ON THE CHOICE OF A PUBLIC GOOD FOR AGENTS WITH DOUBLE-PEAKED PREFERENCES

by

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ABSTRACT

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We study the problem of choosing the location of a public good on a finite interval when agents have double-peaked preferences. A preference relation is double-peaked when an agent has two most preferred spots and her location is his least preferred in between these two spots. We assume that the locations of the agents are observable and agents report only their most preferred spots. We characterize strategy-proof mechanisms, and show that there is no strategy-proof and Pareto efficient mechanism for this problem. We also show that our results still hold when we replace the assumption of a finite interval with the continuity of the mechanism. Additionally, we discuss the consequences of dropping the assumption that the locations are observable, and the possibility of strategy-proof mechanisms that use the whole preference relations of the agents.

ÖZET

ÇİFT TEPELİ TERCİHLERİ OLAN AJANLARLA BİR KAMUSAL MAL İÇİN YER SEÇİMİ ÜZERİNE

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Bireylerin çift tepeli tercihleri olduğu durumda, sonlu bir aralıkta tek bir kamusal mal için yer seçimi problemini çalışıyoruz. Bir birey eğer iki adet en çok tercih ettiği noktaya sahipse ve bulunduğu konum bu iki nokta aralığında en az tercih ettiği noktaysa, tercihi çift tepeli tercihtir. Bireylerin konumlarının gözlemlenebilir olduğunu ve sadece en çok tercih ettikleri noktaları bildirdiklerini varsayıyoruz. Bu problem için manipüle edilemeyen mekanizmaları karakterize ediyoruz ve aynı anda Pareto optimal ve manipüle edilemeyen bir mekanizmanın olamayacağını gösteriyoruz. Aynı zamanda sonlu bir aralık varsayımımızı mekanizmanın sürekliliği ile değiştirdiğimizde sonuçlarımızın hala geçerli olduğunu gösteriyoruz. Ek olarak, gözlemlenebilir konum varsayımımızı kaldırmanın yaratacağı sonuçları ve bireylerin tercih bilgisinin tamamını kullanan manipüle edilemeyen mekanizmaların olasılığını tartışıyoruz.

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CHAPTER 1

INTRODUCTION

Suppose there is only one main street in the town where Mr. A lives, and the municipality is planning to build a school on that street. A nearby school means increased traffic and noise during some hours, which Mr. A would like to avoid. Moreover, suppose that he also has a child at school age, and does not want the school to be too far away either. In this case, Mr. A would prefer the school to be constructed at some distance to his house, in other words, his most preferred spots (which we will call his peaks) for the new school would be two points to the right and to the left of his location (Filos-Ratsikas et al., 2017). These peaks are not necessarily symmetric to the location of Mr. A, as it is possible that one side of the road is going up the hill, and therefore Mr. A wants the distance to be smaller if the school is built on that side.

The above is an example of what we call in this thesis double-peaked preferences. Double-peaked preferences are so that, an agent has two most preferred points on a real line and his location is his least preferred spot in between these two peaks. Double-peaked preferences can also represent political preferences of individuals on certain matters when the policy space is modeled on a real line (Egan, 2014), or individual preferences on government spending for a public good such as education when private alternatives also exist (Flowers, 1975). In the literature review, we discuss those examples in a more detailed manner.

We consider a problem where we need to decide the location of a public good. There is a group of agents, and they all have double-peaked preferences for the possible spots as discussed in the previous paragraph. There are many ways to choose such a location: for instance, we can make one of the agents a dictator, and choose one of his peaks regardless of the preferences of other agents; or we can randomly select a peak, among the ones that the agents report. We call such a process a mechanism. More formally, a mechanism collects information about the problem and provides a solution. Our main purpose for this study is to find "good mechanisms", that is to say, mechanisms that have desirable properties. We talk about these properties later in the introduction.

There are not many studies involving double-peaked preferences in the mechanism design literature. However, related domains such as single-peaked and single-dipped preferences have been studied extensively. Consider the case of a school teacher in our previous example. She might prefer the school to be built just next to her house for convenience. In this case, since she is worse off as the distance between the school and her location increases, her location would be her most preferred spot. Such a preference where an agent has a unique most preferred choice among the set of alternatives is called a single-peaked preference. Under certain institutional constraints as to the location of the public facility, our domain is related to problems with single peaked preferences. For instance, suppose the municipality decides to build the school to the left of Mr. A's location for a reason independent of agents' preferences. Now, Mr. A would actually have only one feasible spot that he prefers the most, and between the end of the street and his location, his preference would be single peaked. Another domain that is related to ours is single-dipped preferences. An example of that would be the construction of a waste treatment facility. Agents would prefer the facility to be as far away from their houses as possible in this case. This would mean the least preferred point (dip) for an agent would be her location, and the most preferred point would be either one of the ends of the street, depending on the distance of the agent to those ends. This domain also shares a similarity to ours in the same way as the single-peaked domain. If the municipality decides that the school will be built close enough to the location of Mr. A, such that the set of alternatives contain the location of Mr. A but not his peaks, Mr. A's preference would be single-dipped in this reduced space.

The two related literatures on social choice and mechanism design focus on designing mechanisms with desirable properties. A very central property in the literature is strategy-proofness. It has been the main focus of many studies since Gibbard (1973) and Satterthwaite (1975). A mechanism is strategy-proof if no agent can obtain a more desirable outcome by misrepresenting her preferences, no matter what the other agents do. It is important since we do not want an agent to be worse off when she reports her true preferences.

Another desirable property in the literature is Pareto efficiency. A ”social choice” is Pareto efficient if by changing the social choice it is not possible to make one agent strictly better off without hurting another. If a mechanism is not Pareto efficient, we might end up with outcomes over which the agents can unanimously improve upon. In the social choice literature, Pareto efficiency is usually treated alongside with strategy-proofness (e.g. see Moulin (1980)).

Lastly, one might like to have a notion of fairness in such collective decisions. The most common fairness concept in the social choice literature is anonymity. Anonymity requires that every agent is treated equally, in the sense that the mechanism bases its choices on the agents' preferences, not their identities. The famous Gibbard-Satterthwaite theorem implies that any nontrivial strategy-proof mechanism must be dictatorial when there are three or more alternatives (Gibbard, 1973; Satterthwaite, 1975). Therefore, special attention was given to restricted domains in the social choice literature to find mechanisms that are both anonymous and strategy-proof.

Our purpose is to find non-trivial mechanisms that satisfy two of these desirable properties mentioned above, namely strategy-proofness and Pareto efficiency. However, this is a demanding task as it is evidenced by Gibbard (1973) and Satterthwaite (1975). Yet, there are also possible results in the literature for restricted domains. For example, in the case of single-peaked preferences, anonymous, strategy-proof and Pareto efficient mechanisms are known to exist (Moulin, 1980). In the case of single-dipped preferences, unanimous, strategy-proof and Pareto efficient mechanisms are known to exist (Manjunath, 2014). In the case of double-peaked preferences with unknown locations, the existence of position invariant, anonymous and strategy-proof mechanisms is also documented (Filos-Ratsikas et al., 2017).

Our strategy is as follows. First we make strong assumptions regarding what is observable and what the mechanism uses as information. We assume that the locations of the agents are observable, hence limiting the room for potential misrepresentation by agents. This is a strong assumption, and when agents have the liberty to report their locations, the class of strategy-proof mechanisms becomes substantially smaller even with further restrictions (Filos-Ratsikas et al., 2017). Moreover, similar to what Moulin (1980) did in the case of single-peaked preferences, we restrict our attention to mechanisms that use only the peak information of individual preferences. Following Moulin (1980) we call such mechanisms as voting mechanisms. Hence, agents report only their peaks, with the only condition that these peaks are at the opposite sides of their locations. Lastly, we assume that the range of the mechanism is a finite interval, since all of the examples about double-peaked preferences we discussed above involves a finite interval of alternatives. Equipped with these assumptions, we first analyze the existence of strategy proof mechanisms. Our main result, Theorem 1, shows that a mechanism is strategy-proof if and only if it is a Generalized Median Mechanism (GMM). We then analyze mechanisms that satisfy Pareto efficiency on top of srategy proofness. We show that there is no strategy-proof mechanism that is also Pareto efficient. Lastly, we analyze the implications of weakening these assumptions. We show that if instead of voting mechanisms, we consider all possible mechanisms there are strategy-proof mechanisms other than GMMs. Alternatively, we know from Filos-Ratsikas et al. (2017) that if the agents' locations are unknown, then only strategy-proof mechanisms are GMMs which guarantee that either the leftmost or rightmost peak is selected.

