STATISTICS OF REAL ROOTS OF RANDOM POLYNOMIALS

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to all innocent war victims.

Statistics of Real Roots of Random Polynomials

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Abstract

In this thesis, we present two approaches in order to study the expected number of real zeros of random univariate polynomials. Namely, the Kac-Rice method and Edelman-Kostlan's geometric approach. We derive a remarkable result called the Kac-Rice formula concerning the expected number of real zeros and apply this result to certain random polynomial ensembles. We also report some basic facts from potential theory in the complex plane and its connection to complex random polynomials. In addition, we consider certain random orthogonal polynomials associated to suitable weight functions supported in the complex plane, and we present some known results in this direction.

Rassal Polinomların Reel Köklerinin Istatistikleri

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Özet

Bu tez çalışmasında, rassal polinomların reel köklerinin beklenen sayısını hesaplamak için biri Kac-Rice metodu ve diğeri Edelman-Kostlan'ın geometrik yaklaşımı olmak üzere iki bakış açısı sunulmuştur. Reel köklerin beklenen sayısı için Kac-Rice formülü olarak bilinen önemli bir sonuç incelenmiştir. Bu sonuç literatürde tanınan bazı rassal polinomlar ailelerine uygulanmıştır. Ayrıca, karmaşık düzlem üzerinde potansiyel teorisinden bazı sonuçlar verilip, bu sonuçların karmaşık rassal polinomlar ile olan ilişkisi gösterilmiştir. Son olarak karmaşık düzlem uzerinde yaşayan, bazı belirli özelliklere sahip olan ölçülere denk gelen rassal ortogonal polinomlar incelenmiştir ve bu doğrultuda bilinen sonuçlar ifade edilmiştir.

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Introduction

Let $\mathcal{P}_n$ be the space of holomorphic polynomials with real coefficients of degree at most n. Then, any inner product on $\mathcal{P}_n$

$$< P_n, Q_n >_\mu = \int_{\mathbb{C}} P_n(z)\overline{Q_n(z)}d\mu$$

associated with suitable measures supported in $\mathbb{C}$ induces a Gaussian probability measure $dProb_n^\mu$ on $\mathcal{P}_n$ as follows: Fix an orthonormal basis p_j^n for $\mathcal{P}_n$ with respect to $<,>_\mu$, then for any polynomial $P_n \in \mathcal{P}_n$

$$P_n(z) = \sum_{j=0}^{n} a_j p_j^n(z). \tag{0.0.1}$$

Now assuming the coefficients of this polynomial are chosen randomly with respect to non-degenerate centred Gaussian distribution with covariance matrix Σ. Then identifying P_n by its coefficients, we obtain

$$dProb_n^\mu = \frac{1}{\sqrt{\det(2\pi\Sigma)}} e^{-\frac{1}{2}<\Sigma^{-1}\mathbf{a},\mathbf{a}>} d\mathbf{a}$$

where $\mathbf{a} = (a_0, ..., a_n) \in \mathbb{R}^{n+1}$ and $d\mathbf{a}$ is Lebesgue measure on $\mathbb{R}^{n+1}$. Therefore, the ensemble $(\mathcal{P}_n, dProb_n^\mu)$ consists of random polynomials of the form (0.0.1) with the Gaussian probability measure $dProb_n^\mu$ [1]. Most of the time

[1] Observe that by the unitary invariance of Gaussian Distribution, $dProb_n^\mu$ is independent of the choice of orthonormal basis

we will assume that a_j's are independent Gaussian random variables of mean zero and variance one . In this case ,

$$dProb_n^{\mu} = \frac{1}{(2\pi)^{\frac{n+1}{2}}} e^{-\frac{\|\mathbf{a}\|^2}{2}} d\mathbf{a}.$$

where $||.||_{\mu}$ is the norm induced by $<,>_{\mu}$. Some of the models that are frequently studied in the literature are the following:

- Kac Polynomials: This model consists of the ensemble where the Gaussian measure is induced by the following inner product

$$< P_n, Q_n >= \frac{1}{2\pi} \int_0^{2\pi} P_n(e^{i\theta}) \overline{Q_n(e^{i\theta})} d\theta \tag{0.0.2}$$

  A typicall random polynomial in this ensemble is of the form 0.0.1 where $p_j(z) = z^j$ and $a_j \sim \mathcal{N}(0,1)$.

- Elliptic Polynomials: In this model the Gaussian measure is induced by

$$< P_n, Q_n >= \int_{\mathbb{C}} P_n \overline{Q_n} \frac{dz}{(1+|z|^2)^{n+2}}$$

  and random polynomials are of the form 0.0.1 with $p_j(z) = \sqrt{\binom{n}{j}} z^j$ and $a_j \sim \mathcal{N}(0,1)$. Equaivalently, they are of the form 0.0.1 where $p_j^n(z) = z^j$ and $a_j \sim \mathcal{N}(0, \binom{n}{j})$.

- Legendre Polynomials: Here the Gaussian measure is induced by

$$< P_n, Q_n >= \frac{1}{2} \int_{-1}^{1} P_n(x) Q_n(x) dx$$

  and random polynomials consists of linear combinations of $p_j(x) = (j+\frac{1}{2})^{1/2} L_j(x)$ where $L_j(x) = \frac{1}{2^j j!} \frac{d^j}{dt^j}(x^2-1)^j$ and the coefficients $a_j \sim \mathcal{N}(0,1)$.

We denote by $N_n(\mathbb{R})$ the number of real zeros of polynomials in $\mathcal{P}_n$. Therefore $N_n(\mathbb{R}) : (\mathcal{P}_n, dProb_n^{\mu}) \to \{0, 1, ..., n\}$ defines a random variable. In this thesis we will be interested in the statistics of $N_n(\mathbb{R})$. Over the years, many scientists have been interested in this problem. The earliest works on this subject dates back to 1930's and it is focused on the Kac Polynomials. One of the first results on this context was provided by Bloch and Polya [1], they showed that $\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n})$ when a_j's uniformly distributed in $\{-1, 0, 1\}$. This problem was also studied by Littlewood and Offord in the serries of papers [2]-[3] for real Gaussians, Bernoulli and Uniform distributions. According to their results $\mathbb{E}[N_n(\mathbb{R})] \sim \log n$ as $n \to \infty$. Subsequently, in [4, 5] Mark Kac established the following explicit formula for $\mathbb{E}[N_n(\mathbb{R})]$, when the coefficients are standart real Gaussians

$$\mathbb{E}N_n(\mathbb{R}) = \frac{4}{\pi} \int_0^1 \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)} dx \tag{0.0.3}$$

where

$$A(x) = \sum_{j=0}^{n} x^{2j}, \ B(x) = \sum_{j=0}^{n} j x^{2j-1}, \ C(x) = \sum_{j=0}^{n} j^2 x^{2j}.$$

In addition, in [6] he also proved the following important asymptotics ,

$$\mathbb{E}N_n(\mathbb{R}) = (\frac{2}{\pi} + o(1)) \log n.$$

More refined versions of this asymptotics were developed by many authors. However, the sharpest known result is given by Wilkins [7], he established an asymptotic series expansion for $\mathbb{E}[N_n(\mathbb{R})]$. On the other hand, Erdos and Offord [8] generalized the asymptotic result to many other distributions. Finally, Ibragimov and Maslova [9, 10] extended the result to all mean-zero distributions in the domain of attraction of the normal law. In contrast,

Edelman&Kostlan [11] considered random functions of the form

$$P_n(z) = \sum_{j=0}^{n} a_j f_j(z)$$

where f_j's are suitable entire functions that take real values on the real line. Using a nice geometric approach they have shown that if $\mathbf{a} = (a_0, ..., a_n) \sim \mathcal{N}(0, \Sigma)$ and $m(t) = (f_0(t), ..., f_n(t))$ is any collection of differentiable functions on $\mathbb{R}$. Then the expected number of real zeros of P_n

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\pi} \int_{\mathbb{R}} \left(\frac{\partial^2}{\partial x \partial y} (\log m(x)^T \Sigma m(y)) \Big|_{x=y=t} \right)^{1/2} dt$$

In particular, if the coefficients are independent identically distriubuted (i.i.d) Standart Gaussians and $f_j = t^j$, this specializes to 0.0.3. As an immediate corrollary of this argument they also proved that $\mathbb{E}[N_n(\mathbb{R})] = \sqrt{n}$ for Kostlan polynomials. The asymptotic results of Kac Polynomials are also generalized in many other directions. For example Das in [12] proved that

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{n}{\sqrt{3}} + o(n) \tag{0.0.4}$$

for random linear combinations Legendre polynomials. Later, Lubinsky Pritsker and Xie [13, 14, 15] generalized this result to random orthogonal polynomials induced by measures with compactly suported weights on the real line. On the otherhand, Bayraktar [16] studied random polynomials where the probability measure on the space $\mathcal{P}_n$ is induced by the following inner product

$$< P_n, Q_n >= \int_{\mathbb{C}} P_n(z) \overline{Q_n(z)} e^{-2n\varphi(z)} dz \tag{0.0.5}$$

where $\varphi : \mathbb{C} \to \mathbb{R}$ is a non-negative smooth circulary-symmetric weight function which satisfies the following growth condition i.e.

$$\varphi(z) \geq (1+\epsilon)\log|z| \quad \text{for some } \epsilon > 0.$$

Assuming that the coefficients are independent copies of a random variable satisfying certain moment condition. He showed that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\pi}\int_{\mathcal{B}_\varphi \cap \mathbb{R}} \sqrt{\frac{1}{2}\Delta\varphi(x)}dx \tag{0.0.6}$$

where $\mathcal{B}_\varphi = \{z \in supp(\mu_{\mathbb{C},\varphi}) : \Delta\varphi > 0\}$ and $\mu_{\mathbb{C},\varphi}$ is the weighted equilibrium measure associated to φ . This result is general in the sense that, if $\varphi(z) = \frac{|z|^2}{2}$ we obtain the so-called Weyl polynomials. Hence it covers the results of [17] for Weyl polynomials. As a result, one should observe that in all the models above changing the inner product in $\mathcal{P}_n$ affects drastically the assymptotics of $\mathbb{E}[N_n(\mathbb{R})]$. In a nutshell, in this draft we will report in details the results of Kac. Namely we derive the Kac-Rice formula in two different ways and apply it to certain random polynomials. We also present some facts from potential theory and the distribution of complex zeros. Finally, we report the results of [16] and provide a conjecture in this direction for the variance of the real roots , which is still an ongoing project.

Chapter 1

Expected distribution of real zeros

In this chapter we present two different approaches in order to study the expected number of real zeros of random univariate polynomials with independent identically distributed (i.i.d) real Gaussian coefficients. In the first approach, we will consider certain *random functions*[1] as the path of a real-valued smooth stochastic process defined over some time interval I. Then, we will investigate the number of level crossings of this stochastic process. In particular, *random polynomials* arise as a special case of random functions, and studying its real roots is equivalent to study the *0*-crossings of such random functions. In addition, we will present a remarkable result due to Kac [4] called the *Kac-Rice* formula for the expected number of u-crossings of this stochastic process. On the otherhand, in the second approach we will obtain the same results by following a nice geometric argument provided by Edelman and Kostlan [11]. This approach will be more elegant and comprehensive.

[1] A function of the form $F_n(t) = F(t) := \sum_{k=0}^{n} a_k f_k(t)$ where the coefficients are random variables defined over the same probability space and f_k's are real valued smooth functions defined over some intervals in $\mathbb{R}$.

1.1 Kac-Rice

1.1.1 Basic ideas and definitions

In this section we will present some definitions and develop some notations that we will use throughout this note.

Definition 1. Let $I \subset \mathbb{R}$ be an interval and $f_0, ..., f_n : I \to \mathbb{R}$ some functions. Then a *random function* $F : I \to \mathbb{R}$ is the finite sum

$$F(t) := F_n(t, \omega) := \sum_{k=0}^{n} a_k(\omega) f_k(t) \tag{1.1.1}$$

where the coefficients $a_k = a_k(\omega)$ are random variables defined over the same probability space $(\Omega, \Sigma, \mathbb{P})$. In particular, if $f_k = t^k$ for $k = 0, 1, ..., n$ then F_n is called a *random polynomial*.

Remark 1. For the sake of simplicity, we will assume the coefficients are Gaussian random variables. In this case (1.1.1) is called a *Gaussian random function.*

Since during this section we will consider a *random function* as the path of a certain stochastic process $\mathcal{F}$ which is defined over some time interval I, i.e. $\mathcal{F} = \{F(t) : t \in I\}$. We need the following definitons and notations.

Definition 2. The *covariance kernel (function)* of a stochastic process $\mathcal{X} = \{X(t) : t \in I\}$ is the function $K : I \times I \to \mathbb{R}$ defined as

$$K(t, s) := Cov(X(t), X(s)) = \mathbb{E}[(X(t) - \mathbb{E}[X(t)])(X(s) - \mathbb{E}[X(s)])]$$

If $\mathcal{X} = \mathcal{F}$ then we denote the covariance kernel by $K_n(s, t)$.

Note that if $\mathcal{X}$ is a *centered* stochastic process (*i.e.* $\mathbb{E}[X(t)] = 0$ for all $t \in I$). Then,

$$K(t, s) = \mathbb{E}[X(t)X(s)]$$

For the centred stochastic process $\mathcal{F}$, the linearity of expectation implies that

$$K_n(x,y) = \mathbb{E}[\sum_{i=0}^{n} a_i f_i(x) \sum_{j=0}^{n} a_j f_j(y)] = \sum_{i=0}^{n}\sum_{j=0}^{n} f_i(x) f_j(y) \mathbb{E}[a_i a_j]$$

Moreover, if the coefficients a_k are independent Gaussian random variables of mean zero and variance σ_k^2, i.e. $a_k \sim \mathcal{N}(0, \sigma_k)$, $k = 1, 2, ...$ We have

$$K_n(x,y) = \sum_{j=0}^{n} f_j(x) f_j(y) \sigma_j^2$$

Notations: Let $f : I \to \mathbb{R}$ be a differentiable function and $u \in f(I)$, then we denote by

- $U_u(f, I) := \{t \in I : f(t) = u, f'(t) > 0\}$ the set of *up-crossings of f.*
- $D_u(f, I) := \{t \in I : f(t) = u, f'(t) < 0\}$ the set of *down-crossings of f.*
- $C_u(f, I) := \{t \in I : f(t) = u\}$ the set of *crossings of f.*
- $N_u(f, I) = |C_u(f, I)|$, if $u = 0$ we denote by $N(f, I)$.

Remark 2. In particular if P_n is a random polynomial of degree n, its number of real zeros on an interval I will be denoted by $N_n(I)$.

1.1.2 Kac Rice Formulas

In this subsection we present the *Kac-Rice* formula for the u-crossings of the random function F. Then as a corrollary we state the *Kac-Rice* formula for the number of real zeros of F. In order to derive this formula we will first prove some lemmas like the *Kac's counting formula.* During this section we will mainly follow ([18],[19]).

Definition 3. A $\mathcal{C}^1$- function $f : [a, b] \to \mathbb{R}$ is said to be *convenient* if the following conditions are satisfied:

- $f(a) \neq u$ and $f(b) \neq u$.
- $\{t \in [a,b] : f(t) = u, f'(t) = 0\} = \varnothing$, i.e. if $f(t) = u$ then $f'(t) \neq 0$.

Lemma 1.1.1. *(Kac's counting formula) Let $f : [a,b] \to \mathbb{R}$ be a* convenient *function, then the number of u-crossings of f in $[a,b]$ is*

$$N_u(f,[a,b]) = \lim_{\epsilon \to 0} N_u^{\epsilon}(f,[a,b]) \tag{1.1.2}$$

where $N_u^{\epsilon}(f,[a,b]) = \frac{1}{2\epsilon}\int_{[a,b]} \mathbb{1}_{\{|f(t)-u|<\epsilon\}}|f'(t)|dt$.

