EXISTENCE OF TRAVELING WAVES SOLUTION FOR CERTAIN NONLOCAL WAVE EQUATIONS

by
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Abstract

In this thesis we investigate the existence of traveling waves solutions for nonlocal wave equations determined by a kernel function. In a series of publications Stefanov and Kevrekidis used the bell-shapedness property of the triangular kernel to study the existence and nature of a traveling wave solution in generalized lattices. In this thesis, we studied their work, and generalized the idea to a certain class of kernels that satisfy some conditions.

YEREL OLMAYAN BAZI DALGA DENKLEMLERİNDE GEZEN DALGA ÇÖZÜMLERİNİN VARLIĞI

Abba Ibrahim Ramadan

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Anahtar Kelimeler: gezen dalgalar, çan şekilli fonksyonlar, yerel olmayan dalga denklemleri, Euler-Lagrange denklemi, varyasyonlar hesabı.

Özet

Bu tezde bir çekirdek fonksıyonu tarafından belirlenen yerel olmayan bazı dalga denklemlerinde gezen dalga çözümlerinin varlığı araştırıldı. Stefanov ve Kevrekidis, bir dizi çalışmada üçgensel çekirdeğin çan şekilli olmasını kullanarak, genelleştirilmiş lattislerde gezen dalga çözümlerini elde ettiler. Bu tezde adı geçen çalışmaları inceledik ve sonuçları bazı uygun koşulları sağlayan bir çekirdek sınıfına genelledik.

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CHAPTER 1

Introduction

A traveling waves solution of a partial differential equation is solutions of the form $u(x,t) = \phi(x-ct)$, where c is some constant. Clearly all solutions $u(x,t)$ of the transport equation

$$u_t + cu_x = 0$$

are traveling waves; whereas for the equation $u_{tt} - c^2 u_{xx} = 0$, all solutions are of the form $u(x,t) = \phi(x-ct) + \psi(x+ct)$, namely a linear combination of two waves traveling in opposite directions. For nonlinear equations traveling waves represent a balance between the nonlinear and dispersion effects; namely high order derivatives. A prominent example of nonlinear wave equations is the Korteweg de Vries equation (KdV equation for short). The history of KdV equation started in 1834 with an experiment conducted by John Scott Russel, a Scottish naval engineer. In his work to determine the most efficient design for canal boats, he discovered a phenomenon called the wave of translation. This was followed by theoretical investigations by Lord Rayleigh and Joseph Boussineq around 1870, then after more than two decades by Korteweg and de Vries in 1895. For about a century the KdV equation was not studied much until Zabusky and Kruskal in 1965. They discovered numerically that the solution of the KdV seemed to decompose over long periods into collection of "solitons" which behave like particles or solutions of linear systems. In other words, these solutions are well separated solitary waves. Moreover, they seem to be almost unaffected in shape by passing through each other [1]. The KdV is a nonlinear, dispersive PDE for a function u of two variables, t denoting time and x space,

$$u_t + uu_x + u_{xxx} = 0.$$

Again, by considering the solution $u(x,t) = \phi(x-ct) = \phi(\xi)$ and substituting in the KdV equation, we have the ordinary differential equation

$$-c\phi' + (\frac{1}{2}\phi^2)' + \phi''' = 0.$$

Assuming ϕ and its derivative vanish at $\pm\infty$, and integrating once, the above ODE results in

$$-c\phi + \frac{1}{2}\phi^2 + \phi'' = 0.$$

The solution of the ODE yields the following hyperbolic function

$$\phi = \frac{c}{2}sech^2(\frac{\sqrt{c}}{2}\xi)$$

and thus for any value of $c \neq 0$,

$$u(x,t) = \frac{c}{2}sech^2(\frac{\sqrt{c}}{2}(x-ct))$$

represent traveling waves of the KdV equation.

The above method is usually referred to as the direct computation method. That is, the reduction of the PDE to ODE and solving it to obtain an explicit solution of the initial PDE. When this fails one needs an abstract method for showing existence of traveling waves. One approach is the variational method via the Euler-Lagrange equation. The variational method is one of the solid basis for the existence theory of PDE and other applied problems. The method is an extension of the method of finding extreme values and critical points in calculus. For instance, consider the abstract form

$$\mathcal{L}[u] = 0 \quad in \quad \Omega, \quad \mathcal{A}[u] = 0 \quad on \quad \partial\Omega \tag{1.1}$$

where $\mathcal{L}[u]$ denotes a given PDE and $\mathcal{A}[u]$ is a given boundary value condition. To study this problem using the calculus of variations, $\mathcal{L}[u]$ can be formulated as a first variation of an energy functional $J(u)$ on a subset Y of a Banach space $X(\Omega)$ incorporating the boundary condition, that is $\mathcal{L}[u] = J'(u)$, so the equation (1.1) can be weakly formulated as

$$\langle J'(u), v \rangle = 0, \forall v \in Y.$$

Solving (1.1) is equivalent to finding the critical point of J on X. A number of steps are taken when solving a PDE problem with variational method.

First show that $J(u)$ is bounded from above(or below) so that $\sup_{u \in Y} J(u)$ (or $\inf_{u \in Y} J(u)$) exists. Then take a maximizing (minimizing) sequence $(u^n) \subset Y$ so that $\lim_{n\to\infty} J(u^n) = \sup_{u\in Y} J(u)$ (or $\lim_{n\to\infty} J(u^n) = \inf_{u\in Y} J(u)$). In an infinite dimensional Banach space, bounded sets are not compact, so passing to a convergence subsequence of the (u^n) is not trivial. For a bounded domain Ω, compactness is usually obtained through a combination of derivative estimates and the Arzela-Ascoli theorem or compactness of Sobolev embeddings. On the contrary, when Ω is not compact, say $\Omega = \mathbb{R}$, the Sobolev embedding $W^{k,p}(\mathbb{R}) \subset L^p(\mathbb{R})$ is not compact. One can apply the Banach-Alaoglu theorem to get weak compactness, but this does not necessarily imply the existence of a maximizer. One approach is to work on a bounded interval $[-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ and then control the "tails" that is to take the limit as $\epsilon \to 0$.

In general when $\Omega = \mathbb{R}$, and L is a constant coefficient operator another problem is that if $\phi(x)$ is a solution then $\phi(x - x_0)$ is also a solution to the optimization problem, that is the minimizing problem does not change under shift. The reason why this is a problem is because the minimizing u^n's may be scattered. So we want to first shift u^n's to $\tilde{u^n}(x) = u^n(x - x^n)$ so that the $\tilde{u^n}$ may converge. But this is not always clear in general. One will need to use the concentration compactness principle [2], or a similar approach.

Stefanov and Kevrekidis in [3, 4] provide a reformulation and illustration of existence of bell-shaped traveling waves in generalized Hertzian lattice,

$$u_{ntt} = [u_{n+1}]^p - 2[u_n]^p + [u_{n-1}]^p$$

and the related traveling wave equation

$$u''(x) = u^p(x+1) - 2u^p(x) + u^p(x-1). \quad x \in \mathbb{R}$$

In the two papers they used a simpler method by introducing the bell-shaped functions which fixes the shift of the solutions and for the tail they used the $\frac{1}{\epsilon}$ approach. In this sense, the main aim of this thesis is to understand the approach of the Stefanov and Kevrekidis [3] and generalize it to a certain bell-shaped kernels that satisfy some reasonable conditions. Precisely, we study the existence of bell-shaped traveling wave solutions in the problem

$$u_{tt} = (\beta * u^p)_{xx} = \quad or \quad u_{tt} - u_{xx} = (\beta * u^p)_{xx} \tag{1.2}$$

where the kernel β is a bell-shaped integrable function. Well posedness and other properties of a particular β kernel problems have been studied in [4]. The traveling waves of (1.2) will then satisfy $c^2 u = \beta * u \quad or \quad (c^2 - 1)u = \beta * u.$
If β is taken to be a triangular kernel, that is for

$$\beta(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases} \tag{1.3}$$

since $\hat{\beta(\xi)} = \frac{4\sin^2(\frac{\xi}{2})}{\xi^2}$, so,

$$(\beta(x) * v)_{xx} = v(x-1) - 2v(x) + v(x+1) \tag{1.4}$$

which is the case similar to the problem studied by Stefanov and Kevrekidis in [3].

The rest o the thesis is organized as follows: In Chapter 2 we introduce some preliminary concepts such as the Kolmogrov-Riesz theorem that are useful in understanding in compactness of $L^p(\mathbb{R})$. In Chapter 3 we study the papers of Stefanov and Kevrekidis. In Chapter 4, we adopt approach in Chapter 3 and generalize the result to bell-shaped kernels.

CHAPTER 2

Preliminaries

In this chapter we provide important definition such as tail and bell-shapedness of a function. We also state and give proofs of some key theorems. More details can be found in L.C Evans [5] and H.Brezis [6].