In the next chapter, we provide a literature review. In Chapter 3, we present our model. In Chapter 4, we demonstrate our results that we discussed above. In Chapter 5, we conclude.

CHAPTER 2

LITERATURE REVIEW

Modern social choice theory starts with Kenneth Arrow's *Social Choice and Individual Values* which includes Arrow's Impossibility Theorem. Arrow (1963) shows that any preference aggregating procedure that has three properties, namely Universal Domain (any type of preference is admissible), Independence of Irrelevant Alternatives (if the order of two alternatives does not change between two different preference profiles, the resulting societal ordering should not change for these two alternatives), and Pareto efficiency (also known as unanimity, meaning that if everyone prefers one alternative over another, it should be the same in the societal ordering), implies dictatorship.

Gibbard (1973) and Satterthwaite (1975) famously show that under the assumption of universal domain, all non-dictatorial mechanisms are manipulable with the exception of some trivial cases. Barberà and Peleg (1990) shows that this impossibility result still holds when the set of alternatives is not finite and agents have continuous preferences. This brings us to the quest of finding restricted domains that are relevant to real life situations, and where non-dictatorial and strategy-proof mechanisms are possible.

We focus on the problem of choosing a location for a single public good. This problem has been the center of a growing literature, following Moulin (1980) which provided a characterization of anonymous, strategy-proof and efficient mechanisms when agents have single-peaked preferences. Moulin (1980) focuses on mechanisms that use only the peak information. However, this does not create a loss of generality as Barberà and Jackson (1994) shows that strategy-proofness implies that the mechanism uses only the peak information when preferences are single-peaked. Moulin (1980) shows that for n-many agents with single-peaked preferences, any anonymous and strategy-proof mechanism can be represented by a median mechanism with n+1 ghost voters, and Pareto efficiency is achieved in addition to these two properties if the number of ghost voters is n-1.

Barberà et al. (1993) shows that under single-peaked domain any strategy-proof mechanism is a Generalized Median Voter Scheme, such that every element of the set of alternatives has a (winning) coalition that can guarantee that element's selection by the mechanism. Barberà et al. (1993)'s analysis is similar to ours in a sense that it focuses on strategy-proofness only, and its characterization contains mechanisms that are not anonymous as a result. Massó and Moreno de Barreda (2011) provides a characterization similar to Moulin (1980) but when the preferences of the agents are symmetric around their peaks, and shows that discontinuous mechanisms are now allowed in the class of strategy-proof rules. It is argued that, this is a significant contribution since in most real life situations the set of alternatives is possibly not a continuous interval (Masso and Moreno de Barreda, 2011).

A stronger notion regarding manipulability is group strategy-proofness, which means no group of agents can collude by misrepresenting their preferences to achieve a more desirable result for themselves. In the single-peaked domain strategy-proofness and group strategy-proofness are equivalent, and Barberà et al. (2010) shows that a property defined for preference relations called indirect sequential inclusion is necessary and sufficient for such equivalence.

There has also been studies that focus on relations between different restricted domains, such as Barberà and Moreno (2011) which shows that single-peaked, single-plateaued, single crossing and order restricted domains all share a common root called top monotonicity. Barberà and Moreno (2011) demonstrates that this property is sufficient to guarantee a voting equilibria at the peak of the median voter. While this study is not directly related to ours, we would like to mention that double-peaked preferences does not satisfy top-monotonicity.

Another domain that is similar to ours is single-dipped preferences, for which Manjunath (2014) provides a characterization of strategy-proof mechanisms. All unanimous and strategy-proof mechanisms belong to a class called voting by extended collection of 0-decisive sets (VEZD), which always chooses either one of the ends of the set of alternatives and is also Pareto efficient (Manjunath, 2014). A VEZD is defined on an extended collection of 0-decisive sets (in other words, winning coalitions), and a tie breaker function for the case when every agent is indifferent between both ends (Manjunath, 2014). An example of a VEZD is the majority rule where the end that more agents prefer over the other is chosen (Manjunath, 2014). We note that, the case of doubled-peaked preferences strongly resembles that of the single-peaked preferences, rather than single-dipped preferences. This is due to the fact that we fix agents' locations (local dips), hence, each agent actually reports a single peak for two separate set of alternatives relative to her location.

Filos-Ratsikas et al. (2017) is the closest study to ours as it also assumes agents have double-peaked preferences. However, in Filos-Ratsikas et al. (2017) agents report their locations, and it is assumed that the distance of peaks to their location is same and symmetric for everyone. In other words, once the agent's location is known, his symmetric preferences around it can be directly constructed from the location information. When agents report their locations, and their peaks are of known distance to their location, only strategy-proof mechanisms are those that select either the leftmost peak or the rightmost peak (Filos-Ratsikas et al., 2017). In the motivating example given at the beginning of the thesis, the agent's location is observable by the mechanism designer. What is not observable is his most preferred locations to the left and right of his location. The fact that we assume agents' locations are known to the mechanism but not their peaks, makes our model distinct from Filos-Ratsikas et al. (2017), and we obtain a larger class of strategy-proof mechanisms as a result.

There are several studies in the literature that mention the observation of double-peaked preferences among individuals. Flowers (1975) mentions that the preferences of the individuals on the level of public spending on education is very likely to exhibit double-peakedness. It is argued that the availability of private schools as a substitute for public schooling creates a possibility for individuals to exhibit a double-peaked preference (Flowers, 1975). This is due to the fact that a parent would want her child to attend either a public or a private school. For the former, she would prefer a high level of government spending on public education, whereas for the latter she would prefer the government spends a very low amount or nothing at all, so that she will not be paying higher taxes for a service she does not intend to use. Clearly, a parent would not prefer a level of spending that is in the middle, such that the quality of public schools is not high enough whereas taxes are also not that low.

There is also experimental evidence that the policy preferences of individuals can be double-peaked, especially in matters that they think the status quo fails to achieve a goal that is desired for both ends of the political spectrum (Egan, 2014). For instance, a lower crime rate or a higher economic growth rate is desirable for any political faction in the society. In such problems where the status quo is not perceived to be an effective policy towards these goals by the public, the local dip that we call the location of an agent becomes the status quo. For instance, in Egan (2014), almost half of the sample exhibits a double-peaked policy preference on the issue of foreign economic competition. It is also suggested that double-peakedness increases in the population as the public deems a policy matter more urgent (Egan, 2014). However, Egan (2014)'s definition of a double-peaked preference, where agents are not necessarily indifferent between their two peaks, is slightly different than ours. Nevertheless, the experimental evidence and the plausibility of the real life scenarios involving double-peaked preferences make it promising to examine the strategy-proof and Pareto efficient mechanisms in this domain.

CHAPTER 3

THE MODEL

Let $N = \{1,2,...,n\}$ be a finite set of agents and let $X = \mathbb{R}$ be the commodity space. Let $\delta = (\delta_1,...,\delta_n) \in X^N$ be a profile of locations. Throughout the paper, δ will be fixed. For agent i with location δ_i, let R_i be a preference relation on X and let P_i and I_i be the strict preference and indifference relations associated with R_i, respectively. For two alternatives $x,y \in X$, we have xP_iy if and only if xR_iy and $y \not R_i x$, and it means agent i is strictly better off when x is chosen over y. We have xI_iy if and only if xR_iy and yR_ix, which means agent i is neither better off or worse off when one of the alternatives is chosen over the other. We first give a formal definition of double-peaked preferences.

