AFFIRMATIVE ACTIONS UNDER THE BOSTON MECHANISM

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Abstract

We consider three popular affirmative action policies in school choice: quota-based, priority-based, and reserve-based affirmative actions. The Boston mechanism (BM) is responsive to the latter two policies in that a stronger priority-based or reserve-based affirmative action makes some minority student better off. However, a stronger quota-based affirmative action may yield a Pareto inferior outcome for the minority under the BM. These positive results disappear once we look for a stronger welfare consequence on the minority or focus on BM equilibrium outcomes.

Keywords: Matching theory, school choice, boston mechanism, affirmative action, minority, welfare.

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Özet

Biz bu çalışmada pozitif ayrımcılık politikalarının Boston mekanizması altında azınlık öğrenci grubu açısından nasıl çalıstıklarını inceledik. Çoğunluk öğrenci grubuna kota koyan pozitif ayrımcılık politikası Boston mekanizması altinda her zaman olumlu sonuç vermezken, öncelik bazlı ve azınlık öğrencilere yer ayıran politikalar Boston mekanizmasında olumlu sonuçlar vermektedirler. Bununla birlikte daha güçlü refah sonuçları istendiğinde ve kişilerin stratejik davrandıkları durumlarda bu üç politika da Boston mekanizması altında azınlık öğrencilerin faydasina çalışmayabilmektedir.

Anahtar Kelimeler: Eşleştirme teorisi, okul seçimi, boston mekanizması, pozitif ayrımcılık, azınlık, refah.

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Contents

Chapter 1

Introduction

Origins of matching theory take their roots from Gale and Shapley (1962)'s attempts to match 'n' number of women and 'n' number of men in 1962. Since human beings are not commodities, they do not have money value. For this reason, money and price mechanism is not able to solve this matching problem. Thus, stable matchings between partners can only be possible with a special mechanism. The concept that solve such matching problems with a special mechanism in markets in which there is no money is matching theory. Gale and Shapley (1962) suggest deferred acceptance algorithm(DAA) to match 'n' number of women and 'n' number of men. The matchings which are produced by DAA are *stable*. In other words, DAA is a stable mechanism. Before going into detail, we can start with a basic definition of stability in this context. Briefly, if any woman and man in a couple wants to match with a person in another couple and if this person also prefers that woman or man to his/her partner, this marriage is not stable. If there is no such a situation, the matchings are stable. We will give the formal definition of stability below. If a mechanism always gives stable matchings, this is a stable mechanism. Among all stable matchings the one that is produced by deferred acceptance mechanism is the most desirable. DAA is an efficient mechanism among all stable mechanisms.

Matching theory which is generated in Gale and Shapley (1962) 's mathematical article is used to solve the malfunctioning problems and to develop better matchings. Today, different version of Gale and Shapley 's *deferred acceptance algorithm*, is used to match residents with hospitals and to assign seats to students in public schools in New York City and Boston. Shapley and Scarf (1974) suggest a basic exchange economy in their paper, namely *On Cores and Indivisibility*, in 1974. In this model agents can trade their indivisible goods with better ones in the market. Every agent is restricted to consume one indivisible good. Thus, each agent can trade their good only once. They suggest a basic version of *Top Trading Cycle*(TTC) mechanism to obtain core allocation among agents. Core allocation is an allocation at which all the coalitions among agents do not want to deviate. We will explain how TTC and DAA works.

As we mentioned above matching theory is used in real life in many areas to solve such efficiency problems. TTC is used to match with donors and patients who are waiting for kidney. Because of this successful application of matching theory some people survive. One of the most important areas among them is *school choice*. In United States, students are assigned public schools in many states with mechanisms that are produced after the attempts of Gale and Shapley. Because of the paper that has been written by Tayfun Sönmez and Atila Abdülkadiroğlu, namely *School choice: A mechanism design*, matching theory is started to use in school choice. Thus, more efficient assignments are produced.

The system that involves preferences which are made by parents to give seats to students in public schools is school choice. It is not possible that every student is assigned with her top choice. Because of that reason schools and students need some mechanisms to obtain most desirable assignments for the sake of welfare of society.

In Turkey students are admitted by central authority. Balinski and Sönmez (1999) explained the shortcomings of this admission system in their paper. They proposed the use of DAA in Turkey for the sake of efficiency.

We mentioned above stability in the marriage context. At same time stability is the central notion in the school choice literature. We can define stability in school choice context as follows : there should be no unmatched student-school pair (i,s) where student i prefers school s to her assignment and she has higher priority than some other student who is assigned a seat at school s.[1]

[1]See Sönmez and Abdulkadiroğlu (2003)

Moreover, it is well known that there exists a stable matching which is preferred to any stable matching by every student in the context of school choice.[2] *Deferred acceptance algorithm* gives students most efficient assignment among all stable matchings. *DAA* works as follows: First every student ranks his/her school preferences. Every School ranks also students according to her priorities. At step one: Each student s applies to her first choice school. Schools keep these applicants and rank them according to their priority order. Schools give tentative acceptance to certain students who have the highest priority up to their total quota and give rejection to the others. At step two: Students who were rejected at first step apply their second choice school. Schools keeps all these applicants (applicants at step one and two) and rank them according to their priority order. Then schools give tentative acceptance to students who have the highest priority up to their total quotas and give rejection to the others. At step k: Students who were rejected at $(k-1)th$ step apply their best choice school at which they were not rejected. Schools keeps all these applicants (applicants at step one, two, ..., and k) and rank them according to their priority order. Schools give tentative acceptance to students who have the highest priority up to their total quotas and give rejection to the others. This procedure terminates at a step at which no rejection occurs. Tentative assignments at that step will be permanent assignments.

Pareto efficiency is crucial notion in school choice. A matching is *pareto efficient*, if there is no other matching that dominates it. In this context domination is *pareto domination*. If every student is weakly better off and some students are strictly better off when they have another matchings, the matchings which students have pareto dominate to the former matchings.

Another important notion in school choice context is *strategy proofness*. The school choice concept in matching theory can be allowed to transform to a preference revelation game. If there is no incentive for students to misreport their true preferences, the equilibrium outcomes of this game are strategy proof. If a mechanism gives always strategy proof matchings to students, this mechanism is strategy proof. *DAA* is a strategy proof mechanism.

[2]See Gale and Shapley (1962)

Although DAA is stable and strategy proof mechanism, it is not perfect in school choice context. It gives matchings that dominate matchings which belong to any stable mechanism, but its outcome can be dominated by another matching. That is why, stability is incompatible with pareto efficiency.

As we mentioned above, TTC is one of the most important mechanisms in matching theory. TTC works as follows:

At step one: Every student chooses his/her best option and points to it. Then, the school which was pointed to the student points to her best option, and this procedure terminates once a cycle has occurred. Since there is a finite number of students and a finite number of schools, there will be at least one cycle. After the cycles have occurred, students are permanently assigned to the schools which they pointed to. Students who are in the cycles are removed. Schools are removed when their capacities are exhausted.

At step two: The remaining students point to their favorite schools among the remaining schools. The remaining schools point to students who have the highest priority among remaining students. The new cycles have occurred. Then students are assigned to the schools which were pointed. This procedure continues in such a fashion. It terminates when the capacity of all schools is exhausted or every student is assigned.

