A SURVEY ON CAUCHY PROBLEMS FOR PERIDYNAMIC EQUATIONS

by

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Abstract

The peridynamic theory, proposed by Silling in 2000, is a nonlocal theory of continuum mechanics based on an integro-differential equation without spatial derivatives. This is seen to be main advantage, because it provides a more general framework than the classical theory for problems involving discontinuities or other singularities in the deformation.

In this thesis, we present a survey on the well-posedness of the Cauchy problems for peridynamic equations with different initial data spaces. These kind of equations can be also viewed as Banach space valued second order ordinary differential equations. So, in the first part of this study, we recall the theorems about local well-posedness of abstract differential equations of second order. Then, nonlinear problems related to the peridynamic model are reduced to abstract ordinary differential equations so that the right conditions can be imposed to imply local well-posedness. In the second part, we study a linear peridynamic problem and discuss the equivalent spaces in which the solution of the problem can take values. We use a functional analytic setting to show the well-posedness of the problem.

PERİDİNAMİK DENKLEMLER İÇİN CAUCHY PROBLEMLER ÜZERİNE BİR DERLEME

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Özet

2000 yılında Silling tarafından ortaya atılan peridinamik teori, sürekli ortamlar mekaniğinin yerel olmayan bir kuramıdır. Peridinamik teorinin belirgin özelliği, türetilen denklemlerin uzaysal türevler içermemesidir. Bu olgu, deformasyonda süreksizlik veya tekillik içeren problemler için klasik teoriye göre daha genel bir çerçeve sunar.

Bu tezde, peridinamik denklemler için Cauchy problemlerinin iyi konulmuş olmaları üzerine değişik sonuçları içeren bir derleme sunduk. Bu tür denklemler, Banach uzayında değer alan, zamanda ikinci derece adi diferansiyel denklemler olarak da düşünülebilir. Dolayısıyla, bu çalışmanın ilk kısmında, ikinci derece soyut adi difereansiyel denklemlerin yerel olarak iyi konulmuş olmalarına ilişkin teoremleri ele aldık. Sonra, peridinamik modele ait lineer olmayan denklemleri, uygun Banach fonksiyon uzaylarında değer alan ikinci derece adi diferansiyel denklemlere indirgeyerek Cauchy problemlerinin çeşitli başlangıç verilerine göre iyi konulmuş olmalarını gerektirecek uygun koşulları belirledik. İkinci kısımda, lineer peridinamik denklemi inceledik ve fonksiyonel analitik bir kurgu içerisinde, problemin başlangıç verileri ile çözümünün yer alabileceği eşdeğer uzaylardan bahsettik.

to my family

&

to whom supported my study

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CHAPTER 1

Introduction and Preliminaries

The peridynamic theory, proposed by Silling [1] in 2000, is a nonlocal theory of continuum mechanics based on an integro-differential equation without spatial derivatives. This is seen to be main advantage, because it provides a more general framework than the classical theory for problems involving discontinuities or other singularities in the deformation. Some applications for problems involving heat conduction in bodies with discontinuities and damage growth in materials can be found in [13] and [14], respectively.

The well-posedness of the linearized problem is first studied in [2] whereas the first results towards the nonlinear model can be found in [3]. The other results for the well-posedness of linear and nonlinear problems are shown in [4] and [5], respectively. On the other hand, some numerical approximation methods of the model are illustrated in [6].

In this thesis, we present a survey on the well-posedness of the Cauchy problems for peridynamic equations with different initial data spaces. For this purpose, we devote the rest of the first chapter to the preliminaries and present the main tools and theorems that will be used throughout this thesis.

Peridynamic equations can be also viewed as Banach space valued second order ordinary differential equations. So, in the second chapter, we recall the theorems about local well-posedness of abstract differential equations of second order.

We begin the third chapter by describing the peridynamic model. Then, nonlinear problems given in [3] and [4] are reduced to abstract ordinary differential equations so that the right conditions can be imposed to imply local well-posedness.

In the last chapter, we study the linear problem given in [5] and discuss the equivalent spaces in which the solution of the problem can take values. We use the same functional setting given there to show the well-posedness of the problem.

1.1. Spaces Involving Time

In this section, we present some function spaces that we will use later. We denote the spaces and the norms by

- $\mathcal{C}_b(\mathbb{R})$, the space of continuous, bounded functions on $\mathbb{R}$ with sup-norm
$$||f||_\infty = \sup_{x\in\mathbb{R}} |f(x)|.$$
- $\mathcal{C}_b^k(\mathbb{R})$, the space of continuous functions whose derivatives up to order k also belong to $\mathcal{C}_b(\mathbb{R})$ with norm
$$||f||_{\mathcal{C}_b^k} = \sum_{i=0}^{k} \left|\left| \frac{d^i f}{dx^i}(x) \right|\right|_\infty .$$
- $L^p(\mathbb{R})$, the set of Lebesgue measurable functions with L^p-norm
$$||f||_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p},$$
for $1 \leq p < \infty$.

- $L^\infty(\mathbb{R})$, the space of Lebesgue measurable functions that are essentially bounded on $\mathbb{R}$, meaning that the complement of the set that f is not bounded has measure 0 with the norm
$$||f||_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

We see that with the chosen norms, the given spaces are Banach Spaces.

Let $(X, ||.||_X)$ be a Banach space. Now, we define the following function spaces.

Definition 1.1.1 *The space $C([0,T],X)$ consists of continuous Banach-valued functions over the closed interval $[0,T]$, that is*
$$C([0,T],X) := \{\mathbf{u} : [0,T] \to X |\ \mathbf{u}(t) \text{ is continuous in } X\}.$$

It is a Banach Space with the following norm
$$||\mathbf{u}||_{C([0,T],X)} = \max_{t\in[0,T]} ||\mathbf{u}(t)||_X.$$

Example 1.1.1 *Let $X = C_b(\mathbb{R})$. Take $\mathbf{u} \in C([0,T], C_b(\mathbb{R}))$. This means that $\mathbf{u}$ is continuous in t and takes values in $C_b(\mathbb{R})$. Thus, $\mathbf{u}(t)$ is continuous in x. On the other hand, $u \in C([0,T] \times \mathbb{R})$ means u is continuous and bounded in both t and x. Then, $\mathbf{u}(t) = u(t,x)$.*

Definition 1.1.2 *$L^p([0,T], X)$, the space of Banach valued L^p functions over $[0,T]$, becomes a Banach Space with the norm*

$$||\mathbf{u}||_{L^p([0,T],X)} = \left(\int_0^T (||\mathbf{u}(t)||_X^p \, dt \right)^{1/p},$$

$1 \leq p < \infty$.

Notice that the Banach valued functions are denoted in bold font. But as far as it is clear from the context, we use "u" instead of "$\mathbf{u}$".

1.2. Hilbert Spaces

In this part, we give brief information about Hilbert Spaces [7].

Let $\mathcal{H}$ be a vector space over $\mathbb{R}$. A linear map from $\mathcal{H}$ to $\mathbb{R}$ is called a *linear functional* on $\mathcal{H}$. If $\mathcal{H}$ is a normed space, the space $\mathcal{L}(\mathcal{H}, \mathbb{R})$ of bounded linear functionals on $\mathcal{H}$ is called a *dual space* of $\mathcal{H}$ and is denoted by $\mathcal{H}^*$. An inner product on $\mathcal{H}$ is a map $(x,y) \to \langle x,y \rangle$ from $\mathcal{H} \times \mathcal{H} \to \mathbb{R}$ such that

i. $(ax + by, z) = a(x,z) + b(y,z)$ for all $x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{R}$.

ii. $(y,x) = \overline{(x,y)}$ for all $x, y \in \mathcal{H}$.

iii. $(x,x) \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$.

A vector space $\mathcal{H}$ that is equipped with an inner product is called an *inner product space*. Moreover, if $\mathcal{H}$ is complete with respect to the norm:

$$||x|| = \sqrt{(x,x)}, \tag{1.1}$$

then $\mathcal{H}$ is said to be *Hilbert Space*. Let $f, g \in L^2(\mathbb{R})$, then $|fg| \leq \frac{1}{2}(|f|^2 + |g|^2)$, so that $fg \in L^1(\mathbb{R})$. It follows that the formula

$$(f,g) = \int f(x)g(x)dx \tag{1.2}$$

defines an inner product space on $L^2(\mathbb{R})$. Now, we will state a well-known theorem concerning relationship between a Hilbert Space $\mathcal{H}$ and its dual $\mathcal{H}^*$ [12].

Theorem 1.2.1 (Riesz Representation Theorem [12]) *For any $f \in \mathcal{H}^*$, there exists a unique element $v \in \mathcal{H}$ such that $f(u) = (u, v)$. Similarly, every function $f(u) = (u, v)$ for $v \in \mathcal{H}$ defines an element of $\mathcal{H}^*$ with $||f||_{\mathcal{H}^*} = ||v||_{\mathcal{H}}$. Consequently, there is a natural isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$.*

1.3. Sobolev Spaces

The notion of well-posedness is related to the requirements that can be expected from solving a differential equation. A given problem for a differential equation is said to be well-posed if

- the problem in fact has a solution;
- the solution is unique; and
- the solution depends continuously on the data given in the problem.

The third condition indicates that the small changes in the initial data should lead to small changes in the solution, in the associated space. However, the requirements of existence and uniqueness for the solution are not clear enough as the exact definition of the related unique solution is not given. It is reasonable to ask for a solution of a differential equation of order k to be at least k times continuously differentiable. In this case, all derivatives in the equation must exist and be continuous. This kind of a solution is call a *classical* solution. Although some equations can be solved in the classical sense, many physical problems may admit solutions that are not differentiable or even not continuous. For this reason, we give different type of solutions that are called *generalized* or *weak* solutions. Such solutions are less smooth. To weaken the notion of partial derivatives, we give the definition of weak derivatives [8].

Let $C_c^\infty(U)$ be the space of infinitely differentiable functions $\phi : U \mapsto \mathbb{R}$, with compact support in $U \subset \mathbb{R}$. These functions are called as *test functions*. Assume $u \in C^1(U)$ and $\phi \in C_c^\infty(U)$. Then integration by parts formula implies that

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx \quad (i = 1, ..., n). \tag{1.3}$$

Let u be k times differentiable function, i.e. $u \in C^k(U)$, and $\alpha = (\alpha_1, ..., \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + ... + \alpha_n = k$. By applying the formula (1.3) $|\alpha|$ times, we have

$$\int_U uD^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u\phi dx, \text{ with } D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} \phi. \tag{1.4}$$

If u is not in $C^k(U)$, then it is meaningful to replace the expression "$D^\alpha u$" on the right hand side of (1.4) by a locally integrable function v:

Definition 1.3.3 *Suppose $u, v \in L^1_{loc}(U)$, and α is a multiindex. We say that v is the α^{th}-weak derivative of u, and write*

$$D^\alpha u = v,$$

provided that

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx$$

for all test functions $\phi \in C_c^\infty(U)$.

Definition 1.3.4 *Let k $1 \leq p \leq \infty$ and let k be a nonnegative integer. The Sobolev space $W^{k,p}(U)$ consists of all integrable functions $u : U \mapsto \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.*

The proof of the following theorem can be found in Section 5.2 of [8].

Theorem 1.3.2 *For each $k = 1, 2, ...$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach Space with the usual norm*

$$||u||_{W^{k,p}(U)} = \sum_{\alpha \leq k} ||D^\alpha u||_{L^p}.$$

Remark 1.3.1 *As $W^{k,2}(U)$ is a Hilbert Space, we use the notation*

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, ...).$$

Moreover, $H^0(U) = L^2(U)$.

For two Banach Spaces $\mathcal{B}_1$, $\mathcal{B}_2$, we say $\mathcal{B}_1$ is continuously embedded to $\mathcal{B}_2$, denoted by $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$, if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and the embedding map is continuous, i.e there exists a nonnegative number C such that

$$||u||_{\mathcal{B}_2} \leq C||u||_{\mathcal{B}_1}. \tag{1.5}$$

Lemma 1.3.3 *$L^\infty(\mathbb{R})$ is continuously embedded in $H^1(\mathbb{R})$.*

Proof: Let $\phi \in C_c^\infty$. Then, for all $x \in \mathbb{R}$,

$$\begin{aligned}(\phi(x))^2 = \int_{-\infty}^x 2\phi'(t)\phi(t)dt &\leq 2\int_{-\infty}^x |\phi'(t)||\phi(t)|dt \\ &\leq 2\int_{-\infty}^\infty |\phi'(t)||\phi(t)||dt \\ &\leq \int_{-\infty} (|\phi'(t)|^2 + |\phi(t)|^2)dt = ||\phi||^2_{H^1}.\end{aligned}$$

Thus $||\phi||^2_{L^\infty} \leq ||\phi||^2_{H^1}$, and $||\phi||_{L^\infty} \leq ||\phi||_{H^1}$. But $C_c^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$. Hence for every $f \in H^1(\mathbb{R})$ there holds $||f||_{L^\infty} \leq ||f||_{H^1}$. □

1.4. Fourier Transform

In this part, we give basic properties of Fourier Transform [8]:

Definition 1.4.5 *If $u \in L^1(\mathbb{R}^n)$, we define the Fourier Transform and the inverse Fourier Transform of u by*

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx, \quad \text{and} \quad \check{u}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{x\cdot\xi} u(\xi)d\xi,$$

respectively.

Since $|e^{\pm ix\cdot\xi}| = 1$ and $u \in L^1(\mathbb{R}^n)$, the integrals above are well-defined.

Now, we extend these definition to functions $u \in L^2(\mathbb{R}^n)$ by the following theorems ([8], [7]).

Theorem 1.4.4 (Plancherel's Theorem) *Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and*

$$||\hat{u}||_{L^2(\mathbb{R}^n)} = ||\check{u}||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\mathbb{R}^n)}.$$

Theorem 1.4.5 *Assume $u, v \in L^2(\mathbb{R}^n)$. Then*

(i) $\int_{\mathbb{R}^n} u\bar{v}dx = \int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}d\xi$,

(ii) $\widehat{D^\alpha u} = (i\xi)^\alpha \hat{u}$,

(iii) $\widehat{(u * v)} = (2\pi)^{n/2}\hat{u}\hat{v}$,

(iv) $u = \check{\hat{u}}$.

Next, we use the Fourier Transform to give an alternate characterization of the spaces $H^k(\mathbb{R})$ [8]. From Plancherel's Theorem, we have

$$||u||^2_{H^k} = ||u||^2_{W^{k,2}} = \sum_{|\alpha|\leq k} ||D^\alpha u||^2_{L^2} = \sum_{|\alpha|\leq k} \left|\left|\widehat{D^\alpha u}\right|\right|^2_{L^2}.$$

On the other hand, Theorem 1.4.5 implies that

$$\sum_{|\alpha|\leq k} \left|\left|\widehat{D^\alpha u}\right|\right|^2_{L^2} = \sum_{|\alpha|\leq k} \left|\left|(i\xi)^\alpha \hat{u}\right|\right|^2_{L^2} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}} |i\xi|^{2\alpha} |\hat{u}(\xi)|^2 d\xi$$
$$= \sum_{|\alpha|\leq k} \int_{\mathbb{R}} |\xi|^{2\alpha} \hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \sum_{|\alpha|\leq k} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 d\xi.$$

We let

$$\sum_{|\alpha|\le k} |\xi|^{2\alpha} = 1 + |\xi|^2 + |\xi|^4 + ... + |\xi|^{2k} = P_k(\xi). \tag{1.6}$$

Lemma 1.4.6 *Assume that $P_k(\xi)$ is defined as in* (1.6). *Then*

(i) $1 + |\xi|^{2k} \le P_k(\xi) \le k(1 + |\xi|^{2k})$

(ii) and there exist $C_1, C_2 > 0$ such that

$$C_1(1 + |\xi|^2)^k \le P_k(\xi) \le C_2(1 + |\xi|^2)^k. \tag{1.7}$$

Proof: (i) It is clear that for every $k \in \mathbb{Z}^+$ and $\xi \in \mathbb{R}^n$, we have

$$1 + |\xi|^{2k} \le 1 + |\xi|^2 + |\xi|^4 + ... + |\xi|^{2k} = P_k(\xi).$$

It remains to show the right hand side of the inequality. For this purpose, we distinguish two cases:
Case 1. Let $|\xi| \ge 1$. Then

$$\begin{aligned} P_k(\xi) = 1 + |\xi|^2 + |\xi|^4 + ... + |\xi|^{2k} &\le 1 + |\xi|^{2k} + |\xi|^{2k} + ... + |\xi|^{2k} \\ &= 1 + k|\xi|^{2k} \\ &\le k(1 + |\xi|^{2k}). \end{aligned} \tag{1.8}$$

Case 2. Let $|\xi| \le 1$. Then

$$\begin{aligned} P_k(\xi) = 1 + |\xi|^2 + |\xi|^4 + ... + |\xi|^{2k} &\le 1 + 1 + 1 + ... + |\xi|^{2k} \\ &= k + |\xi|^{2k} \\ &\le k(1 + |\xi|^{2k}). \end{aligned} \tag{1.9}$$

For all $\xi \in R^n$, (1.8)-(1.9) imply that

$$1 + |\xi|^{2k} \le P_k(\xi) \le k(1 + |\xi|^{2k}).$$

(ii) Let $(1 + |\xi|^2)^k = Q_k(\xi)$. To show (1.7), we first expand $Q_k(\xi)$:
Case 1. Let $|\xi| \ge 1$. Hence,

$$\begin{aligned} Q_k(\xi) = (1 + |\xi|^2)^k &= \binom{k}{0} + \binom{k}{1}|\xi|^2 + \binom{k}{2}|\xi|^4 + ... + \binom{k}{k}|\xi|^{2k} \\ &\le 1 + \left[\binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right]|\xi|^{2k} \\ &\le 1 + (2^k - 1)|\xi|^{2k} \le 2^k(1 + |\xi|^{2k}). \end{aligned} \tag{1.10}$$

Case 2. Let $|\xi| \le 1$. Then,

$$Q_k(\xi) \le \left[1 + \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k-1}\right] + |\xi|^{2k}$$
$$\le 2^k - 1 + |\xi|^{2k} \le 2^k(1 + |\xi|^{2k}). \quad (1.11)$$

For all $\xi \in R^n$, (1.10)-(1.11) imply that

$$1 + |\xi|^{2k} \le Q_k(\xi) \le 2^k(1 + |\xi|^{2k}).$$

Moreover,

$$0 < \lim_{|\xi|\to\infty} \frac{P_k(\xi)}{Q_k(\xi)} = 1,$$

meaning that $\frac{1}{2} \le \frac{P_k(\xi)}{Q_k(\xi)} \le 2$ for $|\xi| \ge 1$, On the other hand, $\frac{P_k(\xi)}{Q_k(\xi)}$ is continuous on $|\xi| \le 1$. Hence, there exist $m > 0, M \ge 0$ so that $m \le \frac{P_k(\xi)}{Q_k(\xi)} \le M$. Therefore,

$$C_1 \le \frac{P_k(\xi)}{Q_k(\xi)} \le C_2 \quad \forall \xi \in R^n,$$

and hence

$$C_1(1 + |\xi|^2)^k \le P_k(\xi) \le C_2(1 + |\xi|^2)^k \quad \forall \xi \in R^n$$

where $C_1 = \min\{m, 1/2\}$ and $C_2 = \max\{M, 2\}$. □

Lemma 1.4.6 suggests an alternative definition for *Sobolev Spaces*:

Definition 1.4.6 *Assume $s \ge 0$ a real number and $u \in L^2(\mathbb{R}^n)$. Then $u \in H^s(\mathbb{R}^n)$ if $(1 + |\xi|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For noninteger s, we set*

$$||u||_{H^s(\mathbb{R}^n)} := ||\sqrt{(1 + |\xi|^{2s})}\hat{u}||_{L^2(\mathbb{R}^n)} \approx ||(1 + |\xi|^2)^{\frac{s}{2}}\hat{u}||_{L^2(\mathbb{R}^n)}. \quad (1.12)$$

From Theorem 1.3.2 and (1.12), H^s is a Hilbert Space with

$$(u, v)_{H^s} = \int_{\mathbb{R}} (1 + |\xi|^2)^s \hat{u}(\xi)\hat{v}(\xi) d\xi.$$

Then, $(H^s)^* \approx H^s$ through $(.,.)_{H^s}$.

Define

$$H^s = \{v : (1 + |\xi|^2)^{\frac{s}{2}} \in L^2\}.$$

Then $(H^s)^* \approx H^{-s}$ through L^2 norm. That is, if $f \in (H^s)^*$, then there exists $v \in H^{-s}$ such that

$$f(u) = \int_{\mathbb{R}} uv dx = (u, v)_{L^2}.$$

That is, v correspons a bounded linear function on H^s.

Lemma 1.5.7 (Gronwall's Inequality [8]) *Let $\phi(t)$ be the nonnegative, continuous function on $[0,T]$ which satisfies almost everywhere t the integral inequality*

$$\phi(t) \leq C_1 \int_0^t \phi(s)ds + C_2,$$

where C_1 and C_2 are nonnegative constants. Then,

$$\phi(t) \leq C_2 e^{C_1 t}$$

for almost all $0 \leq t \leq T$.