2.1 L^p Space and Some Important Theorems

Definition 2.1.1. *Given a measure space* $(X, \mathcal{M}, \mu)$*, if* $1 \leq p < \infty$*, the space* $L^p(X, \mu)$ *consists of all complex valued measurable functions on* X *that satisfy*

$$\int_X |f(x)|^p d\mu(x) < \infty.$$

To simplify the notation, we write $L^p(X)$*, when the underlying measure space has been specified. Then, if* $f \in L^p(X, \mu)$ *we define the* L^p *norm of* f *by*

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

when the measure space is clear from the context we abbreviate this as $\|f\|_{L^p}$*.*
When $p = 1$ *the space* $L^1(X, \mu)$ *consists of all integrable functions on* X*.*

Definition 2.1.2. $L^\infty(\Omega)$ *is the set of* $f : \Omega \to \mathbb{R}, f$ *is measurable and there exists* C *such that* $|f(x)| \leq C$ μ *a.e on* Ω *with*

$$\|f\|_{L^\infty} = inf\{C : |f(x)| \leq C \quad \mu \quad a.e \quad on \quad \Omega\}$$

Definition 2.1.3. *If the two exponents* p *and* p^* *satisfy* $1 \leq p, p^* \leq \infty$*, and the* *relation*

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

holds, we say that p *and* p^* *are* ***conjugate*** *or* ***dual exponents****.*

Theorem 2.1.4. *(**Lebesgue Dominated Convergence Theorem,**). Let* (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*
(a) $f_n(x) \to f(x)$ *a.e. on* Ω*,*
(b) there is a function $g \in L^1(\Omega)$ *such that for all* n*,* $|f_n(x)| \leq g(x)$ *a.e. on* Ω*.*
Then

$$f \in L^1(\Omega) \quad and \quad \|fn - f\|_{L^1(\Omega)} \to 0$$

.

Theorem 2.1.5. *(**Hölder Inequality**) Suppose* $1 < p < \infty$ *and* $1 < p^* < \infty$ *are conjugate exponents. If* $f \in L^p$ *and* $g \in L^{p^*}$*, then* $fg \in L^1$ *and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p^*}}. \tag{2.1}$$

Theorem 2.1.6. *(**Minkowski Inequality**) If* $1 \leq p < \infty$ *and* $f, g \in L^p$*, then* $f + g \in L^p$ *and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

From this point forward $X = \mathbb{R}$ and λ the Lebesgue measure unless stated otherwise. We will also use $L^p = L^p(\mathbb{R})$.

Definition 2.1.7. *Let* f *and* g *be two continuous functions, we define the* ***convolution*** *of* $f(x)$ *and* $g(x)$*, denoted* $f * g$*, as*

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) dy.$$

Theorem 2.1.8. *(**Young's Inequality**) Suppose* $f \in L^p$*,* $g \in L^p$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ *where* $1 \leq p, q, r \leq \infty$ *then,*

$$\|f * g\|_r = \|f\|_p \|g\|_q$$

2.2 Weak and Strong Convergence in L^p

Recall that for a sequence $\{f_n\}$ in L^p if there exists $f \in L^p$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0,$$

then f_n **converges** to f in L^p and we denote this by $f_n \to f \in L^p$.

Definition 2.2.1. *(**Dual**) The vector space of all continuous linear functional on X equipped with a norm $\|.\|_X$ is called the dual space of X and is denoted by X^*. For $1 \leq p < \infty$ the dual of $L^{p*} = L^{p^*}$ where q is the conjugate of p.*

Definition 2.2.2. *For a sequence $\{u_n\}_{n=1}^{\infty} \subset L^p$ we say that u_n converges to $u \in L^p$ **weakly**, denoted as $u_n \rightharpoonup u$ if for each $u^* \in L^q$, we have*

$$\langle u^*, u_n \rangle \to \langle u^*, u \rangle$$

that is

$$\int_{\mathbb{R}} u^*(x)u_n(x)dx \to \int_{\mathbb{R}} u^*(x)u(x)dx.$$

Proposition 2.2.3. *Strong convergence implies weak convergence, that is if*

$$u_n \to u$$

then,

$$u_n \rightharpoonup u$$

Theorem 2.2.4. *For $f_n \in L^p$, if $f_n \rightharpoonup f \in L^p$ the*

$$\|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}.$$

Theorem 2.2.5. *(**Banach Alaoglu Theorem**) Let $1 < p < \infty$, for a given norm space $(L^p, \|.\|_p)$ define*

$$\boldsymbol{B}^* := \{f \in L^{p^*} : \|f\|_{L^{p*}} \leq 1\}$$

as the closed unit ball in L^q, then $\boldsymbol{B}^*$ is a compact space in the weak topology. Here $1 < p, p^* < \infty$ and p^* is the conjugate of p.

Remark: In general the compactness of Banach Alaoglu theorem is in the weak* topology but for $1 < p < \infty, \frac{1}{p} + \frac{1}{p^*} = 1$, $(L^p(\mathbb{R}))^* = L^{p^*}(\mathbb{R})$ and $(L^{p^*}(\mathbb{R}))^* = L^p(\mathbb{R})$ that is $(L^p(\mathbb{R}))^{**} = L^p(\mathbb{R})$ thus L^p is reflexive, thus the weak* is the same as weak topology.

Corollary 2.2.6. *Let $1 < p < \infty$ and (f_n) be bounded sequence in $L^p(\mathbb{R})$; then (f_n) has a weakly convergent subsequence in $L^p(\mathbb{R})$.*

2.3 Fourier Transform

Definition 2.3.1. *Let $f \in L^1$, the **Fourier transform** of f is defined as*

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx.$$

Theorem 2.3.2. *(Inversion Theorem) For a given Fourier transform $\hat{f} \in L^1$ the Inverse Fourier transform is given by*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}.$$

If f and f' are in L^1, then it follows that $\widehat{f'} = i\xi\hat{f}(\xi)$. More generally if $f, f', f', ..., f^(k) \in L^1$ we have

$$\widehat{f^{(k)}}(\xi) = (i\xi)^k\hat{f}(\xi).$$

Theorem 2.3.3. *(**Plancherel's Theorem**) The Fourier transform can be extented to a map on L^2 satisfying for all $f \in L^2$*

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

Remark: The Plancherel's theorem makes several Hilbert space operations easily.

Theorem 2.3.4. *For $f, g \in L^1$,*

$$\widehat{f * g} = \hat{f}.\hat{g}$$

2.4 Sobolev and Some Compactness Theorems

Definition 2.4.1. *(**Weak derivatives**) Suppose f, g are locally integrable functions on U, and α is a multiindex, then g is the α^{th}-weak partial derivative of f, written as*

$$D^\alpha f = g,$$

if for all test functions $\phi \in C_c^\infty(U)$ we have

$$\int_U f D^\alpha \phi dx = (-1)^{|\alpha|} \int_U g \phi dx.$$

Definition 2.4.2. *(**Sobolev Space**) For $U \subset \mathbb{R}$, $p \leq p \leq \infty$ the sobolev space $W^{k,p}(U)$ consists of all locally summable functions $f : U \to \mathbb{R}$ such that for each α with $|\alpha| \leq k$, $D^\alpha f \in L^p(U)$ exists in the weak sense.*

Definition 2.4.3. *If $f \in W^{k,p}(U)$, the norm associated with the Sobolev space is defined for $1 \leq p < \infty$*

$$\|f\|_{W^{k,p}(U)} := \left(\Sigma_{|\alpha| \leq k} \int_U |D^\alpha f|^p dx \right)^{\frac{1}{p}} = \Sigma_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}$$

and for $p = \infty$

$$\Sigma_{|\alpha| \leq k} ess \sup_U |D^\alpha f| = \Sigma_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty}.$$

Theorem 2.4.4. *(**Morrey's Inequality**)Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in c^1(\mathbb{R}^n)$, where $\gamma := 1 - \frac{n}{p}$.

In our case of study we want $n = 1$, $\gamma = 1 - \frac{1}{p}$, thus

$$\|u\|_{C^{0,\gamma}(\mathbb{R})} \leq C \|u\|_{W^{1,p}(\mathbb{R})}.$$

Observe that $C^{0,\gamma}$ is a Hölder space equipped with the norm

$$\|u\|_{C^{0,\gamma}(\mathbb{R})} = \sup_{\mathbb{R}} |u(x)| + \sup_{x,y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

We observe $C_b^1 \subset C_{L,p} \subset C^{0,\gamma} \subset L^\infty$ here and that $C_{L,p}$ is the Lipschitz space.

Another key observation is that by Morrey inequality we have $\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}$ this implies that $W^{1,p}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, while recalling that $W^{1,p} \subset L^p(\mathbb{R})$, yields the following result.

Lemma 2.4.5. *Let all the assumptions in Morrey's inequality hold, then for any* $p \leq r \leq \infty$ *we have* $W^{1,p} \subset L^r$

Proof.

$$
\begin{aligned}
\|u\|_{L^r}^r = \int_{\mathbb{R}} |u(x)|^r dx &= \int_{\mathbb{R}} |u(x)|^p |u(x)|^{r-p} dx \\
&\leq (\|u\|_{L^\infty}^{r-p}) \|u\|_{L^p}^p \\
\|u\|_{L^r} &\leq \|u\|_{L^\infty}^{1-\frac{p}{r}} \|u\|_{L^p}^{\frac{p}{r}} \\
&\leq C\|u\|_{W^{1,p}}^{1-\frac{p}{r}} \|u\|_{W^{1,p}}^{\frac{p}{r}} \\
\|u\|_{L^r} \leq C\|u\|_{W^{1,p}}
\end{aligned}
$$

so, $W^{1,p} \subset L^r$ for $p \leq p \leq \infty$. ☐

The Arzela-Ascoli theorem will play an important role in Kolmogrov-Riesz theorem, which is useful in understanding the compactness in $\mathbb{R}$.