TTC has very nice properties. It is more efficient than DAA. At same time students do not have an incentive to manipulate their preferences when TTC is applied. Thus, TTC is strategy proof. However, TTC is not a stable mechanism.

Some students can be excluded from society because of particular reasons. They cannot be represented in public schools as much as they are in the population. Because of this *affirmative action policies* are needed in the school choice context for the sake of the representation of minorities.

Affirmative action means positive steps are taken to increase the representation of women and minorities in areas of employment, education, and culture from which they have been historically excluded. [3]

Affirmative action policies have been broadly implemented in school choice programs

[3]This passage is from web site of Stanford Encyclopedia of Philosophy. It can be found at http://plato.stanford.edu/entries/affirmative-action/

in the United States. In many school districts of America (such as in Boston prior to 1999, as well as in Columbus and Minneapolis) controlled choice constraints have been implemented by imposing affirmative action policies.[4]

While the ultimate goal of these policies is to have more minority students at desired schools, thereby increasing both the welfare of minorities and diversity, they have received various criticisms. Furthermore, theoretical literature has reached negative conclusions regarding their effectiveness. Specifically, recent studies formally demonstrate that under certain well-known student placement mechanisms, affirmative action policies may hurt minority students, who are the purported beneficiaries. This study contributes to prior literature by comparing the welfare effects of certain popular affirmative action policies on minorities under the well-known and widely used *Boston mechanism* (BM). The BM is used in various school districts such as Minneapolis, Lee Country of Florida, and Seattle.

The Boston mechanism works as follows: Each student submits her/his preferences. Each school ranks students according to its priorities. At step one, each student applies to his/her top choice. Schools collect these applicants and give permanent acceptance to students who have the highest priority according to their quotas. At step two, schools which have exhausted their quotas and students who were accepted by a school are removed from preference lists and priority rankings. Then students who were rejected at the previous step apply to their top choice among remaining schools. Remaining schools collect these applicants and give permanent assignments to students who are the best students for them among remaining students. This procedure continues in such a fashion. When every student is assigned to a school or every school has exhausted its quota, the procedure terminates. BM has both shortcomings and nice properties. BM is not a stable mechanism. However, BM is more efficient than DAA.

We implement affirmative action policies on BM. We consider three types of affirmative action policies: quota-type, priority-type, and reserve-type. Under the quota-type, schools have majority-type specific quotas in addition to their usual total quotas. Schools can not give seats to majority students when majority quotas are exhausted.

[4]See Sönmez and Abdulkadiroğlu (2003)

Under the reserve-type, schools reserve some seats for the minority. At least one school has a certain number of seats for minority students, and if any minority student does not prefer these seats, majority students can be assigned to them. The final affirmative action policy is priority-type. Under this policy, at least one school give more priority to minority students than majority students.

In this study we offer three important definitions which are about the consequences of policies for minorities. The first one among these definitions is *pareto inferiority*. When affirmative action policy is implemented under the Boston mechanism, it gives assignments for minorities. If every student is weakly worse off and some students are strictly worse off when they have another matchings, the matchings which students have pareto inferior to the former matchings. The second definition is *pareto superiority*. If every student is weakly better off and some students are strictly better off when they have another matchings, the matchings which students have pareto superior to the former matchings. The final definition is *responsiveness*. An affirmative action policy is *responsive*, if stronger affirmative action policy does not give pareto inferior assignments to minority students.

We find that the BM is not responsive to the affirmative action with majority quotas in that a stronger affirmative action with majority quotas policy may produce a pareto inferior outcome for the minority. However, the BM is responsive to other two policies. That is, a stronger affirmative action with minority reserves or priority-based affirmative action policy makes some minorities better off (if they ever change the outcome). These results are in contrast with other well-known mechanisms. Kojima (2012) shows that all the Top Trading Cycles and stable mechanisms are non-responsive to affirmative action with majority quotas and priority-based affirmative action. Doğan (2015) finds that no stable mechanism is responsive to affirmative action with minority reserves.

Two issues are to be visited. The positive results above do not hold once we look for a stronger responsiveness by requiring a Pareto superior outcome for the minority after a stronger priority-based affirmative action or stronger affirmative action with minority reserves. Besides, we assume that students are sincere. However, it is well-known that the BM is open to preference manipulations. BM determines assignments according to preferences. Since a student submits a school in her top choice, she can be assigned to this school instead of a student who has more priority than her and who did not submit that school in her preferences as top choice. The Boston mechanism is open to manipulation of preferences. Because of this BM is not strategy proof.

Once students become strategic in their preference submissions, BM equilibrium outcomes may not be responsive to any of the above affirmative actions. The Equilibrium notion that has been just mentioned above is the *Nash equilibrium*. *Nash equilibrium* is a situation in which no player wants to deviate from their current situation. In the school choice concept, no student wants to submit another preference profile.

This study is broadly related to the affirmative action in school choice literature. While there are several related papers, we mention the ones that are closely related. Apart from the already cited ones, Hafalir and et al. (2014) consider affirmative action policies as soft constraints, and the former introduces the affirmative action with minority reserves.

We proceed as follows. First we introduce related studies in the related literature chapter. Second, we describe the school choice problem as a mathematical model. Under this chapter we define three main policies with weak and strong versions of each. Then we explain how the Boston Mechanism works under these policies. Finally, we obtain results. We complete our study by obtaining equilibrium results and giving concluding remarks.

Chapter 2

Related Literature

The paper with which matching theory started is Gale and Shapley (1962). In this paper, the authors try to answer following questions: *How will schools fill their quotas? How will students assign schools which they want?* Gale and Shapley (1962) suggests a mechanism to fill the quotas of school in an efficient way. This suggestion makes schools and students satisfied. From this beginning, the literature on matching theory grew. Stability and optimality notions are also suggested first in this paper. The main question of the paper is: *Is it possible to find a stable assignment for each admission problem?* In order to solve this problem, Gale and Shapley suggest a prototype of this problem, namely *marriage problem*. A question, *'For any pattern of preferences is it possible to find a stable set of marriages?'*, creates *deferred acceptance algorithm (DAA)*. For each college admission problem, *DAA* provides stable and optimal assignments among all stable assignments.

After Gale and Shapley (1962), matching theory starts to be implemented in many important areas in real life. One of the most important areas of the implementation of matching theory is school choice. Gale and Shapley (1962) try to solve the college admission problem. A mechanism which solves this problem is provided in their paper. The mechanism was *DAA*. The central issue of school choice is to create a mechanism which provides assignments to students. It is impossible that every student is assigned to her/his top choice.

The school choice and college admission problem actually is the same problem if we consider the priorities over students in school choice as schools' preferences. Since college admission and school choice are the same problem, DAA can be adopted to school choice.[1] Because of the shortcomings of school choice systems in many districts, Sönmez and Abdulkadiroğlu (2003) suggested two competing mechanisms, namely DAA and TTC.[2] Sönmez and Abdulkadiroğlu (2003) is the first paper to approach the school choice problem from a mechanism design perspective. Sönmez and Abdulkadiroğlu established that these two mechanisms are strategy-proof. They also established that the top trading cycle mechanism is pareto efficient and not stable.