Theorem 1.5.8 (Contraction Mapping Principle) *Suppose that S is a closed subset of a Banach Space, Y, and that $\mathcal{T} : S \to S$ is a mapping on S such that*

$$||\mathcal{T}u - \mathcal{T}v||_Y \leq \alpha||u - v||_Y \quad u, v \in S$$

for some constant $\alpha < 1$. Then $\mathcal{T}$ has a unique fixed point $u \in S$ that satisfies $\mathcal{T}u = u$.

Lemma 1.5.9 (Young's Inequality [7]) *If $f \in L^1$ and $g \in L^p (1 \leq p \leq \infty)$, then $(f * g)(x)$ exists for almost every x, $(f * g)(x) \in L^p$, and*

$$||f * g||_p \leq ||f||_1 ||g||_p \tag{1.13}$$

where

$$(f * g)(x) = \int_{\mathbb{R}} f(y - x)g(y)dy. \tag{1.14}$$

Lemma 1.5.10 (Minkowski's Inequality for Integrals) *If $1 \leq p \leq \infty$, and $\mathbf{u} \in L^p([0,T], L^p(\mathbb{R}))$ for a.e $0 \leq t \leq T$, then*

$$\left|\left|\int_0^T u(.,t)\, dt\right|\right|_p \leq \int_0^T ||u(.,t)||_p \, dt.$$

CHAPTER 2

Abstract Differential Equation of Second Order

2.1. Introduction

Let $(X, ||.||_X)$ be a Banach Space. Recall that if $u \in C([0,T], X)$, then given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $||u(t) - u(t_0)||_X < \epsilon$ whenever $|t - t_0| < \delta$ for every $t_0 \in [0,T]$. Furthermore, the differentiability of a function $u \in C([0,T], X)$ can be defined in the following way.

Definition 2.1.1 ([12]) *A function $u : [0,T] \to X$ is said to be differentiable in $t_0 \in (0,T)$, if there exists a linear transformation $\Lambda \in L([0,T], X)$ such that*

$$\lim_{h \to 0} \frac{||u(t_0 + h) - u(t_0) - \Lambda h||_X}{h} = 0. \tag{2.1}$$

We denote Λ by $u'(t_0)$ if it exists. Moreover, u is said to be differentiable on $(0,T)$, if it is differentiable at all points in $(0,T)$.

Then, by $u \in C^1([0,T], X)$ we mean $u : [0,T] \to X$ is continuous at every $t \in [0,T]$ and differentiable at every $t \in (0,T)$. Consider autonomous system of first order ordinary differential equation

$$u' = \mathcal{G}(u), \quad t \in (0,T), \quad u(0) = u_0, \quad \varphi \in X. \tag{2.2}$$

Remark 2.1.1 *There is no loss of generality of taking the initial point $t_0 = 0$ since we deal with system that does not depend explicitly on t. That is to say if $u(t)$ is a solution, then so is $u(t + t_0)$.*

In the study of ordinary differential equations, some functions $\mathcal{G} : X \to X$ can be taken to be locally Lipschitz continuous:

Definition 2.1.2 *A function $\mathcal{G} : X \to X$ is said to be locally Lipschitz continuous , if for every $R > 0$, there exists $L_R > 0$ such that*

$$||\mathcal{G}(u) - \mathcal{G}(v)||_X \leq L_R||u - v||_X \quad \text{for all } u, v \in \bar{B}_X(0, R). \tag{2.3}$$

It is well-known from Picard-Lindelöf Theorem that if $\mathcal{G}$ is locally Lipschitz continuous, then there exists $T_1 \leq T$ such that the initial value problem 2.2 has a unique solution $u \in C^1([0, T_1], \bar{B}_X(0, R))$.

Remark 2.1.2 *If $\mathcal{G}$ is continuously differentiable, then the condition* (2.3) *is satisfied by the Mean Value Theorem.*

2.2. Abstract Differential Equation of Second Order

In this study, as the equation we have at hand is of second order, we will deal with the well-posedness of the initial value problem of second order abstract differential equation:

$$u'' = \mathcal{G}(u), \quad t \in (0, T), \quad u(0) = \varphi, \quad u'(0) = \psi \tag{2.4}$$

with initial data $\varphi, \psi \in X$.

One can note that if we let $u_1 = u$, $u_2 = u'$, then the second order differential equation (2.4) can be converted to a system of first order differential equation :

$$\frac{du_1}{dt} = u_2, \qquad u_1(0) = \varphi$$
$$\frac{du_2}{dt} = \mathcal{G}(u_1, u_2), \quad u_2(0) = \psi.$$

Therefore,

$$\frac{d\overrightarrow{u}}{dt} = \mathcal{G}(\overrightarrow{u}), \quad \overrightarrow{u}(0) = \overrightarrow{\varphi}.$$

where we let $\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\mathcal{H}(\overrightarrow{u}) = \begin{pmatrix} u_2 \\ \mathcal{G}(u_1, u_2) \end{pmatrix}$, and $\overrightarrow{\varphi} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$.

However, we will state the sufficient conditions for well-posedness of problem 2.4 in Theorem 2.2.1 and prove it directly rather than converting it to a first order system.

Theorem 2.2.1 *Let $\mathcal{G} : X \to X$ be locally Lipschitz continuous. Then, for any $\varphi, \psi \in X$, there exists $T > 0$ such that the initial value problem (2.4) has a unique solution $u \in \mathcal{C}^2([0,T], X)$. The solution u depends continuously on the initial data.*

Proof: We first show the existence of the solution of the problem (2.4). By integrating (2.4) twice, we obtain

$$\begin{aligned} u(t) = \varphi + \int_0^t u'(s)\, ds &= \varphi + \int_0^t \left(\psi + \int_0^s \mathcal{G}(u(\tau)) d\tau \right)\, ds \\ &= \varphi + t\,\psi + \int_0^t \int_0^s \mathcal{G}(u(\tau))\, d\tau\, ds \end{aligned} \tag{2.5}$$

with the initial conditions $u(0) = \varphi$, $u'(0) = \psi$.
By changing the order of the integration in the right hand side, one can obtain

$$\begin{aligned} u(t) &= \varphi + t\,\psi + \int_0^t \int_\tau^t \mathcal{G}(u(\tau))\, ds\, d\tau \\ &= \varphi + t\,\psi + \int_0^t (t-\tau)\mathcal{G}(u(\tau))\, d\tau \end{aligned} \tag{2.6}$$

If we define $\mathcal{S}$ by the right hand side of (2.6). Then, the initial value problem (2.4) is equivalent to finding a fixed point $\mathcal{S}(u) = u$. For some T that will be determined later, we let $X(T) = C([0,T], X)$. Let $M = \sup_{u \in \bar{B}_X(0,R)} ||\mathcal{G}(u)||_X$. Notice that M is finite as $\mathcal{G}$ is Lipschitz on $\bar{B}_X(0,R)$ with Lipschitz constant L_R:

$$\begin{aligned} ||\mathcal{G}(u)||_X &\le ||\mathcal{G}(0)||_X + ||\mathcal{G}(u) - \mathcal{G}(0)||_X \\ &\le ||\mathcal{G}(0)||_X + L_R ||u||_X \\ &\le ||\mathcal{G}(0)||_X + L_R R = M. \end{aligned}$$

Claim 2.2.2 $\mathcal{S} : X(T) \to X(T)$ *is well-defined, i.e.*

(i) $\forall t \in [0,T] \quad \mathcal{S}(u)(t) \in X$,

(ii) $t \to \mathcal{S}(u)(t)$ *is continuous.*

Proof: (i) We know that $u : [0,T] \to X$ and $\mathcal{G} : X \to X$ are continuous. Therefore, $\mathcal{G}u : [0,T] \to X$ is continuous. Hence, keeping in mind that $\varphi, \psi \in X$ we can write

$$\begin{aligned} \mathcal{S}(u)(t) &= \varphi + t\psi + \int_0^t (t-\tau)\, \mathcal{G}(u(\tau)) d\tau \\ &= \varphi + t\psi + \lim_{\Delta\tau \to 0} \sum_{j=0}^N (t - t_j)\mathcal{G}(u(\tau_j))\Delta\tau. \end{aligned}$$

As each $\mathcal{G}(u(\tau_j))$ and their linear combinations are in X, the sum is in X. So the limit is in X.
(ii) We show $\mathcal{S}(u)$ is continuous in t. Let $t_0 \in [0,T]$ be fixed.

$$\begin{aligned}
&\mathcal{S}(u)(t_0+\Delta t) - \mathcal{S}(u)(t_0) \\
&\quad = \Delta t\psi + \int_0^{t_0+\Delta t} (t_0+\Delta t-\tau)\mathcal{G}(u(\tau))d\tau - \int_0^{t_0} (t_0-\tau)\mathcal{G}(u(\tau))d\tau \\
&\quad = \Delta t\psi + \int_0^{t_0+\Delta t} (t_0-\tau)\mathcal{G}(u(\tau))d\tau + \Delta t\int_0^{t_0+\Delta t} \mathcal{G}(u(\tau))d\tau - \int_0^{t_0} (t_0-\tau)\mathcal{G}(u(\tau))d\tau \\
&\quad = \Delta t\psi + \int_{t_0}^{t_0+\Delta t} (t_0-\tau)\mathcal{G}(u(\tau))d\tau + \Delta t\int_0^{t_0} \mathcal{G}(u(\tau))d\tau. \qquad (2.7)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&||\mathcal{S}(u)(t_0+\Delta t) - \mathcal{S}(u)(t_0)||_X \\
&\quad \le \Delta t||\psi||_X + \left|\left| \int_{t_0}^{t_0+\Delta t} (t_0-\tau)\mathcal{G}(u(\tau))d\tau \right|\right|_X + \Delta t\left|\left| \int_0^{t_0} \mathcal{G}(u(\tau))d\tau \right|\right|_X \\
&\quad \le \Delta t||\psi||_X + \int_{t_0}^{t_0+\Delta t} (t_0-\tau)||\mathcal{G}(u(\tau))||_X d\tau + \Delta t\int_0^{t_0} ||\mathcal{G}(u(\tau))||_X d\tau \\
&\quad \le \Delta t||\psi||_X + M\int_{t_0}^{t_0+\Delta t} (t_0-\tau)d\tau + \Delta t M\int_0^{t_0} d\tau \\
&\quad = \Delta t||\psi||_X + \frac{M}{2}((\Delta t)^2 - 2t_0\Delta t) + \frac{M}{2}t_0^2\Delta t
\end{aligned}$$

and $\lim_{\Delta t\to 0} ||\mathcal{S}(u)(t_0+\Delta t) - \mathcal{S}(u)(t_0)||_X = 0$. □

Now, we can go on proving Theorem 2.2.1. Fix $R \ge 2||\varphi||_X$ and choose the set

$$Y(T) = \mathcal{C}([0,T], \bar{B}_X(0,R)) = \{u \in X(T) : \ ||u||_{X(T)} \le R\}.$$

This implies that if $u \in Y(T)$, then $u(t) \in \bar{B}_X(0,R)$ for all $t \in [0,T]$. We begin with showing that $\mathcal{S}$ maps $Y(T)$ into itself for a suitable choice of T:

$$||\mathcal{S}(u)(t)||_X \le ||\varphi||_X + t||\psi||_X + \int_0^t (t-\tau)||\mathcal{G}(u(\tau))||_X \, d\tau.$$

Since $\tau \in [0,t] \subseteq [0,T]$, we have $||u(\tau)||_X \le R$ and $||\mathcal{G}(u(\tau)||_X \le M$. We continue as:

$$\begin{aligned}
&\le ||\varphi||_X + t||\psi||_X + M\int_0^t (t-\tau)\, d\tau \\
&\le ||\varphi||_X + t||\psi||_X + \frac{M}{2}t^2.
\end{aligned}$$

Taking supremum over t yields

$$||\mathcal{S}(u)||_{X(T)} \le ||\varphi||_X + T||\psi||_X + \frac{M}{2}T^2.$$

Choosing T small enough to satisfy $T||\psi||_X + \frac{M}{2}T^2 \leq R/2$ will give $\mathcal{S} : Y(T) \to Y(T)$.

Next, we show that $\mathcal{S}$ is contractive. For all $u, v \in \bar{B}_X(0, R)$ and $\forall \tau \in [0, T]$, we have $u(\tau), v(\tau) \in B(0, R)$, and

$$
\begin{aligned}
||\mathcal{S}(u)(t) - \mathcal{S}(v)(t)||_X &\leq \int_0^t (t-\tau)||\mathcal{G}(u(\tau)) - \mathcal{G}(v(\tau))||_X \, d\tau \\
&\leq L_R \int_0^t (t-\tau)||u(\tau) - v(\tau)||_X \, d\tau.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||\mathcal{S}(u) - \mathcal{S}(v)||_{X(T)} &\leq L_R ||u - v||_{X(T)} \int_0^T (t-\tau) \, d\tau \\
&\leq L_R \frac{T^2}{2} ||u - v||_{X(T)}.
\end{aligned}
$$

For $L_R T^2 \leq 1$, S becomes contractive.

In fact, it is possible to determine T explicitly. Let $P(T) = MT^2 + 2T||\psi||_X - R$. Then $\Delta = 4||\psi||_X^2 + 4MR$. Hence $T = \frac{-2||\psi||_X + 2\sqrt{||\psi||_X^2 + MR}}{2M} = \sqrt{\frac{||\psi||_X^2}{M^2} + \frac{R}{M}} - \frac{||\psi||_X}{M}$. If we choose $T = \min\left\{ \sqrt{\frac{||\psi||_X^2}{M^2} + \frac{R}{M}} - \frac{||\psi||_X}{M}, \frac{1}{\sqrt{L_R}} \right\}$, by Contraction Mapping Principle there exists $u \in Y(T)$ such that $u = \mathcal{S}(u)$.

Now, it remains to show the continuous dependence. Assume u_1, u_2 are two solutions with the initial data (φ_1, ψ_1) and (φ_2, ψ_2), respectively. Choose R with $R \geq 2\max\{||\varphi_1||_X, ||\varphi_2||_X\}$. Then

$$
\begin{aligned}
||u_1(t) - u_2(t)||_X &\leq ||\varphi_1 - \varphi_2||_X + t||\psi_1 - \psi_2||_X + \int_0^t (t-\tau)||\mathcal{G}(u_1(\tau)) - \mathcal{G}(u_2(\tau))||_X \, d\tau \\
&\leq ||\varphi_1 - \varphi_2||_X + T||\psi_1 - \psi_2||_X + L_R T \int_0^t ||u_1(\tau) - u_2(\tau)||_X \, d\tau.
\end{aligned}
$$

Gronwall's Inequality implies that

$$
||u_1(t) - u_2(t)||_X \leq (||\varphi_1 - \varphi_2||_X + T\,||\psi_1 - \psi_2||_X) e^{L_R T t}
$$

and hence

$$
||u_1(t) - u_2(t)||_{X(T)} \leq (||\varphi_1 - \varphi_2||_X + T\,||\psi_1 - \psi_2||_X) e^{L_R T^2}.
$$

This implies that small changes in the initial data lead to small changes in the solution.

Therefore, the problem (2.4) has a unique local solution which depends continuously on the initial data. □

We can also think about the extension of the solution to the maximal time interval. If we consider the problem (2.4), we know that there is some $T_1 > 0$ such that the solution of (2.4) exists uniquely in $[0, T_1]$. Next, we look for the solution for $t \geq T_1$. For this purpose write the shifted version of the problem as follows

$$u'' = \mathcal{G}(u), \quad u(T_1) = \varphi_1, \quad u'(T_1) = \psi_1, \quad t > T_1$$

where φ_1, ψ_1 are obtained from the solution of problem (2.4). Theorem (2.2.1) enables us to say that this shifted problem has a unique solution on $[T_1, T_2]$ for some $T_2 > T_1$. Hence, the solution is extended to $[0, T_2]$. Keeping on this way, one can extend the solution to $[0, T_n]$ provided that all φ_n, ψ_n are in X. In this way, the maximal interval will be $[0, T_{\max})$. If $\lim\limits_{t \to T_{\max}} (u(t), u'(t))$ does not exist, then $T_{\max} < \infty$.

Now, by considering the non-homogenous case, we will write a more general abstract differential equation:

$$u'' = \mathcal{G}(u) + b(t), \quad t \in (0, T), \quad u(0) = \varphi, \, u'(0) = \psi \tag{2.8}$$

for $\varphi, \psi \in X$, where $b \in C([0, T], X)$. We note that the function b is assumed to be continuous only for $t \in [0, T]$.

Thus, the system will be nonautonomus and the sufficient conditions for the well-posedness of the initial value problem 2.8 can be summed up in the next theorem.

Theorem 2.2.3 *Let $\mathcal{G} : X \to X$ be locally Lipschitz continuous and $b \in C([0, \tilde{T}), X)$. Then, for any $\varphi, \psi \in X$, there exists $0 < T \leq \tilde{T}$ such that the initial value problem 2.8 has a unique solution $u \in \mathcal{C}^2([0, T], X)$. The solution u depends continuously on the initial data.*

Proof: The steps of the proof will be similar to the ones in Theorem 2.2.1. So, we only state the main differences.

- First of all, the corresponding operator will be

$$\mathcal{S}(u)(t) = \varphi + t\,\psi + \int_0^t (t - \tau)\mathcal{G}(u(\tau))\, d\tau + \int_0^t (t - \tau) b(\tau)\, d\tau. \tag{2.9}$$

- $\mathcal{S} : X(T) \to X(T)$ becomes well-defined as both $\mathcal{G}$ and b are given to be continuous.

- Again for fix $R \geq 2||\varphi||_X$, we will choose the same set

$$Y(T) = \mathcal{C}([0, T], \bar{B}_X(0, R)) = \{u \in X(T) : \; ||u||_{X(T)} \leq R\}.$$

- However, we will have

$$\begin{aligned} \sup_{(t,u)\in[0,T]\times\bar{B}_X(0,R)} ||\mathcal{G}(u) + b(t)||_X &\leq \sup_{(t,u)\in[0,T]\times\bar{B}_X(0,R)} ||\mathcal{G}u||_X + ||b(t)||_X \\ &\leq ||\mathcal{G}u||_X + ||b||_{X(T)} = M^*. \end{aligned}$$

- In order to show that $\mathcal{S}(Y(T)) \subseteq Y(T)$, we should choose T small enough to satisfy

$$\begin{aligned} ||\mathcal{S}(u)||_{X(T)} &\leq ||\varphi||_X + T||\psi||_X + \frac{M^*}{2}T^2 + \sup_{0\leq t\leq T} \int_0^t (t-\tau)||b(\tau)||_X \, d\tau|| \\ &\leq ||\varphi||_X + T||\psi||_X + \frac{M^*}{2}T^2 + ||b||_{X(T)} \sup_{0\leq t\leq T} \int_0^t (t-\tau) \, d\tau \\ &\leq ||\varphi||_X + T||\psi||_X + \left(\frac{M^*}{2} + ||b||_{X(T)}\right)\frac{T^2}{2} \leq R/2 \end{aligned}$$

in addition to $R \geq 2||\varphi||_X$.

- While showing that $\mathcal{S} : Y(T) \to Y(T)$ is a contraction mapping, the integral in the right hand side of (2.9) will vanish and it will not affect the assumption that $L_R T^2 \leq 1$.

- If we choose $T = \min\left\{\sqrt{\left(\frac{||\psi||_X}{(M^*+2||b||_{X(T)})}\right)^2 + \frac{R}{M^*+2||b||_{X(T)}}} - \frac{||\psi||_X}{M^*+2||b||_{X(T)}}, \frac{1}{\sqrt{L_R}}\right\}$, by Contraction Mapping Principle there exists $u \in Y(T)$ such that $u = \mathcal{S}(u)$.

- We will follow the same steps in Theorem 2.2.1 to show the continuous dependence on the initial data.

□

In [4], one can find the sufficient conditions for the local well-posedness of the general second order non-homogeneous abstract differential equation

$$u''(t) = \mathcal{G}(t,u), \quad u(0) = \varphi, \;\; u'(0) = \psi, \; 0 < t \leq T. \tag{2.10}$$

On the other hand, global in time solutions for (2.10) can be obtained for continuous functions $\mathcal{G} : [0,T] \times X \to X$ where $\mathcal{G}$ is globally Lipschitz continuous in the second variable:

Consider the weighted norm

$$||u||_{X(T)}^{\sim} = e^{-Kt} \max_{t\in[0,T]} ||u(t)||_X$$

where $K > 0$. Then, it is easy to see that the norms $||.||_{X(T)}$ and $||u||_{X(T)}^{\sim}$ are equivalent:

$$e^{-KT}||u||_{C([0,T],X)} \leq ||u||_{X(T)}^{\sim} \leq \max_{t\in[0,T]} ||u(t)||_X.$$

Hence $(C([0,T],X), ||.||_{X(T)})$ and $(C([0,T],X), ||.||_{X(T)}^{\sim})$ are equivalent Banach Spaces. Then, for appropriately chosen $K = K(L)$, one can obtain that

$$\mathcal{S} : (C([0,T],X), ||.||_{X(T)}^{\sim}) \to (C([0,T],X), ||.||_{X(T)}^{\sim})$$

defined by (2.6) is a contraction in case that $\mathcal{G}$ is globally Lipschitz continuous. Hence, there exists a unique $u \in (C([0,T],X), ||.||_{X(T)})$ such that $\mathcal{S}u = u$ and the Cauchy problem (2.10) has a unique global solution $u \in C([0,T],X)$ for $\varphi, \psi \in X$. Moreover, the solution depends continuously on the initial data.