Proof. Observe that the assumption on f being convenient function, implies that f has finite number of u-crossings i.e. $N_u(f,I) = n$. If $n = 0$, then choosing ϵ sufficiently small we get the result. If $n \geq 1$, let $C_u(f,I) = \{c_1, ..., c_n\}$ then since f is convenient $f'(c_k) \neq 0$ for all $k \in \{1, ..., k\}$. Choosing $\epsilon > 0$ sufficiently small, $f^{-1}(u-\epsilon, u+\epsilon)$ is disjoint union of n intervals $I_k = (a_k, b_k)$ such that $c_k \in (a_k, b_k)$ for all k. Now since a_k and b_k are the local extremal points of the intervals I_k, we have $f(a_k) = u \pm \epsilon$ and $f(b_k) = b_k \mp \epsilon$ for all $k = 1, 2, ..., n$. Since $\epsilon > 0$ is sufficiently small I_k doesn't contain extreme points of f, hence f' does not change sign on each I_k. Then by using fundamental theorem of calculus

$$\frac{1}{2\epsilon}\int_a^b \mathbb{1}_{\{|f(t)-u|<\epsilon\}}\ |f'(t)|dt = \frac{1}{2\epsilon}\sum_{k=1}^{n}\int_{a_k}^{b_k}|f'(t)|dt$$

$$= \frac{1}{2\epsilon}\sum_{k=1}^{n}|f(a_k) - f(b_k)| = \frac{1}{2\epsilon}\sum_{k=1}^{n} 2\epsilon = n.$$

□

Remark 3. Lemma holds true also for f polygonal, even though these are not $\mathcal{C}^1$- functions.

One could derive the *Kac's counting formula* also in a different way by approximating the Dirac function δ. For the detailed explanation one may check ([19], §2).

Lemma 1.1.2. *Let $f : I \to \mathbb{R}$ be a* convenient *function such that $f(t)$ has r, u-crossing and s critical points. Then for $\epsilon > 0$ we have*

$$N_u^\epsilon(f, I) \leq r + 2s \tag{1.1.3}$$

Proof. Without lost of generality assume that f has $r-$zeros and $s-$critical points and prove $N_0^\epsilon(f, I) \leq r + 2s$. Observe that since f has finitely many critical points i.e. $f'(t) = 0$ for finitely many $t \in \mathbb{R}$. Then Rolle's theorem implies that $f(t) = c$ has finitely many solutions for any $c \in \mathbb{R}$. Fix $\epsilon > 0$, then since $|f(t)| = \epsilon$ has finitely many solutions, the set $\{|f| < \epsilon\}$ has finitely many connected components of the form $I_j = (a_j, b_j)$, $j = 1, 2, ..., n$ such that $|f(a_j)| = |f(b_j)| = \epsilon$. Now let k_j be the number of the turning points of f in the interval I_j (i.e. the points where f' changes sign in I_j). Then if I_j containts no turning points, f is either increasing or decreasing on this interval I_j, that is $f(a_j)f(b_j) < 0$ and thus I_j contains a unique zero of f. In particular if I_j contains no turning points, then

$$\int_{I_j} |f'(t)|dt = |f(t)|\Big|_{a_j}^{b_j} = 2\epsilon$$

Now let us define the following sets

$$\mathcal{S}_0 = \{j \in \{1, 2, ..., n\} : I_j \text{ contains no turning points}\}$$

$$\mathcal{S}_1 = \{j \in \{1, 2, ..., n\} : I_j \text{ contains turning points}\}$$

Then clearly $|\mathcal{S}_0| = r$, $|\mathcal{S}_1| \leq s$ and we have

$$N_0^\epsilon(f, I) = \frac{1}{2\epsilon} \int_I \mathbb{1}_{\{|f(t)| \leq \epsilon\}} |f'(t)|dt = \frac{1}{2\epsilon} \sum_{j=1}^{n} \int_{a_j}^{b_j} |f'(t)|dt$$

$$= \frac{1}{2\epsilon} \sum_{j \in \mathcal{S}_0} \int_{a_j}^{b_j} |f'(t)|dt + \frac{1}{2\epsilon} \sum_{j \in \mathcal{S}_1} \int_{a_j}^{b_j} |f'(t)|dt = r + \frac{1}{2\epsilon} \sum_{j \in \mathcal{S}_1} \int_{a_j}^{b_j} |f'(t)|dt$$

Now let $j \in \mathcal{S}_1$ and assume that $t_1 < ... < t_{k_j}$ are the turning points of f in I_j. Then

$$\begin{aligned} \int_{a_j}^{b_j} |f'(t)|dt &= |f(a_j) - f(a_1)| + |f(a_1) - f(a_2)| + ... + |f(a_{k_j}) - f(b_j)| \\ &\leq 2\epsilon(k_j + 1) \end{aligned}$$

Thus

$$N_0^\epsilon(f, I) \leq r + \sum_{j \in \mathcal{S}_1} 2\epsilon(k_j + 1) = \sum_{j \in \mathcal{S}_1} k_j + |\mathcal{S}_1| \leq r + 2s$$

where the last inequality follows from the fact that the sum of the number of turning points is equal to s. □

In the following part we will establish the *Kac-Rice formula*, that is the formula for the expected number of the u-crossings of a random function F. The rough idea will be to start from the *Kac's counting formula* and take expectation on both sides.

Let $F : I \to \mathbb{R}$ be the random function as defined in 1.1.1 with the coefficients a_k independent Gaussian random variables of mean zero and variance σ_k^2. Then let us start by assuming that F satisfies the following assumptions
(A1) F is almost surely *convenient.*
(A2) There exists a constant $M > 0$ such that $N_u(F, I) + N_u(F', I) < M$ almost surely.
By using *(A2), Lemma 1.1.2 and Lebesgue's dominated convergence theorem.* We have

$$\mathbb{E}[N_u(F, I)] = \int_\Omega N_u(F, I)d\mathbb{P} = \int_\Omega \lim_{\epsilon \to 0} N_u^\epsilon(F, I)d\mathbb{P}$$

$$= \lim_{\epsilon \to 0} \mathbb{E}[N_u^\epsilon(F, I)] = \lim_{\epsilon \to 0} \mathbb{E}[\frac{1}{2\epsilon} \int_I \mathbb{1}_{\{|F(t)-u|<\epsilon\}} |F'(t)| dt]$$
$$= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_I \mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}} |F'(t)|] dt$$

Note that in the last equality we have interchanged the expectation and the integral. Hence we obtain

$$\mathbb{E}[N_u(F, I)] = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_I \mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}} |F'(t)|] dt. \tag{1.1.4}$$

So in order to calculate the expectation we first have to compute the integrand $\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}} |F'(t)|]$. To do this, first observe that $(F(t), F'(t))$ is a Gaussian[2] random vector with mean $\mu = (\mathbb{E}[F(t)], \mathbb{E}[F'(t)]) = (0, 0)$ and the covariance matrix[3] Σ, which is given by the following symmetric matrix

$$\Sigma = \Sigma(t) := \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{12}(t) & \Sigma_{22}(t) \end{pmatrix}$$

where

$$\Sigma_{11}(t) = \mathrm{Cov}(F, F) = \mathbb{E}[F^2] - \mathbb{E}[F]^2 = \mathbb{E}[F^2] = K_n(t, t).$$

$$\Sigma_{12}(t) = \mathrm{Cov}(F, F') = \mathbb{E}[FF'] - \mathbb{E}[F]\mathbb{E}[F'] = \mathbb{E}[FF'] = K_n^{(1,0)}(x, y)\Big|_{x=y=t}$$

$$\Sigma_{22}(t) = \mathrm{Cov}(F', F') = \mathbb{E}[(F')^2] - \mathbb{E}[F']^2 = \mathbb{E}[(F')^2] = K_n^{(1,1)}(x, y)\Big|_{x=y=t}$$

Here $K(x, y) = \mathbb{E}[F(x)F(y)]$ is the covariance kernel of the random function F, and $K_n^{(1,0)}(x, y) := \frac{\partial K(x,y)}{\partial x}$, $K_n^{(1,0)}(x, y) = \frac{\partial^2 K(x,y)}{\partial x \partial y}$.
Let us define $\Delta(t) := \det(\Sigma(t))$ and suppose that the following assumption holds, that is

[2] A random vector $X = (X_1, X_2, ..., X_n) \in \mathbb{R}^n$ is a Gaussian random vector if for all real numbers $a_1, ..., a_n$, the random variable $a_1X_1 + ... + a_nX_n$ is a Gaussian random variable.

[3] The covariance matrix of a Gaussian random vector X is given by $\Sigma = (\mathrm{Cov}(X_i, X_j))_{ij}$ where $\mathrm{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]$.

(A3) For all $t \in I$, $\Delta(t) = \Sigma_{11}(t)\Sigma_{22}(t) - (\Sigma_{12}(t))^2 > 0$.
Now in order to compute $\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}}|F'(t)|]$, observe that

$$\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}}|F'(t)|] = \mathbb{E}[G(F(t), F'(t))], \tag{1.1.5}$$

where $G(x, y) = \mathbb{1}_{\{|x-u|<\epsilon\}}|y|$. Then using the fact that if X and Y are two random variables and $G : \mathbb{R}^2 \to \mathbb{R}$ is a function then $\mathbb{E}[G(X, Y)] = \int_{\mathbb{R}} \int_{\mathbb{R}} G(x, y)p_{(X,Y)}(x, y)dxdy$, where $p_{(X,Y)}(x, y)$ is the joint density of the random vector (X, Y). We have the following

$$\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}}|F'(t)|] = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{|x-u|<\epsilon\}}|y|p_{(F,F')}(x, y)dxdy$$

where $p_{(F,F')}(x, y)$ is the density[4] of the Gaussian random vector $(F(t), F'(t))$ of mean $\mu = (\mathbb{E}[F], \mathbb{E}[F']) = (0, 0)$, that is

$$p_{(F,F')}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\Delta(t)}} \exp(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{1.1.6}$$

Here $\mathbf{x}^T = (x, y)$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, in addition since *(A3)* holds Σ is invertible with

$$\Sigma^{-1} = \frac{1}{\Delta(t)} \begin{pmatrix} \Sigma_{22}(t) & -\Sigma_{12}(t) \\ -\Sigma_{12}(t) & \Sigma_{11}(t) \end{pmatrix}$$

Hence the density has the form

$$p_{(F,F')}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\Delta(t)}} \exp\left(-\frac{1}{2\Delta(t)}\Sigma_{22}(t)x^2 - 2\Sigma_{12}(t)xy + \Sigma_{11}(t)y^2\right).$$

[4]If $X = (X_1, ..., X_n)$ is a Gaussian random vector with mean μ and non-singular covariance matrix Σ then the density of X is $p_X(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp(-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu))$

Now after some simple algebraic manipulations we obtain

$$-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x} = -\frac{\Sigma_{11}(t)}{2\Delta(t)}\left(y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x\right)^2 - \frac{x^2}{2\Sigma_{11}(t)}$$

plugging this in 1.1.6, we obtain the density of $(F(t), F'(t))$,

$$p_{(F,F')}(x,y) = \frac{1}{2\pi\sqrt{\Delta(t)}}\exp\left[-\frac{\Sigma_{11}(t)}{2\Delta(t)}\left(y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x\right)^2 - \frac{x^2}{2\Sigma_{11}(t)}\right]. \qquad (1.1.7)$$

Substituing this expression in 1.1.2 we have

$$\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}}|F'(t)|] = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{|x-u|<\epsilon\}}|y|p_{(F,F')}(x,y)dxdy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{|x-u|<\epsilon\}}|y|\frac{1}{2\pi\sqrt{\Delta(t)}}\exp\left[-\frac{\Sigma_{11}(t)}{2\Delta(t)}\left(y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x\right)^2 - \frac{x^2}{2\Sigma_{11}(t)}\right]dydx$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{\Delta(t)}}\mathbb{1}_{\{|x-u|<\epsilon\}}\left(\int_{\mathbb{R}}|y|\exp\left[-\frac{\Sigma_{11}(t)}{2\Delta(t)}\left(y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x\right)^2 - \frac{x^2}{2\Sigma_{11}(t)}\right]dy\right)dx$$

$$= \int_{u-\epsilon}^{u+\epsilon} \frac{1}{2\pi\sqrt{\Delta(t)}}\exp\left[-\frac{x^2}{2\Sigma_{11}(t)}\right]\left(\int_{\mathbb{R}}|y|\exp\left[-\frac{\Sigma_{11}(t)}{2\Delta(t)}(y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x)^2\right]dy\right)dx$$

Now setting $\Omega(t)=\frac{\Delta(t)}{\Sigma_{11}(t)}$ and using the fact that

$$\frac{1}{2\pi\sqrt{\Delta(t)}} = \frac{1}{\sqrt{2\pi\Sigma_{11}(t)}}\frac{1}{\sqrt{2\pi\Delta(t)/\Sigma_{11}(t)}} = \frac{1}{\sqrt{2\pi\Sigma_{11}(t)}}\frac{1}{\sqrt{2\pi\Omega(t)}}.$$

we get

$$\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}}|F'(t)|] = \int_{u-\epsilon}^{u+\epsilon}\Phi_t(x)dx \qquad (1.1.8)$$

where $\Phi_t(x) := \frac{1}{\sqrt{2\pi\Sigma_{11}}}\exp\left[-\frac{x^2}{2\Sigma_{11}}\right]\left(\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\Omega}}|y|\exp\left[-\frac{1}{2\Omega}\left(y - \frac{\Sigma_{12}}{\Sigma_{11}}x\right)^2\right]dy\right)$.
One can easily observe that the integrand with respect to y in the expression of Φ_t can be written in a similar form as the density of a Gaussian random

variable say Y of mean $\mathbb{E}[Y] = \frac{\Sigma_{12}(t)x}{\Sigma_{11}(t)}$ and variance $\Omega(t)$, namely

$$\frac{1}{\sqrt{2\pi\Omega}}|y|\exp\left[-\frac{1}{2\Omega}\left(y-\frac{\Sigma_{12}}{\Sigma_{11}}x\right)^2\right] = |y|\frac{1}{\sqrt{2\pi}\sqrt{\Omega}}\exp\left[-\frac{1}{2}\left(\frac{y-\frac{\Sigma_{12}}{\Sigma_{11}}x}{\sqrt{\Omega}}\right)^2\right]$$
$$= |y|\Gamma_{\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x,\Omega(t)}(y)$$

where

$$\Gamma_{\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x,\Omega(t)}(y) = \frac{1}{\sqrt{2\pi}\sqrt{\Omega(t)}}\exp\left[-\frac{1}{2}\left(\frac{y-\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x}{\sqrt{\Omega(t)}}\right)^2\right]$$

Now using this density, Φ_t can be written as

$$\Phi_t(x) = \frac{1}{\sqrt{2\pi\Omega(t)}}\exp\left[-\frac{x^2}{2\Sigma_{11}(t)}\right]\left(\int_{\mathbb{R}}|y|\Gamma_{\frac{\Sigma_{12}}{\Sigma_{11}}x,\Omega}(y)dy\right) \tag{1.1.9}$$

$$= \frac{1}{\sqrt{2\pi\Omega(t)}}\exp\left[-\frac{x^2}{2\Sigma_{11}(t)}\right]\mathbb{E}[|Y|].$$

Hence using 1.1.2 and 1.1.8 in 1.1.4, the expectation $\mathbb{E}[N_u(F,I)]$ becomes

$$\mathbb{E}[N_u(F,I)] = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_I\int_{u-\epsilon}^{u+\epsilon}\Phi_t(x)dxdt. \tag{1.1.10}$$

Now our goal is to apply the limit on the integral with respect to x, that is we will have to interchange the limit with the integral with respect to t. For this reason we need to use the *Lebesgue's dominated convergence theorem.* Thus we need to find an integrable function $\theta(t)$ such that $|\Phi_t(x)| \leq \theta(t)$ on I. In order to do this, note that $|Y| \geq 0$ implies $\mathbb{E}[|Y|] \geq 0$ and by 1.1.2 we obtain that Φ_t is a positive function that is $|\Phi_t(x)| = \Phi_t(x)$. Then by Cauchy-Schwartz inequality

$$\mathbb{E}[|Y|] = \mathbb{E}[1\cdot|Y|] \leq \sqrt{\mathbb{E}[Y^2]} = \sqrt{Var(Y)+\mathbb{E}[Y]^2}$$

$$= \sqrt{\Omega(t) + \left(\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}x\right)^2} \leq \sqrt{\Omega(t)} + \frac{\Sigma_{12}(t)x}{\Sigma_{11}(t)} \tag{1.1.11}$$

where the last inequlity follows from the fact that if $a, b \geq 0$ then $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Hence by using 1.1.11 in the expression of Φ_t , we have

$$\Phi_t(x) = \frac{1}{\sqrt{2\pi\Omega(t)}} \exp\left[-\frac{x^2}{2\Sigma_{11}(t)}\right] \left(\sqrt{\Omega(t)} + \frac{\Sigma_{12}(t)x}{\Sigma_{11}(t)}\right)$$

On the other hand since $e^{-x^2} \leq 1$ for all $x \in \mathbb{R}$, in particullar for $|x| \leq 1$, we get

$$\Phi_t(x) \leq \frac{1}{\sqrt{2\pi\Omega(t)}} \left(\sqrt{\Omega(t)} + \frac{\Sigma_{12}(t)x}{\Sigma_{11}(t)}\right) = \frac{1}{2\pi} \left(\frac{\sqrt{\Delta(t)}}{\Sigma_{11}(t)} + \frac{|\Sigma_{12}(t)|}{\Sigma_{11}(t)^{3/2}}\right) := \theta(t)$$

Thus in order to use *Lebesgue* dominated convergence theorem we need the integrability of $\theta(t)$, that is

(A4). The function $\theta(t) = \frac{1}{2\pi}\left(\frac{\sqrt{\Delta(t)}}{\Sigma_{11}(t)} + \frac{|\Sigma_{12}(t)|}{\Sigma_{11}(t)^{3/2}}\right)$ is integrable on I, i.e. $\int_I \theta(t)dt < \infty$.