One important contribution of this paper which is most related part to our study is controlled choice. For the sake of desegregation in certain districts, controlled choice constraints are implemented.[3] They showed TTC and DAA with controlled choice are strategy proof. They also showed that DAA with controlled choice is stable and TTC with controlled choice is pareto efficient. Controlled choice constraints and affirmative action policies work for the same reason. To destroy racial and ethnic discrimination they are implemented with different variations. Since the first mechanism design approach to school choice is done in this study, we can conclude that the study of affirmative actions in school choice started with this paper. The contribution of this paper to matching theory is crucial. This paper is a pioneer of school choice and school choice with affirmative action policies.

Controlled constraints are first implemented with type-specific quotas. By this policy, the cohort is divided into subgroups and schools have the quotas for each group of students. For the benefit of minority students, type specific quotas do a restriction to the majority. Kojima (2012) studies affirmative action policies in the context of the school choice problem as analyzed by Sönmez and Abdulkadiroğlu (2003). Kojima (2012) looks type-specific quota and priority-based affirmative action for stable mechanisms and TTC. He looks simple environment of controlled choice with two groups: majority

[1] See Sönmez and Abdulkadiroğlu (2003)

[2] In Boston, Columbus, Minneapolis and Seattle, some school choice protocols are used to rank students. According to these protocols, there are some shortcomings. For example, students and their parents are forced play a hard game to submit their school preferences in a very important issue.

[3] In Boston, first controlled choice is used in 1999. In Columbus and Minneapolis, controlled choice is used.

and minority. He suggests that if his claims hold in a simple environment, his claims also hold in a more general model. We inspire from Kojima (2012) 's study and look the welfare effects of affirmative actions on minority students under Boston Mechanism.

In Kojima (2012) 's paper, two important definitions are provided: *respect the spirit of affirmative action* and *respect the spirit of priority based affirmative action.* We use similar versions of two definitions and suggest strong versions of them. According to Kojima (2012) 's definitions, when an affirmative action policy is implemented; if at least one minority student is better off, this policy respects the spirit of affirmative action policy or the spirit of priority-based affirmative action policy. Kojima (2012) finds that there exists no stable mechanism that respects the spirit of affirmative action and priority-based affirmative action. He also looks whether these two definitions hold or not for well-known matching mechanism TTC. According to his results, TTC does not respect the spirit of affirmative action and priority-based affirmative action. He concludes his study with the fact that the caution should be taken into account when affirmative action policy is implemented.

The other important study about affirmative actions is Hafalir et al. (2013). The results of Kojima (2012) show that quota and priority based affirmative action policies hurt minority students. Hafalir et al. (2013) suggests new affirmative action policy, namely *reserves type affirmative action policy.* We also investigate the effects of this policy on minority students under the BM. Briefly, this policy reserves some seats at schools to minority students, while no minority student claims to apply one of these seats at one school and if these seats are not exhausted by other minority students, the majority students can apply and can be admitted to these seats.

According to the results of this paper, reserve type affirmative action policy provides better results than quota based affirmative action under the DAA.[4] According to the *Theorem 2* of this paper, at least one minority student weakly prefers reserve type affirmative action policy to majority quotas. However, there may be some situations at which some minority students are worse off and the others are indifferent.[5] They argue

[4]See Theorem 1 of Hafalir et al. (2013)

[5]Example 2 of this paper shows such a situation

that there should be some assumptions to obtain better results for minority students.[6] Two well-known matching algorithms are investigated with this study. The other one is TTC. According to the results about TTC, there exists at least one minority student who weakly prefers TTC with reserves type to TTC with no affirmative action. Finally, they argue that there is no pareto dominance relationship between the TTC with reserves and TTC with or without majority quotas.

Type-specific quota or quota based affirmative action policy can be interpreted as upper bounds in controlled school choice problem. Hafalir and et al. (2014) study hard bounds and soft bounds in controlled school choice context with the perspective of the fairness and non-wastefulness notions. The main difference between this study and Kojima (2012) 's one is the diversity of type space. In Kojima (2012) 's study there was only two type, namely majority and minority, but in this study Hafalir and et al. (2014) use many types.

In this study, hard bounds are defined as a number of seats for each type of student. For each type of student, there is a floor level of seats and ceilings level of seats at each school that is determined by the school district or law. If there is no violation to these bounds at every school, school assignments to students are feasible. Under hard bounds or strict bounds, they define fairness and non-wastefulness notion. According to their results under hard bounds, the set of feasible assignments that are fair across types may be empty in a controlled school choice. In addition, the set of feasible assignments that are both fair for same types and non-wasteful may be empty in a controlled school choice problem.[7] They provide new form of non-wastefulness, namely *constrained non-wastefulness*. After providing a new definition, they suggest a new algorithm, namely *Student Exchange Algorithm*. This algorithm takes feasible and fair for same types assignments as input and gives fair for same types and constrained non-wastefulness

[6] Two assumptions are suggested by them. Since every school has same priority order over students and each student has same preferences over schools, the matchings under deferred acceptance algorithm with minority reserves are the same with deferred acceptance algorithm with majority quotas and they are weakly preferred by minority students to the matchings of deferred acceptance algorithm with no affirmative action. The second assumption is set the minority reserves smartly for every school. If minority reserves are smartly set at every school, then DAA with minority reserves pareto dominates to DAA with no affirmative action for minorities.

[7] See Theorem 1 in Hafalir and et al. (2014)

assignments as output.

Two issues are visited in this study. First, they relax fairness notion.[8] After relaxing this notion, they obtained results as we mentioned above. Secondly, they suggest to modifying the bounds. Soft bounds are provided by them. By soft bounds, school districts adopt a dynamic priority structure giving highest priority to student types who have not filled their floors; medium priority to student types who have filled their floors, but not filled their ceilings; and lowest priority to student types who have filled their ceilings.[9] When the bounds are modified in such a fashion, there will be no feasibility constraint such as mentioned in the previous part of the paper. According to results under student-proposing deferred acceptance algorithm, they guarantee the existence of an assignment that is non-wasteful under soft bounds and fair under soft bounds. They obtain same results for school-proposing deferred acceptance algorithm.

Finally, they compare hard bounds versus soft bounds. According to the pareto comparison between hard bounds and soft bounds, they argue that all students are weakly better off under soft bounds than under hard bounds in some situations.

Another important study which encouraged us to work on affirmative action policies in school choice problem is Doğan (2015). The crucial contribution of this study is to design a mechanism which works with affirmative action policy for the sake of the benefits of the minority. First, he indicates the impossibilities to do weakly better off minority students under well known affirmative actions, namely quota based affirmative action and reserve type affirmative actions. Clearly, if minority students mostly have priority over majority students, then affirmative action policy works with the intention of benefits of the minority. However, in this situation, there is no need to implement affirmative action policy. Secondly he modifies DAA with minority reserves to obtain better results for minority after a stronger affirmative action. Doğan (2015) is inspired by the working process of DAA with minority reserves. In the working process of the DAA, some minority students might initiate a rejection cycle. Clearly, a minority student can be accepted tentatively a school and a majority student might

[8]The relaxed mode of fairness is defined as fairness for same types.