Examples can be found in [4, 11].

CHAPTER 3

Local Well-posedness of Nonlinear Peridynamic Models

In this chapter, we will study the local well-posedness of peridynamic model. In the first section we introduce the model. Then, we will consider seperable form in one-dimensional case. Next, we will study the general case. In each cases, we will discuss the sufficient conditions to use Theorem 2.2.1.

3.1. Peridynamic Model

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be the unbounded domain of an undeformed body and $[0, T]$ the time interval under consideration. Let $u : \bar{\Omega} \times [0, T] \to \mathbb{R}^n$ be the deformation of the solid body. Then, for $(x, t) \in \Omega \times (0, T)$, the nonlinear peridynamic equation of motion reads

$$\rho(x)u_{tt}(x,t) = \int_{\mathcal{H}(x)} f(y - x, u(y,t) - u(x,t))\, dy + b(x,t). \tag{3.1}$$

Here ρ is the density of the body, b represents external forces and the integration domain $\mathcal{H}(x)$ describes the volume of particles interacting and is the ball of radius δ centered at x intersected with Ω. The radius δ is called peridynamic horizon. The integrand f is called pairwise force function and gives the force that the particle y exerts on particle x. It is considered as 0 beyond the horizon, thus one may consider the integral in (3.1) as

$$\int_{\mathcal{H}(x)} \tilde{f}(y - x, u(y,t) - u(x,t))\, dy \quad \text{with} \quad \tilde{f} = \begin{cases} f, & y \in \mathcal{H}(x) \\ 0, & y \notin \mathcal{H}(x) \end{cases}$$

.

The theory is named after the greek words *peri* (near) and *dynamics* (force).

Identifying $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ with $u : [0,T] \to X$ for a function space X by $u(t)(x) := u(x,t)$, the problems reduces to (2.4).

3.2. Seperable Form

In this section, we consider the following Cauchy problem

$$u_{tt}(x,t) = \int_{\mathbb{R}} \alpha(y-x) w(u(y,t) - u(x,t))\, dy, \quad x \in \mathbb{R}, t > 0 \tag{3.2}$$

with the initial data

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{3.3}$$

given in [3]. This is a one-dimensional peridynamic model where the pairwise force function is taken to be seperable. That is, $f(\xi,\eta) = \alpha(\xi)w(\eta)$ where α is an integrable even function on $\mathbb{R}$ and w is a sufficiently smooth odd function satisfying $w(0) = 0$. Moreover, it is assumed that $\rho = 1$ and that there are no external forces.

Now we will study the local well-posedness of (3.2)-(3.3) for initial data spaces $C_b(\mathbb{R}), L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}), C_b^1(\mathbb{R})$ and $H^1(\mathbb{R})$, respectively. The preceeding theorem suggests that the solution of the system:

$$u'' = \mathcal{G}(u), \quad u(0) = \varphi,\ u'(0) = \psi$$

where

$$\mathcal{G}(u)(x) = \int_{\mathbb{R}} \alpha(y-x) w(u(y) - u(x)) dy \tag{3.4}$$

will be of the form

$$u(t) = \varphi + t\psi + \int_0^t (t-\tau)\mathcal{G}(u(\tau))\, d\tau. \tag{3.5}$$

For each four cases, it remains to find the conditions under which the mapping $\mathcal{G} : X \to X$ is

(i) well-defined, and

(ii) locally Lipschitz continuous on X.

Before concentrating on the following theorems, we will introduce a nondecreasing function D which we will often encounter :

$$D(R) = \max_{|\eta| \le 2R} |w'(\eta)|. \tag{3.6}$$

Theorem 3.2.1 *Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^1(\mathbb{R})$ with $w(0) = 0$. Then there is some $T > 0$ such that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$.*

Proof: Let $X = C_b(\mathbb{R})$. We want to show $\mathcal{G} : C_b(\mathbb{R}) \to C_b(\mathbb{R})$, i.e $\mathcal{G}(u)$ is continuous in x and it is uniformly bounded. Now, assume that $\{x_n\}$ is a Cauchy sequence with $\lim_{n\to\infty} x_n = x$. Let $R > 0$ and take $u \in \bar{B}_X(0, R)$. Let's recall our integral operator:

$$\mathcal{G}(u)(x) = \int_{\mathbb{R}} \alpha(y-x)w(u(y) - u(x))dy$$

and make a substitution $y - x = z$. Then $dy = dz$. Hence

$$= \int_{\mathbb{R}} \alpha(z)w(u(x+z) - u(x))dz. \tag{3.7}$$

Now, consider the following integral

$$\mathcal{G}(u)(x_n) = \int_{\mathbb{R}} \alpha(z)w(u(x_n+z) - u(x_n))dz,$$

and let $h_n(z) = \alpha(z)w(u(x_n + z) - u(x_n))$. Then $\lim_{n\to\infty} h_n(z) = \alpha(z)w(u(x+z) - u(x))$ since u, w are both continuous . Moreover,

$$|h_n(z)| \le |\alpha(z)||w(u(x_n+z) - u(x_n))|.$$

But $w(0) = 0$, and $|u(x_n + z) - u(x)| \le |u(x_n + z)| + |u(x_n)| \le 2||u||_\infty \le 2R$. By (3.6) we have

$$\begin{aligned} |w(u(x_n+z) - u(x))| &= |w(u(x_n+z) - u(x_n)) - w(0)| \\ &\le \sup_{|\eta| \le 2||u||_\infty} |w'(\eta)||u(x_n+z) - u(x_n)| \\ &\le 2D(||u||_\infty)||u||_\infty. \end{aligned} \tag{3.8}$$

Since $||u||_\infty \le R$ and $D(||u||_\infty) \le D(R)$, we conclude that

$$|\alpha(z)w(u(x_n+z) - u(x))| \le 2D(R)|\alpha(z)|||u||_\infty$$

and $2D(R)|\alpha(z)|||u||_\infty \in L^1$. Therefore, by Dominated Convergence Theorem, we can write

$$\begin{aligned} \lim_{n\to\infty} \mathcal{G}(u)(x_n) = \lim_{n\to\infty} \int_{\mathbb{R}} h_n(z) &= \int_{\mathbb{R}} \lim_{n\to\infty} h_n(z)dz \\ &= \int_{\mathbb{R}} \alpha(z)w(u(x+z) - u(x))dz, \end{aligned}$$

and back substitution yields

$$\begin{aligned} &= \int_{\mathbb{R}} \alpha(y-x)w(u(y) - u(x))dy \\ &= \mathcal{G}(u)(x). \end{aligned}$$

Hence $\mathcal{G}(u)$ is continuous in x. Now, we want to show that $\mathcal{G}(u)$ is uniformly bounded.

$$|\mathcal{G}(u)(x)| \leq \int_{\mathbb{R}} |\alpha(y-x)||w(u(y)-u(x))|dy$$

where $|u(y)-u(x)| \leq |u(y)|+|u(x)| \leq 2||u||_\infty \leq 2R$, so we can write

$$\begin{aligned}
&\leq \int_{\mathbb{R}} |\alpha(y-x)| \sup_{|\eta|\leq 2||u||_\infty} |w'(\eta)||w(u(y)-u(x))|dy \\
&\leq \int_{\mathbb{R}} |\alpha(y-x)|D(||u||_\infty)(|u(y)|+|u(x)|)dy \\
&= D(||u||_\infty)\left(\int_{\mathbb{R}} |\alpha(y-x)||u(y)|dy + \int_{\mathbb{R}} |\alpha(y-x)||u(x)|\,dy\right) \\
&= D(||u||_\infty)\left[(|\alpha| * |u|)(x) + ||\alpha||_1|u(x)|\right]. \qquad (3.9)
\end{aligned}$$

After taking supremum over x, we use definition of D and Lemma 1.5.9 to obtain

$$\begin{aligned}
||\mathcal{G}(u)||_\infty &\leq D(R)\,[\,||\alpha||_1||u||_\infty + ||\alpha||_1||u||_\infty] \\
&= 2D(R)||\alpha||_1||u||_\infty \qquad (3.10) \\
&\leq 2D(R)R||\alpha||_1.
\end{aligned}$$

As $2D(R)R||\alpha||_1 < \infty$, we have $\mathcal{G} : C_b(\mathbb{R}) \to C_b(\mathbb{R})$.

One can note that in the following theorems, we will obtain similar estimates.

Now, our aim is to show that $\mathcal{G}$ is locally Lipschitz continuous. Take $u, v \in \bar{B}_X(0,R)$. Then

$$\begin{aligned}
|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| &\leq \int_{\mathbb{R}} |\alpha(y-x)||w(u(y)-u(x)) - w(v(y)-v(x))|dy \\
&\leq \int_{\mathbb{R}} |\alpha(y-x)| \sup_{|\eta|<2||u||_\infty} |w'(\eta)||u(y)-u(x)-v(y)-v(x)|dy \\
&\leq D(||u||_\infty)(|\alpha| * |u-v|)(x) + ||\alpha||_1|u(x)-v(x)|. \qquad (3.11)
\end{aligned}$$

We take supremum over x and use lemma 1.5.9 (3.6) to obtain

$$||\mathcal{G}(u) - \mathcal{G}(v)||_\infty \leq 2D(R)||\alpha||_1||u-v||_\infty. \qquad (3.12)$$

Hence, $\mathcal{G}$ is locally Lipschitz on X with Lipschitz constant $L_R = 2D(R)||\alpha||_1$.

Calculations above show that the requirements of Theorem 2.2.1 are fulfilled. Thus, we can conclude that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$. □

Theorem 3.2.2 *Let* $1 \leq p \leq \infty$. $\alpha \in L^1(\mathbb{R})$ *and* $w \in C^1(\mathbb{R})$ *with* $w(0) = 0$. *Then there is some* $T > 0$ *such that the Cauchy problem* (3.2)-(3.3) *is well-posed with solution in* $C^2([0,T], L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ *for initial data* $\varphi, \psi \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Proof: Take $u \in \bar{B}_X(0, R)$. Then

$$|\mathcal{G}(u)(x)| \leq \int_{\mathbb{R}} |\alpha(y-x)||w(u(y)-u(x))|dy.$$

But $w(0) = 0$, and $|u(y)-u(x)| \leq 2||u||_{L^\infty} \leq 2||u||_{L^\infty} + 2||u||_{L^p} = 2||u||_X \leq 2R$. Hence $|u(y) - u(x)| \leq 2R$. Analogous to uniform norm estimate, by (3.6) and (3.9), one can obtain

$$||\mathcal{G}(u)||_{L^\infty} \leq 2D(R)||\alpha||_1||u||_{L^\infty}. \tag{3.13}$$

We need corresponding L^p estimates. Keeping in mind that $|u(y) - u(x)| \leq 2R$, we take p-th norm of both sides of (3.9) to obtain

$$||\mathcal{G}(u)||_{L^p} \leq D(R)\left(|||\alpha| * |u|||_{L^p} + ||\alpha||_1||u||_{L^p}\right).$$

Moreover, Lemma 1.5.9 implies that

$$||\mathcal{G}(u)||_{L^p} \leq 2D(R)||\alpha||_1||u||_{L^p}. \tag{3.14}$$

Summing up (3.13) and (3.14), we get

$$||\mathcal{G}(u)||_X \leq 2D(R)||\alpha||_1||u||_X.$$

Since $2D(R)||\alpha||_1||u||_X < \infty$, we have $\mathcal{G} : X \to X$. Now it remains to show Lipschitz continuity of $\mathcal{G}$. Using the estimate (3.11) and Lemma 1.5.9, we have L^∞ estimate as

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^\infty} \leq 2D(R)||\alpha||_1||u-v||_{L^\infty} \tag{3.15}$$

Again taking p-th norm of (3.11), and using the fact that

$$|||\alpha| * |u-v|||_{L^p} \leq ||\alpha||_1||u-v||_{L^p},$$

we have

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^p} \leq 2D(R)||\alpha||_1||u-v||_{L^p}. \tag{3.16}$$

Summing up (3.15) and (3.16) yields

$$||\mathcal{G}(u) - \mathcal{G}(v)||_X \leq 2D(R)||\alpha||_1||u-v||_X.$$

This gives that $\mathcal{G}$ is Lipschitz with Lipschitz constant $L_R = 2D(R)||\alpha||_1$.

Calculations above show that the assumptions of Theorem 2.2.1 are satisfied. Thus, we can conclude that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ for initial data $\varphi, \psi \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$. □

Theorem 3.2.3 *Let* $X = C_b^1(\mathbb{R})$. $\alpha \in L^1(\mathbb{R})$ *and* $w \in C^2(\mathbb{R})$ *with* $w(0) = 0$. *Then there is some* $T > 0$ *such that the Cauchy problem* (3.2)-(3.3) *is well-posed with solution in* $C^2([0,T], C_b^1(\mathbb{R}))$ *for initial data* $\varphi, \psi \in C_b^1(\mathbb{R})$.

Proof: Let $X = C_b^1(\mathbb{R})$. Then $||u||_X = ||u||_\infty + ||u'||_\infty$. As we already have the supremum norm estimates of $\mathcal{G}(u)$ and $\mathcal{G}(u) - \mathcal{G}(v)$, all we need is supremum norm estimates of their x derivatives. Take $u \in \bar{B}(0, R)$. Differentiating (3.4) gives

$$\frac{d}{dx}\mathcal{G}(u)(x) = \frac{d}{dx}\int_{\mathbb{R}} \alpha(y-x)w(u(y)-u(x))dy.$$

Change of variables $y = x + z$ yields $dy = dz$, and the equality becomes

$$\begin{aligned} &= \frac{d}{dx}\int_{\mathbb{R}} \alpha(z)w(u(x+z)-u(x))dz. \\ &= \int_{\mathbb{R}} \alpha(z)w'(u(x+z)-u(x))(u_x(x+z)-u_x(x))dz. \end{aligned}$$

By back substitution, we continue as

$$= \int_{\mathbb{R}} \alpha(y-x)w'(u(y)-u(x))(u_x(y)-u_x(x))dy. \tag{3.17}$$

Since $u \in \bar{B}_X(0, R)$, we get $|u(y) - u(x)| \leq 2||u||_\infty \leq 2||u||_\infty + 2||u'||_\infty = 2||u||_X \leq 2R$. Therefore $|u(y) - u(x)| \leq 2R$. Now we can use the definition of D in (3.6) and obtain

$$\begin{aligned} \left|\frac{d}{dx}\mathcal{G}(u)(x)\right| &\leq D(||u||_\infty)\int_{\mathbb{R}} \alpha(y-x)|(u_x(y)-u_x(x))|dy \\ &\leq D(||u||_\infty)\left[(|\alpha| * |u_x|)(x) + ||\alpha||_1|u_x(x)|\right]. \end{aligned} \tag{3.18}$$

Taking sup norm of (3.18) yields

$$||(\mathcal{G}(u))_x||_\infty \leq D(||u||_\infty)\left(|||\alpha| * |u_x|||_\infty + ||\alpha||_1|u_x||_\infty\right),$$

and by Lemma 1.5.9, we get

$$\leq 2D(||u||_\infty)||\alpha||_1||u_x||_\infty. \tag{3.19}$$

Hence, summing up 3.10 and 3.19 gives

$$||(\mathcal{G}(u))_x||_X \leq 2D(R)||\alpha||_1|u_x||_X,$$

and $\mathcal{G} : X \to X$. Next, we show Lipschitz continuity of $\mathcal{G}$. Take $u, v \in B_X(0, R)$.

$$\begin{aligned} |(\mathcal{G}(u))_x - (\mathcal{G}(v))_x| = &\left|\int_{\mathbb{R}} \alpha(y-x)w'(u(y)-u(x))(u_x(y)-u_x(x))dy\right. \\ &\left.- \int_{\mathbb{R}} \alpha(y-x)w'(v(y)-v(x))(v_x(y)-v_x(x))dy\right|. \end{aligned} \tag{3.20}$$

Let $u(y) - u(x) = \eta_1,\ v(y) - v(x) = \eta_2$, and $u_x(y) - u_x(x) = \mu_1,\ v_x(y) - v_x(x) = \mu_2$. Then $|\eta_1| \leq 2||u||_\infty \leq 2R$ and $|\mu_1| \leq 2||u_x||_\infty \leq 2||u||_\infty + 2||u_x||_\infty \leq 2R$. Similar inequalities hold for η_2 and μ_2, respectively. Thus, we have

$$
\begin{aligned}
|w'(\eta_1)\mu_1 - w'(\eta_2)\mu_2| &\leq |w'(\eta_1)\mu_1 - w'(\eta_1)\mu_2 + w'(\eta_1)\mu_2 - w'(\eta_2)\mu_2| \\
&\leq |w'(\eta_1)||\mu_1 - \mu_2| + |w'(\eta_1) - w'(\eta_2)||\mu_2| \\
&\leq (\underset{|\eta|\leq 2||u||_\infty}{max}|w'(\eta)|)|\mu_1 - \mu_2| + 2R(\underset{|\eta|\leq 2||u||_\infty}{max}|w''(\eta)|)|\eta_1 - \eta_2| \\
&\leq D(||u||_\infty)|\mu_1 - \mu_2| + 2RE(||u||_\infty)|\eta_1 - \eta_2| \qquad (3.21)
\end{aligned}
$$

where $E(R) = \underset{|\eta|\leq 2R}{max}|w''(\eta)|$. By plugging (3.21) in (3.20), we obtain

$$
\begin{aligned}
|(\mathcal{G}(u))_x(x) - (\mathcal{G}(v))_x(x)| \leq & D(||u||_\infty)\int_{\mathbb{R}} |\alpha(y-x)||u_x(y) - u_x(x) - v_x(y) + v_x(x)|dy \\
& + 2RE(||u||_\infty)\int_{\mathbb{R}} |\alpha(y-x)||u(y) - u(x) - v(y) + v(x)|dy \\
\leq & D(||u||_\infty)\int_{\mathbb{R}} |\alpha(y-x)|(|u_x(y) - v_x(y)| + |u_x(x) - v_x(x)|)dy \\
& + 2RE(||u||_\infty)\int_{\mathbb{R}} |\alpha(y-x)|(|u(y) - v(y)| + |u(x) - v(x)|)dy \\
= & D(||u||_\infty)(|\alpha| * |u_x - v_x|)(x)) + D(||u||_\infty)||\alpha||_1|(u_x - v_x)(x)| \\
& + 2RE(||u||_\infty)(|\alpha| * |u - v|)(x)) + 2RE(||u||_\infty)||\alpha||_1|(u - v)(x)|. \qquad (3.22)
\end{aligned}
$$

Taking supnorm of (3.22) yields

$$
\begin{aligned}
||(\mathcal{G}(u))_x - (\mathcal{G}(v))_x||_\infty \leq & 2D(R)||\alpha||_1||u_x - v_x||_\infty + 4RE(R)||\alpha||_1||u - v||_\infty \\
\leq & 2D(R)||\alpha||_1||u_x - v_x||_\infty + 4RE(R)||\alpha||_1||u - v||_\infty \\
& + 4RE(R)||\alpha||_1||u_x - v_x||_\infty + 2D(R)||\alpha||_1||u - v||_\infty \\
= & (2D(R) + 4RE(R))||u - v||_X \qquad (3.23)
\end{aligned}
$$

Besides this from (3.12) we have

$$
\begin{aligned}
||\mathcal{G}(u) - \mathcal{G}(v)||_\infty &\leq 2D(R)||\alpha||_1||u - v||_\infty \\
&\leq 2D(R)||\alpha||_1||u - v||_\infty + 2D(R)||\alpha||_1||u_x - v_x||_\infty \\
&= 2D(R)||\alpha||_1||u - v||_X \qquad (3.24)
\end{aligned}
$$

Summing up (3.23) and (3.24) gives

$$||\mathcal{G}(u) - \mathcal{G}(v)||_X \leq 4(D(R) + RE(R))||\alpha||_1||u - v||_X.$$

Therefore, $\mathcal{G}$ becomes Lipschitz with Lipschitz constant $L_R = 4(D(R) + RE(R))||\alpha||_1$.

