An open problem of Corteel, Lovejoy, and Mallet

Shashank Kanade ⋆, Kağan Kurşungöz, and Matthew Russell

This paper is dedicated to Krishna Alladi on the occasion of his 60^{th} birthday.

Abstract Corteel, Lovejoy and Mallet concluded their paper "An extension to overpartitions of the Rogers-Ramanujan identities for even moduli" with an open question of investigating the combinatorial properties of a q-series with two additional parameters. We settle their question, unfortunately in the negative, by showing that the series yields only the known results in overpartitions. However; when one annihilates one of the parameters, the resulting series have nice integer partitions interpretations. Those series appeared in another publication as well. In particular, Corteel, Lovejoy, and Mallet's series involve an index d. This index unifies two classes of overpartition identities for $d = 1$ and $d = 2$, but does not give additional overpartition identities for $d \geq 3$. Upon setting one of the parameters zero, one does get regular partition identities for all d. The proofs are conventional, formal verifications for brevity, but we show how to make the proofs constructive.

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1 Introduction

Andrews' H and J functions [3] are not only a framework for many results hitherto known, but also a source of inspiration for a wave of results after it.

$$H_{k,a}(y;x;q) = \sum_{n\geq 0} \frac{x^{kn}q^{kn^2+n-an}y^n(1-x^aq^{2na})(yxq^{n+1})_\infty(1/y)_n}{(q)_n(xq^n)_\infty} \quad (1.1)$$

$$J_{k,a}(y;x;q) = H_{k,a}(y;xq;q) - yxqH_{k,a-1}(y;xq;q) \quad (1.2)$$

Above and elsewhere, we employ the standard q-series notation [10]

$$(a)_n = (a;q)_n = \prod_{j=1}^{n}(1-aq^{j-1}),$$

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n,$$

$$(a_1,a_2,\ldots,a_s)_n = (a_1,a_2,\ldots,a_s;q)_n = \prod_{j=1}^{s}(a_j)_n.$$

$a, a_1, \ldots, a_s$ are called parameters, and q is called the base. If the base is not specified, it is understood to be q. $|q| < 1$ is enough for absolute convergence of the infinite products. $|x| < 1/|q|$ in addition is required for absolute convergence of (1.1) and (1.2). In this note, however, one does not need to worry about convergence.

The series (1.1) and (1.2) satisfy a number of functional equations. They admit application of Jacobi's triple product identity [10, eq. (1.6.1)] under various substitutions. This yields integer partition identities. Some examples are stated below.

Definition 1.1. A partition of a non-negative integer n is a non-increasing sum of positive integers

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$. The number of parts m is also known as the length of the partition.

Alternatively, one can write

$$n = 1f_1 + 2f_2 + 3f_3 + \cdots$$

for the same partition, where f_i denotes the number of occurrences, or the frequency, of i among λ_1, λ_2, $\ldots$, λ_m.

Obviously, only finitely many of the f_i can be nonzero. For example the non-increasing sum

$$5 + 5 + 3 + 3 + 2 + 1 + 1$$

is a partition of 20, where

$$f_1 = 2, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 0, \quad f_5 = 2, \qquad \text{and } f_i = 0 \text{ for } i \geq 6.$$

Theorem 1.1 (Rogers-Ramanujan identities [16, 12, 17]). *Given a non-negative integer n, the number of partitions of n where $f_i + f_{i+1} < 2$ equals the number of partitions of n where $f_{5j} = f_{5j\pm 2} = 0$.*

The number of partitions of n where $f_i + f_{i+1} < 2$ and $f_1 = 0$ equals the number of partitions of n where $f_{5j} = f_{5j\pm 1} = 0$.

The conventional way to express the first Rogers-Ramanujan identity is that the number of partitions of n into distinct and non-consecutive parts equals the number of partitions of n into parts that are 1 or 4 modulo 5. Notice that the condition $f_i + f_{i+1} < 2$ stipulates that the parts cannot repeat, and i and $i+1$ cannot appear together. Also, the condition $f_{5j} = f_{5j\pm 2} = 0$ amounts to disallowing parts that are 0, 2, or 3 modulo 5, so that only those 1 or 4 modulo 5 can be used.

We stick to the frequency notation so that all results in this paper can be written in a unified and succinct manner [3, Ch. 7].

The specialization here is

$$J_{2,2}(0; x; q) \text{ and } J_{2,1}(0; x; q),$$

respectively. These series appeared in [18] previously.

Theorem 1.2 (Rogers-Ramanujan-Gordon identities [11]). *Given a non-negative integer n, and integers k and a such that $k \geq 2$, $1 \leq a \leq k$, the number of partitions of n where $f_i + f_{i+1} < k$ and $f_1 < a$ equals the number of partitions of n where $f_{(2k+1)j} = f_{(2k+1)j\pm a} = 0$.*

Andrews' proof of Gordon's theorem [1] uses

$$J_{k,a}(0, x; q).$$

Of course, an immediate open problem after Gordon's theorem was the possibility of an even moduli extension (instead of $(2k+1)$). This problem is partially solved by Andrews [2], who found a modulo $(4k+2)$ analog. A full solution was given by Bressoud [5].

Theorem 1.3 (Bressoud's theorem). *Suppose n is a non-negative integer, k and a are integers such that $k \geq 2$, $1 \leq a < k$. Let $A(n)$ be the number of partitions of n where $f_{2k} = f_{2k\pm a} = 0$, Let $B(n)$ be the number of partitions of n where $f_i + f_{i+1} < k$, $f_1 < a$, and if $f_i + f_{i+1} = k-1$, then $if_i + (i+1)f_{i+1} \equiv a - 1 \pmod 2$. Then $A(n) = B(n)$.*

The proof of Bressoud's theorem [5] uses

$$(-xq)_\infty J_{\frac{k-1}{2},\frac{a}{2}}(0;x^2;q^2).$$

Later, Corteel and Lovejoy introduced overpartitions [8].

Definition 1.2. An overpartition of a non-negative integer n is a partition of n in which the first occurrence of each part may be overlined. One can write

$$n = 1f_1 + 1f_{\overline{1}} + 2f_2 + 2f_{\overline{2}} + 3f_3 + 3f_{\overline{3}} + \cdots$$

where f_i denotes the number of occurrences, or the frequency, of i (non-overlined), and $f_{\overline{i}}$ denotes that of $\overline{i}$ (overlined).

Again, only finitely many of f_i or $f_{\overline{i}}$s can be non-zero. In addition, $f_{\overline{i}}$'s may be 0 or 1 only. For example,

$$8+8+7+7+5+5+\overline{4}+3+3+\overline{2}+\overline{1}+1 \tag{1.3}$$

is an overpartition of 54 where

$$\begin{gathered} f_1 = 1,\ f_{\overline{1}} = 1,\ f_2 = 0,\ f_{\overline{2}} = 1,\ f_3 = 2,\ f_{\overline{3}} = 0,\ f_4 = 0,\ f_{\overline{4}} = 1, \\ f_5 = 2,\ f_{\overline{5}} = 0,\ f_6 = 0,\ f_{\overline{6}} = 0,\ f_7 = 2,\ f_{\overline{7}} = 0,\ f_8 = 2,\ f_{\overline{8}} = 0, \\ \text{and } f_i = f_{\overline{i}} = 0 \text{ for } i \geq 9. \end{gathered}$$

Lovejoy gave an overpartition analog of Gordon's theorem for overpartitions in two cases [14]. His proof used

$$J_{k,k}(-1;x;q) \text{ and } J_{k,1}(-1/q;x;q).$$

In fact, Lovejoy considered $J_{k,a}(y;x;q)$ but only the two cases above admitted (single) infinite product representations. Chen, Sang and Shi obtained the general theorem [6].

Theorem 1.4 (Gordon's theorem for overpartitions). *Suppose n is a non-negative intege, and k and a are integers such that $k \geq 2$, and $1 \leq a \leq k$.*

Let $D_{k,a}(n)$ be the number of overpartitions of n where $f_i + f_{\overline{i}} + f_{i+1} < k$ and $f_1 < a$.

For $1 \leq a < k$, let $C_{k,i}(n)$ be the number of overpartitions of n where $f_{2kj} = f_{2kj \pm a} = 0$.

Let $C_{k,k}(n)$ be the number of overpartitions of n where $f_{kj} = f_{\overline{kj}} = 0$.

Then $C_{k,a}(n) = D_{k,a}(n)$.

It should be noted that the $a = k$ case in Theorem 1.4 is different from Lovejoy's $a = k$ case [14]. Chen, Sang and Shi's proof used

$$H_{k,a}(-1/q;q;q),$$

instead of the specializations of J-function.

Corteel, Lovejoy and Mallet extended Bressoud's theorem to overpartitions in one case [9]. They utilized the following statistic.