[9]See Part 4 in Hafalir and et al. (2014)

be rejected from this school because of minority reserves. This rejection initiates a rejection cycle. Finally, the minority student who had tentative acceptance at first step can be rejected at step k. He defined such minority as **interferer** and changed them as the majority. The main principle of *Modified Deferred Acceptance Algorithm with Minority Reserve*(MDA^m) is this. MDA^m has a weakly fairness property and minimally responsive.[10] The bad news about MDA^m is that the algorithm does not have strategy-proofness property. However, there is no assignment rule which is fair with conditional reserve, minimally responsive, and strategy-proof.[11]

We have mentioned the studies which are about affirmative action policies in school choice context so far. These studies investigate the effects of affirmative action policies on students and look with axiomatic perspective. Some studies, namely Doğan (2015), suggest a new algorithm which effectively works with affirmative action with the intention of minority's welfare and some studies, namely Hafalir et al. (2013) suggest a new affirmative action policy. Since our study is about effects of affirmative action policies under BM, it is necessary to look studies which are about the affirmative action policies and the BM.

In the final part of related literature part, we investigate studies which are about the BM. First, we should mention Kojima and Unver (2014). This study is the first study which provides the characterizations of the BM with an axiomatic perspective.

They use the standard modeling of the two-sided matching market in school choice context but they provide a difference: a priority structure in a school choice problem induces a new BM. They also provide as a remark that there is no two distinct priority structure that gives the same BM. They emphasize the welfare property of the BM and define the axiom of the respect of preference ranking. Briefly, this notion says that a student who gives higher preference to a school than other students will be admitted to this school if the quota is not exhausted. At the first glance, this notion can be understood as a characteristic of the working process of the BM. However, they

[10] MDA^m is fair with conditional reserve. At least one minority is better off under this algorithm with a stronger affirmative action policy. For definition of fairness with conditional reserve can be seen in Doğan (2015).

[11] See Theorem 5 of Doğan (2015)

provide a BM which is induced by a priority structure cannot satisfy this axiom.[12]

BM is criticized because of the lack of strategy-proofness. A student can be admitted to a better school by misreporting his preferences under the BM. This axiom is incompatible with the axiom of the respect of preference rankings. They provide a new axiom, namely *rank-respecting invariance (r.r. invariance)*. Before providing the definition of *r.r. invariance*, we should provide the definition of the axiom of monotonic transformation. It is said that P' is a monotonic transformation of P at $c \in C \cup \{\emptyset\}$ if every school that is ranked above c under P' is ranked above c under P. Additionally, if the set of students, who are in competition with i for his assigned school, does not expand when i has a monotonic transformation preferences profile, this monotonic transformation is a rank-respecting monotonic transformation for student i. A mechanism "φ" satisfies rank-respecting invariance if, for any pair of preference profiles P and P', a rank-respecting monotonic transformation does not change the assignments. This axiom can be accepted as a different version of strategy-proofness. According to his results, a mechanism "φ" is BM induced by a priority structure if and only if *respects preference ranking* and satisfies *resource monotonicity*, *consistency* and r.r. invariance.[13]

In the final part of their study, they provide a special environment. According to this environment, every school has one student seat. Their second crucial result is that a mechanism "φ" is the BM induced by a priority structure if and only if "φ" *respects preference rankings* and satisfies *individual rationality*[14], *population monotonicity* and *r.r. invariance* under this special environment. [15]

In contrast of the article of Kojima and Unver (2014), Afacan (2013) suggests a new axiom which holds for every problem in school choice context, namely *respect both preference rankings and priorities.* According to the definition, if a student prefers a

[12]See Example 2 of Kojima and Unver (2014)

[13]If a school increases her quota, every student is weakly better off under a resource monotonic mechanism. If a student is removed from the school choice problem, another students' assignments do not change under consistent mechanisms.

[14]Individual rationality says that if a student prefers a school to her assignment then this school exhausted their quota. If we remove a student from a school choice problem, this school choice problem provides weakly better results to other students under a mechanism which is population monotone.

[15]See Theorem 2 of Kojima and Unver (2014)

school to her assignment, either she is unacceptable at this school or she does not rank this school as high as students who are assigned to this school. If she ranks this school equally with the students who are assigned to this school, her priority is lower than the students who are assigned to this school. Additionally, Afacan (2013) provides a new definition *individually rationality for schools*. An unacceptable student can not be assigned to any school under an individually rational for schools matchings. According to his results, a mechanism is BM if and only if it is *individually rational for schools* and *respects both preference and priorities* at every priority structure. The crucial difference between Afacan (2013) 's study and Kojima and Unver (2014) 's study is the taking priority structure as primitive to the model. Afacan (2013) characterizes BM as individual rational for schools and respects both preference and priorities for every priority structure.

We terminate our literature review with Afacan (2013) 's study. After the literature review, we provide the primitives of school choice problem with affirmative actions.

Chapter 3

The Model

A school choice problem with affirmative actions is a tuple $(S, C, \succeq_{i \in S \cup C}, q, r)$ where

- S and C are finite and disjoint sets of students and schools.
- $S = S^M \cup S^m$ where S^M and S^m are the set of majority and minority students, respectively.
- $\succeq = (\succeq_s)_{s \in S}$ is the preference profile of the students over C and being unassigned, denoted by $\emptyset$. We write $c \succ_s c'$ if and only if $c \succeq_s c'$ but not $c = c'$. A school *acceptable* to student s if $c \succ_s \emptyset$.
- $\succ = (\succ_c)_{c \in C}$ is the priority profile of the schools over S
- $\mathbf{q} = (q_c, q_c^M)$ is the capacity profile of the schools such that q_c is the total quota of the school c, and q_c^M is the majority type-specific quota.
- $r = (r_c)_{c \in C}$ is the minority reserves profile of the schools.

A *matching* μ is a mapping from $C \cup S$ to $C \cup S \cup \emptyset$ such that

1. $\mu(s) \in C \cup \emptyset$
2. For any $s \in S$ and $c \in C$, $\mu(s) = c$ if and only if $s \in \mu(c)$.
3. $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$ for all $c \in C$.

4. $|\mu(c) \cap S^M| \leq q_c^M$ for all $c \in C$.

A *matching* μ is **pareto inferior** to μ', if for every student $s \in S$ $\mu'(s) \succeq_s \mu(s)$ holds , and at least for one $s \in S$ strictly holds. A *matching* μ is **pareto superior** to μ', if for every student $s \in S$ $\mu(s) \succeq_s \mu'(s)$ holds, and for at least one $s \in S$ strictly holds.

The first affirmative action policy is **affirmative action with majority quotas** or simply **majority quotas**. At the school c which implements majority quotas, can not be accepted the number of majority students which is greater than its majority quotas. The maximum number of majority students admitted by c is equal to q_c^M.

The second affirmative action policy is **affirmative action with minority reserves** or simply **minority reserves**. It is implemented by reserving some seats for minority students. Minority students are ranked above all majority students until the reserved seats are exhausted at the school in which minority reserves is implemented. A majority student can be admitted by the school, when minority reserves are not exhausted and no minority student prefers that school to her assigned shcool. However, at a school majority quotas is implemented, a majority student can not be admitted when majority quotas was exhausted.

The final affirmative action policy is **priority-based affirmative action**. At a school that priority-based affirmative action policy is implemented, the ranking of at least one minority student is promoted relative to majority students, but the relative ranking of each student does not change within her own group.