Calculations above show that $\mathcal{G}$ fulfills the assumptions of Theorem 2.2.1. Thus, we can conclude that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], C_b^1(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b^1(\mathbb{R})$. □

Theorem 3.2.4 *Let $1 \leq p \leq \infty$. $\alpha \in L^1(\mathbb{R})$ and $w \in C^2(\mathbb{R})$ with $w(0) = 0$. Then there is some $T > 0$ such that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], H^1(\mathbb{R}))$ for initial data $\varphi, \psi \in H^1(\mathbb{R})$.*

Proof: Let $X = H^1(\mathbb{R})$. Then $||u||_{H^1} = ||u||_{L^2} + ||u'||_{L^2}$. In this proof, we will mostly use results obtained from Theorem 3.2.3. Take $u, v \in \bar{B}_X(0,R)$. Then, we know from (3.9) and (3.18) where $||u||_\infty$ replaced by $||u||_{L^\infty}$ that

$$|\mathcal{G}(u)(x)| \leq D(||u||_{L^\infty}) \left[(|\alpha| * |u|)(x) + ||\alpha||_1 |u(x)|\right] \tag{3.25}$$

and

$$|(\mathcal{G}(u))_x(x)| \leq D(||u||_{L^\infty}) \left[(|\alpha| * |u_x|)(x) + ||\alpha||_1 |u_x(x)|\right]. \tag{3.26}$$

However, we have $||u||_{L^\infty} \leq C||u||_{H^1} \leq CR$ due to Lemma 1.3.3. Definition of the non-increasing function D implies that $D(||u||_{L^\infty}) \leq D(CR)$. Thus, by Lemma 1.5.9, L^2 norms of 3.25 and 3.26 can be estimated as follows

$$||\mathcal{G}(u)||_{L^2} \leq 2D(CR)||\alpha||_1 ||u(x)||_{L^2} \tag{3.27}$$

and

$$||(\mathcal{G}(u))_x||_{L^2} \leq 2D(CR)||\alpha||_1 ||u_x||_{L^2}. \tag{3.28}$$

(3.27) and (3.28) both imply that

$$||(\mathcal{G}(u))_x||_{H^1} \leq 2D(CR)||\alpha||_1 ||u_x||_{H^1}.$$

Hence $\mathcal{G} : X \to X$. Similarly, from (3.11) and (3.22) where $||u||_{L^\infty}$ is replaced by $||u||_{L^\infty}$, we have

$$|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| \leq D(||u||_{L^\infty})(|\alpha| * |u - v|)(x) + ||\alpha||_1 |u(x) - v(x)|. \tag{3.29}$$

and

$$\begin{aligned} |(\mathcal{G}(u))_x(x) - (\mathcal{G}(v))_x(x)| \leq & D(||u||_{L^\infty})(|\alpha| * |u_x - v_x|)(x)) + D(||u||_{L^\infty})||\alpha||_1 |(u_x - v_x)(x)| \\ & + 2RE(||u||_{L^\infty})(|\alpha| * |u - v|)(x)) + 2RE(||u||_{L^\infty})||\alpha||_1 |(u - v)(x)|. \end{aligned} \tag{3.30}$$

L^2 norms of (3.29) and (3.30) can be found as

$$\begin{aligned} ||\mathcal{G}(u) - \mathcal{G}(v)||_{L^2} &\leq 2D(CR)||\alpha||_1 ||u - v||_{L^2} \\ &\leq 2D(CR)||\alpha||_1 ||u - v||_{H^1} \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
||(\mathcal{G}(u))_x - (\mathcal{G}(v))_x||_{L^2} \leq & 2D(CR)||\alpha||_1||u_x - v_x||_{L^2} \\
& + 4RE(CR)||\alpha||_1||u - v||_{L^2} \\
& \leq (2D(CR) + 4RE(CR))||u - v||_{H^1}.
\end{aligned} \tag{3.32}
$$

Summing up (3.31) and (3.32) gives

$$
\begin{aligned}
||\mathcal{G}(u) - \mathcal{G}(v)||_{H^1} & \leq 4D(CR)||\alpha||_1||u_x - v_x||_{L^2} + 4RE(CR)||\alpha||_1||u - v||_{L^2} \\
& \leq 4(D(CR) + RE(CR))||\alpha||_1||u - v||_{H^1}.
\end{aligned}
$$

Thus, $\mathcal{G}$ is Lipschitz with Lipschitz constant $L_R = 4(D(CR) + RE(CR))$.

Calculations above show that assumptions of Theorem 2.2.1 are fulfilled. Thus, we can conclude that the Cauchy problem (3.2)-(3.3) is well-posed with solution in $C^2([0,T], H^1(\mathbb{R}))$ for initial data $\varphi, \psi \in H^1(\mathbb{R})$. □

The above theorems of local well-posedness can be easily adapted to the general peridynamic equation. The next theorem is the extension of Theorem 3.2.1 to the general peridynamic equation

$$
u_{tt} = \int_{\mathbb{R}} f(y - x, u(y) - u(x))dy. \tag{3.33}
$$

Theorem 3.2.5 *Assume that $f(\xi, 0) = 0$ and $f(\xi, \eta)$ is continuously differentiable in η for almost all ξ. Moreover, suppose that for each $R > 0$, there are integrable functions Λ_1^R, Λ_2^R satisfying*

$$
\left|f(\xi, \eta)\right| \leq \Lambda_1^R(\xi), \quad \left|f_\eta(\xi, \eta)\right| \leq \Lambda_2^R(\xi) \tag{3.34}
$$

for almost all ξ and for all $|\eta| \leq 2R$. Then there is some $T > 0$ such that the Cauchy problem (3.33)-(3.3) is well-posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$.

Proof: We first show $\mathcal{G}(u)$ is continuous in x. Take $u \in \bar{B}_X(0, R)$. Let $\{x_n\}$ be a Cauchy sequence with $\lim_{n\to\infty} x_n = x$. Consider the following integral

$$
\mathcal{G}(u)(x) = \int_{\mathbb{R}} f(y - x, u(y) - u(x))dy.
$$

As we don't know whether f is continuous in its first argument or not, we again make a suitable substitution like $y = x + z$. Then we have

$$
\mathcal{G}(u)(x) = \int_{\mathbb{R}} f(z, u(x + z) - u(x))dz.
$$

Then we consider

$$
\mathcal{G}(u)(x_n) = \int_{\mathbb{R}} f(z, u(x_n + z) - u(x_n))dz.
$$

Call $h_n(z) := f(z, u(x_n + z) - u(x_n))$. Then $\lim_{n\to\infty} h_n(z) = f(z, u(x+z) - u(x))$ since $u \in C_b(\mathbb{R})$ and f is continuous in its second argument. As $|u(x_n + z) - u(x_n)| \leq 2||u||_\infty \leq 2R$, (3.34) implies that $|h_n(\xi)| \leq \Lambda_1^R(\xi)$ for almost all ξ. By Dominated Convergence Theorem, we can write

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{G}(u)(x_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} f(z, u(x_n + z) - u(x_n))dz \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} h_n(z)dz \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} h_n(z)dz \\
&= \int_{\mathbb{R}} f(z, u(x+z) - u(x))dz \\
&= \int_{\mathbb{R}} f(y - x, u(y) - u(x))dy \\
&= \mathcal{G}(u)(x).
\end{aligned}$$

Hence $\mathcal{G}(u)$ is continuous in x. Now we show that $\mathcal{G}(u)$ is Lipschitz continuous. Take $u, v \in \bar{B}_X(0, R)$. Then,

$$\begin{aligned}
|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| &\leq \int_{\mathbb{R}} |f(y - x, u(y) - u(x)) - f(y - x, v(y) - v(x))|dy \\
&\leq \int_{\mathbb{R}} \sup_{|\eta|\leq 2R} |f_\eta(y - x, \eta)||u(y) - u(x) - v(y) + v(x)|dy
\end{aligned}$$

as $|u(y) - u(x)| \leq 2R$, and $|v(y) - v(x)| \leq 2R$. So, we continue as

$$\begin{aligned}
&\leq \int_{\mathbb{R}} \sup_{|\eta|\leq 2R} |f_\eta(y - x, \eta)|(|u(y) - v(y)| + |u(x) - v(x)|)dy \\
&\leq \int_{\mathbb{R}} \Lambda_2^R(y - x)|u(y) - v(y)|dy + \int_{\mathbb{R}} \Lambda_2^R(y - x)|u(x) - v(x)|dy \\
&\leq (\Lambda_2^R * |u - v|)(x) + ||\Lambda_2^R||_1 |u(x) - v(x)|. \qquad (3.35)
\end{aligned}$$

Taking supremum of (3.35) gives

$$||\mathcal{G}(u) - \mathcal{G}(v)||_\infty \leq 2||\Lambda_2^R||_1 ||u - v||_\infty$$

and $\mathcal{G}$ is Lipschitz with $L_R = 2||\Lambda_2^R||_1$.

We showed that $\mathcal{G}$ fulfills the assumptions of Theorem 2.2.1. Thus, we can conclude that the Cauchy problem (3.33)-(3.3) is well-posed with solution in $C^2([0, T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$. □

Remark 3.2.1 *[3] The calculations above show that similar extensions can be done for theorems 3.2.2 and 3.2.3.*

3.3. General Form

In this part, we consider the general peridynamic model given in [4]:

$$u_{tt}(x,t) = \int_{\mathcal{H}(x)} f(y-x, u(y,t)-u(x,t))\,dy + b(x,t), \quad x \in \bar{\Omega}, \quad t > 0, \tag{3.36}$$

with the initial data

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{3.37}$$

where we additionally assumed that the density of the body 1. It is more general than (3.33) as the domain of integration is a ball in $\mathbb{R}^d$ and there are some external forces.

We will study the local well-posedness of (3.36)-(3.37) with initial data spaces $C(\bar{\Omega}^d)$, $L^\infty(\Omega)^d$, $L^\infty(\Omega)^d \cap L^p(\Omega)^d$ and $L^p(\Omega)^d$, respectively. But, this time we have

$$\mathcal{G}(u)(x) = \int_{\mathcal{H}(x)} f(y-x, u(y)-u(x))\,dy \tag{3.38}$$

and hence our main goal will be to identify the right conditions so that the assumptions of Theorem 2.2.3 are satisfied. We begin with giving a definition:

Definition 3.3.1 *Let $\bar{B}_1, \bar{B}_2 \subseteq \mathbb{R}^d$. A continuous function*

$$f : \bar{B}_1 \times \bar{B}_2 \to \mathbb{R}^d$$

is said to be Lipschitz continuous in its second argument, if there exists a nonnegative function $L_f \in L^1(B_1)$ such that for all $\xi \in \bar{B}_1$ and all $\eta_1, \eta_2 \in \bar{B}_2$ there holds

$$|f(\xi, \eta_1) - f(\xi, \eta_2)| \leq L_f(\xi)|\eta_1 - \eta_2|.$$

Theorem 3.3.6 *Let $X = C(\bar{\Omega})^d$. Suppose that for some $R > 0$, the pairwise force function*

$$f : \bar{B}_{\mathbb{R}^d}(0,\delta) \times \bar{B}_{\mathbb{R}^d}(0,2R) \to \mathbb{R}^d$$

is continuous and Lipschitz continuous in its second argument. Moreover, if $b \in C([0,\tilde{T}], C(\bar{\Omega})^d)$, then there is some $0 < T \leq \tilde{T}$ such that the Cauchy problem (3.36)-(3.37) *is well-posed with solution in $C^2([0,T], C(\bar{\Omega})^d)$ for initial data $\varphi, \psi \in C(\bar{\Omega})^d$.*

Proof: In order to show that $\mathcal{G}$ is well-defined, we show that $\mathcal{G}(u)$ is continuous for any $u \in \bar{B}_X(0,R)$. Let $x_1, x_2 \in \bar{\Omega}$ and $\varepsilon > 0$ be given. Then, we have

$$\begin{aligned}|\mathcal{G}(u)(x_1) - \mathcal{G}(u)(x_2)| \leq &\bigg| \int_{\mathcal{H}(x_1)} f(y - x_1, u(y) - u(x_1))dy \\ &- \int_{\mathcal{H}(x_2)} f(y - x_2, u(y) - u(x_2))dy \bigg| \\ \leq &\, I_1 + I_2 + I_3\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{\mathcal{H}(x_1)\cap\mathcal{H}(x_2)} |f(y - x_1, u(y) - u(x_1)) - f(y - x_2, u(y) - u(x_2))|dy, \\ I_2 &= \int_{\mathcal{H}(x_1)\setminus\mathcal{H}(x_2)} |f((y - x_1, u(y) - u(x_1))|dy, \\ I_3 &= \int_{\mathcal{H}(x_2)\setminus\mathcal{H}(x_1)} |f(y - x_2, u(y) - u(x_2))|dy.\end{aligned}$$

We are given that f is continuous on the closed ball $\bar{B}_{\mathbb{R}^d}(0,\delta) \times \bar{B}_{\mathbb{R}^d}(0,2R)$. This means that f is uniformly continuous on $\bar{B}_{\mathbb{R}^d}(0,\delta) \times \bar{B}_{\mathbb{R}^d}(0,2R)$. Moreover, there exists $M > 0$ such that $|f(\xi,\eta)| \leq M$ for every $(\xi,\eta) \in \bar{B}_{\mathbb{R}^d}(0,\delta) \times \bar{B}_{\mathbb{R}^d}(0,2R)$. Now, we let

$$\xi_1 = y - x_1, \quad \xi_2 = y - x_2, \quad \eta_1 = u(y) - u(x_1), \quad \text{and} \quad \eta_2 = u(y) - u(x_2). \tag{3.39}$$

We first estimate I_1: $y \in \mathcal{H}(x_1) \cap \mathcal{H}(x_2)$. Recall that $u \in \bar{B}_{\mathbb{R}^d}(0,2R)$. Hence, we have

$$|\xi_1| = |y - x_1| \leq \delta, \quad |\eta_1| = |u(y) - u(x_1)| \leq 2||u||_\infty \leq 2R, \tag{3.40}$$

and

$$|\xi_2| = |y - x_2| \leq \delta, \quad |\eta_2| = |u(y) - u(x_2)| \leq 2||u||_\infty \leq 2R. \tag{3.41}$$

Moreover, from (3.39), we have $\xi_1 - \xi_2 = x_2 - x_1$, and $\eta_1 - \eta_2 = u(x_2) - u(x_1)$. Uniform continuity of f implies that for every $\tilde{\varepsilon}_1 > 0$, there exists $\tilde{\delta} > 0$ such that $|f(\xi_1,\eta_1) - f(\xi_2,\eta_2)| < \tilde{\varepsilon}_1$ whenever $|(\xi_1,\eta_1) - (\xi_2,\eta_2)| < \tilde{\delta}$. We want $|\xi_1 - \xi_2| = |x_2 - x_1| < \frac{\tilde{\delta}}{2}$, and $|\eta_1 - \eta_2| = |u(x_2) - u(x_1)| < \frac{\tilde{\delta}}{2}$. But u is also uniformly continuous. Then, there exists $\tilde{\delta}_1 > 0$ such that $|u(x_2) - u(x_2)| < \frac{\tilde{\delta}}{2}$ whenever $|x_2 - x_1| < \tilde{\delta}_1$. Choose $|x_2 - x_1| < \min\{\frac{\tilde{\delta}}{2}, \tilde{\delta}_1\} = \delta_1$. Then $|f(\xi_1,\eta_1) - f(\xi_2,\eta_2)| < \tilde{\varepsilon}_1$ and we can estimate I_1:

$$\begin{aligned}I_1 &\leq \int_{\mathcal{H}(x_1)\cap\mathcal{H}(x_2)} |f(\xi_1,\eta_1) - f(\xi_2,\eta_2)|dy \\ &\leq \tilde{\varepsilon}_1 \operatorname{vol}(\mathcal{H}(x_1) \cap \mathcal{H}(x_2)).\end{aligned}$$

Now, we estimate I_2: $y \in \mathcal{H}(x_1) \setminus \mathcal{H}(x_2)$. Due to (3.40), f is bounded by M on $\mathcal{H}(x_1) \setminus \mathcal{H}(x_2)$. For $\tilde{\varepsilon}_2 > 0$, there exists $\tilde{\delta}_2 > 0$ such that $\operatorname{vol}(\mathcal{H}(x_1) \setminus \mathcal{H}(x_2)) < \tilde{\varepsilon}_2$ whenever $|x_2 - x_1| < \tilde{\delta}_2$. Now, choose $|x_2 - x_1| < \min\{\delta_1, \tilde{\delta}_2\} = \delta_2$. Then

$$I_2 = \int_{\mathcal{H}(x_1)\setminus\mathcal{H}(x_2)} |f((\xi_1,\eta_1)|dy \leq M \operatorname{vol}(\mathcal{H}(x_1) \setminus \mathcal{H}(x_2)) < M\tilde{\varepsilon}_2.$$

Similarly, f is bounded by M on $\mathcal{H}(x_2) \setminus \mathcal{H}(x_1)$ because of (3.41). For $\tilde{\varepsilon}_3 > 0$ there exists $\tilde{\delta}_3 > 0$ such that $\mathrm{vol}(\mathcal{H}(x_2) \setminus \mathcal{H}(x_1)) < \tilde{\varepsilon}_3$ whenever $|x_2 - x_1| < \tilde{\delta}_3$. Now, choose $|x_2 - x_1| < \min\{\delta_2, \tilde{\delta}_3\} = \delta_3$.

$$I_3 = \int_{\mathcal{H}(x_2)\setminus\mathcal{H}(x_1)} |f((\xi_2, \eta_2)|dy \leq \; M\,\mathrm{vol}(\mathcal{H}(x_2) \setminus \mathcal{H}(x_1)) < M\tilde{\varepsilon}_3.$$

If we choose $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3$ small enough so that ,

$$\mathrm{vol}(\mathcal{H}(x_1) \cap \mathcal{H}(x_2))\tilde{\varepsilon}_1 + M\tilde{\varepsilon}_2 + M\tilde{\varepsilon}_3 < \varepsilon,$$

we will obtain

$$|\mathcal{G}(u)(x_1) - \mathcal{G}(u)(x_2)| < \varepsilon$$

whenever $|x_1 - x_2| < \delta_3$. Hence, $\mathcal{G}(u)$ is continuous in x. Next, we show that $\mathcal{G}$ is Lipschitz continuous. Let $u, v \in \bar{B}(0, R)$. Then,

$$|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| \leq \int_{\mathcal{H}(x)} |f(y - x, u(y) - u(x))dy - f(y - x, v(y) - v(x)|)dy,$$

but $|u(y)-u(x)| \leq 2||u||_\infty \leq 2R$ and $|u(y)-u(x)| \leq 2||u||_\infty \leq 2R$. Lipschitz continuity of f implies that there exists $L_f \in L^1(\bar{B}_{\mathbb{R}^d}(0, \delta))$ such that

$$\begin{aligned}
&\leq \int_{\mathcal{H}(x)} L_f(y - x)|u(y) - u(x) - v(y) + v(x))dy \\
&\leq \int_{\mathcal{H}(x)} L_f(y - x)(|u(y) - v(y)| + |u(x) - v(x)|)dy \\
&\leq \int_{\Omega} \chi_\delta(|y - x|)L_f(y - x)(|u(y) - v(y)| + |u(x) - v(x)|)dy \\
&\leq \int_{\Omega} \chi_\delta(|y - x|)L_f(y - x)(|u(y) - v(y)|dy \\
&\quad + |u(x) - v(x)| \int_{\Omega} \chi_\delta(|y - x|)L_f(y - x)dy \\
&\leq |(\chi_\delta L_f * |u - v|)(x)| + |u(x) - v(x)|||\chi_\delta L_f||_{L^1} \qquad (3.42)
\end{aligned}$$

After taking supremum over x we use Lemma 1.5.9 to obtain

$$\begin{aligned}
||\mathcal{G}(u) - \mathcal{G}(v)||_\infty &\leq 2||\chi_\delta L_f||_{L^1}||u - v||_\infty \\
&= 2||L_f||_{L^1((B_{\mathbb{R}^d}(0,\delta))}||u - v||_\infty. \qquad (3.43)
\end{aligned}$$

Thus, $\mathcal{G} : C(\bar{\Omega})^d \to C(\bar{\Omega})^d$ is locally Lipschitz continuous with Lipschitz constant $L_R = 2||L_f||_{L^1((B_{\mathbb{R}^d}(0,\delta))}$. Besides this b is given to be in $C(\bar{\Omega})^d$). Hence, we have shown that the requirements of Theorem 2.2.3 are fulfilled. Thus, we can conclude that the Cauchy problem (3.36)-(3.37) is well-posed with solution in $C^2([0, T], C(\bar{\Omega})^d)$ for initial data $\varphi, \psi \in C(\bar{\Omega})^d$. □

Definition 3.3.2 *Let $\bar{B}_1, \bar{B}_2 \subseteq \mathbb{R}^d$. A function*

$$f : B_1 \times \bar{B}_2 \to \mathbb{R}^d$$

that is Lebesgue measurable in its first argument is said to be Lipschitz continuous in its second argument, if there exists a nonnegative function $L_f \in L^1(B_1)$ such that for almost all $\xi \in B_1$ and all $\eta_1, \eta_2 \in \bar{B}_2$ there holds

$$|f(\xi, \eta_1) - f(\xi, \eta_2)| \leq L_f(\xi)|\eta_1 - \eta_2|.$$

Theorem 3.3.7 *Suppose there is some $R > 0$ such that the pairwise force function*

$$f : B_{\mathbb{R}^d}(0, \delta) \times \bar{B}_{\mathbb{R}^d}(0, 2R) \to \mathbb{R}^d$$

is Lebesgue measurable in its first argument and Lipschitz continuous in its second argument. If $b \in C([0, \tilde{T}], L^\infty(\Omega)^d)$ and $f(., 0) \in L^1((B_{\mathbb{R}^d}(0, \delta))^d$, then there is some $0 < T \leq \tilde{T}$ such that the Cauchy problem (3.36)-(3.37) is well-posed with solution in $C^2([0, T], L^\infty(\Omega)^d)$ for initial data $\varphi, \psi \in L^\infty(\Omega)^d$.

