ON RANDOM OBJECT ALLOCATION

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Submitted to the Graduate School of Social Sciences
in partial fulfilment of
the requirements for the degree of Master of Arts

Sabancı University
January 2020

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Date of Approval: January 28, 2020

ABSTRACT

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ECONOMICS M.A. THESIS, JANUARY 2020

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Keywords: probabilistic assignment, stability, alignment, rank respecting, discrimination, choice function

In this thesis, we study a linkage between object allocation problems and two-sided matching markets. Our main purpose is to analyse the desirable properties such as efficiency, respect for rank and no-discrimination, and associate them with well-known stability concept. We show that any rank respecting allocation could be interpreted a stable allocation of a specific matching market. Under certain circumstances, the allocation also exhibits no-discrimination. Also, we associate our two-sided matching market derived from an object allocation problem with aggregate efficiency concept. Moreover, we provide a process that yields the PS allocation.

ÖZET

RASSAL NESNE TAHSİSİ ÜZERİNE

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EKONOMİ YÜKSEK LİSANS TEZİ, OCAK 2020

Tez Danışmanı: Doç. Dr. MEHMET BARLO

Anahtar Kelimeler: olasılıksal atama, kararlılık, sıralamaya riayet, ayrımcılık, tercih fonksiyonu

Bu tezde nesne tahsis problemlerinin iki taraflı eşleşme piyasaları ile arasındaki ilişkiyi inceliyoruz. Temel amacımız, verimlilik, sıralamaya riayet, ayrımcılık karşıtlığı gibi arzu edilen özellikleri analiz etmek ve bu özellikleri iyi bilinen kararlılık kavramı ile ilişkilendirmektir. Sıralamaya riayet eden herhangi bir tahsisin, iki taraflı özel bir eşleştirme piyasasının kararlı tahsis sonucu olarak yorumlanabileceğini gösterdik. Belirli koşullar altında, bu tahsisat ayrımcılık karşıtı bir özellik de sergiler. Ayrıca bir atama piyasasından türettiğimiz iki taraflı eşleştirme piyasasını, toplam verimlilik kavramı ile de ilişkilendirdik. Dahası, PS tahsisini sağlayan bir süreç de tanımladık.

To my dear wife and beloved son.

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1. INTRODUCTION

Fair and efficient distribution of the indivisible objects has attracted attention and has been studied for decades. A random assignment problem is defined as a one-sided matching problem. Different solutions to this problem have been proposed and each of them has some strengths and some weaknesses. Characterization of the solution concepts has been developed and different properties for the solution have been added to the literature. Bogomolnaia and Moulin (2001)'s investigation on lack of efficiency in a random priority mechanism[1] has accelerated the debate for the object allocation problems.

This thesis examines an object allocation problem where each individual has a preference ordering over objects, but objects do not hold any preference relation. As a solution, we consider a random allocation matrix where each individual ends up with a combination of probability shares for objects. Each share indicates a probability that the individual is assigned the object. We care about the ex-ante efficiency as well as the notion of respect for rank. According to this concept, if an individual could be made better off by being assigned to an object with more probability and there exists another individual who receives positive probability from that object, then second individual should rank that object at least as high as does the first individual.[2]. We show that, in an object allocation problem where each individual has strict preference over objects whereas objects do not hold any preference ordering, each rank respecting allocation is ordinally efficient.

We find that approaching this problem and solution concept from a different angle widens our vision. For this purpose, we transform an object allocation market into a two-sided matching market. More specifically, we attribute a specific imaginary preference orderings to the objects in order to have a two-sided matching market. With this arrangement, we make our analysis based on a different framework where it will make us look to the problem from a different angle. The driving force behind

[1] See Abdulkadiroglu and Sönmez (1998)

[2] See Harless (2018)

the attribution of a priority ordering to an object is to reflect individuals' preferences to the objects. That is basically to say that objects enjoy being preferred. An object prefers the individual who likes him more over an individual who does not rank this object as high as the former. As we attributed such a preference ordering to objects, we jump on another domain where both sides have preference orderings.

Note that, beginning with the simple object allocation problem, we derive a new market associated with the object allocation problem, where both sides have a preference ordering over each other. It is noteworthy to underline that, attributed preference orderings are weak preference orderings. We also assume that each object has a choice function which corresponds to this weak preference ordering. We observe that the set of stable allocations is singleton if we consider the attributed preferences as choice functions that correspond to these preference orderings. Moreover, we prove that if an allocation is stable, which is a well-known property in matching literature, then it is rank respecting.

The family of choice functions can be thought of as a set of different choice functions that distinguish only in terms of way of choosing an individual among those who are is indifferent between this object. One example of a choice function is the following: Say the choice function assigns equal weight to the individuals in which the object is indifferent between the given individuals. This defines the well-known constraint equal award rule. If we allow the object to only use constrained equal award rule while choosing between the fractions of the individuals in the same preference level, we are safe to say that the resulting allocation will exhibits no-discrimination and be rank respecting.

In the second part of the thesis, given an object allocation problem, we introduce a dynamic process where objects are specialized to follow some rule, called constrained equal loss, and individuals are also taking a specific action. In this process, each individual demands a full fraction of his most favored object. Each object who receives a demand, calculate its excess demands and reduce the demands it receives equally. Upon receiving a rejection, each individual transfers his rejected proportion to the next best object. Note that his demand for the previous objects still valid and can be increased if the object he likes right before this object increases his rejection. We conclude that the sequence of the individuals is not important since the individuals and objects take robotic actions. Therefore, any subset of the individuals can update their demand on any subset of objects with any sequence. They all result in the same allocation.

The dynamic process exhibits a good property which will be useful for the process to converge. There exists a time where an object will issue a loss amount and will never change it again. That is to say that this object will completely distribute itself permanently. Any new demand will be completely rejected. Since a sequence of an individuals' move does not matter, we formulated this property in the sequence where individuals move simultaneously. In this regard, the time where an object is completely exhausted is the time where it has the maximum average excess demand among the other objects who are not exhausted. More clearly, in every step, there is at least one object who has the maximum average excess demand. In a pivot step, if an object has a maximum excess demand among the ones who never have the maximum average excess demand, then it permanently distributes itself at that step. This result guarantees that the process will end. Moreover, as the main result, we prove that the result of the dynamic process coincides with the Probabilistic Serial rule (PS rule)[3].