Consider the following *Boston Mechanism(BM)* adapted to affirmative actions:

Step 1. Each student applies to his best acceptable school. Each offer receiving school c first considers minority applicants and permanently accepts them up to its minority reserve r_c one at a time following its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time following its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than q_c^M.

In general,

Step k. Each rejected student in the previous round applies to his next best acceptable school. Each offer receiving school c which still has an available seat first considers minority applicants and permanently accepts them up to its left minority reserve one at a time following its priority order. School c then considers all the applicants who are yet to be accepted, and one at a time following its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than the remaining majority quotas.

The algorithm terminates whenever each student is either accepted by a school or has all acceptable offers rejected. The assignments at the terminal round realize as the final BM outcome.

A *matching mechanism* ϕ is a systematic procedure which provides assignments to the students for each school choice problem with affirmative actions.

Chapter 4

Consequences of the Affirmative Action Policies Under the Boston Mechanism

In this chapter, we investigate the welfare effects of three affirmative action policies under the BM.

Definition 1 *Given a school choice problem as a market* $G = (S, C, \succ_{c\in C}, \succeq_{s\in S}, \boldsymbol{q}, r)$ *is said to* ***have a stronger affirmative action with majority quotas than*** $\tilde{G} = (S, C, \succ_{c\in C}, \succeq_{s\in S}, \bar{\boldsymbol{q}}, r)$ *if for every school* $c \in C$ $q_c^M = \bar{q}_c^M$ *and* $q_c^M \geq \bar{q}_c^M$.

This definition basically says that stronger affirmative action with majority quotas requires lower majority quotas.

Definition 2 *A mechanism* ϕ *is* ***responsive to the stronger affirmative action with majority quotas*** *if there is no market pair* G *and* $\tilde{G}$ *such that* $\tilde{G}$ *has a stronger affirmative action with majority quotas policy than* G, *and* $\tilde{G}$ *is Pareto inferior to* G *for the minority.*

According to this definition, if the results of the stronger affirmative action policy are strictly better off at least for one minority student, it, then, can be concluded that this stronger affirmative action policy is responsive for minority. The spirit of this definition could be summarized as follows: Responsive affirmative action with majority quotas could be strictly worse off for all minority students expect for one. Clearly, a stronger affirmative action with majority quotas might be beneficial for one minority student and might be hurt for the remaining of minority students.

Example 1 *Let* $C = \{c_1, c_2, c_3\}$, $S = \{s_1, s_2, s_3, s_4\}$. $S^M = \{s_1, s_2\}$, $S^m = \{s_3, s_4\}$.

In the following part of the example, students' preferences and schools' priorities are provided. Students are listed according to the priority ordering of the schools. For instance, for every school, s_1 *represents the student who has highest priority,* s_2 *is the second best student in the priority ordering, and* s_3 *has the lowest priority in the ordering of all schools.*

In students' preferences, schools are listed according to the preferences of the students. For instance, c_1 *is the first choice for* s_1 *and the assignment of* c_1 *gives the highest utility to the first student, the order of the other schools is arbitrary.*

$$\succ_{c_1,c_2,c_3} : s_1, s_2, s_3, s_4.$$

$$\succeq_{s_1} : c_1, ...,$$
$$\succeq_{s_2} : c_1, c_3, ...,$$
$$\succeq_{s_3} : c_1, c_2, ...,$$
$$\succeq_{s_4} : c_1, c_3, c_2, ...$$

In the end of the preferences of students the notation of ... shows that the remaining part of the preferences are arbitrary.

The quotas vectors are as follows. **"q"** *represents initial quotas and "*$\bar{\boldsymbol{q}}$*" represents other quotas vector that corresponds stronger affirmative action with majority quotas vector. First element of this vectors is total quotas of the school c,* q_c*, and second element represents majority quotas,* q_c^M*, at school c. Simply,* $q_c^M > \bar{q}_c^M$.

$\boldsymbol{q}_{c_1} = (2, 2)$	$\bar{\boldsymbol{q}}_{c_1} = (2, 1)$
$\boldsymbol{q}_{c_2} = (1, 1)$	$\boldsymbol{q}_{c_2} = (1, 1)$
$\boldsymbol{q}_{c_3} = (1, 1)$	$\boldsymbol{q}_{c_3} = (1, 1)$

*BM works as follows: At first step, all students apply to c_1. According to **q** of c_1, and the priority structure, c_1 provides permanent acceptance to the s_1 and s_2. At second step, s_3 applies to c_2 and s_4 applies to c_3. Final assignments take place as follows according to the **q**:*

$$BM = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_2 & s_3 & s_4 \end{pmatrix}.$$

After changing the majority quotas vector, at school c_1, from $q^M_{c_1} = 2$ to $\bar{q}^M_{c_1} = 1$, final outcome of stronger affirmative action with majority quotas under the BM, (BM'), occurs as follows:

$$BM' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_3 & s_4 & s_2 \end{pmatrix}.$$

Example 1 shows that stronger affirmative action with majority quotas is useful for one minority student (s_1), while the other minority student(s_4) is being hurt. Our main aim is to provide an answer to whether responsiveness of stronger affirmative action with majority quotas holds for every school choice problem or not. Unfortunately, in contrast of the *Example 1* the negative results might appear.

Theorem 1 *The BM is not responsive to the stronger affirmative action with majority quotas.*

Proof. Consider the following market $G = (S, C, \succeq_{s \in S}, \succ_{c \in C}, \mathbf{q})$ and $\tilde{G} = (S, C, \succeq_{s \in S}, \succ_{c \in C}, \bar{\mathbf{q}})$. $\tilde{G}$ has a stronger affirmative action with majority quotas than G. Let $C = \{c_1, c_2, c_3\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $S^M = \{s_1, s_2\}$, $S^m = \{s_3, s_4\}$, and

$$\mathbf{q}_{c_1} = (q_{c_1}, q_{c_1}^M) = (2,2) \qquad \bar{\mathbf{q}}_{c_1} = (2,1)$$
$$\mathbf{q}_{c_2} = (1,1) \qquad \mathbf{q}_{c_2} = (1,1)$$
$$\mathbf{q}_{c_3} = (1,1) \qquad \mathbf{q}_{c_3} = (1,1)$$

Students' preferences are given by

$$\succeq_{s_1}: c_1, ...,$$
$$\succeq_{s_2}: c_1, c_2, c_3,$$
$$\succeq_{s_3}: c_3, c_2, c_1,$$
$$\succeq_{s_4}: c_3, c_2, c_1.$$

Schools' priorities are given by

$$\succ_{c_1}: s_1, s_2, s_3, s_4,$$
$$\succ_{c_2}: s_2, s_3, s_4, s_1,$$
$$\succ_{c_3}: s_4, s_2, s_3, s_1,$$

According to the initial quotas, $(\mathbf{q})$, the outcome of the BM occurs as the following way:

$$BM = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_2 & s_3 & s_4 \end{pmatrix}.$$

After having a stronger affirmative action with majority quotas, based on new quotas, $(\bar{\mathbf{q}})$, the outcome of BM occurs as follows:

$$BM' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_3 & s_2 & s_4 \end{pmatrix}.$$

Student s_3 is worse off under the BM' than BM when a stronger affirmative action with majority quotas is implemented. The assignment of s_4 do not change. It is concluded that BM' is pareto inferior to BM for the minority. This counter example completes the proof. ■

In the example that presented in the proof, two minority students are weakly worse off. This situation is observed because of the implementation of stronger affirmative action policy. It is resulted that majority students also can be weakly worse off with a stronger affirmative action policy. s_1 has the same assignment, but s_2 is assigned by a worse school than his initial assignment.