Definition 1.3. Given an overpartition and an arbitrary positive integer i that need not occur in the overpartition,

$$V(i) = \sum_{j=1}^{i} f_{\overline{j}}.$$

In other words, $V(i)$ is the number of overlined parts that are less than or equal to i.

For instance, the overpartition (1.3) has

$$V(1) = 1,\ V(2) = 2,\ V(3) = 2,\ \text{ and } V(i) = 3,\ \text{for } i \geq 4.$$

Again, Chen, Sang and Shi proved the remaining cases [7].

Theorem 1.5 (Bressoud's theorem for overpartitions). *Suppose n is a non-negative integer, and k and a are integers such that $k \geq 2$, and $1 \leq a \leq k$.*

Let $D_{k,a}(n)$ be the number of overpartitions of n where $f_i + f_{\overline{i}} + f_{i+1} < k$, $f_1 < a$, and if $f_i + f_{\overline{i}} + f_{i+1} = k-1$, then $if_i + if_{\overline{i}} + (i+1)f_{i+1} \equiv V(i) + a - 1 \pmod 2$.

Let $C_{k,a}(n)$ be the number of overpartitions of n where $f_{(2k-1)j} = f_{(2k-1)j\pm a} = 0$.

Then $C_{k,a}(n) = D_{k,a}(n)$.

Corteel, Lovejoy and Mallet introduced the following variant of the H and J functions.

$$\widetilde{H}_{k,a}(y;x;q)$$
$$= \sum_{n\geq 0} \frac{(-y)^n q^{kn^2 - n(n-1)/2 + n - an} x^{(k-1)n} (1 - x^a q^{2na})(-x, -1/y)_n (-yxq^{n+1})_\infty}{(q^2;q^2)_n (xq^n)_\infty}$$
$$\widetilde{J}_{k,a}(y;x,q) = \widetilde{H}_{k,a}(y;xq;q) + yxq\widetilde{H}_{k,a-1}(y;xq;q)$$

Corteel, Lovejoy, and Mallet's result uses

$$\widetilde{J}_{k,1}(1/q;x;q),$$

whereas Chen, Sang and Shi use

$$\widetilde{H}_{k,a}(1/q;xq;q).$$

in the proofs. For $y = 0$ instead of $y = 1/q$, Theorem 1.5 reduces to Theorem 1.3.

Corteel, Lovejoy and Mallet concluded their paper with the open question of the combinatorial merit of the $J_{k,a,d}(y;x,q)$ series defined below.

$$
\begin{aligned}
&H_{k,a,d}(y;x;q)\\
&\quad=\sum_{n\geq 0}(-y)^n q^{kn^2+n-an-(d-1)n(n-1)/2}x^{k-d+1}(1-x^aq^{2na})\\
&\qquad\times\frac{(-1/y)_n(-yxq^{n+1})_\infty(x^d;q^d)_n}{(q^d;q^d)_n(x)_\infty},
\end{aligned}
\tag{1.4}
$$

$$
J_{k,a,d}(y;x;q)=H_{k,a,d}(y;xq;q)+yxqH_{k,a-1,d}(y;xq;q). \tag{1.5}
$$

All of the above listed results have the same *formal verification* method in their proofs. One starts with the multiplicity conditions imposed on the partitions. The functional equations along with the initial conditions their generating functions satisfy are found. Of course, these functional equations and initial conditions must uniquely determine the generating functions, hence the partition or overpartition enumerants.

Then one verifies that a particular specialization or twist of (1.1) or (1.2) satisfies the same functional equations and the same initial conditions. Therefore, one argues, the series at hand must be the generating function. Finally, one renders all variables but q ineffective (by substituting a power of q, 0, or 1) and applies Jacobi's triple product identity. This yields the congruence condition on the partitions or overpartitions, and hence completes the proof.

Now we turn to the discussion of the case $d \geq 3$ for (1.5). For many well-known classes of (over)partitions, it is easy to derive the recursions satisfied by their generating functions. Conversely, given a set of generating functions along with the recursions they satisfy, one can conceive of naïvely reversing this procedure to guess the (over)partitions counted by the generating functions. Such a process is carried out implicitly in many problems, for instance, in some well-known proofs of Rogers-Ramanujan, Gordon-Andrews, Göllnitz-Gordon-Andrews, Andrews-Bressoud identities, etc. We explore such a reverse-engineering procedure to guess what partitions might be counted by the series $J_{k,a,d}(y;x,q)$. Unfortunately, our naïve explorations for the case $k=5, d=3$ suggest that these series may not have easily deducible combinatorial interpretations. We discuss our findings in Section 4 below. We first explain our procedure by applying it to the well-known example of Rogers-Ramanujan recursion, then we build a formal framework to go beyond. Our results for the specific case $k=5, d=3$ are explained in Section 4.4.

The paper is organized as follows. In section 2 we state and prove the main result, and indicate some implications. Although our proofs are formal verifications as well, we demonstrate how to *construct* those series in section 3. Thus, the proofs may be made into constructive proofs. We also display the constructed generating functions when we alter the characterization of classes of overpartitions slightly. In Section 4 we turn to a concrete exploration done in the case $k=5, d=3$. We conclude with some further exploration topics in section 5.

2 Main results

In this section, we indicate that the answer to Corteel, Lovejoy and Mallet's question is most likely negative for overpartitions. The answer is affirmative for regular partitions, when the parameter y is set to zero [13]. We need another overpartition statistic [13] before we proceed.

Definition 2.1. Given an overpartition and an arbitrary positive integer i that need not occur in the overpartition,

$$\rho(i) = \sum_{j=1}^{i} (-1)^j f_{\overline{j}}.$$

In other words, $\rho(i)$ is the signed sum of number of occurrences of overlined parts that are less than or equal to i.

For instance, the overpartition (1.3) has

$$\rho(1) = -1,\ \rho(2) = 0,\ \rho(3) = 0,\ \text{ and } \rho(i) = 1,\ \text{for } i \geq 4.$$

We will place the series defined by (1.4) in the following class of series.

$$\begin{aligned} H^s_{k,a,d}(y;x;q) =& \frac{(x^d;q^d)_\infty}{(x;q)_\infty} \\ & \times \sum_{n\geq 0} (-1)^n x^{n(k+1-d)} q^{(2k+1-d)n(n-1)/2+(k+1)n-an} \\ & \quad \times \frac{y^n(-1/y;q)_n(-yxq^{n+1};q)_\infty}{(q^d;q^d)_n(x^dq^{nd};q^d)_\infty} \\ & \quad \times q^{-sn}\left[q^{dn}\frac{x^{d-s}-x^d}{1-x^d} + \frac{1-x^{d-s}}{1-x^d}\right] \\ & \quad -(-1)^n x^{n(k+1-d)+a} q^{(2k+1-d)n(n-1)/2+(k+1)n+a(n+1)} \\ & \quad \times \frac{y^n(-1/y;q)_n(-yxq^{n+1};q)_\infty}{(q^d;q^d)_n(x^dq^{nd};q^d)_\infty} \\ & \quad \times q^{sn}\left[q^{-dn}\frac{1-x^s}{1-x^d} + \frac{x^s-x^d}{1-x^d}\right] \end{aligned}$$

$H^d_{k,a,d}(y;x;q) = H^0_{k,a,d}(y;x;q)$ is (1.4).

Theorem 2.1. *Suppose m, n, r are non-negative integers, and k, a, d, and s are integers such that*

$$k \geq 2, \quad 1 \leq a \leq k, \quad 1 \leq d \leq k, \quad 0 \leq s \leq d-1.$$

Let ${}_d\overline{b}^{s}_{k,a}(m,n,r)$ *be the number of overpartitions of* n *into* m *parts,* r *of which are overlined, such that*

$$
\begin{aligned}
&f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 < a,\\
&f_{\overline{1}} = 0, \text{ i.e. 1 cannot be overlined,}\\
&\text{if } f_i + f_{\overline{i+1}} + f_{i+1} = k-\delta \text{ for } \delta = 1,2\ldots,d-1,\\
&\text{then } a+s-1-f_{odd(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0,1,\ldots,\delta-1 \pmod{d}.
\end{aligned}
$$

Here, $odd(i) = i$ *if* i *is odd,* $odd(i) = i+1$ *if* i *is even; and* $\chi_e(\overline{i+1})$ *is 1 if* $(i+1)$ *is even and* $f_{\overline{i+1}} = 1$*,* $\chi_e(\overline{i+1})$ *is 0 otherwise. Then,*

$$H^s_{k,a,d}(y;xq;q) = \sum_{m,n,r\geq 0} {}_d\overline{b}^{s}_{k,a}(m,n,r)x^m q^n y^r$$

for $d=1$ *or* $d=2$*.*

This theorem is a one-parameter extension of Corollary 12 in [13] up to a substitution. The formal verification proofs are more or less the same, but we include the proof here for the sake of completeness.

