TEMPTATION AS A RESULT OF AMBIGUITY

by

ÖZDE ÖZKAYA

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APPROVED BY:

Doç. Dr. Mehmet Barlo ....
(Thesis Supervisor)

Doç. Dr. Mustafa Oğuz Afacan ....

Doç. Dr. Ayça Ebru Giritligil ....

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ABSTRACT

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Özde ÖZKAYA

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Thesis Supervisor: Assoc. Prof. Mehmet BARLO

Keywords: Temptation, Ambiguity, Dual-self, Multiple-selves

Employing the well-accepted axioms of Ghirardato, Maccheroni, and Marinacci (2004) on preferences under ambiguity and extending them to the case with menus while using a mild condition, we obtain a multiple-selves representation. As a result, when evaluating a menu the decision maker can be thought to imagine that with some menu-dependent probability his/her "ego" is in charge and he/she consumes the best alternative, whereas with the remaining probability the decision maker faces an ambiguity about his/her consumption as he/she does not know which one of his/her "alter egos" is to decide. We also show that our multiple-selves representation transforms into a dual-self representation under a more restrictive condition. Finally, the relation of our representation theorem with some models of temptation is analyzed and we show that our representation result delivers their key axioms concerning temptation.

ÖZET

MUĞLAKLIK SONUCUNDA AYARTI

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Anahtar Kelimeler: Ayartı, Muğlaklık, İkili-Benlik, Çoklu-Benlik

Bu tezde, Ghirardato, Maccheroni, ve Marinacci (2004)'ün muğlaklık içeren tercihler üzerine yaptıkları varsayımları kabul edip, bu varsayımları menü içeren durumlara, önermekte olduğumuz ılımlı bir koşul altında genişleterek, bir çoklu-benlik temsil sonucunu kanıtlıyoruz. Böylelikle, karar vericinin şu şekilde davranıyormuş gibi hareket ettiğini meşrulaştırmaktayız: Karar verici birey bir menüyü değerlendirken, menü ile alakalı belirli bir olasılık ile menüdeki en iyi alternatifi seçecek olan benliğinin ortaya çıkacağını, kalan olasılıkla da hangi benliğinin ortaya çıkacağını ve hangi alternatifin tüketileceğini bilememesi sebebi ile muğlaklık durumu ile karşılaşacağını düşünür. Buna ek olarak, daha kısıtlayıcı bir koşul altında, ortaya çıkardığımız çoklu-benlik temsil sonucunun, ikili-benlik tensil sonucuna dönüştüğünü gösteriyoruz. Son olarak, elde ettiğimiz temsil sonuçlarının, bazı alakalı ayartı modelleri ile karşılaştırmalı analizini yapıp, bu ayartı modellerinin en önemli varsayımlarının, bizim temsil sonucumuz ile elde edilebildiğini gösteriyoruz.

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1 INTRODUCTION

Conventional decision making theories put emphasis on well–defined, stationary and coherent preferences and entail the decision maker to choosing always his/her most preferred alternative; as a result of which it is as if he/she maximizes his/her well-being according to his/her preferences. Nevertheless, casual real-world observations suggest that people often succumb to temptation and make decisions which are not compatible with their "rational" preferences. For example, a person who is on diet and prefers low calorie meals may choose to eat a hamburger with French fries at lunch and may regret this decision after his/her short-run craving is gone; or, a diabetic person who is not allowed eat food containing sugar may attempt to eat chocolate cake even though he/she knows of the risks involved. To that regard, experimental psychologists and economists conducted both laboratory and field experiments to understand the dynamics of temptation; and their findings suggest that the dynamic inconsistency in choices (for instance, choice of salad yesterday and choice of burger today) and *preference for commitment* (e.g., a person on a diet making a lunch reservation in the morning to a restaurant serving only low calorie healthy food) constitutes an evidence for temptation's impacts on decision making process.[1]

Such effects on decision making processes have led psychologists and economists to analyze psychological phenomena of self-control and willpower as important reasons of succumbing to temptation. Based on experimental evidence, Baumeister, Vohs, and Tice (2007) states that willpower is a limited cognitive resource whose depletion may lead a decision

[1] We refer our readers to see Houser, Schunk, Winter, and Xiao (2018), Toussaert (2018), Ashraf, Karlan, and Yin (2006),Thaler (1981), Loewenstein and Thaler (1989) and Lipman and Pesendorfer (2011) for related literature.

maker not to exercise self-control, and hence, succumb to temptation.[2] Therefore, decision theory expanded to include issues involving self-control and temptation. Except few studies which model the choice from a menu, e.g. Masatlioglu, Nakajima, and Ozdenoren (2011), the mainstream focus in the literature on temptation is to represent preferences over menus in order to capture the idea of preference for commitment.

Employing well-accepted axioms on preferences under ambiguity and a mild condition used when extending these preferences to menus which demands every acceptable menu to have an unambiguous value, the current study obtains the resulting representation of these preferences and proves that the associated behavior under ambiguity admits a *multiple-selves representation*. That is, the decision maker acts as if with some endogenously determined menu specific probability, he/she gets to consume the best alternative of that menu while he/she does not have any idea and imagines the worst case scenario about the behavior of his/her alter egos in the other contingencies.

When evaluating a menu, the multiple-selves setting calls for the decision maker to presume that with some probability the "ego" (alternatively, long-run self or the rational decision maker him/herself) is decisive and as a result the decision maker consumes the best alternative of that menu, while with the remaining probability the decision maker faces ambiguity about which one of many "alter egos" (alternatively, short-run selves or ids) is in charge. In these ambiguous situations, the decision maker cannot predict the behavior of his/her alter ego as he/she does not know either the particular alter ego that will be deciding or the exact probability distribution on the potential deciding alter egos. This makes the decision maker imagine the worst case scenario in these contingencies.

At this stage, we think that briefly discussing multiple-selves and dual-self models in the psychology and economics literature is useful to capture the intuition behind and contribution of our model. In order to explain the social behavior of an individual, Strack and Deutsch (2004) argues a dual-system model which assumes that behavior is determined by

[2] See the following studies regarding the experimental evidence discussed above: Muraven, Tice, and Baumeister (1998), Baumeister, Bratslavsky, Muraven, and Tice (1998), and Baumeister, Gailliot, DeWall, and Oaten (2006).

two interacting systems which are operating distinctively. The reflective system produces decisions based on knowledge about facts and values whereas the impulsive system produces decisions based associative links and motivational orientations. Furthermore, in a review of judgment and decision making literature, Weber and Johnson (2009) states that dual-process models involving a fast, automatic, effortless, associative and intuitive process and a slower, rule-governed, analytic, deliberate and effortful process are accounted for many decision making phenomena such as valuation of risky options, risk taking and hyperbolic discounting. Therefore, dual-process ideas led economists to consider models in which decisions are explained via interaction of two different selves. For example, Fudenberg and Levine (2006) presents a model in which a decision problem reflects conflict between short-run impulsive-self and long-run patient-self.

On the other hand, the psychology literature includes studies which focuses on the multiple-selves approaches as well. For example, James (1890) claims that human beings carry many different social selves as each of one them is a part of many different groups of people; and Roberts and Donahue (1994) and Markus and Nurius (1986) also propose that individuals have multiple-selves. Furthermore, Schelling (1984) points out that individuals do not always act according to their usual self and behave as if there are other different selves who take turns and act according to their own values.

As our model relies heavily on representation of preferences under ambiguity, we would like to present a standard terminology that is used in the literature of ambiguity by the virtues of one simple example. Imagine that there is a horse race in which three horses (Queen, King and Princess) are competing. Then, there are six states of the natures: Queen is the first, King is the second and Princess is the third; Queen is the first, Princess is the second and King is the third; King is the first, Queen is the second and Princess is the third; King is the first, Princess is the second and Queen is the third; Princess is the first, King is the second and Queen is the third; and finally, Princess is the first, Queen is the second and King is the third. Notice that, many factors including the weather, the performance of the jockey, the health condition of the horse, and so and so forth, make a bettor face ambiguity in a horse race. Furthermore, suppose that a person gets 10 TL if the horse he/she bets on wins and loses 10 TL otherwise. Then, an *act* is a function which assigns each state an outcome; for example, consider two acts, denoted by f and g where f assigns 10 TL to states in which Queen wins, and g assigns 10 TL to states in which King wins. Hence, f maps the first two states to 10 TL and last four states to -10 whereas g maps the third and fourth states to 10 TL and others to -10 TL.

While Schmeidler (1989) aims to lay the ground work to obtain representation of preferences of a decision maker over ambiguous acts, Gilboa and Schmeidler (1989) obtains such a representation, the well-known maxmin expected utility representation. On the other hand, the representation model we build upon relies on the α–maxmin expected utility model of Ghirardato, Maccheroni, and Marinacci (2004) which is regarded as an extension of Hurwicz (1951). In this model, probabilistic scenarios that the decision maker considers concerning possible states of nature are revealed via the *unambiguous preference relation* between acts. We say that an act is unambiguously preferred to another if the expected utility of the first act is greater or equal to the expected utility of the other in each possible probabilistic scenario that the decision maker considers. Hence, their representation model states that α-maxmin expected utility of an act is based on the weighted average of the best case and the worst case scenarios with act specific weights which can be interpreted as an act dependent index of optimism and pessimism, respectively.

In order to adopt the setting of Ghirardato, Maccheroni, and Marinacci (2004) and employ $\alpha-$ maxmin expected utility representation, we constructed the state space with the following properties. In our model, each alternative in a menu is treated as a potential tempting option for the decision maker because cognitive, commercial, social, cultural, professional and financial factors may cause different alter egos to show up at different times. For example, a person who is on diet could be affected by his/her friend's choice of pizza and succumb to temptation by ordering pizza in one instance whereas the same decision maker can be tempted by a hamburger in another instance due to the delicious image of a hamburger on the menu. That is why we have constructed our state space as a Cartesian product of menus so that a particular alter ego amounts to a state, and vice versa; hence, the j^{th} dimension of a given state represents the specific alter ego's choice from the j^{th} menu. One point that we would like to highlight is that we do not necessarily assume that the alter egos are rational in their own right and they can be viewed as "behavioral types" or "machines".

This method also ensures that there exists a state, each dimension of which corresponds to the best alternative from the corresponding menu; hence, the realization of this state generates the best case scenario for a decision maker. In fact, this is the state that corresponds to the rational id, the ego.

The condition we employ to extend preferences under ambiguity to menus and obtain the aforementioned multiple-selves representation demands that each acceptable menu (set of options containing an element that provides strictly higher utilities than the globally worst alternative) is unambiguously strictly preferred to the menu containing only the globally worst option. That is why, we refer to this condition as the *strict unambiguous value of acceptable menus*. In fact, it ensures that among the probability distributions obtained there is one which assigns probability 1 to the state that corresponds to the ego, the rational id. This, then, implies that when menus are added to the consideration, the representation theorem of Ghirardato, Maccheroni, and Marinacci (2004) takes a form where the best case scenario coincides with the behavior of the ego, hence, the decision maker; in turn, delivering us the multiple-selves representation.

Therefore, the main contribution of the current thesis involves employing the construction of the states using menus and showing that the strict unambiguous value of acceptable menus enables us to get rid off the ambiguity concerning the best scenario and to replace it with the behavior of the ego.

We also show that under a more restrictive condition calling for unambiguous preference relations to be sustained only by monotonicity (an axiom ensuring that the act delivering more desired consequences in every possible contingency must be unambiguously preferred), the decision maker's beliefs about potential alter egos is equal to all probability distributions on alter egos. Hence, our multiple-selves representation transforms into a dual-self version in which the worst case scenario regarding the consumption of a menu corresponds to the consumption of the worst alternative from that menu. Therefore, a decision maker evaluates a menu by imagining that the ego will decide and he/she will consume the best alternative of this menu with some menu dependent probability, whereas the alter-ego which represents the most evil-self will be at the helm, resulting in the consumption of the worst alternative in this menu with the remaining probability.

Instead of presenting the model and results of Ghirardato, Maccheroni, and Marinacci (2004) superfluously, this thesis analyzes the construction and proofs of that paper in detail due to the following: First, when menus are added to the consideration, we need to make sure that our state space construction is compatible with theirs and these deliver the α–maxmin expected utility model. Second, we have to check under the hood by going deep into their construction and proofs in order to come up with a clean condition that we need in obtaining multiple-selves representation of preferences on menus under ambiguity.

The studies in the temptation literature closest in spirit to ours are Chatterjee and Krishna (2009), and its working paper version, Chatterjee and Krishna (2005). Chatterjee and Krishna (2009) argues that individuals do not always choose the best alternative from a menu and that study claims that such "mistakes" can be interpreted as an indication of the presence of a virtual alter ego. Therefore, using some axioms they obtain a dual-self representation of preferences of the decision maker over menus in which he/she acts as if he/she has a virtual alter ego. This alter ego appears with some constant probability and makes choices which are not necessarily preferred by the long-run self. Meanwhile, their axioms on preferences on menus ensure that the alter ego is rational in his/her own right since the alter ego's preferences are represented by a von-Neumann Morgenstern utility function. Consequently, they interpret alternatives maximizing the alter ego's utility as the tempting alternatives of a given menu.

By employing slightly different axioms, Chatterjee and Krishna (2005), the working paper version of the aforementioned study, obtains a similar, but menu dependent, dual-self representation one where the only difference is that the probability of the alter ego being decisive depends on the given menu at hand. Furthermore, they show that preferences on menus satisfying the axioms of Gul and Pesendorfer (2001) also admit the menu dependent dual-self representation, whereas the converse is not necessarily true.

Chatterjee and Krishna (2009) and Chatterjee and Krishna (2005) both employ a key axiom to obtain their representation theorem. This, so-called, temptation axiom requires that given any menu, the menu consisting of only the the best alternatives from the given menu must be preferred to the given menu while the menu itself must be preferred to the menu consisting of only the worst alternatives from the given menu.

In the current thesis, we prove that our multiple-selves representation delivers the temptation axioms of Chatterjee and Krishna (2009) and Chatterjee and Krishna (2005).

Gul and Pesendorfer (2001) triggered the temptation and self-control literature with their well-known representation theorem. Their model delivers the decision maker's commitment ranking and temptation ranking over alternatives and the cost of self-control. Therefore, when the decision maker evaluates the worth of a menu, he/she imagines that he/she would be choosing an alternative which brings about the highest utility despite the cost of self-control that is exerted while choosing this alternative. They also claim that their model captures the choice in the second period where the decision maker chooses an item from a menu selected at the first period. In other words, the choice from a menu projects the compromise between choosing the best alternative according to his/her commitment preferences and the cost of self-control due to not choosing the most tempting alternative. Hence, if the decision maker is able to choose the best alternative in a given menu, then he/she is endowed with a self-control which enables him/her to resist temptation.

The key axiom in their representation theorem is set betweenness: A preference relation defined on menus (compact subsets of the simplex formed on the set of alternatives) satisfies set betweenness if for any pair of menus with the first being preferred to the second, the menu consisting of the union of these menus must be preferred to the second menu while the first menu is to be preferred to the menu consisting of the union of the two menus.

We analyze situations which are rich enough to enable our dual-self representation to deliver the set betweenness axiom of Gul and Pesendorfer (2001). Nevertheless, we have to confess that the current thesis does not deliver a definitive answer to this question as the corresponding endeavor requires an additional verification involving the use of Clarke differential which we are not well acquainted with. Ergo, this creates a future task for us that we would like address in the near future.

This thesis is organized as follows: Chapter 2 presents preliminaries and axioms for preferences over acts, the unambiguous preference relation defined on acts, and the α–maxmin expected utility model of Ghirardato, Maccheroni, and Marinacci (2004) along with its proof. In Chapter 3, we present our construction of acts concerning menus and the multiple-selves representation. Next, Chapter 4 introduces the dual-self representation under ambiguity and discusses the constant ambiguity aversion index and the relation of our dual-self model with key axioms of Chatterjee and Krishna (2005), Chatterjee and Krishna (2009) and Gul and Pesendorfer (2001). Finally, Chapter 5 concludes.

2 REPRESENTATION UNDER AMBIGUITY

This chapter follows Gilboa and Schmeidler (1989) and Ghirardato, Maccheroni, and Marinacci (2004) and presents the required arguments and proofs in detail in order to obtain the desired representation theorem under ambiguity and to apply it to the acts concerning menus involving ambiguous choices of the alter egos.

2.1 Preliminaries and axioms

We let X be the set of all *alternatives*, a compact metric space, and $\mathbb{X} \equiv \Delta(X)$, the set of probability measures on the Borel sigma-algebra of X (alternatively, the simplex formed on X) endowed with the weak* topology. Then, $\mathbb{X}$ is non-empty and convex and weak* compact and a convex subset of a vector space (endowed with the variation norm). Next, we define the *state space* S with a generic member, a *state*, $s \in S$.

In order to apply the α–maxmin expected utility model of Ghirardato, Maccheroni, and Marinacci (2004) in our analysis, first we adapt the well-known setup of Anscombe and Aumann (1963) to our model.

Given S and X respectively, which are defined above, let Σ be an algebra sigma of subsets of S, called *events*, and let $\mathcal{F}$ be a set of all the *acts* which are finite valued Σ–measurable functions $f : S \to \mathbb{X}$. It is useful to point out that $\mathcal{F}$ is convex. A convex combination of two acts, also referred to as a mixtures of two acts is defined as follows: For any given $f, g \in \mathcal{F}$

and $\lambda \in [0,1]$, the convex combination of f and g via λ is $\lambda f + (1-\lambda)g \in \mathcal{F}$ where this act delivers $\lambda f(s) + (1-\lambda)g(s) \in \mathbb{X}$ in state $s \in S$. Moreover, an act $h \in \mathcal{F}$ is called a *constant act* if $h(s) = \ell$ for all $s \in S$ and for some $\ell \in \mathbb{X}$; namely, regardless of the state which is realized, h gives the same consequence, the lottery ℓ. We denote $\mathcal{F}_c$ the set of all constant acts.

$B_0(\Sigma)$ denotes the set of all bounded Σ–measurable real valued simple functions. Then, for any $f \in \mathcal{F}$ and define $u_f \in B_0(\Sigma)$ by $u_f(s) \in \mathbb{R}$, i.e. for any $S \in \Sigma$ the set $u_f(S) = \{u_f(s) : s \in S\}$ is measurable and $u_f : S \to \mathbb{R}$ is a simple function.

Let $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ a binary relation which we will refer to as a preference relation over $\mathcal{F}$. Indeed, $f \succsim g$ is pronounced as f is weakly preferred to g. Also, $f \succ g$ is equivalent to $f \succsim g$ while not $g \succsim f$, and is to be read as f is strictly preferred to g. Finally, $f \sim g$ if and only if $f \succsim g$ and $g \succsim f$, a case which we will refer to as f being indifferent to g.

Ghirardato, Maccheroni, and Marinacci (2004) borrowed the following five axioms from Gilboa and Schmeidler (1989):

Axiom 1 (Weak Order) *$\succsim$ is a complete and transitive binary relation; i.e. (i.) for all $f, g \in \mathcal{F}$, either $f \succsim g$ or $g \succsim f$ or both, and (ii.) for all $f, g, m \in \mathcal{F}$, if $f \succsim g$ and $g \succsim m$, then $f \succsim m$.*

Axiom 2 (Certainity Independence) *For all $f, g \in \mathcal{F}$ and for all $h \in \mathcal{F}_c$ and for all $\lambda \in [0,1]$, it must be that $f \succsim g$ if and only if $\lambda f + (1-\lambda)h \succsim \lambda g + (1-\lambda)h$.*

Axiom 3 (Continuity) *For all $f, g, m \in \mathcal{F}$ with $f \succ g$ and $g \succ m$, there exist $\lambda, \lambda' \in (0,1)$ such that $\lambda f + (1-\lambda)m \succ g$ and $g \succ \lambda' f + (1-\lambda')m$.*

Axiom 4 (Monotonicity) *For all $f, g \in \mathcal{F}$ and for any given $s \in S$ letting $h_{f(s)}, h_{g(s)} \in \mathcal{F}_c$ be defined by $h_{f(s)}(s') = f(s)$ and $h_{g(s)}(s') = g(s)$ for all $s' \in S$, the condition $h_{f(s)} \succsim h_{g(s)}$ for all $s \in S$ implies $f \succsim g$.*

Axiom 5 (Non–degeneracy) *There exist $f, g \in \mathcal{F}$ such that $f \succ g$.*

The following result is due to Ghirardato, Maccheroni, and Marinacci (2004) (Lemma 1; p.141)

Theorem 1 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 if and only if there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$ and a non-constant affine function $u : \mathbb{X} \to \mathbb{R}$ such that $f \succsim g$ if and only if $I(u_f) \geq I(u_g)$, $f, g \in \mathcal{F}$, and $\ell \succsim^{\mathbb{X}} \ell'$ if and only if $u(\ell) \geq u(\ell')$, $\ell, \ell' \in \mathbb{X}$. Moreover, functional I is unique and u is unique up to a positive affine transformation.* [1]

Proof. The sufficiency direction of the current Theorem is trivial; hence, omitted.

The regarding the necessity direction is done by mimicking the arguments in the Lemmas 3.1–3.3 of Gilboa and Schimeidler (1989).

Notice that the relation $\succsim$ on $\mathcal{F}$ induces a relation $\succsim^{\mathbb{X}}$ on $\mathbb{X}$ defined as follows: for any $\ell, \ell' \in \mathbb{X}$, $\ell \succsim^{\mathbb{X}} \ell'$ if and only if $h_\ell \succsim h_{\ell'}$ $h_\ell, h_{\ell'} \in \mathcal{F}_c$ are such that $h_\ell(s) = \ell$ and $h_{\ell'}(s) = \ell'$ for all $s \in S$. Then, Axiom 2 ensures that the independence axiom for $\succsim^{\mathbb{X}}$ is satisfied. To see that, consider any $\ell, \ell' \in \mathbb{X}$ with $\ell \succsim^{\mathbb{X}} \ell'$ and any $\lambda \in [0, 1]$ and $\ell'' \in \mathbb{X}$ and notice that $h_\ell, h_{\ell'}, h_{\ell''} \in \mathcal{F}_c$ with $h_\ell \succsim h_{\ell'}$; so by C-independence, we have $\lambda\, h_\ell + (1-\lambda)h_{\ell''} \succsim \lambda\, h_{\ell'} + (1-\lambda)h_{\ell''}$ which implies with the help of the definition of $\succsim^{\mathbb{X}}$ that $\lambda\, \ell + (1-\lambda)\ell'' \succsim^{\mathbb{X}} \lambda\, \ell' + (1-\lambda)\ell''$. Also, letting $\ell = \ell'$ and $\lambda = 1$ establishes $\succsim^{\mathbb{X}}$ is reflexive. As all axioms of von Neumann – Morgenstern expected utility theorem is satisfied we obtain the following result stated as a lemma:

Lemma 1 *There is an affine function $u : \mathbb{X} \to R$ such that for all $\ell, \ell' \in \mathbb{X}$, $\ell \succsim^{\mathbb{X}} \ell'$ if and only if $u(\ell) \geq u(\ell')$. Therefore, for any $h_\ell, h_{\ell'} \in \mathcal{F}_c$ with $h_\ell(s) = \ell$ and $h_{\ell'}(s) = \ell'$ for all $s \in S$, $h_\ell \succsim h_{\ell'}$ if and only if $u(\ell) \geq u(\ell')$. Moreover, u is unique up to positive affine transformation.*

Next, we observe that von Neumann – Morgenstern's construction involving best and worst lotteries, extends to the case of acts:

[1] It may be appropriate to remark that $u(\ell) = u_f(s)$ for any $f \in \mathcal{F}_c$ with $f(s) = \ell \in \mathbb{X}$ for all $s \in S$.

Lemma 2 *There exists* $\overline{f}, \underline{f} \in \mathcal{F}_c$ *such that for any* $f \in \mathcal{F}$ *it must be that* $\overline{f} \succsim f \succsim \underline{f}$. *In fact,* $\overline{f}$ *is defined by* $\overline{f}(s) = \bar{\ell} \in \mathbb{X}$ *with* $\bar{\ell} \succsim^{\mathbb{X}} \ell'$ *for all* $\ell' \in \mathbb{X}$ *while* $\underline{f}$ *is defined by* $\underline{f}(s) = \underline{\ell} \in \mathbb{X}$ *with* $\ell' \succsim^{\mathbb{X}} \underline{\ell}$ *for all* $\ell' \in \mathbb{X}$, *for all* $s \in S$. *Moreover, for any* $f \in \mathcal{F}$ *there exists a unique* $\alpha_f \in [0,1]$ *such that* $f \sim \alpha_f \overline{f} + (1 - \alpha_f)\underline{f}$.