Proof: Take $u, v \in \bar{B}_X(0, R)$. We know that $|u(y) - u(x)| \leq 2||u||_{L^\infty} \leq 2R$ and $|u(y) - u(x)| \leq 2||u||_{L^\infty} \leq 2R$. Also, we are given that f is Lipschitz continuous in its second argument. Then, there exists a nonnegative function $L_f \in L^1(B_{\mathbb{R}^d}(0, \delta))$. Hence, just replacing the uniform norm by L^∞ norm, we can follow the same steps in Theorem 3.3.6 and show that $\mathcal{G} : L^\infty(\Omega)^d) \to L^\infty(\Omega)^d)$ Lipschitz continuous:

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^\infty} \leq L_R||u - v||_{L^\infty}$$

with $L_R = 2||L_f||_{L^1((\bar{B}_{\mathbb{R}^d}(0,\delta))}$. $\mathcal{G}$ is also well-defined. Because Lipschitz continuity of $\mathcal{G}$ implies that

$$||\mathcal{G}(u)||_{L^\infty} \leq ||\mathcal{G}(u) - \mathcal{G}(0)||_{L^\infty} + ||\mathcal{G}(0)||_{L^\infty} \leq L_R||u||_{L^\infty} + ||f(., 0)||_{L^1(B_{\mathbb{R}^d}(0,\delta))^d}.$$

where

$$|\mathcal{G}(0)(x)| \leq \int_{\mathcal{H}(x)} |f(y - x, 0|dy \leq ||f(., 0)||_{L^1(B_{\mathbb{R}^d}(0,\delta))^d} \tag{3.44}$$

and

$$||\mathcal{G}(0)||_\infty \leq ||f(., 0)||_{L^1(B_{\mathbb{R}^d}(0,\delta))^d}. \tag{3.45}$$

Since we also have $b \in C([0, \tilde{T}], L^\infty(\Omega)^d)$. Thus, the requirements of Theorem 2.2.3 are fulfilled. As a conclusion, there is some $T \leq \tilde{T}$ such that the Cauchy problem (3.36)-(3.37) is well-posed with solution in $C^2([0, T], L^\infty(\Omega)^d)$ for initial data $\varphi, \psi \in L^\infty(\Omega)^d$.
□

Theorem 3.3.8 *Suppose there is some $R > 0$ such that the pairwise force function*

$$f : B_{\mathbb{R}^d}(0,\delta) \times \bar{B}_{\mathbb{R}^d}(0,2R) \to \mathbb{R}^d$$

is Lebesgue measurable in its first argument and Lipschitz continuous in its second argument. If $b \in C([0,\tilde{T}], L^\infty(\Omega)^d \cap L^p(\Omega)^d)$ and $f(.,0) \in L^1(B_{\mathbb{R}^d}(0,\delta))^d$, then there is some $0 < T \leq \tilde{T}$ such that the Cauchy problem (3.36)-(3.37) is well-posed with solution in $C^2([0,T], L^\infty(\Omega)^d) \cap L^p(\Omega)^d$ for initial data $\varphi, \psi \in L^\infty(\Omega)^d \cap L^p(\Omega)^d$.

Proof: Let $X = L^\infty(\Omega)^d \cap L^p(\Omega)^d$. Take $u, v \in \bar{B}_X(0,R)$. Then, we know that $|u(y) - u(x)| \leq 2||u||_{L^\infty} \leq ||u-v||_X \leq 2R$ and $|u(y) - u(x)| \leq 2||u||_{L^\infty} \leq ||u-v||_X \leq 2R$. Also, we are given that f is Lipschitz continuous in its second argument. Then, there exists a nonnegative function $L_f \in L^1(B_{\mathbb{R}^d}(0,\delta))$ such that inequality (3.42) is satisfied. Also, L^∞ and L^p norms of (3.42) can be calculated as

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^\infty} \leq 2||L_f||_{L^1(B_{\mathbb{R}^d}(0,\delta))}||u-v||_{L^\infty}, \tag{3.46}$$

and

$$||\mathcal{G}(u) - \mathcal{G}(v)||_{L^p} \leq 2||L_f||_{L^1(B_{\mathbb{R}^d}(0,\delta))}||u-v||_{L^p}, \tag{3.47}$$

respectively. Summing up (3.46) and (3.47) yields

$$||\mathcal{G}(u) - \mathcal{G}(v)||_X \leq 2||L_f||_{L^1(B_{\mathbb{R}^d}(0,\delta))}||u-v||_X. \tag{3.48}$$

Hence $\mathcal{G} : X \to X$ is locally Lipschitz continuous. On the other hand, we can also show that that $\mathcal{G}$ is well-defined. As $\mathcal{G}$ is Lipschitz, from (3.45), we already know that

$$||\mathcal{G}(0)||_{L^\infty} \leq ||f(.,0)||_{L^1(B_{\mathbb{R}^d}(0,\delta))^d}$$

From (3.44), we can obtain

$$||\mathcal{G}(0)||_{L^p} \leq (\text{vol}(\bar{\Omega}))^{1/p}||f(.,0)||_{L^1(B_{\mathbb{R}^d}(0,\delta)^d)}. \tag{3.49}$$

Therefore, from (3.45) and (3.49) we obtain

$$||\mathcal{G}(0)||_X \leq (1 + (\text{vol}(\bar{\Omega}))^{1/p})||f(.,0)||_{L^1(B_{\mathbb{R}^d}(0,\delta)^d)}. \tag{3.50}$$

Therefore,

$$\begin{aligned} ||\mathcal{G}(u)||_X &\leq ||\mathcal{G}(u) - \mathcal{G}(0)||_X + ||\mathcal{G}(0)||_X \\ &\leq L_R||u-v||_X + (1 + (\text{vol}(\bar{\Omega}))^{1/p})||f(.,0)||_{L^1(B_{\mathbb{R}^d}(0,\delta))^d} \end{aligned}$$

and $\mathcal{G} : X \to X$ is well-defined. Moreover, we have $b \in C([0,\tilde{T}], L^\infty(\Omega)^d \cap L^p(\Omega)^d)$ and all the requirements of Theorem 2.2.3 are satisfied. In conclusion, there is some $\tilde{T} > 0$ such that the Cauchy problem (3.36)-(3.37) is well-posed with solution in $C^2([0,T], L^\infty(\Omega)^d) \cap L^p(\Omega)^d$ for initial data $\varphi, \psi \in L^\infty(\Omega)^d \cap L^p(\Omega)^d$. □

Remark 3.3.2 *Instead of* $C(\bar{\Omega})$, $L^\infty(\Omega)^d$ *or* $L^\infty(\Omega)^d \cap L^p(\Omega)^d$, *if we had taken* $L^p(\Omega)^d$ *as a function space, we wouldn't have been able to deduce Lipschitz continuity of* $\mathcal{G}$ *on* $\bar{B}_X(0,R)$ *from local Lipschitz continuity of* f *in the second argument. This is because,* L^p *functions need not to be bounded. Therefore, in* $L^p(\Omega)^d$ *space, to overcome this difficulty, we should consider measurable pairwise force functions that are globally Lipschitz continuous in their second arguments:*

Definition 3.3.3 *Let* $B \subseteq \mathbb{R}^d$. *A function*

$$f : B \times \mathbb{R}^d \to \mathbb{R}^d$$

that is Lebesgue measurable in its first argument is said to be Lipschitz continuous in its second argument, if there exists a nonnegative even function $L_f \in L^1(B)$ *such that for almost all* $\xi \in B$ *and all* $\eta_1, \eta_2 \in \mathbb{R}^d$ *there holds*

$$|f(\xi,\eta_1) - f(\xi,\eta_2)| \leq L_f(\xi)|\eta_1 - \eta_2|.$$

Theorem 3.3.9 *Suppose that the pairwise force function*

$$f : B_{\mathbb{R}^d}(0,\delta) \times \mathbb{R}^d \to \mathbb{R}^d$$

is Lebesgue measurable in its first argument and Lipschitz continuous in its second argument. If $b \in C([0,T], L^p(\Omega)^d)$, *and* $f(.,0) \in L^1((B_{\mathbb{R}^d}(0,\delta))^d)$, *then there is some* $T \leq \tilde{T}$ *such that the Cauchy problem* (3.2)-(3.3) *has a unique global solution in* $C^2([0,T], L^p(\Omega)^d)$ *which depends continuously on the initial data* $\varphi, \psi \in L^p(\Omega)^d$.

Proof: Let $X = L^p(\Omega)^d$ and take $u, v \in \bar{B}_X(0,R)$. Note that $\mathcal{G}(u)$ is measurable. We first show that $\mathcal{G} : X \to X$ is well-defined. Once we show the Lipschitz continuity of $\mathcal{G}$ for $1 \leq p \leq \infty$, we will deduce that

$$||\mathcal{G}(u)||_{L^p} \leq ||\mathcal{G}(u) - \mathcal{G}(0)||_{L^p} + ||\mathcal{G}(0)||_{L^p}.$$

However, we know from (3.45) and (3.49) that $||\mathcal{G}(0)||_{L^p}$ is bounded. To show the local Lipschitz continuity of $\mathcal{G}$, we cannot use the fact imposed on u, v of being a member of L^∞ space, namely $|u(x)|, |v(x)| \leq R$ while estimating the integral

$$|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| \leq \int_{\mathcal{H}(x)} |f(y-x, u(y)-u(x))dy - f(y-x, v(y)-v(x)|)dy.$$

However, f is given to be uniformly Lipschitz continuous on its second argument. Hence, for almost all $\xi \in B(0,\delta)$ and all $\eta_1, \eta_2 \in \mathbb{R}^d$, and in particular for $y-x \in B(0,\delta)$, and $u(y)-u(x)$, $v(y)-v(x) \in \mathbb{R}^d$ there holds

$$|\mathcal{G}(u)(x) - \mathcal{G}(v)(x)| \leq |(\chi_\delta L_f * |u-v|)(x)| + |u(x)-v(x)| ||\chi_\delta L_f||_{L^1}.$$

Hence, the estimations (3.46) and (3.47) obtained in Theorem 3.3.8 remain valid. But in this case $\mathcal{G} : X \to X$ becomes globally Lipschitz continuous. Apart from this, we are given that $b \in C([0,T], L^p(\Omega)^d)$. Hence, the assumptions of Theorem 2.2.1 are satisfied. Therefore, the Cauchy problem (3.36)-(3.37) has a local solution in $C^2([0,T], L^p(\Omega)^d)$ for $\varphi, \psi \in L^p(\Omega)^d$. Moreover, the solution depends continuously on the initial data. □

CHAPTER 4

Linear Peridynamic Model

4.1. Introduction

In this chapter, we will study peridynamic equation where the pairwise force function is linear, i.e

$$f(u(y,t) - u(x,t), y - x) = \mathbf{C}(y - x)u(y,t) - u(x,t)) \tag{4.1}$$

where $\mathbf{C}(x,y) = \mathbf{C}(y,x)$ is the stiffnes tensor given by

$$\mathbf{C}(y - x) = c_\delta \varsigma(|y - x|)(y - x) \otimes (y - x) + F_0(|y - x|)\mathbf{I} \tag{4.2}$$

with $\mathbf{I}$ being the identity matrix and $\varsigma = \varsigma(|y - x|)$ being a scalar-valued function. If $F_0(|y - x|) \equiv 0$, the equation models a spring network system [1] which is the case Du and Zhou considered in [5]. In this case, the linear problem will be

$$u_{tt}(t,x) - L_\delta u(t,x) = b(t,x) \quad \forall t \in (0,T), \forall x \in \mathbb{R}^d \tag{4.3}$$

with initial data

$$u(x,0) = \varphi(x) \quad u_t(x,0) = \psi(x) \quad \forall x \in \mathbb{R}^d, \tag{4.4}$$

where

$$L_\delta u(x) = \int_{B_\delta(x)} \frac{(y - x) \otimes (y - x)}{\sigma(|y - x|)}(u(y) - u(x))\, dy. \tag{4.5}$$

Here, $c_\delta > 0$ is a positive normalization constant, and we call $\sigma = \sigma(|y-x|) = \frac{1}{\varsigma(|y-x|)}$ a kernel function of the peridynamic integral operator. Besides this, we use the notation $B_\delta(x)$ instead of $\mathcal{H}(x)$ for $d = 1$.

In this study, we will consider the linear problem in [5] for one dimensional case to deal with simple calculations. Then the operator defined as in (4.5) will become

$$L_\delta u(x) = c_\delta \int_{B_\delta(x)} \frac{|y-x|^2}{\sigma(|y-x|)}(u(y) - u(x))\, dy. \tag{4.6}$$

Let's find the Fourier Transform of (4.6). To simplify the expression in (4.6), we let

$$\alpha(y-x) = \frac{|y-x|^2}{\sigma(|y-x|)}. \tag{4.7}$$

Moreover, we let

$$\alpha_\delta(x) = \begin{cases} \alpha(x), & x \in B_\delta(x) \\ 0, & x \notin B_\delta(x) \end{cases}$$

Then,

$$\begin{aligned} L_\delta u(x) &= c_\delta \left(\int_{\mathbb{R}} \alpha_\delta(y-x)u(y)dy - \int_{\mathbb{R}} \alpha_\delta(y-x)u(x)dy \right) \\ &= c_\delta \left((\alpha_\delta * u)(x) - u(x) \int_{\mathbb{R}} \alpha_\delta(y-x)dy \right) \\ &= c_\delta \left((\alpha_\delta * u)(x) - \hat{\alpha}_\delta(0)u(x) \right). \end{aligned}$$

Hence

$$\begin{aligned} \widehat{(-L_\delta u)}(\xi) &= c_\delta \left(\hat{\alpha}_\delta(0)\hat{u}(\xi) - \hat{\alpha}_\delta(\xi)\hat{u}(\xi) \right) \\ &= c_\delta \left(\hat{\alpha}_\delta(0) - \hat{\alpha}_\delta(\xi) \right) \hat{u}(\xi) = M_\delta(\xi)\hat{u}(\xi) \end{aligned}$$

where

$$\begin{aligned} M_\delta(\xi) &= c_\delta \left(\int_{\mathbb{R}} \alpha_\delta(x)dx - \int_{\mathbb{R}} \alpha_\delta(y)e^{-i\xi y}\, dy \right) \\ &= \int_{B_\delta(0)} \alpha(y)(1 - \cos(\xi y))dy \\ &= c_\delta \int_{B_\delta(0)} \frac{1 - cos(\xi y)}{\sigma(|y|)} |y|^2\, dy \end{aligned} \tag{4.8}$$

for any $\xi \in \mathbb{R}$ and $\delta > 0$. As a conclusion, by performing the Fourier transform, we could introduce an equivalent definition of peridynamic operator

$$-L_\delta u(x) = \frac{c_\delta}{\sqrt{2\pi}} \int_{\mathbb{R}} M_\delta(\xi)\hat{u}(\xi)e^{ix\xi} d\xi. \tag{4.9}$$

Using the Fourier Transform, we first rewrite the problem (4.3)-(4.4) as

$$\hat{u}_{tt}(t,\xi) + M_\delta(\xi)\hat{u}(t,\xi) = \hat{b}(t,\xi), \tag{4.10}$$

with the initial data

$$\hat{u}(0,\xi) = \hat{\varphi}(\xi), \quad \hat{u}_t(0,\xi) = \hat{\psi}(\xi). \tag{4.11}$$

The problem (4.10)-(4.11) can be considered as a non-homogenous ordinary differential equation where ξ is a parameter. Then, the solution of this problem will be of the following form:

$$\hat{u}(t,\xi) = \hat{u}_h(t,\xi) + \hat{u}_p(t,\xi)$$

where

$$\hat{u}_h(t,\xi) = f(\xi)\cos(\sqrt{M_\delta(\xi)}t) + g(\xi)\sin(\sqrt{M_\delta(\xi)}t). \tag{4.12}$$

If we use the initial conditions (4.11), we see that $f(\xi) = \varphi(\xi)$ and $g(\xi) = \frac{\psi(\xi)}{\sqrt{M_\delta(\xi)}}$. Then the solution of the homogenous equation is given by

$$\hat{u}_h(t,\xi) = \hat{\varphi}(\xi)\cos(\sqrt{M_\delta(\xi)}t) + \frac{\sin(\sqrt{M_\delta(\xi)}t)}{\sqrt{M_\delta(\xi)}}\hat{\psi}(\xi). \tag{4.13}$$

Let us denote $\hat{u}_1(t,\xi) = \cos(\sqrt{M_\delta(\xi)}t)$ and $\hat{u}_2(t,\xi) = \sin(\sqrt{M_\delta(\xi)}t)$. To find the particular solution $\hat{u}_p(\xi,t)$ of (4.10), we apply Variation of Parameters Method [9]. In this method, we seek for a particular solution that satisfy both

$$\begin{aligned} \hat{u}_p(t,\xi) &= \hat{u}_1(t,\xi)y_1(t) + \hat{u}_2(t,\xi)y_2(t) \\ (\hat{u}_p)_t(t,\xi) &= (\hat{u}_1)_t(t,\xi)y_1(t) + (\hat{u}_2)_t(t,\xi)y_2(t) \end{aligned} \tag{4.14}$$

where $y_1(t)$ and $y_2(t)$ are variable functions. In this case, the functions y_1' and y_2' will assure

$$\hat{u}_1(t,\xi)y_1'(t) + \hat{u}_2(t,\xi)y_2'(t) = 0 \tag{4.15}$$

not to violate the formula for the derivatives of two functions. On the other hand, the second derivative of the particular solution will be

$$(\hat{u}_p)_{tt}(t,\xi) = (\hat{u}_1)_{tt}(t,\xi)y_1(t) + (\hat{u}_1)_t(t,\xi)y_1'(t) + (\hat{u}_2)_{tt}(t,\xi)y_2(t) + (\hat{u}_2)_t(t,\xi)y_2'(t)$$

from (4.14). But $\hat{u}_p(t,\xi)$ solves the nonhomogenous equation (4.10) whereas $\hat{u}_1(t,\xi)$ and $\hat{u}_2(t,\xi)$ have to satisfy the homogenous equation. Then, we see that

$$\begin{aligned} \hat{b}(t,\xi) =& (\hat{u}_1)_{tt}(t,\xi)y_1(t) + (\hat{u}_1)_t(t,\xi)y_1'(t) + (\hat{u}_2)_{tt}(t,\xi)y_2(t) + (\hat{u}_2)_t(t,\xi)y_2'(t) \\ &+ M_\delta(\xi)\left[\hat{u}_1(t,\xi)y_1(t) + \hat{u}_2(t,\xi)y_2(t)\right] \\ =& \left[(\hat{u}_1)_{tt}(t,\xi) + M_\delta(\xi)\hat{u}_1(t,\xi)\right]y_1(t) + \left[(\hat{u}_2)_{tt}(t,\xi) + M_\delta(\xi)\hat{u}_2(t,\xi)\right]y_2(t) \\ &+ (\hat{u}_1)_t(t,\xi)y_1'(t) + (\hat{u}_2)_t(t,\xi)y_2'(t) \\ =& (\hat{u}_1)_t(t,\xi)y_1'(t) + (\hat{u}_2)_t(t,\xi)y_2'(t). \end{aligned} \tag{4.16}$$

If we solve the system (4.15)-(4.16) for y_1' and y_2', we see that

$$y_1'(t) = -\frac{b(t,\xi)\hat{u}_2(\xi,t)}{W[\hat{u}_1(t,\xi),\hat{u}_2(t,\xi)]}$$
$$y_2'(t) = \frac{b(t,\xi)\hat{u}_1(\xi,t)}{W[\hat{u}_1(t,\xi),\hat{u}_2(t,\xi)]}$$

where

$$W[\hat{u}_1(t,\xi),\hat{u}_2(t,\xi)] = \hat{u}_1(t,\xi)(\hat{u}_2)_t(t,\xi) - ((\hat{u})_1)_t(t,\xi)\hat{u}_2(t,\xi) = \sqrt{M_\delta(\xi)}.$$

Then

$$\begin{aligned}\hat{u}_p(t,\xi) &= \hat{u}_1(t,\xi)y_1(t) + \hat{u}_2(t,\xi)y_2(t)\\ &= \int_0^t \frac{-\cos(\sqrt{M_\delta(\xi)}t)\sin(\sqrt{M_\delta(\xi)}s) + \sin(\sqrt{M_\delta(\xi)}t)\cos(\sqrt{M_\delta(\xi)}s)}{\sqrt{M_\delta(\xi)}}\hat{b}(s,\xi)\,ds\\ &= \int_0^t \frac{\sin(\sqrt{M_\delta(\xi)}(t-s))}{\sqrt{M_\delta(\xi)}}\hat{b}(s,\xi)\,ds\end{aligned}$$

Hence, we have

$$\hat{u}(t,\xi) = \cos(\sqrt{M_\delta(\xi)}t)\hat{\varphi}(\xi) + \frac{\sin(\sqrt{M_\delta(\xi)}t)}{\sqrt{M_\delta(\xi)}}\hat{\psi}(\xi) + \int_0^t \frac{\sin(\sqrt{M_\delta(\xi)}s)}{\sqrt{M_\delta(\xi)}}\hat{b}(t-s,\xi)\,ds. \tag{4.17}$$

Then by taking the inverse Fourier Transform, of (4.17), we can get

$$\begin{aligned}u(x,t) =& F^{-1}\left(\cos(\sqrt{M_\delta(\xi)}t)\right) * \varphi(x) + F^{-1}\left(\frac{\sin(\sqrt{M_\delta(\xi)}t)}{\sqrt{M_\delta(\xi)}}\right) * \psi(x)\\ &+ \int_0^t F^{-1}\left(\frac{\sin(\sqrt{M_\delta(\xi)}s)}{\sqrt{M_\delta(\xi)}}\right) * b(t-s,x)\,ds\\ =& \int_{\mathbb{R}} \frac{d}{dt}G(t,y-x)\varphi(y)\,dy + \int_{\mathbb{R}} G(t,y-x)\psi(y)\,dy\\ &+ \int_0^t \int_{\mathbb{R}} G(s,y-x)b(t-s,y)dy\,ds\end{aligned} \tag{4.18}$$

where $F^{-1}\left(\frac{\sin(\sqrt{M_\delta(\xi)}t)}{\sqrt{M_\delta(\xi)}}\right) = G(t,y)$.

