WHEN DOES THE CUMULATIVE OFFER PROCESS PRODUCE AN ALLOCATION?

by

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ABSTRACT

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This thesis examines the properties of an algorithm, namely the Cumulative Offer Process (COP), which has been the principal algorithm in the matching with contracts setting. Matching with contracts is an allocation problem which employs contracts as its basic unit of analysis. We examine properties of COP under the substitutes (S) condition as well as the bilateral substitutes (BS) and the unilateral substitutes (US) conditions. These conditions are imposed on the choice functions of hospitals to obtain desirable matchings. In our research, we found that in the absence of IRC, the US, and hence automatically the BS, does not guarantee the existence of a feasible allocation that is produced by COP, yet S guarantees it. Therefore, our study shows that IRC is an essential property of choice functions of hospitals in order for the COP algorithm to be well-defined under BS or US.

Keywords: Matching with Contracts, The Cumulative Offer Process, Substitutes, Unilateral Substitutes, Irrelevance of Rejected Contracts

ÖZET

KÜMÜLATİF TEKLİF SÜRECİ NE ZAMAN BİR TAHSİSAT ÜRETİR?

DİLEK ŞAHİN
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Bu makalede, sözleşmelerle eşleşme sisteminde ana mekanizma olan Kümülatif Teklif Süreci'ni (COP) inceliyoruz. Sözleşmelerle eşleşme, temel analiz birimi olarak sözleşmeleri kullanan bir tahsisat problemidir. COP'un ikame (S) koşulu ile iki taraflı ikame (BS) ve tek taraflı ikame (US) koşulları altındaki özelliklerini inceliyoruz. Bu koşullar, istenen eşleşmeleri elde etmek için hastanelerin seçim işlevlerine dayatılır. Araştırmamızda, IRC'nin yokluğunda US'nin ve böylece otomatik olarak BS'nin COP tarafından üretilen uygun bir tahsisatın varlığını garanti etmediğini, ancak S'nin bunu garanti ettiğini gördük. Bu nedenle, çalışmamız, COP mekanizmasının BS veya US altında tanımlanabilmesi için, IRC'nin hastanelerin seçim işlevlerinin vazgeçilmez bir özelliği olması gerektiğini göstermektedir.

Anahtar Kelimeler: Sözleşmelerle Eşleşme, Kümülatif Teklif Süreci, İkame, Tek Taraflı İkame, Reddedilen Sözleşmelerin İlgisizliği

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TABLE OF CONTENTS

1 INTRODUCTION

The theoretical foundation of matching theory is rooted in the prominent paper of Gale and Shapley, namely, "College Admissions and the Stability of Marriage". This paper investigates two-sided matching markets in which there is bilateral exchange and there are two disjoint groups that agents in these markets can belong to, such as colleges and students, or men and women. In the basic two-sided matching model proposed in Gale and Shapley (1962), each member of the two parties has preferences over the members of the opposite party. Gale and Shapley define the concept of *stability*, which is an essential feature of any assignment since a stable matching cannot be blocked by any agent itself or any pair of agents. If we were to define stability, a matching is *stable* if there is no individual matched with a mate that is unacceptable to him/her, and there are no woman and man who are not matched with each other, yet prefer to be matched with each other. In their search for stable matchings, Gale and Shapley suggested the "Deferred Acceptance Algorithm" (DAA) that always produces a stable matching. If the proposing party is men, the DAA works as follows: Each man m proposes to his first choice among the women acceptable to him. Each woman rejects any unacceptable proposal, and if she receives more than one acceptable proposal, "holds" the most preferred among all the offers she has received and rejects all the others. Each man rejected at the previous step makes a new proposal to his most preferred acceptable woman among the ones who have not yet rejected him (If he has no acceptable choices left, he makes no proposal). Each woman holds her most preferred proposal among the new proposals and the one that she holds (if any), and rejects the rest. The algorithm stops when no further offers are made. At the end, it matches each woman to the man (if any) whose offer she is holding.

When the preferences are strict, as a result of this alghoritm not only is a stable matching produced but also the resulting matching is optimal for the proposing party among all the other possible stable matchings. Therefore, Gale and Shapley proved the existence of stable, man-optimal stable and woman-optimal stable outcomes in two-sided matching markets, for both one-to-one and many-to-one. They formulated a one-to-one marriage model in which any individual in a party can be matched with at most one agent from the other party. They also formulated a many-to-one college admission model in which each student can enroll in at most one college and each college wants to be matched with at most q many students, where q is the *quota* of college c. They extended the DAA that they had defined through the marriage market to the college admission model by allowing for quotas of colleges, and obtained stable matchings in college admission problems as well, restoring the optimality result of the outcome that is produced by DAA.