Theorem 2.4.6. *(**Arzela-Ascoli Theorem**) Let* K *be a compact subset of* $\mathbb{R}$, *then* $\mathcal{F} \subset C(K)$ *is totally bounded if*
(1) $\mathcal{F}$ *is bounded*
(2) $\mathcal{F}$ *is equicontinuous.*

Note that: The above theorem is not true on $\mathbb{R}$.

Definition 2.4.7. *Let* X *and* Y *be Banach space,* $X \subset Y$. *We say that* X *is compactly embedded in* Y, *denoted as*

$$X \subset\subset Y,$$

provided
(i) $\|X\|_Y \leq C\|x\|_X \quad (x \in X)$ *for some constant* C, *and*
(ii) each bounded sequence in X *is precompact in* Y.

Remark: Let U be bounded set, by Morrey inequality we have $W^{1,p}(u) \subset C^{0,\gamma}(U)$ and by the Arzela-Ascoli theorem we have $C^{0,\gamma}(u) \subset\subset C(\bar{U})$. This implies that

$$W^{1,p}(U) \subset\subset C(\bar{U}) \subset L^{\infty}(U) \subset L^{p}(U)$$

Thus, for $p \leq q$ we have

$$W^{1,p}(U) \subset\subset L^{q}(U)$$

this also implies total boundedness in $C_b \subset L^{\infty}$.

Theorem 2.4.8. *A bounded set in $W^{1,p}(I)$ is totally bounded in C_b, L^{∞}, L^{p}. In particular, $W^{1,p} \subset\subset L^{q}(I)$ if $p \leq q$.*

As mentioned above the Arzela-Ascoli theorem doest not work on $\mathbb{R}$ and similarly, the compact embedding $W^{1,p}(K) \subset\subset L^{q}(K)$ only works on a bounded set. The remedy to this problem is when the set has small tails.

Definition 2.4.9. *A subset $\mathcal{F} \subset L^{p}(\mathbb{R})$ is said to have* ***small tails*** *if given $\epsilon > 0$ there exists $R > 0$ such that*

$$\int_{|x|>R} |f(x)|^{p} dx \leq \epsilon^{p},$$

for all $f \in \mathcal{F}$.

Theorem 2.4.10. ***Kolmogrov-Riesz Theorem*** *A subset $\mathcal{N} \subset L^{p}(\mathbb{R}^{n})$ is totally bounded in L^{p} if and only if*
(1) *$\mathcal{N}$ is bounded*
(2) *$\mathcal{N}$ has small tails and*
(3) $\lim_{y \to 0} \int_{\mathbb{R}^{n}} |f(x+y) - f(x)|^{p} dx = 0$ *uniformly for $y \in \mathbb{R}, f \in \mathcal{N}$.*

Theorem 2.4.11. *Suppose for $1 < p < \infty$, $\mathcal{F} \subset L^{p}(\mathbb{R})$ and*
(1) *$\mathcal{F}$ has small tails,*
(2) *$\mathcal{F}$ is bounded in $W^{1,p}(\mathbb{R})$,*
then $\mathcal{F}$ is totally bounded in $L^{p}(\mathbb{R})$.

Proof. We prove this theorem using Kolmogrov-Riesz theorem, we start by the identity

$$
\begin{aligned}
u(x+y)-u(x) &= \int_x^{x+y} u'(s)ds \\
|u(x+y)-u(x)|^p &\leq |\int_x^{x+y} u'(s)ds|^p \\
&\leq^{Hölder} \left(\int_x^{x+y} 1^{p^*} ds\right)^{\frac{p}{p^*}} \left(\int_x^{x+y} |u'(s)|^p ds\right) \\
|u(x+y)-u(x)|^p &\leq y^{\frac{p}{p^*}} \int_x^{x+y} |u'(s)|^p ds
\end{aligned}
$$

where $\frac{1}{p^*}+\frac{1}{p}=1$, so, we have

$$
\int_{\mathbb{R}} |u(x+y)-u(x)|^p dx \leq y^{\frac{p}{p^*}} \int_{-\infty}^{\infty} \int_x^{x+y} |u'(s)|^p ds dx.
$$

When the order of the double integral is changed, this results in $-\infty < s < \infty, s-y < x < s$ and so,

$$
\begin{aligned}
\int_{\mathbb{R}} |u(x+y)-u(x)|^p dx &\leq y^{\frac{p}{p^*}} \int_{-\infty}^{\infty} \int_{s-y}^{s} |u'(s)|^p dx ds \\
&= y^{1+\frac{p}{p^*}} \int_{\mathbb{R}} |u'(s)|^p ds \\
&\leq y^{1+\frac{p}{p^*}} \|u\|_{W^{1,p}}
\end{aligned}
$$

now we have

$$
\|u(x+y)-u(x)\|_{L^p(\mathbb{R})} \leq y^{1+\frac{p}{p^*}} \|u\|_{W^{1,p}} = y^{1+\frac{p}{p^*}} M
$$

where $M > 0$ and thus as $y \to 0$ $\|u(x+y)-u(x)\|_{L^p} \to 0$ uniformly on $\mathcal{F}$ Since the first condition of the theorem coincides with that of the Kolmogrove-Riesz theorem. This implies that $W^{1,p}(\mathbb{R}) \subset\subset L^p(\mathbb{R})$. □

Corollary 2.4.12. *For* $1 \leq p < 2$, $\mathcal{F} \subset L^2(\mathbb{R})$ *and*
(1) $\mathcal{F}$ *has small tails that is given* $\epsilon > 0$ *there exists* $R > 0$ *such that*

$$
\int_{|x|>R} |f(x)|^2 dx \leq \epsilon^2,
$$

for all $f \in \mathcal{F}$, *and*

(2) *$\mathcal{F}$ is bounded in $W^{1,p}(\mathbb{R})$,*
then $\mathcal{F}$ is totally bounded in $L^2(\mathbb{R})$.

Proof. We prove the theorem in two ways
Direct proof: Take $\epsilon > 0$, choose R so that $\int_{|x|>R} |f(x)|^2 dx \leq \frac{\epsilon}{2}$. Let $I_R = [-R, R]$. Now restrict $\mathcal{F}$ to I_R, that is $\mathcal{F}_R = \mathcal{F}|_{[-R,R]}$. This implies that $\mathcal{F}_R$ is totally bounded in $W^{1,p}(I_R)$. By Theorem 2.4.8, $\mathcal{F}_R$ is totally bounded in $L^2(I_R)$, so given $\epsilon > 0$, we can cover $\mathcal{F}_R$ by finitely many $\frac{\epsilon}{2}$-balls. This implies $\exists g_1, g_2 ... g_n \in L^2(I_R)$ so that for $f^R \in \mathcal{F}$, we have $\|f^r - g_j\|_{L^2(\mathbb{R})} < \frac{\epsilon}{2}$ for some j. $g_j, f^R \in L^2(I_R)$.
Now let define

$$\tilde{g}_j := \begin{cases} g_j & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases}$$

Clearly $\tilde{g}_j \in L^2(\mathbb{R})$.
Take $f \in \mathcal{F}$ and define

$$f^R := \begin{cases} f & \text{for} \quad |x| < R \\ 0 & \text{for} \quad |x| > R \end{cases},$$

then $f^R \in \mathbb{F}^R$ implies that there exists g_j such that

$$\|f^R - g_j\|_{L^2(\mathbb{R})} < \frac{\epsilon}{2}.$$

So,

$$\begin{aligned} \|f - \tilde{g}_j\|_{L^2}^2 &= \int_{-\infty}^{\infty} |f(x) - \tilde{g}_j(x)|^2 dx \\ &= \int_{|x|<R} |f(x) - g_j(x)|^2 dx + \int_{|x|>R} |f(x)|^2 dx \\ &= \int_{I_R} |f^R - g_j|^2 dx + \int_{|x|>R} |f|^2 dx \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

this implies that $\bar{\mathcal{F}}$ is compact in L^2 and thus ending the proof.

Alternatively: This proof is based upon theorem 2.4.9, we just need to show that for $1 \leq p < 2$,

$$\int_{\mathbb{R}} |u(x+y)-u(x)|^2 dx = \int_{\mathbb{R}} |u(x+y)-u(x)|^p |u(x+y)-u(x)|^{2-p} dx$$
$$\leq 2\|u\|^{2-p}_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |u(x+y)-u(x)|^p dx$$
$$\leq 2y^{1+\frac{p}{p^*}} \|u\|^{2-p}_{L^\infty(\mathbb{R})} \|u\|_{W^{1,p}(\mathbb{R})}$$

and since when $y \to 0$ we have $\|u(x+y)-u(x)\|^2_{L^2(\mathbb{R})} \to 0$. Thus $\mathcal{F}$ is totally bounded in $L^2(\mathbb{R})$, completing the proof. □

Corollary 2.4.13. *Let $1 \leq p < 2$, Suppose $u^n \in L^2(\mathbb{R})$ and*
(1) *u^n has small tails in L^2*
(2) *$\|u^n\|_{W^{1,p}(\mathbb{R})} \leq C$ for some constant $C > 0$*
then u^n has a convergent subsequence in $L^2(\mathbb{R})$

Remark: The proof can be extented to L^q, where $q > 2$.

Corollary 2.4.14. *Let $1 \leq p < 2$, $q \geq 2$, suppose $u^n \in L^q(\mathbb{R})$ and*
(1) *u^n has small tails in L^q*
(2) *$\|u^n\|_{W^{1,p}(\mathbb{R})} \leq C$ for some constant $C > 0$*
then u^n has a convergent subsequence in $L^q(\mathbb{R})$

2.5 Rearrangement, and Bell-shaped Functions

This section provides information useful in understanding rearrangement and bell-shapedness of functions. We refer to the book of Analysis by Elliott H.Lieb and Micheal Loss [7].