Proof. Let $f \in \mathcal{F}$ and notice that $\bar{\ell} \succsim^{\mathbb{X}} f(s) \succsim^{\mathbb{X}} \underline{\ell}$ for all $s \in S$ implies with the help of Axiom 4 (monotonicity) that $\overline{f} \succsim f \succsim \underline{f}$. Now, define $B^+, B^- \subset [0,1]$ by $B^+ = \{\alpha \in [0,1] : \alpha\overline{f} + (1-\alpha)\underline{f} \succsim f\}$ and $B^- = \{\alpha \in [0,1] : f \succsim \alpha\overline{f} + (1-\alpha)\underline{f}\}$. Due to Axioms 1 and 3 (completeness and continuity), B^+ and B^- are closed and any $\alpha \in [0,1]$ belongs to at least one of them. Thus, the closedness and nonemptiness of B^+ and B^- together with the fact that $[0,1]$ is connected ensures that $B^+ \cap B^- \neq \emptyset$. Hence, there exists at least one α belonging to both of these sets; and it is unique. This follows from Axiom 1 (transitivity) implying that $\mathring{B}^+ = \{\alpha \in [0,1] : \alpha\overline{f} + (1-\alpha)\underline{f} \succ f\}$ and $\mathring{B}^- = \{\alpha \in [0,1] : f \succ \alpha\overline{f} + (1-\alpha)\underline{f}\}$ are such that $\mathring{B}^+ \cap \mathring{B}^- = \emptyset$. ■

The following result will be helpful in the rest of the proof: For any given $\succsim$ and affine function $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$, let $K = u(\mathbb{X}) \subset \mathbb{R}$. Then, by the continuity of u (implied by u being affine), K is compact. Let $B_0(\Sigma, K)$ be subsets of functions in $B_0(\Sigma)$ whose values are in K.

Lemma 3 *The following hold:*

(*i*) *For any* $f \in \mathcal{F}$, *there exists* $f_c \in \mathcal{F}_c$ *such that* $f \sim f_c$; *and*

(*ii*) *For any* $f, g \in \mathcal{F}$ *and resulting* $u_f, u_g \in B_0(\Sigma)$ *and* $\beta \in [0,1]$, $u_{\beta f+(1-\beta)g} \in B_0(\Sigma)$ *and* $u_{\beta f+(1-\beta)g} = \beta u_f + (1-\beta)u_g$; *and*

(*iii*) *For any* $\psi \in B_0(\Sigma, K)$, *there exists* $f \in \mathcal{F}$ *with* $\psi = u_f$.

Proof. Part (*i*) of the above Lemma follows from Lemma 2.

For part (*ii*), take any $f, g \in \mathcal{F}$ and resulting $u_f, u_g \in B_0(\Sigma)$ and $\beta \in [0,1]$. Then, clearly, $u_{\beta f+(1-\beta)g} \in B_0(\Sigma)$. Now, consider act $\beta f + (1-\beta)g \in \mathcal{F}$. By (*i*) of the current Lemma, there is $f_c, g_c \in \mathcal{F}_c$ with $f_c \sim f$ and $g_c \sim g$ and $f_c(s) = \ell_f$ and $g_c(s) = \ell_g$ for all $s \in S$; and notice that $\beta f + (1-\beta)g \sim \beta f_c + (1-\beta)g_c$.[2] Thus, $u_{\beta f+(1-\beta)g} = u_{\beta f_c+(1-\beta)g_c} = u(\beta\ell_f + (1-\beta)\ell_g)$ which, due to (i) of Lemma 3, equals $\beta u(\ell_f) + (1-\beta)u(\ell_g) = \beta u_{f_c} + (1-\beta)u_{g_c} = \beta u_f + (1-\beta)u_g$.

Regarding the last item, part (iii) of the current Lemma, consider any $\psi \in B_0(\Sigma, K)$. Then, for any given $s \in S$, $\psi \in B_0(\Sigma, K)$ implies $\psi(s) \in [u(\underline{\ell}), u(\bar{\ell}))]$, a non-empty, convex, and compact set; thus, due to Lemma 1 u is continuous and so by Brower's Fixed Point Theorem there exists ℓ_s with $u(\ell_s) = \psi(s)$; consequently, defining $f \in \mathcal{F}$ by $f(s) \equiv \ell_s$ for $s \in S$ establishes our observation. ■

Lemma 4 *Given $u : \mathbb{X} \to R$ representing $\succsim^{\mathbb{X}}$, let $J : \mathcal{F} \to \mathbb{R}$ given as follows is well-defined and unique:*

- (i) *$f \succsim g$ if and only if $J(f) \geq J(g)$, $f, g \in \mathcal{F}$, and*
- (ii) *for any $h_\ell \in \mathcal{F}_c$ defined by $h_\ell(s) = \ell$ for all $s \in S$, it must be that $J(h_\ell) = u(\ell)$.*

Proof. By (ii) above, J is uniquely determined on $\mathcal{F}_c$. By Lemma 2, for any $f \in \mathcal{F}$, there exists a unique $\alpha_f \in [0,1]$ such that $f \sim \alpha_f \bar{f} + (1-\alpha_f)\underline{f}$ with $\bar{f}, \underline{f} \in \mathcal{F}_c$. Thus, by (i) and (ii), $J(f) = J(\alpha_f \bar{f} + (1-\alpha_f)\underline{f})$. Therefore, for any $f \in \mathcal{F}$, J clearly satisfied (i) and is uniquely determined. ■

Lemma 5 *For any affine function $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$, there exists a uniquely and well-defined functional $I : B_0(\Sigma) \to \mathbb{R}$ satisfying the following:*

- (i) *For all $f \in \mathcal{F}$, $I(u_f) = J(f)$; and*
- (ii) *I is monotonic, i.e. for any $f, g \in \mathcal{F}$ and resulting $u_f, u_g \in B_0(\Sigma)$, $u_f(s) \geq u_g(s)$ for all $s \in S$ implies $I(u_f) \geq I(u_g)$; and*

[2] This follows from Axiom 2 (certainty independence) as follows: $f \sim f_c$ and $g \sim g_c$ implies for any $\lambda \in [0,1]$ that $\lambda f + (1-\lambda)f_c \sim f_c$ and $\lambda g + (1-\lambda)g_c \sim g_c$ as $f_c, g_c \in \mathcal{F}_c$. So, without loss of generality suppose $\beta f + (1-\beta)g \succ \beta f_c + (1-\beta)g_c$. Then, by Axiom 2 for any $\lambda \in [0,1]$ we have $\lambda(\beta f + (1-\beta)g) + (1-\lambda)(\beta f_c + (1-\beta)g_c) \succ \beta f_c + (1-\beta)g_c$. So, $\lambda(\beta f + (1-\beta)g) + (1-\lambda)(\beta f_c + (1-\beta)g_c) = \beta(\lambda f + (1-\lambda)f_c) + (1-\beta)(\lambda g + (1-\lambda)g_c) \sim \beta f_c + (1-\beta)g_c$ (due to $\lambda f + (1-\lambda)f_c \sim f_c$ and $\lambda g + (1-\lambda)g_c \sim g_c$) we have the desired contradiction.

(iii) *I is homogeneous of degree 1, i.e. for any $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma)$, $I(\beta u_f) = \beta I(u_f)$ for all $\beta \geq 0$.*

(iv) *I is constant additive, i.e. for any $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma)$, $I(u_f + \beta) = I(u_f) + \beta$ for all $\beta \in \mathbb{R}$.*

Proof. Recall that for any given $\succsim$ and affine function $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$, let $K = u(\mathbb{X}) \subset \mathbb{R}$ and $B_0(\Sigma, K)$ be subsets of functions in $B_0(\Sigma)$ whose values are in K.

Claim 1 *Let $I : B_0(\Sigma, K) \to \mathbb{R}$ be defined by for any $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma, K)$, $I(u_f) = J(f)$. Then, I is uniquely and well-defined on $B_0(\Sigma, K)$.*

Proof. The result follows from Lemma 4 ■

Claim 2 *For any $f, g \in \mathcal{F}$ and resulting $u_f, u_g \in B_0(\Sigma)$, $u_f(s) \geq u_g(s)$ for all $s \in S$ implies $I(u_f) \geq I(u_g)$.*

Proof. Let $f, g \in \mathcal{F}$ with $u_f(s) \geq u_g(s)$ for all $s \in S$ which is equivalent to $f(s) \succsim^{\mathbb{X}} g(s)$ for all $s \in S$ as u represents $\succsim^{X}$ and $u_f(s) = u(\ell)$ where $f(s) = \ell$. Thus, by Axiom 4 (monotonicity), $f \succsim g$. Then, by (i) of Lemma 4 $J(f) \geq J(g)$; so, by (i) of Lemma 5 $I(u_f) \geq I(u_g)$. ■

Claim 3 *For any $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma)$, $I(\beta u_f) = \beta I(u_f)$ for all $\beta \geq 0$.*

Proof. Let $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma)$, and consider $\beta \geq 0$. It suffices to restrict attention to the case when $\beta \in [0, 1]$.

Due to Axiom 5 (non-degeneracy) and Lemma 2, we know $\bar{f} \succ \underline{f}$ and by Lemma 1 without loss of generality we can normalize u such that $u(\bar{\ell}) > 1$ and $u(\underline{\ell}) < -1$. Next, notice that by following similar steps presented in the proof of part (iii) of Lemma 3, there exists $\ell_0 \in \mathbb{X}$ such that $u(\ell_0) = 0$; and define $h_0 \in \mathcal{F}_c$ with $h_0(s) = \ell_0$ for all $s \in S$. [3] Then, clearly $u_{h_0}(s) = 0$ for all $s \in S$.

[3] As $\mathbb{X}$ is non-empty convex and compact and $u : \mathbb{X} \to \mathbb{R}$ is continues and $u(\bar{\ell}) > 1$ and $u(\underline{\ell}) < -1$, there exists ℓ_0 with $u(\ell_0) = 0$. Then, let $h_0 \in \mathcal{F}_c$ be defined by $h_0(s) = \ell_0$ for all $s \in S$.

By Lemma 3, there exists $f_c \in \mathcal{F}_c$ with $f \sim f_c$; so, $u_f = u_{f_c}$.

Then, consider the act $\beta f_c + (1-\beta)h_0$; thus, by Lemma 3, $u_{\beta f_c+(1-\beta)h_0} = \beta u_{f_c} + (1-\beta)u_{h_0} = \beta u_{f_c} + \mathbf{0}$.

Therefore, $I(\beta u_f) = I(\beta u_{f_c}) = J(\beta f_c + (1-\beta)h_0)$ which equals (as $\beta f_c + (1-\beta)h_0 \in \mathcal{F}_c$) to $u_{\beta f_c+(1-\beta)h_0}(s) = \bar{u}$ for all $s \in S$ and $u_{\beta f_c+(1-\beta)h_0}(s) = \beta u_{f_c}(s) = \bar{u}$ for all $s \in S$ which implies $\bar{u} = \beta u_{f_c}(s) = \beta J(f_c) = \beta J(f) = \beta I(u_f)$. ■

Claim 4 *For any $f \in \mathcal{F}$ and resulting $u_f \in B_0(\Sigma)$, $I(u_f + \mathbf{b}) = I(u_f) + \beta$ for all $\beta \in \mathbb{R}$ where $\mathbf{b}(s) = \beta$ for all $s \in S$.*

Proof. Without loss of generality let $\beta \in [u(\underline{\ell}), u(\bar{\ell})]$ and for any such β we define $h_\beta \in \mathcal{F}_c$ by $h_\beta(s) = \ell_\beta \in \mathbb{X}$ for all $s \in S$ with $u_{h_\beta} \in B_0(\Sigma)$ be given by $u_{h_\beta}(s) = u(\ell_\beta) = \beta = \mathbf{b}(s)$ for all $s \in S$, so $u_{h_\beta} = \mathbf{b}$. [4]Such a lottery ℓ_β exists as $\mathbb{X}$ is non-empty convex and compact and $u : \mathbb{X} \to \mathbb{R}$ is continuous. Then, $I(u_{h_\beta}) = J(h_\beta) = u(\ell_\beta) = \beta = I(\mathbf{b})$ due to $h_\beta \in \mathcal{F}_c$ and part (ii) of Lemma 4.

By Lemma 3, there exists $f_c \in \mathcal{F}_c$ with $f \sim f_c$; so, $u_f = u_{f_c}$. Let ℓ_{f_c} be such that $\ell_{f_c} = f_c(s)$ for all $s \in S$.

Now, consider act $\frac{1}{2}f + \frac{1}{2}h_\beta$. Notice that $J(\frac{1}{2}f + \frac{1}{2}h_\beta) = I(\frac{1}{2}u_f + \frac{1}{2}u_{h_\beta}) = I(\frac{1}{2}u_{f_c} + \frac{1}{2}u_{h_\beta}) = J(\frac{1}{2}f_c + \frac{1}{2}h_\beta) = u(\frac{1}{2}\ell_{f_c} + \frac{1}{2}\ell_\beta) = \frac{1}{2}u(\ell_{f_c}) + \frac{1}{2}u(\ell_\beta) = \frac{1}{2}u(\ell_{f_c}) + \frac{1}{2}\beta = \frac{1}{2}J(f_c) + \frac{1}{2}\beta = \frac{1}{2}J(f) + \frac{1}{2}\beta = \frac{1}{2}I(u_f) + \frac{1}{2}\beta$. Thus, as $u_{h_\beta} = \mathbf{b}$, $I(\frac{1}{2}u_f + \frac{1}{2}\mathbf{b}) = \frac{1}{2}I(u_f) + \frac{1}{2}\beta$ which by Claim 3 implies that $I(\frac{1}{2}u_f + \frac{1}{2}\mathbf{b}) = I(\frac{1}{2}(u_f + \mathbf{b})) = \frac{1}{2}I(u_f + \mathbf{b}) = \frac{1}{2}I(u_f) + \frac{1}{2}\beta$, hence, delivering the desired result. ■

This concludes the proof of the Lemma, hence, the Theorem. ■ ■

2.2 Unambiguous preferences

In what follows, we need the relation that represents an unambiguous preference between two acts. To that regard, we say that an act $f \in \mathcal{F}$ is *unambiguously preferred* to another

[4]When $\beta \notin [u(\underline{\ell}), u(\bar{\ell})]$, then we can find $\beta' = c\beta$ with $c \in \mathbb{R}$ and $\beta' \in [u(\underline{\ell}), u(\bar{\ell})]$ and define $\mathbf{b}' = c\mathbf{b}$. Then, due to Claim 3, the following proof can be done by employing $cu_f + c\mathbf{b}$ which is in $B_0(\Sigma)$.

act $g \in \mathcal{F}$, denoted by $f \succsim^{UA} g$, whenever $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and for all $m \in \mathcal{F}$. Then, $\succsim^{UA} \subseteq \mathcal{F} \times \mathcal{F}$ is a binary relation which we will refer to as an unambiguous preferences defined over $\mathcal{F}$. Naturally, $f \succ^{UA} g$ is equivalent to $f \succsim^{UA} g$ while not $g \succsim^{UA} f$, and is to be read as f is unambiguously strictly preferred to g. Finally, $f \sim^{UA} g$ if and only if $f \succsim^{UA} g$ and $g \succsim^{UA} f$, a case which we will refer to as f being unambiguously indifferent to g.

The following lemma presents the properties of $\succsim^{UA}$:

Lemma 6 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and $\succsim^{UA}$ on $\mathcal{F}$ is as defined above. Then, each of the following holds for $\succsim^{UA} \subseteq \mathcal{F} \times \mathcal{F}$:*

(i) For any $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ implies $f \succsim g$.

(ii) For any $h, h' \in \mathcal{F}_c$, $h \succsim^{UA} h'$ if and only if $h \succsim h'$.

(iii) $\succsim^{UA}$ is preorder, i.e. reflexive and transitive.

(iv) $\succsim^{UA}$ is monotone. [5]

(v) $\succsim^{UA}$ satisfies the independence axiom. [6]

(vi) $\succsim^{UA}$ is the maximal restriction of $\succsim$ satisfying the independence axiom. [7]

Proof. For the part (i), let $f \succsim^{UA} g$. Then, for any $f, g \in \mathcal{F}$ we must have $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and for all $m \in \mathcal{F}$. If $\lambda = 1$, then clearly $f \succsim g$.

Only if part of (ii) follows from the (i) of current lemma. For the if part of (ii), assume that there are two constant acts $h, h' \in \mathcal{F}_c$ with $h \succsim h'$. Let $\ell, \ell' \in \mathbb{X}$ be two lotteries such that $h(s) = \ell$ and $h'(s) = \ell'$ for all $s \in S$. Then, we must have $\ell \succsim^{\mathbb{X}} \ell'$. Now, let $m \in \mathcal{F}$ be an arbitrary act and for any given $s \in S$ define $m(s) = \ell_{m(s)}$. Therefore, since $\succsim^{\mathbb{X}}$ satisfies

[5]For any given $s \in S$ let $f(s) = \ell_{f(s)}$ and $g(s) = \ell_{g(s)}$ with $\ell_{f(s)}, \ell_{g(s)} \in \mathbb{X}$. Then, $\succsim^{UA}$ is monotone if for any $f, g \in \mathcal{F}$ with $\ell_{f(s)} \succsim^{\mathbb{X}} \ell_{g(s)}$ for all $s \in S$, it must be that $f \succsim^{UA} g$.

[6]$\succsim^{UA}$ satisfies the independence axiom if for any $f, g, m \in \mathcal{F}$ with $f \succsim^{UA} g$ and $\lambda \in [0,1]$ it must be that $\lambda f + (1-\lambda)m \succsim^{UA} \lambda g + (1-\lambda)m$.

[7]That is, if $\succsim^* \subset \succsim$ with the restriction that $\succsim^*$ satisfies the independence axiom, then $\succsim^* \subset \succsim^{UA}$.

the independence axiom, it must be $\lambda \ell + (1-\lambda)\ell_{m(s)} \succsim^{\mathbb{X}} \lambda \ell' + (1-\lambda)\ell_{m(s)}$ for all $s \in S$ and $\lambda \in [0,1]$; so, $\lambda h(s) + (1-\lambda)m(s) \succsim^{\mathbb{X}} \lambda h'(s) + (1-\lambda)m(s)$ for all $s \in S$ and $\lambda \in (0,1]$. This, by Axiom 4 (monotonicity), implies that $\lambda h + (1-\lambda)m \succsim \lambda h' + (1-\lambda)m$ for all $\lambda \in (0,1]$; hence, $h \succsim^{UA} h'$ as $m \in \mathcal{F}$ is selected arbitrarily.

Regarding the third item, first notice that reflexivity of $\succsim^{UA}$ is a direct consequence of Axiom 4 (monotonicity): For any given $f \in \mathcal{F}$ and any arbitrarily selected $\lambda \in (0,1]$ and $m \in \mathcal{F}$, due to the reflexivity of $\succsim^{\mathbb{X}}$, we have $\lambda f(s) + (1-\lambda)m(s) \succsim^{\mathbb{X}} \lambda f(s) + (1-\lambda)m(s)$ for all $s \in S$; thus, by monotonicity, $\lambda f + (1-\lambda)m \succsim \lambda f + (1-\lambda)m$, so $f \succsim^{UA} f$. In order to show that $\succsim^{UA}$ is transitive, assume that $f \succsim^{UA} g$ and $g \succsim^{UA} m$ with $f, g, m \in \mathcal{F}$. Therefore, for any $k \in \mathcal{F}$ and $\lambda \in (0,1]$, we have $\lambda f + (1-\lambda)k \succsim \lambda g + (1-\lambda)k$ and $\lambda g + (1-\lambda)k \succsim \lambda m + (1-\lambda)k$. Then, by Axiom 1, $\lambda f + (1-\lambda)k \succsim \lambda m + (1-\lambda)k$. So, as $\lambda \in (0,1]$ and $m \in \mathcal{F}$ is arbitrary, $f \succsim^{UA} m$.

In order to show that $\succsim^{US}$ is monotonic, we will repeat the similar steps that we took in the proof of (ii): For any $s \in S$ let $\ell_{f(s)} \succsim^{\mathbb{X}} \ell_{g(s)}$ where $f(s) = \ell_{f(s)}$ and $g(s) = \ell_{g(s)}$ and $f, g \in \mathcal{F}$ and $\ell_{f(s)}, \ell_{g(s)} \in \mathbb{X}$. Let $m \in \mathcal{F}$ be arbitrary with $m(s) = \ell_{m(s)}$, $s \in S$. By the independence axiom of $\succsim^{\mathbb{X}}$, it must be $\lambda f(s) + (1-\lambda)m(s) \succsim^{\mathbb{X}} \lambda g(s) + (1-\lambda)m(s)$ for all $s \in S$ and for all $\lambda \in [0,1]$. Then, by Axiom 4 (monotonicity), $\lambda f + (1-\lambda)m \succsim g + (1-\lambda)m$ for all $\lambda \in (0,1]$; so, $f \succsim^{UA} g$.

The proof (v) is as follows: Let $f, g, m \in \mathcal{F}$ with $f \succsim^{UA} g$ and $\lambda \in [0,1]$. We need to show that $\lambda f + (1-\lambda)m \succsim^{UA} \lambda g + (1-\lambda)m$ which is equivalent to $\mu(\lambda f + (1-\lambda)m) + (1-\mu)k \succsim \mu(\lambda g + (1-\lambda)m) + (1-\mu)k$ for any arbitrary $k \in \mathcal{F}$ and $\mu \in (0,1]$. Let $k \in \mathcal{F}$ and $\mu \in (0,1]$ be arbitrary and consider the associated convex combination of two acts $(\lambda f + (1-\lambda)m)$ and $(\lambda g + (1-\lambda)m)$. Notice that $(\lambda f(s) + (1-\lambda)m(s)), (\lambda g(s) + (1-\lambda)m(s)) \in \mathbb{X}$ and as $\succsim^{\mathbb{X}}$ satisfies the independence axiom we have $\theta(\lambda f(s) + (1-\lambda)m(s)) + (1-\theta)k(s) \succsim^{\mathbb{X}} \theta(\lambda g(s) + (1-\lambda)m(s)) + (1-\theta)k(s)$ for all $s \in S$ and $\theta \in [0,1]$. By Axiom 4 (monotonicity), we have $\theta(\lambda f + (1-\lambda)m) + (1-\theta)k \succsim \theta(\lambda g + (1-\lambda)m) + (1-\theta)k$, for all $\theta \in [0,1]$. So, $\mu(\lambda f + (1-\lambda)m) + (1-\mu)k \succsim \mu(\lambda g + (1-\lambda)m) + (1-\mu)k$, for all $\mu \in (0,1]$, which implies $\lambda f + (1-\lambda)m \succsim^{UA} \lambda g + (1-\lambda)m$ as k is arbitrary.

Finally, for the proof of (vi), let $\succsim^* \subset \succsim$ and $\succsim^*$ satisfies the independence axiom. So , for any $f, g \in \mathcal{F}$ with $f \succsim^* g$ we have $\lambda f + (1-\lambda)m \succsim^* \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and $m \in \mathcal{F}$. As $\succsim^* \subset \succsim$, then $f \succsim^* g$ and $\lambda f + (1-\lambda)m \succsim^* \lambda g + (1-\lambda)m$ implies $f \succsim g$ and $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and arbitrary $m \in \mathcal{F}$. Therefore, by the definition of $\succsim^{UA}$ we have $f \succsim^{UA} g$. ■

Now, we will turn our attention to revealed ambiguity. The following result is due to Ghirardato, Maccheroni, and Marinacci (2004) (Proposition 5; p.144) and it justifies the following observation: If an act $f \in \mathcal{F}$ is unambiguously preferred to another act $g \in \mathcal{F}$, then for every probabilistic scenario $P \in \mathcal{C}$ the expected utility of f is higher than the expected utility of g.

Theorem 2 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and $\succsim^{UA}$ on $\mathcal{F}$ is as defined above. Then, there exists a unique nonempty, weak* compact and convex set of $\mathcal{C}$ of probabilities on Σ such that for all $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ if and only if $\int_S u_f \mathrm{d}P \geq \int_S u_g \mathrm{d}P$ for all $P \in \mathcal{C}$.*

Proof. Let K be an arbitrary non-singleton interval in $\mathbb{R}$ (implied by the non-triviality and at this stage we do not insist on K equaling $u(\mathbb{X})$) and define, as above, $B_0(\Sigma, K)$ be a subset of $B_0(\Sigma)$ consisting of all bounded Σ–measurable real valued simple functions that take values in K. Define a binary relation $\trianglerighteq$ on $B_0(\Sigma, K)$ with the following properties:

(i) a preorder, i.e. $\trianglerighteq$ is reflexive and transitive;

(ii) continuous, i.e. for any sequence $\{\psi_n, \phi_n\}_{n \in \mathbb{N}}$ in $B_0(\Sigma, K) \times B_0(\Sigma, K)$ with $\psi_n \trianglerighteq \phi_n$ for all $n \in \mathbb{N}$ and $(\psi_n, \phi_n) \xrightarrow{sup} (\psi, \phi)$, we have that $\psi \trianglerighteq \phi$; [8]

(iii) conic, i.e. for any $\psi, \phi \in B_0(\Sigma, K)$ with $\psi \trianglerighteq \phi$ implies $\beta\psi + (1-\beta)\gamma \trianglerighteq \beta\phi + (1-\beta)\gamma$ for all $\gamma \in B_0(\Sigma, K)$ and $\beta \in [0,1]$;

(iv) monotone, i.e. $\psi(s) \geq \phi(s)$ for all $s \in S$ implies $\psi \trianglerighteq \phi$;

[8] For any ψ and $\{\psi_n\}$, $\psi_n \xrightarrow{sup} \psi$ whenever $\lim_n \sup_{s \in S} |\psi_n(s) - \psi(s)| = 0$.

(v) nontrivial, i.e. there are $\psi, \phi \in B_0(\Sigma, K)$ with $\psi \trianglerighteq \phi$ while not $\phi \trianglerighteq \psi$.