Definition 1 *R_i is a **double-peaked preference relation** on X if there exists $\pi_1(R_i), \pi_2(R_i), \delta_i \in X$ such that $\pi_1(R_i) < \delta_i < \pi_2(R_i)$, $\pi_1(R_i)I_i\pi_2(R_i)$, $\pi_1(R_i)P_i\delta_i$, $\pi_2(R_i)P_i\delta_i$, and for all $x,y \in X$:*

$$x < y \leq \pi_1(R_i) \textit{ implies } yP_ix$$

$$\pi_2(R_i) \leq y < x \textit{ implies } yP_ix$$

$$\pi_1(R_i) \leq y < x \leq \delta_i \textit{ implies } yP_ix$$

$$\delta_i \leq x < y \leq \pi_1(R_i) \textit{ implies } yP_ix.$$

A preference profile is an n-tuple of preference orderings. Let $\mathscr{R}$ be the class of all double-peaked preferences on X. Note that, while the peaks are functions of R_i, we do not use the similar notation for δ_i to emphasize the fact that it is fixed for each agent.

Since $X = \mathbb{R}$, each agent has preferences over the whole real line. However, we are interested in choosing a spot from a finite interval, as it is more applicable to the real life situations that we discussed in the introduction. Therefore, we define the feasible set $\mathbb{F} \subseteq X$ separately.

We define a mechanism as follows.

Definition 2 *A* ***mechanism*** *is a function $M : \mathscr{R}^N \to \mathbb{F}$, such that it assigns a single point $x \in \mathbb{F}$ for each preference profile.*

Next, we provide definitions for strategy-proofness and Pareto-efficiency. If a mechanism is strategy-proof, then reporting true preferences is a weakly dominant strategy for each agent.

Definition 3 *A mechanism $M : \mathscr{R}^N \to \mathbb{F}$ is* ***strategy-proof*** *if for all $i \in N$ and $R_i, R'_i \in \mathscr{R}^N$, $M(R_i, R_{-i}) R_i M(R'_i, R_{-i})$.*

If a mechanism is Pareto efficient, then for any outcome of the mechanism, there exists no other alternative that is weakly preferred by all agents and strictly preferred by one of them.

Definition 4 *A mechanism $M : \mathscr{R}^N \to \mathbb{F}$ is* ***Pareto efficient*** *if for all $R^N \in \mathscr{R}^{\mathscr{N}}$ and for every $x \in X$ such that $x \neq M(R_1, ..., R_n)$: $x P_i M(R_1, ..., R_n)$ for some $i \in N$ implies there exists $j \in N$ such that $M(R_1, ..., R_n) P_j x$.*

We focus on mechanisms which only take the peaks and the locations of agents into account. We call such mechanisms as voting mechanisms. In our setting, locations of agents are known to the mechanism, and therefore agents only report their peaks, and these reported peaks must be consistent with their location.

Definition 5 *M is a* ***voting mechanism*** *if it only responds to the $\pi_1(R_i), \pi_2(R_i), \delta_i$ information in a preference relation R_i. Formally, for each R_i, R'_i such that $\pi_1(R_i) = \pi_1(R'_i), \pi_2(R_i) = \pi_2(R'_i)$, and $\delta_i = \delta'_i$, we have $M(R_i, R_{-i}) = M(R'_i, R_{-i})$.*

Define $\overline{\Pi}(R) = \{\pi_1(R_1), \pi_2(R_1), ..., \pi_1(R_n), \pi_2(R_n)\}$ as the set of all peaks, and let $\Pi(R) \subseteq \overline{\Pi}(R)$ be any subset. We define a class of mechanisms called *Generalized Median Mechanisms (GMM)* that is similar to the one defined in Proposition 3 of Moulin (1980). A *GMM* has a parameter for every subset of $\Pi(R)$. By each subset, it first takes the supremum among the elements of the subset and its parameter. After that, it takes the infimum among these supremums.

In the case of single-peaked preferences, an agent can only be represented in a coalition by its single peak. Since, in our case each agent has two peaks, we define our grand set as the set of peaks. Any subset of this grand set can be the set which a *GMM* works with. For instance, $\Pi(R)$ can contain only one peak of agent i but two peaks of agent j. In this case, agent j is represented in some subsets of $\Pi(R)$ by its left peak and it is represented by its right peak in the others. Since we take the supremum of each subset and its parameter, it does not matter whether a subset contains only the right peak of an agent or both of the peaks, as the right peak is always greater than the left peak.

Definition 6 *A **Generalized Median Mechanism (GMM)** is a voting mechanism such that,*

$$M(R) = \underset{S \subseteq \Pi(R)}{inf} \{ \underset{\pi_j(R_i) \in S}{sup} \{\pi_j(R_i), a_S\}\}.$$

CHAPTER 4

RESULTS

We first analyze strategy-proof mechanisms in Section 4.1. In Section 4.2, we inspect Pareto efficiency on top of strategy-proofness. We later discuss relaxing our assumption of known locations in Section 4.3. Lastly, we discuss mechanisms that use the whole preference relation in Section 4.4.

4.1. Strategy-proofness

Our first theorem characterizes strategy-proof voting mechanisms when $\mathbb{F}$ is a finite interval. Under the assumption that a mechanism's range is a finite interval, we find that strategy-proofness implies the mechanism being a *GMM* and vice versa. This is a very similar result to Moulin (1980), and consists of a wider class of mechanisms than Filos-Ratsikas et al. (2017). In fact the stretegy-proof mechanisms in Filos-Ratsikas et al. (2017) also belong to the class of *GMM*. However, it only includes *GMM*s which always select either the leftmost or the rightmost peak. The similarity with the former and the difference with the latter is due to the fact that in our model the locations of agents are known to the mechanism and the reported peaks have to be consistent with these locations.

Assuming $\mathbb{F}$ is a finite interval is not very restrictive considering the fact that in any facility location problem the feasible set would be a finite interval. This assumption also works for the political decision making models, where the possible stands to take on an issue is represented as a finite interval.

Theorem 1 *Assume the range of M is a finite interval. Also assume that M is a voting mechanism. Then, M is strategy-proof if and only if M is a GMM.*

Proof. We first show the following results which are used to prove the theorem:

Lemma 1 *Assume M is a strategy-proof voting mechanism and* $N = 1$*. Let* $x \leq \delta_i$ *be such that* $x \in Range(M)$*. Let* $\pi_1(R_i), \pi_2(R_i)$ *be peaks of* R_i*. Then,* $M(R_i) \in [x, \pi_1(R_i)]$[1]

Proof. First, note that since $x \in Range(M)$, $x = M(R'_i)$ for some $R'_i \in \mathscr{R}$. Fix an R_i with location δ_i and peaks $\pi_1(R_i) = \pi_1$ and $\pi_2(R_i) = \pi_2$.

Claim 1 $M(R_i) \geq inf\{M(R'_i), \pi_1(R_i)\}$.

Proof of the Claim Suppose not. Let $M(R_i) < inf\{M(R'_i), \pi_1(R_i)\}$. If we have $M(R_i) < M(R'_i) \leq \pi_1(R_i)$, then by the definition of double-peaked preferences, we have $M(R'_i)P_iM(R_i)$, contradicting M being strategy-proof. On the other hand, if we have $M(R_i) < \pi_1(R_i) \leq M(R'_i)$, there exists R_i with the peaks π_1 and π_2 such that $M(R'_i)P_iM(R_i)$ (see Figure 1), again contradicting M being strategy-proof.

Claim 2 $M(R_i) \leq sup\{M(R'_i), \pi_1(R_i)\}$ *or* $M(R_i) = \pi_2(R_i)$.

