) + 1$$

in the following cases:

1. *G is very well-covered graph;[37]*

2. *G is unmixed bipartite graph.[36]*

Note that if G is unmixed bipartite graph, then it is a very-well covered graph. Therefore, (1) implies (2) in the Theorem 2.4.20. To prove (1), in [37], the authors first prove the following.

Theorem 2.4.21. *[37, Theorem 3.2] If G is very-well covered graph with $|V(G)| = 2m$ and G is Cohen-Macaulay. Then* $\operatorname{reg}(I(G)) = \nu(G) + 1$.

In the proof of above theorem, it is first proved that $\operatorname{proj}\dim(I(G)^{\vee}) \leq \nu(G)+1$. Then the Theorem 2.4.21 follows from Theorem 2.3.2 and Theorem 1.2.8. Then, the above result is improved by showing that, given any very-well covered graph G, there exists an associated semidirected graph $\widehat{G}$, such that $\widehat{G}$ is very-well covered, Cohen-Macaulay and $\nu(G) = \nu(\widehat{G})$.

Definition 2.4.22. *A graph G is said to be unicyclic if it is connected and contains exactly one cycle. If C_n is the only cycle of G, then we denote by $\Gamma(G)$ the collection of all neighbors of the roots in the rooted trees attaching C_n.*

In the following two theorems we can see that the regularity for an unicyclic graph G is $\nu(G) + 1 \leq reg(I(G)) \leq \nu(G) + 2$.

Theorem 2.4.23. *If G is an unicyclic graph with cycle C_n, then*

$$\operatorname{reg}(I(G)) = \nu(G) + 1$$

when

1. $n \equiv 0,\ 1 \ (mod\ 3)$*[1, 6, 29] or*
2. $\nu(G \setminus \Gamma(G)) < \nu(G)$*.[1]*

Theorem 2.4.24. *If G is a simple graph, then*

$$\operatorname{reg}(I(G)) = \nu(G) + 2$$

in the following cases:

1. *G is an n-cycle C_n when $n \equiv 2 \ (mod\ 3)$;[6, 29]*
2. *G is a unicyclic graph with cycle C_n when $n \equiv 2 \ (mod\ 3)$ and $\nu(G \setminus \Gamma(G)) = \nu(G)$.[1]*

Chapter 3

Regularity of Powers of Edge Ideals

In this chapter, we will discuss the regularity of powers of edge ideals. It is well known that if a graded ideal has linear resolution, then it's power need to have linear resolution as well. Nevertheless, there are classes of graded ideals which admits the property that if I^s has linear resolution for all $s \geq 1$. In context of edge ideals, it is natural to ask that whether they have also this property. As stated in previous chapter, we know that an edge ideal $I(G)$ has linear resolution if and only if G^c is chordal, that is, G is co-chordal. In [28, Theorem 3.2], Herzog, Hibi and Zheng proved that if a quadratic monomial ideal has linear resolution then all of its powers also have linear resolution. As a consequence of this, we have following result

Theorem 3.0.1. *Let G be a simple graph. The edge ideal $I(G)$ has a linear resolution if and only if $I(G)^s$ has a linear resolution for all $s \geq 1$.*

Therefore, the following corollary is obvious.

Corollary 3.0.2. *A graph G is co-chordal graph if and only if $\operatorname{reg}(I(G)^s) = 2s$ for all $s \geq 1$.*

Now, the next natural question is if there exists a graph G such that $I(G)$ do not have linear resolution, but $I(G)^s$ has linear resolution for some $s > 1$. The answer of this question is positive. This led to investigation of further classes of graphs which have the property that even though their edge ideals does not have linear resolution, but the higher powers of their edge ideals admit linear resolution.

It is mentioned in [40, Proposition1.8] that if $I(G)^s$ has linear resolution for some s, then G^c is C_4-free. Also, in [40], Nevo and Peeva gave an example of a graph whose complement is C_4-free such that $I(G)^2$ does not have a linear resolution. Based on these observaion, they stated an open problem

Question 3.0.3. *[40, Question 1.11] Is it true that G^c is C_4-free if and only if $I(G)^s$ has a linear resolution for every $s \gg 0$?*

In particular, Nevo and Peeva raised the following question:

Question 3.0.4. *[40, Question 1.11(2)] Is it true that $I(G)^s$ has a linear resolution for all $s \geq 2$ if G^c is C_4-free and $\operatorname{reg} I(G) = 3$?*

The following result gives a positive answer of above question for special classes of gap-free graphs.

Theorem 3.0.5. *Let G be a simple graph. Then for $s \geq 2$, $\operatorname{reg}(I(G)^s) = 2s$ in following cases:*

1. *G is a grap-free and cricket-free graph;[2, Theorem 1.2]*
2. *G is a gap-free and diamond-free graph[14, Theorem 4.9]*

In particular, in above cases, $I(G)^s$ has a linear resolution.

3.1 Lower and Upper Bounds

In Chapter 2, we discussed some lower and bounds for $I(G)$ in terms of $\nu(G)$ and $\beta(G)$. These invariants of graph play a vital role in bounds of regularity of higher powers of $I(G)$ as well. These bounds also helps us to study the aysmptotic linear function of $\operatorname{reg}(I^s)$ for $s \gg 1$.