Even though Bogomolnaia and Moulin (2001) proved that PS rule results in a weak strategyproof allocation matrix, we observe that the dynamic process has an incentive problem. However, this dishonesty does not occur while reporting preferences. It occurs when the individuals do not follow the rule and misreport their demand vectors. Sometimes they cannot be sure to have a better allocation vector by misreporting, hence they can gamble. However, there exists some situation where individuals would have certainly the most desirable allocation vectors if they misreport the demand vectors.

The structure of the thesis is as follows: In chapter 2, we discuss the related literature and closest works to our study. In chapter 3, we introduce the model, give examples and provide main definitions and related results. In Chapter 4, we introduce a new process and show a crucial equivalence. Chapter 5 proves the propositions and chapter 6 contains the discussion.

[3] See Bogomolnaia and Moulin (2001).

2. RELATED LITERATURE

Starting with the seminal paper of Gale and Shapley (1962), the matching theory has been increasingly studied by many economists. Their framework captures a two-sided matching market with strict preferences. They constitute a marriage market and establish an algorithm to find a stable matching. However, Hylland and Zeckhauser (1979) map out the probabilistic object allocation problem and propose a solution derived from the competitive equilibrium approach. With an equal income, each individual consumes a probability share of an indivisible object. They are endowed with a von Neumann-Morgenstern utilities over random allocations of indivisible objects. The solution is efficient in the sense of both ex-ante and ex-post, assuming that the individuals report their utility values honestly. Zhou (1990) adds to the literature by proving the incompatibility of efficiency, fairness, and strategyproofness.

On the other hand, Abdulkadiroglu and Sönmez (1998) propose a mechanism called random priority, to achieve a fair and strategyproof allocation in an object allocation problem. Their design orders the individuals randomly and lets them pick an object from the available object set. Drawing an ordering of individuals from uniform distribution catches the fairness and also mechanism results in a strategyproof allocation. However, if the individuals are endowed with von Neumann-Morgenstern utilities over random allocations, the mechanism misses efficiency in ex-ante sense (Bogomolnaia and Moulin 2001). With this investigation, along with the matching theory, object allocation problems took more attention. Bogomolnaia and Moulin (2001) introduce a quite intuitive process for an object allocation problem to reach all the ex-ante efficient (ordinally efficient) allocations. The simultaneous eating algorithm proceeds as individuals simultaneously eat probability shares from objects. If an object is exhausted entirely, then individuals jump to the next best object and the time runs from 0 to 1. What they eat is considered as the probability of assigning this object. Different eating speeds result in different ordinal efficient allocations, assuming that the integration of a speed function is equal to 1. Therefore, the set of all ordinal efficient allocations can be captured by altering the eating speeds of individuals. Moreover, they state that a simultaneous eating algorithm with uniform eating speed (PS rule) results in a envy-free, ordinal efficient and weak strategyproof allocation. Bogomolnaia and Heo (2012) show that probabilistic serial rule (PS) is the unique rule that satisfies ordinal efficiency, envy freeness and bounded invariance.

As a contribution to the Bogomolnaia and Moulin (2001) results, Kojima (2009) generalize their process to allow individuals to have more than one object. He says that if all individuals are allowed to have q objects, then time extension to eating algorithm would still preserve the desired properties. Besides, (Budish et al. 2013) produce a generalization for both Bogomolnaia and Moulin (2001) and Hylland and Zeckhauser (1979) results. They extend the market in many ways such as multi-unit supply, multi-unit demand, the possibility of unassigned individuals and family of real-world constraints. They also allow for complicated preferences in a pseudo-market while generalizing the Hylland and Zeckhauser (1979) results.

A different angle for an object allocation problem is presented by Alkan and Gale (2003)'s schedule matching market. They examine a market with two sides, firms and workers, and allow workers to distribute their working hours between different firms. Moreover, firms can hire a worker for some hours bounded by a quota for firms. Both sides hold a choice function over these schedules. They relax the assumption of responsive preferences and jump to a broader space where substitutable choice functions reside. They prove the existence of stable matching by using the Gale - Shapley algorithm. These schedules can be considered as probability shares and the solution could be a random matching. In this aspect, Kesten and Ünver (2015) also generalize two-sided matching markets, rather than an object allocation problem, to have ordinally efficient solutions. They also use a generalization of the Gale - Shapley algorithm, where fractional acceptances and rejections are allowed. They propose two algorithms, one for not to lose important fairness properties (discrimination), the other for capturing ordinal efficiency.

In a deterministic market, besides all properties studied in the literature, Kojima and Ünver (2014) characterize a new property to improve efficiency. Favoring higher ranks guarantees that if a student prefers a school different than its match, then assigned students to the more preferred school by first student rank this school as higher as the first student. Dogan and Klaus (2018) also emphasize on rank-based axioms and analyze the deterministic markets.

On the other hand, we find that recent works about the rank based axioms on an object allocation problem are closest to our study. Harless (2018) defines a rank respecting axiom, on the same line with Kojima and Ünver (2014). He calls it respect for ranks, which is also similar to the justified envy axiom mentioned in Kesten and Ünver (2015). Harless (2018) characterize a rule to be employed under an algorithm, called immediate division. Each individual points out the most favorable object, and related object immediately distributed between individuals who point at it. This algorithm with a specific distribution rule characterizes immediate division rule. While distributing the objects, constrained equal awards rule is issued and hence no discrimination between individuals is eliminated. Under immediate division rule, objects start to distribute themselves equally between individuals by giving them an equally increasing share starting from 0. When one individual is satisfied, the others continue to get equal shares from the object. When an object is exhausted, if an individual is not satisfied, he brings his remaining part to the next best object. The resulting allocation satisfies ordinal efficiency, respect for rank and exhibits no-discrimination. Also, immediate division rule is the unique rule satisfying these properties.

As we mentioned above, Bogomolnaia and Moulin (2001) find all the ordinal efficient allocations. Besides, Harless (2019) introduces efficient rules and defines a recursive algorithm to find all ordinal efficient allocations. The driving force behind this algorithm is the use of a family of rules for selecting the set of objects to distribute and family of rules to distribute them. Leading rules are introduced and contain constrained equal awards and constraint equal loss rule. He also specifically indicate a pair of selection and distribution rule, which coincides with the result of PS rule.