Proof of the Claim Suppose not. Let $M(R_i) > sup\{M(R'_i), \pi_1(R_i)\}$ and $M(R_i) \neq \pi_2(R_i)$. If we have $\pi_1(R_i) \leq M(R'_i) < M(R_i)$, there exists R_i with the peaks π_1 and π_2 such that $M(R'_i)P_iM(R_i)$ (see Figure 2), contradicting M being strategy-proof. On the other hand, if we have $M(R'_i) \leq \pi_1(R_i) < M(R_i)$, then again there exists R_i with the peaks π_1 and π_2 such that $M(R'_i)P_iM(R_i)$ (see Figure 3), which is a contradiction to M being strategy-proof.

Together, Claim 1 and Claim 2 imply Lemma 1. ■

[1] With an abuse of notation we use simply $[x, \pi_1(R_i)]$ throughout the paper whenever it is ambigous whether it is $x \leq \pi_1(R_i)$ or $\pi_1(R_i) < x$.

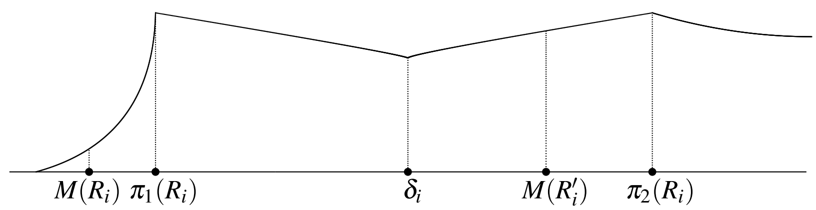

Figure 4.1: A preference relation with $M(R_i')P_iM(R_i)$ when $M(R_i) < \pi_1(R_i) \leq M(R_i')$

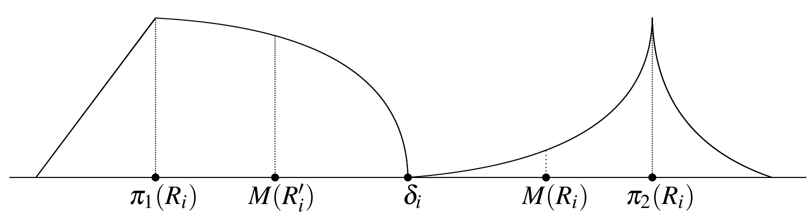

Figure 4.2: A preference relation with $M(R_i')P_iM(R_i)$ when $\pi_1(R_i) \leq M(R_i') < M(R_i)$

Lemma 2 *Assume M is a strategy-proof voting mechanism and $N = 1$. Let $x \geq \delta_i$ be such that $x \in Range(M)$. Let $\pi_1(R_i), \pi_2(R_i)$ be peaks of R_i. Then, $M(R_i) \in [x, \pi_2(R_i)]$ or $M(R_i) = \pi_1(R_i)$.*

Proof. The proof is symmetric to the proof of Lemma 1. ■

Now we prove that every strategy-proof voting mechanism is a *GMM*, using induction. We start with the case where there is only one agent. Suppose $N = 1$. Fix a vector of d. Assume M is a strategy-proof voting mechanism and its range is a finite interval.

Let $\alpha = inf\{M(R_i) \mid R_i \in \mathscr{R}\}$ and $\beta = sup\{M(R_i) \mid R_i \in \mathscr{R}\}$.

Claim 3 *Either $\alpha \leq \beta \leq \delta_i$ or $\delta_i \leq \alpha \leq \beta$.*

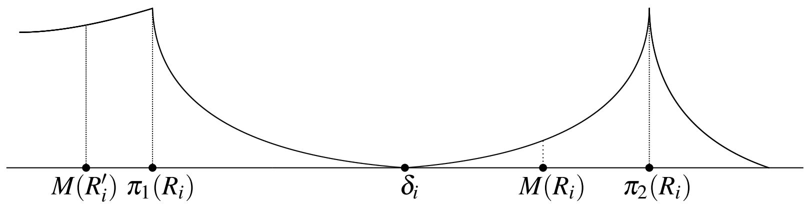

Figure 4.3: A preference relation with $M(R_i')P_iM(R_i)$ when $M(R_i') \leq \pi_1(R_i) < M(R_i)$

Proof of the Claim Suppose not. Let $\alpha < \delta_i < \beta$. Consider R_i such that $\pi_1(R_i) < \alpha < \delta_i < \beta < \pi_2(R_i)$. For such R_i, Lemma 1 implies that, since $\pi_2(R_i) \notin range(M)$, $M(R_i) \in [\pi_1(R_i), \alpha]$, and therefore $M(R_i) = \alpha$. Similarly, Lemma 2 implies that $M(R_i) = \beta$. Since it is not possible for both lemmas to hold at the same time, there exists no one-agent strategy-proof voting mechanism with $\alpha < \delta_i < \beta$.

This means we have only two cases:

If $\alpha \leq \beta \leq \delta_i$, then Lemma 1 implies $M(R_i) \in [\pi_1(R_i), \alpha] \cap [\pi_1(R_i), \beta]$. This would mean, whenever $\alpha \leq \pi_1(R_i) \leq \beta$, $M(R_i) = \pi_1(R_i)$. In the case when $\pi_1(R_i) \leq \alpha \leq \beta$, $M(R_i) = \alpha$ and $\alpha \leq \beta \leq \pi_1(R_i)$ implies $M(R_i) = \beta$. This means that M can be represented as $med\{\alpha, \beta, \pi_1(R_i)\}$ or $inf\{\beta, sup\{\pi_1(R_i), \alpha\}\}$. This is a *GMM* with $\Pi(R) = \{\pi_1(R_i)\}$.

If $\delta_i \leq \alpha \leq \beta$, then Lemma 2 implies $M(R_i) \in [\pi_2(R_i), \alpha] \cap [\pi_2(R_i), \beta]$. Similar to the argument above, this would mean that M can be represented as $med\{\alpha, \beta, \pi_2(R_i)\}$ or $inf\{\beta, sup\{\pi_2(R_i), \alpha\}\}$. This is a *GMM* with $\Pi(R) = \{\pi_2(R_i)\}$.[2]

Next, we assume that our theorem holds for n voters, and show that it holds for $n+1$ voters as well.

Let $M(R_0, R_1, ..., R_n)$ be a strategy-proof voting mechanism for $n+1$ voters. Let R_0 be fixed so that $M(R_0, R_1, ..., R_n)$ is a strategy-proof voting mechanism for n voters, and therefore

[2] M can also be represented as a *GMM* with $\Pi(R) = \{\pi_1(R_i), \pi_2(R_i)\}$, such that $M(R_i) = inf\{\beta, sup\{\pi_2(R_i), \alpha_1\}, sup\{\pi_1(R_i), \alpha_2\}, sup\{\pi_1(R_i), \pi_2(R_i), \alpha_3\}\}$ with $\alpha_2 \geq \beta$ and $\alpha_3 \geq \alpha_1$. Note that, it does not matter for the outcome whether we include both of the peaks of the agent or only one, and in both cases, $\pi_1(R_i)$ is irrelevant for the outcome. However, this situation is unique to the one agent model.

a *GMM* by our supposition. Let $\Pi(R_1,...,R_n)$ be the associated set of peaks with M when R_0 is fixed. By supposition, we can write M as:

$$M(R_0,R_1,...,R_n) = \inf_{S \subseteq \Pi(R_1,...,R_n)} \{ \sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S(R_0)\}\}$$

Let S_0 be a nonempty coalition of peaks such that $\pi_1(R_0), \pi_2(R_0) \notin S_0$. Choose each $\pi_j(R_i)$ such that:

$$\pi_j(R_i) = \begin{cases} \mu_1, & \pi_j(R_i) \in S_0 \textit{ and } j = 1 \\ \mu_2, & \pi_j(R_i) \in S_0 \textit{ and } j = 2 \\ \lambda_1, & \pi_j(R_i) \notin S_0 \textit{ and } j = 1 \\ \lambda_2, & \pi_j(R_i) \notin S_0 \textit{ and } j = 2 \end{cases}$$

Let $\lim\limits_{\substack{\mu_1 \to -\infty \\ \lambda_2 \to +\infty \\ \mu_2, \lambda_1 \to \delta_i}} \sup\limits_{\pi_j(R_i) \in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\} = a'_{S_0}(R_0)$.[3]