The college admission problem that is first outlined in Gale and Shapley (1962) has been widely studied and theoretically formulated up until today. The agents in this model belong to one of two groups; colleges which have preferences over students and enroll at most, say, q_c number of students; and students who have preferences over colleges and can register in at most one college. The dissimilarities in theoretical properties between the marriage market and college admission market were first discovered by Roth (1984), and modeled explicitly by Roth (1985a)[1], and since then the college admission problem has become a prominent study area in matching theory. This problem has been used to study the functioning of real markets, beginning with the market for medical interns that was governed by the National Resident Matching Program through an algorithm called NIMP, National Intern Matching Program. The NIMP algorithm is intiutively similar with DAA, and it was even in use for a decade before the first theoretical paper, (Gale

[1]Roth (1985a) models the college admission problem explicitly by allowing colleges to have preferences over sets of students as well as over individual students. Roth developed a concept called the *responsive preferences*.

and Shapley, 1962) that attempted to model a two-sided matching market.[2] The college admission problem has continued to be used to study more general labor markets.

Economists have also studied labor markets in which a monetary value is created by the agents who are matched with each other. The labor market matching model presented by Kelso and Crawford (1982)[3] can be considered as a more general version of the college admission model. Kelso and Crawford's labor market model contains two types of agents: workers and firms. They compose a general two-sided matching model by incorporating money (wages) explicitly into their model and allowing firms to have a broader range of preferences over the groups of workers (Roth and Sotomayor, 1990). Kelso and Crawford present a condition which they called gross-substitutes, which is imposed on the structure of firms' demand for workers. When all workers are gross-substitutes to firms, Kelso and Crawford were able to obtain the non-emptiness of the core of the labor market. They proved this result via their version of the firm offering Deferred Acceptance Algorithm, which they called the *salary adjustment process*. Therefore, the idea of substitutes condition imposed on preferences of colleges in the college admission model to obtain the existence of stable matchings, both college and student optimal ones, was introduced in Kelso and Crawford. The substitutability of preferences of colleges is weaker than a condition called *responsiveness*, which is also used to obtain non-emptiness of stable outcomes, as well as other properties.[4] The salary adjustment process introduced in Kelso and Crawford's labor market model is an ascending auction mechanism in which firms bid for workers simultaneously in ascending auctions. This process starts with the offers of firms which face a set of salaries to their most preferred set of workers. Each worker who faces more than one offer rejects all except his/her most preferred one. If a worker rejects an offer from firm j, his permitted salary increases by one unit.[5] The unrejected offers remain in force, and firms continue to offer employment to their most preferred sets

[2] Roth (1984) surveys the history of the labor market for medical interns and residents, and the centralized market mechanism whose final version was adopted in 1951. Both the history of this centralized market and the theoretical analysis of the NIMP mechanism was examined throughout the paper. Roth showed the stability of the mechanism and its other properties.

[3] The model presented was developed based upon the model of Crawford and Knoer (1981).

[4] See (Roth, 1985a)

[5] Here we cite the discrete version of the *salary adjustment process*, though Kelso and Crawford (1982) examine both discrete and continuous versions of their labor market model.

of workers. The college offering DAA can be considered as a special case of the salary adjustment process where wage offers are drawn from a singleton set. If we return to the gross substitutes condition after introducing the formal process, we can see its functioning more clearly. Workers being gross substitutes for the firms guarantees that if a worker's salary demand has not increased, firms maintain an offer that has been proposed to that worker even though other workers' salary demands have risen.

Another generalization of the college admission model to study more general labor markets can be found in Alkan and Gale (2003). Alkan and Gale studied schedule matching in which parties decide on the members they will work with as well as how much time of employment will take place in these partnerships. They use choice functions that are partially revealed as the primitives of their model; hence, encompass a broader framework. They define *persistency* which is a generalization the *substitutability* in college admission models. By imposing persistency to the choice functions of both firms and workers, they proved the existence of stable matchings via a method they developed as an extention of Gale and Shapley Algorithm. A similar method used later in the matching with contracts framework of Hatfield and Milgrom (2005). Alkan and Gale also showed that the additional *size monotonicity*[6] condition results in the existence of the set of stable allocations that form a lattice structure.

Connections between general matching and auction models were thoroughly examined by Hatfield and Milgrom (2005). Their encompassing paper called *"Matching with Contracts"* unifies and broadens college admission models, the Kelso and Crawford labor market matching model, and Ausubel and Milgrom's proxy auction by treating these as special cases of the matching with contracts model that they introduced. The unit of analysis is a *contract* in this general framework. They identify a contract with the two parties and the terms of the contract. They modify their identification of a contract in order to underscore the comprehensiveness of their study. The matching with contracts framework can be seen as a college admission model if a contract is defined by a college and a student; as Kelso and Crawford's labor market model if it is defined through a firm,

[6]The choice function C is size monotone if $x \leq y \implies |C(x)| \leq |C(y)| \; \forall x, y \in A$ where A is the range of C and its elements are called acceptable schedules.

a worker and a wage; and as Ausubel and Milgrom's proxy auction if it is defined by a bidder, the package of items that the bidder will obtain, and the price of that package (Hatfield and Milgrom, 2005). They define an algorithm called the *Cumulative Offer Process* (COP), which coincides with the doctor/college offering DAA under the *substitutes* condition they impose upon choice functions of hospitals/colleges. COP also coincides with Ausubel and Milgrom's proxy auction when contracts are not substitutes and there is only one hospital (an "auctioneer") (Hatfield and Milgrom, 2005).