Proof. Let

$${}_d\overline{\mathcal{B}}^{s}_{k,a}(y;x;q) = \sum_{m,n,r\geq 0} {}_d\overline{b}^{s}_{k,a}(m,n,r)x^m q^n y^r.$$

First, we argue that

$$
\begin{aligned}
&{}_d\overline{\mathcal{B}}^{s}_{k,a}(y;x;q) - {}_d\overline{\mathcal{B}}^{s+1}_{k,a-1}(y;x;q)\\
&= x^{a-1}q^{a-1}\,{}_d\overline{\mathcal{B}}^{0}_{k,k-s-a+1}(y;xq;q) + yx^a q^{a+1}\,{}_d\overline{\mathcal{B}}^{0}_{k,k-s-a}(y;xq;q), \qquad (2.1)\\
&{}_d\overline{\mathcal{B}}^{s}_{k,0}(y;x;q) = 0, \qquad (2.2)\\
&{}_d\overline{\mathcal{B}}^{s}_{k,a}(y;0;q) = 1. \qquad (2.3)
\end{aligned}
$$

Notice that the functional equation (2.1) and the initial conditions (2.2) and (2.3) are equivalent to the following recurrence and initial values.

$$
\begin{aligned}
&{}_d\overline{b}^{s}_{k,a}(m,n,r) = {}_d\overline{b}^{s+1}_{k,a-1}(m,n,r)\\
&+ {}_d\overline{b}^{0}_{k,k-s-a+1}(m-a+1,n-a+1,r)\\
&+ {}_d\overline{b}^{0}_{k,k-s-a}(m-a,n-a-1,r-1), \qquad (2.4)\\
&{}_d\overline{b}^{s}_{k,0}(m,n,r) = 0, \qquad (2.5)\\
&{}_d\overline{b}^{s}_{k,a}(0,n,r) = \begin{cases} 1 & \text{if } n=r=0,\\ 0 & \text{otherwise.}\end{cases} \qquad (2.6)
\end{aligned}
$$

It is easy to see that (2.4) - (2.6) uniquely determine ${}_d\overline{b}^{s}_{k,a}(m,n,r)$ because each application of (2.4) decreases one or more parameters. (2.5) and (2.6) are a complete collection of initial conditions. Therefore (2.1), (2.2), and (2.3), uniquely determine ${}_d\overline{\mathcal{B}}^{s}_{k,a}(y;x;q)$.

It appears at first that we also need

$$ {}_d\overline{b}^{s}_{k,a}(m,n,r) = 0 \text{ if } m, n, \text{ or } r < 0. $$

We will momentarily show that the indices cannot go negative. So, the last condition is not essential although it is clearly true.

The initial condition (2.5) is for the fact that there are no overpartitions with $f_1 < 0$. No part can appear a negative number of times. The initial condition (2.6) captures the empty overpartition of zero, which is the only overpartition with no parts.

For the recurrence (2.4) we consider the following collections of overpartitions.

$\mathcal{T}$ = the collection of overpartitions enumerated by ${}_d\overline{b}^{s}_{k,a}(m,n,r)$, i.e. the overpartition of n into m parts, r of which are overlined, such that $f_1 < a$, $f_{\overline{1}} = 0$, $f_i + f_{\overline{i+1}} + f_{i+1} < k$,
and if $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for $\delta = 1, 2, \ldots, d-1$,
then $a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$.

$\mathcal{U}$ = the overpartitions in $\mathcal{T}$ in which $f_1 < a - 1$,

$\mathcal{V}$ = the overpartitions in $\mathcal{T}$ in which $f_1 = a - 1$ and $f_{\overline{2}} = 0$,

$\mathcal{W}$ = the overpartitions in $\mathcal{T}$ in which $f_1 = a - 1$ and $f_{\overline{2}} = 1$.

It is immediate that $\mathcal{T}$ is the disjoint union of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$. Because the conditions $f_1 < a - 1$; $f_1 = a - 1$ and $f_{\overline{2}} = 0$; and $f_1 = a - 1$ and $f_{\overline{2}} = 1$ are mutually exclusive and complementary.

$\mathcal{U}$ is enumerated by ${}_d\overline{b}^{s+1}_{k,a-1}(m,n,r)$, because $f_1 < a - 1$ and if $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for $\delta = 1, 2, \ldots, d-1$, then $(a-1) + (s+1) - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$, which is equivalent to $a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$.

Next, we will show that the overpartitions in $\mathcal{V}$ are in one-to-one correspondence with the overpartitions counted by ${}_d\overline{b}^{0}_{k,k-s-a+1}(m-a+1, n-a+1, r)$. In the course, we will argue that the indices cannot go negative.

If we delete the $(a-1)$ 1's and subtract 1 from the remaining parts, non-overlined and overlined alike, then the parts change parity. The number of parts decreases by $(a-1)$, and becomes $(m-a+1)$. At the beginning, necessarily, $m \geq a-1$ and $n \geq a-1$, so that $m-a+1 \geq 0$ and $n-a+1 \geq 0$ after the transformation.

We know that $f_1 = a-1$ and $f_{\overline{2}} = 0$, so $a - 1 + f_2 < k$ and $\rho(2) = 0$. Moreover, if $a-1+f_2 = k-\delta$ for $\delta = 1, 2, \ldots, d-1$, then $a+s-1-(a-1)+0 \equiv 0, 1, \ldots, \delta-1 \pmod{d}$, or simply $s \equiv 0, 1, \ldots, \delta-1 \pmod{d}$. Taking $0 \leq s \leq d$ and $1 \leq \delta \leq d$ into consideration, the last congruence asserts $s < \delta$. This in turn implies $k-s > k-\delta$, that is $a-1+f_2 = k-\delta < k-s$, so $f_2 < k-s-a+1$. Because $f_2 \geq 0$, $k-s-a+1 > 0$.

Therefore, after the subtraction $f_{\overline{1}} = 0$ and $f_1 < k - s - a + 1$.

Suppose λ is a specific overpartition in $\mathcal{V}$. Call the resulting overpartition $\tilde{\lambda}$ after the removal of $(a-1)$ 1's and subtraction of 1's from the other parts. For arbitrary but fixed $i \geq 1$, when $\rho(i) = A$ in λ, then $\tilde{\rho}(i-1) = -A$ in $\tilde{\lambda}$, since no overlined part is deleted and all parts changed parity. Here, $\tilde{\rho}$ denotes the $\rho-$statistic in $\tilde{\lambda}$. Only one condition remains to verify $\tilde{\lambda}$ is counted by ${}_d\overline{b}^0_{k,k-s-a+1}(m-a+1, n-a+1, r)$. Namely, for $i \geq 2$, if $\tilde{f}_{i-1} + \tilde{f}_{\overline{i}} + \tilde{f}_i = k - \delta$ for $\delta = 1, 2, \ldots, d-1$, then

$$(k-s-a+1)+0-1-\tilde{f}_{\mathrm{odd}(i-1)} - \chi_e(\overline{i}) + \tilde{\rho}(i) \overset{?}{\equiv} 0, 1, \ldots, \delta - 1 \pmod{d}, \quad (2.7)$$

where $\tilde{f}$ denotes the frequencies in $\tilde{\lambda}$.

We know that $\tilde{f}_i = f_{i+1}$ and $\chi_e(\overline{i}) + \chi_e(\overline{i+1}) = \tilde{f}_{\overline{i}} = f_{\overline{i+1}}$, so that $\tilde{f}_{\mathrm{odd}(i-1)} + \chi_e(\overline{i}) + \chi_e(\overline{i+1}) + f_{\mathrm{odd}(i)} = k - \delta$. We saw that $\tilde{\rho}(i) = -\rho(i+1)$ as well. Thus, (2.7) is equivalent to

$$k-s-a+1+0-1-(k-\delta-f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1})) - \rho(i+1) \overset{?}{\equiv} 0, 1, \ldots, \delta - 1 \pmod{d},$$

or, after some rearrangement, to

$$-a - s + 1 + f_{\mathrm{odd}(i)} + \chi_e(\overline{i+1}) - \rho(i+1) \overset{?}{\equiv} 0, -1, \ldots, -\delta + 1 \pmod{d},$$

that is, after negating both sides,

$$a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \overset{?}{\equiv} 0, 1, \ldots, \delta - 1 \pmod{d}.$$

The last condition is satisfied by λ. Therefore (2.7) is satisfied by $\tilde{\lambda}$. It follows that the number of overpartitions in $\mathcal{V}$ is equal to ${}_d\overline{b}^0_{k,k-s-a+1}(m-a+1, n-a+1, r)$.