Claim 4 *We can replace* $a_{S_0}(R_0)$ *with* $a'_{S_0}(R_0)$ *without changing the behavior of the mechanism.*

Proof of the Claim If we have $a_{S_0}(R_0) = a'_{S_0}(R_0)$, the statement is trivial. So suppose, $a_{S_0}(R_0) \neq a'_{S_0}(R_0)$. Note that, for any $\pi_1(R_i) \in S_0$ we have $\pi_1(R_i) = -\infty$. Since $a_{S_0}(R_0) \geq -\infty$, there exists at least one $\pi_2(R_i) \in S_0$ such that $\lim \pi_2(R_i) = \delta_i > a_{S_0}(R_0)$. Let $\delta_i^* = \max\limits_{\pi_2(R_i) \in S_0} \{\delta_i\}$. Clearly, $\lim \sup\limits_{\pi_j(R_i) \in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\} = \delta_i^* = a'_{S_0}(R_0)$. Let $\pi_2^*(R_i)$ be the right peak of the agent with location δ_i^*. Since $\pi_2^*(R_i) \in S_0$, for all $R^N \in \mathscr{R}$, $\sup\limits_{\pi_j(R_i) \in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\} > \delta_i^* > a_{S_0}(R_0)$. Therefore, replacing the real parameter $a_{S_0}(R_0)$ with $\delta_i^* = a'_{S_0}(R_0)$ does not change the outcome of the mechanism for any $R^N \in \mathscr{R}^N$, as neither of them will ever be the outcome of this supremum.

Thanks to Claim 4, we replace $a_{S_0}(R_0)$ with $\lim \sup\limits_{\pi_j(R_i) \in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\}$ without interfering with the mechanism.

Now let $\lim\limits_{\substack{\mu_1 \to -\infty \\ \lambda_2 \to +\infty \\ \mu_2, \lambda_1 \to \delta_i}} M(R_0,R_1,...,R_n) = a'_{S_0}(R_0)$

[3] From now on we will simply write lim to refer this limit operation, omitting $\mu_1 \to -\infty$, $\lambda_2 \to +\infty$, $\mu_2 \to \delta_i$, $\lambda_1 \to \delta_i$.

Claim 5 *We can replace $a_{S_0}(R_0)$ with $a'_{S_0}(R_0)$ without changing the behavior of the mechanism.*

Proof of the Claim If we have $a_{S_0}(R_0) = a'_{S_0}(R_0)$, the statement is trivial. So, suppose $a_{S_0}(R_0) \neq a'_{S_0}(R_0)$. Since we have already replaced $a_{S_0}(R_0)$ with $\lim \sup_{\pi_j(R_i)\in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\}$, $a_{S_0}(R_0) = a'_{S_0}(R_0)$ implies that there exists a T such that $\lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T(R_0)\} < \lim \sup_{\pi_j(R_i)\in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\} = a_{S_0}(R_0)$. Note that, $\lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T(R_0)\} < a_{S_0}(R_0)$ implies that, for any $\pi_2(R_i) \in T$, we have $\pi_2(R_i) \in S_0$, otherwise the left-hand term would be equal to $+\infty$. So the set $T - S_0$ consists of only left peaks. Let $\max_{\pi_1(R_i)\in T-S_0} \{\delta_i\} = \hat{\delta}_i$. Clearly, $\hat{\delta}_i$ is an upper bound for any $\pi_j(R_i) \in T - S_0$, as $\pi_1(R_i) < \delta_i$ for all $i \in N$ by the definition of double-peaked preferences. Moreover, $\lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T(R_0)\} < a_{S_0}(R_0)$ implies that both $\hat{\delta}_i < a_{S_0}(R_0)$ (since for any $\pi_1(R_i) \in T - S_0, \pi_1(R_i) = \delta_i$ in the limit) and $a_T(R_0) < a_{S_0}(R_0)$. Since all the other elements in T are also in S_0, for any $R^N \in \mathscr{R}^N$ we have $\sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T(R_0)\} \leq \sup_{\pi_j(R_i)\in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\}$. This means $\sup_{\pi_j(R_i)\in S_0} \{\pi_j(R_i), a_{S_0}(R_0)\}$ is never selected. Moreover, if we replace $a_{S_0}(R_0)$ with the $\lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T\}$ this inequality would continue to hold for all $R^N \in \mathscr{R}^N$, since for all $R^N \in \mathscr{R}^N$ we have $\pi_j(R_i) \leq \lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T(R_0)\}$[4] for any $\pi_j(R_i) \in T - S_0$. Therefore, the behaviour of the mechanism would not change.

By Claim 5, we update $a_{S_0}(R_0)$ with $\lim M(R_0, R_1, ..., R_n)$, without changing the behavior of mechanism. Note that, in the limit we have $R_1, ...R_n$ fixed, and with fixed $R_1, ...R_n$, M is a one-agent strategy-proof voting mechanism. Therefore, the conditions in the initial stage should hold for $\lim M(R_0, R_1, ..., R_n)$, and therefore they should hold for $a_{S_0}(R_0)$. Then, we have two cases: either (i) $a_{S_0}(R_0) = inf\{\beta_{S_0}, sup\{\pi_1(R_0), \alpha_{S_0}\}\}$ or (ii) $a_{S_0}(R_0) = inf\{\beta_{S_0}, sup\{\pi_2(R_0), \alpha_{S_0}\}\}$ with $\alpha_{S_0} \leq \beta_{S_0}$. Since S_0 was arbitrary, we can make use of this limit operation for all $S \subseteq \Pi(R_1, ..., R_N)$. Therefore, each $a_S(R_0)$ must take either of these forms.

[4] Since for any $\pi_1(R_i) \in T \cap S_0$, $\pi_j(R_i) = -\infty$, $\lim \sup_{\pi_j(R_i)\in T} \{\pi_j(R_i), a_T\}$ must be equal to either $\max_{\pi_2(R_i)\in T\cap S_0} \{\delta_i\} = \delta_i^*$, $\max_{\pi_1(R_i)\in T-S_0} \{\delta_i\} = \hat{\delta}_i$, or a_T.

Without loss of generality, suppose the first case holds. Then, with fixed R_0, we can write M as follows:

$$M(R_0,...,R_n) = \inf_{S\subseteq\Pi(R_1,...,R_n)}\{\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}\}$$

Now, we will show that this mechanism is indeed can be written as a *GMM*.

For any $S\subseteq\Pi(R_1,...,R_n)$, let $a_S = inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}$. For any $S' = S\cup\{\pi_1(R_0)\}$, let $a_{S'} = \alpha_S$. Let $\mathbb{S} = \{S_1,...S_k\}$ and $\mathbb{S}' = \{S'_1,...,S'_k\}$. Also let $\Pi(R_1,...,R_n)\cup\{\pi_1(R_0)\} = \Pi(R_0,R_1,...,R_n)$. Note that, $\mathbb{S}$ and $\mathbb{S}'$ is a partition of $2^{\Pi(R_0,R_1,...,R_n)}$.

Now let $\mathscr{M}(R_0,...,R_n) = \inf_{S\subseteq\Pi(R_0,R_1,...,R_n)}\{\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), a_S\}\}$. Note that $\mathscr{M}$ is a GMM. We now prove that, for all $R^N\in\mathscr{R}^N$, M and $\mathscr{M}$ are equivalent.

Claim 6 *If $\mathscr{M}(R_0,...,R_n) = \sup_{\pi_j(R_i)\in S^*}\{\pi_j(R_i), a_{S^*}\}$ for some $S^*\in\mathbb{S}$, then $M(R_0,...,R_n) = \mathscr{M}(R_0,...,R_n)$.*

Proof of the Claim By the construction of a_S, for any $S\in\mathbb{S}$ we have $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), a_S\} = \sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}$ since $a_S = inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}$. Moreover, $\sup_{\pi_j(R_i)\in S^*}\{\pi_j(R_i), a_{S^*}\}$ is the infimum of all such $S\in\mathbb{S}$ (otherwise it would not have been selected from $\mathscr{M}$), and $\mathbb{S} = 2^{\Pi(R_1,...,R_n)}$ by definition. Hence, $M(R_0,...R_n) = \sup_{\pi_j(R_i)\in S^*}\{\pi_j(R_i), a_{S^*}\}$.