3. OBJECT ALLOCATION PROBLEM AND ASSOCIATED MATCHING MARKETS

3.1 The Model

In this paper, our primary interest is to solve an object allocation problem. Let $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, n\}$ be two distinct sets of individuals and P_i be a strict preference ordering for every $i \in I$ over J. An *object allocation problem* over (I, J) is described by $\mathcal{A} = (P_i)$.

An **allocation** for $\mathcal{A}$ is a matrix $Q = (q_{ij})_{i \in I, j \in J}$ where $\sum_i q_{ij} = \sum_j q_{ij} = 1$ and every q_{ij} is a non-negative real number. q_{ij} denotes the individual i's probability of receiving j. Q_i and Q^j denote the i^{th} row and the j^{th} column of the allocation Q respectively.

Based on an object allocation problem, we will construct a two-sided matching market. First, we will endow every individual j a weak preference ordering R_j over I, in which j prefers i to i' if i ranks j higher than i' does and j is indifferent between i and i' if they both rank j at the same level. We call this behaviour as *preference for being preferred*. Second, we assume that each individual holds a "choice function".

Let $U^n \in \mathbb{R}^n$ be a unit box and denote each element of U^n by x, which will be called as a choice vector. A *choice function* is a map $C : U^n \to U^n$ such that $C(x) \leq x$ for every $x \in U^n$. We denote the j^{th} coordinate of the chosen vector as $C(x)_j$. We assume that every individual has a quota of 1, which bounds the size of the chosen vectors.

As introduced in Alkan and Gale (2003), a *generalized matching market* is described over (I, J) by $\mathcal{M} = ((C^i), (C^j))$ where C^i, C^j are choice functions describing individuals i and j respectively. Since every individual $i \in I$ has a strict preference ordering and every individual $j \in J$ has a weak preference ordering, C^i respond to P_i and C^j respond to R_j, which will be defined formally.

Suppose P_i is constructed as $j \; P_i \; j+1$ for an individual i. Given a choice vector x with $|x| > 1$, let j be the individual such that

$$z = \sum_{j'=1}^{j} x_{j'} \leq 1 \text{ and } z + x_{j+1} > 1$$

Then, we say that C^i is P_i**-responsive** if

$$C^i(x) = (x_1, \ldots, x_j, 1-z, 0, \ldots, 0).$$

In R_j, let us call each indifference class as the rank of R_j and denote the rank of i in R_j as r_i^j [1]. Given a choice vector x with $|x| > 1$, let r_*^j be the rank in R_j such that

$$z = \sum_{i'} x_{i'} \leq 1 \text{ where } r_{i'}^j < r_*^j$$

$$z + \sum_{i} x_i > 1 \text{ where } r_i^j = r_*^j$$

A choice function C^j is R_j**-responsive** if

$$C^j(x)_{i'} = x_{i'}, \qquad \textstyle\sum_i C^j(x)_i = 1 - z, \qquad C^j(x)_{i''} = 0$$

for every i'' where $r_{i''}^j > r_*^j$. We call r_*^j the **border** of R_j.

An **allocation** for a generalized matching market is a matrix $Q = (q_{ij})_{i \in I, j \in J}$ where $\sum_i q_{ij} = \sum_j q_{ij} = 1$ and every q_{ij} is a non-negative real number. An allocation Q is **stable** if there is no pair (i,j) such that

$$C^i(Q_i + zu_j)_j = q_{ij} + z \text{ and } C^j(Q^j + zu_i)_i = q_{ij} + z \text{ for some } z > 0$$

where u_j and u_i are the j^{th} and the i^{th} unit vectors respectively. If this is the case, we say that i **likes** j and j **likes** i.

In this chapter, our main goal is to associate allocations for $\mathcal{A}$, which have desirable properties, with stable allocations of $\mathcal{M}$ derived from $\mathcal{A}$. To guarantee that a stable allocation exists for $\mathcal{M}$, substitutability of choice functions is the key assumption.[2].

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[1] Note that there is a single individual at each class in P_i since the ordering is strict.

[2] See Alkan - Gale (2003) for more detail.

Note that strict responsive choice functions satisfy substitutability and consistency. However, it is not certain how C^j behave at the border. Since each C^j is responsive to a weak preference ordering, which exhibits a preference for being preferred, we can easily conclude that the Gale - Shapley algorithm becomes a greedy algorithm. That is to say that each acceptance is immediate and permanent. Therefore, even though C^j may violate substitutability condition at the border, there exists a stable matching since the algorithm is greedy and no cycle can occur because of the non-substitutability of choice functions.

As we described above, R_j are derived from P_i and exhibits preference for being preferred. To be more concise and formal, we say that a matching market $\mathcal{M}$, where C^i are P_i-responsive and C^j are R_j-responsive, is **aligned** if R_j exhibits preference for being preferred, i.e. $r_i^j = r_j^i$ for every i and j.

By the following lemma, we present our first observation on aligned matching markets, without the proof.

Lemma 3.1. *Each aligned matching market has a single stable allocation.*

An aligned matching market $\mathcal{M} = ((C^i), (C^j))$ and an object allocation problem $\mathcal{A} = (P_i)$ are **associated** with each other if every C^i are P_i-responsive.

An object allocation problem has associated with itself a family of aligned matching markets whereas an aligned matching market has associated with itself a single object allocation problem.

3.2 Examples

Let $\mathcal{A} = (P_i)$ be an object allocation problem. Below we give examples of aligned matching markets $\mathcal{M} = ((C^i), (C^j))$ which are associated with $\mathcal{A}$. Let us define P_i as

P_1	P_2	P_3	P_4	P_5
1	2	2	1	1
2	1	3	3	3
3	3	1	4	4
4	4	4	2	2
5	5	5	5	5

We will construct two different matching market based on $\mathcal{A}$ and compute their (unique) stable allocations when C^j are one of the following well-known choice functions.