The correspondence between overpartitions in $\mathcal{W}$ and overpartitions counted by ${}_d\overline{b}^0_{k,k-s-a}(m-a, n-a-1, r-1)$ is constructed likewise. The difference is that there is a $\overline{2}$ in overpartitions in $\mathcal{W}$, so we delete it alongside the $(a-1)$ 1's. A particular overpartition λ in $\mathcal{W}$ after the deletions and subtraction of 1 from the remaining parts becomes $\tilde{\lambda}$. $\tilde{\lambda}$ has $m-a$ parts, $r-1$ of which are overlined, and it yields an overpartition of $n-a-1$. If $\rho(i+1) = A$ in λ, then $\tilde{\rho}(i) = -(A-1)$ because of the deleted $\overline{2}$.

The above arguments establish (2.4), and consequently (2.1), (2.2), and (2.3).

Next, we investigate when $H^s_{k,a,d}(y; xq; q)$ satisfies (2.1), (2.2), and (2.3). For convenience, set

$$\begin{aligned}&\overline{c}_n(y; xq; q)\\ &= \frac{(-1)^n x^{n(k+1-d)} q^{(2k+1-d)n(n+1)/2+n} y^n (-1/y; q)_n (-yxq^{n+2}; q)_\infty}{(q^d; q^d)_n (x^d q^{(n+1)d}; q^d)_\infty},\end{aligned}$$

so that

$$
\begin{aligned}
H^s_{k,a,d}(y;xq;q) &= \frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty} \\
&\times \sum_{n\geq 0} \overline{c}_n(y;xq;q) q^{-an} q^{-sn} \left[q^{dn} \frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d} + \frac{1-(xq)^{d-s}}{1-(xq)^d} \right] \\
&- \overline{c}_n(y;xq;q) x^a q^{a(n+1)} q^{sn} \left[q^{-dn} \frac{1-(xq)^s}{1-(xq)^d} + \frac{(xq)^s-(xq)^d}{1-(xq)^d} \right].
\end{aligned}
$$

Observe that $\overline{c}_n(y;xq;q)$ depends on k and d, but not on a or s. The series in (1.4) are

$$
\begin{aligned}
&H^d_{k,a,d}(y;xq;q) = H^0_{k,a,d}(y;xq;q) \\
=&\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty} \sum_{n\geq 0} \overline{c}_n(y;xq;q) q^{-an} - \overline{c}_n(y;xq;q) x^a q^{a(n+1)}.
\end{aligned}
$$

It is clear that

$$
\overline{c}_n(y;0;q) = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases}
$$

So,

$$
H^s_{k,a,d}(y;0;q) = \left[\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d} + \frac{1-(xq)^{d-s}}{1-(xq)^d} \right]_{x=0} = 1. \tag{2.8}
$$

Then we examine

$$
\begin{aligned}
H^s_{k,0,d}(y;xq;q) &= \frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty} \sum_{n\geq 0} \frac{\overline{c}_n(y;xq;q)}{1-(xq)^d} \\
&\times \left[q^{(d-s)n} \left((xq)^{d-s}-(xq)^d\right) + q^{-sn}\left(1-(xq)^{d-s}\right) \right. \\
&\left. -q^{(s-d)n}\left(1-(xq)^s\right) - q^{sn}\left((xq)^s-(xq)^d\right) \right].
\end{aligned}
$$

The expression inside brackets in the last two lines vanishes for $s = 0$ or $2s = d$, i.e. $2s \equiv 0 \pmod{d}$. Empirical evidence suggests that $H^s_{k,0,d}(y;xq;q)$ is nonzero in all other cases, but we do not have a proof of this. So we have to be content with saying

$$
H^s_{k,0,d}(y;xq;q) = 0 \quad \text{if} \quad 2s \equiv 0 \pmod{d}. \tag{2.9}
$$

It is worth noting that there are no missing cases for $d = 2$.

We finally argue that

$$
H^s_{k,a,d}(y;xq;q) - H^{s+1}_{k,a-1,d}(y;xq;q) = (xq)^{a-1} H^0_{k,k-s-a+1,d}(y;xq^2;q)
$$

$$+\,yx^aq^{a+1}H^0_{k,k-s-a,d}(y;xq^2;q). \quad (2.10)$$

(2.10) is implied by the following relations.

$$\begin{aligned}
&\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}\overline{c}_n(y;xq;q)\left[q^{-an}q^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\right.\\
&\left.-q^{-(a-1)n}q^{-(s+1)n}\left[q^{dn}\frac{(xq)^{d-(s+1)}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-(s+1)}}{1-(xq)^d}\right]\right]\\
&=\frac{\left((xq^2)^d;q^d\right)_\infty}{(xq^2;q)_\infty}\overline{c}_{n-1}(y;xq^2;q)\\
&\times\left(-(xq)^{a-1}(xq^{n+1})^{k-s-a+1}+yx^aq^{a+1}(xq^{n+1})^{k-s-a}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\left((xq)^d;q^d\right)_\infty}{(xq;q)_\infty}\overline{c}_n(y;xq;q)\\
&\times\left[-(xq^{n+1})^aq^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\right.\\
&\left.+(xq^{n+1})^{a-1}q^{-(s+1)n}\left[q^{dn}\frac{(xq)^{d-(s+1)}-(xq)^d}{1-(xq)^d}+\frac{1-(xq)^{d-(s+1)}}{1-(xq)^d}\right]\right]\\
&=\frac{\left((xq^2)^d;q^d\right)_\infty}{(xq^2;q)_\infty}\overline{c}_n(y;xq^2;q)\\
&\times\left((xq)^{a-1}(q^{-n})^{k-s-a+1}-yx^aq^{a+1}(q^{-n})^{k-s-a}\right).
\end{aligned}$$

These are straightforward verifications.

The left-hand side of (2.10) suggests that the quantity $(a+s)$ is an invariant. On the other hand, (2.9) imposes

$$2(a+s)\equiv 0 \pmod{d}.$$

Applying this to the right-hand side of (2.10), we have

$$2(k-s-a+1)\equiv 0 \pmod{d}, \qquad \text{and} \qquad 2(k-s-a)\equiv 0 \pmod{d},$$

which forces $2\equiv 0 \pmod{d}$. In other words, $d=1$ or $d=2$.

Since (2.1), (2.2), and (2.3) uniquely determine ${}_d\overline{\mathcal{B}}^s_{k,a}(y;x;q)$, we conclude that

$$H^s_{k,a,d}(y;xq;q)={}_d\overline{\mathcal{B}}^s_{k,a}(y;x;q) \quad \text{for } d=1 \text{ or } d=2,$$

by (2.8), (2.9), (2.10), and the above congruences. That is,

$$H^s_{k,a,d}(y;xq;q)=\sum_{m,n,r\geq 0}{}_d\overline{b}^s_{k,a}(m,n,r)x^mq^ny^r$$

for $d = 1$ or $d = 2$.

Corollary 2.1. *Let ${}_d\overline{\eta}^s_{k,a}(m,n,r)$ be the number of overpartitions of n into m parts, r of which are overlined, such that*

$$\begin{aligned} & f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 + f_{\overline{1}} < a, \\ & \textit{if } f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta \textit{ for } \delta = 1, 2 \ldots, d-1, \\ & \textit{then } a + s - 1 - f_{odd(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}. \end{aligned}$$

Then,

$$\begin{aligned} J^s_{k,a,d}(y;x;q) &:= H^s_{k,a,d}(y;xq;q) + xyqH^s_{k,a-1,d}(y;xq;q) \\ &= \sum_{m,n,r \geq 0} {}_d\overline{\eta}^s_{k,a}(m,n,r) x^m q^n y^r \end{aligned}$$

for $d = 1$ or $d = 2$.

Proof. Let λ be an overpartition enumerated by ${}_d\overline{\eta}^s_{k,a}(m,n,r)$. If λ has no $\overline{1}$, then it is also counted by ${}_d\overline{b}^s_{k,a}(m,n,r)$.

If λ has an $\overline{1}$, then erase it to obtain $\tilde{\lambda}$. $\tilde{\lambda}$ is an overpartition of $n-1$ into $m-1$ parts, $r-1$ of which are overlined, because of the deleted $\overline{1}$. $\tilde{\rho}(i) = \rho(i) + 1$ for the same reason, where $\tilde{\rho}$ is the ρ-statistic in $\tilde{\lambda}$. Also, $f_1 < a - 1$ in $\tilde{\lambda}$, because $f_1 + f_{\overline{1}} < a$ and $f_{\overline{1}} = 1$ in λ.