Hence using *(A4)* and *Lebesgue* dominated convergence theorem on 1.1.4 we have

$$\mathbb{E}[N_u(F, I)] = \int_I \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{u-\epsilon}^{u+\epsilon} \phi_t(x)dxdt = \int_I \Phi_t(u)dt \tag{1.1.12}$$

where

$$\Phi_t(u) = \frac{1}{\sqrt{2\pi\Omega(t)}} \exp\left[-\frac{u^2}{2\Sigma_{11}(t)}\right] \left(\int_{\mathbb{R}} |y| \Gamma_{\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)}u, \Omega(t)}(y)dy\right).$$

We have just proved the following theorem.

Theorem 1.1.1. *(*Kac-Rice formula for u-crossings*) Let $f_j : I \to \mathbb{R}$, $j = 0, 1, 2, .., n$ be smooth functions and a_j's independent Gaussian random variables defined over the same probability space $(\Omega, \Sigma, \mathbb{P})$, with mean zero and variance σ_j^2. If the random function*

$$F(t) = \sum_{j=0}^{n} a_j f_j(t)$$

satisfies the assumptions (A1) - (A4)*, then*

$$\mathbb{E}[N_u(F, I)] = \int_I \frac{1}{\sqrt{2\pi\Omega(t)}} \exp\left[-\frac{u^2}{2\Sigma_{11}(t)}\right] \left(\int_{\mathbb{R}} |y| \Gamma_{\frac{\Sigma_{12}}{\Sigma_{11}} u, \Omega(t)}(u) dy\right) dt,$$

where

$$\Gamma_{\frac{\Sigma_{12}(t)}{\Sigma_{11}(t)} x, \Omega(t)}(y) = \frac{1}{\sqrt{2\pi\Omega(t)}} \exp\left[-\frac{1}{2}\left(\frac{y - \frac{\Sigma_{12}(t)}{\Sigma_{11}(t)} x}{\sqrt{\Omega(t)}}\right)^2\right]$$

$$\Sigma_{11}(t) = K_n(t,t), \quad \Sigma_{12}(t) = K_n^{(1,0)}(x,y)\Big|_{x=y=t}, \quad \Sigma_{22}(t) = K_n^{(1,1)}(x,y)\Big|_{x=y=t}$$

$$\Omega(t) = \frac{\Delta(t)}{\Sigma_{11}(t)} = \frac{\Sigma_{11}(t)\Sigma_{22}(t) - \Sigma_{12}(t)^2}{\Sigma_{11}(t)}.$$

Remark 4. Observe that

$$\mathbb{E}[\mathbb{1}_{\{|F(t)-u|<\epsilon\}} |F'(t)|] = \int_{u-\epsilon}^{u+\epsilon} \mathbb{E}[F'(t)|F(t) = x] \cdot p_{F(t)}(x) dx$$

then under certain assumptions on F and using convergence theorems one can show that

$$\begin{aligned}\mathbb{E}[N_u(F, I)] &= \lim_{\epsilon \to 0} \int_I \frac{1}{2\epsilon} \int_{u-\epsilon}^{u+\epsilon} \mathbb{E}[F'(t)|F(t) = x] p_{F(t)}(x) dx dt \\ &= \int_I \mathbb{E}[F'(t)|F(t) = u] p_{F(t)}(u) dt \qquad (1.1.13)\end{aligned}$$

which gives rise to an equivalent form of the Kac-Rice formula. For the details and Kac-Rice formulas in a more general setting see ([18], Chapter 2).

Since in this note we deal with the real roots of random polynomials, henceforth we will restrict ourselves to the zero crossings. Thus, let $u = 0$ then we have

$$\Phi_t(0) = \frac{1}{\sqrt{2\pi\Sigma_{11}(t)}} \left(\int_{\mathbb{R}} |y| \Gamma_{0,\Omega(t)}(y) dy \right) \tag{1.1.14}$$

where

$$\Gamma_{0,\Omega}(y) = \frac{1}{\sqrt{2\pi}\sqrt{\Omega(t)}} \exp\left[-\frac{1}{2} \left(\frac{y}{\sqrt{\Omega(t)}} \right)^2 \right].$$

By computing the integral in 1.1.14, we get

$$\int_{\mathbb{R}} |y| \Gamma_{0,\Omega}(y) dy = \int_0^{+\infty} y \exp\left[-\frac{1}{2} \left(\frac{y}{\sqrt{\Omega(t)}} \right)^2 \right] dy$$

$$= \frac{2\Omega(t)}{\sqrt{2\pi}\sqrt{\Omega(t)}} = 2\sqrt{\frac{\Omega(t)}{2\pi}},$$

Now since $\Omega(t) = \Delta(t)/\Sigma_{11}(t)$, it follows that

$$\Phi_t(0) = \frac{1}{\sqrt{2\pi\Sigma_{11}(t)}} \frac{2\sqrt{\Omega(t)}}{\sqrt{2\pi}} = \frac{\sqrt{\Delta(t)}}{\pi\Sigma_{11}(t)} := \frac{1}{\pi}\rho_n(t)$$

Here,

$$\rho_n(t) = \left(\frac{\Sigma_{11}(t)\Sigma_{22}(t) - \Sigma_{12}(t)^2}{\Sigma_{11}(t)^2} \right)^{1/2} = \left(\frac{K_n(t,t)K_n^{(1,1)}(t,t) - (K_n^{(1,0)}(t,t))^2}{(K_n(t,t))^2} \right)^{1/2}$$

$$= \left(\frac{\partial^2}{\partial x \partial y} \log K_n(x,y) \Big|_{x=y=t} \right)^{1/2}.$$

Therefore we have established the Kac-Rice theorem for 0-crossings, that is

Theorem 1.1.2. *(*Kac-Rice formula for 0-crossings*) Let $f_j : I \to \mathbb{R}$, $j = 0, 1, 2, .., n$ be smooth functions and a_j's independent Gaussian random variables defined over the same probability space $(\Omega, \Sigma, \mathbb{P})$, with mean zero and variance σ_j^2. Then, if the random function*

$$F(t) = \sum_{j=0}^{n} a_j f_j(t)$$

satisfies (A1) - (A4)*, with u=0, then the expected number of real zeros of F in the interval I is given by*

$$\mathbb{E}[N(F, I)] = \frac{1}{\pi} \int_I \rho_n(t) dt,$$

where

$$\rho_n(t) = \left(\frac{K_n(t,t) K_n^{(1,1)}(t,t) - (K_n^{(1,0)}(t,t))^2}{(K_n(t,t))^2} \right)^{1/2}$$

or equivalently in logarithmic derivative form

$$\rho_n(t) = \left(\frac{\partial^2}{\partial x \partial y} \log K_n(x,y) \Big|_{x=y=t} \right)^{1/2}.$$

Remark 5. The term $\frac{1}{\pi}\rho_n(t)$ in the expression above represents the expected density of real zeros of F at the point $t \in \mathbb{R}$.

A note on the factorial moments: In general it's a demanding problem to estimate the higher moments of crossings of a random process, then sometimes we prefer to investigate it's factorial moments. Having this motivation in mind, we state an analogue result of Rice formula for the factorial moments of the crossings of the random process $\mathcal{F}$. For the details and more general treatments of Rice formulas see ([18], Chapter.2).

Theorem 1.1.3. *(Gaussian Rice formula). Let $I \subset \mathbb{R}$ be an interval and $\mathcal{F} = \{F(t) : t \in I\}$ a Gaussian stochastic process which has $\mathcal{C}^1$-paths. Let*

$k \geq 1$ be an integer. Assume that for pairwise distinct points $t_1, ..., t_k$ in I, the random variables $F(t_1), ..., F(t_k)$ have non-degenerate joint distribution. Then

$$\mathbb{E}[N_u^{[k]}(F, I)] = \int_{I^k} \mathbb{E}[|F'(t_1)...F'(t_k)|F(t_1) = ... = F(t_k) = u] \cdot p_{(F(t_1),...,F(t_k))}(u, ..., u)dt_1...dt_k$$

where $p_{(F(t_1),...,F(t_k))}$ is the joint density of the random vector $(F(t_1), ..., F(t_k))$ and $N_u^{[k]} = N_u(N_u - 1)...(N_u - k + 1)$.

Remark 6. Under the assumptions above one can write the Rice formula for k-factorial moment of level crossings also in the following form

$$\mathbb{E}[N_u^{[k]}] = \int_{I^k} \int_{\mathbb{R}^k} |x_1 \cdots x_k| \cdot p_{(F(t_1),...,F(t_k),F'(t_1),...,F'(t_k))}(u, ..., u, x_1, ..., x_k) \cdot dx_1...dx_k dt_1...dt_k.$$

1.2 Edelman-Kostlan

In this section we will follow a different path in order to obtain the Kac-Rice formula for the 0-crossings of a random function F. In this approach we use an elegant geometric argument which is provided by Edelman and Kostlan in [11].

1.2.1 Basic Geometric Arguments and Its relation to zeros of certain functions

Here we present some basic geometric arguments and show their relation with real roots of certain deterministic smooth functions. Throughout the section we will denote by S^n the surface of the unit sphere centered at the origin in $\mathbb{R}^{n+1}$.

Definition 4. Let P be a point on the sphere S^n, the *corresponding equator* $P_\perp$ is the set of points of S^n which lie on the plane through origin that is perpendicular to the line passing through the origin and the point P.

Remark 7. This definiton is the generalization of the usual earth's equator which is equal to $(\text{north pole})_\perp$, equivalently $(\text{south pole})_\perp$

Definition 5. Let $\gamma(t)$ be a rectifiable curve on the sphere S^n parametrized by $t \in \mathbb{R}$, then $\gamma_\perp := \{P_\perp | P \in \gamma\}$ is the set of equators of the curve γ.

Remark 8. (i) If the curve γ is a small part of a great circle, then the region formed by $\gamma_\perp$ is a "lune" denoted by $\cup\gamma_\perp$, and the following proportion is true

$$\frac{\text{area}(\cup\gamma_\perp)}{\text{area of } S^n} = \frac{|\gamma|}{\pi} \tag{1.2.1}$$

(ii) If γ is not a part of a great circle, the same argument is still applicable since we may approximate γ by small great circular arcs.

(iii) If γ is more than just half of a great circle or spirals many times around a point then the lunes will overlap.

These observations require the following definitions.

Definition 6. The multiplicity of a point $Q \in \cup\gamma_\perp$ is the number of equators in $\gamma_\perp$ containing Q, i.e.

$$\text{mult}_{\cup\gamma_\perp}(Q) := \#\{t \in \mathbb{R} | Q \in \gamma(t)_\perp\} \tag{1.2.2}$$

Definition 7. We define $|\gamma_\perp|$ to be the area of the "lune" (that is the area swept out by $\gamma(t)_\perp$) counting multiplicities i.e.

$$|\gamma_\perp| := \int_{\cup\gamma_\perp} \text{mult}_{\cup\gamma_\perp}(Q) d\sigma(Q) \tag{1.2.3}$$

where $d\sigma$ is the surface area measure on the sphere.

Hence, *Remark 7.* and *Definition 4.* implies the following lemma.

Lemma 1.2.1. *If γ is a rectifiable curve then*

$$\frac{|\gamma_\perp|}{\text{area of } S^n} = \frac{|\gamma|}{\pi}$$

After providing these interesting geometric arguments, in the subsequent we show the connection of these results with the real roots of a deterministic smooth function.
Let

$$f(x) = a_0 f_0(x) + a_1 f_1(x) + ... + a_n f_n(x) \tag{1.2.4}$$

be a non-zero deterministic function where $f_k : \mathbb{R} \to \mathbb{R}$, $k = 0, 1, 2, ..., n$ are smooth functions such that $f_k \equiv c \neq 0$ for some k, and $a_k \in \mathbb{R}$. Then we define its *moment curve* to be the curve $m(t) = (f_0(t), f_1(t), f_2(t), ..., f_n(t))$ in $\mathbb{R}^{n+1}$ where t runs over the real numbers. Now for the function $f(x)$ fix $t \in \mathbb{R}$ and define the vectors $a = (a_0, a_1, ..., a_n)$, $m(t) = (f_0(t), f_1(t), f_2(t), ..., f_n(t)) \in \mathbb{R}^{n+1}$ and $\mathbf{a} = \frac{a}{\|a\|}$, $\gamma(t) = \frac{m(t)}{\|m(t)\|}$. Then the condition that $x = t$ is a zero of the function $f(x)$ is precisely the condition that a is perpendicular to $m(t)$. Equivalently, $\mathbf{a} \perp \gamma(t)$ or $\mathbf{a} \in \gamma(t)_\perp$ for fixed $t \in \mathbb{R}$. Therefore, $\gamma(t)_\perp$ corresponds to all functions of the form 1.2.4 which have t as a root. Moreover, the multiplicity of $\mathbf{a}$ in $\gamma_\perp$ is exactly the number of real zeros of the corresponding function $f(x)$.