Equal Award Choice Function: $C^j(x)_i = \min\{\lambda, x_i\}$ and $\lambda \in \mathbb{R}_+$ is chosen so that $\sum_{i' \in I} C^j(x)_{i'} = \min\{\sum_{i \in I} x_i, \hat{c}_j\}$

Equal Loss Choice Function: $C^j(x)_i = \max\{x_i - \lambda, 0\}$ and $\lambda \in \mathbb{R}_+$ is chosen so that $\sum_{i' \in I} C^j(x)_{i'} = \min\{\sum_i x_i, \hat{c}_j\}$

where $\hat{c}_j$ is the amount of j available. If the matching market is aligned, in the Gale - Shapley algorithm, $\hat{c}_j$ can be thought as the quota minus what the more preferable individuals receive from j, that is $\hat{c}_j = 1 - \sum_{i'} q_{i'j}$ where $r^j_{i'} < r^j_i$.

According to preference orderings of individuals, R_j can be constructed as

R_1	R_2	R_3	R_4	R_5
1,4,5	2,3			
2	1	3,4,5		
3		1,2	4,5	
	4,5		1,2,3	
				1,2,3,4,5

Now, let C^j are equal award for $\mathcal{M}_1$ and equal loss for $\mathcal{M}_2$. The stable allocations of $\mathcal{M}_1$ and $\mathcal{M}_2$ can be found by the generalization of the Gale - Shapley Algorithm and respectively yield the following allocations;

$$\begin{pmatrix} 1/3 & 0 & 0 & 1/9 & 5/9 \\ 0 & 1/2 & 0 & 1/9 & 7/18 \\ 0 & 1/2 & 1/3 & 1/9 & 1/18 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \end{pmatrix} \quad \begin{pmatrix} 1/3 & 0 & 0 & 11/36 & 13/36 \\ 0 & 1/2 & 0 & 5/36 & 13/36 \\ 0 & 1/2 & 2/9 & 0 & 5/18 \\ 1/3 & 0 & 7/18 & 5/18 & 0 \\ 1/3 & 0 & 7/18 & 5/18 & 0 \end{pmatrix}$$

We have shown that, given an object allocation problem, different aligned matching markets, which have different stable solutions, can be constructed. Note that not all of different matching markets have different stable solutions[3]. In the next section, we will show that the stable allocation for $\mathcal{M}$ has desirable properties which are defined for a solution of an object allocation problem.

[3]Consider a problem with two individuals. Let i and i' both prefer j to j'. Then, j and j' are both indifferent between two individuals. Consider two different associated matching markets, where one assumes C^j are equal award, the other assumes C^j are equal loss for every j. The stable allocation for both matching markets are the same, in which i and i' both assign to j with probability 1/2 and j' with probability 1/2.

3.3 Respect-For-Rank and No-Discrimination

Let $\mathcal{A} = (P_i)$ and $\mathcal{M} = ((C_i),(C_j))$ be the markets that are associated with each other. Given an allocation Q, we say that j is **bottom** of Q_i if $q_{ij} > 0$ and $q_{ij'} = 0$ for every j' such that $j \ P_i \ j'$.

Definition: An allocation Q is rank respecting if $q_{i'j} = 0$ for every i, i' and j such that j is not bottom of Q_i and $r_j^{i'} > r_j^i$.

Proposition 3.1. *If Q is the stable allocation for $\mathcal{M}$ then Q is a rank respecting allocation for $\mathcal{A}$.*

The proposition helps us to find a rank respecting allocation for a one-sided market $\mathcal{A}$ by using generalization of the Gale - Shapley algorithm which is defined in two-sided markets.

Definition: An allocation Q exhibits **discrimination** against i if $r_j^i = r_j^{i'}$ and j is not bottom of Q_i but $q_{ij} < q_{i'j}$.

The following proposition is the analog of Theorem 1 of Harless (2018). The way we have constructed the market helps us to simplify proofs of the results and makes use of the well-known Gale - Shapley Algorithm.

Proposition 3.2. *An allocation Q for $\mathcal{A}$ exhibits no-discrimination if and only if Q is the stable allocation of $\mathcal{M}$ and C^j are equal award.*

The properties above are well-known properties and studied in the literature. The following definition is a new property of a solution of an object allocation problem.

Definition: An allocation Q exhibits **cumulative discrimination** against i if $r_j^i = r_j^{i'}$, j is not bottom of Q_i and $q_{i'j} > 0$ but $q_{ij}^* < q_{i'j}^*$ where $q_{ij}^* = \sum_{j'} q_{ij'} + q_{ij}$ for every j' such that $j' \ P_i \ j$.

Consider the matching market $\mathcal{M}_2$ and its stable allocation in the previous section. Here, cumulative discrimination can only be against individuals 3,4 and 5 for object 3 and individuals 1,2 and 3 for object 4. However, for the individuals 3,4 and 5 we have;

$$\textstyle\sum_{j=2,3} q_{3j} = \sum_{j=1,3} q_{4j} = \sum_{j=1,3} q_{5j}$$

For the individuals 1,2 and 3 we have;

$$\textstyle\sum_{j=1,2,3,4} q_{3j} > \sum_{j=1,2,3,4} q_{1j} = \sum_{j=1,2,3,4} q_{2j}.$$

In this example, we observe that the allocation for the matching market $\mathcal{M}_2$, where C^j are equal loss, exhibits no cumulative discrimination. We believe that this result can be generalized for all matching markets, where C^j are equal loss, associated with an object allocation problem. However, we do not provide proof for this result.

3.4 Ordinal Efficiency and Aggregate Efficiency

Let $\mathcal{A} = (P_i)$ be an object allocation problem. Given an allocation Q of $\mathcal{A}$, a **cycle** is a sequence of pairs (i,j) such that each consecutive pairs have either the same i or the same j but not both and each cycle starts and ends with the same pair (i,j). An **improvement cycle** is a cycle in which for all i in the cycle, either $j_n \; P_i \; j_{n+1}$ and $q_{i_n j_{n+1}} > 0$, or $j_{n+1} \; P_i \; j_n$ and $q_{i_n j_n} > 0$ are true. Therefore, there exists $\epsilon \in (0,1]$ such that $\Delta q_{i_n j_n} = +\epsilon$ and $\Delta q_{i_n j_{n+1}} = -\epsilon$, or the opposite, for all pairs (i_n, j_n) in the cycle.