The following is stated without a proof as Proposition A.2 in Ghirardato, Maccheroni, and Marinacci (2004), and we thank Paolo Ghirardato for providing us with its proof.

Lemma 7 *A binary relation $\trianglerighteq \subseteq B_0(\Sigma, K) \times B_0(\Sigma, K)$ is a nontrivial, continuous, conic and monotonic preorder if and only if there exists a convex and weak* closed nonempty set $\mathcal{C}$ of probabilities such that $\psi \trianglerighteq \phi$ if and only if $\int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P$ for all $P \in \mathcal{C}$ and $\psi, \phi \in B_0(\Sigma, K)$.*

Proof. Define K^{o} as the interior of K, i.e. the largest open set which is contained in K. Let $k_0 \in K^{\text{o}}$ and notice that 0 is contained in the interval $K - k_0$. Define $\trianglerighteq^{\text{o}}$ on $B_0(\Sigma, K - k_0)$ such that for any $\psi, \phi \in B_0(\Sigma, K - k_0)$, $\psi \trianglerighteq^{\text{o}} \phi$ if and only if $\psi + k_0 \trianglerighteq \phi + k_0$.

Claim 5 *Let $k_0 \in K^{\text{o}}$. Then, $\trianglerighteq^{\text{o}}$ defined on $B_0(\Sigma, K - k_0)$ is nontrivial, continuous, conic and monotonic preorder.*

Proof. We will first show that $\trianglerighteq^{\text{o}}$ is reflexive and transitive. Let $\psi \in B_0(\Sigma, K - k_0)$ be such that $\psi + k_0 \in B_0(\Sigma, K)$. Then, it must be $\psi + k_0 \trianglerighteq \psi + k_0$, as $\trianglerighteq$ is reflexive. Therefore, we observe that $\psi \trianglerighteq^{\text{o}} \psi$, so $\trianglerighteq^{\text{o}}$ is reflexive. For transitivity, suppose $\psi \trianglerighteq^{\text{o}} \phi$ and $\phi \trianglerighteq^{\text{o}} \gamma$ with $\psi, \phi, \gamma \in B_0(\Sigma, K - k_0)$; then, it must be that $\psi \trianglerighteq^{\text{o}} \gamma$. First, notice that $\psi \trianglerighteq^{\text{o}} \phi$ and $\phi \trianglerighteq^{\text{o}} \gamma$ imply $\psi + k_0 \trianglerighteq \phi + k_0$ and $\phi + k_0 \trianglerighteq \gamma + k_0$, respectively. Since $\trianglerighteq$ is transitive, it must be $\psi + k_0 \trianglerighteq \gamma + k_0$; so $\psi \trianglerighteq^{\text{o}} \gamma$.

Next, we will show that $\trianglerighteq^{\text{o}}$ is continuous. Take any sequence $\{\psi_n, \phi_n\}_{n \in \mathbb{N}}$ in $B_0(\Sigma, K - k_0) \times B_0(\Sigma, K - k_0)$ with $\psi_n \trianglerighteq^{\text{o}} \phi_n$ for all $n \in \mathbb{N}$ and $(\psi_n, \phi_n) \to (\psi, \phi)$. Then, it must be $\psi_n + k_0 \trianglerighteq \phi_n + k_0$ for all $n \in \mathbb{N}$. It is obvious that $\psi_n + k_0 \to \psi + k_0$ and $\phi_n + k_0 \to \phi + k_0$. Since $\trianglerighteq$ is continuous, we must have $\psi + k_0 \trianglerighteq \phi + k_0$, which implies $\psi \trianglerighteq^{\text{o}} \phi$.

$\trianglerighteq^{\text{o}}$ is conic if for any $\psi, \phi \in B_0(\Sigma, K - k_0)$ with $\psi \trianglerighteq^{\text{o}} \phi$ implies $\beta\psi + (1-\beta)\gamma \trianglerighteq^{\text{o}} \beta\phi + (1-\beta)\gamma$ for all $\gamma \in B_0(\Sigma, K - k_0)$ and $\beta \in [0, 1]$. Now, notice that $\beta\psi + (1-\beta)\gamma, \beta\phi + (1-\beta)\gamma \in B_0(\Sigma, K - k_0)$ and that $\psi \trianglerighteq^{\text{o}} \phi$ implies $\psi + k_0 \trianglerighteq \phi, +k_0$ and $\psi + k_0, \phi + k_0 \in B_0(\Sigma, K)$. Since $\trianglerighteq$ is conic for all $\beta \in [0, 1]$ and for all $\gamma + k_0 \in B_0(\Sigma, K)$ the following will hold:

$\beta(\psi + k_0) + (1-\beta)(\gamma + k_0) \trianglerighteq \beta(\phi + k_0) + (1-\beta)(\gamma + k_0)$ which is the same as $\beta\psi + (1-\beta)\gamma + k_0 \trianglerighteq \beta\psi + (1-\beta)\gamma + k_0$. Therefore, for all $\gamma \in B_0(\Sigma, K - k_0)$ and $\beta \in [0,1]$ it must be that $\beta\psi + (1-\beta)\gamma \trianglerighteq^{\text{o}} \beta\phi + (1-\beta)\gamma$.

For monotonicity of $\trianglerighteq^{\text{o}}$, let $\psi, \phi \in B_0(\Sigma, K - k_0)$ (and observe that $\psi + k_0, \phi + k_0 \in B_0(\Sigma, K)$) which satisfy $\psi(s) \geq \phi(s)$ for all $s \in S$. Then, $\psi(s) + k_0 \geq \phi(s) + k_0$ will hold for all $s \in S$ as well. Since $\trianglerighteq$ is monotone we have $\psi + k_0 \trianglerighteq \phi, +k_0$, so $\psi \trianglerighteq^{\text{o}} \phi$.

Finally, $\trianglerighteq^{\text{o}}$ is nontrivial since K is a non-singleton interval on real line and $\trianglerighteq^{\text{o}}$ satisfies monotonicity: Define $\psi, \phi \in B_0(\Sigma, K - k_0)$ by $\psi(s) = \max_{k \in K} k - k_0$ and $\phi(s) = \min_{k \in K} k - k_0$ for all $s \in S$ and observe that $\psi + k_0 \trianglerighteq \phi + k_0$, so $\psi \trianglerighteq^{\text{o}} \phi$ and not $\phi \trianglerighteq^{\text{o}} \psi$. ■

Claim 6 *Following statements are equivalent for any $\psi, \phi \in B_0(\Sigma, K - k_0)$:*

(i) $\psi \trianglerighteq^{\text{o}} \phi$*;*

(ii) There is $\alpha > 0$ with $\alpha\psi, \alpha\phi \in B_0(\Sigma, K - k_0)$ and $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$;

(iii) For all $\alpha > 0$ with $\alpha\psi, \alpha\phi \in B_0(\Sigma, K - k_0)$, it must be $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$.

Proof. Taking $\alpha = 1$ is enough to show that (i) implies (ii); and (iii) implies (i).

In order to show that (ii) implies (iii), suppose that there exist $\alpha > 0$ with $\alpha\psi, \alpha\phi \in B_0(\Sigma, K - k_0)$ and $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$. Let $\alpha' > 0$ be such that $\alpha \geq \alpha' > 0$, then $\alpha'\psi = \frac{\alpha'}{\alpha}\alpha\psi + (1 - \frac{\alpha'}{\alpha})\mathbf{0} \trianglerighteq^{\text{o}} \frac{\alpha'}{\alpha}\alpha\phi + (1 - \frac{\alpha'}{\alpha})\mathbf{0} = \alpha'\phi$ and notice that $\alpha'\psi, \alpha'\phi, \mathbf{0} \in B_0(\Sigma, K - k_0)$ because 0 is in the interval $K - k_0$ and $\trianglerighteq^{\text{o}}$ is conic. Now, if $\alpha' > \alpha$ (for a contradiction) assume that $\alpha'\psi, \alpha'\phi \in B_0(\Sigma, K - k_0)$ and not $\alpha'\psi \trianglerighteq^{\text{o}} \alpha'\phi$. Since $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$, it must be $\frac{\alpha}{\alpha'}\alpha'\psi \trianglerighteq^{\text{o}} \frac{\alpha}{\alpha'}\alpha'\phi$. Let $\bar{\delta} = \sup\{\delta \in [0,1] : \delta\alpha'\psi \trianglerighteq^{\text{o}} \delta\alpha'\phi\}$ and notice that $\{\delta \in [0,1] : \delta\alpha'\psi \trianglerighteq^{\text{o}} \delta\alpha'\phi\} \neq \emptyset$ (as δ may equal $\frac{\alpha}{\alpha'} \in (0,1)$) and because that $\trianglerighteq^{\text{o}}$ is continuous, it must be $\bar{\delta}\alpha'\psi \trianglerighteq^{\text{o}} \bar{\delta}\alpha'\phi$. Due to the fact that $\trianglerighteq^{\text{o}}$ is conic and $\bar{\delta} \in [0,1]$ and $\alpha'\psi, \alpha'\phi \in B_0(\Sigma, K - k_0)$, by mixing $\alpha'\psi$ and $\alpha'\phi$ by a weight of $\bar{\delta}/(1+\bar{\delta})$ it must be that

$$\frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\psi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\psi \trianglerighteq^{\text{o}} \frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\phi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\psi, \tag{2.1}$$

$$\frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\psi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\phi \trianglerighteq^{\text{o}} \frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\phi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\phi. \tag{2.2}$$

As the right hand side of (2.1) is the same as the left hand side of (2.2), by the transitivity of $\trianglerighteq^{\text{o}}$, $\frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\psi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\psi \trianglerighteq^{\text{o}} \frac{1}{1+\bar{\delta}}\bar{\delta}\alpha'\phi + \frac{\bar{\delta}}{1+\bar{\delta}}\alpha'\phi$ which is the same as $\frac{2\bar{\delta}}{1+\bar{\delta}}\alpha'\psi \trianglerighteq^{\text{o}} \frac{2\bar{\delta}}{1+\bar{\delta}}\alpha'\phi$. By the definition of $\bar{\delta}$, it must be that $\bar{\delta} \geq \frac{2\bar{\delta}}{1+\bar{\delta}}$, so $\bar{\delta}(1+\bar{\delta}) \geq 2\bar{\delta}$, so $\bar{\delta}^2 - \bar{\delta} \geq 0$. However, since $\bar{\delta} > 0$ requires this inequality to hold only when $\bar{\delta} = 1$ and $\bar{\delta}\alpha'\psi \trianglerighteq^{\text{o}} \bar{\delta}\alpha'\phi$, delivering the desired contradiction. ■

Define $\trianglerighteq^{\Sigma}$ on $B_0(\Sigma)$ as follows: for all $\psi, \phi \in B_0(\Sigma)$, $\psi \trianglerighteq^{\Sigma} \phi$ if and only if $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$ for some (all) $\alpha > 0$ with $\alpha\psi, \alpha\phi \in B_0(\Sigma, K - k_0)$. We observe that, by Claim 6: $\psi, \phi \in B_0(\Sigma, K - k_0)$ implies $\psi \trianglerighteq^{\Sigma} \phi$ if and only if $\psi \trianglerighteq^{\text{o}} \phi$.

Claim 7 $\trianglerighteq^{\Sigma}$ *defined on* $B_0(\Sigma)$ *is nontrivial, continuous, conic and monotonic preorder.*

Proof. By following similar steps presented in the proof of reflexivity and transitivity of $\trianglerighteq^{\text{o}}$ of Claim 5, it is easy to establish that $\trianglerighteq^{\Sigma}$ is reflexive and transitive.

In order to show that $\trianglerighteq^{\Sigma}$ is continuous, take any sequence $\{\psi_n, \phi_n\}_{n\in\mathbb{N}}$ in $B_0(\Sigma) \times B_0(\Sigma)$ with $\psi_n \trianglerighteq^{\Sigma} \phi_n$ for all $n \in \mathbb{N}$ and $(\psi_n, \phi_n) \to (\psi, \phi) \in B_0(\Sigma) \times B_0(\Sigma)$. We need to show that $\psi \trianglerighteq^{\Sigma} \phi$. Now, $\psi_n \trianglerighteq^{\Sigma} \phi_n$ implies that there exists $\alpha_n > 0$ with $\alpha_n\psi_n \trianglerighteq^{\text{o}} \alpha_n\phi_n$ and $\alpha_n\psi_n, \alpha_n\phi_n \in B_0(\Sigma, K - k_0)$ for all $n \in \mathbb{N}$. As $\psi, \phi \in B_0(\Sigma)$, by the Archimedean Proporty, there exists $\alpha^* > 0$ such that $\alpha^*\psi, \alpha^*\phi \in B_0(\Sigma, K - k_0)$; so, and without loss of generality (by focusing on n sufficiently high) we may restrict attention to $\{\alpha_n\}$ living in a compact subset of $\mathbb{R}_+$ and converging to $\alpha^* > 0$. Then, by the Lebesgue Dominated Convergence Theorem (as $\psi_n, \psi, \phi_n, \phi$ are in $B_0(\Sigma)$), $\lim_{n\in\mathbb{N}} \alpha_n\psi_n = \alpha^*\psi$ and $\lim_{n\in\mathbb{N}} \alpha_n\phi_n = \alpha^*\phi$ and as $\alpha^* > 0$ and $\alpha^*\psi \trianglerighteq^{\text{o}} \alpha^*\phi$, we conclude that $\psi \trianglerighteq^{\Sigma} \phi$.

When attention is focused on $K = \mathbb{R}$, the definition of a conic preoder (as presented in the footnote 1 of Ghirardato, Maccheroni, and Marinacci (2002) and footnote 19 of Ghirardato, Maccheroni, and Marinacci (2004)) is as follows: $\trianglerighteq^{\Sigma}$ is conic if and only if $\psi \trianglerighteq^{\Sigma} \phi$ implies $\beta\psi + \theta \trianglerighteq^{\Sigma} \beta\phi + \theta$ for all $\beta \geq 0$ and $\theta \in B_0(\Sigma)$. For showing that $\trianglerighteq^{\Sigma}$ is conic, let $\psi \trianglerighteq^{\Sigma} \phi$ and $\beta \geq 0$ and $\theta \in B_0(\Sigma)$. For any $\theta \in B_0(\Sigma)$, by the Archimedean Proporty, there exists $\lambda \in (0, 1]$ such that $\lambda\theta \in B_0(\Sigma, K - k_0)$. As $\psi \trianglerighteq^{\Sigma} \phi$, it must be that for some (all) $\alpha > 0$ we have $\alpha\psi \trianglerighteq^{\text{o}} \alpha\phi$ and $\alpha\psi, \alpha\phi \in B_0(\Sigma, K - k_0)$. Since $\trianglerighteq^{\text{o}}$ is conic, for any $\lambda > 0$ with

$\lambda\,\theta \in B_0(\Sigma, K-k_0)$, it must be that for all $\mu \in [0,1]$, $(1-\mu)\alpha\psi + \mu\,\lambda\,\theta \trianglerighteq^{o} (1-\mu)\alpha\phi + \mu\,\lambda\,\theta$ (and note that $(1-\mu)\alpha\psi + \mu\,\lambda\,\theta, (1-\mu)\alpha\phi + \mu\,\lambda\,\theta \in B_0(\Sigma, K-k_0)$). Thus, $(1-\mu)\alpha\psi + \mu\,\lambda\,\theta = \mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\psi) + \mu\,\lambda\,\theta \trianglerighteq^{o} (1-\mu)\alpha\phi + \mu\,\lambda\,\theta = \mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\phi) + \mu\,\lambda\,\theta$ which is the same as $\mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\psi + \theta) \trianglerighteq^{o} \mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\phi + \theta)$. Furthermore, notice that $\frac{(1-\mu)\alpha}{\mu\,\lambda}\psi + \theta, \frac{(1-\mu)\alpha}{\mu\,\lambda}\phi + \theta \in B_0(\Sigma)$ and $\mu\,\lambda > 0$ whenever $\mu \in (0,1]$. Hence, $\mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\psi + \theta) \trianglerighteq^{o} \mu\,\lambda(\frac{(1-\mu)\alpha}{\mu\,\lambda}\phi + \theta)$ implies $\frac{(1-\mu)\alpha}{\mu\,\lambda}\psi + \theta \trianglerighteq^{\Sigma} \frac{(1-\mu)\alpha}{\mu\,\lambda}\phi + \theta$ for all $\mu \in (0,1]$, and $\lambda\,\theta \in B(\Sigma, K-k_0)$. Since $\mu, \lambda \in (0,1]$ and $\alpha > 0$, we have $\frac{(1-\mu)\alpha}{\mu\,\lambda} > 0$. Then, for any given $\beta > 0$ and $\alpha > 0$ and $\lambda > 0$ with $\lambda\,\theta \in B_0(\Sigma, K-k_0)$, by letting $\mu \in (0,1]$ (arbitrary small if needed) be such that $\beta = \frac{(1-\mu)\alpha}{\mu\,\lambda}$, we have $\beta\psi + \theta \trianglerighteq^{\Sigma} \beta\phi + \theta$.

In order to show that $\trianglerighteq^{\Sigma}$ is monotonic, suppose that $\psi(s) \geq \phi(s)$ for all $s \in S$ and $\psi, \phi \in B_0(\Sigma)$. Then, $\alpha\psi(s) \geq \alpha\phi(s)$ for all $\alpha > 0$ and $s \in S$. There exist $\lambda_1, \lambda_2 \in (0,1]$ such that $\lambda_1\,\psi \in B_0(\Sigma, K-k_0)$ and $\lambda_2\,\phi = \phi^* \in B_0(\Sigma, K-k_0)$. Without loss of generality, suppose that $\lambda_1 = \min\{\lambda_1, \lambda_2\}$; then, $\lambda_1\,\phi \in B_0(\Sigma, K-k_0)$ as we have shown in the proof of the Claim 6. As $\psi(s) \geq \phi(s)$ for all $s \in S$, it must be that $\lambda_1\,\psi(s) \geq \lambda_1\,\phi(s)$ for all $s \in S$ from which we can obtain $\lambda_1\,\psi \trianglerighteq^{o} \lambda_1\,\phi$, since $\trianglerighteq^{o}$ is monotone. Hence, $\lambda_1\,\psi \trianglerighteq^{o} \lambda_1\,\phi$ for $\lambda_1 > 0$ with $\lambda_1\,\phi, \lambda_1\,\psi \in B_0(\Sigma, K-k_0)$ implies $\psi \trianglerighteq^{\Sigma} \phi$.

As noted in the paragraph before the statement of the current Lemma, $\psi \trianglerighteq^{\Sigma} \phi$ if and only if $\psi \trianglerighteq^{o} \phi$ whenever $\psi, \phi \in B_0(\Sigma)$ are such that $\psi, \phi \in B_0(\Sigma, K-k_0)$. Thus, nontriviality of $\trianglerighteq^{\Sigma}$ follows trivially from nontriviality of $\trianglerighteq^{o}$: By the nontriviality of $\trianglerighteq^{o}$, there are $\psi, \phi \in B_0(\Sigma, K-k_0)$ such that $\psi \trianglerighteq^{o} \phi$ and not $\phi \trianglerighteq^{o} \psi$; so, since the interval $K-k_0$ is trivially contained in $\mathbb{R}$, this means there are $\psi, \phi \in B_0(\Sigma)$ such that $\psi \trianglerighteq^{\Sigma} \phi$ and not $\phi \trianglerighteq^{\Sigma} \psi$. ■

$B_0(\Sigma)$, the set of all bounded Σ–measurable real valued simple functions, can equivalently be viewed as the vector space generated by the simple functions on Σ. Then, $B(\Sigma)$, the closure of $B_0(\Sigma)$ with the supnorm, also called the uniform norm, is a subset of the set of all bounded functions on the state space S which is known to be a Banach space, a complete normed vector space. Also, $B_0(\Sigma)$ is dense in $B(\Sigma)$. Now, we let $ba(\Sigma)$ denote the set of all bounded, finitely additive set functions on Σ (i.e. signed measures on Σ) and it is known that $ba(\Sigma)$ is also a Banach space equipped with the variation norm. Moreover, $pc(\Sigma)$ denotes the set of all probability measures in $ba(\Sigma)$ and is a convex subset of a Banach space $ba(\Sigma)$ (with the variation norm). Due to the duality of Banach spaces, $ba(\Sigma)$ is isometrically isomorphic to $B(\Sigma)$ (and also to $B_0(\Sigma)$ as it is dense in $B(\Sigma)$.[9] This result enables us to view a measure (in $ba(\Sigma)$) as a linear functional on measurable functions (mapping $B_0(\Sigma)$ into $\mathbb{R}$); so, one can define the integral using a finitely additive measure. Thus, for any $L \in ba(\Sigma)$ and $\psi, \phi \in B_0(\Sigma)$ it must be that $L(\psi) \geq L(\phi)$ if and only if $\int_S \psi \mathrm{d}L \geq \int_S \phi \mathrm{d}L$. This is also what will done in our construction of $\mathcal{L}$, the set of all non-negative measures on Σ with the property that for all $\psi \in B_0(\Sigma)$ it must be that $\psi \trianglerighteq^{\Sigma} \mathbf{0}$, which is defined as follows:

$$\mathcal{L} = \{L \in ba(\Sigma) : L(\psi) \geq 0, \text{ for all } \psi \text{ with } \psi \trianglerighteq^{\Sigma} \mathbf{0}\}.$$

Trivially, $\mathbf{0} \in \mathcal{L}$ and $\mathcal{L}$ is a convex cone.[10]

In order to use the Hahn-Banach (Hyperplane Separation) Theorem, we need to show that $\mathcal{L} \subset ba(\Sigma)$ is a Banach space with the variation norm. To that regard, we show that $\mathcal{L}$ is closed in the topology $\tau(ba(\Sigma), B_0(\Sigma))$ as $\tau(ba(\Sigma), B_0(\Sigma))$ coincides with $\tau(ba(\Sigma), B(\Sigma))$ in the weak* topology. Let $\{L_\alpha\}$ be a net in $\mathcal{L}$ with limit L in the weak* topology. That is, for all $\psi \in B_0(\Sigma)$ we have $L_\alpha(\psi)$ converging to $L(\psi)$ (in real numbers). Thus, if $\psi \trianglerighteq^{\Sigma} \mathbf{0}$ and as $L_\alpha \in \mathcal{L}$ for all α it must be that $L_\alpha(\psi) \geq 0$ for all α; therefore, $L(\psi) \geq 0$. Hence, $L \in \mathcal{L}$.

Claim 8 *$\psi \trianglerighteq^{\Sigma} \phi$ if and only if $L(\psi) \geq L(\phi)$ for all $L \in \mathcal{L} \setminus \{\mathbf{0}\}$.*

Proof. For necessity, notice that for any $\psi, \phi \in B_0(\Sigma)$, $\psi \trianglerighteq^{\Sigma} \phi$ implies, as $\trianglerighteq^{\Sigma}$ is conic and $(-\phi) \in B_0(\Sigma)$, $\psi - \phi \trianglerighteq^{\Sigma} \mathbf{0}$; thus, for all $L \in \mathcal{L}$, it must be that $L(\psi - \phi) \geq 0$ which trivially implies that for all $L \in \mathcal{L} \setminus \{\mathbf{0}\}$, it must be that $L(\psi - \phi) \geq 0$. As any such L is a finitely additive set function, $L(\psi - \phi) = L(\psi) - L(\phi) \geq 0$ implying $L(\psi) \geq L(\phi)$.

For sufficiency, suppose that $L(\psi) \geq L(\phi)$ for all $\mathcal{L} \setminus \{\mathbf{0}\}$ but not $\psi \trianglerighteq^{\Sigma} \phi$. Then, $\zeta = \psi - \phi \in B_0(\Sigma)$ but not $\zeta \trianglerighteq^{\Sigma} \mathbf{0}$. Therefore, by Hahn-Banach (Hyperplane Separation) Theorem, there

[9] A space is isometrically isomorphic to another space if there is a distance preserving continuous function with a continuous inverse.