1.2.2 The expected number of real zeros of a random function.

So far we have not discussed any randomness. Here we will use the previous geometric arguments to find an explicit formula for the expected number of

real roots of certain random functions. Concerning this, we need the following lemma from the probability theory that is

Lemma 1.2.2. *Let $X = (X_1, X_2, ..., X_n)$ be a random vector in $\mathbb{R}^n$ such that each X_i is a Standart Gaussian random variable. Then the random vector $\boldsymbol{X} = \frac{X}{\|X\|}$, where $\|X\| = \sqrt{X_1^2 + ... + X_n^2}$ is unifromly distributed on S^n.*

Proof. Let A be any open set in S^{n-1}, and $\hat{A} = \bigcup_{r>0} rA$. Then,

$$\mathbb{P}(\mathbf{X} \in A) = \mathbb{P}(X \in \hat{A}) = \int_{\hat{A}} \frac{1}{(2\pi)^{\frac{n}{2}}} \exp^{-\frac{\|x\|^2}{2}} dx$$

By the polar change of coordinates,

$$\mathbb{P}(\mathbf{X} \in A) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_A \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1} dr d\sigma = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{2^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)}{n} \sigma(A)$$

$$= \frac{\Gamma(\frac{n}{2}+1)}{n\pi^{\frac{n}{2}}} \sigma(A) = \frac{\sigma(A)}{\sigma(S^{n-1})}$$

□

This lemma shows that if the a_i, $i = 0, 1, 2, ..., n$ are independent standart normal random variables, then the vector $\mathbf{a} = \frac{a}{\|a\|}$ is uniformly distributed on the unit sphere S^n.

Now letting $a_k \sim \mathcal{N}(0, 1)$ in 1.2.4, we consider the random function

$$F(x) = F_\omega(x) = a_0 f_0(x) + a_1 f_1(x) + ... + a_n f_n(x). \tag{1.2.5}$$

Identifying this *Random Function* with the random vector $\mathbf{a}$ (the vector generated by its coefficients). We establish that $F(x)$ corresponds to a uniformly distributed random point on the unit sphere S^n. By the previous section we know that $N(F, \mathbb{R}) = \text{mult}_{\gamma_\perp}(\mathbf{a})$. Using this fact, the expected number of real zeros of F is

$$\mathbb{E}[N(F, \mathbb{R})] = \int_{S^n} \text{mult}_{\cup\gamma_\perp}(\mathbf{a}) \frac{d\sigma(\mathbf{a})}{\text{area of} S^n} = \frac{|\gamma|}{\pi}$$

where $d\sigma$ is the surface area measure and $|\gamma|$ is the arc-length of the curve $\gamma(t)$ (recall: $\gamma(t)$ is the projection of the moment curve on the unit sphere S^n). Therefore, in order to calculate the expectation one has to compute the length of the curve γ. To do this, firstly observe that [5]

$$m(x)\cdot m(y) = K_n(x,y),\ m'(x)\cdot m(y) = K_n^{(1,0)}(x,y),\ \ m'(x)\cdot m'(y) = K_n^{(1,1)}(x,y)$$

where $m(t) = (f_0(t), f_1(t), ..., f_n(t))$ and $K_n(x,y)$ is the covariance kernel of the random function F. By the standart arclength formula we know that

$$|\gamma| = \int_{-\infty}^{+\infty} \|\gamma'(t)\|\, dt$$

Now in order to calculate the norm we may proceed in two different ways.
(I) Using some basic calculus, it is not hard to show that

$$\gamma'(t) = \left(\frac{m(t)}{\sqrt{m(t)\cdot m(t)}}\right)' = \frac{[m(t)\cdot m(t)]m'(t) - [m'(t)\cdot m'(t)]m(t)}{[m(t)\cdot m(t)]^{3/2}},$$

hence

$$\begin{aligned}\|\gamma'(t)\|^2 &= \frac{[m(t)\cdot m(t)][m'(t)\cdot m'(t)] - [m(t)\cdot m'(t)]^2}{[m(t)\cdot m(t)]^2} \\ &= \frac{K_n(t,t)K_n^{(1,1)}(t,t) - (K_n^{(1,0)}(t,t))^2}{(K_n(t,t))^2}.\end{aligned}$$

Hence, we obtain the analogue result of Kac-Rice formula

$$\mathbb{E}N_n(\mathbb{R}) = \mathbb{E}[N(F,\mathbb{R})] = \frac{1}{\pi}\int_{\mathbb{R}} \rho_n(t)dt \tag{1.2.6}$$

where

$$\rho_n(t) = \left(\frac{K_n(t,t)K_n^{(1,1)}(t,t) - (K_n^{(1,0)}(t,t))^2}{(K_n(t,t))^2}\right)^{1/2}.$$

[5] Here $\cdot$ is the usual dot product in $\mathbb{R}^{n+1}$

(II) Alternative way to express the expected number of real zeros is given by introducing a logarithmic derivative. In this case we can avoid the messy algebra in (I). It is easy to check that

$$\|\gamma'(t)\|^2 = \frac{\partial^2}{\partial x \partial y} \log\left[m(x) \cdot m(y)\right]\Big|_{x=y=t} = \frac{\partial^2}{\partial x \partial y} \log K_n(x,y)\Big|_{x=y=t}.$$

Hence

$$\mathbb{E}N_n(\mathbb{R}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \sqrt{\frac{\partial^2}{\partial x \partial y} \log K_n(x,y)\Big|_{x=y=t}}\, dt, \tag{1.2.7}$$

Remark 9. Observe that, one can obtain the same results also when the coefficients a_k are Gaussian random variables of mean zero and variance σ_k^2. In this case we simply define the moment curve as $m(t) = (f_0(t)\sigma_0, f_1(t)\sigma_1, ..., f_n(t)\sigma_n)$ and proceed in the same way.

1.3 Random Algebraic Polynomials

In this section we will apply the previous results (i.e. Kac-Rice formula) to certain random polynomial ensembles that are frequently studied in the literature. Recall that, if $I \subset \mathbb{R}$ is an interval, then a random algebraic polynomial of degree n, is a function $P_n : I \to \mathbb{R}$ given by

$$P_n(t) := \sum_{k=0}^{n} a_k t^k \tag{1.3.1}$$

where the coefficients a_k are random variables defined on the same probability space $(\Omega, \Sigma, \mathbb{P})$. In particular if $a_k \sim \mathcal{N}(0,1)$ we call Gaussian polynomial.

Remark 10. The asumptions *(A1) - (A4)* of the Kac-Rice theorem holds true automatically for Gaussian random polynomials e.g *(A4)* is true because in this case $\Sigma_{11}(t)$ is a polynomial of degree $2n$, $\Sigma_{12}(t)$ is a polynomial of degree $2n-1$, $\Sigma_{22}(t)$ is a polynomial of degree $2n-2$ and $\Delta(t)$ is a polynomial of

degree at most $4n-4$. Therefore

$$\frac{\sqrt{\Delta(t)}}{\Sigma_{11}(t)} + \frac{|\Sigma_{12}(t)|}{\Sigma_{11}(t)^2} = O\left(\frac{1}{t^2}\right) \quad \text{as} \quad t \to \infty.$$

In section 1.1 we derived the Kac-Rice formula for an interval I. Now in order to extend it to the real line $\mathbb{R}$ we need the following lemma.

Lemma 1.3.1. *Let $P_n : I \to \mathbb{R}$ be a Gaussian random polynomial such that $a_k \sim \mathcal{N}(0, \sigma_k)$. Then the following results hold*
(i) $\mathbb{E}[N(P_n, \mathbb{R}^{\geq 0})] = \mathbb{E}[N(P_n, \mathbb{R}^{\leq 0})]$.
(ii) If the variances satisfy the symmetry condition $\sigma_k^2 = \sigma_{n-k}^2$ for all k. Then $\mathbb{E}[N(P_n, (0,1))] = \mathbb{E}[N(P_n, (1,\infty))]$. *Moreover,*

$$\mathbb{E}[N(P_n, \mathbb{R})] = 4\mathbb{E}[N(P_n, (0,1))] = 4\mathbb{E}[N(P_n, (1,\infty))].$$

Proof. (i) Observe that $P_n(-t) = \sum_{k=0}^{n} (-1)^k a_k t^k$. The random variables $(-1)^k a_k$ are i.i.d Gaussian random variables of mean 0 and variance σ_k^2 since $\mathbb{E}[(-1)^k a_k] = (-1)^k \mathbb{E}[a_k] = 0$ and $Var((-1)^k a_k) = Var(a_k) = \sigma_k^2$. Hence $P_n(t)$ and $P_n(-t)$ have the same law, which implies that $\mathbb{E}[N(P_n, \mathbb{R}^{\leq 0})] = \mathbb{E}[N_0(P_n, \mathbb{R}^{\geq 0})]$.
(ii) Let $\widetilde{P}_n(t) := t^n P_n(t^{-1})$, then

$$\widetilde{P}_n(t) = t^n \sum_{k=0}^{n} a_k t^{-k} = \sum_{k=0}^{n} a_k t^{n-k} = \sum_{k=0}^{n} a_{n-k} t^k$$

The random variables a_{n-k} have mean zero and variance σ_k^2, since $\mathbb{E}[a_k] = 0$ and $Var(a_{n-k}) = Var(a_k) = \sigma_k^2$ (by the symetry condition). Hence the random polynomials $\widetilde{P}_n$ and P_n have the same law, thus $\mathbb{E}[N(P_n, (0,1))] = \mathbb{E}[N(P_n, (1,\infty))]$. Additionally, by (i), we have

$$\mathbb{E}[N(P_n, \mathbb{R})] = \mathbb{E}[N(P_n, \mathbb{R}^{\leq 0})] + \mathbb{E}[N(P_n, \mathbb{R}^{\geq 0})] = 2\mathbb{E}[N(P_n, \mathbb{R}^{\geq 0})]$$

Now since $\mathbb{R}^{\geq 0} = [0,1] \cup [1,\infty]$ then using (ii) we have

$$\begin{aligned}\mathbb{E}[N(P_n,\mathbb{R})] &= 2\mathbb{E}[N(P_n,\mathbb{R}^{\geq 0})] = 2\left(\mathbb{E}[N(P_n,(0,1))] + \mathbb{E}[N(P_n,(1,\infty))]\right) \\ &= 4\mathbb{E}[N(P_n,(0,1))] = 4\mathbb{E}[N(P_n,(1,\infty))]\end{aligned}$$

□

1.3.1 Kac Polynomials

A random algebraic polynomial of the form

$$P_n(t) := \sum_{k=0}^{n} a_k t^k$$

where the coefficients a_k are i.i.d. Gaussian random variables of mean zero and variance one is called a *Kac Polynomial*.[6]. The covariance kernel for the Kac polynomials is given by

$$K_n(x,y) = \sum_{i=0}^{n} x^i y^i = \frac{1-(xy)^{n+1}}{1-xy}$$

In order to calculate $\mathbb{E}N(P_n,\mathbb{R})$ we will use the Kac-Rice formula with the density in the logarithmic derivative form 1.1.2. Then

$$\log K_n(x,y) = \log(1-(xy)^n) - \log(1-xy)$$

$$\frac{\partial}{\partial x}\log K_n(x,y) = \frac{y}{1-xy} - \frac{(n+1)(xy)^n y}{1-(xy)^{n+1}},$$

$$\frac{\partial^2}{\partial x \partial y}\log K_n(x,y) = \frac{1}{(1-xy)^2} - \frac{(n+1)^2(xy)^n}{(1-(xy)^{n+1})^2}.$$

[6]Note that Kac polynomials satisfy the symmetry condition of lemma 1.3.1(ii)

Hence

$$\frac{\partial^2}{\partial x \partial y} \log K_n(x,y)\Big|_{x=y=t} = \frac{1}{(1-t^2)^2} - \frac{(n+1)^2 t^{2n}}{(1-t^{2n+2})^2} := \rho_n^2(t)$$

Therefore the expected number of real zeros of P_n is given by

$$\mathbb{E}N_n(\mathbb{R}) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \rho_n(t)dt = \frac{4}{\pi}\int_1^{\infty} \rho_n(t)dt \tag{1.3.2}$$

Thus we have proved the following theorem:

Theorem 1.3.1. *(Kac formula) The expected number of real zeros of the Kac polynomial P_n is*

$$\mathbb{E}N_n(\mathbb{R}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2 t^{2n}}{(t^{2n+2}-1)^2}}dt \tag{1.3.3}$$

$$= \frac{4}{\pi}\int_0^{1} \sqrt{\frac{1}{(1-t^2)^2} - \frac{(n+1^2 t^{2n})}{(1-t^{2n+2})^2}}dt. \tag{1.3.4}$$

Remark 11. $\rho_n(t)$ is the expected density of real zeros of Kac polynomials. Plotting its graph we see that it has two peaks at $t = -1$ and $t = 1$, which shows that the real zeros of Kac polynomials tend to concetrate near $t = \pm 1$ (see Fig. 1.1).

Note that this result was first obtained by Kac [4]. Kac in [6] also showed that $\mathbb{E}N_n(\mathbb{R}) \sim \frac{2}{\pi}\log n$. But several researchers have sharpend the Kac's original estimate, one can see [20] for a rigorous treatment of this estimate. Now we will provide a theorem without proof on the asymptotics of $\mathbb{E}N_n(\mathbb{R})$, for the detailed proof of this theorem we refer to ([11],§3.1).

Theorem 1.3.2. *Let P_n be the Kac polynomial of degree n. Then as $n \to \infty$,*

$$\mathbb{E}N_n(\mathbb{R}) = \mathbb{E}N(P_n, \mathbb{R}) = \frac{2}{\pi}\log n + C + \frac{2}{n\pi} + O(\frac{1}{n^2}),$$

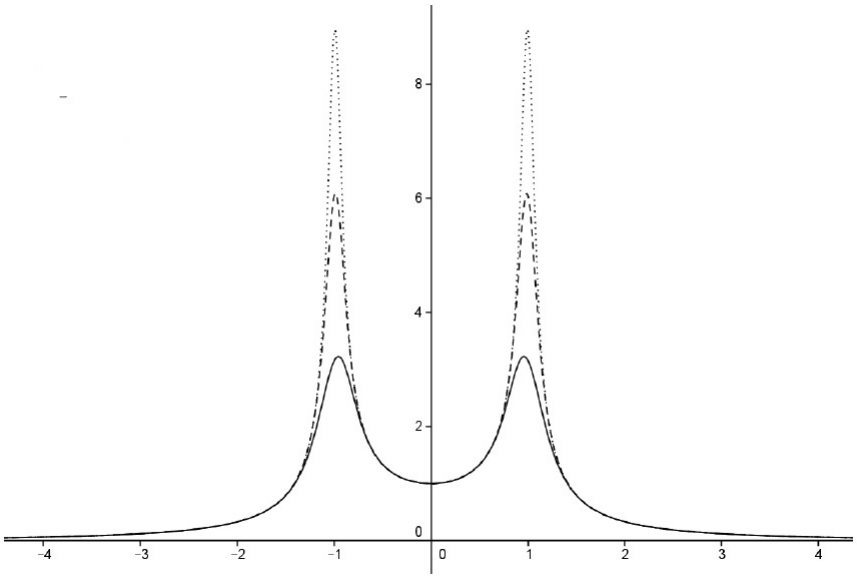

Figure 1.1: Density of real zeros for increasing degree $n = 10, 20, 30$.

where

$$C = 0.6257358072.....$$

On the other hand Ibragimov&Maslova[9][10] established the asymptotics for the variance of real zeros. They showed that

$$\mathrm{Var}(N_n(\mathbb{R})) = \mathrm{Var}[N(P_n, \mathbb{R})] \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)\log n. \tag{1.3.5}$$

Apart of this, they also established a CLT for the number of real roots of Kac polynomials.

1.3.2 Kostlan-Shub-Smale Polynomials

A random algebraic polynomial of the form

$$P_n(t) := \sum_{k=0}^{n} a_k t^k$$

where the coefficients a_k are Gaussian random variables of mean zero and variance $\sigma_k^2 = Var(a_k) = \binom{n}{k}$ is called the *Kostlan-Shub-Smale polynomial (KSS).*

Remark 12. Also in this ensemble the variances of the coefficients satisfy the symmetry condition of Lemma 1.3.1 since $\binom{n}{k} = \binom{n-k}{k}$ for all $k \in \{0, 1, ..., n\}$.