That is to say that an improvement cycle is a sequence of individual pairs in an allocation Q, in which at least two individuals can profitably exchange the probability shares of some objects. Therefore, if an allocation contains an improvement cycle, there exist some individuals who can be better off. After the exchange occurs in the improvement cycle, the new allocation, say Q', is said to stochastically dominates the allocation Q.

Definition: An allocation is said to be **ordinally efficient** if there exists no improvement cycle. In other words, it is not stochastically dominated by another allocation.

Proposition 3.3 (Bogomolnaia and Moulin (2001))**.**

An allocation Q is ordinally efficient if and only if it exhibits no improvement cycle.

Proposition 3.4. *Any rank respecting allocation Q for $\mathcal{A}$ is ordinally efficient.*

Note that every stable allocation of $\mathcal{M}$ associated with $\mathcal{A}$ is ordinally efficient, also rank respecting by Proposition 3.4.

Now, we refer to another efficiency definition. Let us denote the set of most preferred k objects by individual $i \in I$ as $r_i(k)$. Given any allocation Q for $\mathcal{A}$, define $w^Q = (w_1^Q, \dots, w_n^Q)$ as the aggregate efficiency vector of the allocation Q where $w_k^Q = \sum_{i \in I} \sum_{j \in r_i(k)} q_{ij}$. w^Q is a vector in which the coordinates of the vector show the sum of probabilities for each individual to be assigned his first best, first two bests, first three best and so on.

Definition: An allocation Q is said to **aggregate stochastically dominate** an allocation Q' in $\mathcal{A}$, if $w^Q \geq w^{Q'}$.

In an object allocation problem $\mathcal{A}$, $R1$ mechanism, proposed by Alioğulları, Barlo, and Tuncay (2013), can be considered as the specification of the well-known PS rule. Different than the PS rule, assume that each individual has the right to reserve his first best object so that no individual, who does not rank it best, can receive a positive probability share. Alioğulları, Barlo, and Tuncay (2013) proved that the resulting allocation of the $R1$ mechanism aggregate stochastically dominates the PS allocation.

Proposition 3.5. *The stable allocation of a matching market $\mathcal{M} = ((C^i), (C^j))$ associated with $\mathcal{A}$ where C^j are equal award, aggregate stochastically dominates the resulting allocation of $R1$ mechanism.*

4. THE DYNAMIC PROCESS THAT YIELDS PS ALLOCATION

In an object allocation market $\mathcal{A}$, PS allocation is generated by the well-known uniform speed simultaneous eating algorithm introduced by Bogomolnaia - Moulin (2001). We mentioned that PS rule is the unique procedure resulting in a unique random assignment that satisfies ordinal efficiency, envy freeness, and bounded invariance properties [1].

In the literature, recently, Harless (2019) shows that PS allocation can be obtained by a different algorithm, which is called ordered-claims-algorithm. Here, we study if PS allocation can be obtained by a different and more "decentralized" process than eating algorithm and ordered-claims-algorithm. The driving force behind this study is to investigate which underlying assumptions is required to construct such a process.

Consider an object allocation problem $\mathcal{A} = (P_i)$ over (I, J), where Q is an allocation matrix described as in Chapter 3.1.

Let d_i be the individual i's demand vector and d_{ij} denote the amount of i's demand on j. Similarly, let r_j be the object j's rejection vector and r_{ij} denote the amount of j's rejection for i. Define $\mathbf{D} = \{d \in \mathbb{R}^J_+ \mid 0 \leq d_{ij} \leq 1\}$ and $\mathbf{R} = \{r \in \mathbb{R}^I_+ \mid 0 \leq r_{ij} \leq 1\}$ as the two sets, namely the sets of demand and rejection vectors. A demand function is a map $D : \mathbf{R} \to \mathbf{D}$ and a rejection function is a map $R : \mathbf{D} \to \mathbf{R}$. For each $j \in J$ and each preference ordering P_i, define $U(j, P_i) = \{j' \in J \mid j' \, P_i \, j\}$ as the upper counter set of j in P_i. Additionally, let $j_i^*(j) \in U(j, P_i)$ be the least favoured object by i among the upper counter set of j.

The demand function for an individual i is described by;

$$D_i(r) = d_i \text{ where } d_{ij} = \begin{cases} 1 & U(j, P_i) = \emptyset \\ r_{ij_i^*(j)} & otherwise \end{cases}$$

for all j. In other words, each i demands full probability from her best favourite object and transfer the rejection to the next best object as her demand. An indi-

[1]See Bogomolnaia - Heo (2012) for more detail.

vidual's demand corresponds to her preference ordering with one restriction: Any rejection from an object cannot be offered as a demand to an object which is ranked higher than the first object. We call this behaviour **no-going-back condition**. Rejection function for an object j is described by;

$R_j(d) = r_j$ where $r_{ij} = \min\{\lambda_j, d_{ij}\}$ and $\lambda_j \in \mathbb{R}_+$ is chosen so that $\sum_{i' \in I} r_{i'j} = \max\{0, \sum_{i \in I} d_{ij} - 1\}$.

In other words, each j has a equal loss choice function described in Chapter 3.2 [2]. However, since demand functions are described by rejections, we choose to use rejection functions for every j.

Define the sequences $(\mathbf{D}^t)$, $(\mathbf{R}^t)$ and (Q^t) by the following recursive process; [3]

$$\mathbf{R}^0 = 0$$

$$\mathbf{D}^t = D_I(\mathbf{R}^t)$$

$$\mathbf{R}^t = R_J(\mathbf{D}^t)$$

$$Q^t = \mathbf{D}^t - \mathbf{R}^t$$

Basically, we call $\mathbf{R}^t$ as a rejection matrix and individuals choose their demand according to $\mathbf{R}^t$. $\mathbf{D}^t$ is called demand matrix of individuals, and objects issue the rejection vector according to $\mathbf{D}^t$. Q is the allocation matrix, which is difference between demands and rejections. The process ends where there is no change in $\mathbf{R}^t$.

Remark: d_i and r_j are monotonically increasing vectors for every i and j.

The matrices can be updated simultaneously by all individuals or some set of individuals. At the one extreme, it can be updated only one individual. In between, any subset of individuals can update the demand vectors for any subset of objects at any step of the recursive process.