[10] $\mathcal{L}$ is a convex cone if for all $\alpha, \beta \geq 0$ and $L, L' \in \mathcal{L}$, $\alpha L + \beta L' \in \mathcal{L}$.

exists $L' \in ba(\Sigma)$ such that $L'(\theta) \geq 0 > L'(\zeta)$ for all $\theta \in B_0(\Sigma)$ with $\theta \trianglerighteq^{\Sigma} \mathbf{0}$. Because that $L' \in ba(\Sigma)$ is such that $L'(\theta) \geq 0$ for all $\theta \in B_0(\Sigma)$ with $\theta \trianglerighteq^{\Sigma} \mathbf{0}$, $L' \in \mathcal{L}$. Thus, $L'(\zeta) = L'(\psi - \phi) < 0$, as not $\psi \trianglerighteq^{\Sigma} \phi$, implies $L' \neq \mathbf{0}$ and (due to the finite additivity of L') $L'(\psi) < L'(\phi)$ contradicting to the hypothesis of $L(\psi) \geq L(\phi)$ for all $\mathcal{L} \setminus \{\mathbf{0}\}$. ■

Since L is a bounded, finitely additive set function on Σ, the Claim 8 implies

$$\psi \trianglerighteq^{\Sigma} \phi \Leftrightarrow \frac{L(\psi)}{L(S)} \geq \frac{L(\phi)}{L(S)} \text{ for all } L \in \mathcal{L} \setminus \{\mathbf{0}\}.$$

We define $\mathcal{C}$ as $\mathcal{C} = \mathcal{L} \cap pc(\Sigma)$, then $\mathcal{C}$ is the set of all probability charges in $\mathcal{L}$. Moreover, notice that $\mathcal{C}$ is weak* closed and convex since $\mathcal{L}$ and $pc(\Sigma)$ are weak* closed and convex sets. Hence, we conclude that

$$\psi \trianglerighteq^{\Sigma} \phi \Leftrightarrow P(\psi) \geq P(\phi) \text{ for all } P \in \mathcal{C}.$$

Furthermore, observe that for any $\psi, \phi \in B_0(\Sigma, K)$ we have $\psi - k_0, \phi - k_0 \in B_0(\Sigma, K - k_0)$ and $\psi - k_0, \phi - k_0 \in B_0(\Sigma)$. Hence, for any $\psi, \phi \in B_0(\Sigma, K)$ the following must be true:

$$\psi \trianglerighteq \phi \Leftrightarrow \psi - k_0 \trianglerighteq^{\mathrm{o}} \phi - k_0 \Leftrightarrow \psi - k_0 \trianglerighteq^{\Sigma} \phi - k_0 \Leftrightarrow P(\psi - k_0) \geq P(\phi - k_0) \text{ for all } P \in \mathcal{C}$$
$$\Leftrightarrow P(\psi) \geq P(\phi) \text{ for all } P \in \mathcal{C} \Leftrightarrow \int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P \text{ for all } P \in \mathcal{C}.$$

In turn, this concludes the proof of Lemma 7. ■

Lemma 8 *For $i = 1, 2$, let $\mathcal{C}_i$ be non-empty sets of probabilities on Σ and binary relations $\trianglerighteq_i \subseteq B_0(\Sigma, K) \times B_0(\Sigma, K)$ be defined by $\psi \trianglerighteq_i \phi$ if and only if $\int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P$ for all $P \in \mathcal{C}_i$. Then,*

$$\psi \trianglerighteq_i \phi \textit{ if and only if } \int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P \textit{ for all } P \in \overline{co}^{w^*}(\mathcal{C}_i).^{11}$$

Furthermore, the following statements are equivalent:

[11] For a set Y, $\overline{co}^{w^*}(Y)$ is the closure of the convex hull of the set Y in weak* topology.

(*i*) *For all* $\psi, \phi \in B_0(\Sigma, K)$*,* $\psi \trianglerighteq_1 \phi$ *implies* $\psi \trianglerighteq_2 \phi$*;*

(*ii*) $\overline{co}^{w^*}(\mathcal{C}_2) \subseteq \overline{co}^{w^*}(\mathcal{C}_1)$.

Proof. For any $\psi, \phi \in B_0(\Sigma, K)$ and $i \in \{1, 2\}$, by Lemma 7, $\psi \trianglerighteq_i \phi$ implies there exists a convex and weak* closed nonempty $\mathcal{C}_i$ with $\int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P$ for all $P \in \mathcal{C}_i$. Notice that due to these $\mathcal{C}_i = \overline{co}^{w^*}(\mathcal{C}_i)$. Now, for any $P \in \mathcal{C}_i$, there exists a net $\{P_\alpha\} \subset \mathcal{C}_i$ such that $P_\alpha(\psi) \to P(\psi)$ for all $\psi \in B_0(\Sigma)$. Therefore, if $\psi \trianglerighteq_i \phi$ for any $\psi, \phi \in B_0(\Sigma)$, then (by Lemma 7) it must be $P_\alpha(\psi) \geq P_\alpha(\phi)$ for all α and for all $P_\alpha \in \mathcal{C}_i$; thus, $\psi \trianglerighteq_i \phi$ implies $P(\psi) \geq P(\phi)$ (alternatively, $\int_S \psi \mathrm{d}P \geq \int_S \phi \mathrm{d}P$). As $P \in \mathcal{C}_i$ is arbitrary, the proof of the necessity concludes. For the converse direction, if $P(\psi) \geq P(\phi)$ for all $P \in \overline{co}^{w^*}(\mathcal{C}_i)$; then, $P'(\psi) \geq P'(\phi)$ for all $P' \in \mathcal{C}_i$ since $\mathcal{C}_i \subseteq \overline{co}^{w^*}(\mathcal{C}_i)$; hence, by Lemma 7, $\psi \trianglerighteq_i \phi$.

In what follows, we will show that (*i*) implies (*ii*). First, for a contradiction suppose that $\psi \trianglerighteq_1 \phi$ implies $\psi \trianglerighteq_2 \phi$ for any $\psi, \phi \in B_0(\Sigma)$, but $\overline{co}^{w^*}(\mathcal{C}_2) \nsubseteq \overline{co}^{w^*}(\mathcal{C}_1)$; so $\mathcal{C}_2 \nsubseteq \mathcal{C}_1$. Thus, there is $\tilde{P} \in \mathcal{C}_2 \setminus \mathcal{C}_1$.

Let us define the set $M_1 = \{\alpha P : \alpha \geq 0, P \in \mathcal{C}_1\}$. Trivially, (by setting $\alpha = 1$), $\mathcal{C}_1 \subset M_1$. Notice that M_1 is a convex cone since for any $\alpha P, \alpha' P' \in M_1$, if $\alpha = \alpha' = 0$, then $\beta \alpha P + \beta' \alpha P' = \mathbf{0} \in M_1$ for all $\beta, \beta' \in \mathbb{R}_+$; and if $\beta = \beta' = 0$, then $\beta \alpha P + \beta' \alpha P' = \mathbf{0} \in M_1$ for all $\alpha, \alpha' \in \mathbb{R}_+$; lastly, without loss of generality, suppose that $\alpha > 0$, $\alpha' \geq 0$, $\beta > 0$ and $\beta' \geq 0$, then $\beta \alpha P + \beta' \alpha' P' = \beta\alpha + \beta'\alpha'(\frac{\beta\alpha}{\beta\alpha+\beta'\alpha'} P + \frac{\beta'\alpha'}{\beta\alpha+\beta'\alpha'} P') \in M_1$ since $\beta\alpha + \beta'\alpha' > 0$ and $(\frac{\beta\alpha}{\beta\alpha+\beta'\alpha'} P + \frac{\beta'\alpha'}{\beta\alpha+\beta'\alpha'} P') \in \mathcal{C}_1$ (as $\mathcal{C}_1$ is weak* closed and convex). Next, we will show that M_1 is weak* closed. Suppose that a net $\{\beta_\alpha P_\alpha\} \subset M_1$ with $\{\beta_\alpha P_\alpha\} \xrightarrow{w*} L$. We need to show that $L \in M_1$. As $\beta_\alpha P_\alpha \in M_1$, $\beta_\alpha \geq 0$ and $P_\alpha \in \mathcal{C}_1$ for all α. If $L = \mathbf{0}$, then $L \in M_1$. So suppose $L \neq \mathbf{0}$. Then, by the Archimedean Property, $L = \beta P$ for some $\beta > 0$ and $P \in pc(\Sigma)$. So, $\beta_\alpha P_\alpha(S)$ converging to $\beta P(S)$, implies β_α converges to β. Hence, $P_\alpha = \frac{1}{\beta_\alpha}(\beta_\alpha P_\alpha) \xrightarrow{w*} \frac{1}{\beta}(\beta P) = P$ from which we can infer as $\{P_\alpha\} \subset \mathcal{C}_1$ and $\mathcal{C}_1$ is weak* closed, $P \in \mathcal{C}_1$ which implies $L = \beta P \in M_1$.

Furthermore, notice that $\tilde{P} \notin \mathcal{C}_1 \subset M_1$. As M_1 is a convex weak* closed cone (implying that is a Banach space), by the Hahn-Banach (Hyperplane Separation) Theorem, there exists

$\psi \in B_0(\Sigma) \setminus \mathbf{0}$ such that:

$$L(\psi) \geq 0 > \tilde{P}(\psi) \text{ for all } L \in M_1 \tag{2.3}$$

As K is a non-singleton interval there exist $k_0 \in K^{\mathrm{o}}$ and $\mu > 0$ such that $\mu\psi + k_0 \in B_0(\Sigma, K)$. Furthermore, notice that (2.3) implies $P(\psi) \geq 0$ for all $P \in \mathcal{C}_1$ as $\mathcal{C}_1 \subset M_1$. Thus, by the construction presented in the proof of Lemma 7, the following holds:

$$\begin{aligned}
P(\psi) \geq 0 \text{ for all } P \in \mathcal{C}_1 \;&\Leftrightarrow\; \alpha P(\psi) \geq 0 \text{ for all } P \in \mathcal{C}_1 \\
&\Leftrightarrow\; P(\alpha\psi) \geq 0 \text{ for all } P \in \mathcal{C}_1 \\
&\Leftrightarrow\; P(\alpha\psi) + k_0 \geq k_0 \text{ for all } P \in \mathcal{C}_1 \\
&\Leftrightarrow\; P(\alpha\psi + k_0) \geq k_0 \text{ for all } P \in \mathcal{C}_1 \\
&\Leftrightarrow\; \alpha\psi + k_0 \trianglerighteq_1 k_0.
\end{aligned}$$

On the other hand, $0 > \tilde{P}(\psi)$, (2.3) implies $k_0 > \tilde{P}(\alpha\psi + k_0)$, hence, not $\alpha\psi + k_0 \trianglerighteq_2 k_0$ since $\tilde{P} \in \mathcal{C}_2$; in turn, delivering the desired contradiction. Thus, (i) implies (ii).

In order to show that (ii) implies (i), suppose that $\overline{co}^{w^*}(\mathcal{C}_2) \subseteq \overline{co}^{w^*}(\mathcal{C}_1)$ and $\psi \trianglerighteq_1 \phi$ for any $\psi, \phi \in B_0(\Sigma)$. Then, $P(\psi) \geq P(\phi)$ for all $P \in \overline{co}^{w^*}(\mathcal{C}_1)$ which implies $P(\psi) \geq P(\phi)$ for all $P \in \overline{co}^{w^*}(\mathcal{C}_2) \subseteq \overline{co}^{w^*}(\mathcal{C}_1)$; and hence, $\psi \trianglerighteq_2 \phi$. ■

$f \succsim^{UA} g$ for any f, g if and only if $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for any $\lambda \in (0,1]$ and $m \in \mathcal{F}$. Then, by Theorem 1, $I(u_{\lambda f+(1-\lambda)m}) \geq I(u_{\lambda g+(1-\lambda)m})$ which can be written as $I(\lambda u_f + (1-\lambda)u_m) \geq I(\lambda u_g + (1-\lambda)u_m)$ due to Lemma 3.

Define a binary relation $\trianglerighteq^{UA}$ on $B_0(\Sigma, K)$ with $K = u(\mathbb{X})$ such that for any $f, g \in \mathcal{F}$, $u_f \trianglerighteq^{UA} u_g$ if and only if $I(\lambda u_f + (1-\lambda)u_m) \geq I(\lambda u_g + (1-\lambda)u_m)$ for all $m \in \mathcal{F}$ and $\lambda \in (0,1]$. Therefore, $f \succsim^{UA} g$ if and only if $u_f \trianglerighteq^{UA} u_g$.

Claim 9 $\trianglerighteq^{UA} \subseteq B_0(\Sigma, K) \times B_0(\Sigma, K)$ *nontrivial, continuous, conic and monotonic preorder.*

Proof. Reflexivity and transitivity of $\succsim^{UA}$ established in Lemma 6 implies trivially that $\trianglerighteq^{UA}$ is reflexive and transitive.

By Claim 3 there exists $\bar{\ell}, \underline{\ell} \in \mathbb{X}$ such that $u(\bar{\ell}) > 1$ and $u(\underline{\ell}) < -1$. Let $\bar{h}, \underline{h} \in \mathcal{F}_c$ be two acts with $\bar{h}(s) = \bar{\ell}$ and $\underline{h}(s) = \underline{\ell}$ for all $s \in S$. Clearly $\bar{h} \succ \underline{h}$, and then by (ii) of Lemma 6 we have $\bar{h} \succ^{UA} \underline{h}$ which is equivalent to $\bar{h} \succsim^{UA} \underline{h}$ while not $\underline{h} \succsim^{UA} \bar{h}$. Hence, for resulting $u_{\bar{h}}, u_{\underline{h}} \in B(\Sigma, K)$ we have $u_{\bar{h}} \trianglerighteq^{UA} u_{\underline{h}}$ while not $u_{\underline{h}} \trianglerighteq^{UA} u_{\bar{h}}$ due to the definition of $\trianglerighteq^{UA}$ which shows that it is nontrivial.

Let $f, g \in \mathcal{F}$ with $f(s) = \ell_{f(s)}$ and $g(s) = \ell_{g(s)}$ be two acts such that resulting $u_f, u_g \in B_0(\Sigma, K)$ satisfies $u_f(s) \geq u_g(s)$ for all $s \in S$. Then, $u(\ell_{f(s)}) \geq u(\ell_{g(s)})$ which implies $\ell_{f(s)} \succsim^{\mathbb{X}} \ell_{g(s)}$ for all $s \in S$. Therefore, due to the (iv) of Lemma 6, we have $f \succsim^{UA} g$ which implies $u_f \trianglerighteq^{UA} u_g$, i.e. monotonicity.

In order to show that $\trianglerighteq^{UA}$ is conic suppose that we have $u_f \trianglerighteq^{UA} u_g$ for any $f, g \in \mathcal{F}$. Then, by the definition of $\trianglerighteq^{UA}$ it must be $f \succsim^{UA} g$ which implies $\lambda f + (1-\lambda)m \succsim^{UA} \lambda g + (1-\lambda)m$ for all $m \in \mathcal{F}$ and for all $\lambda \in (0,1]$ due to the (v) of Lemma 6. Therefore, it must be $u_{\lambda f+(1-\lambda)m} \trianglerighteq^{UA} u_{\lambda g+(1-\lambda)m}$ for all $\lambda \in (0,1]$. By (ii) of Lemma 3, we know that $u_{\lambda f+(1-\lambda)m} = \lambda u_f + (1-\lambda)u_m$ and $u_{\lambda g+(1-\lambda)m} = \lambda u_g + (1-\lambda)u_m$. Thus, $u_{\lambda f+(1-\lambda)m} \trianglerighteq^{UA} u_{\lambda g+(1-\lambda)m}$ implies $\lambda u_f + (1-\lambda)u_m \trianglerighteq^{UA} \lambda u_g + (1-\lambda)u_m$ for all $\lambda \in (0,1]$. Clearly, if $\lambda = 0$ then the previous observation holds trivial.

For the continuity of $\trianglerighteq^{UA}$ suppose that we have $u_{f_n} \trianglerighteq^{UA} u_{g_n}$ for all $n \in \mathbb{N}$ with $u_{f_n} \xrightarrow{sup} u_f$ and $u_{g_n} \xrightarrow{sup} u_g$ then $I(\lambda u_{f_n} + (1-\lambda)u_m) \geq I(\lambda u_{g_n} + (1-\lambda)u_m)$ for all $u_m \in B(\Sigma, K)$, $\lambda \in (0,1]$ and $n \in \mathbb{N}$. [12] As I is continuous under the supnorm convergence (in the sense that when $\psi_n \xrightarrow{sup} \psi$ for $\psi, \psi_n \in B_0(\Sigma)$, then $\{I(\psi_n)\} \subset \mathbb{R}$ converges to $I(\psi)$), we have $I(\lambda u_f + (1-\lambda)u_m) \geq I(\lambda u_g + (1-\lambda)u_m)$ for all $u_m \in B(\Sigma, K)$, $\lambda \in (0,1]$ which implies $u_f \trianglerighteq^{UA} u_g$. ■

As we have shown that $\trianglerighteq^{UA}$ nontrivial, continuous, conic and monotonic preorder on $B_0(\Sigma, K)$, by Lemma 7 and Lemma 8 there exists a unique nonempty, weak* compact and convex set of $\mathcal{C}$ of probabilities on Σ such that for all $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ if and only if $\int_S u_f \mathrm{d}P \geq \int_S u_g \mathrm{d}P$ for all $P \in \mathcal{C}$. Therefore, the proof of Theorem 2 is concluded. ■

The following analysis concerning the sure-thing principle is of independent interest and

[12] For recalling the definition of the supnorm continuity, please check footnote 8.

will not be employed in our results. Savage (1954) introduced the sure-thing principle which, in words, states that an agent who would take a certain action when he/she knew whether event E is achieved or not, should take the same action even if he/she does not know anything about event E. Formally, for any $f, m \in \mathcal{F}$ and $E \in \Sigma$ define fEm by

$$fEm = \begin{cases} f(s) & \text{if} \quad s \in E \\ m(s) & \text{if} \quad s \notin E. \end{cases}$$

Then, $\succsim^{UA}$ satisfies the sure-thing principle if for all $f, g, m, m' \in \mathcal{F}$ and $E \in \Sigma$, $fEm \succsim^{UA} gEm$ if and only if $fEm' \succsim^{UA} gEm'$.

Lemma 9 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and $\succsim^{UA}$ on $\mathcal{F}$ is as defined above. Then, $\succsim^{UA}$ satisfies the sure-thing principle.*

Proof. We know that for all $f, g, m, m' \in \mathcal{F}$ and $E \in \Sigma$ if $fEm \succsim^{US} gEm$ then it must be $\int_S u_{fEm} dP \geq \int_S u_{gEm} dP$ for all $P \in \mathcal{C}$ by Theorem 2. Then, we can write that inequality as $\int_{s \in E} u_f dP + \int_{s \notin E} u_m dP \geq \int_{s \in E} u_g dP + \int_{s \notin E} u_m dP$ for all $P \in \mathcal{C}$. Therefore, the following observation must be true for all $f, g, m, m' \in \mathcal{F}$ and $E \in \Sigma$:

$$\begin{aligned} fEm \succsim^{UA} gEm &\Leftrightarrow \int_{s \in E} u_f dP + \int_{s \notin E} u_m dP \geq \int_{s \in E} u_g dP + \int_{s \notin E} u_m dP \text{ for all } P \in \mathcal{C} \\ &\Leftrightarrow \int_{s \in E} u_f dP + \int_{s \notin E} u_{m'} dP \geq \int_{s \in E} u_g dP + \int_{s \notin E} u_{m'} dP \text{ for all } P \in \mathcal{C} \\ &\Leftrightarrow fEm' \succsim^{UA} gEm'. \end{aligned}$$

■

The next Theorem establishes an important implication that will be used in our main results.

Theorem 3 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and $\succsim^{UA}$ on $\mathcal{F}$ is as defined above. Then, for all $f, g \in \mathcal{F}$, $f \succsim_1^{UA} g$ implies $f \succsim_2^{UA} g$ if and only if u_1 is a positive affine transformation of u_2 and $\mathcal{C}_2 \subseteq \mathcal{C}_1$.*

Proof. For the proof of this Theorem, we will first introduce the following Lemma:

Lemma 10 *Suppose that Z is a vector space and u, v are two nonzero linear functionals on Z. One and only one of the following statements is true:*

(i) $u(z) = av(z)$ *for some* $a > 0$ *and for any* $z \in Z$*;*

(ii) *There exists* $z \in Z$ *such that* $u(z)v(z) < 0$.

Proof. If (i) is true, then clearly $u(z)$ and $v(z)$ must have the same sign for any $z \in Z$; therefore, (ii) can not be true. If (ii) is true, then $u(z)$ and $v(z)$ has opposite signs for a $z \in Z$ and for any $z' \in Z$, $u(z') = av(z')$ is possible only when $a < 0$; so, (i) cannot be true. Therefore, (i) and (ii) cannot hold at the same time.

Next, we will show that only one of them must hold. For a contradiction, suppose that both of them are false. Then, it must be $u(z) \neq av(z)$ for all $a > 0$ and for all $z \in Z$ and $u(z)v(z) \geq 0$ for all $z \in Z$. Therefore, $Z = \{z \in Z : u(z)v(z) > 0\} \cup \{z \in Z : u(z) = 0\} \cup \{z \in Z : v(z) = 0\}$; and observe that $\{z \in Z : u(z) = 0\}$ and $\{z \in Z : v(z) = 0\}$ are equivalent to the $\ker u$ and $\ker v$, respectively; hence, $Z = \{z \in Z : u(z)v(z) > 0\} \cup \ker u \cup \ker v$. $\ker u$ and $\ker v$ are maximal subspaces of Z[13]. To see this, suppose that Z' with $\ker u \subsetneq Z' \subset Z$ is a subspace of Z and $z' \in Z' \setminus \ker u$, thus, $u(z') \neq 0$. Any arbitrary $z \in Z$ can be written as $z = \frac{u(z)}{u(z')}z' + (z - \frac{u(z)}{u(z')}z')$. Furthermore, notice that $u(z - \frac{u(z)}{u(z')}z') = u(z) - u(\frac{u(z)}{u(z')}z') = u(z) - \frac{u(z)}{u(z')}u(z') = u(z) - u(z) = 0$ since u is linear functional; hence, $z - \frac{u(z)}{u(z')}z' \in \ker u$. Due to the fact that each subspace is also a vector space, then Z' is endowed with the same operations as Z; so for any $z \in Z'$ and $c \in \mathbb{R}$ we have $cz \in Z'$; and if $z', z'' \in Z'$, then it must be $z' + z'' \in Z'$. Therefore, $\frac{u(z)}{u(z')}z' \in Z'$ (as $z' \in Z' \setminus \ker u$ and $\frac{u(z)}{u(z')}$ is a real number); and $z = \frac{u(z)}{u(z')}z' + (z - \frac{u(z)}{u(z')}z') \in Z$ (as Z' also contains $\ker u$). Thus, since the $z \in Z$ was chosen arbitrarily, $Z \subseteq Z'$ which, together with $Z' \subseteq Z$, implies $Z' = Z$, so $\ker u$ is a maximal subspace of Z. Similar steps can be followed to see that $\ker v$ is also maximal subspace of Z.

[13] A supspace V is called a maximal subspace of a vector space Z if for any subspace W with $V \subsetneq W \subseteq Z$, then $W = Z$.

First, suppose that $\ker u = \ker v$. Since u is non-zero linear functionals on Z, then there exist a $z' \in Z$ such that $u(z') \neq 0$. Let $z_1 \in Z$ be such that $z_1 = \frac{z'}{u(z')}$, and observe that $u(z_1) = u(\frac{z'}{u(z')}) = \frac{u(z')}{u(z')} = 1$. Therefore, for all $z \in Z$ the following must be true: $u(z - u(z)z_1) = u(z) - u(u(z)z_1) = u(z) - u(z)u(z_1) = 0$ since u is linear functional. Then, $z - u(z)z_1 \in \ker u$ which implies $z - u(z)z_1 \in \ker v$ and, hence, $v(z - u(z)z_1) = 0$ (as $\ker u = \ker v$). Since v is also linear functional, $v(z - u(z)z_1) + v(u(z)z_1) = v(z) - v(u(z)z_1) + v(u(z)z_1) = v(z)$ for all $z \in Z$. Furthermore, notice that $v(z - u(z)z_1) + v(u(z)z_1) = v(u(z)z_1) = v(u(z))v(z_1) = u(z)v(z_1)$ (as $v(z - u(z)z_1) = 0$) from which we can obtain $v(z) = v(z_1)u(z)$. However, this contradicts with $u(z) \neq av(z)$ for all $a > 0$ and $u(z)v(z) \geq 0$ for all $z \in Z$.

Secondly, suppose that $\ker u \neq \ker v$. Then, let $z_u \in \ker u \setminus \ker v$ and $z_v \in \ker v \setminus \ker u$ (as $\ker u$ and $\ker v$ are maximal subspace of z, it can not be $\ker u \subset \ker v$ or $\ker v \subset \ker u$) be such that $u(z_u) = 0$ and $v(z_u) > 0$; and $v(z_v) = 0$ and $u(z_v) < 0$. Thus, $u(z_u + z_v)v(z_u + v_u) = u(z_v)v(z_u) < 0$ which contradicts with $u(z)v(z) \geq 0$ for all $z \in Z$. ■

Lemma 11 *Suppose that X is a nonempty convex subset of a vector space Z and u, v are two non-constant affine functionals on Z. Then, there exist $a > 0$ and $b \in \mathbb{R}$ such that $u = av + b$ if and only if $u(x) \geq u(x')$ implies $v(x) \geq v(x')$ for all $x, x' \in \mathbb{X}$.*

Proof. For the only if part of the proof, suppose that $u = av + b$ for some $a > 0$ and $b \in \mathbb{R}$, so for any $x, x' \in X$ we have $u(x) = av(x) + b$ and $u(x') = av(x') + b$. If $u(x) \geq u(\ell')$, then $av(x) + b \geq av(x') + b$. Therefore, $v(x) \geq v(x')$.

For the if part of the proof, let $W = \{t(x_1 - x_2) : t > 0, x_1, x_2 \in X\}$. Then, W is vector space because vector addition and scalar multiplication operations are defined on W. To see that let $w, w' \in W$ be such that $w = t(x_1 - x_2)$ and $w' = (x_3 - x_4)$ where $x_1, x_2, x_3, x_4 \in X$ and $t > 0$, then $w + w' = t(x_1 + x_3 - x_2 - x_4)$. Since $x_1 + x_3, x_2 + x_4 \in X$ (as X is a vector space), $w + w' = t((x_1 + x_3) - (x_2 + x_4)) \in W$. Furthermore, let $w^* \in W$ be such that $w^* = t(x_1 - x_2)$ where $x_1, x_2 \in X$ and $t > 0$; then, $cw^* = ct(x_1 - x_2) = t(cx_1 - cx_2)$ and since $cx_1, cx_2 \in X$, we have $cw^* \in W$.