The covariance kernel for the KSS polynomials is given by

$$K_n(x, y) = \sum_{i=0}^{n} \binom{n}{i} x^i y^i = (1 + xy)^n$$

The corresponding derivates are

$$\frac{\partial}{\partial x} \log K_n(x, y) = n \frac{\partial}{\partial x} \log(1 + xy) = \frac{ny}{1 + xy}$$

$$\frac{\partial^2}{\partial x \partial y} \log K_n(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \log K_n(x, y) \right) = \frac{n}{(1 + xy)^2}$$

Then the density of real zeros is

$$\rho_n(t) = \sqrt{\frac{\partial^2}{\partial x \partial y} \log K(x, y) \Big|_{x=y=t}} = \sqrt{\frac{n}{(1 + t^2)^2}} = \frac{\sqrt{n}}{1 + t^2}$$

Therefore, the Kac-Rice formula implies that expected number of real zeros of the KSS polynomials of degree n is

$$\mathbb{E}[N(P_n, \mathbb{R})] = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sqrt{n}}{1 + t^2} dt = \sqrt{n}. \tag{1.3.6}$$

Remark 13. One need to observe that the KSS polynomials have on average more real zeros than the Kac polynomials. In addition, we have an exact value for the expected number of real zeros.

On the other hand Dalmao in [21] provided an asymptotic estimate for the variance of real roots of KSS polynomials and developed a CLT. More

precisely he showed that

$$\frac{\mathrm{Var}(N_n(\mathbb{R}))}{\sqrt{n}} \to C^2 \tag{1.3.7}$$

where $C^2 = \frac{2}{\pi}\int_0^\infty \left(B(t)\left[\sqrt{1-A^2(t)} + A(t)\arctan\left(\frac{A(t)}{\sqrt{1-A^2(t)}}\right) - 1\right]\right) dt + 1$ and $A(t) = \frac{1-(1+t^2)e^{-t^2}}{(1-e^{-t^2})^{3/2}}$, $B(t) = \frac{1-t^2-e^{-t^2}}{1-e^{-t^2}-t^2e^{-t^2}}e^{-t^2/2}$.

1.3.3 Weyl Polynomials

A random algebraic polynomial of the form,

$$P_n(t) := \sum_{k=0}^{n} a_k t^k$$

where the coefficients a_k are Gaussian random variables of mean zero and variance $\sigma_k^2 = \frac{1}{k!}$ is called a *Weyl polynomial*.
The covariance kernel for this type of polynomials is

$$K_n(x,y) = \sum_{i=0}^{n} \frac{1}{i!}x^i y^i$$

Then using the Kac-Rice formula together with the Stirling's formula, one can show that (see [22],[11] for details)

$$\mathbb{E}N_n(\mathbb{R}) = \left(\frac{2}{\pi} + o(1)\right)\sqrt{n} \tag{1.3.8}$$

Moreover, Do&Vu [17] provided variance estimates and a CLT for the number of real roots. More explicitly, they proved that

$$\mathrm{Var}(N_n(\mathbb{R})) = (2C + o(1))\sqrt{n} \tag{1.3.9}$$

where $C = 0.1819...$ is a positive consant.

1.3.4 Random Legendre Polynomials

Let μ be a Borel measure on the real line such that $d\mu(x) = dx$ on $[-1, 1]$ where dx is the Lebesgue measure. Applying Gram-Schmidt to monomials $\{1, t, t^2, ...\}$ with respect to inner product $< f, g >:= \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx$ we obtain the normalized Legendre polynomials

$$p_k(t) = (k + \frac{1}{2})^{1/2} L_k(t), \quad \text{where} \quad L_k(t) = \frac{1}{2^k k!}\frac{d^k}{dt^k}(t^2 - 1)^k \tag{1.3.10}$$

Using $\{p_k(t)\}$ we consider the following ensemble of random polynomials

$$P_n(t) = \sum_{k=0}^{n} a_k p_k(t), \qquad n \in \mathbb{N} \tag{1.3.11}$$

where a_j's are i.i.d random variables. This ensemble of random polynomials are called Random Legendre Polynomials. In this setting the covariance kernel is given by the so-called Christoffel-Darboux formula which states that

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x)p_k(y) = \frac{n+1}{2}\frac{L_{n+1}(x)L_n(y) - L_{n+1}(y)L_n(x)}{x - y} \tag{1.3.12}$$

As a result Das in [23] considered this type of random polynomials and he showed that

$$\mathbb{E}[N_n(-1, 1)] \sim \frac{n}{\sqrt{3}}$$

Later on Wilkins [24] improved this result by showing that $\mathbb{E}[N_n(-1, 1)] = \frac{n}{\sqrt{3}} + o(n^\epsilon)$ for any $\epsilon > 0$. Finally, Lubinsky, Pritsker and Xie in [13] generalized the result of Das by proving that for compactly supported weights on the real line the corresponding random orthogonal polynomials have $\frac{n}{\sqrt{3}} + o(n)$ expected number of real zeros, under suitable conditions.

Chapter 2

Distribution of Complex zeros

2.1 Basics of Potential Theory in $\mathbb{C}$

Since potential theory in $\mathbb{C}$ plays an important role on the zero distribution of complex random polynomials, we present some basic facts that we will use later.

Definition 8. Let $D \subset \mathbb{C}$ be a domain in $\mathbb{C}$ and $u : D \to [-\infty, \infty)$. We say that the function u is *subharmonic on* D if :
(i) u is upper-semicontinous on D. i.e. $\{z \in D : u(z) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$. Equivalently for each $z_0 \in D$, $\limsup_{z \to z_0} u(z) \leq u(z_0)$.
(ii) u satisfies the *submean value inequatliy* on D, that is, given $z_0 \in D$ and $r > 0$ with $\{z : |z - z_0| < r\} \subset D$,

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

We say that u is *superharmonic* if $-u$ is subharmonic.

Examples: (1) If f is holomorphic on D, then $u = |f|$ is subharmonic on D. (2) If f is holomorphic on D, then $u = \log |f|$ is subharmonic on D.

Theorem 2.1.1. *(Properties of subharmonic functions)*

(1) Let u and v be subharmonic functions on a domain D. Then
a) $\max\{u, v\}$ *is subharmonic on* D.
b) If $\alpha, \beta > 0$, *then* $\alpha u + \beta v$ *is a subharmonic function on* D.
(2) If $\{u_n\}$ *is a decreasing sequence of subharmonic functions on a domain* D *in* $\mathbb{C}$. *Then* $u = \lim_{n\to\infty} u_n$ is subharmonic on D.
(3) $u \in \mathcal{C}^2(D)$, *then* u *is subharmonic on* D *if and only if* $\Delta u \geq 0$ *on* D.
(4) Let $\{u_\alpha\}$ *be any family of subharmonic functions of* D *which is uniformly bounded on compact subsets of* D, *and let* $u(z) = \sup_\alpha u_\alpha(z)$. *Then the upper semicontinous regularization of* u *is subharmonic on* D, *i.e.*

$$u^*(z) = \limsup_{\xi\to z} u(\xi) \quad \text{is subharmonic on } D.$$

Similarly if $\{v_n\}$ *is subharmonic on* D *which is uniformly bounded on compact subsets of* D *and* $v(z) = \limsup_{n\to\infty} v_n(z)$. *Then*

$$v^*(z) := \limsup_{\xi\to z} v(\xi) \qquad \text{is subharmonic on } D.$$

(5) (Max.Principle) Let u *be a subharmonic function on a domain* $D \subset \mathbb{C}$.
a) If u *attains global max in* D *then* u *is constant. i.e. if* $u(z_0) = \sup_{z\in D} u(z)$ *for some* $z_0 \in D$, *then* $u(z) \equiv u(z_0)$.
b) If for all $\xi \in \partial D$,

$$\limsup_{z\to\xi} u(z) \leq 0$$

then $u \leq 0$ *on* D. *(If* D *is unbounded, this boundary condition includes* $\xi = \infty$*)*

(6) (Gluing) Let u *be a subharmonic function on an open set* $U \subset \mathbb{C}$, *and let* v *be a subharmonic function on an open subset* V *of* U *such that*

$$\limsup_{z\to\xi} v(z) \leq u(\xi) \quad \text{for all } \xi \in U \cap \partial V. \text{ Then the function}$$

$$\tilde{u} = \begin{cases} \max(u, v) & \text{on } V \\ u & \text{on } U \setminus V \end{cases}$$

is subharmonic on U.

(7) (Integrability) Let u be a subharmonic function on a domain $D \in \mathbb{C}$, with $u \not\equiv -\infty$ on D. Then u is locally integrable on D, that is

$$\int_K |u(z)| dm(z) < \infty \quad \text{for each compact subset } K \subset D.$$

(8) (Convex increasing function of a subharmonic function is subharmonic) Let $-\infty \leq a \leq b \leq \infty$, and let $u : U \to [a, b)$ be a subharmonic function on an open set $U \in \mathbb{C}$. Let $\psi : (a, b) \to \mathbb{R}$ be an increasing convex function. Then $\psi \circ u$ is subharmonic on U, where we define $\psi(a) = \lim_{t \to a} \psi(t)$.

Proof. (cf. [25], Chp.2). □

Remark 14. i) Note that (2) is not true if u_n is increasing, because if $u_n(z) = \frac{1}{n} \log |z|$ on $\Delta(0, 1)$. Then

$$u_n(z) \to u(z) = \begin{cases} 0, & \text{if } 0 < |z| < 1 \\ -\infty, & \text{if } z = 0 \end{cases} \text{ which is not usc at 0.}$$

ii) In (5) u can attain a global minimum without being constant on D. For example for $D = \mathbb{C}$ and $u(z) = \max\{\Re(z), 0\}$. Clearly u is subharmonic on D by (1.a) and u attains a global minimum at any point in the closed left-half plane $Re z \leq 0$.

ii) If u is a subharmonic function on $D \subset \mathbb{C}$, then e^u is subharmonic on D because $e^x : \mathbb{R} \to \mathbb{R}^{>0}$ is incresing convex function. Moreover, $|f|^p$, for $p > 0$ is subharmonic, since

$$|f|^p = \exp(p \log |f|) = \psi(\log |f|) \text{ where } \psi(t) = \exp(pt),$$

is a convex increasing function. One may also generate other example of

subharmonic functions using the properties above.

Now using some basic multivariable calculus in particular *Green's identity*, one can prove a fundametal result on the Laplace operator in $\mathbb{R}^2 = \mathbb{C}$ ([25] §3.7), which will be quite useful.

Theorem 2.1.2. $\Delta(\frac{1}{2\pi}\log|z|) = \delta_0$ *in the sense of distributions. That is*

$$\int_D \Delta\phi(z)\cdot(\frac{1}{2\pi}\log|z|)dm(z) = \phi(0), \quad \text{for all } \phi \in \mathcal{C}_0^\infty$$

where D is a neighborhood of the origin and dm- is the Lebesgue measure.

Note, since the function $u(z) = \log|z|$ is subharmonic i.e. locally integrable, then for a given measure μ of finite total mass and compact support, one can form the following convolution

$$V_\mu(z) := -p_\mu(z) := (u * \mu)(z) := \int_{\mathbb{C}} \log|z-\omega|d\mu(\omega).$$

Then

$$\Delta V_\mu = \Delta(u*\mu) = \Delta u * \mu = 2\pi\delta_0 * \mu = 2\pi\mu \tag{2.1.1}$$

since δ_0 acts like identity[1] under convolution.

Definition 9. Let μ be a finite Borel measure on $\mathbb{C}$ with compact support, the function $p_\mu : \mathbb{C} \to \mathbb{R}$ defined by

$$p_\mu(z) = \int_{\mathbb{C}} \log\frac{1}{|z-\omega|}d\mu(\omega) = -\int_{\mathbb{C}} \log|z-\omega|d\mu(\omega).$$

is called the *logarithmic potential of* μ.

Remark 15. The name arises from physical considerations. If we think of μ representing a charge distribution, then p_μ represents the potential at the point z due to charge μ.

[1]For measures μ, ν of finite total mass, their convolution is defined as $\mu * \nu(B) := \int \nu(B-x)d\mu(x)$ for any measurable set B.

Theorem 2.1.3. *(Properties of logarithmic potential)*
Let V_μ be as above. Then
a) V_μ is subharmonic in $\mathbb{C}$, (i.e. p_μ is superharmonic).
b) V_μ is harmonic in $\mathbb{C} \setminus supp[\mu]$.
c) As $|z| \to \infty$,

$$V_\mu = \mu(\mathbb{C}) \log |z| + O(\frac{1}{|z|}).$$

d) $\Delta V_\mu = 2\pi\mu$.
e) Let K=supp[μ], then for $z_0 \in K$,

$$\liminf_{z \to z_0} V_\mu(z) = \liminf_{z \to z_0 z \in K} V_\mu(z).$$

In particular, if $V_\mu|_K$ is continous, then V_μ is continous on $\mathbb{C}$.

Proof. ([25], §3). □

Examples:
(1) If $\mu = \delta_0$, then $V_\mu = \log |z|$.
(2) If $\mu = \frac{1}{n} \sum_{j=1}^n \delta_{z_j}$, then

$$V_\mu(z) = \frac{1}{n} \sum_{j=1}^n \log |z - z_j| = \frac{1}{n} \log \prod_{j=1}^n |z - z_j| = \frac{1}{n} \log |p_n(z)|$$

where $p_n(z) = \prod_{j=1}^n (z - z_j)$.
(3) If $\mu = \frac{d\theta}{2\pi}$ on $|z| = 1$, then $V_\mu(z) = \log^+ |z| = \max\{\log |z|, 0\}$.
Sol: If $|z| > 1$, consider the function $\log |z - t|$ which is harmonic for $|t| \leq 1$, then by mean-value property at $t = 0$ we have $\frac{1}{2\pi} \int_0^{2\pi} \log |z - e^{i\theta}| d\theta = \log |z|$.
If $|z| < 1$ then using the previous part we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |z - e^{i\theta}| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |1 - ze^{i\theta}| d\theta = \log 1 = 0$$

Now if $|z| = 1$ then for $0 < r < 1$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|z - e^{i\theta}|d\theta = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \log|z - re^{i\theta}|d\theta = \log 1 = 0.$$

Therefore

$$V_\mu(z) = \frac{1}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi} \log|z - e^{i\theta}|d\theta = \log^+|z|.$$

Now let us introduce the notion of energy corresponding to a measure μ.

Definition 10. (Energy Integrals) Let μ be a finite Borel measure in $\mathbb{C}$ with compact support. Then its *logarithmic energy* $I(\mu)$ is given by

$$I(\mu) := -\int\int \log|z - \omega|d\mu(\omega)d\mu(z) = \int p_\mu(z)d\mu(z)$$

Remark 16. Physically $I(\mu)$ means the total energy due to the charge μ.

Definition 11. (Polar Set)
a) A subset E of $\mathbb{C}$ is called *polar* if $I(\mu) = +\infty$ for every finite Borel measure μ with compact support in E.
b) A propery **P** is said to hold *quasi-everywhere (q.e)* on a set $S \subset \mathbb{C}$, if it holds on S except a polar subset of S.

Remark 17. The simplest example of a polar set is a set with finitely many points. For simplicity, consider a set with single point $E = \{a\}$. If μ is a finite Borel measure on E that is not a zero measure i.e. $\mu(E) > 0$. Then

$$I(\mu) = -\int\int \log|z - \omega|d\mu(z)d\mu(\omega) = -(\log|a - a|)(\mu(\{a\}))^2 = \infty.$$

which shows that E is polar.

One can prove the following theorem which characterizes polar sets ([25],§3.5).

Theorem 2.1.4. *Let $E \subset \mathbb{C}$, then E is a polar set if there exists a subharmonic function $u \not\equiv -\infty$, with $E \subset \{u(z) = -\infty\}$.*

Since $u(z) = \log|f(z)|$ is subharmonic for f holomorphic, then using the fact that the zeros of holomorphic functions are discrete, we obtain that any discrete set in $\mathbb{C}$ is polar.