Now, λ satisfies

$$a + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$$

when $f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta$ for some $\delta = 1, 2, \ldots, d$ and $i \in \mathbb{Z}^+$. So,

$$(a-1) + s - 1 - f_{\mathrm{odd}(i)} - \chi_e(\overline{i+1}) + \tilde{\rho}(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}$$

for $\tilde{\lambda}$. Therefore, $\tilde{\lambda}$ is enumerated by ${}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$. Conversely, we can append an $\overline{1}$ to any overpartition counted by ${}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$, and obtain one counted by ${}_d\overline{\eta}^s_{k,a}(m,n,r)$.

Having or lacking $\overline{1}$ are mutually exclusive and complementary cases for λ. We have shown that

$${}_d\overline{\eta}^s_{k,a}(m,n,r) = {}_d\overline{b}^s_{k,a}(m,n,r) + {}_d\overline{b}^s_{k,a-1}(m-1, n-1, r-1)$$

which implies the corollary.

In [14], Lovejoy states that $J^0_{k,a,1}(1;1;q)$ is not an infinite product, but a combination of two infinite products. One still has a partition identity in this case, because we can interpret both infinite products as partition enumerants, and obtain an identity in the form of

$$A(n) - A(n-*) = B(n) - B(n-*) + C(n) - C(n-*),$$

which admittedly is not as elegant as the classical partition identities. Above, $*$ stands for various fixed positive integers, not necessarily the same in each occurence.

Lovejoy, however, observes also that $J^0_{k,a,1}(1/q;1;q)$ is a combination of two infinite products. In this case, as in the proof of the above Corollary, one subtracts another 1 from all overlined parts, and a possibility of an overlined zero arises. Then, an overpartition identity cannot be obtained since some partitions will be counted twice. One needs further work to eliminate the occurrence of overlined zero.

It is interesting to substitute $y = 0$ in the series $H^s_{k,a,d}(y;xq;q)$ and see what one obtains:

$$\begin{aligned}
&J^s_{k,a,d}(0;x;q) = H^s_{k,a,d}(0;xq;q) \\
=&\frac{((xq)^d;q^d)_\infty}{(xq;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n x^{n(k+1-d)} q^{(2k+2-d)n(n+1)/2-an}}{(q^d;q^d)_n (x^d q^{nd};q^d)_\infty} \\
&\times q^{-sn}\left[q^{dn}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d} + \frac{1-(xq)^{d-s}}{1-(xq)^d}\right] \\
-&\frac{(-1)^n x^{n(k+1-d)+a} q^{(2k+2-d)n(n+1)/2+a(n+1)}}{(q^d;q^d)_n (x^d q^{nd};q^d)_\infty} \\
&\times q^{sn}\left[q^{-dn}\frac{1-(xq)^s}{1-(xq)^d} + \frac{(xq)^s-(xq)^d}{1-(xq)^d}\right].
\end{aligned}$$

This the exact same series as [13, Lemma 11]. There, the series was constructed from scratch.

3 Constructions

In this section, we will show that all results stated above can be proven linearly and constructively. In other words, there is no need for formal verifications that a proposed series is indeed the generating function sought for. Given the description of partition classes, we will *construct* their generating function as a series. We will carry out computations for one example in detail.

Let's recall the partition enumerant in Corollary 2.1. Let ${}_d\overline{\eta}^s_{k,a}(m,n,r)$ be the number of overpartitions of n into m parts, r of which are overlined, such that

$$\begin{aligned}
&f_i + f_{\overline{i+1}} + f_{i+1} < k, \quad f_1 + f_{\overline{1}} < a, \\
&\text{if } f_i + f_{\overline{i+1}} + f_{i+1} = k - \delta \text{ for } \delta = 1, 2 \ldots, d-1, \\
&\text{then } a + s - 1 - f_{\text{odd}(i)} - \chi_e(\overline{i+1}) + \rho(i+1) \equiv 0, 1, \ldots, \delta - 1 \pmod{d}.
\end{aligned}$$

It is possible (as in the proof of Theorem 2.1) to justify the following recurrences and initial conditions.

$$\begin{aligned} {}_d\overline{\eta}^s_{k,a}(m,n,r) = {}_d\overline{\eta}^{s+1}_{k,a-1}(m,n,r) \\ + {}_d\overline{\eta}^0_{k,k-a-s+1}(m-a+1, n-a+1, r) \\ + {}_d\overline{\eta}^0_{k,k-a-s+2}(m-a+1, n-a+1, r-1), \end{aligned} \tag{3.1}$$

$${}_d\overline{\eta}^s_{k,a}(0,n,r) = \begin{cases} 1 & \text{if } n = r = 0, \\ 0 & \text{otherwise ,} \end{cases} \tag{3.2}$$

$${}_d\overline{\eta}^s_{k,1}(m,n,r) = {}_d\overline{\eta}^0_{k,k-s}(m,n,r). \tag{3.3}$$

It is fairly clear that equations (3.1) for $a = 2, 3, \ldots, k$, (3.2), and (3.3) uniquely determine ${}_d\overline{\eta}^s_{k,a}(m,n,r)$. The reason we did not use ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$ is that the equation (3.1) already needs a reinterpretation for $a = 1$, and the reinterpretation (3.3) implies that ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$. Another reason for not explicitly stating ${}_d\overline{\eta}^s_{k,0}(m,n,r) = 0$ is that the computations will not yield it explicitly. Still we will be able to construct the series.

Set

$$Q^s_a(x) := Q^s_{k,a,d}(y; x, q) = \sum_{m,n,r \geq 0} {}_d\overline{\eta}^s_{k,a}(m,n,r) x^m y^r q^n.$$

We suppress writing d, k, y, and q because they are unchanged throughout the computations.

The conditions (3.1)-(3.3) are translated as the following.

$$Q^s_a(x) - Q^{s+1}_{a-1} = (xq)^{a-1} Q^0_{k-a-s+1}(xq) + y(xq)^{a-1} Q^0_{k-a-s+2}(xq) \tag{3.4}$$

for $a = 2, 3, \ldots, k$,

$$Q^s_a(0) = 1, \tag{3.5}$$

$$Q^s_1(x) = Q^0_{k-s}(xq), \tag{3.6}$$

and these functional equations and the initial conditions uniquely determine $Q^s_a(x)$.

The next step is taking Andrews's analytic proof of Gordon's theorem [1] as a black box, and assuming that $Q^s_a(x)$ is of the form

$$Q^s_a(x) = \sum_{n \geq 0} \alpha^s_n(x) q^{-na} + \beta^s_n(x)(xq^{n+1})^a. \tag{3.7}$$

$\alpha^s_n(x)$ and $\beta^s_n(x)$ depend on d, k, y and q; but not on a. Again, we imitate the mechanism in [1] to assert that

$$\alpha^s_n(x) q^{-na} - \alpha^{s+1}_n(x) q^{-na+n}$$

$$= (xq)^{a-1}\beta_{n-1}^0(xq)(xq^{n+1})^{k-a-s+1}$$
$$+ y(xq)^{a-1}\beta_{n-1}^0(xq)(xq^{n+1})^{k-a-s+2}, \tag{3.8}$$

and

$$\beta_n^s(x)(xq^{n+1})^a - \beta_n^{s+1}(x)(xq^{n-1})^{a-1}$$
$$= (xq)^{a-1}\alpha_n^0(xq)(q^{-n})^{k-a-s+1} + y(xq)^{a-1}\alpha_n^0(xq)(q^{-n})^{k-a-s+2}. \tag{3.9}$$

It is useful to keep in mind that although $s = 0, 1, \ldots, d-1$, it is interpreted as a residue class $\pmod d$. So, $\alpha_n^d(x) = \alpha_n^0(x)$, and $\beta_n^d(x) = \beta_n^0(x)$.

The recurrences (3.8) and (3.9) imply (3.4). The idea is to discover α's and β's first, then imposing the initial conditions (3.5) and (3.6).

The reader can check that if one tries to make α's and β's independent of s as well, one either encounters inconsistent equations, or has to adjust the format of (3.7). In the latter case, the adjustment is more difficult to come up with, and the resulting equations are much harder to solve. Empirical evidence shows that this is a convenient way to proceed.