Claim 7 *For all $R^N\in\mathscr{R}^N$, $\mathscr{M}(R_0,...,R_n) = \sup_{\pi_j(R_i)\in S^*}\{\pi_j(R_i), a_{S^*}\}$ for some $S^*\in\mathbb{S}$.*

Proof of the Claim Since $\mathbb{S}$ and $\mathbb{S}'$ is a partition of $2^{\Pi(R_0,R_1,...,R_n)}$, proof of this claim is equivalent to showing that $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), a_S\} \leq \sup_{\pi_j(R_i)\in S'}\{\pi_j(R_i), a_{S'}\}$ where $S' = S\cup\{\pi_1(R_0)\}$ for any $S\subseteq\Pi(R_1,...,R_n)$. By construction, instead of a_S we can write $inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}$ in the left hand term, and α_S in the right hand term, so what we need to show is for all $R^N\in\mathscr{R}^N$, $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), inf\{\beta_S, sup\{\pi_1(R_0),\alpha_S\}\}\} \leq \sup_{\pi_j(R_i)\in S}\{\pi_j(R_i),\pi_1(R_0),\alpha_S\}$. We have three cases:

Case 1: $\pi_1(R_0)\leq\alpha_S$.

In this case the left hand side becomes $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i),\alpha_S\}$. Since $\pi_1(R_0)\leq\alpha_S$ by supposition, we can remove $\pi_1(R_0)$ from the right hand term without changing the outcome. Therefore, the claim holds with equality.

Case 2: $\alpha_S < \pi_1(R_0) \leq \beta_S$.

In this case the left hand side becomes $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), \pi_1(R_0)\}$. Since $\alpha_S < \pi_1(R_0)$ by supposition, we can remove α_S from the right hand term without changing the outcome. Hence, the claim again holds with equality.

Case 3: $\beta_S < \pi_1(R_0)$

In this case the left hand side becomes $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), \beta_S\}$. Since $\alpha_S < \pi_1(R_0)$, we can further simplify right hand side as $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), \pi_1(R_0)\}$, and since $\beta_S < \pi_1(R_0)$, we have $\sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), \beta_S\} \leq \sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), \pi_1(R_0)\}$.

Together, Claim 6 and Claim 7 establish that for all $R^N \in \mathscr{R}^N$, $M(R_0, ..., R_n) = \mathscr{M}(R_0, ..., R_n)$. An identical proof can be constructed for a *GMM* with $\Pi(R_0, R_1, ..., R_n) = \Pi(R_1, ..., R_n) \cup \{\pi_2(R_0)\}$. A *GMM* with $\Pi(R_0, R_1, ..., R_n) = \Pi(R_1, ..., R_n) \cup \{\pi_1(R_0), \pi_2(R_0)\}$ can be constructed by adding a *faux* agent with its preference identical to R_0, and repeating each step to include the other peak in the last step.

In the second part of the proof, we show that any *GMM* is strategy-proof. Assume M is a *GMM* with n agents.

Fix some agent $k \in N$ with location δ_k. Let R_k be the preference relation of k and let $\pi_j(R_k)$ be the peaks of R_k. Suppose $M(R_1, ..., R_n) = x$ with $x \neq \pi_1(R_k)$ and $x \neq \pi_2(R_k)$. Since $M(R_1, ..., R_n) = x$, there exists an $S \subseteq \Pi(R_1, ..., R_n)$ such that $x = \sup_{\pi_j(R_i)\in S}\{\pi_j(R_i), a_S\} \leq \sup_{\pi_j(R_i)\in S'}\{\pi_j(R_i), a_{S'}\}$ for any $S' \subseteq \Pi(R_1, ..., R_n)$. We have four cases:

Case 1: $x < \pi_1(R_k)$

Then clearly $\pi_1(R_k), \pi_2(R_k) \notin S$. Since x is already inside the infimum operation, for any $R'_k \in \mathscr{R}$, $M(R'_k, R_{-k}) \leq M(R_k, R_{-k})$. $M(R'_k, R_{-k}) \leq M(R_k, R_k) < \pi_1(R_k)$ implies for any $R'_k \in \mathscr{R}$ we have $M(R_k, R_{-k}) R_k M(R'_k, R_{-k})$.

Case 2: $\pi_2(R_k) < x$

$\pi_1(R_k) < \pi_2(R_k) < x$ implies that for any $S' \subseteq \Pi(R_1, ..., R_n)$, there exists either a $\pi_j(R_i) \in S'$ with $\pi_2(R_k) < x \leq \pi_j(R_i)$ or $\pi_2(R_k) < x \leq a_{S'}$, otherwise x would not have been selected. Therefore, for all $S' \subseteq \Pi(R_1, ..., R_n)$, for any $R'^N \in \mathscr{R}^N$ with $R'_{-k} = R_{-k}$, $x \leq \sup_{\pi_j(R_i)\in S'}\{\pi_j(R_i), a_{S'}\} \leq \sup_{\pi_j(R'_i)\in S'}\{\pi_j(R'_i), a_{S'}\}$. Since each supremum inside the infimum operation is weakly greater than x for any $R'_k \in \mathscr{R}$, the infimum of them will be weakly greater as well, implying $M(R_k, R_{-k}) \leq$

$M(R'_k, R_{-k})$. Therefore, by the definition of double-peaked preferences and by $\pi_2(R_k) < M(R_k, R_{-k}) \leq M(R'_k, R_{-k})$, for any $R'_k \in \mathscr{R}$ we have $M(R_k, R_{-k}) R_k M(R'_k, R_{-k})$.

Case 3: $\pi_1(R_k) < x \leq \delta_k$

Then clearly $\pi_2(R_k) \notin S$. If $\pi_1(R_k) \notin S$, we have for all $R'^N \in \mathscr{R}^N$ with $R'_{-k} = R_{-k}$, $\sup_{\pi_j(R'_i) \in S} \{\pi_j(R'_i), a_S\} = x$. If $\pi_1(R_k) \in S$, since for all $R'_k \in \mathscr{R}$, $\pi_1(R'_k) < \delta_k$, for all $R'^N \in \mathscr{R}^N$ with $R'_{-k} = R_{-k}$, we have $\sup_{\pi_j(R'_i) \in S} \{\pi_j(R'_i), a_S\} \in [x, \delta_k)$. Therefore, there exists no $R'_k \in \mathscr{R}$ such that $\delta_k \leq M(R'_k, R_{-k})$ as $\sup_{\pi_j(R'_i) \in S} \{\pi_j(R'_i), a_S\}$ is inside the infimum operation. Moreover, since x is selected in the first place, we know that for all $S' \neq S$, $x = \sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S\} \leq \sup_{\pi_j(R_i) \in S'} \{\pi_j(R_i), a_{S'}\}$, therefore there exists no $R'_k \in \mathscr{R}$ such that $M(R'_k, R_{-k}) < x$. Since for all $R'_k \in \mathscr{R}$ we have $\pi_1(R_k) < M(R_k, R_{-k}) \leq M(R'_k, R_{-k}) < \delta_k$, by the definition of double-peaked preferences we have $M(R_k, R_{-k}) R_k M(R'_k, R_{-k})$.