Definition 2.5.1. *For a measurable function $f : \mathbb{R} \to \mathbb{R}$, the* ***distribution of*** *f is define as*

$$d_f(s) = \lambda(\{x \in \mathbb{R} : |f(x)| > s\})$$

here λ stands for the Lebesgue Measure.

Lemma 2.5.2. *For every $\varphi \in C^1(\mathbb{R})$ we have the equality*

$$\int_{\mathbb{R}} \varphi\lambda((f(x)))dx = \int_0^\infty \varphi'(\alpha) d_f(\alpha) d\alpha$$

Definition 2.5.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be measurable function. Then we define $f^* : [0, \infty) \to [0, \infty),$ as*

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\},$$

and f^* is called the ***non-increasing rearrangement of*** *f*

Some obvious properties of f^*

(1) f^* is non-negative
(2) $f^*(x)$ is a measurable function
(3) $f^*(x)$ is decreasing
(4) Suppose $0 \le f(x) \le g(x) \quad \forall x \in \mathbb{R}^n$, and varnish at infinity, then $f^*(x) \le g^*(x) \quad \forall x \in \mathbb{R}^n$.

Lemma 2.5.4. *Suppose $\phi = \phi_1 - \phi_2$, where ϕ_1, ϕ_2 are monotone functions. If either one of $\int_{\mathbb{R}^n} \phi_1(|f(x)|)dx$ or $\int_{\mathbb{R}^n} \phi_2(|f(x)|)dx$ is finite, then*

$$\int_{\mathbb{R}^n} \phi(|f(x)|)dx = \int_{\mathbb{R}^n} \phi(|f^*(x)|)dx.$$

In particular for $f \in L^p(\mathbb{R}^n)$ we have

$$\|f\|_p = \|f^*\|_p$$

for all $1 \le p \le \infty$.

Theorem 2.5.5. *Let $f, g \ge 0 \in \mathbb{R}^n$, vanish at infinity, then*

$$\int_{\mathbb{R}^n} f(x)g(x)dx \le \int_{\mathbb{R}^n} f^*(x)g^*(x)dx$$

Lemma 2.5.6. *(**Riesz's rearrangement inequality**) Let f, g and h be non negative functions on a real line,vanishing at infinity, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y)dxdy \le \int_{\mathbb{R}} \int_{\mathbb{R}} f^*(x)g^*(x-y)h^*(y)dxdy$$

Definition 2.5.7. *A function f is said to be* ***bell-shaped*** *if $\forall x \in \mathbb{R} \quad f(x) \ge 0$, $f(x) = f(-x)$, and f is non-increasing in $[0, \infty)$ and non-decreasing in $(-\infty, 0]$.*

For us to easily characterize bell-shapedness of a function, consider the definition below

Definition 2.5.8. *Let f be a measurable function. We define $f^{\#}(t) = f^*(2|t|)$.*

Corollary 2.5.9. *(**Characterization of bell-shapedness**) A function f is bell-shaped if and only if $f^{\#} = f$.*

Corollary 2.5.10. *Let f, g and h be non negative functions on a real line,vanishing at infinity, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^{\#}(x)g^{\#}(x-y)h^{\#}(y)dxdy$$

More information on this concepts explained in this chapter can be found in references [5, 6, 8–12].

Remark: $|f|^* = f^*$, and $|f|^{\#} = f^{\#}$.

CHAPTER 3

Stefanov and Kevrekidis's Result

In this chapter we provide alternative approaches to the work performed by Stefanov and Kevrekidis [3, 4]. The issue of compactness in L^p is of great importance for the attainability of the maximizer of our optimization problem. [3, 4] uses the approach of considering the problem within a specific interval $(\frac{-1}{\epsilon}, \frac{1}{\epsilon})$ and later taking the limit as ϵ approaches infinity. We get our compactness result via Corollary 2.4.11 controlling the tails. Our proof via the Kolmogrov compactness theorem simplifies the approach in [3]. We also give an alternative proof using the Lebesgue Dominated Convergence theorem as is suggested in [4].

3.1 Setting of the Problem

Atanas Stefanov and Panayotis Kevrekidis provide a reformulation and illustration of the existence of bell-shaded traveling waves in generalized lattices [3]. Their work is based on iterative schemes that have been previously presented in [13, 14] for the computation of the traveling waves in such chains of the form

$$\ddot{v} = [v_{n-1} - v_n]_+^p - [v_n - v_{n+1}]_+^p. \tag{3.1}$$

where v_n is the displacement of the n-th bead from its equilibrium position. The spacial case of Hertzian contacts is for $p = 3/2$. The construction of the traveling waves and the derivation of their monotonicity properties will be based on the strain variant of the equation for $u_n = v_{n-1} - v_n, u > 0$ such that:

$$\ddot{u} = \ddot{v}_{n-1} - \ddot{v}_n, \quad \ddot{v}_n = u_n^p - u_{n+1}^p, \quad \ddot{v}_{n-1} = u_{n-1}^p - u_n^p, \quad \ddot{u}_n = u_{n-1}^p - u_n^p - (u_n^p - u_{n+1}^p)$$

$$\ddot{u}_n = [\delta_0 + u_{n+1}]^p - 2[\delta_0 + u_n]^p + [\delta_0 + u_{n-1}]^p. \tag{3.2}$$

where δ_0 is a given positive number. When $\delta = 0$, in continuous form, this becomes

$$u_{tt} = \Delta_{disc}(u) \tag{3.3}$$

where

$$\Delta_{disc} f(x) := f(x+1) - 2f(x) + f(x+1). \tag{3.4}$$

Using the definition of $\Delta_{disc} f(x)$, we have that the above equation becomes

$$c^2 u'' = \Delta_{disc}[u^p]. \tag{3.5}$$

Observe that we can also write

$$\Delta_{disc} f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)(e^{i\xi} + e^{-i\xi} - 2) e^{ix\xi} d\xi.$$

Since $\cos\xi = \frac{e^{i\xi} + e^{-i\xi}}{2}$ and using half angles identities we have

$$\Delta_{disc} f(x) = -4 \int_{-\infty}^{\infty} \sin^2(\frac{\xi}{2}) \hat{f}(\xi) e^{ix\xi} d\xi$$

that is the Fourier transform of the operator Δ_{disc} is

$$\widehat{\Delta_{disc} f}(\xi) = -4 \sin^2(\frac{\xi}{2}) \hat{f}(\xi)$$

After taking the Fourier transform of both sides in (3.5) we have,

$$u(\hat{\xi}) = \frac{4\sin^2(\frac{\xi}{2})}{\xi^2} \hat{u^p}(\xi).$$

Setting $\widehat{\Lambda}(\xi) = \frac{4\sin^2(\frac{\xi}{2})}{\xi^2}$, the problem becomes

$$c^2u(x) = \Lambda * u^p(x) = \int_{-\infty}^{\infty} \Lambda(x-y)u^p(y)dy =: M[u^p](x). \tag{3.6}$$

It is clear that after taking the inverse Fourier transform of $\Lambda(\hat{\xi}) = \frac{4\sin^2(\frac{\xi}{2})}{\xi^2}$ the result becomes

$$\Lambda(x) = \begin{cases} 1-|x| & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

Note that we have the following formula for the convolution $\Lambda * f$

$$Mf = \Lambda * f(x) = \int_{x-1}^{x+1} (1-|x-y|)f(y)dy. \tag{3.7}$$

$$c^2u = M(u^p). \tag{3.8}$$

3.2 Solution of the Problem

We will also consider the following multiplier

$$\widehat{Qf}(\xi) = \frac{\sin(\frac{\xi}{2})}{\xi} f(\hat{\xi}).$$

It easily follows that since $\widehat{\chi_{[-\frac{1}{2},\frac{1}{2}]}}(\xi) = \frac{\sin(\xi/2)}{\xi}$, we have also the representation

$$Qf(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(y)dy. \tag{3.9}$$

Based on the definition of the operator M, we have $M = Q^2$.

Theorem 3.2.1. *The equation* (3.1.3) *has a bell-shaped solution* u.

To prove theorem 3.2.1, we take the following steps.
Considering a different representation of (3.6) by introducing a positive function

$w : w^{\frac{1}{p}} = u$, (3.6) reduces to

$$c^2 w^{\frac{1}{p}} = \Lambda * w, \tag{3.10}$$

then we need to find a solution w to solve (3.10), as is stated in theorem 3.2.1.

Let $q = 1 + \frac{1}{p}$, and multiply 3.8 by w and integrate over $\mathbb{R}$ to get

$$\begin{aligned} c^2 \int w^q dx &= \int_{-\infty}^{\infty} (\Lambda * w) w dx \\ &= \langle Q^2 w, w \rangle = \langle Qw, Qw \rangle = \|Qw\|_{L^2}^2 \end{aligned}$$

this leads to the following constraint optimization problem.

$$J_{max} = \sup\{J[v] = \|Qv(x)\|_{L^2}^2 : \|v(x)\|_{L^q} = 1, \quad v \quad even\} \tag{3.11}$$

We show that the above energy functional in (3.11) of the problem (3.4) is bounded from above. This will guarantee the existence(existence of the supremum). Next we then choose a maximizing sequence v^n that satisfy the constraint in (3.11). According to the Alaoglu theorem, we have that $v^n \rightharpoonup v$ for some $v \in L^q$ in L^q.the next step is showing that this maximizer is attained. we do this by considering two different approaches ; the Lebesgue dominated convergence theorem and the compactness theorem2.4.11. Finally, we derive the Euler-Lagrange equation of (3.11) shows that the maximizer solves the original problem (proving theorem 3.2.1).