Theorem 2.1.5. *(Properties of Polar sets)*
(1) Let μ be a finite Borel measure in $\mathbb{C}$ with compact support, and $I(\mu) < \infty$. Then, $\mu(E) = 0$ for every polar set E.
(2) Every polar set has Lebesgue measure zero.
(3) A countable union of polar sets is polar.

Proof. [25], §3.2 □

Remark 18. In fact polar sets are much thinner than sets of Lebesgue measure. For example, the Cantor ternary set in $[0, 1]$ has zero Lebesgue measure, but it is not polar.

It's of both mathematical and physical importance to define the following energy minimization problem.

Definition 12. Let $K \subset \mathbb{C}$ be a compact set and let $\mathcal{P}(K)$ denote the set of all probability measures on K. If there exists $\mu_K \in \mathcal{P}(K)$ such that

$$I(\mu_K) = \inf_{\mu \in \mathcal{P}(K)} I(\mu)$$

then μ_K is called an *equilibrium measure for K.*

Theorem 2.1.6. *If K is a non-polar compact set in $\mathbb{C}$, then there exists a unique equilibrium measure $\mu_K \in \mathcal{P}(K)$ i.e. $\inf_{\mu \in \mathcal{P}(K)} I(\mu) = I(\mu_K) < \infty$.*

Proof. [25], §3.3. □

In order to characterize the measure μ_K for a non-polar compact set K. We present the following result due to Frostman which is known as one of the most fundamental theorems of potential theory,

Theorem 2.1.7. *(Frostman) Let $K \subset \mathbb{C}$ be a non-polar compact set. Then*
(1) $p_{\mu_K}(z) \leq I(\mu_K)$ for all $z \in \mathbb{C}$;
(2) $p_{\mu_K}(z) = I(\mu_K)$ q.e on K.

Proof. [25], §3.3. □

Now let us define the following important class of subharmonic functions in $\mathbb{C}$, that is the *global subharmonic functions of at most logarithmic growth* i.e.

$$\mathcal{L}(\mathbb{C}) := \{u \text{ shm on } \mathbb{C} : u(z) - \log|z| = O(1), |z| \to \infty\} \tag{2.1.2}$$

Also let $\mathcal{L}^+(\mathbb{C}) := \{u \in L(\mathbb{C}) : u(z) \geq \log^+|z| + C_u\}$ be the restricted subclass of $\mathcal{L}(\mathbb{C})$. A typical example in this class is the function $u(z) = \frac{1}{n}\log|p_n(z)|$ where $p_n(z) = \sum_{j=1}^n a_j z^j$. Moreover, if $\mu(\mathbb{C}) = 1$ then $V_\mu \in \mathcal{L}(\mathbb{C})$, in particular the examples above belong to this class.

Let us also state the *global domination principle*, which plays an essential role in the relation of V_K^* and p_{μ_K}. The proof of this principle can be found in ([26], Chp.2 Theorem 3.2).

Proposition 2.1.1. *(GDP) Let $u \in \mathcal{L}(\mathbb{C})$ and $v \in \mathcal{L}^+(\mathbb{C})$, if $u \leq v$ a.e. Δv. Then $u \leq v$ on $\mathbb{C}$.*

Definition 13. Let K be a compact subset of $\mathbb{C}$, and let

$$V_K(z) := \sup\{u(z) \in \mathcal{L}(\mathbb{C}), u \leq 0 \text{ on } K\}$$

then $V_K^*(z) := \limsup_{\xi \to z} V_K(\xi)$ is called the *global extremal function of K.*

Remark 19. (i) V_K also can be obtained as follows

$$V_K(z) = \sup\{\frac{1}{\deg(p)}\log|p(z)| : p \text{ is a polynomial and } \|p\|_K = \sup_K |p| \leq 1\}$$

where the supremum is taken over all non-constant holomorphic polynomials. (ii) Either $V_K^* \equiv +\infty$ (which happens if K is polar), or else $V_K^* \in \mathcal{L}^+(\mathbb{C})$. (for details see [25], §4).

Observe that for a non-polar compact set K Frostman's theorem implies that $p_{\mu_K} = I(\mu_K)$ q.e on K. Thus ,

$$V_{\mu_K} + I(\mu_K) = 0 \quad \text{q.e on } K$$

So the function $V_{\mu_K} + I(\mu_K) \in \mathcal{L}(\mathbb{C})$, indeed by Frostman theorem since $V_{\mu_K} \geq -I(\mu_K)$ on $\mathbb{C}$ we have $V_{\mu_K} + I(\mu_K) \in \mathcal{L}^+(\mathbb{C})$. Then it turns out that (see [27] for details)

$$V_{\mu_K}(z) + I(\mu_K) = V_K^*(z) \quad \text{on } \mathbb{C}$$

Moreover,

$$\mu_K = \frac{1}{2\pi}\Delta V_K^* \quad \text{and} \quad \text{supp}(\mu_K) = \text{supp}(\Delta V_K^*) \subset K.$$

Remark 20. Often g_K is used for V_K^*, known as the *Green function* for K, that is the unique subharmonic function in $\mathbb{C}$ which is in $\mathcal{L}^+(\mathbb{C})$, harmonic in $\mathbb{C} \setminus K$, and equals 0 q.e on K. In particular if $g_K = 0$ on all of K we say that K has a *classical Green function*.

Examples:
(1) If $K = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ we have $V_K(z) = \max[\log|z|, 0]$ and $\mu_K = \frac{d\theta}{2\pi}$. To see this, let $u(z) := \log^+|z| \in \mathcal{L}^+(\mathbb{C})$. Then u is a harmonic function outside of K and vanishes on K. Moreover, Δu is supported on K. For any $v \in \mathcal{L}(\mathbb{C})$ with $v \leq 0$ on K we have $v \leq u$ on $K = supp(\Delta u)$, hence by GDP $v \leq u$ on $\mathbb{C}$ which implies that $V_K^* = \log^+|z|$. Note that here $V_K = V_K^*$ since $\log^+|z|$ is continous.
(2) If $K = \overline{B(a,r)} = \{z : |z-a| \leq r\}$ or $K = \partial B(a,r)$, then $V_K(z) = V_K^*(z) = \log^+\frac{|z-a|}{r}$.

2.2 Basics of Weighted Potential theory in $\mathbb{C}$

Let $K \subset \mathbb{C}$ be a closed set, then the function $w : K \to [0, \infty)$ is said to be the weight function of K.

Definition 14. A weight function w on K is said to be *admissible* if it satisfies the following conditions:
(i) w is upper semi-continous;
(ii) $K_0 := \{z \in K : w(z) > 0\}$ is non-polar;
(iii) if K is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in K$;

Writing $\varphi := -\log w$, then we have $\varphi : K \to (-\infty, \infty]$ and
(i) φ is lower semi-continous;
(ii) $\varphi < \infty$ on a non-polar set;
(iii) If K is unbounded, then

$$\varphi(z) - \log|z| \to \infty \text{ as } |z| \to \infty, z \in K.$$

Denote the collection of such functions φ by $\mathcal{A}(K)$.

Let $\mathcal{P}(K)$ be the collection of probability measures ν with $\text{supp}(\nu) \subset K$, then we define the *weighted energy integral*

$$I_\varphi(\nu) := \int\int \log \frac{1}{|z-t|w(z)w(t)} d\nu(t)d\nu(z) = I(\nu) + 2\int \varphi d\nu.$$

Note that the classical case corresponds to choosing K compact and $w = 1$ on K. The existence and uniqueness of a weighted energy minimizer measure $\mu_{K,\varphi} \in \mathcal{P}(K)$ that is, the measure which satisfies

$$I_\varphi(\mu_{K,\varphi}) = \inf_{\nu \in \mathcal{P}(K)} I_\varphi(\nu) =: V_\varphi$$

follows like in the unweighted case and it is called the *weighted equilibrium or extremal measure*. Moreover, condition (iii) implies that $\text{supp}(\mu_{K,\varphi})$ is compact and $\text{supp}(\mu_{K,\varphi}) \subset K_0$, in particular $\text{supp}(\mu_{K,\varphi}) \subset K_\epsilon$ where $K_\epsilon =$

$\{z : w(z) \geq \epsilon\}$ for some $\epsilon > 0$. The problem of finding the probability measure that minimizes the weighted energy integral in literature is known is known as the *logarithmic energy minimization in the presence of an external field* φ. Next we present the weighted version of Frostman's theorem. Given a closed set $K \subset \mathbb{C}$ and $\varphi \not\equiv 0 \in \mathcal{A}(K)$, let us define

$$F_\varphi := I_\varphi(\mu_{K,\varphi}) - \int_K \varphi d\mu_{K,\varphi} = V_\varphi - \int_K \varphi d\mu_{K,\varphi}.$$

Then we have

$$\begin{aligned} F_\varphi &= I(\mu_{K,\varphi}) + 2\int_K \varphi d\mu_{K,\varphi} - \int_K \varphi d\mu_{K,\varphi} \\ &= I(\mu_{K,\varphi}) + \int_K \varphi d\mu_{K,\varphi} = \int_K [p_{\mu_{K,\varphi}} + \varphi] d\mu_{K,\varphi}. \end{aligned}$$

The constant F_φ is called the *modified Robin constant* for φ. Note that if $\varphi \equiv 0$, then for K compact and non-polar $F = I(\mu_K)$.

Theorem 2.2.1. *Let $K \subset \mathbb{C}$ be a closed set and $\varphi \in \mathcal{A}(K)$. Then*

$$p_{\mu_{K,\varphi}} + \varphi \geq F \quad q.e \text{ on } K;$$

$$p_{\mu_{K,\varphi}} + \varphi \leq F \text{ on } S_\varphi := supp(\mu_{K,\varphi}).$$

In particular, $p_{\mu_{K,\varphi}} + \varphi = F$ q.e on S_φ.

We note that $F - p_{\mu_{K,\varphi}} = F + V_{\mu_{K,\varphi}} \in \mathcal{L}(\mathbb{C})$ and $F + V_{\mu_{K,\varphi}} = \varphi$ q.e on K. Now as in the unweighted case we define the *weighted extremal function*

$$V_{K,\varphi}(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}),\ u \leq \varphi \text{ on } K\}$$

Let $K \subseteq \mathbb{C}$ compact, we say that K is *locally regular* if for each $z \in K$ the unweighted Green function for the sets $K \cap \overline{B(z,r)}$, $r > 0$ are continous at z. Here $B(z,r)$ is the Euclidean disk with center at z and radius r. In the one variable case the local regularity of K is equivalent to global regularity

that is $V_K = V_K^*$ is continous. If K is regular and φ is continous one can show that $V_{K,\varphi}$ is continous, that is $V_{K,\varphi} = V_{K,\varphi}^*(z) \leq \varphi$ on K. In addition, by GDP applied to $V_{K,\varphi}^*, F + V_{\mu_K,\varphi}$ turns out that (see [26]for details)

$$V_{K,\varphi}^* = F + V_{\mu_{K,\varphi}} \text{ on } \mathbb{C} \text{ and } V_{K,\varphi}^* \in \mathcal{L}(\mathbb{C})$$

Moreover,

$$\mu_{K,\varphi} = \frac{1}{2\pi}\Delta V_{K,\varphi}^* \text{ and } supp(\Delta V_{K,\varphi}^*) \subset \{z \in K : V_{K,\varphi}^* \geq \varphi(z)\}$$

Indeed $V_{K,\varphi}^* = \varphi$ on $S_\varphi = supp(\mu_{K,\varphi})$ except perhaps a polar set.
Now let's look at some concrete examples.
Examples:
(1) Let $K = \overline{B(0,1)}$. In this case we know that $\mu_K = \frac{d\theta}{2\pi}$ on $S^1 = \partial B(0,1)$. Now take $\varphi(z) = |z|^2$, i.e., $w(z) = e^{-|z|^2}$.Then $V_{K,\varphi}^* = V_{K,\varphi}$ this follows since φ is continous on K and K is locally regular. Then

$$V_{K,\varphi}(z) = \begin{cases} |z|^2 & \text{if } |z| \leq \frac{1}{\sqrt{2}} \\ \log|z| + \frac{1}{2} - \log\frac{1}{\sqrt{2}} & \text{if } |z| \geq \frac{1}{\sqrt{2}} \end{cases}$$

In particular $supp(\mu_{K,\varphi}) = \overline{B(0,\frac{1}{\sqrt{2}})}$. To see this let $V(z)$ be the function defined as above. Then $V \leq V_{K,\varphi}$ since $V \in \mathcal{L}(\mathbb{C})$ and $V \leq \varphi$ on K. Now since ΔV is supported on $\overline{B(0,\frac{1}{\sqrt{2}})}$ and $V = \varphi$ on this set, we have $V_{K,\varphi} \leq V$ on $supp(\Delta V)$. Then by GDP, $V_{K,\varphi} \leq V$ on $\mathbb{C}$, hence $V = V_{K,\varphi}$. Indeed, taking $K = \mathbb{C}$ and $\varphi(z) = |z|^2$, one obtains the same result.
More generally for radially symmetric weight functions of super-logarithmic growth we have the following.

- Let $\varphi(z) = \varphi(|z|) = \varphi(r)$ be a weight function on $\mathbb{C}$ which is convex on $r > 0$. Let r_0 be the smallest number for which $\varphi'(r) > 0$ for all $r > r_0$, and let R_0 be the smallest solution of $R_0\varphi'(R_0) = 1$. Then $S_\varphi = \{z : r_0 \leq |z| \leq R_0\}$ and $d\mu_{K,\varphi} = \frac{1}{2\pi}(r\varphi'(r))'drd\theta$.

This result is a part of Theorem 6.2 pg.265, [26].

Now let $K \subset \mathbb{C}$ be a closed set and φ an admissible weight on K. We define the weighted sup-norm on $\mathcal{P}_n$ as follows

$$\|p_n\|_{K,\varphi} := \left\|p_n e^{-n\varphi}\right\|_K := \sup_{z\in K} |p_n(z)e^{-n\varphi(z)}|$$

It is easy to check that $\|p_n\|_{K,\varphi} < \infty$ for $p_n \in \mathcal{P}_n$ simply by using the fact that $\lim_{|z|\to\infty z\in K} |z|e^{-\varphi(z)} = 0$ implies that $\lim_{|z|\to\infty z\in K} |z|^n e^{-n\varphi(z)} = 0$.

Example. Let $K = \mathbb{C}$, $\varphi(z) = |z|^2$ and consider $p_n(z) = z^n$. Then

$$\|p_n\|_{K,\varphi} = \sup_{z\in\mathbb{C}} |z|^n e^{-n|z|^2} = \frac{1}{\sqrt{2^n}} e^{-\frac{n}{2}} < \infty, \quad n \in \mathbb{N}$$

Because if $g(r) = r^n e^{-nr^2}$ then $g'(r) = nr^{n-1}e^{-nr^2} - 2nr^{n+1}e^{-nr^2} = 0$ implies that $r = 0, \frac{1}{\sqrt{2}}$. That is $\max_{r\geq 0} g(r) = g(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2^n}} e^{-\frac{n}{2}}$. It follows that the supremum is attained on $|z| = \frac{1}{\sqrt{2}}$, which shows that $\|p_n\|_{K,\varphi} = \|p_n\|_{S_\varphi,\varphi}$.