In the *Example* 1, we give an example in which at least one minority student is better off followed by a stronger affirmative action policy. However, this is not the case is realized for every school choice problem.

We have seen that stronger affirmative action with majority quotas is not helpful for the minority for every school choice problem. The second policy that we investigate welfare effects on minority student is *minority reserves.*

Definition 3 *Given a school choice problem as a market* $G = (S, C, \succ_{c \in C}, \succeq_{s \in S}, \boldsymbol{q}, r)$ *is said to* ***have stronger affirmative action with minority reserves than*** $\tilde{G} = (S, C, \succ_{c \in C}, \succeq_{s \in S}, \boldsymbol{q}, \tilde{r})$ *if, for every school* $c \in C$ $\tilde{r}_c \geq r_c$.

This definition basically states that a stronger affirmative action with minority reserves requires increasing minority reserves at certain schools and keeping the same at the other schools.

Definition 4 *A mechanism* ϕ *is* ***responsive to the stronger affirmative action with minority reserves*** *if there is no market pair* G *and* $\tilde{G}$ *such that* $\tilde{G}$ *has a stronger affirmative action with minority reserves policy than* G*, and* $\tilde{G}$ *is Pareto inferior to* G *for the minority.*

We look at the responsiveness of majority quotas policy and obtain a negative result for minority. The intuition behind the negative result is the rejection cycle that occurs after stronger restriction for majority students at schools. In the example presented in the proof, increasing majority quotas at school c_1 has caused to the rejection of majority student s_2. At the second step of BM, s_2 has applied to the same school with s_3 and this has caused a rejection of a minority student at the second step. Finally, this rejection cycle has caused a worse assignment for minority student s_3 while the assignment of other minority student is not changed. Clearly, a weakly worse assignment for every minority student has occurred due to the increasing of strict restriction of majority students at school c_1.

Minority reserves policy does not have a strict restriction on reserved seats for minority. If there is not satisfactory demand for the reserved seats from minority students, then these seats can be open for majority students. This is the main difference between these two affirmative action policies. Because of this, the positive results appear for minority students after having a stronger affirmative action with minority reserves.

Theorem 2 *The BM is* responsive to the stronger affirmative action with minority reserves.

Proof. There will be two cases:

(i) There is no change in the assignments. At this case, having a stronger affirmative action with minority reserves does not give pareto inferior assignments to the minority students and it trivially satisfies the responsiveness.

(ii) There is at least one change in the assignments of students. Consider the first change will occur at step k at school c. $(S, C, \succ_{c\in C}, \succeq_{s\in S}, \mathbf{q}_{c\in C})$ are the same when a stronger affirmative action with minority reserves policy is implemented. The only primitive that is changed is r. r represents the initial minority reserves and $\tilde{r}$ represents a stronger affirmative action with minority reserves parameter.

Suppose that $A(c, r)$ represents the set of applicants of c with the r reserves for minority. $A(c, \tilde{r})$ is the set of applicants of school c with the $\tilde{r}$ reserves for minority. Since the preferences of students do not change, $A(c, r) = A(c, \tilde{r})$ for every $s \in S$. Let

$r_c^\star$ and $q_c^\star$ represent the remaining minority reserves seats and the remaining total seats at k^{th} step at school c, respectively. Since the first change of assignments occurs at the k^{th} step after passing stronger affirmative action with minority reserves, the remaining quotas of schools before k^{th} step are the same.

Claim 1. $\tilde{r} > r$ at k^{th} step.

Proof of Claim 1. Suppose not.

(1) Let $\tilde{r} = r$. At school c, priority ordering over students and total quotas do not change when a stronger affirmative action with minority reserves policy is implemented. If minority reserves are equal to the each other, $\tilde{r} = r$. There will be no change in assignments of students. However, we assume that there is a change at school c at k^{th} step. There is a contradiction.

(2) Let $\tilde{r} < r$. According to the definition of stronger affirmative action with minority reserves, $\tilde{r} < r$ creates a contradiction. This completes the proof of $Claim 1$.

Claim 2. $\exists s \in A(c,r) = A(c,\tilde{r})$ such that $s \in S^m$.

Proof of Claim 2. Suppose any $s \in A(c,r)$, $s \in S^M$.As the only thing that changes at school c is minority reserves, there is no change in assignments at that school. On the other hand, we suppose that there is a change at school c at the k^{th} step. This creates a contradiction. Therefore, there is at least one minority student s, say, $s \in A(c,r)$. This completes the proof of $Claim$ 2.

Let $\hat{A}(c,r)$ be the set of students who are admitted by the school c. We create a set, titled N, as follows: $N = \{s \in A(c,r) = A(c,\tilde{r}) | s \notin \hat{A}(c,r) \text{ and } s \in \hat{A}(c,\tilde{r})\}$

Claim 3. $\exists s \in N$ such that $s \in S^m$

Proof of Claim 3. Suppose for any $s \in N$, $s \in S^M$. $s \in A(c,r) = A(c,\tilde{r})$ and $s \notin \hat{A}(c,r)$. We create a set as follows: $G = \{s \in S | s \in A(c,r) \text{ and } s \succ_c s' \text{ for any } s' \in N\}$. Let $q_c^{\star\prime}$ represents the total remaining quotas for school c after changing reserves from r to $\tilde{r}$. We know that $q_c^\star = q_c^{\star\prime}$ at k^{th} step for school c and for every $s \in S^M$, $|G| \geq q_c^\star$. From $s \in S^M$, $s \notin \hat{A}(c,r)$, and $|G| \geq q_c^\star$ implies $s \notin \hat{A}(c,\tilde{r})$. However, $s \in N$. A contradiction is appeared here. At least one minority student must be in N. This completes the proof. ■

In the proof we have completed above, the only thing which is changed is the number of minority reserves after having a stronger affirmative action with minority reserves. We assume that this change occurs at school c, and at the step k at first time. The fundamental idea behind the proof is that, as we have changed minority reserves, minority students are affected by this change. Since the all other primitives of the school choice problem are the same after having stronger affirmative action with minority reserves, at least one minority student have a better assignment under the BM.

Next, we introduce two definitions which are related to priority-based affirmative action policy.

Definition 5 *Given a school choice problem as a market* $G = (S, C, \succ_{c\in C}, \succeq_{s\in S}, \boldsymbol{q}, r)$ *is said to be* ***have a stronger priority-based affirmative action than*** $\tilde{G} = (S, C, \tilde{\succ}_{c\in C}, \succeq_{s\in S}, \boldsymbol{q}, r)$ *if, for every school* $c \in C$ *and* s, s' *where* $s \in S^m$, $s \succ_c s'$, *then* $s \tilde{\succ}_c s'$.