3.3 Constructing a Maximizer

We first we show that $J(v)$ is bounded from above .

Lemma 3.3.1. *If v satisfies the constraint of* (3.11) *then $J(v)$ is bounded from above .*

Proof. using Young's inequality we have

$$\begin{aligned} J(v) = \|Qv\|_{L^2}^2 &= \|\chi_{[-\frac{1}{2},\frac{1}{2}]} * v(x)\|_{L^2}^2 \\ &\leq \|\chi_{[-\frac{1}{2},\frac{1}{2}]}\|_{L^r}\|v\|_{L^q}, \end{aligned}$$

so, since for $r = \frac{2p+2}{p+3}$ we have

$$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q},$$

by Young's inequality,

$$J(v) \leq \|\chi_{[-\frac{1}{2},\frac{1}{2}]}\|_{L^{\frac{2p+2}{p+3}}}\|v\|_{L^q}$$

since, $\|v\|_{L^q} = 1$, let us look at

$$\|\chi_{[-\frac{1}{2},\frac{1}{2}]}\|_{L^{\frac{2p+2}{p+3}}} = \int_{\mathbb{R}} \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)^{\frac{2p+2}{p+3}}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1dx = 1.$$

Thus,

$$J(v) \leq 1.$$

□

This boundedness of $J(v)$ from above guarantees the existence of the supremum of (3.11). However since bounded sets in $L^q(\mathbb{R})$ are not compact, we do not have the assurance that this supremum, say J^{max} of (3.11), is actually attained. To achieve this, we consider another maximization problem

$$J^{\#}_{max} = \sup\{J(\omega) : \|\omega\|_{L^q} = 1, \quad \omega \quad bell - shaped\}$$

.

Proposition 3.3.2.

$$J^{\#}_{max} = J_{max}.$$

Proof. Clearly, we need to show that $v^{\#}$ is a solution to (3.11) and that the other properties hold for any non-increasing rearrangement. We first show that it satisfies the constraint. i.e

$$\|v^{\#}\|_{L^q(\mathbb{R})} = 1$$

however, we know that rearrangement preserves L^p-norms, thus

$$1 = \|v\|_{L^q(\mathbb{R})} = \|v^{\#}\|_{L^q(\mathbb{R})} = 1,$$

hence $v^{\#}$satisfies the constraint. It is also clear that

$$J_{max} \geq J^{\#}_{max}.$$

Conversely, we need to show that

$$\int_{\mathbb{R}} |Qv(x)|^2 dx \leq \int_{\mathbb{R}} |Qv^{\#}(x)|^2 dx.$$

To this end, for any test function v, using both the Riesz Convolution-reaarangement inequality Corollary 2.5.10 and Lemma 2.5.4, we have

$$\begin{aligned}
\|Qv(x)\|^2_{L^2} &= \int\int \Lambda(x-y)v(x)v(y)dxdy \\
&\leq \int\int \Lambda(x-y)|v(x)||v(y)|dxdy \\
&\leq \int\int \Lambda(x-y)|v|^{\#}(x)|v|^{\#}(y)dxdy \\
&\leq \|Q|v|^{\#}\|_{L^2}
\end{aligned}$$

thus, $J(v) \leq J(|v|^{\#})$. □

Now by proposition 3.3.2, we can reduce the set of allowable v to the set of bell-shaped functions. Let $\mathcal{F}$ be the set of all v bell-shaped functions that satisfies the given constraint i.e $\|v\|_{L^q} = 1$. We first recall some properties of $\mathcal{F}$.

Lemma 3.3.3. *Let $v \in \mathcal{F}$ then for all $x_0 \neq 0$ we have*

$$v(|x_0|) \leq \frac{1}{|x_0|^{1/q}}.$$

Proof.

$$1 = \int_{\mathbb{R}} |v^q(x)|dx \geq \int_{|x| \leq x_0} |v(x)|^q dx \geq 2x_0 v^q(x_0)$$

so,

$$v^q(|x_0|) \leq \frac{1}{2|x_0|}$$

hence,

$$v(|x_0|) \leq \frac{1}{2^q}|x_0|^{-\frac{1}{q}} \leq |x_0|^{-\frac{1}{q}},$$

thus, we have for all $x \neq 0 \quad v(|x|) \leq |x|^{-\frac{1}{q}}$. □

Lemma 3.3.4. *let $v \in \mathcal{F}$ then for $|x| > 1$ we have*

$$Qv(x) \leq \frac{1}{(|x| - 1/2)^{\frac{p}{p+1}}},$$

and for $|x| < 1$ we have

$$Qv(x) \leq 1.$$

Proof. Since for $v \in \mathcal{F}$ by Lemma 3.3.3 for all $x > 0$ we have,

$$v \leq x^{-\frac{p}{p+1}}.$$

and for all $|x| > 1$ and for any $|x| - 1/2 \leq y \leq |x| + 1/2$ using the decreasing property of v we have

$$v \leq v(|x| - 1/2) \leq (|x| - 1/2)^{-\frac{p}{p+1}}$$

so,

$$Qv(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} v(y)dy \leq (|x| - \frac{1}{2})^{-\frac{p}{p+1}} \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} dy = (|x| - \frac{1}{2})^{-\frac{p}{p+1}}$$

and for all $|x| < 1$, we have

$$Qv(x) \leq (\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} v^{1+\frac{1}{p}}(y)dy)^{\frac{p}{p+1}} \leq \|v\|_{L^q} = 1$$

□

Since we have shown in Proposition 3.2.2 that $J_{max} = J^{\#}_{max}$, there exists a bell-shaped maximizing sequence $\{v^n\}$, that is $\|v^n\|_{L^q} = 1$ and $\lim J(v^n) = J_{max}$

Lemma 3.3.5. *For given maximizing sequence v^n there exist a subsequence v^{n_k} such that $J(v^{n_k}) \to J(v)$ and $\|v\|_{L^q} = 1$*

Proof. Since $\|v\|_{L^q} = 1$, then by Alaoglu's theorem we have $v^{n_k} \rightharpoonup v$, for some $v \in L^q$. Clearly for all x, we have

$$Qv^{n_k}(x) = \int \chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)v^{n_k}(y)dy = \langle\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x-\cdot), v^{n_k}\rangle \to \langle\, \chi_{[-\frac{1}{2},\frac{1}{2}]}(x-\cdot), v\rangle = Qv(x)$$

hence $Q(v^{n_k})$ converge pointwise to $Q(v)$. For strong convergence of $J(v^{n_k})$ we have two different proofs.

Proof 1: For a given v^n we have $\lim J(v^n) = J_{max}$. We need to show that for some $v \in L^q$,

$$\lim_{n_k \to \infty} J(v^{n_k}) = J(v)$$

i.e $\int_{\mathbb{R}} |Qv^{n_k}|^2 dx \to \int_{\mathbb{R}} Qv(x)dx$. Recall $Qv(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} v(x)dx$, then we have

$$\lim_{n} \|Qv^{n_k}\|^2_{L^2} = \lim_{n\to\infty} \int_{\mathbb{R}} |\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} v^{n_k}(x)dy|^2 dx.$$

and using the bound for $Qv^n(x)$ from Lemma 3.3.4 we have

$$|Qv^n(x)|^2 \leq \Phi^2(x)$$

where

$$\Phi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1 \\ \frac{1}{(|x|-\frac{1}{2})^{\frac{1}{q}}} & \text{for} \quad |x| \geq 1 \end{cases}$$

and clearly, $\Phi(x) \in L^1$, s0 the problem $J(v^{n_k}) \to J(v)$ is a limit-integral interchange problem. By the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} |Qv^{nk}(x)|^2 dx = \int_{\mathbb{R}} |\lim_{n\to\infty} Qv^{nk}|^2 dx = \int_{\mathbb{R}} |Qv|^2 dx.$$

Thus,

$$\lim_{n_k\to\infty} J(v^{n_k}) = J(v) = J_{max}.$$

Which ends the proof.