They introduce the *substitutes* and *the law of aggregate demand*[7] conditions that are imposed on choice functions of hospitals and apply these properties throughout their analysis. The *substitutes* condition is a generalized version of the demand theory substitutes condition, which allows them to include an analysis of models both with money and without money (Hatfield and Milgrom, 2005). In matching with contracts settings, contracts are *substitutes* if the set of rejected contracts does not shrink whenever the firms' choice set expands.[8] Their substitutes condition is equivalent to the demand theory substitutes; hence, their analysis covers the model of Kelso and Crawford. They use a general version of Gale and Shapley's DAA to show that the set of stable allocations forms a non-empty lattice.[9] This generalized algorithm allows doctors/hospitals to choose from an expanded set of contracts in each step. As a result, they were able to show the existence of doctor-optimal/hospital-pessimal and doctor-pessimal/hospital-optimal points[10] in the set of stable contracts. By doing so, they connected their study to the college admission problem and Kelso and Crawford's model.

[7]The law of aggregate demand condition is equivalent to the size monotonicity condition in Alkan and Gale (2003).

[8]In demand theory this condition applies using terminology of prices (wages). The hospitals demand for any doctor d_i is nondecreasing in the wage of every other doctor d_{-i}(Hatfield and Milgrom, 2005). Also *substitutes* coincides with the *substitutable preferences* condition in college admission models in a contracts setting (Hatfield and Milgrom (2005) proved the equivalence between these two conditions in their proof of Theorem 2.) The concept of contracts being substitutes is equivalent to *Sen's alpha* condition in the study of social choice.

[9]Hatfield and Milgrom (2005) exploit Tarski's fixed point theorem (see (Tarski, 1955)) to achieve this result, and hence they connect their work to Lattice Theorem for the marriage market. (See Gale (2001) for the definition of the Lattice Theorem and its applications to the marriage model.)

[10]Hospital-optimal allocation is the stable allocation that is weakly preferred to every stable allocation by all hospitals. Remaining concepts can be defined with a similar logic.

Aygün and Sönmez (2013) underlined the fact that throughout the analysis in Hatfield and Milgrom, *the irrelevance of rejected contracts* was assumed implicitly. Aygün and Sönmez showed that if choice functions of hospitals are considered as primitives of the matching with contracts model, IRC needs to be assumed explicitly to restore some results in Hatfield and Milgrom. The existence of a stable allocation is the main result that fails to hold in the absence of IRC. Intuitively, IRC says that choice sets must remain unaltered when the rejected contracts are removed. We note that whenever the hospitals choices are derived from underlying strict preferences of hospitals, they automatically satisfy IRC; however, this structure might limit the scope of analysis with the substitutes condition.[11]

They also investigate the rural hospitals theorem and strategy proofness property in their model of matching with contracts. They introduce the condition called *the law of aggregate demand* (LAD) which requires that hospitals' chosen sets of contracts do not shrink if hospitals choose from an expanded set of contracts (decrease in some doctors' wages can be also seen as an expanding choice set of hospitals). By imposing LAD and substitutes conditions on hospitals' choices, they proved the fact that each hospital signs the same number of contracts in every stable matching.[12] They also showed that truthful reporting is a dominant strategy for doctors in the doctor-offering algorithm.[13]

Under the substitutes condition, the matching with contracts model and the college admission models become isomorphic to each other.[14] However, the matching with contracts framework opened a possibility of weakening the substitutes condition imposed on choice functions of hospitals, and hence of broadening the domain of choice functions while maintaining some desirable results. Therefore, this new framework opened new market design possibilities by incorporating a broader class of choice functions.

[11] According to Aygün and Sönmez (2013) and Aygün and Sönmez (2012).

[12] See Theorem 8 and 9 of (Hatfield and Milgrom, 2005) for proofs. Roth (1986) proved previously the Rural Hospitals Theorem for the two-sided matching market without money whenever hospitals have responsive preferences.

[13] See Theorems 10, 11 and 12 of (Hatfield and Milgrom, 2005) for a detailed analysis of strategy proofness in contract setting.

[14] Echenique (2012) shows that under the substitutes condition, Hatfield and Milgrom's model can be embedded into Kelso and Crawford's framework.

The fact that substitutes is not a necessary[15] condition for the stability result initiated the weakening this condition in subsequent papers. Hatfield and Kojima (2010) developed the concepts of unilateral and bilateral substitutes, and investigated which of the previous results continued to hold and which of them had failed to hold under these weakened conditions. We introduce these concepts briefly for a better understanding of the results presented below. Hatfield and Kojima explain these concepts as follows: Contracts are *bilateral substitutes (BS)* for hospitals when none of the hospitals receiving an offer from a doctor who it does not currently employ, wishes also to hire another doctor who it does not currently employ at a contract it previously rejected. Contracts are *unilateral substitutes (US)* for hospitals when none of the hospitals which received a new offer from a doctor (this doctor can also be a doctor that hospital employs currently), wishes to employ a doctor whom the hospital does not currently employ at a contract which was previously rejected by the hospital. It can be understood that the substitutes implies the US and the US implies the BS.[16]

The analysis of Hatfield and Kojima along with the detailed analysis in Aygün and Sönmez (2013) and Aygün and Sönmez (2012) about the role of IRC in the matching with contracts setting demonstrates important results. If the choice functions of hospitals are treated as primitives in the model, substitutes and IRC guarantee the existence of stable allocations. The weakest condition which is the bilateral substitutes with additional IRC also guarantees the existence of a stable allocation while unilateral substitutes and IRC are needed to guarantee the existence of a doctor-optimal stable allocation. Also, Hatfield and Kojima demonstrates that under US and LAD, number of the contracts signed by each doctor and each hospital are the same in every stable matching and the doctor-optimal stable mechanism is group strategy proof.