Theorem 3.1.1. *[6, Theorem 4.5] Let G be a simple graph with the edge ideal $I = I(G)$. Then*

$$\operatorname{reg}(I^s) \geq 2s + \nu(G) - 1,$$

for all $s \geq 1$.

Proof. Let $\{e_1, \ldots, e_r\}$ be an induced matching in G. Also, set $e_i := \{x_1, y_1\}$. Let H be the induced subgraph of G such that $V(H) = \{\{x_i, y_i\} : i = 1, \ldots, r\}$. Then,

$I(H) = (x_1y_1, \dots, x_ry_r)$ is a complete intersection i.e., it is generated by regular elements. So, from Lemma 1.2.5 we have:

$$\operatorname{reg}(I(H)^s) = 2s + (2-1)(r-1) = 2s + r - 1 = 2s + \nu(G) - 1.$$

As shown in Corollary 4.3 of [6] that if H is an induced subgraph of G then $\operatorname{reg} I(H)^s \leq \operatorname{reg} I(G)^s$, we get the following

$$\operatorname{reg}(I(G)^s) \geq \operatorname{reg}(I(H)^s) = 2s + \nu(G) - 1.$$

which completes the proof. □

Then we continue with the upper bound with is similar to that just given, but reduced to only bipartite graphs.

Theorem 3.1.2. *[31, Theorem 1.1] Let G be a bipartite graph with edge ideal $I = I(G)$. Then*

$$\operatorname{reg}(I^s) \leq 2s + \textit{co-chord}(G) - 1,$$

for all $s \geq 1$.

In relation to Question 3.0.4, Banerjee stated the following upper bound for gap-free graphs.

Theorem 3.1.3. *[2, Theorem 6.19] Let G be gap-free graph with edge ideal $I = I(G)$. Then*

$$\operatorname{reg}(I^s) \leq 2s + \operatorname{reg}(I) - 1,$$

for all $s \geq 2$.

In a recent paper [27], Herzog and Hibi gave another interesting upperbound.

Theorem 3.1.4. *Let G be a graph and $\Delta(G)$ be its independence complex. Then $\operatorname{reg}(I(G)^s) \leq 2s + \dim \Delta(G)$.*

Let G be a simple graph where $|V(G)| = n$ and $|E(G)| = m$. Hansen in [24] has proved that the size of a maximum independent set is bounded by $\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2m} \rfloor$. This gives the following corollary.

Corollary 3.1.5. *[27] Let G be a simple graph where $|V(G)| = n$ and $|E(G)| = m$. Then,*

$$\operatorname{reg}(I(G)^s) \leq 2s + \lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2m} \rfloor - 1,$$

for all s.

Many other upper bounds of size of maximal independent sets of G are known, see [51]. One of them is known as Kwok's upper bound given as exercise in [50], which is $n - \frac{m}{D}$, where D is the maximal degree among the degrees of vertices of G. So, we have the following Corollary.

Corollary 3.1.6. *Let G be a simple graph where $|V(G)| = n$ and $|E(G)| = m$. Also, let D be the maximal degree among the degrees of vertices of G. Then,*

$$\operatorname{reg}(I(G)^s) \leq 2s + n - \frac{m}{D} - 1,$$

for all s.

3.2 Exact Values for $\operatorname{reg}(I(G)^s)$ for some special classes of graphs

The asymptotic linear function of $\operatorname{reg}(I^s)$ for $s \gg 1$ is of special interest. Due to a theorem of Herzog, Cutkosky and Trung [10] and Kodiyalam [35], it is known that if I is a graded ideal of a standard graded K-algebra, then the regularity of I^s is asymptotically a linear function of s. In simple words, there exists constants a and b such that $\operatorname{reg}(I^s) = as + b$ when s is large enough. In last couple of years, several articles have appeared where authors studied this function. We point out some prominent results in this direction.

When G is a cycle, then following theorem gives the regularity of $I(G)$ and its powers. Note that $\nu(C_n) = \lfloor \frac{n}{3} \rfloor$.

Theorem 3.2.1. *[6, Theorem 5.2] Let $G = C_n$ with edge ideal $I = I(G)$. Then*

$$\operatorname{reg}(I) = \begin{cases} \nu(G) + 1 \ if \ n \equiv 0, 1 \ (mod\ 3) \\ \nu(G) + 2 \ if \ n \equiv 2 \ (mod\ 3) \end{cases}$$

for all $s \geq 2$, we have

$$\operatorname{reg}(I^s) = 2s + \nu(G) - 1.$$

The above result was generalized for unicyclic graphs, which include whiskered cycle graph as a particular case.

Theorem 3.2.2. *[1, Theorem 1.2] Let G be an unicyclic graph with edge ideal $I = I(G)$. Then*

$$\operatorname{reg}(I^s) = 2s + \operatorname{reg}(I) - 2,$$

for all $s \geq 1$.

Finally, we give an other list of some special graphs, for which $\operatorname{reg} I(G)^s$ is determined in terms of induced matching number of G.

Theorem 3.2.3. *Let G be a simple graph. Then, for any power $s \geq 1$*

$$\operatorname{reg}(I^s) = 2s + \nu(G) - 1,$$

in the following cases:

1. *G is a forest;[6, Theorem 4.7]*
2. *G is a very well-covered graph;[30, Theorem 5.3]*

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