Now, define functionals $\tilde{u}, \tilde{v}$ on W as follows: $\tilde{u}(t(x_1 - x_2)) = t(u(x_1) - u(x_2))$; and $\tilde{v}(t(x_1-x_2)) = t(v(x_1)-v(x_2))$. Notice that $\tilde{u}$ and $\tilde{v}$ are well-defined, non-empty and linear on W (as u and v are non-empty, linear functionals). If, for any $t(x_1-x_2) \in W$ with $x_1, x_2 \in X$ and $t > 0$ $\tilde{u}(t(x_1 - x_2)) = t(u(x_1) - u(x_2)) = tu(x_1) - tu(x_2) \geq 0$, then $u(x_1) \geq u(x_2)$ which implies $v(x_1) \geq v(x_2)$; so $\tilde{v}(t(x_1 - x_2)) \geq 0$. Hence, there is no $w \in W$ such that $\tilde{u}(w)\tilde{v}(w) < 0$. Then, due to the Lemma 10, there is $a > 0$ such that $\tilde{u} = a\tilde{v}$. Next, we will fix $x_0 \in X$; so that for all $x \in X$ we have $\tilde{u}(t(x - x_0)) = t(u(x) - u(x_0)) = at(v(x) - v(x_0))$ which implies $(u(x) - u(x_0)) = a(v(x) - v(x_0))$, and $u(x) = av(x) - av(x_0) + u(x_0)$. By setting $b = u(x_0) - av(x_0)$, we can obtain $u(x) = av(x) + b$. ■

For the proof of Theorem 3, first, suppose that $u_1 = au_2 + b$, where $a > 0$ and $b \in \mathbb{R}$, and $\mathcal{C}_2 \subseteq \mathcal{C}_1$; and $f \succsim_1 g$. Then, due to the Theorem 2, $\int_S u_{1_f} \mathrm{d}P \geq \int_S u_{1_g} \mathrm{d}P$ for all $P \in \mathcal{C}_1$. If $u_1 = au_2 + b$, where $a > 0$ and $b \in \mathbb{R}$; then, we have $\int_S (au_{2_f} + b)\mathrm{d}P = a\int_S u_{2_f}\mathrm{d}P + b \geq a\int_S u_{2_g}\mathrm{d}P + b = \int_S (au_{2_g} + b)\mathrm{d}P$ for all $P \in \mathcal{C}_1$. Since $\mathcal{C}_2 \subseteq \mathcal{C}_1$, we can observe that $\int_S u_{2_f}\mathrm{d}P \geq a\int_S u_{2_g}\mathrm{d}P$ for all $P \in \mathcal{C}_2$; so $f \succsim_2 g$.

For the converse direction, suppose that for any $f, g \in \mathcal{F}$ $f \succsim_1^{UA} g$ implies $f \succsim_2^{UA} g$. Let $\ell, \ell' \in \mathbb{X}$ such that $u_1(\ell) \geq u_1(\ell')$; then, by the Theorem 1 $\ell \succsim^{\mathbb{X}} \ell'$ which implies $h \succsim_1^{UA} h'$ where $h, h' \in \mathcal{F}_c$ are such that $h(s) = \ell$ and $h'(s) = \ell'$ for all $s \in S$ due to the (iv) of Lemma 6. Therefore, we have $h \succsim_2^{UA} h'$; and $u_2(\ell) \geq u_2(\ell')$. By the Lemma 11, we can conclude that u_1 is a positive affine transformation of u_2. Furthermore, if $f \succsim_1^{UA} g$ implies $f \succsim_2^{UA} g$ for any $f, g \in \mathcal{F}$, then by the Theorem 2, $\int_S u_{1_f}\mathrm{d}P = a\int_S u_{2_f}\mathrm{d}P + b \geq a\int_S u_{2_g}\mathrm{d}P + b = \int_S u_{1_g}\mathrm{d}P$ for all $P \in \mathcal{C}_1$ must imply $\int_S u_{2_f}\mathrm{d}P \geq \int_S u_{2_g}\mathrm{d}P$ for all $P \in \mathcal{C}_2$. After rearranging the terms, we can obtain $\int_S u_{2_f}\mathrm{d}P \geq \int_S u_{2_g}\mathrm{d}P$ for all $P \in \mathcal{C}_1$ implies $\int_S u_{2_f}\mathrm{d}P \geq \int_S u_{2_g}\mathrm{d}P$ for all $P \in \mathcal{C}_2$. Due to the Lemma 8, we have $\mathcal{C}_2 \subseteq \mathcal{C}_1$. ■

2.3 The representation theorem

The next result concerns the best and worst case scenarios and establishes that the "answer" must be somewhere in between.

Theorem 4 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and let $f \in \mathcal{F}$. Then, for the functional I and utility function u that we have obtained through Theorem 1 and the set of $\mathcal{C}$ of probabilities that we have obtained through Theorem 2, the following holds: $\max_{P\in\mathcal{C}} P(u_f) \geq I(u_f) \geq \min_{P\in\mathcal{C}} P(u_f)$, where $\max_{P\in\mathcal{C}} P(u_f)$ and $\min_{P\in\mathcal{C}} P(u_f)$ correspond to best-case and worst-case scenario evaluations in $\mathcal{C}$.*

Proof. For simplicity, we will denote for every $u_f \in B_0(\Sigma, K)$, $\underline{P}(u_f) = \min_{P\in\mathcal{C}} P(u_f)$ and $\bar{P}(u_f) = \max_{P\in\mathcal{C}} P(u_f)$.

Lemma 12 *For all $f \in \mathcal{F}$, we have the following equalities:*

(*i*) $\max_{P\in\mathcal{C}} P(u_f) = \sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = \sup\{I(u_f + \psi) - I(\psi) : \psi \in B_0(\Sigma)\}$

(*ii*) $\min_{P\in\mathcal{C}} P(u_f) = \inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = \inf\{I(u_f + \psi) - I(\psi) : \psi \in B_0(\Sigma)\}$

Proof. First, notice that $\{\frac{1-\lambda}{\lambda}u_g : g \in \mathcal{F} \text{ and } \lambda \in (0,1]\} \subseteq B_0(\Sigma)$. Next, for all $\psi \in B_0(\Sigma)$ there exists $\phi \in B_0(\Sigma, K)$ and $\beta \in (0,1)$ where $K = u(\mathbb{X})$ such that $\phi = \beta\psi$. Furthermore, from the (*iii*) of Lemma 3 we know that for any $\phi \in B_0(\Sigma, K)$, there exists $g \in \mathcal{F}$ such that $\phi = u_g$. Thus, there also exists $g \in \mathcal{F}$ with $\phi = u_g = \beta\psi$ which implies $\psi = \frac{1}{\beta}u_g$. Given that $\lambda \in (0,1]$ and $\beta \in (0,1)$, it must be $\frac{1-\lambda}{\lambda} \in [0,\infty)$ and $\frac{1}{\beta} \in (1,\infty)$. Then, there is a $\lambda^* \in (0,1]$ such that $\frac{1-\lambda^*}{\lambda^*} = \frac{1}{\beta}$ hence $\psi = \frac{1-\lambda^*}{\lambda^*}u_g$. Therefore, we have $B_0(\Sigma) \subseteq \{\frac{1-\lambda}{\lambda}u_g : g \in \mathcal{F} \text{ and } \lambda \in (0,1]\}$ which, together with $\{\frac{1-\lambda}{\lambda}u_g : g \in \mathcal{F} \text{and}\, \lambda \in (0,1]\} \subseteq B_0(\Sigma)$, implies $B_0(\Sigma) = \{\frac{1-\lambda}{\lambda}u_g : g \in \mathcal{F} \text{and}\, \lambda \in (0,1]\}$. As a result, for any $f \in \mathcal{F}$ we have that $\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = \{I(u_f + \psi) - I(\psi) : \psi \in B(\Sigma)\}$ as well as $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = \sup\{I(u_f + \psi) - I(\psi) : \psi \in B(\Sigma)\}$ and $\inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = \inf\{I(u_f + \psi) - I(\psi) : \psi \in B(\Sigma)\}$.

For any $f \in \mathcal{F}$ take $\ell_{min} \in \mathbb{X}$ with $u(\ell_{min}) = \underline{P}(u_f)$. Notice that due to the Lemma 2 we know that there exists $\bar{h}, \underline{h} \in \mathcal{F}_c$ with $\bar{h}(s) = \bar{\ell} \in \mathbb{X}$, $\underline{h}(s) = \underline{\ell} \in \mathbb{X}$ such that $\bar{l} \succsim^{\mathbb{X}} \ell \succsim^{\mathbb{X}} \underline{\ell}$ for all $\ell \in \mathbb{X}$ and for all $s \in S$; therefore, for any $f \in \mathcal{F}$ it must be $\bar{\ell} \succsim^{\mathbb{X}} f(s) \succsim^{\mathbb{X}} \underline{\ell}$ for all $s \in S$.

Then, by Axiom 4 and (iv) of Lemma 6 we must have $\bar{h} \succsim f \succsim \underline{h}$ and $\bar{h} \succsim^{UA} f \succsim^{UA} \underline{h}$. By the Theorem 2 it must be $\int_S u_{\bar{h}} dP \geq \int_S u_f dP \geq \int_S u_{\underline{h}} dP$ for all $P \in \mathcal{C}$. Then, it must be $u(\bar{\ell}) \geq P(u_f) \geq u(\underline{\ell})$ for all $P \in \mathcal{C}$. So we can write $u(\bar{\ell}) \geq \underline{P}(u_f) \geq u(\underline{\ell})$. Since u is continuous, there exists a $\ell_{min} \in \mathbb{X}$ such that $u(\ell_{min}) = \underline{P}(u_f)$ by the Intermediate Value Theorem. Furthermore, we have $h_{min} \in \mathcal{F}_c$ with $f \succsim^{UA} h_{min}$ where $h_{min}(s) = \ell_{min}$ for all $s \in S$ since we have $P(u_f) \geq \underline{P}(u_f) = u(\ell_{min}) = P(u_{h_{min}})$ for all $P \in \mathcal{C}$. Hence, for all $\lambda \in (0,1]$ and $g \in \mathcal{F}$ we have the following inequality $I(u_{(\lambda f + (1-\lambda) g)}) \geq I(u_{(\lambda h_{min} + (1-\lambda) g)})$ which can be written as $I(\lambda u_f + (1-\lambda) u_g) \geq I(\lambda u_{h_{min}} + (1-\lambda) u_g)$ due to the (ii) of Lemma 3. Since $\lambda u_{h_{min}}$ is a constant which is equal to $\lambda u(\ell_{min})$, and functional I is constant additive as we have shown in (iv) of Lemma 5, we can write $I(\lambda u_f + (1-\lambda) u_g) \geq \lambda u(\ell_{min}) + I((1-\lambda) u_g)$ which implies $I(\lambda u_f + (1-\lambda) u_g) - I((1-\lambda) u_g) \geq \lambda u(\ell_{min})$. Then, by the (iii) of Lemma 5 we know that I is homogeneous of degree 1, so we can obtain $I(u_f + \frac{(1-\lambda)}{\lambda} u_g) - I(\frac{(1-\lambda)}{\lambda} u_g) \geq u(\ell_{min})$ from the previous inequality. As $u(\ell_{min}) = \underline{P}(u_f)$ and $I(u_f + \frac{(1-\lambda)}{\lambda} u_g) - I(\frac{(1-\lambda)}{\lambda} u_g) \geq u(\ell_{min})$ holds for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$, we can write $\inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq \underline{P}(u_f)$.

Next, suppose that $\ell_{inf} \in \mathbb{X}$ is such that $u(\ell_{inf}) = \inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. First of all, due to definition of infimum, for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$ it must be $(I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g)) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. Then, by taking $g \in \mathcal{F}$ with $u_g = \mathbf{0}$, or $\lambda = 0$, we have that $I(u_f) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. [14] We have already shown that $\inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq \underline{P}(u_f) = u(\ell_{min})$ and there exist $\bar{h}, \underline{h} \in \mathcal{F}_c$ such that $\bar{h} \succsim f \succsim \underline{h}$ which implies $u_{\bar{h}} \geq I(u_f) \geq u_{\underline{h}}$. Hence, we have the following inequalities: $u_{\bar{h}} \geq I(u_f) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq u(\ell_{min})$. Again, since u is continuous, there exists $\ell_{inf} \in \mathbb{X}$ such that $u(\ell_{inf}) = \inf\{I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$ due to the Intermediate Value Theorem. Therefore, we can write $(I(u_f + \frac{1-\lambda}{\lambda} u_g) - I(\frac{1-\lambda}{\lambda} u_g)) \geq u(\ell_{inf})$; or equivalently $(I(u_f + \frac{1-\lambda}{\lambda} u_g) \geq u(\ell_{inf}) + I(\frac{1-\lambda}{\lambda} u_g))$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$. Let $h_{inf} \in \mathcal{F}_c$ be defined by $h_{inf}(s) = \ell_{inf}$ for all $s \in S$. Since $u(\ell_{inf}) = u_{h_{inf}}(s)$ for all $s \in S$

[14] Such u_g exists as we have shown in the proof of Claim 3.

is constant and functional I is constant additive due to the (iv) of Lemma 5, we can obtain $I(u_f + \frac{1-\lambda}{\lambda}u_g) \geq I(u_{h_{inf}} + \frac{1-\lambda}{\lambda}u_g)$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$. Then, by using the (iii) of Lemma 5, we have the following inequality: $I(\lambda\, u_f + (1-\lambda)u_g) \geq I(\lambda\, u_{h_{inf}} + (1-\lambda)u_g)$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$ which implies $f \succsim^{UA} h_{inf}$. By the Theorem 2, we have $\int_S u_f dP \geq \int_S u_{h_{inf}} dP = u(\ell_{inf})$ for all $P \in \mathcal{C}$. Hence, $P(u_f) \geq u(\ell_{inf})$ for all $P \in \mathcal{C}$ from which we can obtain $\underline{P}(u_f) \geq u(\ell_{inf})$; or $\underline{P}(u_f) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$ as $\inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = u(\ell_{inf})$. Therefore, we have $\min_{P\in\mathcal{C}} P(u_f) = \underline{P}(u_f) = \inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$.

In what follows, we will show that $\max_{P\in\mathcal{C}} P(u_f) = \sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. Firstly, for any $f \in \mathcal{F}$, let $\ell_{max} = \bar{P}(u_f)$. Existence of such ℓ_{max} can be easily seen with the argument that we have used for existence of ℓ_{min}. Moreover, we have $h_{max} \in \mathcal{F}_c$ with $h_{max} \succsim^{UA} f$ where $h_{max}(s) = \ell_{max}$ for all $s \in S$ since $\bar{P}(u_f) = u(\ell_{max}) = P(u_{h_{max}}) \geq P(u_f)$ for all $P \in \mathcal{C}$; so, $I(u_{(\lambda\, h_{max}+(1-\lambda)g)}) \geq I(u_{(\lambda\, f+(1-\lambda)g)})$, or equivalently, $I(\lambda\, u_{h_{max}} + (1-\lambda)u_g) \geq I(\lambda\, u_f + (1-\lambda)u_g)$ due to the (ii) of Lemma 3, for all $\lambda \in (0,1]$ and $g \in \mathcal{F}$. Then, again by using (iii) and (iv) of Lemma 5 we can obtain $u(\ell_{max}) \geq I(u_f + \frac{(1-\lambda)}{\lambda}u_g) - I(\frac{(1-\lambda)}{\lambda}u_g)$ for all $\lambda \in (0,1]$ and $g \in \mathcal{F}$ which implies $\bar{P}(u_f) \geq \sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$.

Secondly, let $\ell_{sup} \in \mathbb{X}$ be such that $u(\ell_{sup}) = \sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. The existence of such ℓ_{sup} can be easily seen by using the same argument for the existence of ℓ_{inf}. Furthermore, we should note that, due to the definition of the supremum it must be $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq (I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g))$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$. Hence, $u(\ell_{sup}) \geq (I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g))$, and $u(\ell_{sup}) + I(\frac{1-\lambda}{\lambda}u_g)) \geq (I(u_f + \frac{1-\lambda}{\lambda}u_g)$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$. Let $h_{sup} \in \mathcal{F}_c$ be defined by $h_{sup}(s) = \ell_{sup}$ for all $s \in S$, then we have $u(\ell_{sup}) = u_{h_{up}}(s)$. Then, again by applying (iii) and (iv) of Lemma 5 the following result can be obtained : $I(\lambda\, u_{h_{sup}} + (1-\lambda)u_g) \geq I(\lambda\, u_f + (1-\lambda)u_g)$ for all $g \in \mathcal{F}$ and $\lambda \in (0,1]$, which implies $h_{inf} \succsim^{UA} f$. Theorem 2 ensures that $u(\ell_{sup}) = \int_S u_{h_{sup}} dP \geq \int_S u_f dP$ for all $P \in \mathcal{C}$ which implies $u(\ell_{sup}) \geq \bar{P}(u_f)$ as well as $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq \bar{P}(u_f)$ since we set $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} = u(\ell_{sup})$. Therefore, we have $\max_{P\in\mathcal{C}} P(u_f) = \sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$ which concludes the proof of the current lemma. ■

As it was mentioned in the proof of Lemma 12, due to the definition of infimum and supremum, for a given $f \in \mathcal{F}$, for all $g \in \mathcal{F}$ and for all $\lambda \in (0,1]$ we must have: $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq (I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g)) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. By using a specific $\tilde{g} \in \mathcal{F}$ with $u_{\tilde{g}} = \mathbf{0}$ in the previous inequality, we obtain that $\sup\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\} \geq I(u_f) \geq \inf\{I(u_f + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g) : g \in \mathcal{F}, \lambda \in (0,1]\}$. Then, due to the Lemma 12, we have $\max_{P\in\mathcal{C}} P(u_f) \geq I(u_f) \geq \min_{P\in\mathcal{C}} P(u_f)$; in turn, concluding the proof of Theorem 4. ■

At this stage, it maybe useful to remind that for any $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ whenever $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and for all $m \in \mathcal{F}$; and $f \sim^{UA} g$ if and only if $f \succsim^{UA} g$ and $g \succsim^{UA} f$, a case which we will refer to as f being unambiguously indifferent to g.

The ambiguity concerning two acts $f, g \in \mathcal{F}$ are equivalent, denoted by $f \asymp g$, whenever there are $h, h' \in \mathcal{F}_c$ and $\lambda, \lambda' \in (0,1]$ which satisfy $\lambda f + (1-\lambda)h \sim^{UA} \lambda' g \ + (1-\lambda')h'$. For any given $f \in \mathcal{F}$, we denote the set of acts $g \in \mathcal{F}$ whose ambiguity is equivalent to that of f, by $\mathcal{F}(f)$. That is, $\mathcal{F}(f) = \{g \in \mathcal{F} : g \asymp f\}$. Moreover, the family of ambiguity equivalent sets is defined by $\mathfrak{F} = \{\mathcal{F}(f) : f \in \mathcal{F}\}$.

The following lemma establishes that $\asymp$ defined on $\mathcal{F}$ is, indeed, an equivalence relation:

Lemma 13 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5 and $\asymp$ on $\mathcal{F}$ is as defined above. Then, the following statements are equivalent for any $f, g \in \mathcal{F}$:*

(i) *$f \asymp g$.*

(ii) *For all $P, P' \in \mathcal{C}$, we have $P(u_f) \geq P'(u_f)$ if and only if $P(u_g) \geq P'(u_g)$. In other words, $\{P(u_f) : P \in \mathcal{C}\}$ and $\{P(u_g) : P \in \mathcal{C}\}$ are isotonic.*

(iii) *There exist $a > 0$ and $b \in \mathbb{R}$ which satisfies following equality : $P(u_f) = aP(u_g) + b$ for all $P \in \mathcal{C}$. In other words, $\{P(u_f) : P \in \mathcal{C}\}$ and $\{P(u_g) : P \in \mathcal{C}\}$ are positive affine* *transformation of each other;*

Proof. First, we will first show that (i) implies (ii). Let $f, g \in \mathcal{F}$ be two acts such that $f \asymp g$, then there exist $\lambda', \lambda'' \in (0,1]$ and $h', h'' \in \mathcal{F}_c$ with $h'(s) = \ell' \in \mathbb{X}$ and $h''(s) = \ell'' \in \mathbb{X}$ for all $s \in S$, and $\lambda' f + (1-\lambda')h' \sim^{UA} \lambda'' g \ + (1-\lambda'')h''$. Then, by Theorem 2 it must be $P(u_{\lambda' f+(1-\lambda')h'}) = P(u_{\lambda'' g+(1-\lambda'')h''})$ for all $P \in \mathcal{C}$. Due to the (ii) of Lemma 3, we can write $P(\lambda' u_f + (1-\lambda')u_{h'}) = P(\lambda'' u_g + (1-\lambda'')u_{h''})$ for all $P \in \mathcal{C}$. Since expected utility mapping P is homogeneous of degree 1 and constant additive and $u_{h'} = u(\ell')$, $u_{h''} = u(\ell'')$ are constant. [15] Then, we can rewrite the previous inequality as $\lambda' P(u_f) + (1-\lambda')u(\ell') = \lambda'' P(u_g) + (1-\lambda'')u(\ell'')$ for all $P \in \mathcal{C}$ and for some $\lambda', \lambda'' \in (0,1]$ and for some $\ell', \ell'' \in \mathbb{X}$. Without loss of generality, suppose that for any $P, P' \in \mathcal{C}$ we have $P(u_f) \geq P'(u_f)$ which implies $\lambda P(u_f)+(1-\lambda)u(\ell) \geq \lambda P'(u_f)+(1-\lambda)u(\ell)$ must hold for all $\lambda \in (0,1]$ and for all $\ell \in \mathbb{X}$. Then, clearly $\lambda' P(u_f) + (1-\lambda')u(\ell') \geq \lambda' P'(u_f) + (1-\lambda')u(\ell')$ for $\lambda' \in (0,1]$ and for $\ell' \in \mathbb{X}$. Therefore, it must be $\lambda' P(u_f) + (1-\lambda')u(\ell') = \lambda'' P(u_g) + (1-\lambda'')u(\ell'') \geq \lambda' P'(u_f)+(1-\lambda')u(\ell') = \lambda'' P'(u_g)+(1-\lambda'')u(\ell'')$ for any $P, P' \in \mathcal{C}$ and for some $\lambda', \lambda'' \in (0,1]$ and for some $\ell', \ell'' \in \mathbb{X}$; so, we have $P(u_g) \geq P'(u_g)$ for any $P, P' \in \mathcal{C}$.

In order to show that (ii) implies (iii), let $P, P' \in \mathcal{C}$ and since both P and P' are positive affine functionals mapping $B_0(\Sigma)$ into $\mathbb{R}$, $\{P(u_f) : P \in \mathcal{C}\}$ and $\{P(u_g) : P \in \mathcal{C}\}$ are isotonic (i.e. every probability measure on u_f order the elements in the same way as it order u_g) if and only if u_f and u_g in $B_0(\Sigma)$ are positive affine transformations of one another.

Finally, to show that (iii) implies (i), suppose that for any $f, g \in \mathcal{F}$ there exist $a > 0$ and $b \in \mathbb{R}$ such that $P(u_f) = aP(u_g) + b$ for all $P \in \mathcal{C}$. First, if $0 < a < 1$, then take $\lambda' = a$ and suppose that $\frac{b}{1-\lambda'} \in u(\mathbb{X})$. [16] Therefore, we can write $P(u_f) = \lambda' P(u_g) + (1-\lambda')\frac{b}{1-\lambda'}$ for all $P \in \mathcal{C}$ which implies $P(u_f) = P(\lambda' u_g + (1-\lambda')\frac{b}{1-\lambda'})$ for all $P \in \mathcal{C}$. Then, we can write $P(\lambda u_f + (1-\lambda)u_h) = P(\lambda' u_g + (1-\lambda')u_{h'})$ for all $P \in \mathcal{C}$, for $\lambda = 1$ and $\lambda' = a$,

[15] For any $f \in \mathcal{F}$, $h \in \mathcal{F}_c$ with $h(s) = \ell$ for all $s \in S$, and for a given $P \in \mathcal{C}$, $P(u_f)$ is defined by $\int_S u_f dP$. Therefore, due to the properties of integral, we have $P(\lambda u_f + (1-\lambda)u_h) = \int_S (\lambda u_f + (1-\lambda)u_h)dP = \lambda \int_S u_f dP + (1-\lambda)u_h = \lambda P(u_f) + (1-\lambda)u(\ell)$.