Therefore, if we are to summarize, the substitutes, the US or the BS alone does not guarantee the existence of a stable allocation. Once IRC is assumed, the set of stable outcomes become non-empty under IRC and BS. The *Cumulative Offer Process* (COP)

[15] This was shown by Hatfield and Kojima (2008).

[16] The full characterization between these conditions are presented in Hatfield and Kojima (2010) and Afacan and Turhan (2015).

introduced by Hatfield and Milgrom (2005) produces a stable allocation under IRC and BS; hence, under IRC and US as well. Our main contribution is to show that solely under US the COP fails to produce an allocation, i.e. the outcome produced by the COP includes at least two distinct contracts that are signed by the same doctor. Since each doctor can sign at most one contract in an assignment, the COP is not well-defined under US. However, once IRC is assumed together with US, the COP becomes well-defined.[17] We additionally prove that contracts being *substitutes* for hospitals guarantees the COP to be well-defined. Therefore, in the attempts of market design applications with matching with contracts, the necessity of IRC to the COP algorithm should be taken into consideration.[18]

The rest of this paper is organized as follows: In section 2, we talk about the related literature. In Section 3, we formally introduce the matching with contracts framework along with the definitions of the desirable properties of choice functions and we present our results. In Section 4, we conclude.

[17]Since under US and IRC, the COP produces a stable outcome.

[18]There are several applications on school choice with soft caps and cadet-branch matching that automatically satisfy IRC. See (Hafalır et al., 2013) for the former and (Sönmez, 2013) and (Sönmez and Switzer, 2011) for the latter.

2 RELATED LITERATURE

Matching with contracts is a three-dimensional allocation problem which employs contracts as its basic unit of analysis. Hatfield and Milgrom (2005) formalized a general matching with contracts framework. In this framework, a contract is fully identified by a doctor, a hospital and possibly a wage (this might represent other terms of relations such as working hours, employment benefits or responsibilities of a doctor within a hospital etc.). The Kelso-Crawford labor market matching, package auctions and the college admission problem are embodied in this unified framework as its special cases. We use doctor-hospital terminology throughout the paper; although matching with contracts setting incorporates various other matching problems as its applications. Hatfield and Milgrom (2005) introduced the substitutes condition that is imposed on choice functions of hospitals. This condition is equivalent to Roth and Sotomayor's substitutable preferences in the college admissions problem. They demonstrated that a stable allocation always exists whenever contracts are substitutes for hospitals. Afterwards, Aygün and Sönmez (2013) showed that the irrelevance of rejected contracts (IRC) condition is needed if hospitals' choices are not generated by hospitals' preferences. The formal definiton of IRC will be presented in the subsequent section.

Substitutes and IRC conditions are the two properties that guarantee the existence of a stable allocation whenever they are imposed together on choice functions of hospitals. However, the fact that substitutes is not a necessary condition for the existence of a stable allocation was shown by Hatfield and Kojima (2008). This fact resulted in the creation of weaker substitutes conditions under which a stable allocation is still guaranteed to exist (when they are imposed together with the IRC condition like in the case of substitutes).

These weaker conditions, namely, bilateral substitutes (BS) and unilateral substitutes (US) conditions were introduced in Hatfield and Kojima (2010). Since US and BS are weaker than the substitutes, IRC is still explicitly needed to guarantee the existence of a stable allocation.

Although the weakest conditions BS and IRC are sufficient for the existence of a stable allocation, they are neither necessary nor sufficient conditions for other well-known properties of stable allocations in the standard matching problems. For instance, the doctor-optimal or hospital-pessimal allocation does not necessarily exist under only BS and IRC. In order to restore doctor optimality, US along with IRC is needed. However, even under US and IRC, the set of stable allocations does not necessarily form a lattice; hence, the doctor-pessimal/hospital-optimal allocation might not exists. There are various dynamics between these conditions, and certain desirable properties are obtained under various combinations of these conditions imposed on choice functions.

Hatfield and Milgrom also introduced a mechanism that coincides with the doctor-offering Gale-Shapley algorithm under the substitutes condition. This algorithm, namely, the Cumulative Offer Process (COP) allows hospitals to choose among all the offers they have received previously including the current offers. Using this algorithm, Hatfield and Kojima (2010) showed that along with the law of aggregate demand condition, the unilateral substitutes guarantees the group strategy proofness of the doctor-optimal stable mechanism and a version of the rural hospital theorem.

These previously mentioned properties and conditions are imposed on the choice functions or the preferences of the hospitals in order to obtain desirable matchings and allocations. IRC condition is also a condition that turned out to be desirable even for the existence of a stable allocation. Also, the fact that the majority of the theorems in Hatfield and Kojima (2010) are not hold without the irrelevance of rejected contracts condition (if it was not implicitly assumed) was shown by Aygün and Sönmez (2012).