Proof 2: Alternatively we can also prove $J(v^{n_k}) \to J(v)$. by the following approach. We will start with the following claim

Claim 3.3.6. *The sequence $\{Qv^n\}$ is precompact in L^2*

Proof. To prove this we need to show that the sequence Qv^n satisfies the conditions in Corollary 2.4.13. For the first condition we have a bound for Qv^n from Lemma 3.3.4. So, for a given $\epsilon > 0$ and $R > 1$

$$\int_{|x|>R} |Qv^n(x)|^2 dx \leq \int_{|x|>R} \frac{dx}{(x-\frac{1}{2})^{\frac{2p}{p+1}}} \leq 2\int_R^\infty \frac{dx}{(x-\frac{1}{2})^{\frac{2p}{p+1}}} = \frac{1+p}{(R-\frac{1}{2})^{\frac{p-1}{p+1}}(p+1)},$$

$\epsilon = \frac{1+p}{(R-\frac{1}{2})^{\frac{p-1}{p+1}}(p+1)}$ namely, $R = (\frac{1+p}{\epsilon^2(p-1)})^{\frac{p+1}{p-1}} + \frac{1}{2}$,
and thus we obtain a small tail. Finally,

$$\|Qv^n\|_{W^{1,q}} = \|Qv^n(x)\|_{L^q} + \|\partial_x Qv^n(x)\|_{L^q}.$$

Using the definition of Q in (3.9) and also using the Leibniz integral rule we have

$$\|\partial_x Qv^n\|_L^q = \|v^n(x+1/2) - v^n(x-1/2)\|_{L^q} \leq 2\|v^n\|_{L^q} = 2$$

and

$$\|Qv^n(x)\|_{L^q}^q \leq \int_{-\infty}^{\infty} \left(\int_{x-1/2}^{x+1/2} v^n(y)dy\right)^q dx \leq \|v\|_{L^q}^q = 1$$

which implies that

$$\|Qv^n\|_{W^{1,q}} \leq 3.$$

thus, by Corollary 2.4.13 we have Qv^n is precompact in L^2. $\square$

From Lemma 3.3.5 there exists a subsequence $Qv^{n_k} \to w_0$ in L^2 and by the uniqueness of the weak limit we have that $w_0 = Qv$, so

$$\lim J(v^{n_k}) = \lim \|Qv^{n_k}\|^2 = \|Qv\|^2 = J(v) = J_{max}.$$

The final step is to show that v satisfies the constrain. To that end, we know that by lower semi-continuity of norms we have

$$\|v\|_{L^q} \leq 1.$$

Since $J(v) = J^{max} > 0$ we have $v \neq 0$.We will show that in fact $\|v\|_{L^q} = 1$, Assume the opposite $0 < \rho = \|v\|_{L^q} < 1$ and consider the function $v/\rho : \|v/\rho\|_{L^q} = 1$. Observe that $J(\frac{v}{\rho}) = J(v)\rho^{-2} = J^{max}\rho^{-2} > J^{max}$. Thus, $\|v\|_{L^q} = 1$ (otherwise, we get a contradiction with the constrained maximization problem). This implies that $J(v) = J^{max}$, otherwise, we get a contradiction with definition of J^{max}. Thus, we have shown that the limit v is indeed a maximizer for our problem. $\square$

After showing that the energy functional has a maximizer and that such a maximizer can be attained, what is left to derive the Euler-Lagrange equation of (3.11).

Lemma 3.3.7. (3.11) *has a bell shaped maximizer.*

Proof. If v is not bell shaped then consider $|v|^\#$

$$J_{max} = J(v) \leq J(|v|^\#) \leq J_{max}$$

which says that $|v|^\#$ is a bell shaped maximizer $\square$

3.4 Euler-Lagrange Equation

We have already shown that problem (3.11) has a maximizer v , next is to derive the Euler-Lagrange equation. Consider perturbations of v of the form $v+\lambda z$, where z a is fixed even $C_0^\infty(\mathbb{R})$ function. Clearly, for each z, there exists $\lambda_0 = \lambda_0(z)$ such that for all $0 < \lambda < \lambda_0$, $\frac{(v+\lambda z)}{\|v+\lambda z\|_{L^q}}$ satisfies all the constraints. The binomial expansion of $(v(x) + \lambda z)^q$ up to order 2 provides,

$$\begin{aligned}\|v + \lambda z\|_{L^q} = \int_{-\infty}^{\infty} (v(x) + \lambda z)^q dx &= \int_{-\infty}^{\infty} v^q(x)dx + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2) \\ &= \|v\|_{L^q} + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2) \\ &= 1 + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2).\end{aligned}$$

In addition, by the definition of $J(v)$,the self-adjointness of Q on $L^2(\mathbb{R})$ and the properties of dot the product, we have

$$\begin{aligned}J(v + \lambda z) &= \langle\, Q(v + \lambda z), Q(v + \lambda z)\rangle_{L^2(\mathbb{R},\mathbb{R})} \\ &= \langle\, Qv, Qv\rangle + \langle\, Qv, \lambda Qz\rangle + \langle\, \lambda Qz, Qv\rangle + \langle\, \lambda Qz, \lambda Qz\rangle \\ &= \langle\, Qv, Qv\rangle + 2\lambda\langle\, Q^2 v, z\rangle + O(\lambda^2) \\ &= J(v) + 2\lambda\langle\, Q^2 v, z\rangle + O(\lambda^2).\end{aligned}$$

then

$$J(\frac{v + \lambda z}{\|v + \lambda z\|_{L^q}}) = \frac{J(v + \lambda z)}{\|v + \lambda z\|^2_{L^q}} = \frac{J^{max} + 2\lambda\langle\, Mv, z\rangle + O(\lambda^2)}{(1 + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2))^{2/q}}$$

again expanding the denominator to $O(\lambda^2)$ order again, we have

$$= \frac{J^{max} + 2\lambda\langle\, Mv, z\rangle + O(\lambda^2)}{1 + 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2)},$$

we now multiply the whole expression by $\frac{1-2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx}{1-2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx}$, this gives

$$= \frac{J^{max} + 2\lambda\langle\ Mv, z\rangle + O(\lambda^2)}{1 + 2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx + O(\lambda^2)}.\frac{1-2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx}{1-2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx}$$

$$= J^{max} + 2\lambda\langle\ M, z\rangle - J^{max}2\lambda q\int_{-\infty}^{\infty}v^{q-1}(x)z(x)dx + O(\lambda^2)$$

$$= J^{max} + 2\lambda(\langle\ Mv, z\rangle - J^{max}\langle\ v^{q-1}, z\rangle) + O(\lambda^2).$$

Since $J(\frac{v+\lambda z}{\|v+\lambda z\|_{L^q}}) \leq J^{max}$, then for all $z \in C_0^\infty(\mathbb{R})$ we have

$$\langle\ Mv - J^{max}v^{q-1}, z\rangle \leq 0.$$

and also since there are no restrictions on z (other that even with compact support), it follows that $\langle\ Mv - J^{max}v^{q-1}, z\rangle = 0$ and thus,

$$Mv - J^{max}v^{q-1} = 0. \tag{3.12}$$

This is our Euler-Lagrange equation.

3.4.1 Conclusion

We obtained a bell-shaped traveling wave $u = v^{\frac{1}{p}}$ with $c^2 = J_{max} = c_0^2 > 0$. But what about the other values of c? To answer the this question we consider $u_\lambda = \lambda u_0$ and plug it into the initial problem

$$c^2u = \mathcal{M}(u^p)$$

so we have,

$$\mathcal{M}(u_\lambda) = \mathcal{M}(\lambda^p u_0^p)$$

$$= \lambda^p\mathcal{M}(u_0^p)$$

and since we know that $c_0^2 u_0 = \mathcal{M}(u_0^p)$ then we have

$$= \lambda^p c_0^2 u_0$$

$$= \lambda^{p-1} c_0^2 u_\lambda.$$

If we set $c^2 = \lambda^{p-1}$ this gives

$$\lambda^{p-1} = \frac{c^2}{c_0^2}$$

and thus for any $c > 0$

$$\lambda_c = \left(\frac{c}{c_0}\right)^{\frac{2}{p-1}}$$

and $u = \lambda_c v$ is a traveling wave solution with velocity c.

CHAPTER 4

Generalization to Bell Shaped Kernels

In [15], the authors considered a non local model for one dimensional elasticity leading to the equation

$$u_{tt} = (\beta * f(u))_{xx}, \tag{4.1}$$

where $u = u(x,t)$, f is a non linear function of u, and β is an even integrable function. They proved local well posedness of the related Cauchy problem under the assumption that the Fourier transform satisfies

$$0 \leq \beta(\xi) \leq C(1+\xi^2)^{-r/2}$$

here $r \geq 2$, and other properties. The triangular kernel, the exponential kernel and the Gaussian kernel are examples of the most commonly used kernels functions. Note that from (4.1) if we replace $f(u)$ with u^p we get

$$u_{tt} = (\beta * u^p)_{xx}$$

which gives the Stefanov-Kevrekidis's problem that was studied in Chapter 3 for the given kernel $\beta = \Lambda$. In [12], using the concentration compactness principle it was shown that these equations have traveling wave solutions. In this chapter adopting the approach in [3, 4], we will generalize the result in Chapter 3 to bell shaped kernels.

4.1 Setting of the Problem

Traveling wave solution $u = u(x - ct)$ of (4.1) will satisfy

$$u'' = (\beta * u^p)''.$$

Assuming that u, u' vanishes at infinity, this gives

$$c^2 u = \beta * u^p. \tag{4.2}$$

Note that the problem studied in Chapter 3 could be expressed as

$$c^2 u(x) = \Lambda * u^p(x),$$

where Λ is the triangular kernel.This is the motivation to generalise the results of Chapter 3.

Throughout this section, we assume that an integrable bell-shaped kernel β satisfies:
$|x|^{1/q}\beta \in L^{q^*}$, where $q = 1 + \frac{1}{q}$, q^* is the dual exponent.

Theorem 4.1.1. *The equation* (4.2) *has a bell shaped solution* u*.*

The theorem can be proven with the following steps.
We first consider a different representation of (4.1) by introducing a positive function $w : w^{\frac{1}{p}}$. This reduces the equation to $c^2 w^{\frac{1}{p}} = \beta * w$. Thus, (4.2) becomes

$$c^2 w^{\frac{1}{p}} = \beta * w = \int_{-\infty}^{\infty} \beta(x - y) w(y) dy. \tag{4.3}$$

To find a solution for (4.3) as stated in theorem 4.1.1 , let $q = 1 + \frac{1}{p}$ and q^* be its dual exponent. As in Chapter 3 multiplying (4.3) by w and integrating over $\mathbb{R}$

yields the constraint optimization problem

$$J_{max} = \sup\{J(v) = \langle \beta * v, v\rangle : \|v\|_{L^q} = 1, v \quad even\} \tag{4.4}$$

where the bracket shows L^q, L^{q*} duality. We will fellow the same steps in Chapter3.