Lemma 4.1. *The Dynamic Process is sequence independent.*

As we mentioned before, Harless (2019) introduced a new algorithm that yields PS allocation, namely the ordered-claims-algorithm. The algorithm proceeds as follows: Suppose there is a "coordinator" in an object allocation problem, who follows a selection rule at each round and a distribution rule for each object at each round. To obtain the PS allocation, Harless introduces a particular selection and distribution rule. If the coordinator selects the objects that have the maximum average excess

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[2] Different than the Chapter 3, the choice functions does not respond to any preference orderings.

[3] Note that $D_i(0) = u_{\bar{j}_i}$ for every i where $\bar{j}_i$ is the first best object for every i and $u_{\bar{j}_i}$ is the $\bar{j}_i^{th}$ unit vector.

demand and distribute them according to the equal loss rule, the algorithm results in the PS allocation. Our main result here is the following proposition and we will prove it by showing the equivalence between the ordered-claims-algorithm with particular selection and distribution rule and the Dynamic Process with simultaneous sequence.

Proposition 4.1. *the Dynamic Process generates the PS allocation.*

As in ordered-claims-algorithm, PS rule also requires a coordinator who informs individuals when an object is exhausted. However, under two assumptions, the Dynamic Process is more decentralized and sequence independent. These assumptions are characteristics of demand and rejection functions. Objects choice functions are equal loss, which is the distribution rule that Harless uses to have PS allocation. Our main investigation here is the no-going-back condition, which is the key assumption to have more decentralized process than PS rule and ordered-claims-algorithm.

Even though the Dynamic Process generates the PS allocation and the PS algorithm is weak strategyproof, the recursive structure of the Dynamic Process creates a new incentive problem. Weak strategyproofness guarentees that individuals cannot be better off by misreporting their preference orderings. However, no-going-back condition is a strong assumption on individuals' behaviours. If we relax this assumption, individuals can be better off by misreporting their demand vectors. More specifically, an individual has an incentive to transfer the rejection by an object to the more preferred ones. Since objects have equal loss choice functions, higher amount of demand will generate higher amount of acceptance.

5. PROOFS

Proof of Proposition 3.1. Suppose Q is the stable allocation but not rank respecting. Then, if j is not bottom of Q_i, there exists at least one individual i' with $q_{i'j} > 0$ and $r^{i'}_j > r^i_j$. Since j is not bottom of Q_i, i likes j. However, since the market is aligned, $q_{i'j} > 0$ and $r^{i'}_j > r^i_j$ imply that j also likes i. Hence, it contradicts with Q being stable.

□

Proof of Proposition 3.2. **If part:** Suppose $r^i_j = r^{i'}_j$ and j is not bottom for i. Since Q exhibits no-discrimination, we know that $q_{ij} \geq q_{i'j}$ and also since market is aligned we know that j is indifferent between i and i'. Since j is not bottom for i, there exists a j' where $q_{ij'} > 0$ and jP_ij'.

Suppose C^j are not equal award. Then, in the Gale - Shapley algorithm i must be partially rejected by j. If i' is bottom for j, then C^j does not contradict with equal award. Suppose i' is not bottom for j and $q_{ij} = q_{i'j}$, then again C^j does not contradict with equal award. Now, suppose $q_{ij} > q_{i'j}$. Then, it contradicts with Q exhibiting no-discrimination.

Only if part: Suppose C^j are equal award and the stable allocation for $\mathcal{M}$ exhibits discrimination against i. Then, if j is not bottom for i and $r^i_j = r^{i'}_j$, we have $q_{ij} < q_{i'j}$.

Since j is not bottom for i, i likes j. Also, since market is aligned, $r^i_j = r^{i'}_j$ implies j is indifferent between i and i'. Therefore, $q_{ij} < q_{i'j}$ implies that j does not equally award i and i'. Hence, it contradicts with C^j being equal award.

□

Proof of Proposition 3.4. Take an allocation Q that is rank respecting and suppose it is not ordinally efficient. Therefore, there must be an improvement cycle where individuals can profitably exchange some probability shares. Since the allocation respects rank, for at least one pair in the improvement cycle i_{n-1}, $i_n \in I$ and j_n, $j_{n+1} \in J$, we have that i_n strictly prefers j_n to j_{n+1}, $q_{i_nj_{n+1}} > 0$ and $q_{i_{n-1}j_n} > 0$

and this implies that object j_n is ranked at the same level in $P_{i_{n-1}}$ and P_{i_n}, or at the higher level in $P_{i_{n-1}}$. Therefore, individual i_n desires more probability shares of j_n, which can be taken from any i' with $q_{i'j_n} > 0$. Therefore, i' must be in the improvement cycle, without loss of generality let us call him as individual i_{n-1}. Then, he desires more probability of j_{n-1} where $j_{n-1}P_{i_{n-1}}j_n$, which can be exchanged with an i_{n-2} with $q_{i_{n-2}j_{n-1}} > 0$. Similarly, i_{n-2} is also in the improvement cycle. Since the cycle end with the initial pair of individuals, there exists some i_m who desires more probability of object j_{n+1} from i_n. There must be the case that $q_{i_n j_{n+1}} > 0$, hence i_n must rank this object as higher as the i_m. However, by continuing in this regard, it must be the case that j_{n+1} must be ranked higher in P_{i_m} than P_{i_n}. This contradiction concludes the proof.

□

Proof of Proposition 3.5. Suppose Q is the stable allocation of an aligned matching market $\mathcal{M}$ associated with $\mathcal{A}$ and Q' is the resulting allocation of $R1$ mechanism. We can easily say that $w_1^Q = w_1^{Q'}$. Notice that since Q is rank respecting, there exists no individual who receives positive probability from j even though individuals who rank j higher than the first individual are not completely satisfied. However in the $R1$ mechanism, the opposite can occur. Therefore, we can guarantee that $w_k^Q \geq w_k^{Q'}$. By the example given in Chapter 3.2, we can conclude the proof.