One may show that the weighted extremal function $V_{k,\varphi}$ can be obtained as an upper envelope of certain polynomials only i.e.,

$$V_{K,\varphi} = \sup\{\frac{1}{deg(p)} \log|p(z)| : p \text{ polynomial and } \|p_n\|_{K,\varphi} = \left\|w^{deg(p)}p\right\|_K \leq 1\}.$$

Let μ be a measure on K such that

$$supp(\mu) \cap \{z \in K : e^{-\varphi(z)} > 0\}$$

contains infinitely many points. Then for each $n \in \mathbb{N}$ we can define the weighted $L^2(\mu)$ norms on $\mathcal{P}_n$ as follows

$$\|p_n\|^2_{L^2_{\varphi,n}(K)} := \left\|e^{-n\varphi}p_n\right\|^2_{L^2(K,\mu)} := \int_K |p_n(z)|^2 e^{-2n\varphi(z)} d\mu(z)$$

provided that it is finite.

Note: If K is not compact we assume strong decay condition on $w(z) = e^{-\varphi(z)}$ as $|z| \to \infty$, to ensure that $\|p_n\|_{L^2_{\varphi,n}(K)} < \infty$

Example. If $K = \mathbb{C}$, $\varphi(z) = |z|^2$ and $d\mu = dm$ wehere dm is the Lebesgue measure on $\mathbb{C}$. Then $\mu(\mathbb{C}) = \infty$, but for each $n \in \mathbb{N}$ the measures $d\mu_n = e^{-2n|z|^2} dm(z)$ have finite total mass and $\|p_n\|_{L^2_{\varphi,n}(K)} < \infty$ for $p_n \in \mathcal{P}_n$.

Note:We obtained two different norms on $\mathcal{P}_n$, but since it is finite dimensional vector space there exists a possitive constant M_n such that $\|p_n\|_{K,\varphi} \leq M_n \|p_n\|_{L^2_{\varphi,n}(K)}$ for all $p_n \in \mathcal{P}_n$.

2.3 Random Polynomials in $\mathbb{C}$

Let $P_n(z) = \sum_{j=0}^{n} a_j z^j$ be a random polynomial, where the coefficients $a_0, ..., a_n$ are i.i.d complex Gaussian random variables with $\mathbb{E}[a_j] = \mathbb{E}[a_j a_k] = 0$ and $\mathbb{E}[a_j \bar{a_k}] = \delta_{jk}$; i.e. each a_j has distribution

$$\phi(t)dm(t) = \frac{1}{\pi} e^{-|t|^2} dm(t)$$

where dm is the Lebesgue measure in $\mathbb{C}$. Let $\mathcal{P}_n$ be the vector space of polynomials of degree at most n. Now identifying p_n with a random vector $(a_0, ..., a_n) \in \mathbb{C}^{n+1}$, we endow $\mathcal{P}_n$ with a probability measure $Prob_n$ that is for $G \subset \mathbb{C}^{n+1}$

$$Prob_n(G) = \int_G \phi(a_0)...\phi(a_n) dm(a_0)...dm(a_n)$$

$$= \frac{1}{\pi^{n+1}} \int_G e^{-\sum_{j=0}^{n} |a_j|^2} dm(a_0)...dm(a_n).$$

Thus $(\mathcal{P}_n, Prob_n)$ forms a probability space whose elements are random polynomials of degree n. We also form the product probability space of sequences

of polynomials:

$$\mathcal{P} := \bigotimes_{n=1}^{\infty}(\mathcal{P}_n, Prob_n) = \bigotimes_{n=1}^{\infty}(\mathbb{C}^{n+1}, Prob_n)$$

For each $P_n \in \mathcal{P}_n$ we may write $P_n(z) = a_n \prod_{j=1}^{n}(z - \xi_j)$ where $(\xi_1, ..., \xi_n)$ are zeros of P_n, then we define the new measure valued random variable $\mathcal{Z}_{P_n} := \frac{1}{n}\sum_{j=1}^{n}\delta_{\xi_j}$ and call the *normalized zero measure of* P_n. Note that $\mathcal{Z}_{P_n} = \Delta(\frac{1}{n}\log|p_n|)$ where $\Delta \log|z| = \delta_0$ and $\Delta = \frac{1}{2\pi}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. We are interseted in the asymptotics of:

- $\mathbb{E}[\mathcal{Z}_{P_n}]$
- $\{\frac{1}{n}\log|P_n|\}$ for random sequences of polynomials $\{P_n\} \in \mathcal{P}$.

Let us start start with defining $\mathbb{E}[\mathcal{Z}_{P_n}]$.
$\mathbb{E}[\mathcal{Z}_{P_n}]$ is a measure on $\mathbb{C}$ defined as follows, for $\psi \in \mathcal{C}_c(\mathbb{C})$

$$(\mathbb{E}[\mathcal{Z}_{P_n}], \psi) := \int_{\mathbb{C}^{n+1}}(\mathcal{Z}_{p_n}, \psi)dProb_n(\mathbf{a}_n)$$

where $\mathbf{a}_n = (a_0, ..., a_n)$ and $(\mathcal{Z}_{P_n}, \psi) = \frac{1}{n}\sum_{j=1}^{n}\psi(\xi_j)$. Thus,

$$(\mathbb{E}[\mathcal{Z}_{P_n}], \psi) = \frac{1}{\pi^{n+1}}\int_{\mathbb{C}^{n+1}}\frac{1}{n}\sum_{j=1}^{n}\psi(\xi_j)e^{-\sum_{j=0}^{n}|a_j|^2}dm(a_0)...dm(a_n).$$

Now in order to investigate the asymptotics stated above we need the notion of Bergman Kernels.
Bergman Kernel: Let $L^2(\mu_{S^1}) = \{f : S^1 \to \mathbb{C} : \int_0^{2\pi}|f|^2\frac{d\theta}{2\pi} < \infty\}$ be the space of square integrable functions on the unit circle S^1 in $\mathbb{C}$. Also let $\mathcal{P}_n = span\{1, z, ..., z^n\}$ then, since $\mathcal{P}_n$ is a closed subspace the orthogonal projection operator $\Pi_n : L^2(\mu_{S^1}) \to \mathcal{P}_n$ is bounded and linear. However,

since $\{z^j\}_{j=0}^n$ forms an orthonormal basis for $\mathcal{P}_n$. We have

$$\Pi_n f(z) = \sum_{j=0}^{n} < f, z^j > z^j = \int_{S^1} f(w) \left(\sum_{j=0}^{n} z^j \overline{w}^j \right) d\mu_{S^1}(w)$$

$$= \int_{S^1} f(w) K_n(z,w) d\mu_{S^1}(w).$$

Here, $K_n(z,w) = \sum_{j=0}^{n} z^j \overline{w}^j$ is called *Bergman Kernel (or Reproducing kernel for point evaluation at z on $\mathcal{P}_n$)*. On the diagonal $\Delta = \{z = w\}$ if :
(1) $z = w = e^{i\theta}$, then $K_n(e^{i\theta}, e^{i\theta}) = n + 1 = dim(\mathcal{P}_n)$.
(2) $z = w \neq e^{i\theta}$, then $K_n(z,z) = \sum_{j=0}^{n} |z|^{2j} = \frac{|z|^{2n+2}-1}{|z|-1}$.
Thus we have

$$\frac{1}{2n} \log K_n(z,z) \to \log^+ |z| \tag{2.3.1}$$

locally uniformly in $\mathbb{C}$. Hence

$$\Delta(\frac{1}{2n} K_n(z,z)) \to \mu_{S^1} \tag{2.3.2}$$

So we have the following result.

Theorem 2.3.1. $\mathbb{E}[\mathcal{Z}_{P_n}] \to \mu_{S^1} = \frac{d\theta}{2\pi}$ *as* $n \to \infty$.

Proof. Let us start by writing

$$P_n(z) = |\sum_{j=0}^{n} a_j z^j| =: | < \mathbf{a}_n, \mathbf{z}_n > | = K_n(z,z)^{1/2} | < \mathbf{a}_n, \mathbf{u}_n(z) > |$$

where $\mathbf{a}_n = (a_0, ..., a_n)$, $\mathbf{z}_n = (1, z, ..., z^n)$ and $\mathbf{u}_n(z) = \frac{\mathbf{z}_n(z)}{\|\mathbf{z}_n(z)\|} = \frac{\mathbf{z}_n(z)}{K_n(z,z)^{1/2}}$
Then for $\psi \in \mathcal{C}_c(\mathbb{C})$

$$(\mathbb{E}[\mathcal{Z}_{P_n}], \psi) = \int_{\mathbb{C}^{n+1}} (\Delta \frac{1}{n} \log |P_n(z)|, \psi(z)) dProb_n(\mathbf{a}_n)$$

$$= \int_{\mathbb{C}^{n+1}} (\Delta \frac{1}{2n} \log |K_n(z,z)|, \psi(z)) dProb_n(\mathbf{a}_n)$$

$$+\int_{\mathbb{C}^{n+1}}(\frac{1}{n}\log|<\mathbf{a}_n,\mathbf{u}_n(z)>|,\psi(z))dProb_n(\mathbf{a}_n)$$

The first term in the expression above goes to $\int_{S^1}\psi d\mu_{S^1}$ as $n\to\infty$ and the second term can be written as

$$\int_{\mathbb{C}^{n+1}}(\frac{1}{n}\log|<\mathbf{a}_n,\mathbf{u}_n(z)>|,\Delta\psi(z))dProb_n(\mathbf{a}_n)$$

$$=\int_{\mathbb{C}}\Delta\psi(z)[\frac{1}{n}\int_{\mathbb{C}^{n+1}}\log|<\mathbf{a}_n,\mathbf{u}_n(z)>|dProb_n(\mathbf{a}_n)]dm(z)$$

Note that in the last equality we used Fubini. Now since Gaussian measure $dProb_n(\mathbf{a}_n)$ is unitary invariant then by mapping $\mathbf{u}_n\to(1,0,...,0)$ we have

$$\int_{\mathbb{C}^{n+1}}\log|<\mathbf{a}_n,\mathbf{u}_n(z)>|dProb_n(\mathbf{a}_n)$$

$$=\int_{\mathbb{C}^{n+1}}\frac{1}{\pi^{n+1}}\log|<\mathbf{a}_n,\mathbf{u}_n(z)>|e^{-\sum_{j=0}^{n}|a_j|^2}dm(a_0)...dm(a_n)$$

$$-\frac{1}{\pi}\int_{\mathbb{C}}\log|a_0|e^{-|a_0|^2}dm(a_0)$$

Thus the second term is $O(\frac{1}{n})$. Therefore $\mathbb{E}[\mathcal{Z}_{P_n}]\to\frac{d\theta}{2\pi}$. □

Note that the main point in the proof of the theorem is the fact that $\frac{1}{2n}\log K_n(z,z)\to\log^+|z|$ locally uniformly in $\mathbb{C}$. Now for $K\subset\mathbb{C}$ compact we know that the extremal function is

$$V_K(z)=\sup\{\frac{1}{deg(p)}\log|p(z)|:\|p\|_K\le 1\}\text{ and }\ \mu_K=\Delta V_K.$$

Then if K is regular that is, V_K is continous i.e. $V_K^*=V_K$, then defining

$$\phi_n(z):=\sup\{|p(z)|:p\in\mathcal{P}_n\text{ and }\ \|p\|_K\le 1\}$$

we have

$$\frac{1}{n}\log\phi_n(z)\to V_K(z)\ \text{ locally uniformly on }\mathbb{C}\qquad(2.3.3)$$

(If $K = S^1$ then $V_K = \log^+ |z|$ and $\mu_K = \mu_{S^1} = \frac{d\theta}{2\pi}$). We can recover V_K and μ_K using L^2- methods. For this we need the following definiton

Definition 15. (Bernstein Markov measure) Let τ be a measure on K such that for $n = 1, 2, ...$

$$\|p\|_K \leq M_n \|p\|_{L^2(\tau)} \quad \text{for all } p \in \mathcal{P}_n \text{ with } \limsup_{n\to\infty} M_n^{1/n} = 1$$

Then we say τ is a Bernstein Markov (BM) measure for K.

Remark 21. A simple example is the normalized arc-length measure $d\mu_{S^1} = \frac{d\theta}{2\pi}$ on the unit circle $K = S^1 = \{z : |z| = 1\}$ in the complex plane. Indeed for any compact set K in the complex plane there is a finite measure which satisfies the *Bernstein-Markov* property on K check ([28], Corrollary 3.5) for details. In particular for any non-polar compact set the corresponding equilibrium measure μ_K satisfies the Bernstein-Markov inequality.

Lemma 2.3.1. *If τ is a (BM) measure on K. Then*

$$\frac{1}{n+1} \leq \frac{K_n(z,z)}{\phi_n(z)^2} \leq M_n^2(n+1) \tag{2.3.4}$$

where $K_n(z,w) := \sum_{j=0}^n p_j(z)\overline{p_j(w)}$ and $\{p_j(z)\}_{j=0}^n$ orthonormal basis for $\mathcal{P}_n$ with the inner product comming from $L^2(\tau)$.

Remark 22. The left-hand side of this inequality follows from the reproducing property of the Bergman Kernel and the Cauchy-Schwarz inequalty. But for the right-hand side one needs to use the (BM)- property (cf. Lemma 2.2, [29]) for details.

Then by 2.3.3 and 2.3.4 we have the following result.

Corollary 2.3.1. *If V_K is continous, $\lim_{n\to\infty} \frac{1}{2n} \log K_n(z,z) = V_K(z)$ locally uniformly on $\mathbb{C}$.*

Thus we have the following universality result which a is generalization of Theorem 2.3.1 .

Theorem 2.3.2. *Let τ be a (BM) measaure on a compact set K with V_K continous. Consider random polynomials of the form $P_n(z) = \sum_{j=0}^{n} a_j p_j(z)$ where $p_j(z)$ form an orthonormal basis for $\mathcal{P}_n$ in $L^2(\tau)$ and $a_0, ..., a_n$ are i.i.d standart complex Gaussian random variables. Then*

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{Z}_{p_n}] = \mu_K.$$

Additionally, the same asymptotics holds for the sequence of normalized measures.

Theorem 2.3.3. *Let K be a compact set with V_K continous and let τ be a (BM) measure on K. Consider the random polynomials $P_n(z) = \sum_{j=0}^{n} a_j p_j(z)$ where $\{p_j(z)\}_{j=0}^{n}$ is an orthonormal basis for $\mathcal{P}_n$ in $L^2(\tau)$ and the coefficients $\{a_j\}_{j=0}^{n}$ are i.i.d complex Gaussian random variables. Then almost surely in $\mathcal{P}$ we have*

$$(\limsup_{n\to\infty} \frac{1}{n} \log |P_n(z)|)^* = V_K^*(z)$$

pointwise for all $z \in \mathbb{C}$ and $\frac{1}{n} \log |P_n(z)| \to V_K(z)$ in $L^1_{loc}(\mathbb{C})$. Hence almost surely

$$\Delta(\frac{1}{n} \log |P_n|) \to \mu_K.$$

Remark 23. This result holds true also for more general distributions satisfying certain tail condition, for details see (cf. [27], Theorem 4.2 and [30], §2.2).

Now in the weighted setting for $\varphi \in \mathcal{A}$ we also have.

Definition 16. The measure τ is said to be weighted Bernstein-Markov(BM) measure for K, φ if for all $p_n \in \mathcal{P}_n$

$$\|p_n\|_{K,\varphi} \leq M_n \|p_n\|_{L^2_{\varphi,n}(K)} \text{ with } \limsup_{n\to\infty} M_n^{\frac{1}{n}} = 1$$

Example. For $K = \mathbb{C}$ and $\varphi(z) = |z|^2$, $d\tau = dm$, which is the Lebesgue measure in $\mathbb{C}$ is a weighted BM measure for K, φ.