4.2 Constructing a Maximizer

Lemma 4.2.1. *If v satisfies the constraint in* (4.5), *then $J(v)$ is bounded from above.*

Proof. Using the Hölder inequality and Young's inequality (Theorem 2.1.8), we have

$$\begin{aligned} J(v) = |\langle \beta * v, v\rangle| &= \int \beta * v(x) v(x) dx \\ &\leq \|\beta * v\|_{L^{q*}} \|v\|_{L^q} \\ &\leq \|\beta\|_{L^{\frac{p+1}{2}}} \|v\|_{L^q}^2 \leq \|\beta\|_{L^\infty} \|\beta\|_{L^1} \|v\|_{L^q}^2 \leq C. \end{aligned}$$

Note that β being bell shaped implies that it is bounded. Then $\beta \in L^1 \cap L^\infty$, so $\beta \in L^r$ for any $1 \leq r \leq \infty$. Thus the supremum of above J_{max} exists. □

To attain the supremum of the maximization problem (4.5), consider another optimization problem

$$J^{\#}_{max} = \sup\{J(w) : \|w\|_{L^q} = 1, w \quad bell - shaped\}$$

Proposition 4.2.2. $J^{\#}_{max} = J_{max}$

Proof. We clearly need to show that $v^{\#}$ is a solution to (4.5), the other properties hold for any non-increasing rearrangement. Firstly, we check whether it satisfies the constraint. i.e

$$\|v^{\#}\|_{L^q(\mathbb{R})} = 1$$

we also know that rearrangement preserves L^p-norms, thus

$$1 = \|v\|_{L^q(\mathbb{R})} = \|v^{\#}\|_{L^q(\mathbb{R})} = 1,$$

hence $v^{\#}$satisfies the constraint. It is also clear that

$$J_{max} \geq J^{\#}_{max}.$$

For the converse, we need to show that $J(v) \leq J(|v|^{\#})$.

$$|\int_{\mathbb{R}}\int_{\mathbb{R}} \beta(x-y)v(x)v(y)dydx| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \beta(x-y)|v|^{\#}(x)|v|^{\#}(y)dydx.$$

To this end, using the Riesz convolution-rearrangement inequality Lemma 2.5.6, provides

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v|(x)\beta(x-y)|v|(y)dxdy \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |v|^{\#}(x)\beta^{\#}(x-y)|v|^{\#}(y)dxdy.$$

Clearly $\beta(x-y) = \beta^{\#}(x-y)$. Therefore, the last expression is equivalent to $\langle \beta * |v|^{\#}, |v|^{\#}\rangle$ and thus, $J(v) \leq J(|v|^{\#})$. □

Based on proposition 4.2.2, we can finally reduce the set of allowable v to the set of bell-shaped functions.

Lemma 4.2.3. *Suppose that $\beta \in L^1$ and $|x|^{\frac{1}{q}}\beta \in L^{q^*}$. Then there is some $C > 0$ so that for all bell-shaped v with $\|v\|_{L^q} = 1$,*

$$\beta * v(x) \leq C(|x|-1)^{\frac{-1}{q}}$$

for $|x| \geq 2$, and $\beta * v(x) \leq C$ for $|x| \leq 2$.

Proof. $\beta * v(x)$ is even because

$$\beta * v(x) = \int_{\mathbb{R}} \beta(x-y)v(y)dy = \int_{\mathbb{R}} \beta(y-x)v(-y)dy = \beta * v(-x)$$

here we use that fact that both β and v are bell-shaped. So it suffices to consider $x \geq 2$. By Lemma 2.3.3 $v(x) \leq |x|^{-\frac{1}{q}}$ for $x \neq 0$. Then $v(x) \leq |x|^{\frac{-1}{q}}$

$$\beta * v(x) = \int_{-1}^{1} \beta(x-y)v(y)dy + \int_{|x|>1} \beta(x-y)v(y)dy = I + II.$$

Then

$$I = \int_{x-1}^{x+1} \beta(x-y)v(y)dy \leq v(x-1) \int_{-\infty}^{\infty} \beta(x-y)dy \leq \|\beta\|_{L^1}(x-1)^{\frac{-1}{q}}$$

$$\begin{aligned}
II &= \int_{|x|>1} |x-y|^{\frac{-1}{q}} |x-y|^{\frac{1}{q}} \beta(x-y)v(y)dy \\
&\leq (x-1)^{\frac{-1}{q}} \int_{-\infty}^{\infty} |x-y|^{\frac{1}{q}} \beta(x-y)v(y)dy \\
&\leq (x-1)^{\frac{-1}{q}} \left(\int_{-\infty}^{\infty} \left(|x-y|^{\frac{1}{q}} \beta(x-y) \right)^{q^*} dy \right)^{\frac{1}{q^*}} \left(\int_{-\infty}^{\infty} v^q(y)dy \right)^{\frac{1}{q}} = \||x|^{\frac{1}{q}}\beta\|_{L^{q^*}}(x-1)^{\frac{-1}{q}}.
\end{aligned}$$

Adding up I and II gives

$$\beta * v(x) \leq C(|x|-1)^{\frac{-1}{q}}.$$

And for $|x| \leq 2$ we have

$$\begin{aligned}
\beta * v(x) &= \int \beta(x-y)v(y)dy \\
&\leq \left(\int (\beta(x-y)^{q^*} \right)^{\frac{1}{q^*}} \|v\|_{L^q} \\
&\leq \|\beta\|_{L^{q^*}}^{\frac{1}{q^*}} \|v\|_{L^q} \leq C.
\end{aligned}$$

□

We then pick a bell shaped maximizing sequence $\{v^n\}$, that is $\lim J(v^n) = J_{max}$ and $\|v^n\|_{L^q} = 1$.

Corollary 4.2.4. *For given maximizing sequence v^n,*

$$\beta * v^n(x) \le \Theta(x) = \begin{cases} C & for \quad |x| \le 2 \\ C(|x|-1)^{\frac{-1}{q}} & for \quad |x| \ge 2 \end{cases}$$

Lemma 4.2.5. *For given maximizing sequence v^n there exists a subsequence v^{n_k} such that $J(v^{n_k}) \to J(v)$ and $\|v\|_{L^q} = 1$*

Proof. Since $\|v\|_{L^q} = 1$, then by Alaoglu's theorem we have $v^{n_k} \rightharpoonup v$, for some $v \in L^q$. Then for each $x \in \mathbb{R}$, we have

$$\beta * v^{n_k}(x) = \int \beta(x-y)v^{n_k}(y)dy = \langle \beta(x-\cdot), v^{n_k}\rangle \to \langle \beta(x-\cdot), v\rangle = \beta * v(x)$$

so, $\beta * v^{n_k}(x)$ converges pointwise to $\beta * v(x)$.

The next is to show that $J(v^{n_k})$ converges to $J(v)$. We provide two different proofs for this convergence.

Proof 1: From Corollary 4.2.4 we have,

$$\beta * v^n(x) \le \Theta(x).$$

Passing to the pointwise limit this gives

$$\beta * v(x) \le \Theta(x).$$

Then

$$|\beta * v^{n_k}(x) - \beta * v(x)|^{q^*} \le 2^{q^*}(\Theta(x))^{q^*}.$$

But for $|x| \ge 2$

$$(\Theta(x))^{q^*} \le C(|x|-1)^{\frac{-q^*}{q}}.$$

And since $\frac{q^*}{q} > 1$, $(\Theta(x))^{q^*} \in L^1$ then the problem $J(v^{nk}) \to J(v)$ is a limit-integral interchange problem. Then by Lebesgue Dominated Convergence Theorem we have $\lim_{n_k \to \infty} \|\beta * v^{n_k} - \beta * v\|_{L^{q*}} = 0$, so that

$$\beta * v^{n_k} \to \beta * v \quad in \quad L^{q*}$$

Then we have

$$J(v^{n_k}) = \langle \beta * v^{n_k}, v^{n_k} \rangle = \langle\ \beta * v^{n_k} - \beta * v, v^{n_k} \rangle + \langle \beta * v, v^{n_k} \rangle.$$

since we have

$$\langle \beta * v, v^{n_k} \rangle \to \langle\ \beta * v, v \rangle$$

and by Hölder Inequality we have

$$|\langle\ \beta * v^{n_k} - \beta * v, v^{n_k} \rangle| \leq \|\beta * v^{n_k} - \beta * v\| \|v^{n_k}\|$$

Since $\|v^{n_k}\| = 1$ and $\|\beta * v^{n_k} - \beta * v\| \to 0$ we then have

$$J(v^{n_k}) = \langle \beta * v^{n_k}, v^{n_k} \rangle \to \langle\ \beta * v, v \rangle = J(v)$$

Thus,

$$J(v^{n_k}) \to J(v).$$

This proves that $J(v) = J_{max}$.