Case 4: $\delta_k < x < \pi_2(R_k)$

Then again clearly $\pi_2(R_k) \notin S$. Moreover, $\delta_k < x$ implies there exists $\pi_j(R_i) \in S$ with $\pi_j(R_i) = x > \delta_k$ or $a_S = x > \delta_k$. Since $\pi_1(R'_k) < \delta_k$ for all $R'_k \in \mathscr{R}$, we have $\sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S\} = x$ for all $R'_k \in \mathscr{R}$ and $R'_{-k} = R_{-k}$. Therefore, x is always inside the infimum operation, which implies there exists no $R'_k \in \mathscr{R}$ such that $x < M(R'_k, R_{-k})$. Moreover, for all $\hat{S}$ with $\pi_2(R_k) \notin \hat{S}$, there exists either a $\pi_j(R_i) \in \hat{S}$ with $\pi_j(R_i) \geq x$ or $a_{\hat{S}} \geq x$, otherwise x would not have been selected. For all S' with $\pi_2(R_k) \in S'$, since for all $R'_k \in \mathscr{R}$, $\delta_k < \pi_2(R'_k)$, we have $\delta_k < \sup_{\pi_j(R_i) \in S'} \{\pi_j(R_i), a_{S'}\}$. Since all the values inside the infimum operation are greater than δ_k, for all $R'_k \in \mathscr{R}$ we have $\delta_k < M(R'_k, R_{-k}) \leq M(R_k, R_{-k}) < \pi_2(R_k)$. This implies $M(R_k, R_{-k}) R_k M(R'_k, R_{-k})$ by the definition of double-peaked preferences.

Since for all $R'_k \in \mathscr{R}$, we always have $M(R_k, R_{-k}) R_k M(R'_k, R_{-k})$, *GMM* is strategy-proof, concluding the proof of Theorem 1. ■

In Theorem 2, we show that our characterization works when the assumption that $\mathbb{F}$ is a finite interval is replaced with continuity. Theorem 1 and Theorem 2 also show that having a finite interval as range and strategy-proofness together imply continuity for a voting mechanism, since *GMM* is continuous. However, continuity and strategy-proofness does not necessarily imply that the range of the mechanism is a finite interval, hence these two properties are not completely equivalent for strategy-proof voting mechanisms in our model.

Theorem 2 *Assume that M is a voting mechanism. Then, M is continuous and strategy-proof if and only if M is a GMM.*

Proof. The proof of this theorem follows from the proof of Theorem 1, except Claim 3 where we use the assumption that M's range is a finite interval. Therefore, to prove Theorem 2, it is sufficient to show that (i) Claim 3 still holds when the assumption of M's range being a finite interval is replaced with continuity, and (ii) *GMM* is continuous.

First we restate and prove Claim 3:

Assume M is a continuous and strategy-proof voting mechanism. Let $N = 1$. Let $\alpha = inf\{M(R_i) \mid R_i \in \mathscr{R}\}$ and $\beta = sup\{M(R_i) \mid R_i \in \mathscr{R}\}$.

Claim 8 *Either $\alpha \leq \beta \leq \delta_i$ or $\delta_i \leq \alpha \leq \beta$.*

Proof of the Claim Suppose not. Let $\alpha < \delta_i < \beta$. Consider the case $\alpha \leq \pi_1(R_i) < \delta_i < \pi_2(R_i) \leq \beta$. By Lemma 1 and Lemma 2 we have either $M(\pi_1, \pi_2) = \pi_1$[5] or $M(\pi_1, \pi_2) = \pi_2$. Suppose $M(\pi_1, \pi_2) = \pi_1$ is true. Fix $\pi_2 = \pi_2^*$.

Claim 8.1 $M(\pi_1, \pi_2^*) = \pi_1$ for all $\pi_1 \in [\alpha, \delta_i)$.

Proof Suppose not. Fix a $\pi_1 = \pi_1^*$ such that $M_d(\pi_1^*, \pi_2^*) = \pi_1^*$. Let $\hat{\pi}_1 = inf\{\pi_1 \in (\pi_1^*, d) : M(\pi_1, \pi_2^*) = \pi_2^*\}$. (If it doesn't exist choose $\hat{\pi}_1 = sup\{\pi_1 \in (\alpha, \pi_1^*) : M(\pi_1, \pi_2^*) = \pi_2^*\}$ and modify the steps below accordingly.)

Take a sequence $(\pi_1)_n \to \hat{\pi}_1$ such that $\pi_1^n < \hat{\pi}_1$ for all n with $\pi_1^0 > \pi_1^*$. Note that since $\hat{\pi}_1 = inf\{\pi_1 \in (\pi_1^*, d) : M(\pi_1, \pi_2^*) = \pi_2^*\}$, $M(\pi_1^n, \pi_2^*) = \pi_1^n$ for all n. Then by continuity $M_d(\hat{\pi}_1, \pi_2^*) = \hat{\pi}_1$. Now take a sequence $(\pi_1)_k \to \hat{\pi}_1$ such that for all k, $\pi_1^k \in \{\pi_1 \in (\pi_1^*, d) : M(\pi_1, \pi_2^*) = \pi_2^*\}$. Then by continuity $M(\hat{\pi}_1, \pi_2^*) = \pi_2^*$. Contradiction.

Now fix a $\pi_1 = \pi_1^*$. Note that, by the above claim $M(\pi_1^*, \pi_2^*) = \pi_1^*$.

Claim 8.2 $M(\pi_1^*, \pi_2) = \pi_1^*$ for all $\pi_2 \in (\delta_i, +\infty)$

Proof Suppose not. Let $\hat{\pi}_2 = inf\{\pi_2 \in (\pi_2^*, +\infty) : M(\pi_1^*, \pi_2) = \pi_2\}$. (If it doesn't exist, choose $\hat{\pi}_2 = sup\{\pi_2 \in (\delta_i, \pi_2^*) : M(\pi_1^*, \pi_2) = \pi_2\}$ and modify the following steps accordingly.)

Take a sequence $(\pi_2)_n \to \hat{\pi}_2$ such that $\pi_2^n < \hat{\pi}_2$ for all n with $\pi_2^0 > \pi_2^*$. Note that since $\hat{\pi}_2 = inf\{\pi_2 \in (\pi_2^*, +\infty) : M(\pi_1^*, \pi_2) = \pi_2\}$, $M(\pi_1^*, \pi_2^n) = \pi_1^*$ for all n. Then by continuity,

[5] To make the proof easier to follow we wil use $M(\pi_1, \pi_2)$ instead of $M(R_i)$ as we will only be changing the peaks and M is a voting mechanism.

$M(\pi_1^*, \hat{\pi}_2) = \pi_1^*$. Now take a sequence $(\pi_2)_k \to \hat{\pi}_2$ such that for all k, $\pi_2^k \in \{\pi_2 \in (\pi_2^*, +\infty) : M(\pi_1^*, \pi_2) = \pi_2\}$. Then by continuity, $M_d(\pi_1^*, \hat{\pi}_2) = \hat{\pi}_2$. Contradiction.

If $\alpha = -\infty$, Claim 8.1 and 8.2 together implies $M(\pi_1, \pi_2) = \pi_1$ for all π_1, π_2. Therefore, there exists no R_i with location δ_i such that $M(\pi_1(R_i), \pi_2(R_i)) = \pi_2(R_i)$ and therefore the $range(M) = (-\infty, \delta_i)$ which is a contradiction to our supposition $\alpha < \delta_i < \beta$. In the next claim, we suppose α is finite and show that this result still holds.

Claim 8.3 $M(\pi_1, \pi_2) = \alpha$ for all $\pi_1 \in (-\infty, \alpha)$.

Proof Suppose not. Since we assume $\alpha < \delta_i < \beta$ and α is finite, $\beta = +\infty$ by Claim 3 of Theorem 1. Fix an arbitrary $\pi_2 = \pi_2^*$. Note that, for all $\pi_1 \in (-\infty, \alpha)$: $M(\pi_1, \pi_2^*) = \pi_2^*$ by Lemma 1 and Lemma 2 since $\pi_1 \notin range(M)$.

Take a sequence $(\pi_1)_n \to \alpha$ such that $\pi_1^n < \alpha$ for all n. Note that since $\pi_1^n \in (-\infty, \alpha)$ for all n, $M(\pi_1^n, \pi_2^*) = \pi_2^*$ for all n. Then by continuity $M(\alpha, \pi_2^*) = \pi_2^*$. However, by Claim 8.1, $M(\alpha, \pi_2^*) = \alpha$. Contradiction.