We want to examine the importance of IRC condition for the COP algorithm in the matching with contracts setting further. We examine under which conditions the Cumulative Offer Process is well-defined i.e. is able to produce an allocation. We found that while US does not necessarily assure that the COP to be well-defined without the IRC, substitutes condition is sufficient for the COP to be well-defined.

3 MODEL AND RESULTS

There are finite sets D and H of doctors and hospitals, and a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Each doctor can sign at most one contract. The null contract, meaning that the doctor has no contract, is denoted by $\emptyset$. Given a set of contracts $Y \subseteq X$, let Y_D denotes the set of doctors who have contracts in Y. A set of contracts $X' \subseteq X$ is an **allocation** if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. This means, a set of contracts is an **allocation** if each doctor signs at most one contract.

For each doctor $d \in D$, P_d is a strict preference relation on $\{x \in X \mid x_D = d\} \cup \{\emptyset\}$. A contract is **acceptable** if it is strictly preferred to the null contract and it is otherwise **unacceptable**. For every $d \in D$ and $X' \subseteq X$, the **chosen set** $C_d(X')$ is defined as:

$$C_d(X') = \max_{P_d} \Big[\{x \in X' \mid x_D = d\} \cup \{\emptyset\} \Big]$$

For a given set of contracts, we denote $C_D(X') = \bigcup_{d \in D} C_d(X')$ for the set of contracts chosen from X' by some doctor D.

Each hospital h has a choice function which is not necessarily induced by a preference relation. The **choice function** of hospital h is the function that maps each set of contracts to a chosen set. Each hospital can sign multiple contracts. The chosen set of hospital h is defined as, for any $X' \subseteq X$,

$$C_h(X') \in \Big\{ Y \subseteq X' \cap X_h \mid y, y' \in Y \text{and } y \neq y' \Longrightarrow y_D \neq y'_D \Big\}$$

For a given set of contracts, we denote $C_H(X') = \bigcup_{h \in H} C_h(X')$ for the set of contracts chosen from X' by some hospitals *H*.

The preference profile of doctors is denoted by $P_D = (P_d)_{d \in D}$. P_{-d} denotes $(P'_d)_{d' \in D \setminus \{d\}}$ for $d \in D$, P'_D denotes $(P_d)_{d \in D'}$ and $P_{-D'}$ denotes $(P_d)_{d \in D \setminus D'}$ for $D' \subset D$.

Definition 1. *A set of conracts $X' \subseteq X$ is a **stable allocation** if*

1. $C_D(X') = C_H(X') = X'$
2. *There exist no hospital $h \in H$ and a set of contracts $X'' \neq C_h(X')$ such that*

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'')$$

Definition 2. *Contracts are **substitutes** for hospital h if there do not exist a set of contracts $Y \subset X$ and a pair of contracts $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}) \textit{ and } z \in C_h(Y \cup \{x, z\})$$

Definition 3. *Contracts satisfy the **irrelevance of rejected contracts (IRC)** for h if*

$$\forall\, Y \subset X,\ \forall\, z \in X \setminus Y \qquad z \notin C_h(Y \cup \{z\}) \Longrightarrow C_h(Y) = C_h(Y \cup \{z\})$$

IRC condition means that the chosen sets are remained uneffected from the removal of rejected contracts. This condition along with the substitutes condition is a sufficient condition for the existence of a stable allocation.

The Cumulative Offer Process was introduced in Hatfield and Milgrom (2005). The COP is a generalization of the Deferrred Acceptance mechanism of Gale and Shapley's to the matching with contracts framework. The COP allows the offer-receiving party (in our definition hospitals) to choose from cumulatively expanding set of contracts. In this regard, we denote *the cumulative offer set* of a hospital h at step t as $A_h(t)$.

The **COP** is defined as:

Step 1: One of the doctors offers her first choice, say contract x_1. The hospital $h_1 = (x_1)_H$ which has received the offer x_1, keeps the contract if it is acceptable and rejects it otherwise. Let $A_{h_1}(1) = \{x_1\}$ and $A_{h'}(1) = \emptyset$ for all $h' \neq h$.

In general,

Step t $\geq$ 2: One of the doctors who have no contract that is currently held by any hospital, offers his most preferred contract, say x_t, which has not been rejected in previous steps. The hospital $h_t = (x_t)_H$ who have received the offer, holds the contracts in $C_{h_t}(A_{h_t}(t-1) \cup \{x_t\})$ and rejects the others. Let $A_{h_t}(t) = A_{h_t}(t-1) \cup \{x_t\}$ and $A_{h'}(t) = A_{h'}(t-1)$, for all $h' \neq h_t$.

The algorithm terminates when either every doctor is matched to a hospital or every unmatched doctor has all acceptable contracts rejected. Since there are finite number of contracts, the algorithm terminates at a finite step T. The final outcome is $\bigcup_{h \in H} C_h(A_h(T))$.

Definition 4. *Contracts are* ***bilateral substitutes*** *for h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X \backslash Y$,*

$$z \notin C_h(Y \cup \{z\}) \textit{ and } z \in C_h(Y \cup \{x, z\}) \Longrightarrow z_D \in Y_D \textit{ or } x_D \in Y_D$$

The bilateral substitutes condition requires rejection of a contract z whenever contracts with new doctors are added to the choice set, if z is rejected when all available contracts include seperate doctors. This condition along with the subsequent unilateral substitutes condition presented below were introduced in Hatfield and Kojima (2010).