The equations (3.8) and (3.9) can be simplified as

$$\alpha_n^s(x) - q^n\alpha_n^{s+1}(x) = (xq^{n+1})^k q^n (xq^{n+1})^{-s}(1 + yxq^{n+1})\beta_{n-1}^0(xq),$$

and

$$xq^{n+1}\beta_n^s(x) - \beta_n^{s+1}(x) = (q^{-n})kq^{ns}yq^{-n}(1 + q^n/y)\alpha_n^0(xq).$$

We can collect equations for various s's and write them in matrix form.

$$\begin{bmatrix} 1 & -q^n & & \\ & 1 & -q^n & \\ & & \ddots & \\ -q^n & & & 1 \end{bmatrix} \begin{bmatrix} \alpha_n^0(x) \\ \alpha_n^1(x) \\ \vdots \\ \alpha_n^{d-1}(x) \end{bmatrix}$$
$$= (xq^{n+1})^k q^n (1 + yxq^{n+1})\beta_{n-1}^0(xq) \begin{bmatrix} 1 \\ (xq^{n+1})^{-1} \\ \vdots \\ (xq^{n+1})^{-d+1} \end{bmatrix},$$

and

$$\begin{bmatrix} xq^{n+1} & -1 & & \\ & xq^{n+1} & -1 & \\ & & \ddots & \\ -1 & & & xq^{n+1} \end{bmatrix} \begin{bmatrix} \beta_n^0(x) \\ \beta_n^1(x) \\ \vdots \\ \beta_n^{d-1}(x) \end{bmatrix}$$

$$= (q^{-n})^k q^{-n} y (1 + q^n/y) \alpha_n^0(xq) \begin{bmatrix} 1 \\ q^n \\ \vdots \\ (q^n)^{d-1} \end{bmatrix}.$$

The displayed matrices have the following respective inverses.

$$\frac{1}{(1-q^{dn})} \begin{bmatrix} 1 & q^n & q^{2n} & \cdots & q^{dn-n} \\ q^{dn-n} & 1 & q^n & \cdots & q^{dn-2n} \\ & & \vdots & & \\ q^n & q^{2n} & q^{3n} & \cdots & 1 \end{bmatrix},$$

and

$$\frac{(-1)}{(1-(xq^{n+1})^d)} \begin{bmatrix} (xq^{n+1})^{d-1} & (xq^{n+1})^{d-2} & \cdots & 1 \\ 1 & (xq^{n+1})^{d-1} & \cdots & (xq^{n+1}) \\ & \vdots & & \\ (xq^{n+1})^{d-2} & (xq^{n+1})^{d-3} & \cdots & (xq^{n+1})^{d-1} \end{bmatrix}.$$

Multiplying by the corresponding inverse matrix on both sides, and performing the matrix-vector multiplication, we obtain

$$\alpha_n^s(x) = \frac{(xq^{n+1})^k q^n (xq)^{1-d} (1 + yxq^{n+1})(1-(xq)^d)}{(1-q^{dn})(1-xq)} \beta_{n-1}^0(xq) \\ \times \left[q^{(d-s)n} \frac{(xq)^{d-s} - (xq)^d}{1-(xq)^d} + q^{-sn} \frac{1-(xq)^{d-s}}{1-(xq)^d} \right],$$

and

$$\beta_n^s(x) = \frac{(-1)(q^{-n})^k q^{-n} (q^{-n})^{1-d} y (1 + q^n/y)(1-(xq)^d)}{(1-(xq^{n+1})^d)(1-xq)} \alpha_n^0(xq) \\ \times \left[q^{(s-d)n} \frac{1-(xq)^s}{1-(xq)^d} + q^{sn} \frac{(xq)^s - (xq)^d}{1-(xq)^d} \right].$$

Please notice that the only part involving s in both recurrences is inside the brackets on the right hand sides, and both brackets evaluate to 1 for $s = 0$ or $s = d$.

Unfolding the last two equations, we first find

$$\alpha_n^0(x) = (-1)(xq^2)^{k+1-d} q^{n(d-1)+1} y \\ \times \frac{(1+q^{n-1}/y)(1+yxq^{n+1})(1-(xq)^d)(1-(xq^2)^d)}{(1-q^{dn})(1-(xq^{n+1})^d)(1-xq)(1-xq^2)} \alpha_{n-1}^0(xq^2),$$

and then

$$\alpha_n^0(x) = (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+n} y^n$$

$$\times \frac{(-1/y;q)_n(-yxq^{n+1};q)_n((xq)^d;q^d)_{2n}}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_n(xq;q)_{2n}}\alpha_0^0(xq^{2n}).$$

Defining

$$\widetilde{\alpha}_0^0(x) = \frac{((xq)^d;q^d)_\infty(xq;q)_\infty}{(-yxq;q)_\infty((xq)^d;q^d)_\infty}\alpha_0^0(x),$$

we finally get

$$\alpha_n^0(x) = (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+n} y^n \times \frac{(-1/y;q)_n(-yxq^{n+1};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty}\widetilde{\alpha}_0^0(xq^{2n}).$$

Then, in the order given below, we find

$$\beta_n^0(x) = -(-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^{n+1} \times \frac{(-1/y;q)_{n+1}(-yxq^{n+2};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty}\widetilde{\alpha}_0^0(xq^{2n+1}),$$

$$\alpha_n^s(x) = (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+n} y^n \times \frac{(-1/y;q)_n(-yxq^{n+1};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty} \times \left[q^{(d-s)n}\frac{(xq)^{d-s}-(xq)^d}{1-(xq)^d} + q^{-sn}\frac{1-(xq)^{d-s}}{1-(xq)^d}\right]\widetilde{\alpha}_0^0(xq^{2n}),$$

and

$$\beta_n^s(x) = -(-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^{n+1} \times \frac{(-1/y;q)_{n+1}(-yxq^{n+2};q)_\infty((xq)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty} \times \left[q^{(s-d)n}\frac{1-(xq)^s}{1-(xq)^d} + q^{sn}\frac{(xq)^s-(xq)^d}{1-(xq)^d}\right]\widetilde{\alpha}_0^0(xq^{2n+1}).$$

The first initial condition (3.5) is easily seen to hold as long as $\widetilde{\alpha}_0^0(0) = 1$. The subsequent computations will show that there is no harm in taking $\widetilde{\alpha}_0^0(x) = 1$.

There are two options for the other initial condition. The more obvious

$$Q_0^s(x) = 0$$

does not seem to hold, unfortunately. Therefore, one has to resort to (3.6), which is

$$Q_1^s(x) = Q_{k-s}^0(xq).$$

$Q_1^s(x)$ will be used as is. $Q_{k-s}^0(xq)$ needs a small transformation.

$$\begin{aligned}
Q_{k-s}^0(xq) &= \sum_{n\geq 0} (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(s+2-d)n} y^n \\
&\quad\times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
&\quad - (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(k+1-d)n} y^{n+1} \\
&\quad\times \frac{(-1/y;q)_{n+1}(-yxq^{n+3};q)_\infty((xq^2)^d;q^d)_\infty(xq^{n+2})^{k-s}}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
&= \sum_{n\geq 0} (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}+(s+2-d)n} y^n \\
&\quad\times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+2})^d;q^d)_\infty(xq^2;q)_\infty} \\
&\quad + (-1)^n x^{(k+1-d)n+d-1-s} q^{(2k+1-d)\binom{n+1}{2}-sn+d-s-1} y^n \\
&\quad\times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_{n-1}((xq^{n+1})^d;q^d)_\infty(xq^2;q)_\infty}
\end{aligned}$$

In particular, we shifted the index $n \leftarrow (n-1)$ in the second term. The introduction of the $n=-1$ term in the sum is no problem since $1/(q^d;q^d)_{-1} = 0$. Noticing the common factor

$$\begin{aligned}
\mathcal{C}_n :=& (-1)^n x^{(k+1-d)n} q^{(2k+1-d)\binom{n+1}{2}} y^n \\
&\times \frac{(-1/y;q)_n(-yxq^{n+2};q)_\infty((xq^2)^d;q^d)_\infty}{(q^d;q^d)_n((xq^{n+1})^d;q^d)_\infty(xq;q)_\infty},
\end{aligned}$$

the series at hand become

$$\begin{aligned}
Q_1^s(x) =& \sum_{n\geq 0} \mathcal{C}_n \left\{(1+yxq^{n+1})\right. \\
&\times \left[q^{(d-s)n}\left((xq)^{d-s}-(xq)^d\right) + q^{-sn}\left(1-(xq)^d\right)\right] \\
&- xq^{n+1}y(1+q^n/y) \\
&\times \left.\left[q^{(s-d)n}\left(1-(xq)^s\right) + q^{sn}\left((xq)^s-(xq)^d\right)\right]\right\}, \\
Q_{k-s}^0(xq) =& \sum_{n\geq 0} \mathcal{C}_n \left\{q^{(s+2-d)n}\left(1-(xq^{n+1})^d\right)(1-xq)\right. \\
&\left. + x^{d-1-s}q^{-s(n+1)+d-1}(1-q^{dn})(1-xq)\right\}.
\end{aligned}$$

Now one can use a computer algebra system to examine the difference of the expressions in curly braces. Their difference is zero for $d = 1, 2$ and all corresponding s. It is empirically nonzero for $d \geq 3$ and various s. This ends the construction along with the proof of Corollary 2.1.