This definition says that, the relative ranking of minority students over majority students at every school are the same or are promoted when a stronger priority-based affirmative action policy is implemented.

Definition 6 *A mechanism* ϕ *is* ***responsive to the stronger priority-based affirmative action****, if there is no market pair* G *and* $\tilde{G}$ *such that* $\tilde{G}$ *has a stronger priority-based affirmative action policy than* G*, and* $\tilde{G}$ *is Pareto inferior to* G *for the minority.*

Theorem 3 *The* BM *is responsive to the stronger priority-based affirmative action.*

The idea behind the proof will be presented below is very similar to the logic of the proof of $Theorem$ 2.

Proof.

There will be two cases:

(i) There is no change in the assignments. At this case having stronger priority-based affirmative action will not give Pareto inferior assignments to the minority students and it trivially satisfies the responsiveness.

(ii) There is at least one change in the assignments of students. Consider the first change that will be identified at step k at school c. $(S, C, \succeq_{s\in S}, \mathbf{q}_{c\in C}, r)$ are the same when a stronger priority-based affirmative action policy is implemented.

Suppose that $A(c, \succ)$ represents the set of applicants of c with the $\succ$ priority order. $A(c, \tilde{\succ})$ is the set of applicants of the school c with the $\tilde{\succ}$ priority order. Since the preferences of students do not change, $A(c, \succ) = A(c, \tilde{\succ})$ for every $s \in S$. Let $q_c^\star$ represents the remaining total seats at the k^{th} step at the school c. Since the first change of assignments occurs at the k^{th} step after passing stronger priority-based affirmative action, the remaining quotas of schools before k^{th} step are the same. Let $q_c^{\star\prime}$ represents the remaining total seats of school c after passing stronger priority-based affirmative action policy.

Claim 1. $\exists s \in A(c, \succ) = A(c, \tilde{\succ})$ such that $s \in S^m$.

Proof of Claim 1. Suppose that $s \in A(c, \succ) = A(c, \tilde{\succ})$ and $s \in S^M$. Since $q_c^\star = q_c^{\star\prime}$, $\succeq_c$, and the priority ordering of majority students within their own group do not change at school c, at step k there will be no change. However, we assume that there will a change at first time at k^{th} step. There is a contradiction. Therefore, at least one minority student will apply to the school c. This completes the proof of $Claim 1$.

We construct a set as follows: $\hat{A}(c, \succ) = \{s \in S | s \in A(c, \succ)$ and s is placed at school c at step k under $\succ\}$

We create second set as follows: $N = \{s \in A(c, \succ) = A(c, \tilde{\succ})$ — $s \notin \hat{A}(c, \succ)$ and $s \in \hat{A}(c, \tilde{\succ})\}$

Claim 2. $\exists s \in N$ such that $s \in S^m$.

Proof of Claim 2. Suppose that for any $s \in N$, $s \in S^M$. $s \in A(c, \succ) = A(c, \tilde{\succ})$ and $s \notin \hat{A}(c, \succ)$.

Let we construct two sets as follows: $EB = \{s \in S | s \in A(c, \succ)$ and $s \succ_c s'$ for any $s' \in N\}$, $EB' = \{s \in S | s \in A(c, \tilde{\succ})$ and $s \tilde{\succ}_c s'$ for any $s' \in N\}$.

We know that $q_c^\star = q_c^{\star\prime}$ at k^{th} step for school c, $|EB| \geq q_c^{\star\prime}$ and $|EB'| \geq q_c^{\star\prime}$.

From $s \in S^M$, $s \notin \hat{A}(c, \succ)$, $|EB| \geq q_c^{\star\prime}$, and $|EB'| \geq q_c^{\star\prime}$ s must be $s \notin \hat{A}(c, \tilde{\succ})$. This contradicts with $s \in N$. Therefore, there must be at leats one minority student which is $s \in N$. ■

It is beneficial to remind that, we have investigated responsiveness of BM to the stronger affirmative action policies. Responsiveness is a weak requirement. The situations in which every minority student is made(weakly) worse off by a stronger affirmative action policy is only excluded by it. Now, we define stronger version of responsiveness and control whether BM is strongly responsive to the priority-based affirmative action and affirmative action with minority reserves. We do not investigate that the strongly responsiveness of affirmative action with majority quotas because this policy mentioned before does not satisfy the weak version of responsiveness.

Definition 7 *A mechanism ϕ is* ***strongly responsive to the priority-based affirmative action or affirmative action with minority reserves****, if for every market pair G and $\tilde{G}$ such that $\tilde{G}$ has a stronger priority-based affirmative action policy or stronger affirmative action with minority reserves than G, and $\tilde{G}$ is Pareto superior to G for the minority.*

Theorem 4 *The BM is not strongly responsive to the priority-based affirmative action policy.*

Proof. Consider the following problem $G = (S, C, \succ_{c \in C}, \succeq_{s \in S}, \mathbf{q}, r)$ such that $C = \{c_1, c_2, c_3, c_4\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $S^M = \{s_1, s_2\}$, $S^m = \{s_3, s_4, s_5\}$, $q_{c_1} = (2, 2)$, $q_{c_2} = (1, 1)$, $q_{c_3} = (1, 1)$ and $q_{c_4} = (1, 1)$.

Students' preferences and schools' priorities are given by

$\succeq_{s_1}: c_1, c_3, c_4, c_2,$ $\succeq_{s_2}: c_1, c_3, c_4, c_2,$

$\succeq_{s_3}: c_1, c_2, c_3, c_4,$ $\succeq_{s_4}: c_1, c_3, c_4, c_2,$

$\succeq_{s_5}: c_1, c_3, c_2, c_4.$

$\succ_{c_1}: s_1, s_2, s_3, s_4, s_5,$

$\succ_{c_2}: s_2, s_5, s_3, s_4, s_1,$

$\succ_{c_3}: c_1, c_2, c_5, c_4, s_3,$

$\succ_{c_4}: c_4, c_2, c_3, c_1, s_5.$

Let G' has a stronger priority-based affirmative action with $\succeq'_{c_1}: s_3, s_4, s_5, s_1, s_2$. The remaining part of the problem G is same as G'. The BM outcomes of G and G' are as follows:

$$BM = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_1, s_2 & s_3 & s_5 & s_4 \end{pmatrix} \qquad BM' = \begin{pmatrix} c_1 & c_2 & c_3 & c_2 \\ s_3, s_4 & s_5 & s_1 & s_2 \end{pmatrix}$$

Although G' has a stronger priority-based affirmative action policy than G, the outcome of G' is not Pareto superior to G. This completes the proof. ■

Theorem 5 *The BM is not strongly responsive to the affirmative action with minority reserves.*

Proof. Consider the following problem $G = (S, C, \succ_{c \in C}, \succeq_{s \in S}, \mathbf{q}, r)$ such that $C = \{c_1, c_2, c_3\}$, $S = \{s_1, s_2, s_3, s_4\}$, $S^M = \{s_1, s_2\}$, $S^m = \{s_3, s_4\}$, $q_{c_1} = (2, 0)$, $q_{c_2} = (1, 0)$ and $q_{c_3} = (1, 0)$ where $q = (q_c, r)$.