[16] If $\frac{b}{1-\lambda'} \notin u(\mathbb{X})$, then we can renormalize the von-Neumann Morgenstern utility function via taking an appropriate affine transformation.

for any $h' \in \mathcal{F}_c$ with $h'(s) = \ell' \in \mathbb{X}$ for all $s \in S$ where $u(\ell') = \frac{b}{1-\lambda'}$, and for all $h \in \mathcal{F}$. Thus, there exist $\lambda, \lambda' \in (0,1]$ and $h, h' \in \mathcal{F}$ such that $\lambda f + (1-\lambda)h \sim^{UA} \lambda' g + (1-\lambda')h'$, so $f \asymp g$. Secondly, if $a > 1$, we will rewrite $P(u_f) = aP(u_g) + b$ as $\frac{1}{a}P(u_f) - \frac{b}{a} = P(u_g)$ for all $P \in \mathcal{C}$. Then, by taking $\lambda = \frac{1}{a}$ and supposing that $\frac{1-\lambda}{a}b \in u(\mathbb{X})$ we can write $\lambda P(u_f) + (1-\lambda)\frac{1-\lambda}{a}b = P(u_g)$ for all $P \in \mathcal{C}$. [17] By following the similar steps above, we can obtain $\lambda f + (1-\lambda)h \sim^{UA} \lambda' g + (1-\lambda')h'$ for $\lambda = \frac{1}{a}$, for $\lambda' = 0$, for any $h \in \mathcal{F}_c$ with $h(s) = \ell \in \mathbb{X}$ for all $s \in S$ where $u(\ell) = \frac{1-\lambda}{a}b$, and for all $h' \in \mathcal{F}_c$. Thus, we have $f \asymp g$. Finally, if $a = 1$, then $P(u_f) = P(u_g) + b$, or equivalently $\frac{1}{2}P(u_f) = \frac{1}{2}P(u_g) + \frac{1}{2}b$ for all $P \in \mathcal{C}$. Suppose that $b \in u(\mathbb{X})$, so we can write $P(\frac{1}{2}u_f + \frac{1}{2}u_h) = P(\frac{1}{2}u_g + \frac{1}{2}u_{h'})$ for all $P \in \mathcal{C}$, for any $h, h' \in \mathcal{F}_c$ with $h(s) = \ell$ and $h'(s) = \ell'$ where $\ell' \in \mathbb{X}$ is such that $u(\ell') = b$ and $\ell \in \mathbb{X}$ such that $u(\ell) = 0$ which is exist as we have shown in the proof of Claim 3. [18] Then, for $\lambda, \lambda' = \frac{1}{2}$ and for $h, h' \in \mathcal{F}_c$ we have $\frac{1}{2}f + \frac{1}{2}h \sim \frac{1}{2}g + \frac{1}{2}h'$, so $f \asymp g$. ■

In order to obtain the desired representation, we need to handle acts whose evaluation does not have an effect on the ambiguity, so-called crisp acts: An act $m \in \mathcal{F}$ is a *crisp act* if $f \sim g$ implies $(1-\lambda)f + \lambda m \sim (1-\lambda)g + \lambda m$ for all $f, g \in \mathcal{F}$ with $f \sim g$ and $\lambda \in (0,1)$. Furthermore, we will denote $\mathcal{M}$ for the set of all crisp acts.

The following displays some properties of crisp acts:

Lemma 14 *Suppose that a binary relation $\succsim$ on $\mathcal{F}$ satisfies Axiom 1–5. Then, the following statements are equivalent:*

(i) *m is a crisp act;*

(ii) $\int u_m \mathrm{d}P = \int u_m \mathrm{d}P'$ *for all* $P, P' \in \mathcal{C}$*;*

(iii) $I(u_{\lambda m + (1-\lambda)f}) = \lambda I(u_m) + (1-\lambda)I(u_f)$ *for all* $f \in \mathcal{F}$ *and* $\lambda \in [0,1]$*;*

(iv) $m \asymp h$ *for some* $h \in \mathcal{F}_c$*.*

[17] If needed, a renormalization discussed in Footnote 16 can be done in this case as well.

[18] In this case, a more careful renormalization (using a selected affine transformation) of the von Neumann Morgenstern utility u is in order so that both $b \in u(\mathbb{X})$ and $u(\underline{\ell}) < 0$ and $u(\overline{\ell}) > 0$.

Proof. Firstly, we will show that (i) implies (ii). Suppose that $m \in \mathcal{F}$ is a crisp act; i.e. for all $f, g \in \mathcal{F}$ with $f \sim g$ and $\lambda \in (0,1)$ we have $(1-\lambda)f + \lambda\, m \sim (1-\lambda)g + \lambda\, m$. Thus, $I(u_f) = I(u_g)$ and $I(u_{(1-\lambda)f+\lambda\, m}) = I(u_{(1-\lambda)g+\lambda\, m})$ by Theorem 1. Due to the (ii) of Lemma 3, $I(u_{(1-\lambda)f+\lambda\, m}) = I(u_{(1-\lambda)g+\lambda\, m})$ implies $I((1-\lambda)u_f + \lambda\, u_m) = I((1-\lambda)u_g + \lambda\, u_m)$ which can be written as $I(u_m + \frac{1-\lambda}{\lambda}u_f) = I(u_m + \frac{1-\lambda}{\lambda}u_g)$. Since $I(u_f) = I(u_g)$, $I(u_m + \frac{1-\lambda}{\lambda}u_f) - I(\frac{1-\lambda}{\lambda}u_f) = I(u_m + \frac{1-\lambda}{\lambda}u_g) - I(\frac{1-\lambda}{\lambda}u_g)$ must be true.

From the proof of Lemma 12, we know that $B_0(\Sigma) = \{\frac{1-\lambda}{\lambda}u_g : g \in \mathcal{F} \text{ and } \lambda \in (0,1]\}$, then for any $\psi, \phi \in B_0(\Sigma)$ with $I(\psi) = I(\phi)$ we have $I(u_m + \psi) - I(\psi) = I(u_m + \phi) - I(\phi)$. On the other hand, let $I(\psi) = I(\phi) + b$ where $b \in \mathbb{R}$ for any $\psi, \phi \in B_0(\Sigma)$ with $I(\psi) \neq I(\phi)$; clearly $I(\psi) = I(\phi + b)$ since I is constant additive. Therefore, we have $I(u_m + \psi) - I(\psi) = I(u_m + \phi + b) - I(\phi + b)$ which implies $I(u_m + \psi) - I(\psi) = I(u_m + \phi) + b - I(\phi) - b = I(u_m + \phi) - I(\phi)$ again since I is constant additive. Hence, for all $\psi, \phi \in B_0(\Sigma)$ we have $I(u_m + \psi) - I(\psi) = I(u_m + \phi) - I(\phi)$ which implies $\sup\{I(u_m + \psi) - I(\psi) : \psi \in B_0(\Sigma)\} = \inf\{I(u_m + \psi) - I(\psi) : \psi \in B_0(\Sigma)\}$. Due to the Lemma 12, $\max_{P \in \mathcal{C}} P(u_m) = \min_{P \in \mathcal{C}} P(u_m)$ and therefore, $\int u_m dP = \int u_m dP'$ for all $P, P' \in \mathcal{C}$.

Next, we will show that (ii) implies (iii). If (ii) is true, $\max_{P \in \mathcal{C}} P(u_m) = \min_{P \in \mathcal{C}} P(u_m)$. Therefore, by the Lemma 12, $\sup\{I(u_m + \frac{1-\lambda}{\lambda}u_f) - I(\frac{1-\lambda}{\lambda}u_f) : f \in \mathcal{F}, \lambda \in (0,1]\} = \inf\{I(u_m + \frac{1-\lambda}{\lambda}u_f) - I(\frac{1-\lambda}{\lambda}u_f) : f \in \mathcal{F}, \lambda \in (0,1]\}$. Hence, the result of $I(u_m + \frac{1-\lambda}{\lambda}u_f) - I(\frac{1-\lambda}{\lambda}u_f)$ must be the same for all $f \in \mathcal{F}$ and $\lambda \in (0,1]$. In fact, we have $I(u_m + \frac{1-\lambda}{\lambda}u_f) - I(\frac{1-\lambda}{\lambda}u_f) = I(u_m)$ for all $f \in \mathcal{F}$ and all $\lambda \in (0,1]$ which can be easily seen by taking $\lambda = 1$. By multiplying both sides of the equation with λ, we can obtain $I(\lambda\, u_m + (1-\lambda)u_f) - I((1-\lambda)u_f) = I(\lambda\, u_m)$ for all $f \in \mathcal{F}$ and all $\lambda \in (0,1]$. Hence, arranging the terms $I(\lambda\, u_m + (1-\lambda)u_f) = I(\lambda\, u_m) + I((1-\lambda)u_f)$ so as $I(u_{\lambda\, m+(1-\lambda)f}) = I(\lambda\, u_m + (1-\lambda)u_f)$ (due to Lemma 3) we have $I(u_{\lambda\, m+(1-\lambda)f}) = \lambda\, I(u_m) + (1-\lambda)I(u_f)$ for all $f \in \mathcal{F}$ and all $\lambda \in (0,1]$. When $\lambda = 0$, $I(\lambda\, u_m + (1-\lambda)u_f) = \lambda\, I(u_m) + (1-\lambda)I(u_f)$ holds for all $f \in \mathcal{F}$ trivially.

Next, we will show that (iii) implies (i). Let $f, g \in \mathcal{F}$ be such that $f \sim g$, then $I(u_f) = I(u_g)$. As (iii) is true, for any $f, g, m \in \mathcal{F}$ with $f \sim g$ and $\lambda \in [0,1]$ we have $I(\lambda\, u_m + (1-\lambda)u_f) = \lambda\, I(u_m) + (1-\lambda)I(u_f) = \lambda\, I(u_m) + (1-\lambda)I(u_g) = I(\lambda\, u_m) + I((1-\lambda)u_g) =$

$I(\lambda\, u_m + (1-\lambda)u_g)$ from which we obtain $I(u_{\lambda\, m+(1-\lambda)f}) = I(u_{\lambda\, m+(1-\lambda)g})$ which follows from Lemma 3. Hence, $\lambda\, m + (1-\lambda)f \sim \lambda\, m + (1-\lambda)g$ for all $f, g, m \in \mathcal{F}$ with $f \sim g$ and $\lambda \in (0,1)$, in turn, establishing that $m \in \mathcal{F}$ is a crisp act.

So far, we have proved that $(i), (ii)$ and (iii) are equivalent. In what follows, we will show that (ii) is true if and only if (iv) is true. For the if direction ((ii) implying (iv)), suppose that for any $m \in \mathcal{F}$ we have $\int u_m dP = c$ for all $P \in \mathcal{C}$, and let $h \in \mathcal{F}_c$ be such that $h(s) = \ell \in \mathbb{X}$ for all $s \in S$ and $u(\ell) = c$. Thus, $\int_S u_m dP = c = \int_S u_h dP$ for all $P \in \mathcal{C}$ which implies $m \succsim^{UA} h$ due to Theorem 2. Then, clearly we have $\lambda\, f + (1-\lambda)h' \sim^{UA} \lambda'\, h\ + (1-\lambda')h''$ for $\lambda, \lambda' = 1$ and any $h', h'' \in \mathcal{F}_c$, so $m \asymp h$. Conversely, for the only if direction, suppose that $m \asymp h$ for some $h \in \mathcal{F}_c$. Then, there exist some $\lambda, \lambda' \in (0,1]$ and $h', h'' \in \mathcal{F}_c$ such that $\lambda\, m + (1-\lambda)h' \sim^{UA} \lambda'\, h + (1-\lambda')h''$. Due to the Theorem 2, we have $P(u_{\lambda\, m+(1-\lambda)h'}) = P(u_{\lambda'\, h+(1-\lambda')h''})$ for all $P \in \mathcal{C}$. Then by the (ii) of Lemma 3, we have $P(\lambda\, u_m + (1-\lambda)u_{h'}) = P(\lambda'\, u_h + (1-\lambda')u_{h''})$ which implies (due to constant additivity and homogeneity of degree 1) $\lambda\, P(u_m) + (1-\lambda)u(\ell') = \lambda'\, u(\ell) + (1-\lambda')u(\ell'')$ for all $P \in \mathcal{C}$ where $u(\ell) = u_h(s)$ and $u(\ell') = u_{h'}(s)$ and $u(\ell'') = u_{h''}(s)$ for all $s \in S$ as $u_h, u_{h'}, u_{h''} \in B_0(\Sigma, K)$ are constant. Hence, $P(u_m)$ equals a constant for all $P \in \mathcal{C}$ which concludes the proof of the statement that (iii) and (iv) are equivalent. ■

Before presenting the desired representation result concerning $\succsim$, we need to define the family of equivalence sets restricted on non-crisp acts: $\mathfrak{F}^* = \{\mathcal{F}(f) : f \in \mathcal{F} \setminus \mathcal{M}\}$.

Theorem 5 *A binary relation $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ satisfies axioms 1-5 if and only if there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$, and a non-empty and unique and weak* compact and convex set $\mathcal{C}$ of probabilities on Σ, and a function $\alpha : \mathfrak{F} \to [0,1]$ with a uniquely defined restriction on $\mathfrak{F}^*$ such that $f \succsim g$ if and only if*

$$\begin{aligned} I(u_f) &= \alpha_{\mathcal{F}(f)} \max_{P \in \mathcal{C}} \int u_f \mathrm{d}P + (1-\alpha_{\mathcal{F}(f)}) \min_{P \in \mathcal{C}} \int u_f \mathrm{d}P \\ &\geq \alpha_{\mathcal{F}(g)} \max_{P \in \mathcal{C}} \int u_g \mathrm{d}P + (1-\alpha_{\mathcal{F}(g)}) \min_{P \in \mathcal{C}} \int u_g \mathrm{d}P = I(u_g) \end{aligned} \tag{2.4}$$

and u_f and u_g and $\mathcal{C}$ are as in the representation theorem of $\succsim^{UA}$, Theorem 2, i.e. $f \succsim^{UA} g$ if and only if $\int_S u_f \mathrm{d}P \geq \int_S u_g \mathrm{d}P$ for all $P \in \mathcal{C}$.

Proof. The sufficiency direction of the current Theorem is trivial; hence, omitted. Suppose that $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ satisfies Axioms 1-5; and let I and u be the monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$ and the non-constant affine function $u : \mathbb{X} \to \mathbb{X}$, respectively, that we have obtained in the Theorem 1. By the same Theorem, we know that I is unique and represents $\succsim$; and u is unique up to positive affine transformation and represents $\succsim^{\mathbb{X}}$. Furthermore, let $\mathcal{C}$ be the nonempty and unique weak* compact and convex set of probabilities on Σ that represents $\succsim^{UA}$ which is obtained in Theorem 2.

Due to the Theorem 4, we know that for any $f \in \mathcal{F}$ we have

$$\bar{P}(u_f) \equiv \max_{P \in \mathcal{C}} P(u_f) \geq I(u_f) \geq \min_{P \in \mathcal{C}} P(u_f) \equiv \underline{P}(u_f).$$

In what follows, for any given $f \in \mathcal{F}$ we will identify $\alpha_{\mathcal{F}(f)} \in [0,1]$ such that

$$I(u_f) = \alpha_{\mathcal{F}(f)} \bar{P}(u_f) + (1 - \alpha_{\mathcal{F}(f)}) \underline{P}(u_f).$$

In Lemma 14, we have observed that f is a crisp act (i.e. $f \in \mathcal{M}$) if and only if $\bar{P}(u_f) = I(u_f) = \underline{P}(u_f)$ and we may choose $\alpha_{\mathcal{F}(f)} \in [0,1]$ arbitrarily. On the other hand, if $f \in \mathcal{F} \backslash \mathcal{M}$, then $\bar{P}(u_f) \geq I(u_f) \geq \underline{P}(u_f)$ with one of the inequalities holding strictly and we set

$$\alpha_{\mathcal{F}(f)} = \frac{I(u_f) - \underline{P}(u_f)}{\bar{P}(u_f) - \underline{P}(u_f)},$$

which is in $[0,1]$ as $\bar{P}(u_f) - \underline{P}(u_f) > 0$. Notice that this defines $\alpha_{\mathcal{F}(f)}$ uniquely for all $f \in \mathcal{F} \setminus \mathcal{M}$.

Next, we will show that for any $f, g \in \mathcal{F} \setminus \mathcal{M}$ with $f \asymp g$, it must be that $\alpha_{\mathcal{F}(f)} = \alpha_{\mathcal{F}(g)}$. For any $f, g \in \mathcal{F} \backslash \mathcal{M}$ with $f \asymp g$, it must be that $\lambda f + (1-\lambda) h \sim^{UA} \lambda' g + (1-\lambda') h'$ for some $\lambda, \lambda' \in (0,1]$ and some $h, h' \in \mathcal{F}_c$. By (i) of Lemma 6, we have $\lambda f + (1-\lambda) h \sim \lambda' g + (1-\lambda') h'$ which implies, by Lemma 3 and Theorem 1, $I(\lambda u_f + (1-\lambda) u_h) = I(\lambda' g + (1-\lambda') u_{h'})$. Since

I is constant additive and homogeneous of degree 1, the previous equality is equivalent to $\lambda\, I(u_f) + (1-\lambda)u_h = \lambda'\, I(u_g) + (1-\lambda')u_{h'}$ where $h(s) = \ell \in \mathbb{X}$ with $u_h = u(\ell)$ and $h'(s) = \ell' \in \mathbb{X}$ with $u_{h'} = u(\ell')$ for all $s \in S$. After dividing both sides to $\lambda > 0$ and rearranging terms, we obtain $I(u_f) = \frac{\lambda'}{\lambda} I(u_g) + \frac{(1-\lambda')}{\lambda} u_{h'} - \frac{(1-\lambda)}{\lambda} u_h = \frac{\lambda'}{\lambda} I(u_g) + \frac{1}{\lambda}[(1-\lambda')u_{h'} - (1-\lambda)u_h]$. Let $a = \frac{\lambda'}{\lambda} > 0$ and $b = \frac{1}{\lambda}[(1-\lambda')u_{h'} - (1-\lambda)u_h]$, then $I(u_f) = aI(u_g) + b$. On the other hand, due to the Theorem 2, $\lambda\, f + (1-\lambda)h \sim^{UA} \lambda'\, g + (1-\lambda')h'$ implies (by Lemma 3) $P(\lambda\, u_f + (1-\lambda)u_h) = P(\lambda'\, g + (1-\lambda')u_{h'})$ for all $P \in \mathcal{C}$ from which we can obtain $\lambda\, P(u_f) + (1-\lambda)u_h = \lambda'\, P(u_g) + (1-\lambda')u_{h'}$ for all $P \in \mathcal{C}$ since P is also constant additive and homogeneous of degree 1. After rearranging these terms as above, we have $P(u_f) = \frac{\lambda'}{\lambda} P(u_g) + \frac{(1-\lambda')}{\lambda} u_{h'} - \frac{(1-\lambda)}{\lambda} u_h = \frac{\lambda'}{\lambda} P(u_g) + \frac{1}{\lambda}[(1-\lambda')u_{h'} - (1-\lambda)u_h]$; so, $P(u_f) = aP(u_g) + b$. Hence, for any $f, g \in \mathcal{F} \setminus \mathcal{M}$ with $f \asymp g$ we have:

$$\alpha_{\mathcal{F}(f)} = \frac{I(u_f) - \underline{P}(u_f)}{\bar{P}(u_f) - \underline{P}(u_f)} = \frac{(aI(u_g) + b) - (a\underline{P}(u_g) + b)}{(a\bar{P}(u_g) + b) - (a\underline{P}(u_g) + b)} = \frac{I(u_g) - \underline{P}(u_g)}{\bar{P}(u_g) - \underline{P}(u_g)} = \alpha_{\mathcal{F}(g)}.$$

This concludes the proof of Theorem 5. ■

3 ACTS CONCERNING MENUS

For reasons of exposition and in order to avoid technical and notational complexities, we restrict attention to cases in which X is a finite set of alternatives and $\mathbb{X}$ is the simplex on X (which is non-empty and convex and compact) and we let $\mathcal{X}$ be set of all non-empty subsets of X. Then, a menu $A \in \mathcal{X}$ is given by $\mathbb{A} \equiv \Delta(A)$ a non-empty and convex and compact subset of $\mathbb{X}$ and we let $\mathcal{A}$ to be the set of all menus on X. When the meaning is clear, we will abuse notation and refer to a generic member of the set of menus by $A \in \mathcal{A}$ while formally $\mathbb{A} \in \mathcal{A}$. [1] It should be pointed out that given a finite X, there exists $J \in \mathbb{N}$ such that $\mathcal{A} = \{A_1, \ldots, A_J\}$. [2]

In this study, we justify *the multiple-selves setting* where with a probability $p \in [0, 1]$ that we identify endogenously from the given underlying preferences of the decision maker,

[1] It needs to be told that, in general, we do not need the assumption concerning the finiteness of X. In fact, as in Chapter 2, we can let X denote a compact metric space of alternatives and $\mathbb{X}$ identifies the simplex formed on X endowed with the weak* topology and $\mathbb{X}$ is well-known to be non-empty and convex and weak* compact. Then, any $A \in \mathcal{X}$, where $\mathcal{X}$ denotes the set of all non-empty and compact and measurable subsets of X (which can be equipped with the Hausdorff metric), identifies the set of discreet alternatives in menu $\mathbb{A} \equiv \Delta(A)$, the set of probability measures on the Borel sigma-algebra of A: thus, $\mathbb{A}$ is non-empty and convex and weak* compact. Again, the set of all menus would be denoted by $\mathcal{A}$. This method of defining menus, on the other hand, is not standard. In fact, as was done in Gul and Pesendorfer (2001) and Chatterjee and Krishna (2009), a menu is often defined as a compact subset of $\mathbb{X}$ and letting $\mathscr{A}$ be the non-empty and compact subsets of $\mathbb{X}$ endowed with the Hausdorff metric: Then, a menu is a member of $\mathscr{A}$. Even though this formulation can also be adopted in our setting, our way of defining a menu, we feel, is more convenient as menus are required to be defined via taking the simplex on some discreet alternatives. Then, using the standard linearity properties of the resulting von Neumann − Morgenstern expected utilities, in representation theorems one obtains the luxury of restricting attention to (discreet) alternatives rather than working with some exogenously given abstract set of lotteries.

[2] For each $A \in \mathcal{A}$, the exogenously given *temptation set* of A could be identified by Γ_A and in order to avoid non-fruitful complexities in notation we restrict attention to cases where the temptation set equals the menu; i.e. for any given menu $A \in \mathcal{A}$, let $\Gamma_A = A$.

the “ego” (alternatively, the long-run or the rational decision maker) gets to choose, a case to which we will refer to as the *ego contingency*; while with probability $(1-p)$, one of the many potential “alter egos” (alternatively, the ids or the short run selves) will choose and these cases will be referred to as the *alter ego contingencies*. In fact, given a menu $A \in \mathcal{X}$, the decision maker's ambiguity attitude involving the choice from this menu implies that the associated decision making process is as if with a probability p_A the ego is at the helm and the best alternative of the menu is consumed, while with probability $(1-p_A)$ the decision maker faces the ambiguity about which one of the alter egos gets to make a decision about which alternative to go for.

In order to apply the representation result under ambiguity presented and discussed in detail in Chapter 2 to the case with menus and obtain a multiple-selves representation, we need some extra structure on the set of states. Indeed, this addition does not tamper with any of the results of Chapter 2.

Given X and $J \in \mathbb{N}$ with menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ with $A_j \in \mathcal{X}$, if the decision maker has chosen menu A_j, then “a particular id” may choose any one of $a_j \in A_j$, $j = 1, \ldots, J$. Therefore, our state space is a Cartesian product of menus, $S \equiv A_1 \times \ldots \times A_J$ and a state $s \in S$ is equal to $(a_1, \ldots, a_J)$ with $a_j \in A_j$ for all $j = 1, \ldots, J$. So, in our formulation a particular id corresponds to a state, and vice versa: for any given particular id $s = (a_1, \ldots, a_J) \in S$, it must be that s's choice from A_j denoted by $a_{js} \in A_j$, equals to a_j, the jth dimension of s; therefore, $s = (a_{js})_{j=1}^{J}$. The cardinality of S, alternatively, the number of distinct and particular ids, equals some number $\bar{S} = \prod_{j=1}^{J} \#|A_j|$. [3]

The decision maker knows neither the particular id, $s \in S$, he/she faces, nor the exact

[3]The essence of the technical difficulties discussed in footnote 1 can clearly be seen at this stage, the construction of states of nature with menus. In fact, below we discuss how to handle cases when X is countably infinite: without loss of generality the set of menus (the set of all non-empty subsets of X) could be indexed using $[0,1]$; thus, we can write $\mathcal{A} = \{A_r : r \in [0,1]\}$. Then, a state $s \in S$ is a function mapping $[0,1]$ (i.e. menus) into an element of the menu under analysis; that is, for any A_r we have $s(r) \in A_r$, with $r \in [0,1]$. Thus, the states of nature S are all such functions. Still the resulting function space is well-behaved for us to construct a sigma algebra on, as was done in Chapter 2. However, when X is not necessarily countable, then we currently lack the technical training in order to identify the specific properties to be insisted on so that the resulting state space is well-behaved for us to be able obtain the construction and the representation theorem presented in Chapter 2.