As we have seen the Cauchy Problem (4.3)-(4.4) has a unique solution, so our main goal will be to determine the proper space that the solution belongs to. By the equivalent definition of the peridynamic operator in (4.9), we can define the following functional space,

$$\mathcal{M}_\sigma(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \sqrt{1+M_\delta}\hat{u} \in L^2\}. \tag{4.19}$$

Naturally, the associated norm will be

$$||u||_{\mathcal{M}_\sigma} = ||\sqrt{1+M_\delta}\hat{u}||_{L^2} = \left(\int_{\mathbb{R}} (1+M_\delta(\xi))|\hat{u}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}. \tag{4.20}$$

For any $u, v \in \mathcal{M}_\sigma$, we also define the corresponding inner product by

$$(u,v)_{\mathcal{M}_\sigma} = (\sqrt{1+M_\delta}\hat{u}, \sqrt{1+M_\delta}\hat{v}). \tag{4.21}$$

Lemma 4.1.1 *$\mathcal{M}_\sigma(\mathbb{R})$ is a Hilbert Space corresponding to the inner product $(.,.)_{\mathcal{M}_\sigma}$.*

Proof: We have to show every Cauchy sequence in $\mathcal{M}_\sigma(\mathbb{R})$ has limit in $\mathcal{M}_\sigma(\mathbb{R})$. Let $\{u_n\}$ be an arbitrary Cauchy sequence in $\mathcal{M}_\sigma(\mathbb{R})$. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $||u_n - u_m||_{\mathcal{M}_\sigma} < \varepsilon$ when $n, m \geq N$. However, by definition (4.20), we have

$$||u_n - u_m||_{\mathcal{M}_\sigma} = ||\sqrt{1+M_\delta}(\hat{u}_n - \hat{u}_m)||_{L^2} < \varepsilon$$

when $n, m \geq N$. Then $\{\sqrt{1+M_\delta}\hat{u}_n\}$ is a Cauchy sequence in $L^2(\mathbb{R})$. But $L^2(\mathbb{R})$ is a complete Banach space. Then there exists $v \in L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} ||\sqrt{1+M_\delta}\hat{u}_n - v||_{L^2} = 0.$$

We claim that $\hat{u}(\xi)(\sqrt{1+M_\delta}) = v(\xi)$ with $v \in \mathcal{M}_\sigma(\mathbb{R})$:

$$\begin{aligned}\lim_{n\to\infty} ||u_n - u||_{\mathcal{M}_\sigma} &= \lim_{n\to\infty} ||\sqrt{1+M_\delta}(\hat{u}_n - \hat{u})||_{L^2} \\ &= \lim_{n\to\infty} ||\sqrt{1+M_\delta}\hat{u}_n - \sqrt{1+M_\delta}\hat{u}||_{L^2} \\ &= \lim_{n\to\infty} ||\sqrt{1+M_\delta}\hat{u}_n - v||_{L^2} = 0.\end{aligned}$$

Hence, $\mathcal{M}_\sigma(\mathbb{R})$ is a Hilbert Space. □

Lemma 4.1.2 *The space defined by*

$$\mathcal{M}_\sigma^{-1}(\mathbb{R}) = \{u : (1+M_\delta)^{-\frac{1}{2}}\hat{u} \in L^2\}. \tag{4.22}$$

equipped with the norm

$$||u||_{\mathcal{M}_\sigma^{-1}} = ||(1+M_\delta)^{-\frac{1}{2}}\hat{u}||_{L^2}. \tag{4.23}$$

is the dual space of $\mathcal{M}_\sigma(\mathbb{R})$.

Proof: Let $f = f(u)$ be a bounded linear functional on $\mathcal{M}_\sigma(\mathbb{R})$. Then by Riesz Representation Theorem there exists a unique $w \in \mathcal{M}_\sigma(\mathbb{R})$ such that

$$f(u) = (u,w)_{\mathcal{M}_\sigma} \quad \text{and} \quad ||f||_{\mathcal{M}_\sigma^*} = ||w||_{\mathcal{M}_\sigma}, \tag{4.24}$$

for every $u \in \mathcal{M}_\sigma(\mathbb{R})$. Using the inner product given in (4.21) we have

$$\begin{aligned} f(u) = (u, w)_{\mathcal{M}_\sigma} &= \int_{\mathbb{R}} \sqrt{1 + M_\delta(\xi)}\hat{u}(\xi)\sqrt{1 + M_\delta(\xi)}\hat{w}(\xi)\, d\xi \\ &= \int_{\mathbb{R}} \hat{u}(\xi)(1 + M_\delta(\xi))\hat{w}(\xi)\, d\xi \end{aligned}$$

Let $(1 + M_\delta)\hat{w}(\xi) = \hat{v}(\xi)$. Thus

$$(1 + M_\delta)^{-\frac{1}{2}}\hat{v}(\xi) = \sqrt{1 + M_\delta}\hat{w}(\xi) \in L^2$$

since $w \in \mathcal{M}_\sigma(\mathbb{R})$. Hence $(1 + M_\delta)^{-\frac{1}{2}}\hat{v} \in L^2$ and $v \in \mathcal{M}_\sigma^{-1}(\mathbb{R})$. Moreover,

$$f(u) = \int_{\mathbb{R}} \hat{u}(\xi)(1 + M_\delta(\xi))\hat{w}(\xi)\, d\xi \tag{4.25}$$

$$= \int_{\mathbb{R}} \hat{u}(\xi)\hat{v}(\xi)\, d\xi. \tag{4.26}$$

Thus from (4.24), we have

$$\begin{aligned} ||f||^2_{\mathcal{M}_\sigma^*} &= ||w||^2_{\mathcal{M}_\sigma} \\ &= \int_{\mathbb{R}} \sqrt{1 + M_\delta(\xi)}\hat{w}(\xi))\sqrt{1 + M_\delta(\xi)}\hat{w}(\xi)\, d\xi \\ &= \int_{\mathbb{R}} (1 + M_\delta(\xi))^{-\frac{1}{2}}\hat{v}(\xi)(1 + M_\delta(\xi))^{-\frac{1}{2}}\hat{v}(\xi)\, d\xi \\ &= ||v||^2_{\mathcal{M}_\sigma^{-1}}. \end{aligned}$$

and $||f||_{\mathcal{M}_\sigma^{-1}} = ||v||_{\mathcal{M}_\sigma^{-1}}$. Besides this, if $v \in \mathcal{M}_\sigma^{-1}(\mathbb{R})$, then for any $u \in \mathcal{M}_\sigma(\mathbb{R})$,

$$\begin{aligned} |f(u)| &= \left| \int_{\mathbb{R}} \hat{u}(\xi)\hat{v}(\xi)\, d\xi \right| \\ &= \left| \int_{\mathbb{R}} \hat{u}(\xi)(1 + M_\delta(\xi))(1 + M_\delta(\xi))^{-\frac{1}{2}}\hat{v}(\xi)\, d\xi \right| \\ &= \left|((1 + M_\delta)\hat{u}, (1 + M_\delta)^{-\frac{1}{2}}\hat{v})\right| \\ &\leq ||\sqrt{1 + M_\delta}\hat{u}||_{L^2}||(1 + M_\delta)^{-\frac{1}{2}}\hat{v}||_{L^2} \\ &= ||u||_{\mathcal{M}_\sigma}||v||_{\mathcal{M}_\sigma^{-1}}. \end{aligned}$$

from which we conclude that any $v \in \mathcal{M}_\sigma^{-1}(\mathbb{R})$ corresponds to a continuous and hence a bounded linear functional on $\mathcal{M}_\sigma(\mathbb{R})$. □

Lemma 4.1.3 *(i) The peridynamic operator $-L_\delta$ is self-adjoint on $\mathcal{M}_\sigma(\mathbb{R})$.*

(ii) The operator $-L_\delta + I$ is also an isometry from $\mathcal{M}_\sigma(\mathbb{R})$ to $\mathcal{M}_\sigma^{-1}(\mathbb{R})$.

(iii) The norm and inner product in $\mathcal{M}_\sigma(\mathbb{R})$ can also be formulated as

$$\begin{aligned} ||u||_{\mathcal{M}_\sigma} &= [(u,u)_{\mathcal{M}_\sigma}]^{\frac{1}{2}} \\ &= \left[||u||_{L^2}^2 + \frac{c_\delta}{2} \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(y)-u(x))^2 \, dy \, dx \right]^{\frac{1}{2}} \end{aligned} \tag{4.27}$$

for any $u \in \mathcal{M}_\sigma(\mathbb{R})$.

Proof: (i) Recall that $\widehat{(-L_\delta u)}(\xi) = M_\delta(\xi)\hat{u}(\xi)$. For any $u, v \in \mathcal{M}_\sigma(\mathbb{R})$,

$$\begin{aligned} (-L_\delta u, v)_{\mathcal{M}_\sigma} &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) \widehat{(-L_\delta u)}(\xi) \hat{v}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) M_\delta(\xi) \hat{u}(\xi) \hat{v}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) \hat{u}(\xi) M_\delta(\xi) \hat{v}(\xi) \, d\xi \\ &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) \hat{u}(\xi) \widehat{(-L_\delta v)}(\xi) \, d\xi = (u, -L_\delta v)_{\mathcal{M}_\sigma}. \end{aligned}$$

(ii) We want to show $-L_\delta + I : \mathcal{M}_\sigma(\mathbb{R}) \to \mathcal{M}_\sigma^{-1}(\mathbb{R})$ is an isometry. Then

$$\begin{aligned} ||(-L_\delta + I)u||_{\mathcal{M}_\sigma^{-1}}^2 = \int_{\mathbb{R}} (1 + M_\delta(\xi))^{-1} |(1 + M_\delta(\xi))\hat{u}(\xi)|^2 &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) |\hat{u}(\xi)|^2 \\ &= ||u||_{\mathcal{M}_\sigma}^2 \end{aligned}$$

and result follows.
(iii) Let $u \in \mathcal{M}_\sigma(\mathbb{R})$. Then $u \in L^2$ and we have

$$\begin{aligned} ||u||_{\mathcal{M}_\sigma}^2 &= \int_{\mathbb{R}} (1 + M_\delta(\xi)) |\hat{u}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}} |\hat{u}(\xi)|^2 \, d\xi + \int_{\mathbb{R}} M_\delta(\xi) |\hat{u}(\xi)|^2 \, d\xi \\ &= (\widehat{-L_\delta u}, \hat{u}) + (\hat{u}, \hat{u}) = (-L_\delta u, u) + (u, u) \end{aligned} \tag{4.28}$$

by Plancherel's Theorem. On the other hand,

$$\begin{aligned} (-L_\delta u, u) &= \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(x) - u(y)) u(x) \, dy \, dx \\ &= \frac{c_\delta}{2} \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(x) - u(y)) u(x) \, dy \, dx \\ &\quad + \frac{c_\delta}{2} \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(x) - u(y)) u(x) \, dy \, dx. \end{aligned}$$

Now, we change the order of integration and switch the variables x, y in the last integral to obtain

$$\begin{aligned} (-L_\delta u, u) = &\frac{c_\delta}{2} \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(x) - u(y)) u(x) \, dy \, dx \\ &+ \frac{c_\delta}{2} \int_{\mathbb{R}} \int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)} (u(y) - u(x)) u(y) \, dy \, dx. \end{aligned}$$

Combining these last two integrals gives

$$(-L_\delta u, u) = \frac{c_\delta}{2}\int_{\mathbb{R}}\int_{B_\delta(x)} \frac{(y-x)^2}{\sigma(|y-x|)}(u(y)-u(x))^2\, dy\, dx. \tag{4.29}$$

Finally, if we plug (4.29) in (4.28), we obtain (4.27). □

Remark 4.1.1 *If L_δ is the Laplace operator Δ , then we have the classical result:*

$$(\widehat{-\Delta u})(t,\xi) = (\widehat{-u_{xx}})(t,\xi) = \xi^2 \hat{u}(t,\xi).$$

So, we have

$$||(-\Delta + I)u||_{H^{-1}}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^{-1}|(1+|\xi|^2)\hat{u}(\xi)|^2 = \int_{\mathbb{R}} (1+|\xi|^2)|\hat{u}(\xi)|^2 = ||u||_{H^1}^2$$

and $-\Delta + I : H^1 \to H^{-1}$. Moreover, $-\Delta + I$ is self adjoint on H^1. This is because

$$\begin{aligned}
(-\Delta u, v)_{H^1} &= \int_{\mathbb{R}} (1+|\xi|^2)(\widehat{-\Delta u})(\xi)\hat{v}(\xi)\, d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)|\xi|^2\hat{u}(\xi)\hat{v}(\xi)\, d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)\hat{u}(\xi)|\xi|^2\hat{v}(\xi)\, d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)\hat{u}(\xi)(\widehat{-\Delta v})(\xi)\, d\xi = (u, -\Delta v)_{H^1}.
\end{aligned}$$

To discuss the regularity of the weak solutions, we also need to define the following space

$$\mathcal{M}_\sigma^k(\mathbb{R}) = \{u : (1+M_\delta)^k \hat{u} \in L^2\}, \tag{4.30}$$

with the dual space

$$\mathcal{M}_\sigma^{-k}(\mathbb{R}) = \{u : (1+M_\delta)^{-k} \hat{u} \in L^2\}. \tag{4.31}$$

Remark 4.1.2 ([5]) *$\mathcal{M}_\sigma^k(\mathbb{R})$ and $\mathcal{M}_\sigma^{-k}(\mathbb{R})$ share the similar properties discussed in Lemma 4.1.1, Lemma 4.1.2 and Lemma 4.1.3.*

Claim 4.1.4 *Let n be a positive integer. Then*

(i) $\widehat{(L_\delta^2 u)}(\xi) = M_\delta^2(\xi)\hat{u}(\xi)$.

(ii) $[(I-L_\delta)^n u]\hat{}(\xi) = [1+M_\delta(\xi)]^n \hat{u}(\xi)$.

Proof: We know that $\widehat{L_\delta u}(\xi) = M_\delta(\xi)\hat{u}(\xi)$. Then

(i) $[L_\delta(L_\delta u(x))]\hat{} = M_\delta(\xi)\widehat{(L_\delta u)}(\xi) = M_\delta^2(\xi)\hat{u}(\xi)$.

(ii) We use Binomial expansion:

$(I - L_\delta)^n u(x) = \binom{n}{0}u(x) - \binom{n}{1}L_\delta u(x) + \binom{n}{2}L_\delta^2 u(x) + ... + (-1)^n\binom{n}{n}L_\delta^n u(x)$.
After taking Fourier Transform of both sides of the latter equation, we use (i) to obtain:

$$\begin{aligned}[(I - L_\delta)^n u]\hat{}(\xi) &= \binom{n}{0}\hat{u}(\xi) + \binom{n}{1}M_\delta(\xi)\hat{u}(\xi) + \binom{n}{2}M_\delta^2(\xi)\hat{u}(\xi) + \binom{n}{n}M_\delta^n(\xi)\hat{u}(\xi) \\ &= [1 + M_\delta(\xi)]^n\hat{u}(\xi).\end{aligned}$$

□

Hence, we have the following lemma:

Lemma 4.1.5 $(-L_\delta + I)^n : \mathcal{M}_\sigma^k(\mathbb{R}) \to \mathcal{M}_\sigma^{k-2n}(\mathbb{R})$ *is an isometry.*

Proof: Let $u \in \mathcal{M}_\sigma^k$. Then,

$$\begin{aligned}||(I - L_\delta)^n u||_{\mathcal{M}_\sigma^{k-2n}} &= \int_{\mathbb{R}} (1 + M_\delta(\xi)^{k-2n}(I + M_\delta(\xi))^{2n}|\hat{u}(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}} (1 + M_\delta(\xi)^k|\hat{u}(\xi)|^2\, d\xi = ||u||_{\mathcal{M}_\sigma^k}.\end{aligned}$$

□

Corollary 4.1.6 *If $n = k$, then we have an isometry between $\mathcal{M}_\sigma^k(\mathbb{R})$ and its dual:*

$$(-L_\delta + I)^k : \mathcal{M}_\sigma^k(\mathbb{R}) \to \mathcal{M}_\sigma^{-k}(\mathbb{R}).$$

We have shown that the problem (4.10)-(4.11) has a representation solution $\hat{u}$ in (4.17) depending on $M_\delta(\xi)$. In the next theorem, we give the conditions for which the solution u of Cauchy Problem (4.3)-(4.4) lies in $C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$.