In a similar way, we construct an orthonormal basis $\{p_j^n\}_{j=0}^n$ for $\mathcal{P}_n$ in the weighted L^2- space namely in $L^2_{K,\varphi}(\tau) = L^2(e^{-2n\varphi}d\tau)$. Then

$$K_n^\varphi(z,w) := \sum_{j=0}^{n} p_j^n(z)\overline{p_j^n(w)}$$

is the weighted Bergman Kernel or reproducing kernel for point evaluation at z on $\mathcal{P}_n$. That is for $P_n \in \mathcal{P}_n$,

$$P_n(z) = \int_K P_n(w)K_n^\varphi(z,w)e^{-2n\varphi(z)}d\tau(w)$$

Let

$$\phi_{\varphi,n} := \sup\{|p(z)| : p \in \mathcal{P}_n, \|p\|_{K,\varphi} = \left\|e^{-n\varphi}p\right\|_K \leq 1\}$$

then as in the unweighted case if $V_{K,\varphi}$ is continous. Then

$$\lim_{n\to\infty} \frac{1}{n}\log \phi_{\varphi,n}(z) = V_{K,\varphi}(z) \text{ locally uniformly in } \mathbb{C}.$$

Moreover, the analogue of 2.3.1 holds true also in this setting, that is

$$\frac{1}{n+1} \leq \frac{K_n^\varphi(z.z)}{\phi^2_{\varphi,n}} \leq M_n^2(n+1)$$

As a result we have the analogue of the Theorems 2.3.2 and 2.3.3 in the weighted setting.

Example. (Scaled Weyl Polynomial) Let us consider the scaled Weyl polynomial

$$W_n(z) = \sum_{j=0}^{n} a_j \frac{z^j\sqrt{n^j}}{\sqrt{j!}}$$

where a_j are i.i.d complex or real Gaussian random variables. Let $\varphi(z) = \frac{|z|^2}{2}$

on $K = \mathbb{C}$ be our weight function and let $d\tau = dz$ be the Lebesgue measure in $\mathbb{C}$. Then since the function $\varphi(z)$ is super-logarithmic and radial by the result we had pg. 46, the weighted equilibrium measure is the normalized Lebesgue mesure on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Applying Gram-Schmidt to monomials in $L^2(\mathbb{C}, e^{-2n\varphi}dz)$ we obtain an orthonormal basis $p_j^n(z) = \sqrt{\frac{n^{j+1}}{\pi j!}} z^j$. Hence by the weighted version of Theorem 2.3.3, we obtain that zeros of $W_n(z)$ are equidistributed with respect to the Lebesgue measure on the unit disc in the complex plane.

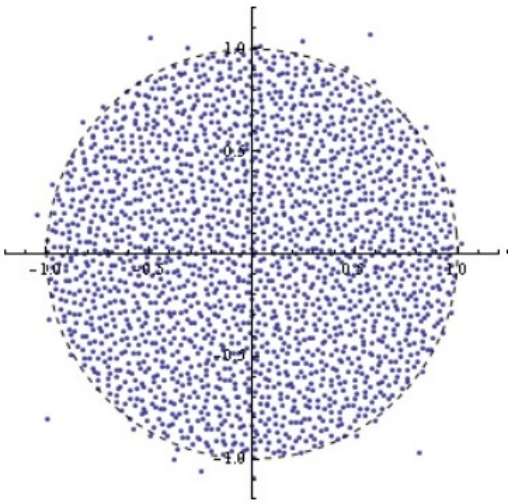

Figure 2.1: The zeros of Weyl Polynomials scaled by $1/\sqrt{n}$ for degree $n = 2000$.

Remark 24. In fact this is an example of how one can modify random polynomials so that the zeros are uniformly distributed on compact sets K with distribution other than μ_K.

Chapter 3

Variance of the number of real zeros

In this chapter we set up an ensemble of random polynomials arising from certain weight functions supported in the complex plane. We state some statistical results for the real roots of random polynomials in this ensemble. In particular we investigate the asymptotics concerning the variance of real roots and we provide a conjecture in this direction.

3.1 Setting the problem and Bergman Kernel Asymptotics

3.1.1 Setting of the problem

Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a $\mathcal{C}^2$- weight function satisfying the following conditions
(1) φ is radially symetric i.e. $\varphi(z) = \varphi(|z|)$ for all $z \in \mathbb{C}$.
(2) $\varphi(z) \geq (1+\epsilon)\log|z|$ for $|z| >> 1$ and $\epsilon > 0$.
Then for each fixed $n \in \mathbb{N}$ define the corresponding weighted L^2-space of

square integrable functions. Namely

$$L^2_{\varphi,n}(\mathbb{C}) = L^2(\mathbb{C}, e^{-2n\varphi}dz) = \{f : \mathbb{C} \to \mathbb{R} : \int_{\mathbb{C}} |f|^2 e^{-2n\varphi(z)}dz < \infty\}$$

where dz is te Lebesgue measure in $\mathbb{C}$. Now let $\mathcal{P}_n = span\{1, z, ..., z^n\}$ be the space of polynomials of degree at most n. The growth condition (2) ensures that every polynomial in $\mathcal{P}_n$ has finite weighted L^2-norm. Hence $(\mathcal{P}_n, ||.||_{L^2_{\varphi,n}(\mathbb{C})})$ has a Hilbert space structure. Applying Gram-Schmidt orthogonalization algorithm to the monomials $\{z^j\}_{j=0}^n$ we obtain an orthonormal basis for $\mathcal{P}_n$, denoted by $\{p_j^n(z)\}_{j=0}^n$. Therefore, the ensemble that we will consider consists of the random linear combinations of $\{p_j^n(z)\}$, that is

$$P_n(z) = \sum_{j=0}^{n} a_j p_j^n(z) \tag{3.1.1}$$

where $a_j \sim \mathcal{N}_{\mathbb{R}}(0, 1)$. In particular, if $\varphi(z) = \frac{|z|^2}{2}$, then $p_j^n(z) = \sqrt{\frac{n^{j+1}}{\pi j!}} z^j$ which gives rise to scaled version of Weyl ensemble. Moreover if $\varphi(z) = \frac{1}{2}\log(1 + |z|^2)$ then $p_j^n(z) = \sqrt{\binom{n}{j}} z^j$ which gives the Elliptic polynomials. Thus the ensemble above is more general in the sense that it covers ensemles like (Kac, Weyl, Elliptic). According to Bayraktar (cf. [16], Theorem 1.1), the asymptotics for the expected number of real zeros in this setting is known even for more general distributions. In contrast of this we will investigate the asymptotics of the variance of real roots.

3.1.2 Bergman Kernel Asymptotics

Consider $\mathcal{P}_n$ the space of polynomials of degree at most n. We know that it is a closed subspace of $L^2_{\varphi,n}(\mathbb{C})$. Hence the orthogonal projection $\Pi_n : L^2_{\varphi,n}(\mathbb{C}) \to \mathcal{P}_n$ is a bounded and linear operator. In particular, for a fixed

orthonormal basis $\{p_j^n(z)\}$, and a function $f \in L^2_{\varphi,n}(\mathbb{C})$ we have

$$\Pi_n f(z) = \sum_{j=0}^{n} < f, p_j^n(z) > p_j^n(z) = \sum_{j=0}^{n} \int_{\mathbb{C}} f(w)\overline{p_j^n(w)}e^{-2n\varphi(w)}dw p_j^n(z)$$

$$= \int_{\mathbb{C}} f(w) \left(\sum_{j=0}^{n} p_j^n(z)\overline{p_j^n(w)} \right) e^{-2n\varphi(w)}dw = \int_{\mathbb{C}} f(w)K_n(z,w)e^{-2n\varphi(w)}dw$$

The integral kernel $K_n(z,w)$ of the projection operator Π_n is called the *Bergman Kernel* or *Reproducing Kernel for point evaluation on* $\mathcal{P}_n$. We denote its derivatives by

$$K_n^{(1,0)}(z,w) = \sum_{j=0}^{n} (p_j^n(z))'(\overline{p_j^n(w)}) \text{ and } K_n^{(1,1)}(z,w) = \sum_{j=0}^{n} (p_j^n(z))'(\overline{p_j^n(w)})'$$

We also denote the Bergman function by

$$B_n(z) := K_n(z,z)e^{-2n\varphi(z)} = \sum_{j=0}^{n} |P_j^n(z)|^2 e^{-2n\varphi(z)} \tag{3.1.2}$$

Note, B_n has the following extremal property,

$$B_n(z) = \sup_{p_n \in \mathcal{P}_n} \frac{|p_n(z)|^2 e^{-2n\varphi(z)}}{||p_n||^2_{\varphi,n}}$$

Moreover, it also has the following dimensional density property

$$\int_{\mathbb{C}} B_n(z)dz = dim(\mathcal{P}_n) = n+1$$

On the otherhand, for the weight function φ satisfying $(1), (2)$ we define its *Bulk*, B_φ as

$$B_\varphi := \{z \in S_\varphi : \Delta\varphi(z) > 0\} \tag{3.1.3}$$

where $S_\varphi = \{z \in \mathbb{C} : V_{\mathbb{C},\varphi}(z) = \varphi(z)\}$ is the compact set where the corresponding equilibrium measure $\mu_{\mathbb{C},\varphi}$ is supported.
Now, by the results of ([31], Theorem 2.2), we have the following near diagonal asymptotics of Bergman Kernel.

Theorem 3.1.1. *If $\varphi : \mathbb{C} \to \mathbb{R}$ is a $\mathcal{C}^2-$weight function satisfying (1),(2) and z is a fixed point in the Bulk B_φ. Then as $n \to \infty$*

$$\frac{1}{n}K_n(z+\frac{u}{\sqrt{n}}, z+\frac{v}{\sqrt{n}})I_n(u,v) \to \frac{1}{2\pi}\Delta\varphi(z)\exp\left(\frac{1}{2}\Delta\varphi(z)u\overline{v}\right) \tag{3.1.4}$$

in $\mathcal{C}^\infty$ topology on compact sets in $\mathbb{C}_u \times \mathbb{C}_v$.

Remark 25. Here, $I_n(u,v) = \exp\left(-n\left[g\left(z+\frac{u}{\sqrt{n}}\right) + \overline{g\left(z+\frac{v}{\sqrt{n}}\right)}\right]\right)$ and $g : \mathbb{C} \to \mathbb{C}$ is the holomorphic function $g(w) = \varphi(z) + 2\frac{\partial\varphi}{\partial w}(z)(w-z)$ where $z \in \mathcal{B}_\varphi$ is a fixed point.

Now using the theorem above, simply by differentiating we obtain the following diagonal asymptotics for $K_n(z,z)$ and its derivatives.

Corollary 3.1.1. *Let φ be the weight function as above, then*

$$K_n(z,z) \sim \left(\frac{1}{2\pi}\Delta\varphi(z)\right)ne^{2n\varphi(z)} \tag{3.1.5}$$

$$K_n^{(1,0)}(z,z) \sim \left(\frac{1}{\pi}\Delta\varphi(z)\frac{\partial\varphi}{\partial z}\right)n^2e^{2n\varphi(z)} \tag{3.1.6}$$

$$K_n^{(1,1)}(z,z) \sim \left(\frac{2}{\pi}\Delta\varphi(z)\frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial\overline{z}}\right)n^3e^{2n\varphi(z)} + \left(\frac{1}{4\pi}(\Delta\varphi(z))^2\right)n^2e^{2n\varphi(z)} \tag{3.1.7}$$

uniformly on compact subsets of B_φ.

On the otherhand by adapting the results of [31] in our setting, in particular Theorem 2.4. We have the following off-diagonal asymptotics of K_n.

Theorem 3.1.2. *Assume that φ is as above and $\Delta\varphi(z) > c$ for some $c > 0$. Then for any $z, w \in K \subset\subset B_\varphi$*

$$\frac{1}{n}|K_n(z,w)|e^{-n\varphi(z)-n\varphi(w)} \leq Ce^{-T\sqrt{n}|z-w|} \tag{3.1.8}$$

where $C, T > 0$ are independent of n.

3.2 Asymptotics of Variance

In this setting due to ([16], *Theorem 1.1*) it is know that

$$\frac{1}{\sqrt{n}}\mathbb{E}[N(P_n, \mathbb{R})] = \frac{1}{\pi}\int_{B_\varphi \cap \mathbb{R}} \sqrt{\frac{1}{2}\Delta\varphi(x)}dx \tag{3.2.1}$$

Now, using the fact that $Var(X) = \mathbb{E}[X(X-1)] - \mathbb{E}[X]^2 + \mathbb{E}[X]$ for any random variable X. We see that it's enough to study the factorial moment term i.e. $\mathbb{E}[X(X-1)]$[1]. According to Fradhmand ([32],Theorem 2.9) we have the following result for the expectation of factorial moments

Lemma 3.2.1. *Let $[a, b] \in \mathbb{R}$, then*

$$\mathbb{E}[N_n(a,b)(N_n(a,b)-1)] = \lim_{\epsilon\to 0}\iint_{D(\epsilon)}\int_{\mathbb{R}}\int_{\mathbb{R}} |x_1x_2|p_{(x,y)}(0,0,x_1,x_2)dx_1dx_2dxdy$$

where $p(t_1, t_2, x_1, x_2)$ is the density of the random vector $(P_n(x), P_n(y), P_n'(x), P_n'(y))$, and $D(\epsilon) = \{(x,y) \in \mathbb{R}^2 : x, y \in (a,b), |x-y| > \epsilon\}$.

Computing the density $p_{(x,y)}$ and performing the integration over $\mathbb{R}^2$ with respect to dx_1dx_2, we obtain the following explicit formula for the factorial moment (cf. [33], §5.3) for details.

[1]We also denote as $\mathbb{E}[X^{[2]}]$

Theorem 3.2.1. *Let (a,b) and $D(\epsilon)$ be as above. Then*

$$\mathbb{E}[N_n^{[2]}(a,b)] = \frac{1}{\pi^2}\lim_{\epsilon\to 0}\iint_{D(\epsilon)}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sqrt{AC-B^2}+B\arcsin\left(\frac{B}{\sqrt{AC}}\right)\right)\frac{dxdy}{\sqrt{\Delta}}$$

where

$$\Delta(x,y) := K_n(x,x)K_n(y,y) - (K_n(x,y))^2$$

and

$$\Sigma = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

is the covariance matrix of the random vector $(P_n(x), P_n'(y))$, conditioned on $P_n(x) = P_n(y) = 0$, with

$$\begin{aligned} A(x,y) = K_n^{(1,1)}(x,y) &-\frac{1}{\Delta}[K_n(y,y)(K_n(x,x))^2 \\ &-2K_n(x,y)K_n^{(0,1)}(x,y)K_n^{(0,1)}(y,x) + K_n(x,x)(K_n^{0,1}(y,y))^2], \end{aligned}$$

$$\begin{aligned} C(x,y) = K_n^{(1,1)}(y,y) &-\frac{1}{\Delta}[K_n(y,y)(K_n^{(0,1))}(x,y)^2 \\ &-2K_n(x,y)K_n^{(0,1)}(x,y)K_n^{(0,1)}(y,y) + K_n(x,x)(K_n(y,y))^2], \end{aligned}$$

$$\begin{aligned} B(x,y) = K_n^{(1,1)}(x,y) \\ &- \frac{1}{\Delta}[K_n(y,y)K_n^{(0,1)}(x,x)K_n^{(0,1)}(x,y)K_n(x,y)K_n^{(0,1)}(x,y)K_n^{(0,1)}(y,x) \\ &- K_n(x,y)K_n(x,y)K_n^{(0,1)}(x,x)K_n^{(0,1)}(y,y) + K_n(x,x)K_n^{(0,1)}(y,x)K_n^{(0,1)}(y,y)]. \end{aligned}$$

Remark 26. Observe that in order to find the asymptotics of $\mathrm{Var}[N(P_n,\mathbb{R})]$, we need to study the asymptotics of Bergman Kernel and its derivatives.

The asymptotics of variance in this ensemble is ongoing project of this masters thesis. However, the following conjecture is expected, but the constant C below is not determined yet, explicitly.
Conjecture: $\frac{1}{\sqrt{n}}\mathrm{Var}[N(P_n,\mathbb{R})] \to C$.

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