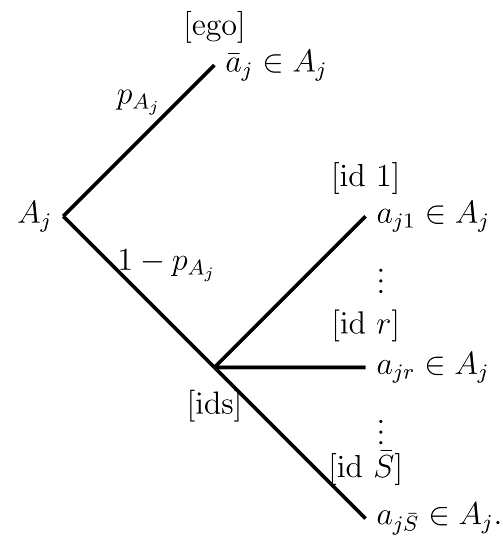

Figure 3.1: Our decision chart.

probability distribution on the potential ids he/she could face. Consequently, the implication of our representation result is portrayed in figure 3.1 where the ambiguity attitude of the decision maker considering menu A_j implies a setting as if the decision maker knows that the best alternative of that menu, $\bar{a}_j$, will be chosen by the ego (with a probability of p_{A_j}) while he/she does not have any idea what his/her alter ego will do.

Formally, given the sets of states and alternatives, S and X respectively, recall that $\mathcal{F}$ is a set of all acts (the set of functions mapping S in to $\mathbb{X}$) which is known to be convex while the set of constant acts is given by $\mathcal{F}_c$. This, in turn, translates to the following in the case with menus and allows us to define acts on menus: Given X and $J \in \mathbb{N}$ with menus $\mathcal{A} = A_1, \ldots, A_J$ with $A_j \in \mathcal{X}$, the act concerning an alternative a from menu A_j, denoted by $f_{(a,A_j)}$, is a function mapping S into A_j with the property that $f_{(a,A_j)}(s) = a_{js} \in A_j$ (the jth dimension of s) (which does not have to equal a). We denote the set of each act concerning an alternative from a menu by $\mathcal{F}_M$.

Below, we provide an example aiming to provide a tangible understanding for the current setting.

Example 1 *Let the set of alternatives be $X = \{a, b, c\}$. Then, menus $\mathcal{A}$ given by (simplices* *formed on the following sets)* $A_1 = \{a\}$ *and* $A_2 = \{b\}$ *and* $A_3 = \{c\}$ *(singleton menus), and* $A_4 = \{a, b\}$ *and* $A_5 = \{a, c\}$ *and* $A_6 = \{b, c\}$ *(doubleton menus), and* $A_7 = \{a, b, c\}$*. So, a state maps every menu into itself. That is* $s_1 = (a, b, c, a, a, b, a)$*, and* $s_2 = (a, b, c, b, a, b, a)$*, and* $s_3 = (a, b, c, a, c, b, a)$*, and* $s_4 = (a, b, c, a, a, b, b)$*, and* $\ldots$*, and* $s_{24} = (a, b, c, b, c, c, c)$*, that is* $\bar{S} = 24$*. Now, when say* A_5 *is chosen, then the third id,* s_3*, would choose* c*; and when* A_6 *is chosen,* s_3 *would choose* b*.*

In this case, $f_{(a,A_5)} \in \mathcal{F}_M$ *is a function mapping* S *into* A_5 *such that* $f_{(a,A_5)}(s) = a$ *whenever the fifth dimension of* s *equals* a *and* $f_{(a,A_5)}(s) = c$ *otherwise. On the other hand,* $f_{(c,A_5)} \in \mathcal{F}_M$ *is a function mapping* S *into* A_5 *such that* $f_{(c,A_5)}(s) = a$ *whenever the fifth dimension of* s *equals* a *and* $f_{(c,A_5)}(s) = c$ *otherwise. We will show below that* $f_{(a,A_5)} = f_{(c,A_5)}$ *is no coincidence.*

The reader will benefit to see an example of a constant act in the current setting. Let $h \in \mathcal{F}_c$ *be defined by* $h(s) = \frac{1}{2}a + \frac{1}{2}c$ *for all* $s \in S$*. Then, as* $f_{a,A_1}(s) = a$ *for all* s *and* $f_{c,A_3}(s) = c$ *for all* s*, we can let* $h(s) = \frac{1}{2}f_{a,A_1} + \frac{1}{2}f_{c,A_3}$*. Thus, this displays that any constant act can be obtained by employing a convex combination of acts concerning singleton menus.*

The last example concerns a non-constant act that cannot be obtained by any convex combination of acts concerning menus: Let $f \in \mathcal{F} \setminus \mathcal{F}_M$ *be given by* $f(s_1) = b$ *and* $f(s_2) = c$ *and* $f(s_3) = a$ *while* $f(s)$ *is arbitrary for other states. As for any* $a \in A$ *and* $f_{a,A}(s) \in A$ *for all* s*, notice that* $f(s_1) = b$ *and* $f(s_2) = c$ *and* $f(s_3) = a$ *cannot be obtained by any convex combination of acts concerning menus.*

The following is an immediate, yet important, implication of the construction of the states in the current setting:

Lemma 15 *For any* $a, b \in A$ *with* $A \in \mathcal{A}$*,* $f_{(a,A)} = f_{(b,A)} = f_A$*; in particular,* $f_{(a,A)} \asymp f_A$*.* [4]

Proof. As for any given $a, b \in A_j$ with $A_j \in \mathcal{A}$, $f_{(a,A_s)}(s) = a_{js} = f_{(b,A_j)}(s)$ for all $s \in S$, the result follows. ■

[4] Recall that the ambiguity concerning two acts $f, g \in \mathcal{F}$ are equivalent, denoted by $f \asymp g$, whenever there are $h, h' \in \mathcal{F}_c$ and $\lambda, \lambda' \in (0, 1]$ which satisfy $\lambda f + (1 - \lambda)h \sim^{UA} \lambda' g\ + (1 - \lambda')h'$.

Due to Lemma 15, without loss of generality we can restrict attention to acts associated with menus, $f_A \in \mathcal{F}_M$. Therefore, for any alternative from a given menu, the ambiguity of the act concerning the choice of an alternative from a menu is equivalent to the ambiguity of the associated menu. In other words, the ambiguity of the act concerning the choice of an alternative from a menu depends only on the menu.

Because of this observation, given X and $J \in \mathbb{N}$ with menus $\mathcal{A} = A_1, \ldots, A_J$ with $A_j \in \mathcal{X}$, it must be that for any $f_{A_j}, g_{A_j} \in \mathcal{F}_M$ we have $f_{A_j} = g_{A_j}$. Thus, given profile of menus $A_1, \ldots, A_J$ with $A_j \in \mathcal{X}$ for all $j = 1, \ldots, J$, the set of acts concerning menus is uniquely identified and $\mathcal{F}_M = \{f_{A_1}, \ldots, f_{A_J}\}$.

In this study, our aim is to obtain a utility representation on acts concerning menus involving ambiguity resulting from id's non-predictable behavior. While the construction of the states performs well in terms of handling acts associated with menus, one has to be careful with the interpretation that this construction implies on non-constant acts which are not associated with any convex combination of acts on menus.

Therefore, in the current simple setting our underlying hypothesis is that the construction of states using menus suffices to provide enough richness to handle all relevant states of nature. If that is not the situation, the following construction of states will take care of the inconvenience: Let S^* be the set representing all possible states of nature when there are no menus under consideration, and for the profile of menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ with $A_j \in \mathcal{X}$ for all $j = 1, \ldots, J$ we define $S \equiv \times_{j=1}^{J} A_j \times S^*$. Then, a state $s = (a_1, \ldots, a_j, s^*)$ represents the id who chooses alternative a_j from menu A_j in the state of nature s^*.

Even though this method of constructing the states of nature is guaranteed to provide enough dimensions to handle all possible states of nature including those with menus, for reasons of exposition we restrict attention to states of nature using menus.

Recall that $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ is a preference relation over $\mathcal{F}$ and $\succsim^{\mathbb{X}}$ a preference relation on $\mathbb{X}$ with an associated utility function $u : \mathbb{X} \to \mathbb{R}$ while the unambiguously preference relation is given by $\succsim^{UA}$ as follows: $f \succsim^{UA} g$, whenever $\lambda f + (1-\lambda)m \succsim \lambda g + (1-\lambda)m$ for all $\lambda \in (0,1]$ and for all $m \in \mathcal{F}$.

The state that corresponds to the vector listing the best alternatives from each associated menu, indeed, is nothing but the ego: among all the possible ids, one of them is the ego. Using that observation, the following condition ensures that the state corresponding to the ego is assigned a probability of 1 in some probability distribution among the ones obtained in the representation theorem, Theorem 5. This, in turn, allows us to employ the α-maxmin expected utility representation when evaluating menus and conclude that the resulting behavior under ambiguity is as if the following holds: with some endogenously determined menu specific probability, the decision maker consumes the best alternative of that menu while he/she does not have any idea and imagines the worst case scenario about the behavior of his/her alter egos in the other cases.

We need to have the following concerning the best elements of a given menu prior to the statement and formal discussion of our condition. When $\succsim$ on $\mathcal{F}$ satisfies Axioms 1–5, by Theorem 1, we know there is a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ representing the preference relation $\succsim^{\mathbb{X}}$ defined on $\mathbb{X}$. For any given finite X and resulting $J \in \mathbb{N}$ with menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ with $A_j \in \mathcal{X}$, it must be that $\mathbb{A}_j$ is a non-empty convex and compact subset of $\mathbb{X}$; ipso facto, $\bar{A}_j = \{a_j \in A_j : u(a_j) \geq u(a'_j) \text{ for all } a'_j \in A_j\}$ is non-empty and contains the best (discreet) alternatives in A_j, while $\underline{A}_j = \{a_j \in A_j : u(a_j) \leq u(a'_j) \text{ for all } a'_j \in A_j\}$ is also non-empty and contains the worst (discreet) alternatives in A_j.

For any given finite X and resulting $J \in \mathbb{N}$ with menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ with $A_j \in \mathcal{X}$, define $J^+ = \{j : u(\bar{a}_j) > u(\underline{\ell}) \text{ with } \bar{a}_j \in \bar{A}_j\}$; the menus whose best alternatives provide strictly higher utilities than those provided by the worst lottery. Due to non-triviality, J^+ is non-empty. Notice that for any menu $r \notin J^+$, by definition it must be that $u(a) = u(\underline{\ell})$ for all $a \in A_r$; thus, we argue that menu r is unacceptable. That is why, we refer to J^+ as the *set of acceptable menus.*

The following condition requires that the best alternative of each acceptable menu provides strictly higher utility than the worst lottery, and each one of the acts concerning acceptable menus must be unambiguously strictly preferred to the worst (constant) act, obtaining the worst lottery in every possible state of nature. Therefore, we refer to this condition as *the strict unambiguous value of acceptable menus.*

Condition 1 *For any preference relation $\succsim$ defined on $\mathcal{F}$ and for any given finite set of alternatives X and resulting $J \in \mathbb{N}$ with menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ uniquely determining $\mathcal{F}_M = \{f_{A_1}, \ldots, f_{A_J}\}$ and for each non-constant affine function that is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ representing the preference relation $\succsim^{\mathbb{X}}$ defined on $\mathbb{X}$, there exists $k_0 \in (u(\underline{\ell}), u(\bar{\ell}))$ such that for all $j \in J^+$*

1. *$u(\bar{a}_j) - k_0 > 0$, for all $\bar{a}_j \in \bar{A}_j$, and*
2. *$f_{A_j} \succsim^{UA} h_{k_0}$ where $h_{k_0} \in \mathcal{F}_c$ such that $h_{k_0}(s) = \ell_{k_0} \in \mathbb{X}$ for all $s \in S$ with $u(\ell_{k_0}) = k_0$.*

Before presenting our main result, it maybe a good idea to remind that the ambiguity concerning two acts $f, g \in \mathcal{F}$ are equivalent, denoted by $f \asymp g$, whenever there are $h, h' \in \mathcal{F}_c$ and $\lambda, \lambda' \in (0, 1]$ which satisfy $\lambda f + (1-\lambda)h \sim^{UA} \lambda' g + (1-\lambda')h'$; and for any given $f \in \mathcal{F}$, the ambiguity equivalence class of f is $\mathcal{F}(f) = \{g \in \mathcal{F} : g \asymp f\}$ while $\mathfrak{F} = \{\mathcal{F}(f) : f \in \mathcal{F}\}$ identifies the set of ambiguity equivalence classes. Meanwhile, the the set of ambiguity equivalence classes obtained via non-crisp acts is given by $\mathfrak{F}^* = \{\mathcal{F}(f) : f \in \mathcal{F} \backslash \mathcal{M}\}$ where the set of crisp acts are $\mathcal{M} = \{m \in \mathcal{F} : f \sim g$ implies $(1-\lambda)f + \lambda m \sim (1-\lambda)g + \lambda m$ for all $\lambda \in (0, 1)\}$.

Now, we present our main result providing a utility notion for menus under ambiguity, in turn, a multiple-selves representation:

Theorem 6 *Given finite X and resulting menus $\mathcal{A} = \{A_1, \ldots, A_J\}$ and preference relation $\succsim$ defined on $\mathcal{F}$ satisfying Axioms 1–5 and Condition 1 implies there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$, and a non-empty and unique and weak* compact and convex set $\mathcal{C}$ of probabilities on Σ, and a function $p_{A_j} : \mathfrak{F} \to [0, 1]$ with a* *uniquely defined restriction on $\mathfrak{F}^*$ such that $f_{A_j} \succsim f_{A_{j'}}$ with $j, j' \in \{1, \ldots J\}$ if and only if*

$$I(u_{f_{A_j}}) = p_{A_j} u(\bar{a}_j) + (1 - p_{A_j}) \min_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P \tag{3.1}$$
$$\geq p_{A_{j'}} u(\bar{a}_{j'}) + (1 - p_{A_{j'}}) \min_{P \in \mathcal{C}} \int u_{f_{A_{j'}}} \mathrm{d}P = I(u_{f_{A_{j'}}})$$

where $\bar{a}_j \in \bar{A}_j, \bar{a}_{j'} \in \bar{A}_{j'}$.

Proof. By Theorem 1, we know that $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ satisfies Axiom 1–5 if and only if there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$ $f \succsim g$ if and only if $I(u_f) \geq I(u_g)$. Furthermore, by Theorem 2 we know that if $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ satisfies Axioms 1–5, then there exists a unique nonempty, weak* compact and convex set of $\mathcal{C}$ of probabilities on Σ such that for all $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ if and only if $\int_S u_f \mathrm{d}P \geq \int_S u_g \mathrm{d}P$ for all $P \in \mathcal{C}$. The particular acts we will restrict attention to when employing these theorems are acts concerning menus and satisfying Condition 1. To that regard, let $f_{A_1}, \ldots, f_{A_J}$ be the associated acts on menus and assume $k_0 \in (u(\underline{\ell}), u(\bar{\ell}))$, and h_{k_0} with $h_{k_0}(s) = \ell_{k_0} \in \mathbb{X}$ for all $s \in S$ and $u(\ell_{k_0}) = k_0$ be as given in Condition 1. By the second item of that condition, we have that $f_{A_j} \succsim^{UA} h_{k_0}$ for all $j \in J^+$, and by Theorem 2, this is equivalent to for all $j \in J^+$

$$\int_S u_{f_{A_j}} \mathrm{d}P \geq \int_S u_{h_{k_0}} \mathrm{d}P = k_0 \text{ for all } P \in \mathcal{C}.$$

Let $\bar{s} = (\bar{a}_1, \ldots, \bar{a}_J)$ with $\bar{a}_j \in \bar{A}_j$ for all $j \in J$. Notice that for any $r \notin J^+$, as was discussed above $u(a) = u(\underline{\ell})$ for all $a \in A_r$; thus, $A_r = \bar{A}_r$. It is clear that $f_{A_j}(\bar{s}) = \bar{a}_j$ by construction. Therefore, the key defining property of $\bar{s}$ concerns $j \in J^+$ as the others can be selected arbitrarily. Then, the following is a consequence of Condition 1:

Lemma 16 *$\mathcal{C}$ contains a distribution that assigns probability 1 to $\bar{s} \in S$.*

Proof. Recall that $B_0(\Sigma)$ is the set of all bounded Σ–measurable real valued simple functions. For the non-singular interval $K = [u(\bar{\ell}), u(\underline{\ell})]$ and $k_0 \in K^{\text{o}}$ as given in Condition

1, $B_0(\Sigma, K)$ (and $B_0(\Sigma, K - k_0)$) denotes the set of all bounded Σ–measurable real valued simple functions with a range given by K (and $K - k_0$, respectively). Moreover, $ba(\Sigma)$ is the set of all bounded, finitely additive set functions on Σ and $pc(\Sigma)$ denotes the set of all probability measures in $ba(\Sigma)$ where Σ is an algebra sigma of subsets of S. It maybe useful to point out that these notions were defined and discussed and extensively used in the proof of Theorem 2 (in particular, Lemma 7) on page 22.

Let $P^* \in pc(\Sigma)$ assign probability of 1 to $\bar{s}$ and probability of 0 to all other $s \in S \setminus \bar{s}$.

First, we remind that for all $f, g \in \mathcal{F}$, $f \succsim^{UA} g$ if and only if $u_f \trianglerighteq^{UA} u_g$ where $\trianglerighteq^{UA}$ is a nontrivial, continuous, conic and monotonic preorder on $B_0(\Sigma, K)$. Furthermore, as was observed in the proof of Lemma 7 (which is employed to prove Theorem 2), we have $u_f \trianglerighteq^{UA} u_g \Leftrightarrow u_f - k_0 \trianglerighteq^{\text{o}} u_g - k_0 \Leftrightarrow u_f - k_0 \trianglerighteq^{\Sigma} u_g - k_0$ where k_0 as given above; $\trianglerighteq^{\text{o}}$ and $\trianglerighteq^{\Sigma}$ are nontrivial, continuous, conic and monotonic preorders on $B_0(\Sigma, K - k_0)$ and $B_0(\Sigma)$, respectively. Therefore, by Condition 1, we have for all $j \in J^+$ we have

$$f_{A_j} \succsim^{UA} h_{k_0} \quad \Leftrightarrow \quad u_{f_{A_j}} \trianglerighteq^{UA} k_0 \quad \Leftrightarrow \quad u_{f_{A_j}} - k_0 \trianglerighteq^{\text{o}} \mathbf{0} \quad \Leftrightarrow \quad u_{f_{A_j}} - k_0 \trianglerighteq^{\Sigma} \mathbf{0}.$$

Next, we wish to remind that the construction of the set of probabilities $\mathcal{C}$ presented in the proof of Lemma 7 calls for $P \in \mathcal{C}$ whenever $P \in \mathcal{L} \cap pc(\Sigma)$ where $\mathcal{L} = \{L \in ba(\Sigma) : L(\psi) \geq 0, \text{ for all } \psi \in B_0(\Sigma) \text{ with } \psi \trianglerighteq^{\Sigma} \mathbf{0}\}$. In this case, notice that $L \in ba(\Sigma)$ is in $\mathcal{L}$ if and only if $L(u_{f_{A_j}} - k_0) \geq 0$ for all $j \in J^+$ since (by Condition 1) $u_{f_{A_j}} - k_0 \trianglerighteq^{\Sigma} \mathbf{0}$ for all such j. Clearly $P^* \in ba(\Sigma)$. So, we need to show that $P^* \in \mathcal{L}$; i.e. $P^*(u_{f_{A_j}} - k_0) \geq 0$ for all $j \in J^+$. Since P^* is constant additive, we have $P^*(u_{f_{A_j}} - k_0) = P^*(u_{f_{A_j}}) - k_0$; and as P^* assigns probability of 1 to $\bar{s} \in S$, we have $P^*(u_{f_{A_j}}) = u(\bar{a}_j)$; hence, $P^*(u_{f_{A_j}} - k_0) = P^*(u_{f_{A_j}}) - k_0 = u(\bar{a}_j) - k_0 > 0$ for all $j \in J^+$ which follows from Condition 1. Therefore, $P^* \in \mathcal{L}$. Thus, $P^* \in \mathcal{L} \cap pc(\Sigma)$ which implies $P^* \in \mathcal{C}$. ■

In what follows, we will show that $I(u_{f_{A_j}}) = p_{A_j} u(\bar{a}_j) + (1 - p_{A_j}) \min_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P$, for all $j = 1, \ldots, J$. By Theorem 5, we know that $\succsim$ satisfies Axioms 1–5 if and only if there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$, and a non-empty and unique and weak* compact and convex set $\mathcal{C}$ of probabilities on Σ (I, u and $\mathcal{C}$ are obtained through Theorem 1 and Theorem 2), and a function $\alpha : \mathfrak{F} \to [0,1]$ with a uniquely defined restriction on $\mathfrak{F}^*$ such that for any $f \in \mathcal{F}$, $I(u_f) = \alpha_{\mathcal{F}(f)} \max_{P\in\mathcal{C}} \int u_f \mathrm{d}P + (1 - \alpha_{\mathcal{F}(f)}) \min_{P\in\mathcal{C}} \int u_f \mathrm{d}P$. Therefore, for f_{A_j}, $j \in J^+$, we have:

$$I(u_{f_{A_j}}) = \alpha_{\mathcal{F}(f_{A_j})} \max_{P\in\mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P + (1 - \alpha_{\mathcal{F}(f_{A_j})}) \min_{P\in\mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P$$

where $\alpha_{\mathcal{F}(f_{A_j})} \in [0,1]$. Since $P^* \in \mathcal{C}$ and $u_{f_{A_j}}(\bar{s}) \geq u_{f_{A_j}}(s)$ for all $s \in S \setminus \bar{s}$, it must be that $\int u_{f_{A_j}} \mathrm{d}P^* = u(\bar{a}_j) \geq \int u_{f_{A_j}} \mathrm{d}P$ for all $P \in \mathcal{C}$ and for all $j \in J^+$. The same observation trivially holds for all $r \notin J^+$, as for all such r we have $u(a) = u(\underline{\ell})$ for all $a \in A_r$. Therefore, $\max_{P\in\mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P = u(\bar{a}_j)$; thus, we set $p_{A_j} = \alpha_{\mathcal{F}(f_{A_j})}$; then, we have:

$$I(u_{f_{A_j}}) = p_{A_j} u(\bar{a}_j) + (1 - p_{A_j}) \min_{P\in\mathcal{C}} \int u_f \mathrm{d}P,$$

for all $j = 1, \ldots, J$; concluding the proof of Theorem 6. ■

4 Temptation

In this chapter, we establish that the implications of our representation result concerning menus under ambiguity under a regularity condition. After presenting the resulting representation result and elaborating on its implications, we show that what we obtain becomes compatible with earlier models of temptation

4.1 Dual-self representation

The regularity condition we wish to employ considers the set of probabilities obtained in the proof of Theorem 2: Restrict attention to unambiguous preference, $\succsim^{UA}$, be sustained only by monotonicity. That is, we would require the existence of $k_0 \in (u(\underline{\ell}), u(\bar{\ell}))$ such that the only unambiguous preference relation allowed concerns $f_{A_j} \succsim^{UA} h_{k_0}$ with j such that $A_j = \{a_j\}$ (a singleton menu) with $u(a_j) - k_0 > 0$. Then, $\mathcal{C} = pc(\Sigma)$; i.e., $\mathcal{C}$ includes all probability distributions on Σ.

As a result, our representation result for menus under ambiguity simplifies to the following:

Corollary 1 *Suppose a preference relation $\succsim$ defined on $\mathcal{F}$ satisfies Axioms 1–5 and the set of probability distributions $\mathcal{C} = pc(\Sigma)$. Then, there exists a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$ and a function $p : \mathcal{X} \to [0,1]$ such that for any $A, A' \in \mathcal{A}$ we have $f_A \succsim f_{A'}$ if and only if*

$U(A) \geq U(A')$ where

$$U(A) = p_A u(\bar{a}) + (1 - p_A) u(\underline{a}) \tag{4.1}$$

with $\bar{a} \in \bar{A} = \{a \in A : u(a) \geq u(a')$ for all $a' \in A\}$ and $\underline{a} \in \underline{A} = \{a \in A : u(a') \geq u(a)$ for all $a' \in A\}$.