It should be possible to examine the aforementioned differences, and prove that the condition $d = 1, 2$ in the results is not only sufficient, but also necessary.

4 The case of $d \geq 3$

In this section we report on an exploration that shows that a naïve approach to finding interpretations of the series (1.5) yields complicated (or fascinating, as per ones taste) results. At the end, for concreteness, we work with $d = 3, k = 5$.

The main idea of this exploration is to "reverse engineer" the process of deducing recurrences satisfied by generating functions of a certain class of partitions. Such a process is an important step in the motivated proof of Rogers-Ramanujan identities as given by Andrews and Baxter [4].

4.1 Motivating example

As an example of what we mean, let us explain this reverse engineering process applied to the familiar Rogers-Ramanujan identities. Suppose that one is presented with a formal series

$$F(x, q) = \sum_{m,n \geq 0} f_{m,n} x^m q^n$$

with integral coefficients with the following conditions:

$$f_{0,0} = 1 \tag{4.1}$$

$$f_{m,n} = 0 \quad \text{if } m > n \tag{4.2}$$

$$F(x, q) = F(xq, q) + xqF(xq^2, q). \tag{4.3}$$

These conditions tell us that the coefficients of F are non-negative. Now, the first two conditions hint at the fact that perhaps $f_{m,n}$ counts certain partitions of n with m parts. With this ansatz, we can now make additional guesses. The transformation $x \mapsto xq^j$ corresponds to adding j to every part of the partition, and multiplication by xq^j corresponds to inserting the part j in the partition. Now we can start "building" the partitions possibly counted by F using the recurrence (4.3).

Let us call the class of partitions of n with exactly m parts counted in F by $\pi_{m,n}$. Let us denote the null partition by $\mathbf{0}$. Thus, $a_{0,0} = 1$ counts this null partition. Note that by definition, we let $\pi_{m,n} = \{\}$ if $(m,n) \notin \{(x,y) \,|\, x \geq 1, y \geq 1, x \leq y\} \cup \{(0,0)\}$. Then, (4.3) written with partitions in mind reads as follows:

To find $\pi_{m,n}$, take the union of the following two sets:

1. For each partition appearing in $\pi_{m,n-m}$, add 1 to every part (corresponds to the term $F(xq,q)$).
2. For each partition appearing in $\pi_{m-1,n-1-2(m-1)}$, add 2 to every part and then adjoin the part 1 to each of the resulting partitions. (corresponds to the term $xqF(xq^2,q)$).

Doing this process, we arrive a "partition generating function" $\Pi(x,q)$ of the sets of partitions $\pi_{n,m}$ as follows:

$$
\begin{aligned}
&\Pi(x,q) \\
&= \{\mathbf{0}\}x^0q^0 \\
&+ \{(1)\}xq + \{(2)\}xq^2 + \{(3)\}xq^3 + \{(4)\}xq^4 + \{(5)\}xq^3 + \{(6)\}xq^4 + \cdots \\
&+ \{(1,3)\}x^2q^4 + \{(1,4)\}x^2q^5 + \{(1,5),(2,4)\}x^2q^6 + \cdots \\
&+ \{(1,3,5)\}x^3q^9 + \{(1,3,6)\}x^3q^{10} + \{(1,3,7),(1,4,6)\}x^3q^{11} + \cdots \\
&+ \cdots .
\end{aligned}
$$

Doing this for sufficiently high powers x^iq^j, one can see a pattern emerging:

$f_{m,n}$ counts the number of partitions of n with exactly m parts in which adjacent parts differ by at least 2.

What we have done is a naïve enrichment of F to a "partition generating function" and a naïve enrichment of (4.3) to a recurrence of "partition generating functions." However, for an arbitrary recurrence, the following problems could arise:

1. The coefficients $f_{m,n}$ may not be (manifestly) non-negative.
2. The sets of partitions that arise from various summands may not be disjoint.

These necessitate that we instead look at "partition generating functions" with integral weights attached to the partitions. We therefore formalize the reverse engineering process given above in an algebraic language as given in the next subsection.

4.2 An algebraic formalism

One can avoid such an algebraic language altogether, however, it facilitates a succinct exposition of our ideas.

Definition 4.1. Let $\mathcal{P}$ denote the set of all partitions and $\mathcal{P}_{m,n}$ denote the set of all partitions of n with exactly m parts. By convention, $\mathcal{P}_{0,0} = \{\mathbf{0}\}$.

Definition 4.2. Let $\mathbf{P}$ denote the free $\mathbb{Z}$-module generated by $\mathcal{P}$. Similarly, let $\mathbf{P}_{m,n}$ denote the free $\mathbb{Z}$-module generated by $\mathcal{P}_{m,n}$.

Definition 4.3. Let $\mathbf{P}[[x,q]]$ denote the space of two variable generating functions with coefficients in $\mathbf{P}$. We say that $f \in \mathbf{P}[[x,q]]$ is a partition generating function if the coefficient of $x^m q^n$ of f lies in $\mathbf{P}_{m,n}$. We denote the space of partition generating functions by $\mathbf{F}$. It is clear that $\mathbf{F}$ is a $\mathbb{Z}$-submodule of $\mathbf{P}[[x,q]]$.

Henceforth, we shall employ the following convention.

Convention 4.1 *We shall write the partitions as tuples of positive integers in a non-decreasing order. For instance, $\pi = (1,1,2,3,4,15) \in \mathcal{P}_{6,26}$. We shall also think of $\mathcal{P}_{m,n}$ as a subset of $\mathbf{P}_{m,n}$. Given $f \in \mathbf{F}$, we will denote the coefficient of $x^m q^n$ by $f_{m,n}$.*

We define the following $\mathbb{Z}$-linear maps:

Definition 4.4. Let $\sigma : \mathbf{P}_{m,n} x^m q^n \to \mathbf{P}_{m,n+m} x^m q^{n+m}$ be the unique map such that $\sigma(\pi x^m q^n) = \tilde{\pi} x^m q^{n+m}$ where $\pi \in \mathcal{P}_{m,n}$ and $\tilde{\pi} \in \mathcal{P}_{m,n+n}$ is obtained by adding 1 to every part of π. Note that the null partition $\mathbf{0}$ does not have any parts, and hence $\sigma(\mathbf{0}) = \mathbf{0}$.

Definition 4.5. Let $\alpha : \mathbf{P}_{m,n} x^m q^n \to \mathbf{P}_{m+1,n+1} x^{m+1} q^{n+1}$ be the unique map such that $\sigma(\pi x^m q^n) = \tilde{\pi} x^{m+1} q^{n+1}$ where $\pi \in \mathcal{P}_{m,n}$ and $\tilde{\pi} \in \mathcal{P}_{m,n+n}$ is obtained by adjoining 1 to π.

Definition 4.6. Let $\chi : \mathbf{P}_{m,n} x^m q^m \to \mathbb{Z} x^m q^n$ be the unique map such that $\chi(\pi x^m q^n) = x^m q^n$ where $\pi \in \mathcal{P}_{m,n}$.

We may and do extend the maps σ, α and χ to the space $\mathbf{F}$ of partition generating functions.

Proposition 4.1. *Let $f \in \mathbf{F}$. Then the following hold.*

$$(\chi(\sigma(f)))(x,q) = (\chi(f))(xq,q)$$
$$(\chi(\alpha(f)))(x,q) = xq \cdot (\chi(f))(x,q).$$

Now we can lift the recurrence (4.3) as a recurrence of partition generating function:

$$\Pi(x,q) = (\sigma\Pi)(x,q) + (\alpha\sigma^2\Pi)(x,q).$$

With the help of computers, it is a trivial matter to generate enough data for such a generating function.