Proof 2: Alternatively, we can also prove $J(v^{n_k}) \to J(v)$. We do this by showing that $\beta * v^n$ is totally bounded in L^{q^*} then using that compactness to get convergent subsequence and that prove the lemma. We start with a claim

Claim 4.2.6. *Suppose $\beta_x \in L^1$ then the sequence $\beta * v^n$ is precompact in L^{q^*}*

Proof. To prove this we need to show that the sequence $\beta * v^n$ satisfies the conditions in Theorem 2.4.11. For the first condition we have a bound for $\beta * v^n$ from Lemma

4.2.3. So, for a given $\epsilon > 0$

$$\int_{|x|>R} |\beta * v^n(x)|^{q^*} dx \leq \int_{|x|>R} \frac{dx}{C^2(|x|-1)^{\frac{q^* p}{p+1}}} \leq \frac{2}{C^2} \int_R^\infty \frac{dx}{(x-1)^{\frac{q^* p}{p+1}}} = \frac{1}{(R-1)^{p+1}(p+1)},$$

$\epsilon^{q^*} = \frac{1}{(R-1)^{p+1}(p+1)}$ so, $R = (\frac{1}{\epsilon^{q^*}(p+1)})^{\frac{1}{p+1}} + 1$

and thus we have a small tail. We then use the fact that $q < q^*$ to show that

$$\|\beta * v^n\|_{W^{1,q^*}} = \|\beta * v^n(x)\|_{L^{q^*}} + \|\beta_x * v^n(x)\|_{L^{q^*}} \leq (\|\beta\|_{L^1} + \|\beta_x\|_{L^1})\|v^n\|_{L^{q^*}} = C$$

which implies that

$$\|\beta * v^n\|_{W^{1,q^*}} \leq C.$$

Thus, by Theorem 2.4.11 we have $\beta * v^n$ is precompact in L^{q^*}. □

From the above there exists a subsequence $\beta * v^{n_k} \to w_0 = \beta * v$. And again since $v^{n_k} \rightharpoonup v$, we have

$$J(v^{n_k}) = \langle \beta * v^{n_k}, v^{n_k} \rangle = \langle\ \beta * v^{n_k} - \beta * v, v^{n_k} \rangle + \langle \beta * v, v^{n_k} \rangle$$

since $v^{n_k} \rightharpoonup v$ we have

$$\langle \beta * v, v^{n_k} \rangle \to \langle\ \beta * v, v \rangle$$

and by Hölder Inequality we obtain

$$|\langle\ \beta * v^{n_k} - \beta * v, v^{n_k} \rangle| \leq \|\beta * v^{n_k} - \beta * v\| \|v^{n_k}\|$$

Since $\|v^{n_k}\| = 1$ and $\|\beta * v^{n_k} - \beta * v\| \to 0$ we then have

$$J(v^{n_k}) = \langle \beta * v^{n_k}, v^{n_k} \rangle \to \langle\ \beta * v, v \rangle = J(v)$$

so by the uniqueness of the weak limit we have that $J(v) = J_{max}$.
And we have same proof as in Chapter 3 for $\|v\|_{L^{q^*}} = 1$. □

Note: How can it be certain that v is infact bell shaped?

Lemma 4.2.7. *There is a bell shaped maximizer v.*

Proof. If v is not bell shaped then consider $|v|^{\#}$,

$$J_{max} = J(v) \leq J(|v|^{\#}) \leq J_{max}$$

which says that $|v|^{\#}$ is a bell shaped maximizer. □

4.3 Euler-Lagrange Equation

We have shown that (4.5) has a maximizer v , next is to derive the Euler-Lagrange equation. Consider perturbations of v in the form $v + \lambda z$, where z a is fixed even $C_0^\infty(\mathbb{R})$ function. Clearly, for each such z, there exists $\lambda_0 = \lambda_0(z)$ such that for all $0 < \lambda < \lambda_0$, $\frac{(v+\lambda z)}{\|v+\lambda z\|_{L^q}}$ satisfy all the constraints. Thus after the binomial expansion of $(v(x) + \lambda z)^q$ up to order 2 we have,,

$$\begin{aligned}\|v + \lambda z\|_{L^q} = \int_{-\infty}^{\infty} (v(x) + \lambda z)^q dx &= \int_{-\infty}^{\infty} v^q(x)dx + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2) \\ &= \|v\|_{L^q} + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2) \\ &= 1 + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2),\end{aligned}$$

and also by the definition of $J(v)$,the self-adjointness of β and the properties of dot product we have,

$$\begin{aligned}J(v + \lambda z) &= \langle\ \beta * (v + \lambda z), (v + \lambda z)\rangle \\ &= \langle\ \beta * v, v\rangle + \langle\ \beta * v, \lambda z\rangle + \langle\ \beta * \lambda z, v\rangle + \langle\ \beta * \lambda z, \lambda z\rangle \\ &= J(v) + 2\lambda\langle\ \beta * v, z\rangle + O(\lambda^2).\end{aligned}$$

then

$$J\left(\frac{v + \lambda z}{\|v + \lambda z\|_{L^q}}\right) = \frac{J(v + \lambda z)}{\|v + \lambda z\|_{L^q}^2} = \frac{J^{max} + 2\lambda\langle\ \beta * v, z\rangle + O(\lambda^2)}{(1 + \lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2))^{2/q}}$$

expanding the denominator to $O(\lambda^2)$ order we have

$$= \frac{J^{max} + 2\lambda\langle\ \beta * v, z\rangle + O(\lambda^2)}{1 + 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2)},$$

and now multiply the whole expression by $\frac{1-2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx}{1-2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx}$, we then have

$$= \frac{J^{max} + 2\lambda\langle\ \beta * v, z\rangle + O(\lambda^2)}{1 + 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2)} . \frac{1 - 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx}{1 - 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx}$$

$$= J^{max} + 2\lambda\langle\ \beta * v, z\rangle - J^{max} 2\lambda q \int_{-\infty}^{\infty} v^{q-1}(x)z(x)dx + O(\lambda^2)$$

$$= J^{max} + 2\lambda(\langle\ \beta * v, z\rangle - J^{max}\langle\ v^{q-1}, z\rangle) + O(\lambda^2).$$

Since $J(\frac{v+\lambda z}{\|v+\lambda z\|_{L^q}}) \leq J^{max}$, then for all $z \in C_0^\infty(\mathbb{R})$ we have

$$\langle\ \beta * v - J^{max} v^{q-1}, z\rangle \leq 0.$$

and again since there are no restrictions on z (other that compact support), it follows that $\langle\ \beta * v - J^{max} v^{q-1}, z\rangle = 0$ and thus,

$$\beta * v - J^{max} v^{q-1} = 0. \tag{4.5}$$

Equation (4.6) is now the Euler-Lagrange equation.
Similar to Conclusion 3.4.1 if we consider $u_\lambda = \lambda u_0$ and set $c^2 = \lambda^{p-1}$ we observe that for any $c \neq 0$, $\lambda_c = \left(\frac{c}{c_0}\right)^{\frac{2}{p-1}}$, and $c^2 u_\lambda = (\beta * u_\lambda^p)$

4.4 Examples

Example 1: Consider $\beta(x) = \frac{1}{2}e^{-|x|}$, β is clearly even Additionally, $\beta(x)$ is non increasing in $[0, \infty)$, and for all x $\beta(x) \geq, 0$ thus $\beta(x)$ is bell-shaped.
Secondly,

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} dx = 1,$$

thus $\beta \in L^1$, and also

$$|x|^{\frac{1}{q}}.\frac{1}{2}e^{-|x|} \in L^{L^{q^*}}.$$

And finally $\beta'(x) = \frac{1}{2} \pm e^{-|x|} \in L^1$.

This shows that $\beta(x) = \frac{1}{2}e^{-|x|}$ satisfy all the assumptions on β.

Example 2: For $\beta(x) = e^{-x^2}$, obviously β is bell-shaped, integrable, $|x|^{\frac{1}{q}}\beta \in L^{q*}$, and since $\beta'(x) = -2xe^{-x^2} \in L^1$ all conditions are satisfied.

Example 3: Consider the triangular kernel

$$\Lambda(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

$\beta \in L^1$, bell-shaped and $\beta'(x) = \frac{x}{|x|}$, and the derivatives is zero outside $(-1, 1)$. So $\int_{-1}^{1} |\beta'(x)|dx = 2$, this means that $\beta'(x) \in L^1$.

Example 4: $\beta(x) = \frac{1}{1+x^2}$. Again $\beta(x)$ is bell-shaped and also

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

and hence $\beta(x) \in L^1$ and also $\int_{\mathbb{R}} \left(\frac{|x|^{\frac{1}{q}}}{1+x^2} \right)^{\frac{q}{q-1}} dx = p+1$. This shows that $|x|^{\frac{1}{q}}\beta \in L^1$. $\beta'(x) = \frac{-2x}{(1+x^2)^2}$ and $\int_{\mathbb{R}} |\beta|'(x)dx = \|\beta\|_{L^1}$. This also implies that $\beta'(x) \in L^1$. Showing that all the conditions are satisfied.

Example 5:

$$\beta(x) = \begin{cases} 1 - x^2 & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

In this case also, $\beta(x)$ is bell-shaped and clearly $\int_{\mathbb{R}} \beta(x)dx = \frac{4}{3}$, this gives $\beta \in L^1$ and $|x|^{\frac{1}{q}}\beta \in L^{q*}$. We also have that

$$\beta'(x) = \begin{cases} -2x & \text{for} \quad |x| \leq 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

This clearly shows that $\beta'(x) \in L^1$. In conclusion this kernel also satisfies all the conditions.

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