Theorem 4.1.7 *If $\varphi \in \mathcal{M}_\sigma(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$, and $b \in L^2([0,T], L^2(\mathbb{R}))$, then the Cauchy Problem* (4.3)-(4.4) *has a unique solution $u \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$ for some $T > 0$. Moreover, $u_t \in L^2([0,T], L^2(\mathbb{R}))$.*

Proof: We want to show that $u(t,x)$ is uniformly bounded in $C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$. For this reason, we have to find the $\mathcal{M}_\sigma(\mathbb{R})$ norm estimation of $u(t,x)$. If we recall the related norm, we see that it depends on the Fourier Transform of u:

$$||u||_{\mathcal{M}_\sigma}^2 = \int_{\mathbb{R}} (1 + M_\delta(\xi))|\hat{u}(\xi)|^2\, d\xi.$$

As $u(t,x)$ in (4.18) is the sum of three integrals, we let

$$u(t,x) = u_1(t,x) + u_2(t,x) + u_3(t,x)$$

with

$$\hat{u}_1(t,\xi) = \cos(\sqrt{M_\delta(\xi)}t)\hat{\varphi}(\xi),$$

$$\hat{u}_2(t,\xi) = \frac{\sin(\sqrt{M_\delta(\xi)}t)}{\sqrt{M_\delta(\xi)}}\hat{\psi}(\xi),$$

$$\hat{u}_3(t,\xi) = \int_0^t \frac{\sin(\sqrt{M_\delta(\xi)}(t-s)}{\sqrt{M_\delta(\xi)}}\hat{b}(s,\xi)\,ds.$$

Then

$$||u|_{\mathcal{M}_\sigma} \leq ||u_1||_{\mathcal{M}_\sigma} + ||u_2||_{\mathcal{M}_\sigma} + ||u_3||_{\mathcal{M}_\sigma}.$$

We show that each term is uniformly bounded in $C([0,T],\mathcal{M}_\sigma(\mathbb{R}))$. We are given that $\varphi \in \mathcal{M}_\sigma(\mathbb{R})$. Then

$$\begin{aligned}||u_1||^2_{\mathcal{M}_\sigma} &= \int_{\mathbb{R}} (1+M_\delta(\xi))|\cos^2(\sqrt{M_\delta(\xi)}t)\hat{\varphi}(\xi)|^2\,d\xi \\ &\leq \int_{\mathbb{R}} (1+M_\delta(\xi))|\hat{\varphi}(\xi)|^2 d\xi = ||\varphi||^2_{\mathcal{M}_\sigma}.\end{aligned}$$

For the estimate of the second integral

$$||u_2||^2_{\mathcal{M}_\sigma} = \int_R (1+M_\delta(\xi))\frac{\sin^2(\sqrt{M_\delta(\xi)}t)}{M_\delta(\xi)}|\hat{\psi}(\xi)|^2 d\xi,$$

we have to be more precise. The reason is that the bound of $\sin(\sqrt{M_\delta(\xi)}t)$ depends on the argument. Notice that $0 \leq t \leq T$ and $|\sin(\sqrt{M_\delta(\xi)}t)| \leq |\sqrt{M_\delta(\xi)}t|$ for small values of $\sqrt{M_\delta(\xi)}$ whereas $|\sin(\sqrt{M_\delta(\xi)}t)| \leq 1$ for large values of $\sqrt{M_\delta(\xi)}$. For this reason, we will follow the same method used in [10] and split the integral into two parts: We observe that the sets $\{\xi : \sqrt{M_\delta(\xi)} \geq 1\}$ and $\{\xi : \sqrt{M_\delta(\xi)} < 1\}$ are measurable since $M_\delta(\xi)$ defined in (4.8) is a measurable function. Hence, we have

$$\begin{aligned}||u_2||^2_{\mathcal{M}_\sigma} =& \int_{\{\xi:\sqrt{M_\delta(\xi)}\geq 1\}} \left(1+\frac{1}{M_\delta(\xi)}\right)\sin^2(\sqrt{M_\delta(\xi)}t)|\hat{\psi}(\xi)|^2 d\xi \\ &+ \int_{\{\xi:\sqrt{M_\delta(\xi)}<1\}} (1+M_\delta(\xi))\frac{\sin^2(\sqrt{M_\delta(\xi)}t)}{M_\delta(\xi)}|\hat{\psi}(\xi)|^2 d\xi.\end{aligned}$$

On the set $\{\xi : \sqrt{M_\delta(\xi)} \geq 1\}$, we have $1+\frac{1}{M_\delta(\xi)} \leq 2$ and $\sin^2(\sqrt{M_\delta(\xi)}t) \leq 1$. However, $1+M_\delta(\xi) \leq 2$ and $\frac{\sin^2(\sqrt{M_\delta(\xi)}t)}{M_\delta(\xi)} \leq \frac{M_\delta(\xi)t^2}{M_\delta(\xi)} = t^2$ on $\{\xi : \sqrt{M_\delta(\xi)} < 1\}$. Thus, we continue as

$$\begin{aligned}||u_2||^2_{\mathcal{M}_\sigma} &\leq 2\int_{\{\xi:\sqrt{M_\delta(\xi)}\geq 1\}} |\hat{\psi}(\xi)|^2 d\xi + 2t^2\int_{\{\xi:\sqrt{M_\delta(\xi)}\geq 1\}} |\hat{\psi}(\xi)|^2 d\xi \\ &\leq 2\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 d\xi + 2t^2\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 d\xi \\ &= (2+2t^2)\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 d\xi = 2(1+t^2)||\hat{\psi}||^2_{L^2} = 2(1+t^2)||\psi||^2_{L^2}\end{aligned}$$

since $\psi \in L^2(\mathbb{R})$.
Similarly, we can evaluate the $\mathcal{M}_\sigma(\mathbb{R})$ norm of the last term in the following way:

$$||u_3||^2_{\mathcal{M}_\sigma} = \int_{\mathbb{R}} (1 + M_\delta(\xi))|\hat{u}_3(t,\xi)|^2 \, d\xi = ||\sqrt{1 + M_\delta}\hat{u}_3||^2_{L^2}.$$

Therefore,

$$\begin{aligned} ||\sqrt{1 + M_\delta}\hat{u}_3||_{L^2} &= \left|\left| \int_0^t \frac{\sqrt{1 + M_\delta(\xi)}}{\sqrt{M_\delta(\xi)}} \sin(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi) \, ds \right|\right|_{L^2} \\ &\le \int_0^t \left|\left| \frac{\sqrt{1 + M_\delta(\xi)}}{\sqrt{M_\delta(\xi)}} \sin(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi) \right|\right|_{L^2} ds \\ &= \int_0^t ||B(s)||_{L^2} \, ds \end{aligned} \tag{4.32}$$

Now, we estimate

$$||B(s)||^2_{L^2} = \int_{\mathbb{R}} \frac{1 + M_\delta(\xi)}{M_\delta(\xi)} \sin^2(\sqrt{M_\delta(\xi)}(t-s))|\hat{b}(s,\xi)|^2 \, d\xi.$$

Similar to what we have done while estimating the $\mathcal{M}_\sigma$ norm of $u_2(t,x)$, we can obtain

$$\begin{aligned} ||B(s)||^2_{L^2} =& \int_{\{\xi : \sqrt{M_\delta(\xi)} \ge 1\}} \left(1 + \frac{1}{M_\delta(\xi)}\right) \sin^2(\sqrt{M_\delta(\xi)}(t-s))|\hat{b}(s,\xi)|^2 d\xi \\ &+ \int_{\{\xi : \sqrt{M_\delta(\xi)} < 1\}} (1 + M_\delta(\xi)) \frac{\sin^2(\sqrt{M_\delta(\xi)}(t-s))}{M_\delta(\xi)} |\hat{b}(s,\xi)|^2 d\xi \\ \le& 2 \int_{\{\xi : \sqrt{M_\delta(\xi)} \ge 1\}} |\hat{b}(t-s,\xi)|^2 d\xi + \int_{\{\xi : \sqrt{M_\delta(\xi)} \ge 1\}} 2(t-s)^2 |\hat{b}(s,\xi)|^2 d\xi. \end{aligned}$$

But $0 \le (t-s)^2 \le t^2$. So we continue as

$$\begin{aligned} ||B(s)||^2_{L^2} \le 2(1+t^2) \int_{\mathbb{R}} |\hat{b}(s,\xi)|^2 d\xi = 2(1+t^2)||\hat{b}(s)||^2_{L^2} &\le 2(1+t)^2 ||\hat{b}(s)||^2_{L^2} \\ &= 2(1+t)^2 ||b(s)||^2_{L^2}, \end{aligned}$$

and

$$||B(s)||_{L^2} \le 2(1+t)||b(s)||_{L^2}. \tag{4.33}$$

If we use (4.33) in (4.32), we get

$$\begin{aligned} ||u_3||_{\mathcal{M}_\sigma} &\le 2 \int_0^t (1+t)||b(s)||_{L^2} \\ &\le 2(1+T) \left(\int_0^T 1^2 ds \right)^{\frac{1}{2}} \left(\int_0^T ||b(s)||^2_{L^2} \, ds \right)^{\frac{1}{2}} \\ &= 2(1+T)\sqrt{T}||b||_{L^2([0,T],L^2(\mathbb{R}))} \end{aligned} \tag{4.34}$$

by Cauchy-Schwartz Inequality. Therefore,

$$\max_{0\le t\le T} ||u||_{\mathcal{M}_\sigma} \le ||u_1||_{C([0,T],\mathcal{M}_\sigma} + ||u_2||_{C([0,T],\mathcal{M}_\sigma)} + ||u_3||_{C([0,T],\mathcal{M}_\sigma)}$$
$$\le ||\varphi||_{\mathcal{M}_\sigma} + 2(1+T^2)||\psi||^2_{L^2} + 2(1+T)\sqrt{T}||b||_{L^2([0,T],L^2(\mathbb{R}))} < \infty.$$

So, $u \in C([0,T],\mathcal{M}_\sigma(\mathbb{R}))$. Next, we differentiate $\hat{u}$ with respect to t to obtain

$$\begin{aligned}\hat{u}_t(t,\xi) =& -\sqrt{M_\delta(\xi)}\sin(\sqrt{M_\delta(\xi)}t)\hat{\varphi}(\xi) + \cos(\sqrt{M_\delta(\xi)}t)\hat{\psi}(\xi)\\ &+ \int_0^t \cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)\,ds\\ =& v_1(t,\xi) + v_2(t,\xi) + v_3(t,\xi).\end{aligned}$$

Then,

$$||v_1||^2_{L^2} = \int_{\mathbb{R}} M_\delta(\xi)\sin^2(\sqrt{M_\delta(\xi)}t)|\hat{\varphi}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{\varphi}(\xi)|^2\,d\xi = ||\varphi||^2_{\mathcal{M}_\sigma}, \tag{4.35}$$

$$||v_2||^2_{L^2} = \int_{\mathbb{R}}\cos^2(\sqrt{M_\delta(\xi)}t)|\hat{\psi}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}|\hat{\psi}(\xi)|^2 = ||\psi||^2_{L^2}, \tag{4.36}$$

$$\begin{aligned}||v_3||_{L^2} &= \left|\left|\int_0^t \cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)\,ds\right|\right|_{L^2}\\ &\le \int_0^t ||\cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)||_{L^2}\,ds\end{aligned}$$

and

$$\begin{aligned}||\cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)||^2_{L^2} &= \int_{\mathbb{R}}\cos^2(\sqrt{M_\delta(\xi)}(t-s))|\hat{b}(s,\xi)|^2 d\xi\\ &\le \int_{\mathbb{R}}|\hat{b}(s,\xi)|^2 d\xi = ||b(s)||^2_{L^2}.\end{aligned}$$

Then

$$||v_3||_{L^2} \le \int_0^t ||b(s)||_{L^2}\,ds \le \sqrt{T}||b||_{L^2([0,T],L^2(\mathbb{R}))}. \tag{4.37}$$

On the other hand,

$$||u_t||^2_{L^2([0,T],L^2(\mathbb{R}))} = \left(\int_0^T ||u_t(t)||^2_{L^2}dt\right) = \left(\int_0^T ||\hat{u}_t(t)||^2_{L^2}dt\right)$$

by Plancherel's Theorem. Then,

$$\begin{aligned}||u_t||_{L^2([0,T],L^2(\mathbb{R}))} &\le ||v_1||_{L^2([0,T],L^2(\mathbb{R}))} + ||v_2||_{L^2([0,T],L^2(\mathbb{R}))} + ||v_3||_{L^2([0,T],L^2(\mathbb{R}))}\\ &= \left(\int_0^T ||v_1(t)||^2_{L^2}dt\right)^{\frac{1}{2}} + \left(\int_0^T ||v_2(t)||^2_{L^2}dt\right)^{\frac{1}{2}} + \left(\int_0^T ||v_3(t)||^2_{L^2}dt\right)^{\frac{1}{2}}\\ &\le \left(\int_0^T ||\varphi||^2_{\mathcal{M}_\sigma}dt\right)^{\frac{1}{2}} + \left(\int_0^T ||\psi||^2_{L^2}dt\right)^{\frac{1}{2}} + \left(\int_0^T T||b||^2_{L^2([0,T],L^2(\mathbb{R}))}dt\right)^{\frac{1}{2}}\\ &= \sqrt{T}||\varphi||_{\mathcal{M}_\sigma} + \sqrt{T}||\psi||_{L^2} + T||b||_{L^2([0,T],L^2(\mathbb{R}))}\end{aligned}$$

where we use (4.35), (4.36) and (4.37). Then, $u_t \in L^2([0,T], L^2(\mathbb{R}))$. □

Remark 4.1.3 *In fact, we can obtain a new result from Theorem 4.1.7.*

First, we notice that the following holds:

Claim 4.1.8 $(L_\delta u)_t = L_\delta u_t$.

Proof: We use the definition in (4.6)

$$\begin{aligned}
(L_\delta u)_t(x) =& \lim_{h\to 0} \frac{L_\delta(u(x,t+h)) - L_\delta(u(x,t))}{h} \\
=& \lim_{h\to 0} c_\delta \int_{B_\delta(x)} \frac{|y-x|^2}{\sigma(|y-x|)} \frac{[u(y,t+h) - u(x,t+h) - u(y,t) + u(x,t)]}{h}\, dy \\
=& \lim_{h\to 0} c_\delta \int_{B_\delta(x)} \frac{|y-x|^2}{\sigma(|y-x|)} \frac{[u(y,t+h) - u(y,t)}{h}\, dy \\
&- \lim_{h\to 0} c_\delta \int_{B_\delta(x)} \frac{|y-x|^2}{\sigma(|y-x|)} \frac{[u(x,t+h) - u(x,t)]}{h}\, dy \\
=& c_\delta \int_{B_\delta(x)} \frac{|y-x|^2}{\sigma(|y-x|)} [u_t(y) - u_t(x)]\, dy \\
=& L_\delta u_t(x).
\end{aligned}$$

□

Similarly, we have $(L_\delta u)_{tt}(x) = L_\delta u_{tt}(x)$.

If we apply $-L_\delta + I$ to the left side of the peridynamic equation given in (4.3), we will obtain

$$\begin{aligned}
(-L_\delta + I)(u_{tt} - L_\delta u) &= (-L_\delta + I)u_{tt} - (-L_\delta + I)L_\delta u \\
&= ((-L_\delta + I)u)_{tt} - L_\delta(-L_\delta + I)u = (-L_\delta + I)b.
\end{aligned}$$

Now, let $(-L_\delta + I)u = v$, and $(-L_\delta + I)b = \tilde{b}$, then we derive a new equation

$$v_{tt}(x,t) - L_\delta v(x,t) = \tilde{b}(x,t) \tag{4.38}$$

with shifted initial data:

$$v(x,0) = (-L_\delta + I)\varphi(x) = \tilde{\varphi}(x), \quad v_t(x,0) = (-L_\delta + I)\psi(x) = \tilde{\psi}(x) \tag{4.39}$$

which also represents a peridynamic equation with different external force $\tilde{b}$. Because of Lemma 4.1.5, we know that $\tilde{\varphi} \in \mathcal{M}_\sigma^{-1}(\mathbb{R})$, $\tilde{\psi} \in \mathcal{M}_\sigma^{-2}(\mathbb{R})$, and $\tilde{b} \in L^2([0,T], \mathcal{M}_\sigma^{-2}(\mathbb{R}))$ where $n = 1$. Moreover, $v \in C([0,T], \mathcal{M}_\sigma^{-1}(\mathbb{R}))$. Hence, by using the fact that $I - L_\delta : \mathcal{M}_\sigma(\mathbb{R}) \to \mathcal{M}_\sigma^{-1}(\mathbb{R})$ is an isometry, we could obtain a new theorem:

Theorem 4.1.9 *If* $\varphi \in \mathcal{M}_\sigma^{-1}(\mathbb{R})$, $\psi \in \mathcal{M}_\sigma^{-2}(\mathbb{R})$, *and* $b \in L^2([0,T], \mathcal{M}_\sigma^{-2}(\mathbb{R}))$, *then the Cauchy Problem* (4.3)-(4.4) *has a unique solution* $u \in C([0,T], \mathcal{M}_\sigma^{-1}(\mathbb{R}))$. *Moreover,* $u_t \in L^2([0,T], \mathcal{M}_\sigma^{-2}(\mathbb{R}))$.

Now, we want to derive a general result from Theorem 4.1.7. That is, we look for conditions for which the equation 4.3 has a solution in $C([0,T],\mathcal{M}_\sigma^k(\mathbb{R}))$. Calculations above suggest that applying right power of $I-L_\delta$ to equation (4.3) will work. Recall that we have

$$(I-L_\delta)^{-n}:\mathcal{M}_\sigma^{k-2n}(\mathbb{R})\to\mathcal{M}_\sigma^k(\mathbb{R})$$

is an isometry. Then, we solve $k-2n=1$ for n, and obtain $n=\frac{k-1}{2}$. Hence if we apply $(I-L_\delta)^{\frac{1-k}{2}}$ to the equation (4.3), we will obtain (4.38) with $(I-L_\delta)^{\frac{1-k}{2}}u=v$. Thus, the data are swithched. As we already have results in Theorem 4.1.7, we get the following:

Theorem 4.1.10 *If $\varphi\in\mathcal{M}_\sigma^k(\mathbb{R})$, $\psi\in\mathcal{M}_\sigma^{k-1}(\mathbb{R})$, and $b\in L^2([0,T],\mathcal{M}_\sigma^{k-1}(\mathbb{R}))$, then the Cauchy Problem* (4.3)-(4.4) *has a unique solution $u\in C([0,T],\mathcal{M}_\sigma^k(\mathbb{R}))$. Moreover, $u_t\in L^2([0,T],\mathcal{M}_\sigma^{k-1}(\mathbb{R}))$.*

Here, we give the detailed proof for $k=2$ to verify the result obtained in Theorem 4.1.10.

Theorem 4.1.11 *If $\varphi\in\mathcal{M}_\sigma^2(\mathbb{R})$, $\psi\in\mathcal{M}_\sigma(\mathbb{R})$, and $b\in L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))$, then the Cauchy Problem* (4.3)-(4.4) *has a unique solution $u\in C([0,T],\mathcal{M}_\sigma^2(\mathbb{R}))$. Moreover, $u_t\in L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))$.*

Proof: This time, we would like to estimate $\mathcal{M}_\sigma^2(\mathbb{R})$ norm of u. In this case

$$\begin{aligned}||u_1||^2_{\mathcal{M}_\sigma^2}&=\int_{\mathbb{R}}(1+M_\delta(\xi))^2|\cos^2(\sqrt{M_\delta(\xi)}t)\hat{\varphi}(\xi)|^2\,d\xi\\&\le\int_{\mathbb{R}}(1+M_\delta(\xi))^2|\hat{\varphi}(\xi)|^2d\xi=||\varphi||^2_{\mathcal{M}_\sigma^2}.\end{aligned}$$

Then, we see that the integrand is multiplied by $1+M_\delta(\xi)$ and this only affect the norm of φ. Similarly,

$$\begin{aligned}||u_2||^2_{\mathcal{M}_\sigma}&\le 2\int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{\psi}(\xi)|^2d\xi+2t^2\int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{\psi}(\xi)|^2d\xi\\&=(2+2t^2)\int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{\psi}(\xi)|^2d\xi=2(1+t^2)||\hat{\psi}||^2_{\mathcal{M}_\sigma}\end{aligned}$$

and

$$||u_3||_{\mathcal{M}_\sigma^2}\le\int_0^t||B(s)||_{L^2}\,ds$$

with

$$||B(s)||^2_{L^2}=\int_{\mathbb{R}}\frac{(1+M_\delta(\xi))^2}{M_\delta(\xi)}\sin^2(\sqrt{M_\delta(\xi)}(t-s)|\hat{b}(s,\xi)|^2\,d\xi.$$

Similar to what we have done while estimating the $\mathcal{M}_\sigma$ norm of $u_2(t,x)$, we can obtain

$$||B(s)||^2_{L^2}\le 2(1+t^2)\int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{b}(s,\xi)|^2d\xi=2(1+t^2)||b(s)||^2_{\mathcal{M}_\sigma}$$

with

$$||B(s)||_{L^2} \leq 2(1+t)||b(s)||_{\mathcal{M}_\sigma}. \tag{4.40}$$

Then, we get

$$\begin{aligned}
||u_3||_{\mathcal{M}^2_\sigma} &\leq 2\int_0^t (1+t)||b(s)||_{\mathcal{M}_\sigma} \\
&\leq 2(1+T)\left(\int_0^T 1^2 ds\right)^{\frac{1}{2}}\left(\int_0^T ||b(s)||^2_{\mathcal{M}_\sigma}\, ds\right)^{\frac{1}{2}} \\
&= 2(1+T)\sqrt{T}||b||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))}.
\end{aligned}$$

So, $u \in C([0,T], \mathcal{M}^2_\sigma(\mathbb{R}))$. Furthermore,

$$||v_1||^2_{\mathcal{M}_\sigma} \leq \int_{\mathbb{R}} (1+M_\delta(\xi))^2|\hat{\varphi}(\xi)|^2\, d\xi = ||\varphi||^2_{\mathcal{M}^2_\sigma},$$

$$||v_2||^2_{\mathcal{M}_\sigma} \leq \int_{\mathbb{R}} (1+M_\delta(\xi))|\hat{\psi}(\xi)|^2 = ||\psi||^2_{\mathcal{M}_\sigma},$$

On the other hand,

$$||v_3||_{\mathcal{M}_\sigma} \leq \int_0^t ||\cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)||_{\mathcal{M}_\sigma}\, ds$$

and

$$||\cos(\sqrt{M_\delta(\xi)}(t-s))\hat{b}(s,\xi)||^2_{\mathcal{M}_\sigma} \leq \int_{\mathbb{R}} (1+M_\delta(\xi))|\hat{b}(s,\xi)|^2 d\xi = ||b(s)||^2_{\mathcal{M}_\sigma}.$$

Then

$$\begin{aligned}
||v_3||_{\mathcal{M}_\sigma} &\leq \int_0^t ||b(s)||_{\mathcal{M}_\sigma}\, ds \\
&\leq \sqrt{T}||b||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))}.
\end{aligned} \tag{4.41}$$

and

$$||v_3||^2_{\mathcal{M}_\sigma} \leq T||b||^2_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))}.$$

On the other hand,

$$||u_t||^2_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))} = \left(\int_0^T ||u_t(t)||^2_{\mathcal{M}_\sigma} dt\right) = \left(\int_0^T ||\hat{u}_t(t)||^2_{L^2} dt\right)$$

$$\begin{aligned}
||u_t||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))} &= ||\hat{u}_t||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))} \\
&\leq ||v_1||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))}||v_2||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))} + ||v_3||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))} \\
&\leq \sqrt{T}||\varphi||_{\mathcal{M}^2_\sigma} + \sqrt{T}||\psi||_{\mathcal{M}_\sigma} + T||b||_{L^2([0,T],\mathcal{M}_\sigma(\mathbb{R}))}
\end{aligned}$$

and $u_t \in L^2([0,T], \mathcal{M}_\sigma(\mathbb{R}))$. □

Now, we look for the sufficient conditions so that the Cauchy Problem (4.3)-(4.4) has a unique solution $u \in C^2([0,T], \mathcal{M}_\sigma(\mathbb{R}))$. Thus, we should have $u \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$ and $u_t \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$. Calculations in Theorem 4.1.7 and 4.1.11 show that the term v_3 determines the space where the function u_t lies. On the other hand, it is controlled by b. Besides this, b and u_t lie in the same space.