4.3 Recurrence for $J_{k,a,d}$

In this subsection, we derive the recurrences followed by $J_{k,a,d}$. The following statements are easy generalizations of the results in [9]. First, recall from [9], with $[d]_x = (1 + x + \cdots + x^{d-1})$:

$$J_{k,a,d}(y, xq, q) = H_{k,a,d}(y, xq, q) + yxqH_{k,a-1,d}(y, xq, q) \tag{4.4}$$

$$H_{k,0,d}(y, x, q) = 0 \tag{4.5}$$

$$H_{k,-a,d}(y, x, q) = -x^{-a}H_{k,a,d}(y, x, q) \tag{4.6}$$

$$H_{k,a,d}(y, x, q) - H_{k,a-d,d}(y, x, q) = x^{a-d}[d]_x J_{k,k-a+1,d}(y, x, q) \tag{4.7}$$

Invoking (4.7) with $a = t$ and $a = d - t$ and dropping the implicit arguments y, x, q, we have:

$$H_{k,t,d} - H_{k,t-d,d} = x^{t-d}[d]_x J_{k,k-t+1,d}$$

$$H_{k,d-t,d} - H_{k,-t,d} = x^{-t}[d]_x J_{k,k-d+t+1,d}.$$

Using equation (4.6), rearranging:

$$H_{k,t,d} + x^{t-d}H_{k,d-t,d} = x^{t-d}[d]_x J_{k,k-t+1,d}$$

$$x^{-t}H_{k,t,d} + H_{k,d-t,d} = x^{-t}[d]_x J_{k,k-d+t+1,d}.$$

Solving, we get:

$$(1 - x^{-d})H_{k,d-t,d} = [d]_x(x^{-t}J_{k,k+t+1-d,d} - x^{-d}J_{k,k-t+1,d})$$

The equation one gets for $H_{k,t,d}$ is just $t \mapsto d - t$. Simplifying,

$$(x^d - 1)H_{k,d-t,d} = [d]_x(x^{d-t}J_{k,k+t+1-d,d} - J_{k,k-t+1,d})$$

Hence,

$$H_{k,d-t,d} = \frac{x^{d-t}J_{k,k+t+1-d,d} - J_{k,k-t+1,d}}{x - 1}$$

Letting $t \mapsto d - t$:

$$H_{k,t,d} = \frac{x^{t}J_{k,k-t+1,d} - J_{k,k+t-d+1,d}}{x - 1}$$

We can now deduce the following:

1. For $a = 1$, we have that:

$$J_{k,1,d}(y, x, q)$$
$$= H_{k,1,d}(y, xq, q) + yxqH_{k,0,d}(y, xq, q) = H_{k,1,d}(y, xq, q)$$

$$= \frac{1}{xq-1}\left\{xqJ_{k,k,d}(y,xq,q) - J_{k,k+2-d,d}(y,xq,q)\right\} \tag{4.8}$$

We get the correct (2.4) from [9] with $d = 2$.

2. Let $1 < a < d$:

$$\begin{aligned} &J_{k,a,d}(y,x,q)\\ &\quad= H_{k,a,d}(y,xq,q) + yxqH_{k,a-1,d}(y,xq,q)\\ &\quad= \frac{1}{xq-1}\left\{(xq)^a J_{k,k-a+1,d}(y,xq,q) - J_{k,k+a-d+1,d}(y,xq,q)\right.\\ &\qquad\left. + y(xq)^a J_{k,k-a+2,d}(y,xq,q) - yxqJ_{k,k+a-d,d}(y,xq,q)\right\}. \end{aligned} \tag{4.9}$$

3. Letting $a = d$ and then using the expression for H in terms of J:

$$\begin{aligned} &J_{k,d,d}(y,x,q)\\ &= H_{k,d,d}(y,xq,q) + yxqH_{k,d-1,d}(y,xq,q)\\ &= H_{k,d,d}(y,xq,q) - H_{k,0,d}(y,xq,q) + axqH_{k,d-1,d}(y,xq,q)\\ &= [d]_{xq}J_{k,k-d+1,d}(y,xq,q) + yxqH_{k,d-1,d}(d,xq,q)\\ &= [d]_{xq}J_{k,k-d+1,d}(y,xq,q) + yxq\frac{(xq)^{d-1}J_{k,k+2-d,d}(y,xq,q) - J_{k,k,d}(y,xq,q)}{xq-1} \end{aligned} \tag{4.10}$$

Note that when $d = 2$, this specializes to (2.5) of [9].

4. For $d + 1 \leq a \leq k$,

$$\begin{aligned} &J_{k,a,d}(y,x,q) - J_{k,a-d,d}(y,x,q)\\ &= H_{k,a,d}(y,xq,q) + yxqH_{k,a-1,d}(y,xq,q)\\ &\quad - H_{k,a-d,d}(y,xq,q) - yxqH_{k,a-d-1,d}(y,xq,q)\\ &= (xq)^{a-d}[d]_{xq}\left(J_{k,k-a+1,d}(y,xq,q) + yJ_{k,k-a+2,d}(y,xq,q)\right) \end{aligned} \tag{4.11}$$

Note that for $d = 2$, we correctly get (2.6) of [9].

4.4 A concrete exploration

For our explorations, we let $d = 3$, $k = 5$ and $y \mapsto 0$, to begin with. We have the following recurrences. For convenience, we shall abbreviate $J_{5,a,3}(0,x,q)$ by $J_a(x,q)$.

$$(xq-1)J_1(x,q) = xqJ_5(xq,q) - J_4(xq,q) \tag{4.12}$$

$$(xq-1)J_2(x,q) = (xq)^2J_4(xq,q) - J_5(xq,q) \tag{4.13}$$

$$J_3(x,q) = X_d(xq)J_3(xq,q) \tag{4.14}$$
$$J_4(x,q) - J_1(x,q) = xqX_d(xq)J_2(xq,q) \tag{4.15}$$
$$J_5(x,q) - J_2(x,q) = (xq)^2X_d(xq)J_1(xq,q). \tag{4.16}$$

We deduce the following functional equations for the "partition generating functions" (we drop the implicit arguments x, q):

$$F_1 = \alpha F_1 - \alpha\sigma F_5 + \sigma F_4 \tag{4.17}$$
$$F_2 = \alpha F_2 - \alpha^2\sigma F_4 + \sigma F_5 \tag{4.18}$$
$$F_3 = (1 + \alpha^1 + \alpha^2)\sigma F_3 \tag{4.19}$$
$$F_4 = F_1 + (\alpha + \alpha^2 + \alpha^3)\sigma F_2 \tag{4.20}$$
$$F_5 = F_2 + (\alpha^2 + \alpha^3 + \alpha^4)\sigma F_1. \tag{4.21}$$

The partition generating functions hold some nice patterns for small partitions, but one quickly gets complicated coefficients. A computer search reveals the following coefficients in the expansion of F_1:

$$2{\cdot}(1,2,2,2,2,3,3,4)x^8q^{19}$$
$$3{\cdot}(1,2,2,2,2,3,3,4,4,5)x^{10}q^{28}$$
$$2{\cdot}(1,2,2,2,2,3,3,4,4,5,5,6)x^{12}q^{39}$$
$$-2{\cdot}(1,2,2,2,2,3,3,4,4,5,5,6,6,7)x^{14}q^{52}$$
$$-8{\cdot}(1,2,2,2,2,3,3,4,4,5,5,6,6,7,7,8)x^{16}q^{67}$$
$$-12{\cdot}(1,2,2,2,2,3,3,4,4,5,5,6,6,7,7,8,8,9)x^{18}q^{84}.$$

Of course, for each x^mq^n appearing above, there are also other partitions of n with length m besides the ones mentioned above that yield non-zero coefficients.

One may very easily do explorations with y not specialized to 0. In this case, we assume that we are working with two-colored partition where parts may appear overlined, with y counting overlined parts and x counting total number of parts.

Looking at equations (4.8)–(4.11), observe the following: Whenever a new overlined part is introduced, that is, whenever we have a factor of y on the right-hand sides, the newly introduced overlined part is always a $\overline{1}$ (a term of the sort $y(xq)^t$ corresponds to introducing one $\overline{1}$ and $t-1$ non-overlined 1s). Moreover, the term with y on the right-hand sides of equations (4.8)–(4.11) is always multiplied with a shifted, (that is, $x \mapsto xq$) generating function. This implies that we get non-zero coefficients in the corresponding partition generating functions only if the overlined parts do not repeat.

However, further computer search reveals interesting patterns; we have the following terms as a sample:

$$2{\cdot}(1,2,2,3,3,\overline{4})\, ax^6q^{15}$$
$$2{\cdot}(1,1,2,2,3,3,\overline{4})\, ax^7q^{16}$$
$$2{\cdot}(1,1,1,2,2,3,3,\overline{4})\, ax^8q^{17},$$

and

$$-2{\cdot}(1,2,3,3,3,\overline{3})\, ax^6q^{15}$$
$$-2{\cdot}(1,1,2,3,3,3,\overline{3})\, ax^7q^{16}$$
$$-2{\cdot}(1,1,1,2,3,3,3,\overline{3})\, ax^8q^{17},$$

etc.

5 Further research

We suggest the following directions for further research:

1. Carry out the explorations in Subsection 4.2 with other values of d and k.
2. First lesson to be learnt from Section 2 is that maybe it is too much to hope that partition generating functions like F_i count something meaningful. Instead, it will be worthwhile to explore if linear combinations of F_i hold interesting information.
3. Second lesson to be learnt is that may be only certain combinations of values (k, a, d) yield interesting results. However, which values of k, a to choose when $d \geq 3$ is not clear yet.
4. Write a computer algebra program to automate the construction in §3. This is a partial converse to the theory developed in [15] in the context of Rogers-Ramanujan generalizations. The $WZ-$theory constructs recurrences given series. In contrast, §3 constructs q-series given functional equations.

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