In the example, Q and Q' are the following matrices, respectively

$$\begin{pmatrix} 1/3 & 0 & 0 & 1/9 & 5/9 \\ 0 & 1/2 & 0 & 1/9 & 7/18 \\ 0 & 1/2 & 1/3 & 1/9 & 1/18 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \end{pmatrix} \quad \begin{pmatrix} 1/3 & 0 & 4/15 & 1/5 & 1/5 \\ 0 & 1/2 & 1/10 & 1/5 & 1/5 \\ 0 & 1/2 & 1/10 & 1/5 & 1/5 \\ 1/3 & 0 & 4/15 & 1/5 & 1/5 \\ 1/3 & 0 & 4/15 & 1/5 & 1/5 \end{pmatrix}$$

Hence, $w^Q = \{2, 3, \frac{11}{3}, 4, 5\}$ and $w^{Q'} = \{2, \frac{79}{30}, \frac{17}{5}, 4, 5\}$. Since $w^Q > w^{Q'}$, Q aggregate stochastically dominates Q'.

□

Proof of Lemma 4.1. Suppose there exist two sequences, S and S', in which at least one individual has a different allocation, meaning that she has a different demand for at least one object. Let $d_{ij}^S \neq d_{ij}^{S'}$ for a pair (i, j) at the end of the each sequence. j cannot be the first best object since the demand for the first best objects is always 1 in every sequence. Therefore, $r_{ij_i^*(j)}^S \neq r_{ij_i^*(j)}^{S'}$, which implies the rejection vector of $j_i^*(j)$ is different at the end of each sequence. Hence, there exists at least one individual whose demand for $j_i^*(j)$ is different in each sequence. Let $d^S_{i'j_i^*(j)} \neq d^{S'}_{i'j_i^*(j)}$. By continuing this process, we can conclude that the rejection from the first ranked object for an individual differs at the end of each sequence, let denote the object as $\bar{j}_i$. Since the demand for the first best objects does not differ, there exists an individual whose demand is different for $\bar{j}_i$ at S and S'. Following the same fashion, we can conclude that the rejection from the first ranked object for an individual differs at S and S'.

Since the set of individuals is finite, every object that is first ranked gives different rejection at the end of S and S'. Therefore, they have different demand vectors at each sequence. Since each individual's demand for her first best object is 1, we can say that each first ranked object accepts at least one demand from an individual who does not rank it best.

Let j^* be the first ranked such that the number of individuals who rank it best is more than any other first ranked objects. There exists a time in S and S' such that j^* receives all of the demands from each individual who ranks it best. Let denote this time as t_* We claim that, at t_*, j^* would reject all the demand that receives from an individual who does not rank it best and rejects all the new demands after t_*. Since the sequences are different, we have $t_*^S \neq t_*^{S'}$. Suppose the number of individuals who rank j^* best is m.

Remark: Since rejection for an individual cannot be higher than her demand, an individual's demand is decreasing as the rank of objects is increasing.

Let i be the individual who is accepted by j^* and not ranks j^* best. Note that i's maximum demand for an object that is not best ranked by i must be less than $1-\frac{1}{m}$. Also, λ_{j^*} would be $1-\frac{1}{m}$, if j^* receives demands only from the individuals who rank j^* best. Therefore, after t_*, j^* will reject every individual who does not first rank j^*. Contradiction.

□

Proof of Proposition 4.1. We will prove the proposition by showing the equivalence between the Dynamic Process with the sequence where the agents move simultaneously and structured ordered claim algorithm with average loss selection order and constrained equal loss distribution rule, $f^{al,CEL}$ introduced by Harless (2019). By Lemma 6.1, the result can generalize for every sequence.

First we prove that the Dynamic Process will converge. In other words, there will be a set of object which will be permanently distribute itself at every t. Let us define

$$M^t = \{j \in J \mid j = \underset{j' \in I \setminus \bigcup_{t'=1}^{t-1} M^{t-1}}{\texttt{argmax}} \lambda_{j'}^t\}$$

be the set of objects which issued the maximum amount of equal rejection for the first time at step t. Notice that M^t is a non-empty set. For this process, following lemma will guarantee that the process will converge.

Lemma 5.1. *Suppose $j \in M^t$. Then, $\lambda_j^t = \lambda_j^{t+1} = \cdots = \lambda_j^T$*

Proof. Begin with λ_j^t is the maximum amount among the all objects, any demanded probability share in the $t+1$ must be less than or equal to λ_j^t. That is, $d_{ij'}^{t+1} < \lambda_j^t$ for all $j' \in J$. Additionally, to have $\lambda_j^t \neq \lambda_j^{t+1}$, it must be the case that there exists at least one object j such that $d_{ij}^{t+1} > \lambda_j^{t+1}$. Then, we have $\lambda_j^{t+1} < d_{ij}^{t+1} < \lambda_j^t$, which contradicts with λ_j being an increasing vector.

□

Now, we will show that the set of selected objects at step t in $f^{al,CEL}$, is the same with the objects in M^t.

Let us denote the i's claim for object j in $f^{al,CEL}$ as c_{ij}. In $f^{al,CEL}$, firstly algorithm selects a set of objects, then distribute the selected objects. The others do not take any action. However in the Dynamic Process, each object and each agent is active. Remember that starting demand vector is the one where each agent demand 1 from the most favoured objects. Similarly, starting claim vector is the one where each object receives 1 from the related individuals in $f^{al,CEL}$. However, differently, distribution is limited to set of some objects in $f^{al,CEL}$. In the second step, some agents take action in $f^{al,CEL}$, who might have already made his action in the first step of Dynamic Process. Regarding this fashion, we know that the demand vector carries all claims from the claim vector, even more.

Now, let us separate the demand vector in two part: demands which are already in $f^{al,CEL}$ as a claim and the others. We will use the same notation, c, in $f^{al,CEL}$ for the first component and we will denote the demands different than the claims in $f^{al,CEL}$ as k.

Let us denote the rejection occurs in $f^{al,CEL}$ at step t as f^t, which is the average excess claim for an object and the object has the maximum amount of f^t is selected to distribute. In $f^{al,CEL}$, if an object is selected, they leave the market and become unavailable for the next steps. Let us call the others as available objects.

Claim: Suppose j is a selected object in $f^{al,CEL}$ at step t, then $c_j^t = d_j^t$.

In other words, the claim vector for the selected objects in $f^{al,CEL}$ is same with the demand vector of the same objects in Dynamic Process.