Definition 5. *Contracts are* ***unilateral substitutes*** *for h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X \backslash Y$,*

$$z \notin C_h(Y \cup \{z\}) \textit{ and } z \in C_h(Y \cup \{x, z\}) \Longrightarrow z_D \in Y_D$$

Unilateral substitutes is satisfied whenever a hospital rejects z when available contracts with z_D is only z, that hospital still rejects z when the choice set expands.

The substitutes condition implies the unilateral substitutes and the latter implies the bilateral substitutes condition by definition. The axiomatic characterization between S and US was shown by Hatfield and Kojima (2010), and same between US and BS was shown by Afacan and Turhan (2015).

The fact that even the stronger *substitutes* condition is not sufficient for the existence of stable allocation, and IRC is needed to restore stability was shown by Aygün and Sönmez (2013). Additionally, Hatfield and Kojima (2010) and Aygün and Sönmez (2012) showed that the COP produces stable allocation under BS and IRC conditions; thereby, the COP is a stable mechanism under US and IRC. Since IRC restores the stability, it might affect the COP algorithm to be well-defined too. We found that US alone does not guarantee the existence of an allocation produced by the COP.

Theorem 1. *Let there are n doctors and m many hospitals. Assume that the contracts are unilateral substitutes for hospitals. The outcome of the COP does not have be an allocation. That is, the COP is not well-defined under US.*

The next example shows that under US solely, the COP does not even produce an allocation.

Example 1 Let there are two hospitals (h_1 and h_2) and three doctors (d_1,d_2 and d_3) with acceptable contracts each. The two contracts x, y are made with h_2, and the remaining contracts x', x'', y', y'', z' are made with h_1. Doctors' preferences are as given below:

$$P_{d_1} : x' \succ x'' \succ x \succ \emptyset$$

$$P_{d_2} : y' \succ y \succ y'' \succ \emptyset$$

$$P_{d_3} : z' \succ \emptyset$$

Hospitals' choice functions are given by the following table:

h_1's choice function is defined as

$C_{h_1}(\{x'\}) = \{x'\}$	$C_{h_1}(\{x', x'', y', y''\}) = \{x'', y'\}$
$C_{h_1}(\{x''\}) = \{x''\}$	$C_{h_1}(\{x', x'', y', z'\}) = \{z'\}$
$C_{h_1}(\{y'\}) = \{y'\}$	$C_{h_1}(\{x', x'', y'', z'\}) = \{y'', z'\}$
$C_{h_1}(\{y''\}) = \{y''\}$	$C_{h_1}(\{x', y', y'', z'\}) = \{z'\}$
$C_{h_1}(\{z'\}) = \{z'\}$	$C_{h_1}(\{x'', y', y'', z'\}) = \{x'', z'\}$
	$C_{h_1}(\{x', x'', y', y'', z'\}) = \{x'', y'', z'\}$

$C_{h_1}(\{x', x''\}) = \{x'\}$	$C_{h_1}(\{x', x'', y'\}) = \{y'\}$
$C_{h_1}(\{x', y'\}) = \{y'\}$	$C_{h_1}(\{x', x'', y''\}) = \{y''\}$
$C_{h_1}(\{x', y''\}) = \{y''\}$	$C_{h_1}(\{x', x'', z'\}) = \{z'\}$
$C_{h_1}(\{x', z'\}) = \{z'\}$	$C_{h_1}(\{x', y', y''\}) = \{y'\}$
$C_{h_1}(\{x'', y'\}) = \{x'', y'\}$	$C_{h_1}(\{x'', y', y''\}) = \{x'', y'\}$
$C_{h_1}(\{x'', y''\}) = \{x'', y''\}$	$C_{h_1}(\{x', y', z'\}) = \{z'\}$
$C_{h_1}(\{x'', z'\}) = \{x'', z'\}$	$C_{h_1}(\{x', y'', z'\}) = \{y'', z'\}$
$C_{h_1}(\{y', y''\}) = \{y'\}$	$C_{h_1}(\{x'', y', z'\}) = \{x'', z'\}$
$C_{h_1}(\{y', z'\}) = \{z'\}$	$C_{h_1}(\{x'', y'', z'\}) = \{x'', y'', z'\}$
$C_{h_1}(\{y'', z'\}) = \{y'', z'\}$	$C_{h_1}(\{y', y'', z'\}) = \{z'\}$

h_2's choice function is defined as

$$C_{h_2}(\{x\}) = \{x\}$$
$$C_{h_2}(\{y\}) = \{y\}$$
$$C_{h_2}(\{x, y\}) = \{x\}$$

Observe that the choice functions of hospitals do satisfy the unilateral substitutes condition since none of the hospitals have a new offer from a doctor (including an offer from doctors currently employs), wants to hire a doctor it does not currently employ at a contract that the hospital previously rejected. Also, substitutes fail to hold. For example, we have $x'' \notin C_{h_1}(\{x', x'', y', z'\})$ but we also have $x'' \in C_{h_1}(\{x', x'', y', y'', z'\})$.