Since for all π_1, π_2, $M(\pi_1, \pi_2) = \pi_1$, we have $range(M) = (-\infty, \delta_i)$, concluding the proof of Claim 8.

Now we prove that GMM is continuous:

Let M be a *GMM* for n agents. Let R^N, $R'^N \in \mathscr{R}^N$ be two preference profiles such that for any $j \in \{1,2\}$ and for all $i \in N$, $|\pi_j(R_i) - \pi_j(R'_i)| \leq \varepsilon$ for some $\varepsilon > 0$. Clearly, for any $S \subseteq \Pi(R)$, $|\sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S\} - \sup_{\pi_j(R'_i) \in S} \{\pi_j(R'_i), a_S\}| \leq \varepsilon$. Hence, $|\inf_{S \subseteq \Pi(R)} \{\sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S\}\} - \inf_{S \subseteq \Pi(R')} \{\sup_{\pi_j(R'_i) \in S} \{\pi_j(R'_i), a_S\}\}| \leq \varepsilon$ implying $|M(R_0, ..., R_n) - M(R'_0, ..., R'_n)| \leq \varepsilon$. ■

There are indeed strategy-proof voting mechanisms that are not continuous and not GMM. An example of such mechanism is given below:

Example 1 *Let $N = 1$. Let M be a voting mechanism given $a > 0$ such that:*

$$M(R_i) = \begin{cases} \pi_2(R_i), & \pi_2(R_i) \leq a \\ \pi_1(R_i), & o/w \end{cases}$$

Observe that $range(M) = (-\infty, \delta_i) \cup (\delta_i, a]$. Moreover, since this mechanism will assign either $\pi_1(R_i)$ or $\pi_2(R_i)$ to any preference, it is strategy-proof. However, there is a jump at $\pi_2(R_i) = a$, therefore it is not continuous.

4.2. Pareto Efficiency

It turns out that the class of GMM does not contain any Pareto efficient mechanism. We show that in Theorem 3. This implies an impossibility result such that if the range of the mechanism is a finite interval or the mechanism is continuous, then it cannot be both strategy-proof and Pareto efficient.

Theorem 3 *There exists no continuous voting mechanism that is strategy-proof and Pareto efficient at the same time.*

Proof. By Theorem 2, any strategy-proof and continuous voting mechanism is a *GMM*. Hence, to prove Theorem 3, it is sufficient to show that *GMM* is not Pareto efficient. Let $N = \{1, ..., n\}$ be the set of agents with locations $\delta_i = d^*$ for all $i \in N$. For all $i \in N$, let $R_i \in \mathscr{R}$ be the associated preference relations such that there exists $i, k \in N$ with $\pi_1(R_i) \neq \pi_1(R_k)$, and $\pi_2(R_i) \neq \pi_2(R_k)$. Let M be a *GMM* and let $M(R_1, ..., R_n) = x$. Note that if $x = d^*$, M cannot be Pareto efficient, since there exists $\varepsilon > 0$ such that $(x+\varepsilon)P_i x$ for all $i \in N$. Hence, there are two cases to look at: either $x < d^*$ or $d^* < x$.

Case 1: $x < d^*$.

$M(R_1, ..., R_n) = x$ implies there exists $S \subseteq \Pi(R_1, ..., R_n)$ such that $x = \sup_{\pi_j(R_i) \in S} \{\pi_j(R_i), a_S\}$. Since $x < d^* < \pi_2(R_i)$, there exists no $i \in N$ such that $\pi_2(R_i) \in S$. This implies that right peaks are irrelevant to the mechanism. Let $R'^N \in \mathscr{R}^{\mathscr{N}}$ be such that, for all $i \in N$, $\pi_1(R'_i) = \pi_1(R_i)$ and $\pi_2(R'_i) = \pi_2(R_1)$. Note that, since we only changed the right peaks, $M(R'_1, ..., R'_n) = x$.

Since there exists $i, k \in N$ with $\pi_1(R'_i) \neq \pi_1(R'_k)$, x cannot be the left peak for every agent. Therefore, there exists $i \in N$ such that $\pi_2(R_1)P'_i x$. Moreover, since $\pi_2(R_1)$ is the right peak for every agent, there exists no $j \in N$ such that $xP'_j\pi_2(R_1)$. Hence, M is not Pareto efficient.

The proof for Case 2 is symmetric. ■

Theorem 4 *There exists no voting mechanism with a finite interval as its range and which is strategy-proof and Pareto efficient at the same time.*

Proof. The proof of this theorem is the same with the proof of Theorem 3. ■

4.3. Unobservable Locations

When δ is not fixed, and is also a function of R_i, not every *GMM* is strategy-proof, as it is possible for agents to manipulate the mechanism by misreporting their locations (Filos-Ratsikas et al., 2017). We demonstrate this fact with an example:

Example 2 *Let* $N = 2$ *and* $X = [0, 100]$. *Let M be a GMM with the parameters,*

$$a_S = \begin{cases} 100, & |S| = 0 \\ 80, & |S| = 1 \\ 20, & |S| \geq 2 \end{cases}$$

Let R_i *be the preference relation of agent i with* $\pi_1(R_i) = 10$ *and* $\pi_2(R_i) = 40$. *Let* R_j *be the preference relation of agent j with* $\pi_1(R_j) = 30$ *and* $\pi_2(R_j) = 70$. *It is easily verifiable that* $M(R_i, R_j) = 30$. *Now consider* R'_i *with* $\pi_1(R'_i) = 40$ *and* $\pi_2(R'_i) = 60$. *Notice that,* $M(R'_i, R_j) = 40$. *Since* $\pi_2(R_i) = 40$, $M(R'_i, R_j) P_i M(R_i, R_j)$, *violating strategy-proofness.*

Filos-Ratsikas et al. (2017) shows that even when the distance of peaks to the agent's location is assumed same and symmetric for everyone (in other words, when the voting mechanism only needs the location information), the strategy-proof mechanisms are only those who select either leftmost or rightmost peak. Removing the assumption of fixed locations does not change the impossibility result in theorems 3 and 4 either, since the strategy-proof mechanisms in this case are again *GMM*s.

4.4. Mechanisms that Use the Whole Preference Relation

When we relax the assumption of a voting mechanism, the class of strategy-proof mechanisms are larger than *GMM*. To illustrate this, we present an example of such a mechanism.

Example 3 *Let* $N = 1$. *Let M be a mechanism such that:*

$$M(R_i) = \begin{cases} \pi_1(R_i), & [\delta_i + \pi_1(R_i)]/2 \succeq_i [\delta_i + \pi_2(R_i)]/2 \\ \pi_2(R_i), & o/w \end{cases}$$

M is strategy-proof since it always chooses a peak of the agent. However, it uses information other than the peaks and the location, hence, it is not a voting mechanism.

This example also demonstrates that, in our model strategy-proofness does not imply that the mechanism uses only the peak information, contrary to the case with single-peaked preferences as demonstrated in Barberà and Jackson (1994).

CHAPTER 5

CONCLUSION

We define a class of mechanisms called Generalized Median Mechanisms (GMM), in a very similar manner to Moulin (1980). We show that any strategy-proof mechanism can be represented as a *GMM* when agents have double-peaked preferences with their locations observable. We demonstrate that our assumption of observable locations is fundamental to our characterization, and when agents are free to report their locations not every *GMM* is strategy-proof. For our characterization to work, we also need to assume either the mechanism's range is a finite interval or it is continuous.

Additionally, we show that *GMM* is not Pareto efficient, which implies an impossibility result for a mechanism to be both Pareto efficient and strategy-proof in our model. Lastly, we show that strategy-proofness does not imply the mechanism uses only the peaks as input, and provide an example of a strategy-proof mechanism that uses more information.

Our characterization includes non-anonymous mechanisms as well, and we consider adding anonymity as a possible direction for further study. Another possibility for future research would be assuming symmetricity around the peaks, and checking whether there is a wider class of strategy-proof mechanisms in that setting.

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