Proof. Suppose not. Let object j is selected at step t in $f^{al,CEL}$. Then, we know that for all j' in the set of available objects at step t;

$$f^t_{j'} < f^t_j$$

Suppose in the Dynamic Process, object j at step t receives the demand vector which contains all claims in $f^{al,CEL}$ but additionally has at least one demand, k^t_{ij}. Since all k^t_{ij} must be feasible, meaning that greater than the λ^t_j, we have;

$$f^t_{j'} < f^t_j \leq \lambda^t_j \leq k_{ij} = \lambda^{t-1}_{j'}$$

where we use the fact that each demand is some amount of rejection coming from an object at previous step. Observe that each k^t_{ij} added to f^t_j will increase this ratio and notice that object j' must have not been selected in $f^{al,CEL}$, otherwise it would be included in claim vector, c_j or j' is not in the set of available objects. Now consider following cases:

Case I In the Dynamic Process, the demand vector of object j' received at $t-1$ is the same with the claim vector in $f^{al,CEL}$, i.e. $d^{t-1}_{j'} = c^{t-1}_{j'}$. Then, immediately we can conclude that $\lambda^{t-1}_{j'} < f^t_{j'} < f^t_j$. However, previously we concluded that

$$f^t_j \leq \lambda^t_j$$

Therefore, following equations gives us contradiction.

$$f^t_j \leq \lambda^t_j \leq k_{ij} = \lambda^{t-1}_{j'} < f^t_{j'} < f^t_j$$

Case II: Suppose $d^{t-1}_{j'} \neq c^{t-1}_{j'}$. Therefore, there exists new demands, $k^{t-1}_{ij'}$, which must be greater than $\lambda^{t-1}_{j'}$, that is, $\lambda^{t-1}_{j'} < k^{t-1}_{ij'}$. We know that $k^{t-1}_{ij'}$ is some amount of rejection from a different object at step $t-2$, which is not a selected object in $f^{al,CEL}$. Otherwise it would be contained in $c^{t-1}_{ij'}$. Therefore we can conclude that $\lambda^{t-1}_{j'} < k^{t-1}_{ij'} = \lambda^{t-2}_{j''}$. Here, again we have two cases. If the demand vector of object j'' receives at t-2 is the same with the claim vector in $f^{al,CEL}$, we are at the same position as in Case I. Therefore suppose they are not the same. Then, with the same logic we can conclude that

$$l^{t-1}_{j'} < k^{t-1}_{ij'} = \lambda^{t-2}_{j''} < k^{t-1}_{ij''} = \lambda^{t-2}_{j'''} < \cdots < k^2_{ij^*} = \lambda^1_{j^{**}}$$

where in the very first step, j^{**} is not selected in $f^{al,CEL}$. Moreover, we know that a claim vector and a demand vector is the same in the first step. Therefore, $\lambda^1_{j^{**}}$ is equal to the loss occurred in $f^{al,CEL}$, which is less than or equal to the losses occurred at the next steps. Therefore, we can conclude that

$f^t_j \leq \lambda^t_j \leq k_{ij} = \lambda^{t-1}_{j'} < k^{t-1}_{ij'} = \lambda^{t-2}_{j''} < k^{t-2}_{ij''} = \lambda^{t-3}_{j'''} < \cdots < k^2_{ij^*} = \lambda^1_{j^{**}} = f^1_{j^{**}} \leq f^2_{j^{**}} \leq \ldots f^t_{j^{**}} < f^t_j$

where we use the fact that none of the objects are selected in $f^{al,CEL}$ so that we can say that the average excess demand is increasing as we increase the steps. Contradiction. □

Claim: The set of selected objects at step t in $f^{al,CEL}$, is the same with the objects in M^t.

Proof. Suppose not. By the previous claim, we know that selected objects in $f^{al,CEL}$ has the same demand vector with Dynamic Process. Suppose an object j is selected in $f^{al,CEL}$ but not in M^t. Then there exists a j' such that $\lambda^t_j < \lambda^t_{j'}$ where $j' \not\in M^t$. Since the claim vectors resulting f^t_j is the same with λ^t_j we have;

$$f^t_{j'} < f^t_j = \lambda^t_j < \lambda^t_{j'} = f^t_{j'}$$

Conradiction.

□

We conclude that permanently exhausted objects at step t are the same objects in $f^{al,CEL}$ and the Dynamic Process; also they have the same claim and demand vectors. Then, each object leaves the market with the same claim vector and rejection amount. Therefore, their resulting allocation is the same.

□

6. CONCLUDING REMARKS

Any given allocation problem, we can derive a matching market by endogenously attributing priority orderings for objects and by assuming that they are endowed with a choice function. The attribution bases on the given preference profile of the individuals and the choice functions correspond to these preference orderings. We know that any strict responsive choice function is substitutable. Moreover, the preference for being preferred property for objects makes the Gale - Shapley algorithm greedy. Therefore, substitutability of choice functions corresponding to weak preference ordering is not necessary condition to have the existence of stable matchings. Hence, we can examine the desirable properties such as respect for rank, no-discrimination, ordinal and aggregate efficiency by relating them to the stability concept.

We show that any rank respecting allocation for an object allocation problem is the stable allocation for a matching market associated with an object allocation problem. The converse is also true, however deriving a matching market from an object allocation problem is much intuitive. Also, if we let the choice functions be equal award, we observe that the stable allocation exhibits no-discrimination and aggregate stochastically dominates the resulting allocation of $R1$ mechanism. Regarding ordinal efficiency, we show that in an object allocation problem, any rank respecting allocation is ordinally efficient.

When we derive a matching market, we remark that there exist infinitely many aligned matching markets associated with an object allocation problem. A specific choice functions in that family might result in different properties, such as aggregate efficiency and no cumulative discrimination.

The Dynamic Process can be thought of as an uncoordinated process that yields PS allocation. However, it obeys some rules in terms of individuals' and objects' behaviors. The process endogenously determines which object would be permanently distributed. However, this creates a new door for an incentive to manipulate. PS allocation is weakly strategy-proof, meaning that the individuals cannot achieve better allocation by misreporting their preference ordering. Even though the Dynamic Process yields PS allocation, it creates a spot for individuals to deviate. When they choosing from a rejection vector, if their choice function is not fixed, they can achieve better allocation by misreporting their demand vectors.

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