Furthermore, the irrelevance of rejected contracts is violated. Consider, for example, $C_{h_1}(\{x', x'', y'\}) = \{y'\}$ and $C_{h_1}(\{x'', y'\}) = \{x'', y'\}$. We have $x' \notin C_{h_1}(\{x', x'', y'\})$, but $\{y'\} = C_{h_1}(\{x', x'', y'\}) \neq C_{h_1}(\{x'', y'\}) = \{x'', y'\}$. Hence IRC does not hold.

Now consider a COP algorithm as described earlier. Let the algorithm starts arbitrarily from a doctor, say d_1, to make the first offer to his most preferred contract x'. h_1 holds x' since $C_{h_1}(\{x'\}) = \{x'\}$. Then, another arbitrarily chosen doctor who does not have a contract that is held by any hospital, say d_2, offers y' to h_1. h_1 keeps y' and drops x' since $C_{h_1}(\{x', y'\}) = \{y'\}$. Then, let the turn is at d_1 again to make the next offer, x''. h_1 continues to keep y' since $C_{h_1}(\{x', x'', y'\}) = \{y'\}$. Next, let d_1 continues his offers since he has no contract that is held by any hospital, and she offers his next best, x. h_2 keeps x, since x is acceptable for it. Then, the only doctor who has no contract held, d_3 offers z' to h_1. Now, h_1 holds z' and drops y' since $C_{h_1}(\{x', x'', y', z'\}) = \{z'\}$. Now, the only lonely doctor is d_2 and she makes the offer y to h_2. h_2 continues to keep x and rejects y since $C_{h_2}(\{x, y\}) = \{x\}$. Still lonely d_2 offers y'' to h_1 in that case, and gets acceptance from it finally, because $C_{h_1}(\{x', x'', y', y'', z'\}) = \{x'', y'', z'\}$. At the end of this algorithm h_2 employs d_1 and h_1 employs d_1, d_2 and d_3. However, notice that this result is not an allocation since both h_1 and h_2 employ d_1 when the algorithm terminates and all the doctors have matched to a hospital.

This example shows that US is not sufficient without IRC for a COP to be well-defined. The fact that BS is also not sufficient to obtain the same result follows automatically, since US is a stronger condition than BS and implies BS.

We note that the choice functions of hospitals satisfy the condition gamma (γ) from the literature of social choice. Therefore the COP is not well-defined under US and γ.

Definition 6. *A choice rule $C(.)$ satisfies condition gamma (γ) if $\forall\ A, B \subseteq X$*

$$x \in (C(A) \cap C(B)) \implies x \in C(A \cup B).$$

Remark 1. *Condition γ is irrelevant for the COP being well-defined under choice functions satisfy US but not S and IRC. In above example; although the choice functions of hospitals satisfy condition γ , the COP is not well-defined.*

Here, we also want to note that the previously mentioned LAD condition might have an impact on the COP producing an allocation.[19] In this example choice functions of hospitals do not satisfy LAD condition. Whether LAD and US is sufficient for the COP to be well-defined might be a topic of potential future research.

Then, we wonder whether the stronger *substitutes* is sufficient or not by itself to guarantee the COP to produce an allocation. Below, we show that if contracts are substitutes for hospitals then the COP is well-defined.

Theorem 2. *Suppose hospitals choice functions satify substitutes condition. Then COP is well-defined.*

Proof. Let hospitals choice functions satisfy the substitutes condition. Suppose the COP is not well-defined under the substitutes. This means there exists a COP yields an outcome such that in this outcome, there exists at least one doctor who has at least two different contracts that contain two different hospitals. Without loss of generality, assume that there are two hospitals h_1 and h_2 and n many doctors. Let the cumulative offer set of h_1 at step s is denoted by Y_s and h_2's by Z_s. Let a COP terminates at step t in which h_1's cumulative offer set is Y_t, and h_2's is Z_t. Suppose, $\exists\ d \in D$ s.t. $[x, z]_D = d$ and $x \in C_{h_1}(Y_t), z \in C_{h_2}(Z_t)$. Hence, d must have been offered some contracts including x and z until step t, and had at least one contract that was previously rejected by at least one of the hospitals before period t. d must have offer one of the contracts x or z before the other because he cannot offer these two at the same time since he have strict preferences. Without loss of generality, let d offers x at step k, before she offers z to h_2 at step m. In order d to offer z to h_2, her previously offered contract x must have been rejected by h_1 at a step l where $k \leq l \leq m - 1 < m \leq t$, in order d to have no contract that is held by any hospital at step $m - 1$. That is $x \notin C_{h_1}(Y_l)$ and $d \notin [Ch_1(Y_{m-1})]_D$ in order d to

[19]Note that Aygün and Sönmez (2013) showed that BS + LAD is not sufficient for the COP to produce a stable allocation and BS + IRC guarantee the existence of a stable allocation.

offer z to h_2 at step m. This means $x \notin C_{h_1}(Y_{m-1})$. Notice that we have also assumed $x \in C_{h_1}(Y_t)$.