Students' preferences and schools' priorities are given by

$\succeq_{s_1}: c_1, c_2, c_3$, $\succeq_{s_2}: c_1, c_3, c_2$, $\succ_{c_1}: s_1, s_2, s_3, s_4$, $\succ_{c_2}: s_2, s_3, s_4, s_1$,

$\succeq_{s_3}: c_1, c_3, c_2$, $\succeq_{s_4}: c_1, c_2, c_3$. $\succ_{c_3}: c_2, c_3, c_1, c_4$.

Let G' has a stronger affirmative action with minority reserves with $q'_{c_1} = (2, 1)$. The remaining part of the problem G is same as G'. The BM outcomes of G and G' are as follows:

$$BM = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_2 & s_4 & s_3 \end{pmatrix} \qquad BM' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1, s_4 & s_3 & s_2 \end{pmatrix}$$

Although G' has a stronger affirmative action with minority reserves than G, the outcome of G' is not Pareto superior to G. This completes to the proof. ■

We prove that stronger requirement for responsiveness does not hold for with the priority-based affirmative action and affirmative action with minority reserves.

Chapter 5

Strategic Results

A school choice problem can be converted in to a preference revelation game. According to this game, students' preferences over schools are the strategies and the payoffs of the students are determined by the outcomes of the BM. The equilibrium concept of this game is *Nash equilibrium*. The formal definition of *Nash equilibrium* is as follows : [1]

- Let S_1 be the strategy profile of *Player* 1.
- Let s_1 be an element of the strategy profile of *Player* 1.
- Let u_1 be the payoff of *Player* 1.

Definition 8 *In the n-player normal-form game* $G = \{S_1, ..., S_n; u_1, ..., u_n\}$*, the strategies* $(s_1^\star, ..., s_n^\star)$ *are a **Nash equilibrium**(NE) if, for each player* i*,* $s_i^\star$ *is player* i*'s best response to the strategies specified for the n - 1 other players,* $(s_1^\star, ..., s_{i-1}^\star, s_{i+1}^\star, ..., s_n^\star)$ *:*

$$u_i(s_1^\star, ..., s_{i-1}^\star, s_i^\star, s_{i+1}^\star, ..., s_n^\star) \geq u_i(s_1^\star, ..., s_{i-1}^\star, s_i, s_{i+1}^\star, ..., s_n^\star)$$

for every feasible strategy s_1 *in* S_1 *; that is,* $s_i^\star$ *solves*

$$\max_{s_i \in S_i} u_i(s_1^\star, ..., s_{i-1}^\star, s_i, s_{i+1}^\star, ..., s_n^\star)$$

To be more clear, we provide an example of reveal preference game.

Example 2 *Let* $C = \{c_1, c_2\}$ *and* $S = \{s_1, s_2\}$ *be the sets of students and schools. The set of strategies is* { (c_1c_2),(c_1c_2) }*. The game matrix is provided as follows:*

[1] This formal definition is from Gibbons (1992)

Student s_2

		c_1c_2	c_2c_1
Student s_1	c_1c_2	$(1,0)$	$(1,1)$
	c_2c_1	$(1,1)$	$(1,0)$

The Nash equilibrium of this game:

$$NE_1 = \{(c_2c_1);(c_1c_2)\}$$
$$NE_2 = \{(c_1c_2);(c_2c_1)\}$$

So far now, we assume that students are sincere. However, the BM is manipulable; thereby, students may strategically misreport their preferences. Once we allow for that and focus on the equilibrium outcomes of the BM, the positive results would no longer hold. Moreover, the negative affirmative action with majority quotas result carries over to the strategic setting. To see this, let $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2\}$, with $q_{c_1} = (1,1)$, $q_{c_2} = (1,1)$ where $q_c = (q_c, q_c^M)$. Let $S^M = \{s_3\}$. Students' preferences and schools' priorities are as follows:

	s_1	s_2	s_3
U_{c_1}	1	1	2
U_{c_2}	0	2	1
$U_\emptyset$	0	0	0

$\succeq_{s_1}: c_1, \emptyset,\ \succeq_{s_2}: c_2, c_1, \emptyset,\ \succeq_{s_3}: c_1, c_2, \emptyset.$

$\succ_{c_1} = \succ_{c_2}: s_3, s_2, s_1.$

Let us first consider the non-affirmative action case, that is, $r_{c_1} = r_{c_2} = 0$. The game matrixes are provided by

Student s_3 plays (c_1c_2)

	c_1c_2	c_2c_1
c_1	$0,2,2$	$0,2,2$

Student s_3 plays (c_2c_1)

	c_1c_2	c_2c_1
c_1	$0,1,1$	$1,0,1$

There are two NE

$$NE_1 = \{(c_1); (c_1c_2); (c_1c_2)\}$$
$$NE_2 = \{(c_1); (c_2c_1); (c_1c_2)\}$$

The NE outcome is

$$BM = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_2 & c_1 \end{pmatrix}$$

Now, let us consider that $r_{c_1} = 1$ while keeping $r_{c_2} = 0$. Then, the unique NE and outcome is

$$NE = \{(c_1); (c_1c_2); (c_2c_1)\}$$

$$BM' = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & c_1 & c_2 \end{pmatrix}$$

If we keep $r_{c_1} = r_{c_2} = 0$, and instead let $\succ_{c_1}: s_2, s_1, s_3$. The unique equilibrium outcome is BM' again, which is pareto inferior to BM for the minority. For the affirmative action with majority quotas case, let $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, with $q_{c_1} = (1, 1)$, $q_{c_2} = (1, 1)$, $q_{c_3} = (1, 1)$ where $q_c = (q_c, q_c^M)$. Let $S^M = \{s_2\}$. Students' preferences and schools' priorities are as follows:

$$\succeq_{s_1}: c_1, \emptyset, \; \succeq_{s_2}: c_3, c_2, \emptyset, \; \succeq_{s_3}: c_1, c_2, c_3, \emptyset.$$

$$\succ_{c_1} = \succ_{c_2}: s_1, s_2, s_3,$$
$$\succ_{c_3}: s_3, s_2, s_1.$$

There are six NE and each of them provides the following outcome

$$BM' = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_3 & c_2 \end{pmatrix}$$

Now, let us consider that $q^M_{c_3} = 0$ while keeping $q^M_{c_1} = q^M_{c_2} = 1$. Then, the unique equilibrium outcome is

$$BM' = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

The negative results for affirmative action with majority quotas case carry over to the strategic setting.

Chapter 6

Conclusion

We investigate the welfare and strategic results of three affirmative action policies under the BM. According to our results, affirmative action with majority quotas does not work for the sake of minority students all the while. The other two affirmative action policies, namely priority-based and minority reserves, provide better assignment to at least one minority student. When we analyze the strongly responsiveness and strategic results of the affirmative action policies, it is basically seen that three affirmative action policies do not work for the sake of minority students under the BM.

School choice is an important field in which matching theory. Determining welfare consequences of the affirmative action policies for minority is important in an effort to implement successful policies. The welfare consequences of the affirmative action policies under BM for minority are investigated first with this study. In our results, affirmative action with majority quotas is contrary to the main goal of the policy. This implies that the authorities should be cautious when they employ affirmative action policies. In contrast of the DAA and TTC, BM is responsive to the minority reserves and priority-based affirmative action policy.

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