From Theorem 4.1.11 and equality (4.41), we have

$$\begin{aligned} ||v_1||_{\mathcal{M}_\sigma} &\leq ||\varphi||_{\mathcal{M}^2_\sigma(\mathbb{R})}, \ \text{ and } \ v_1 \in C([0,T], \mathcal{M}_\sigma(\mathbb{R})), \\ ||v_2||_{\mathcal{M}_\sigma} &\leq ||\psi||_{\mathcal{M}_\sigma(\mathbb{R})}, \ \text{ and } \ v_2 \in C([0,T], \mathcal{M}_\sigma(\mathbb{R})), \\ ||v_3||_{\mathcal{M}_\sigma} &\leq \int_0^t ||b(s)||_{\mathcal{M}_\sigma}\, ds \\ &\leq T||b||_{C([0,T],\mathcal{M}_\sigma(\mathbb{R}))} \end{aligned}$$

If $\varphi \in \mathcal{M}^2_\sigma(\mathbb{R})$, $\psi \in \mathcal{M}_\sigma(\mathbb{R})$, and $b \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$, then $u_t \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$. On the other hand,

$$\mathcal{M}^2_\sigma(\mathbb{R}) \subseteq \mathcal{M}_\sigma(\mathbb{R}) \subseteq L^2(\mathbb{R}).$$

Thus $\varphi \in \mathcal{M}_\sigma(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$, $b \in C([0,T], L^2(\mathbb{R}))$, and $b \in L^2([0,T], L^2(\mathbb{R}))$ and the assumptions of Theorem 4.1.7 are satisfied.

Keeping all these in mind, we can sum up the sufficient conditions in the following theorem.

Theorem 4.1.12 *If $\varphi \in \mathcal{M}^2_\sigma(\mathbb{R})$, $\psi \in \mathcal{M}_\sigma(\mathbb{R})$, and $b \in C([0,T], \mathcal{M}_\sigma(\mathbb{R}))$, then the Cauchy Problem* (4.3)-(4.4) *has a unique solution $u \in C^2([0,T], \mathcal{M}_\sigma(\mathbb{R}))$.*

4.2. Embeddings of $\mathcal{M}^k_\sigma(\mathbb{R})$

Next, we focus on some kernel functions with special properties to establish the relations between $\mathcal{M}^k_\sigma(\mathbb{R})$ for $k = -2, -1, 1, 2$ and the more conventional H^s Spaces.

Lemma 4.2.13 *Let the kernel function σ satisfy*

$$\sigma(x) > 0, \ \ \forall x \in B_\delta(0), \quad \text{and} \quad \tau_\delta := c_\delta \int_{B_\delta(0)} \frac{|x|^4}{\sigma(|x|)}\, dx < \infty. \tag{4.42}$$

Then,

(i) $H^1(\mathbb{R}) \hookrightarrow \mathcal{M}_\sigma(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$,

(ii) $H^2(\mathbb{R}) \hookrightarrow \mathcal{M}^2_\sigma(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$,

(iii) $L^2(\mathbb{R}) \hookrightarrow \mathcal{M}^{-1}_\sigma(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$,

(iv) $L^2(\mathbb{R}) \hookrightarrow \mathcal{M}^{-2}_\sigma(\mathbb{R}) \hookrightarrow H^2(\mathbb{R})$.

Proof: We first find the relationship between the weights $1+\xi^2$ and $1+M_\delta(\xi)$. For this purpose we estimate $M_\delta(\xi)$. From the definition of $M_\delta(\xi)$ in (4.8), we have

$$M_\delta(\xi) = c_\delta \int_{B_\delta(0)} \frac{1-cos(\xi y)}{\sigma(|y|)}|y|^2\,dy \le \frac{\xi^2}{2}c_\delta \int_{B_\delta(0)} \frac{|y|^4}{\sigma(|y|)}\,dy = \frac{\tau_\delta}{2}\xi^2 \tag{4.43}$$

where we use the fact that $1-cos(\xi y) \le (\xi^2 y^2)/2$ and (4.42). If we add 1 to both sides of inequality (4.43), we obtain

$$1+M_\delta(\xi) \le 1+\frac{\tau_\delta}{2}\xi^2 \le C(1+\xi^2) \tag{4.44}$$

where $C=\max\{1,\frac{\tau_\delta}{2}\}$. Then, we are ready to show the embeddings:
(i) From (4.44)

$$\begin{aligned} ||u||_{L^2} &= \int_{\mathbb{R}} |\hat{u}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{u}(\xi)|^2\,d\xi = ||u||^2_{\mathcal{M}_\sigma} \\ ||u||^2_{\mathcal{M}_\sigma} &= \int_{\mathbb{R}}(1+M_\delta(\xi))|\hat{u}(\xi)|^2\,d\xi \le C\int_{\mathbb{R}}(1+\xi^2)|\hat{u}(\xi)|^2\,d\xi = C||u||^2_{H^1}. \end{aligned} \tag{4.45}$$

(ii) Inequality (4.44) implies that $(1+M_\delta(\xi))^2 \le C^2(1+\xi^2)^2$. Then

$$\begin{aligned} ||u||_{L^2} &\le \int_{\mathbb{R}}(1+M_\delta(\xi))^2|\hat{u}(\xi)|^2\,d\xi = ||u||^2_{\mathcal{M}^2_\sigma} \\ ||u||^2_{\mathcal{M}^2_\sigma} &= \int_{\mathbb{R}}(1+M_\delta(\xi))^2|\hat{u}(\xi)|^2\,d\xi \le C^2\int_{\mathbb{R}}(1+\xi^2)^2|\hat{u}(\xi)|^2\,d\xi = C^2||u||^2_{H^2}. \end{aligned} \tag{4.46}$$

(iii) $(1+\xi^2)^{-1} \le C(1+M_\delta(\xi))^{-1}$ and

$$\begin{aligned} ||u||^2_{H^{-1}} &= \int_{\mathbb{R}}(1+\xi^2)^{-1}|\hat{u}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}(1+M_\delta(\xi))^{-1}|\hat{u}(\xi)|^2\,d\xi = ||u||^2_{\mathcal{M}^{-1}_\sigma} \\ ||u||^2_{\mathcal{M}^{-1}_\sigma} &= \int_{\mathbb{R}}(1+M_\delta(\xi))^{-1}|\hat{u}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}|\hat{u}(\xi)|^2\,d\xi = ||u||_{L^2}. \end{aligned} \tag{4.47}$$

(iv) $(1+\xi^2)^{-2} \le C^2(1+M_\delta(\xi))^{-2}$. Then

$$\begin{aligned} ||u||^2_{H^{-2}} &= \int_{\mathbb{R}}(1+\xi^2)^{-2}|\hat{u}(\xi)|^2\,d\xi \le C^2\int_{\mathbb{R}}(1+M_\delta(\xi))^{-2}|\hat{u}(\xi)|^2\,d\xi = C^2||u||^2_{\mathcal{M}^{-2}_\sigma} \\ ||u||^2_{\mathcal{M}^{-2}_\sigma} &= \int_{\mathbb{R}}(1+M_\delta(\xi))^{-2}|\hat{u}(\xi)|^2\,d\xi \le \int_{\mathbb{R}}|\hat{u}(\xi)|^2\,d\xi = ||u||_{L^2}. \end{aligned} \tag{4.48}$$

Inequalities (4.45),(4.45)(4.45) and (4.45) imply what we want to show. □

Calculations above illustrate that once we find an embedding between two function spaces, the embedding between their duals is straightforward.

Lemma 4.2.14 *Let the kernel function σ satisfy*

$$\sigma(x) > 0, \ \forall x \in B_\delta(0), \quad \textit{and} \quad \tau_{\delta_2} := c_\delta \int_{B_\delta(0)} \frac{|x|^2}{\sigma(|y|)}\, dy < \infty, \tag{4.49}$$

we then have

$$\mathcal{M}^2_\sigma(\mathbb{R}) = \mathcal{M}_\sigma(\mathbb{R}) = L^2(\mathbb{R}) = \mathcal{M}^{-1}_\sigma(\mathbb{R}) = \mathcal{M}^{-2}_\sigma(\mathbb{R}).$$

Proof: Let $u \in L^2$. We only show reverse directions of the embeddings stated in Lemma 4.2.13 under the condition (4.49). However,

$$M_\delta(\xi) = c_\delta \int_{B_\delta(0)} \frac{1 - cos(\xi y)}{\sigma(|y|)} |y|^2\, dy \leq 2c_\delta \int_{B_\delta(0)} \frac{|y|^2}{\sigma(|y|)}\, dy = 2\tau_{\delta_2}$$

with

$$(1 + M_\delta(\xi))^k \leq (1 + 2\tau_{\delta_2})^k \quad \text{and} \quad 1 \leq (1 + 2\tau_{\delta_2})^k (1 + M_\delta(\xi))^{-k} \quad \text{for} \quad k = 1, 2. \tag{4.50}$$

Hence,

$$L^2(\mathbb{R}) \hookrightarrow \mathcal{M}^k_\sigma(\mathbb{R}), \quad \text{and} \quad \mathcal{M}^{-1}_\sigma(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}), \quad \text{for} \quad k = 1, 2.$$

Therefore, $\mathcal{M}^2_\sigma(\mathbb{R}) = \mathcal{M}_\sigma(\mathbb{R}) = L^2(\mathbb{R}) = \mathcal{M}^{-1}_\sigma(\mathbb{R}) = \mathcal{M}^{-2}_\sigma(\mathbb{R})$ follows. □

Inequality (4.50) actually implies that all $\mathcal{M}^k_\sigma(\mathbb{R})$ spaces are equivalent to $L^2(\mathbb{R})$ space since $1 + M_\delta(\xi)$ is uniformly bounded in ξ.

Lemma 4.2.15 *Let the kernel function $\sigma = \sigma(|y|)$ satisfy the condition*

$$\sigma(|y|) \leq \gamma_1 |y|^{3+2\beta}, \quad \forall |y| \leq \delta \tag{4.51}$$

for some exponent $\beta \in (0, 1)$ and positive constant γ_1, then we have

(i) $\mathcal{M}_\sigma(\mathbb{R}) \hookrightarrow H^\beta(\mathbb{R})$,

(ii) $\mathcal{M}^2_\sigma(\mathbb{R}) \hookrightarrow H^{2\beta}(\mathbb{R})$,

(iii) $H^{-\beta}(\mathbb{R}) \hookrightarrow \mathcal{M}^{-1}_\sigma(\mathbb{R})$,

(iv) $H^{-2\beta}(\mathbb{R}) \hookrightarrow \mathcal{M}^{-2}_\sigma(\mathbb{R})$.

Proof: For each four cases, we will try to find an estimate for M_δ. From (4.53) we know that

$$\frac{1}{\sigma(|y|)} \geq \frac{1}{\gamma_1 |y|^{3+2\beta}}.$$

Then,

$$M_\delta(\xi) = c_\delta \int_{B_\delta(0)} \frac{1-cos(\xi y)}{\sigma(|y|)}|y|^2\,dy \geq \frac{c_\delta}{\gamma_1}\int_{B_\delta(0)} \frac{1-cos(\xi y)}{|y|^{3+2\beta}}|y|^2\,dy$$
$$= \frac{c_\delta}{\gamma_1}\int_{B_\delta(0)} \frac{1-cos(\xi y)}{|y|^{1+2\beta}}\,dy = \frac{c_\delta}{2\gamma_1}\int_{B_\delta(0)} \frac{\sin^2\left(\frac{\xi y}{2}\right)}{|y|^{1+2\beta}}\,dy.$$

Let $z = \frac{\xi y}{2}$. Then $\frac{2}{\xi}dz = dy$. On the other hand, $\lim_{z\to 0}\frac{\sin z}{z} = 1$. Thus

$$(1-\varepsilon)z \leq \sin z \leq (\varepsilon+1)z.$$

Therefore,

$$\int_{B_{\frac{\xi}{2}\delta(0)}} \frac{\sin^2 z}{|2z|^{1+2\beta}}|\xi|^{1+2\beta}\frac{2}{\xi}dz = \frac{|\xi|^{2\beta}}{4^\beta}\int_{B_{\frac{\xi}{2}\delta(0)}} \frac{\sin^2 z}{|z|^{1+2\beta}}dz$$
$$\geq \frac{|\xi|^{2\beta}(1-\varepsilon)^2}{4^\beta}\int_{B_{\frac{\xi}{2}\delta(0)}} \frac{z^2}{|z|^{1+2\beta}}dz$$
$$= \frac{|\xi|^{2\beta}(1-\varepsilon)^2}{4^\beta}\int_{B_{\frac{\xi}{2}\delta(0)}} \frac{1}{|z|^{2\beta-1}}\,dz = |\xi|^{2\beta}K_\beta$$

since $0<\beta<1$ and $2\beta-1<1$. Therefore,

$$1 + M_\delta(\xi) \geq \left(1 + \frac{c_\delta}{2\gamma_1}K_\beta|\xi|^{2\beta}\right) \geq C_\beta(1+|\xi|^{2\beta}) \tag{4.52}$$

where $C_\beta = \min\{1, \frac{c_\delta}{2\gamma_1}K_\beta\}$. Thus $\mathcal{M}_\sigma(\mathbb{R})$ is continuously embedded to $H^\beta(\mathbb{R})$. From (4.52) and (1.12) we have

$$(1+M_\delta(\xi))^2 \geq C_\beta^2(1+|\xi|^{2\beta})^2 \approx C_\beta^2(1+|\xi|^2)^{2\beta}$$

$$(1+M_\delta(\xi))^{-1} \leq C_\beta^{-1}(1+|\xi|^{2\beta})^{-1} \approx C_\beta^{-1}(1+|\xi|^2)^{-\beta}$$

$$(1+M_\delta(\xi))^{-2} \leq C_\beta^{-2}(1+|\xi|^{2\beta})^{-2} \approx C_\beta^{-2}(1+|\xi|^2)^{-2\beta}$$

and the continuous embeddings that are claimed are shown. □

Lemma 4.2.16 *Let the kernel function $\sigma = \sigma(|y|)$ satisfy the condition*

$$\sigma(|y|) \geq \gamma_2|y|^{3+2\alpha}, \quad \forall |y| \leq \delta \tag{4.53}$$

for some exponent $\alpha \in (0,1)$ and positive constant γ_2, then we have

(i) $H^\alpha(\mathbb{R}) \hookrightarrow \mathcal{M}_\sigma(\mathbb{R})$,

(ii) $H^{2\alpha}(\mathbb{R}) \hookrightarrow \mathcal{M}_\sigma^2(\mathbb{R})$,

(iii) $\mathcal{M}_\sigma^{-1}(\mathbb{R}) \hookrightarrow H^{-\alpha(}\mathbb{R}),$

(iv) $\mathcal{M}_\sigma^{-2}(\mathbb{R}) \hookrightarrow H^{-2\alpha}(\mathbb{R}).$

Proof: Again, we will try to find an estimate for M_δ. From (4.53) we know that

$$\frac{1}{\sigma(|y|)} \leq \frac{1}{\gamma_2 |y|^{3+2\alpha}}.$$

$$M_\delta(\xi) = c_\delta \int_{B_\delta(0)} \frac{1 - cos(\xi y)}{\sigma(|y|)} |y|^2 \, dy \leq \frac{c_\delta}{2\gamma_2} \int_{B_\delta(0)} \frac{\sin^2\left(\frac{\xi y}{2}\right)}{|y|^{1+2\alpha}} \, dy.$$

By making the same substitution as $z = \frac{\xi y}{2}$, we will obtain:

$$M_\delta(\xi) \leq \frac{c_\delta}{2\gamma_2} \frac{|\xi|^{2\alpha}(1-\varepsilon)^2}{4^\alpha} \int_{B_{\frac{\xi}{2}\delta(0)}} \frac{1}{|z|^{2\alpha-1}} \, dz \leq |\xi|^{2\alpha} \frac{c_\delta}{2\gamma_2} K_\alpha$$

since $0 < \alpha < 1$. Then

$$1 + M_\delta(\xi) \leq \left(1 + \frac{c_\delta}{2\gamma_2} K_\alpha |\xi|^{2\alpha}\right) \leq C_\alpha (1 + |\xi|^{2\alpha}) \tag{4.54}$$

where $C_\alpha = \max\{1, \frac{c_\delta}{2\gamma_2} K_\alpha\}$. Thus $H^\alpha(\mathbb{R})$ is continuously embedded to $\mathcal{M}_\sigma(\mathbb{R})$. Moreover, from (4.54) and (1.12) we have

$$(1 + M_\delta(\xi))^2 \leq C_\alpha^2 (1 + |\xi|^{2\alpha})^2 \approx C_\alpha^2 (1 + |\xi|^2)^{2\alpha}$$

$$(1 + M_\delta(\xi))^{-1} \geq C_\alpha^{-1} (1 + |\xi|^{2\alpha})^{-1} \approx C_\alpha^{-1} (1 + |\xi|^2)^{-\alpha}$$

$$(1 + M_\delta(\xi))^{-2} \geq C_\alpha^{-2} (1 + |\xi|^{2\alpha})^{-2} \approx C_\alpha^{-2} (1 + |\xi|^2)^{-2\alpha}$$

and the continuous embeddings that are claimed are shown. □

Consequently, we see that under suitable conditions on the kernel function, the space $\mathcal{M}_\sigma(\mathbb{R})$ is equivalent to some standard *fractional* Sobolev Spaces:

Lemma 4.2.17 *Assume the kernel function satisfy*

$$\gamma_2 |y|^{3+2\alpha} \leq \sigma(|y|) \leq \gamma_1 |y|^{3+2\alpha}, \quad \forall |y| \leq \delta \tag{4.55}$$

for some exponent $\alpha \in (0,1)$ *and positive constant* γ_1, *and* γ_2. *Then, we have*

$$\mathcal{M}_\sigma(\mathbb{R}) = H^\alpha, \quad \mathcal{M}_\sigma^2(\mathbb{R}) = H^{2\alpha}.$$

$$\mathcal{M}_\sigma^{-1}(\mathbb{R}) = H^{-\alpha}, \quad \mathcal{M}_\sigma^{-2}(\mathbb{R}) = H^{-2\alpha}.$$

Proof: This is a direct consequence of Lemma 4.2.15 and Lemma 4.2.16.

Corollary 4.2.18 *Assume $\varphi \in L^2(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$, and $b \in L^2([0,T], L^2(\mathbb{R}))$,. If the kernel function $\sigma = \sigma(|y|)$ satisfies the condition (4.49), then the Cauchy Problem (4.3)-(4.4) has a unique solution $u \in C([0,T], L^2(\mathbb{R}))$ and $u_t \in L^2([0,T, L^2(\mathbb{R}))$ for some $T > 0$.*

Remark 4.2.4 *$u \in C([0,T], L^2(\mathbb{R}))$ means $u \in L^2([0,T], L^2(\mathbb{R}))$ and we also have $u_t \in L^2([0,T], L^2(\mathbb{R}))$. Now, recall the definition of $H^1([0,T], L^2(\mathbb{R}))$ norm:*

$$||u||^2_{H^1([0,T],L^2(\mathbb{R}))} = ||u||^2_{L^2([0,T],L^2(\mathbb{R}))} + ||u_t||^2_{L^2([0,T],L^2(\mathbb{R}))}$$

Then, in fact, $u \in C([0,T], L^2(\mathbb{R})) \cap H^1([0,T], L^2(\mathbb{R}))$.

Corollary 4.2.19 *Assume $\varphi \in H^\alpha(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$, and $b \in L^2([0,T], L^2(\mathbb{R}))$,. If the kernel function $\sigma = \sigma(|y|)$ satisfies the condition (4.55), then the Cauchy Problem (4.3)-(4.4) has a unique solution $u \in C([0,T], H^\alpha(\mathbb{R}))$ and $u_t \in L^2([0,T], L^2(\mathbb{R}))$.*

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