Proof. Given X and $\mathcal{A} = \{A_1, \ldots, A_J\}$, by Theorem 5, we know that there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$, and a non-empty and unique and weak* compact and convex set $\mathcal{C}$ of probabilities on Σ, and a function $p_{A_j} : \mathfrak{F} \to [0, 1]$ with a uniquely defined restriction on $\mathfrak{F}^*$ such that the following function represents $\succsim$ on $\mathcal{F}_M$: for any $f_{A_j} \in \mathcal{F}_M$

$$I(u_{f_{A_j}}) = \alpha_{\mathcal{F}(f_{A_j})} \max_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P + (1 - \alpha_{\mathcal{F}(f_{A_j})}) \min_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P$$

where $\alpha_{\mathcal{F}(f_{A_j})} \in [0, 1]$. Since $\mathcal{C} = pc(\Sigma)$, there exist $P_{\bar{s}}, P_{\underline{s}} \in \mathcal{C}$ which assign probability 1 to $\bar{s} \in S$ and $\underline{s} \in S$, respectively, for $\bar{s} = (\bar{a}_1, \ldots, \bar{a}_J)$ with $\bar{a}_j \in \bar{A}_j$ and $\underline{s} = (\underline{a}_1, \ldots, \underline{a}_J)$ with $\underline{a}_j \in \underline{A}_j$; so, $f_{A_j}(\bar{s}) = \bar{a}_j$ and $f_{A_j}(\underline{s}) = \underline{a}_j$ for all $j = 1, \ldots, J$. Therefore, $\int u_{f_{A_j}} \mathrm{d}P_{\bar{s}} = u(\bar{a}_j) \geq \int u_{f_{A_j}} \mathrm{d}P$ and $\int u_{f_{A_j}} \mathrm{d}P \geq \int u_{f_{A_j}} \mathrm{d}P_{\underline{s}} = u(\underline{a}_j)$ for all $P \in \mathcal{C}$ and for all $j = 1, \ldots, J$. Hence, $\max_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P = u(\bar{a}_j)$ and $\min_{P \in \mathcal{C}} \int u_{f_{A_j}} \mathrm{d}P = u(\underline{a}_j)$ which, together with setting $U(A_j) = I(u_{f_{A_j}})$ and $p_{A_j} = \alpha_{\mathcal{F}(f_{A_j})}$ for all j, implies $U(A_j) = p_{A_j} u(\bar{a}_j) + (1 - p_{A_j}) u(\underline{a}_j)$. ■

Theorem 6 in Chapter 2 presents our representation result justifying the multiple-selves setting, in which with some probability the decision maker will face the ambiguity about which one of the specific alter egos (multiple-selves) will make the decision. The decision maker when faced with this ambiguity knows neither the particular alter ego who will be making the decision, nor the exact distribution on all the alter egos that will select the deciding one. In that representation, our key condition involved a very mild requirement on the behavior of unambiguous preference relations concerning acceptable menus: Condition 1 allows for any unambiguous preference relation such that each one of the acts concerning acceptable menus must be unambiguously strictly preferred to the worst (constant) act. Then, given such an unambiguous preference relation we show that the best case scenario when evaluation a menu corresponds to the consumption of the best alternative in that menu while the worst case scenario involves the aforementioned ambiguity and does not necessarily coincide with the consumption of the worst alternative of that menu. Therefore, when the unambiguous preference relation displays some richness, we obtain a multiple-selves representation, rather than a dual-self.

When unambiguous preference relations are those implied only by monotonicity, Corollary 1 displays that the resulting ambiguity is one which makes the decision maker imagine that the worst case scenario amounts to the consumption of the worst alternative of that menu. This, in turn, provides the dual-self representation presented above: When the decision maker evaluates a menu, he/she imagines that the ego (or rational-self) is at the helm and the best alternative of the menu will be consumed with some probability whereas with the remaining probability the alter-ego (evil-self) is making the decision and the worst alternative of the menu will be consumed.

Hence, when unambiguous preferences become more restricted, the multiple-selves model turns into a dual-self version.

One notable point of our findings is that the implications of our representation results do not involve any rationality on the alter-egos behavior; e.g. we do not impose an independence axiom on menus. In essence, alter-egos in our model can be viewed as "behavioral types" or "machines".

We would like emphasize that our result stated in Corollary 1 resembles the Hurwicz α–criterion which is introduced by Hurwicz (1951). This criterion states that a decision maker's pessimism level is indexed with $\alpha \in [0, 1]$; whenever he/she has to take ambiguous actions, he/she assigns coefficient α to the worst outcome and $(1-\alpha)$ to the best outcome in order to determine an associated value. Then, our representation can be seen as Hurwicz α–criterion by interpreting the probability of alter ego contingencies of a menu $A \in \mathcal{A}$; i.e., $(1 - p_A)$ as pessimism index. Nevertheless, one important difference between Hurwicz α–criterion and our model is that probability $(1-p_A)$ of alter egos making the decision is menu dependent for all menus $A \in \mathcal{A}$ whereas pessimism index is constant for all actions concerning ambiguity.

4.2 Constant ambiguity-aversion index

At this stage it is useful to point that Ghirardato, Maccheroni, and Marinacci (2004) tries to obtain the α-maxmin expected utility representation of Hurwicz (1951) by coming up with an axiom in order to sustain a constant ambiguity-aversion index. Below we present this axiom. Given any $f \in \mathcal{F}$, define

$$\mathbb{X}^{UA}(f) = \{\ell \in \mathbb{X} : h_{\ell'} \succsim^{UA} f \Rightarrow h_{\ell'} \succsim^{UA} h_\ell \text{ and } f \succsim^{UA} h_{\ell'} \Rightarrow h_\ell \succsim^{UA} h_{\ell'}, \forall \ell' \in \mathbb{X}\},$$

and notice $\mathbb{X}^{UA}(f)$ provides the certainty equivalent (lotteries) with respect to unambiguous preference relation $\succsim^{UA}$ associated with the act f. Then, the axiom they use is:

Axiom 6 $\mathbb{X}^{UA}(f) = \mathbb{X}^{UA}(g)$ *implies* $f \sim g,\ f, g \in \mathcal{F}$.

In words, this axiom demands that any two acts f and g having the same unambiguous certainty equivalent sets must be indifferent to one another.

Using this axiom on top of Axioms 1–5, in Proposition 19 Ghirardato, Maccheroni, and Marinacci (2004) obtains the following α-maxmin representation:

A binary relation $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ satisfies axioms 1-6 if and only if there exist a monotone and constant linear functional $I : B_0(\Sigma) \to \mathbb{R}$, and a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$, and a non-empty and unique and weak* compact and convex set $\mathcal{C}$ of probabilities on Σ, and $\alpha \in [0,1]$ such that $f \succsim g$ if and only if

$$\alpha \max_{P \in \mathcal{C}} \int u_f \mathrm{d}P + (1-\alpha) \min_{P \in \mathcal{C}} \int u_f \mathrm{d}P \geq \alpha \max_{P \in \mathcal{C}} \int u_g \mathrm{d}P + (1-\alpha) \min_{P \in \mathcal{C}} \int u_g \mathrm{d}P$$

and u_f and u_g and $\mathcal{C}$ are as in the representation theorem of $\succsim^{UA}$, Theorem 2, i.e. $f \succsim^{UA} g$ if and only if $\int_S u_f \mathrm{d}P \geq \int_S u_g \mathrm{d}P$ for all $P \in \mathcal{C}$.

Eichberger, Grant, Kelsey, and Koshevoy (2011), in a recent paper, shows that when the state space, S, is finite, a preference relation $\succsim$ defined on $\mathcal{F}$ satisfying Axioms 1-6 implies that the constant ambiguity-aversion index, α, equals either 0 or 1. This, clearly, is bad news for the α-maxmin expected utility representation with finite state spaces, because in these cases we get either maxmax or maxmin expected utility representation.

4.3 Related temptation models

Using some axioms Chatterjee and Krishna (2009) obtains a "dual-self" representation theorem for menu preferences and as a result the decision "could be interpreted as being made by an "alter ego" who appears randomly". As in Gul and Pesendorfer (2001), this study considers that elements of menus are lotteries and employs independence on the set of menus. According to their representation theorem, (by restricting attention to degenerate lotteries by using standard linearity properties of von Neumann – Morgenstern utilities) the utility of a menu $A \in \mathcal{A}$ is given by

$$U^{CK}(A) = p \max_{a \in A} \tilde{u}(a) + (1-p) \max_{a' \in B_v(A)} \tilde{u}(a'),$$

where $(1-p)$ is probability of the alter ego making the choice, $\tilde{u}$ is the ego's von Neumann – Morgenstern expected utility function over lotteries and $B_{\tilde{v}}(A) = \text{argmax}_{a' \in A}\, \tilde{v}(a')$ the best alternatives in the menu according to the alter-ego's point of view which is represented by $\tilde{v}$, the alter-ego's von Neumann – Morgenstern expected utility function over lotteries.

The key axiom of Chatterjee and Krishna (2009), the *temptation axiom*, is: Given $A \in \mathcal{A}$, there exists $a, a' \in A$ such that $\{a\} \succsim^{CK} A \succsim^{CK} \{a'\}$.

In Chatterjee and Krishna (2005), the working paper version of Chatterjee and Krishna (2009), they present a representation theorem which employs a menu dependent dual-self model such that the utility of a menu is given by:

$$U^{CK'}(A) = p_A \max_{a \in A} \tilde{u}(a) + (1-p_A) \max_{a' \in B_{\tilde{v}}(A)} \tilde{u}(a'),$$

where probability of the alter ego deciding, $(1 - p_A)$, is menu dependent.

The axiom of temptation in Chatterjee and Krishna (2005) is slightly different from the one defined above. For any given menu $A \in \mathcal{A}$, let $\bar{A} \equiv \{a \in A : \{a\} \succsim^{CK'} \{a'\}$ for all $a' \in A\}$ and $\underline{A} \equiv \{a \in A : \{a'\} \succsim^{CK'} \{a\}$ for all $a' \in A\}$. $\succsim^{CK'}$ satisfies the temptation axiom if $\bar{A} \succsim^{CK'} A \succsim^{CK'} \underline{A}$ for any $A \in \mathcal{A}$.

Below, we show that our representation theorem of acts concerning menus delivers the temptation axioms of Chatterjee and Krishna (2005) and Chatterjee and Krishna (2009).

Corollary 2 *Suppose that $\succsim$ defined on $\mathcal{F}$ satisfies Axioms 1–5 and the set of probability distributions $\mathcal{C} = pc(\Sigma)$. Then, there exists a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$ and a function $p : \mathcal{A} \to [0, 1]$ such that for any $A \in \mathcal{A}$ it must be that*

1. *$h_{\bar{a}} \succsim f_A \succsim h_{\underline{a}}$, and*

2. *$f_{\bar{A}} \succsim f_A \succsim f_{\underline{A}}$,*

where $h_{\bar{a}}, h_{\underline{a}} \in \mathcal{F}_c$ are defined as $h_{\bar{a}}(s) = \bar{a}$ and $h_{\underline{a}}(s) = \underline{a}$ for all $s \in S$ and $\bar{a} \in \bar{A} = \{a \in A : u(a) \geq u(a')$ for all $a' \in A\}$ and $\underline{a} \in \underline{A} = \{a \in A : u(a') \geq u(a)$ for all $a' \in A\}$.

Proof. By Corollary 1, we know that $U(f_A) = p_A u(\bar{a}) + (1 - p_A)u(\underline{a})$.

Then, $u(\bar{a}) \geq p_A u(\bar{a}) + (1 - p_A)u(\underline{a}) \geq u(\underline{a})$ for any $p_A \in [0, 1]$; hence, $h_{\bar{a}} \succsim f_A \succsim h_{\underline{a}}$.

Now, as for any $\bar{a} \in \bar{A}$ and $a_* \in \underline{A}$ we have $u(\bar{a}) = u(\ell)$ for all $\ell \in \Delta(\bar{A})$ and $u(a_*) = u(\ell')$ for all $\ell' \in \Delta(\underline{A})$, $U(\bar{A}) = u(\bar{a})$ and $U(\underline{A}) = u(\underline{a})$; ipso facto, $u(\bar{A}) \geq p_A u(\bar{A}) + (1-p_A)u(\underline{A}) \geq u(\underline{A})$ for any $p_A \in [0, 1]$; hence, $f_{\bar{A}} \succsim f_A \succsim f_{\underline{A}}$. ■

A remark that the interested reader may find helpful is that the conclusion of Corollary 2 can be obtained without restricting attention to the situation with $\mathcal{C} = pc(\Sigma)$; but, using Condition 1. As for any $A \in \mathcal{A}$, $I(u_{f_{\bar{A}}}) = u(\bar{a})$ for $\bar{a} \in \bar{A}$ and $I(u_{f_{\underline{A}}}) = u(\underline{a})$ for $\underline{a} \in \underline{A}$ and $u(\bar{a}) \geq \max_{P \in \mathcal{C}} \int u_{f_A} \mathrm{d}P$ and $\min_{P \in \mathcal{C}} \int u_{f_A} \mathrm{d}P \geq u(\underline{a})$ and $\alpha_A \in [0, 1]$, the conclusion would be obtained; delivering the following:

Remark 1 *Suppose that $\succsim$ defined on $\mathcal{F}$ satisfies Axioms 1–5 and Condition 1. Then, there exists a non-constant affine function which is unique up to a positive affine transformation $u : \mathbb{X} \to \mathbb{R}$ representing $\succsim^{\mathbb{X}}$ and a function $p : \mathcal{A} \to [0,1]$ such that for any $A \in \mathcal{A}$ it must be that $h_{\bar{a}} \succsim f_A \succsim h_{\underline{a}}$, and $f_{\bar{A}} \succsim f_A \succsim f_{\underline{A}}$ where $h_{\bar{a}}, h_{\underline{a}} \in \mathcal{F}_c$ are defined as $h_{\bar{a}}(s) = \bar{a}$ and $h_{\underline{a}}(s) = \underline{a}$ for all $s \in S$ and $\bar{a} \in \bar{A}$ and $\underline{a} \in \underline{A}$.*

On the other hand, Gul and Pesendorfer (2001) pioneered the temptation and self-control literature with their well-known representation theorem. Our model with menus corresponds to a version of theirs in which attention is restricted to lotteries that can be obtained as a convex combination of some discreet lotteries; in turn, enabling us to confine our consideration to discreet lotteries when identifying best/worst alternatives of a menu (due to standard linearity properties of the von Neumann – Morgenstern utilities).[1] The preferences that Gul and Pesendorfer (2001) consider on menus, $\succsim^{GP} \subseteq \mathcal{A} \times \mathcal{A}$, are binary relations defined on the simplex formed on non-empty compact sets of alternatives, i.e. $\mathcal{A} = \{\Delta(A) : A \in \mathcal{A}\}$. Utilizing some set of axioms (including independence on the set of menus), Gul and Pesendorfer (2001) obtains a representation theorem using the following utility of a menu:

$$U^{GP}(A) = \max_{a \in A} \big(\tilde{u}(a) + \tilde{v}(a)\big) - \max_{a' \in A} \tilde{v}(a'),$$

where $\tilde{u}, \tilde{v}$ are von Neumann – Morgenstern expected utility functions over lotteries and $\tilde{u}$ represents the decision maker's commitment ranking while $\tilde{v}$ represents his/her temptation ranking over lotteries. Therefore, given $a^*, \tilde{a} \in A$ with $a^* \in \arg\max_{a \in A}(\tilde{u}(a) + \tilde{v}(a))$ and $\tilde{a} \in \arg\max_{a' \in A} \tilde{v}(a')$, the utility of a menu $U(A)$ equals $\tilde{u}(a^*) - (\tilde{v}(\tilde{a}) - \tilde{v}(a^*))$ where the cost of self-control is given by $(\tilde{v}(\tilde{a}) - \tilde{v}(a^*))$.

An interesting remark concern the fact that Chatterjee and Krishna (2005) shows that preferences which satisfy the axioms needed for the representation of Gul and Pesendorfer

[1] It needs to be emphasized that our setting is, in fact, rich enough to handle that of Gul and Pesendorfer (2001) which defines the set of menus as weak* compact subsets of $\mathbb{X}$, thus, allowing menus to consist of only non-discreet lotteries. While this situation can be handled with the technicalities presented in the current study (see footnote 1 of Chapter 3), we argue that our formulation is better suited for preferences on menus as in representation theorems we can confine interest to (discreet) alternatives.

(2001) also admit menu dependent dual-self representations as defined above.

In order to obtain the representation theorem of Gul and Pesendorfer (2001), that study introduced and employed a key axiom, *set betweenness*, which is stated as follows: For any menu $A, A' \in \mathcal{A}$ with $A \succsim^{GP} A'$ implies $A \succsim^{GP} A \cup A' \succsim^{GP} A'$.

For any given pair of menus $A_r, A_t \in \mathcal{A}$, we denote $A_{r\cup t} = A_r \cup A_t \in \mathcal{A}$.

In what follows, we analyze situations when our representation theorem of acts concerning menus under ambiguity is rich enough to deliver the set betweenness axiom of Gul and Pesendorfer (2001). In fact, we wish to obtain the following result, currently a conjecture:

Conjecture 1 *Let X be a finite set of alternatives and $\mathcal{A} = \{A_1, \dots, A_J\}$ with $A_j \in \mathcal{A}$ for all $j = 1, \dots, J$ be the resulting menus. Then, there exists a preference relation $\succsim$ defined on $\mathcal{F}$ satisfying Axioms 1–5 and the set of probability distributions $\mathcal{C} = pc(\Sigma)$ and $p : \mathcal{A} \to [0, 1]$ such that $f_{A_r} \succsim f_{A_t}$ implies $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$.*

In order to establish that result, we would suppose that a preference relation $\succsim$ defined on $\mathcal{F}$ satisfies Axioms 1–5 and the set of probability distributions $\mathcal{C} = pc(\Sigma)$ (which can be obtained by a condition on unambiguous preferences, e.g. the regularity condition we discussed in the beginning of Chapter 4). Then, by Corollary 1, for any $A \in \mathcal{A}$, $U(A) = p_A u(\bar{a}) + (1 - p_A)u(\underline{a})$ with $\bar{a} \in \bar{A} = \{a \in A : u(a) \geq u(a') \text{ for all } a' \in A\}$ and $\underline{a} \in \underline{A} = \{a \in A : u(a') \geq u(a) \text{ for all } a' \in A\}$.

The remaining task is to establish that there exists $p : \mathcal{A} \to [0, 1]$ such that $f_{A_r} \succsim f_{A_t}$ implies $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$ and the resulting unambiguous preference relation $\succsim^{UA}$ is such that $\mathcal{C}$ is equal to $pc(\Sigma)$.

So, let $f_{A_r} \succsim f_{A_t}$.

If $u(\bar{a}_r) \geq u(\bar{a}_t)$ and $u(\underline{a}_r) \geq u(\underline{a}_t)$, then let $p_{A_{r\cup t}}$ satisfy

$$\frac{p_{A_r}\left(u(\bar{a}_r) - u(\underline{a}_r)\right) + \left(u(\underline{a}_r) - u(\underline{a}_t)\right)}{\left(u(\bar{a}_r) - u(\underline{a}_t)\right)} \geq p_{A_{r\cup t}} \geq \frac{p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t))}{(u(\bar{a}_r) - u(\underline{a}_t))}. \tag{4.2}$$

Then $u(\bar{a}_{r\cup t}) = u(\bar{a}_r)$ and $u(\underline{a}_{r\cup t}) = u(\underline{a}_t)$ which implies $U(A_{r\cup t}) = p_{A_{r\cup t}}\left(u(\bar{a}_r) - u(\underline{a}_t)\right) + u(\underline{a}_t)$ due to Corollary 1. Thus, by (4.2), $p_{A_r}\left(u(\bar{a}_r) - u(\underline{a}_r)\right) + \left(u(\underline{a}_r) - u(\underline{a}_t)\right) \geq p_{A_{r\cup t}}(u(\bar{a}_r) -$

$u(\underline{a}_t)) \geq p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t))$ implies $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$.

If $u(\bar{a}_r) \geq u(\bar{a}_t)$ and $u(\underline{a}_t) \geq u(\underline{a}_r)$, then let $p_{A_{r\cup t}}$ satisfy

$$p_{A_r} \geq p_{A_{r\cup t}} \geq \frac{p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t)) + (u(\underline{a}_t) - u(\underline{a}_r))}{(u(\bar{a}_r) - u(\underline{a}_r))}. \tag{4.3}$$

So $u(\bar{a}_{r\cup t}) = u(\bar{a}_r)$ and $u(\underline{a}_{r\cup t}) = u(\underline{a}_r)$ bringing about $U(A_{r\cup t}) = p_{A_{r\cup t}}(u(\bar{a}_r) - u(\underline{a}_r)) + u(\underline{a}_r)$ by Corollary 1. Thus, by (4.3), $p_{A_r}(u(\bar{a}_r) - u(\underline{a}_r)) \geq p_{A_{r\cup t}}(u(\bar{a}_r) - u(\underline{a}_r)) \geq p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t)) + (u(\underline{a}_t) - u(\underline{a}_r))$ implies $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$.

If $u(\bar{a}_t) \geq u(\bar{a}_r)$ and $u(\underline{a}_t) \geq u(\underline{a}_r)$, then let $p_{A_{r\cup t}}$ with

$$\frac{p_{A_r}(u(\bar{a}_r) - u(\underline{a}_r))}{(u(\bar{a}_t) - u(\underline{a}_r))} \geq p_{A_{r\cup t}} \geq \frac{p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t)) + (u(\underline{a}_t) - u(\underline{a}_r))}{(u(\bar{a}_t) - u(\underline{a}_r))}. \tag{4.4}$$

Now, $u(\bar{a}_{r\cup t}) = u(\bar{a}_t)$ and $u(\underline{a}_{r\cup t}) = u(\underline{a}_r)$ so $U(A_{r\cup t}) = p_{A_{r\cup t}}(u(\bar{a}_t) - u(\underline{a}_r)) + u(\underline{a}_r)$ by Corollary 1. Ipso facto, by (4.4), $p_{A_r}(u(\bar{a}_r) - u(\underline{a}_r)) \geq p_{A_{r\cup t}}(u(\bar{a}_t) - u(\underline{a}_r)) \geq p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t)) + (u(\underline{a}_t) - u(\underline{a}_r))$ implies $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$.

Finally, if $u(\bar{a}_t) \geq u(\bar{a}_r)$ and $u(\underline{a}_r) \geq u(\underline{a}_t)$, then let $p_{A_{r\cup t}}$ with

$$\frac{p_{A_r}(u(\bar{a}_r) - u(\underline{a}_r)) + u(\underline{a}_r) - u(\underline{a}_t)}{(u(\bar{a}_t) - u(\underline{a}_t))} \geq p_{A_{r\cup t}} \geq p_{A_t}. \tag{4.5}$$

Thence, $u(\bar{a}_{r\cup t}) = u(\bar{a}_t)$ and $u(\underline{a}_{r\cup t}) = u(\underline{a}_t)$ so $U(A_{r\cup t}) = p_{A_{r\cup t}}(u(\bar{a}_t) - u(\underline{a}_t)) + u(\underline{a}_t)$ by Corollary 1. Hence, by (4.5), $p_{A_r}(u(\bar{a}_r) - u(\underline{a}_r)) + u(\underline{a}_r) - u(\underline{a}_t) \geq p_{A_{r\cup t}}(u(\bar{a}_t) - u(\underline{a}_t)) \geq p_{A_t}(u(\bar{a}_t) - u(\underline{a}_t))$ implying $f_{A_r} \succsim f_{A_{r\cup t}} \succsim f_{A_t}$.

What remains to be analyzed is whether or not the following holds: Given X and $\mathcal{A} = \{A_1, \ldots, A_J\}$ and p_f, $f \in \mathcal{F}$, obtained by employing Theorem 5 with the particular restrictions given in 4.2 - 4.5, define for any $f \in \mathcal{F}$

$$I(u_f) = p_f \max_{a \in X(f)} u(a) + (1 - p_f) \min_{a \in X(f)} u(a), \tag{4.6}$$

where $X(f) = \{a \in X : f(s) = \ell \text{ for some } s \in S \text{ with } \ell(a) > 0\}$ (i.e. the support of f in alternatives).

The essence of this problem is due to the following fixed point issue at hand: Given a fixed $\mathcal{C}$ and p_f, $f \in \mathcal{F}$, representation equation of Theorem 5, equation 2.4, given by

$$I(u_f) = p_f \max_{P \in \mathcal{C}} \int u_f \mathrm{d}P + (1 - p_f) \min_{P \in \mathcal{C}} \int u_f \mathrm{d}P \tag{4.7}$$

identifies a preference relation $\succsim$ and $\succsim^{UA}$ both defined on $\mathcal{F}$. Then, the resulting unambiguous preference relation $\succsim^{UA}$ implies $\tilde{\mathcal{C}}$ as shown in the proof of Theorem 2. How do we know that $\tilde{\mathcal{C}} = \mathcal{C}$ the set of probability distributions that we started with?

This issue is raised and addressed in Section 5 of Ghirardato, Maccheroni, and Marinacci (2004) and it is shown that if $\mathcal{C}$ is equal to the Clarke differential of I, as given in equation 4.7, at 0.

Therefore, in order to prove the aforementioned conjecture what needs to be done is to show that the Clarke differential of I given in equation 4.6 at 0 is equal to $pc(\Sigma)$; a task to be addressed in the near future.

5 CONCLUDING REMARKS

In order to obtain the desired multiple-selves representation, first we associate each alter ego with a state of nature the realization of which make the associated alter ego decide what to consume in a given menu; and second, we extend the preferences under ambiguity to menus. Then, by employing the well-accepted axioms of Ghirardato, Maccheroni, and Marinacci (2004) on preferences on menus under ambiguity and introducing a mild condition, we obtain the resulting representation for those preferences justifying that the associated behavior under ambiguity admits a multiple-selves representation: When evaluating a menu the decision maker imagines that with some menu-dependent probability the "ego" who consumes the best alternative of that menu will be in charge, whereas with remaining probability the decision maker imagines the worst case scenario as he/she faces ambiguity about which one of many "alter egos" will be present and choose what to consume in that menu.

We also show that the multiple-selves representation model transforms to the dual-self representation if we employ more restrictive condition. Furthermore, this dual-self representation justifies temptation axioms in Chatterjee and Krishna (2009) and Chatterjee and Krishna (2005) which are the closest studies to ours in the spirit of representation. We analyze the cases where our representation model delivers the set betweenness axiom of Gul and Pesendorfer (2001) which leads us to direct our further studies to focus on the Clarke differential of functional I.

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