Since the COP allows hospital's offer sets to expand over time, notice that $Y_k \subset Y_l \subseteq Y_{m-1} \subset Y_m \subseteq Y_t$ and $x \in Y_k$ since d offers x to h_1 at step k. We know that the elements of X are substitutes for a hospital h i.e. for all $X', X'' \in X$ s.t. $X' \subset X'' \subset X$ we have $R_h(X') \subset R_h(X'')$.

We assumed that contracts are ***substitutes*** for hospitals in our assumption, and we have $Y_{m-1} \subset Y_t$. However, we also have $R_{h_1}(Y_{m-1}) \not\subset R_{h_1}(Y_t)$, since $x \in R_{h_1}(Y_{m-1})$ but $x \notin R_{h_1}(Y_t)$. Contradiction. $\square$

4 CONCLUSION

The concepts of US and BS enrich the matching with contracts framework and expand its theoretical aspects as well as market design possibilities using these concepts since they are considered as natural concepts that might stand out in reality. Therefore, it is important to understand if a widely used mechanism in this setting, the COP is well-defined under the BS, the US or the substitutes.

Previously, it was shown that while the COP is not a stable mechanism under US, it is stable under US and IRC conditions imposed on choice functions of hospitals. We show by an example that the COP is not well-defined only under US. In this case, a COP might produce an outcome in which some doctors have more than one accepted contracts. Hence, the employment status of such doctors is ambiguous. Once IRC is assumed explicitly, the COP becomes a well-defined mechanism. We also prove that the *substitutes* condition is sufficient for the existence of a feasible allocation produced by the COP. Hence, we show that the COP is well-defined under the substitutes condition. Our work emphasizes the importance of IRC condition in the matching with contracts model and clarifies the role of IRC further.

REFERENCES

Afacan, M. O. (2016), 'Characterizations of the cumulative offer process', *Social Choice and Welfare* **47**(3), 531–542.

Afacan, M. O. and Turhan, B. (2015), 'On relationships between substitutes conditions', *Economics Letters* **126**, 10–12.

Alkan, A. and Gale, D. (2003), 'Stable schedule matching under revealed preference', *Journal of Economic Theory* **112**, 289–306.

Ausubel, L. M. and Milgrom, P. R. (2002), 'Ascending auctions with package bidding', *Advances in Theoretical Economics* **1**(1), 1534–5963.

Aygün, O. and Sönmez, T. (2012), Matching with contracts: The critical role of irrelevance of rejected contracts. Boston College Working Papers in Economics.

Aygün, O. and Sönmez, T. (2013), 'Matching with contracts: Comment', *American Economic Review* **103**(5), 2050–51.

Crawford, V. P. and Knoer, E. M. (1981), 'Job matching with heterogeneous firms and job matching with heterogeneous firms and workers', *Econometrica* **49**, 437–450.

Echenique, F. (2012), 'Contracts versus salaries in matching', *American Economic Review* **102**(1), 594–601.

Gale, D. (2001), 'The two-sided matching problem. origin, development and current issues', *International Game Theory Review* **3**, 237–252.

Gale, D. and Shapley, S. (1962), 'College admissions and the stability of marriage', *The American Mathematical Monthly* **69**(1), 9–15.

Hafalır, I. E., Yenmez, M. B. and Yıldırım, M. A. (2013), 'Effective affirmative action in school choice', *Theoretical Economics* **8**(2), 325–363.

Hatfield, J. W. and Kojima, F. (2008), 'Matching with contracts: Comment', *American Economic Review* **98**(3), 1189–1194.

Hatfield, J. W. and Kojima, F. (2010), 'Substitutes and stability for matching with contracts', *Journal of Economic Theory* **145**(5), 1704–1723.

Hatfield, J. W. and Milgrom, P. R. (2005), 'Matching with contracts', *American Economic Review* **95**(4), 913–935.

Kelso, A. S. and Crawford, V. P. (1982), 'Job matching, coalition formation, and gross substitutes', *Econometrica* **50**(6), 1483–1504.

Roth, A. E. (1984), 'The evolution of the labor market for medical interns and residents: A case study in game theory', *Journal of Political Economy* **92**, 991–1016.

Roth, A. E. (1985a), 'The college admissions problem is not equivalent to the marriage problem', *Journal of Economic Theory* **36**, 277–288.

Roth, A. E. (1986), 'On the allocation of residents to rural hospitals: A general property of on the allocation of residents on the allocation of residents to rural hospitals: A general property of two-sided matching markets', *Econometrica* **54**(2), 425–27.

Roth, A. E. and Sotomayor, M. (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press.

Shapley, L. S. and Scarf, H. (1974), 'On cores and indivisibility', *Journal of Mathematical Economics* **1**(23-28).

Shapley, L. S. and Shubik, M. (1972), 'The assignment game I: The core', *International Journal of Game Theory* **1**(111-130).

Sönmez, T. (2013), 'Bidding for army career specialties: Improving the ROTC branching mechanism', *Journal of Political Economy* **121**(1), 186–219.

Sönmez, T. and Switzer, T. B. (2011), Matching with (branch-of-choice) contracts at United States Military Academy, Boston College Working Paper in Economics 782, Boston College Department of Economics.

Tarski, A. (1955), 'A lattice theoretical fixpoint theorem and its applications', *Pacific Journal of Mathematics* **5**(2).