A STATIC OVERBOOKING MODEL IN SINGLE LEG FLIGHT REVENUE MANAGEMENT

by

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TEK AYAK UÇUŞ GELİR YÖNETİMİNDE STATİK BİR KAPASİTE ÜSTÜ REZERVASYON MODELİ

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Özet

Bu tezde, tek ayak uçuş gelir yönetiminde statik bir kapasite üstü rezervasyon modeli sunulmaktadır. Modelde, farklı uçuş sınıflarına ait bilet taleplerinin birbirlerinden bağımsız ve homojen olmayan Poisson süreçlerine göre geldiği varsayılmıştır. Kabul edilen her rezervasyon uçuş tarihinden önce iptal edilebilir ve kalkış tarihinde bazı yolcular uçağa gelmeyebilirler. Bu durumda statik bir strateji, bileşenleri bilet sınıflarının kapanış zamanlarını veren deterministik bir vektör ile ifade edilmektedir. Tezde, bu tarz statik stratejiler içeresinden en yüksek beklenen geliri veren bir strateji belirlenmiştir. Model bu haliyle, zamanda sürekli benzer bir dinamik kapasite üstü rezervasyon modelinin statik versiyonu olarak görülebilir. İyi bilinen EMSR metodlarına alternatif olarak da görülebilir. Tezde, sözü edilen optimal statik stratejinin performansı da nümerik olarak incelenmiş ve EMSR ve dinamik stratejilerle karşılaştırılmıştır.

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Abstract

In this thesis, we present a static single leg airline revenue management model with overbooking. In this model it is assumed that the requests for different fare class tickets arrive according to independent nonhomogeneous Poisson processes. Each accepted request may cancel its reservation before the departure, and at the departure time no-shows may occur. In this setup, a static strategy is represented by a deterministic vector whose components give the closing times of the fare classes. Among those strategies we determine one with the highest expected revenue. As such this model can be seen as the static counter part of a dynamic continuous-time airline overbooking model. It can also be considered as an alternative to the well-known EMSR heuristics. In the thesis, we also study the performance of the optimal static strategy numerically and compare it with those of EMSR and dynamic strategies.

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Chapter 1

Introduction

In this thesis, we are studying a single leg airline revenue management problem with no-shows, cancellations and multiple fare classes. In the proposed mathematical model, the arrival process of the different fare class requests are independent nonhomogeneous Poisson processes, while the probability of no-show and the probability distributions of the random times to cancellation are fare class dependent. Due to the occurrence of no-shows and cancellation overbooking of seats is allowed. In the overbooking literature within the field of airline revenue management one distinguishes static and dynamic models. In a dynamic model, when deciding upon the acceptance or rejection of a request for a ticket, one takes into account the realisation of the cancellation and arrival processes up to the point in time that this request occurs. In a static model, the decision to accept or reject is only based on the probability measures of these random processes. In this thesis, the proposed model belongs to the class of static models. In particular, we determine at the starting time of the sales for the differently priced tickets representing the different fare classes, the optimal times to close down the sales for each of these differently priced tickets under the objective of expected revenue maximization. In this decision we need to balance the expected revenue to be gained by accepting all the arriving requests against the expected overbooking costs of accepting too many of those requests. As similar booking decisions are repeated millions of times per year, the objective function is to find that static booking strategy/policy that maximizes the expected revenue (see McGill and Van Ryzin [21]). In 1995, Durham [14] reports that a large computer reservation system must handle five thousand such transactions per second at peak times. Consequently, next to giving a realistic description of reality, our algorithm to compute the optimal static strategy should also be fast in generating the optimal strategy.

In general, static overbooking models, not considering the dynamics of the random arrival and cancellation processes, are much easier to analyse than the dynamic overbooking models. Due to omitting the time dimension of the arrival and cancellation processes, these static models can be analysed under much less stringent conditions. For an overview on the early static models in the literature we refer the reader to Talluri and Van Ryzin [29] and for a recent overview to Aydin et al. [2]. A new feature of our proposed static model, distinguishing it from all the other known static models, is the inclusion in the model of the time dimension aspects of the arrival and cancellation processes.

At the same time dynamic overbooking models under fare class independent show-ups and the more stringent Markovian assumptions on the (fare class independent) cancellation processes are also analysed in the literature. To simplify the mathematical analysis, most of the considered dynamic models assume a discrete time arrival process. To obtain a complete overview on these models for the special case of no cancellations and perfect showup (hence no overbooking) one should consult McGill and Van Ryzin [21] or Lautenbacher and Stidham [18]. McGill and Van Ryzin report that most early seat inventory control research required most or all of the following simplifying assumptions: 1) sequential booking classes; 2) low-before-high fare booking arrival pattern; 3) statistical independence of demands between booking classes; 4) no cancellations or no-shows (hence, no overbooking); 5) single flight leg with no consideration of network effects; and, 6) no batch booking.

Overbooking is reserving more seats than the physical capacity of the airplane. Given the existence of customers cancelling before the departure time of the plane or not showing up at the departure time, the overbooking strategy may help airlines to improve their revenue and hence possible profit by filling up otherwise empty seats. However, applying overbooking may be risky, since we do not know in advance which customers will show up and so there might be denied boardings for which one should pay a penalty. Hence overbooking can increase revenue if the cost to be paid for overbooked customers is not too high.

Generally, the dynamic problem with overbooking is analysed using similar techniques as the dynamic models with no overbooking. Due to the (discrete time) nonhomogenous Bernoulli arrival process it can be modeled as a Markov Decision Process (MDP). Rothstein [25] and Alstrup et al. [1] use MDP to study the overbooking model with one and two fare classes respectively. Among studies dealing with multiple fare classes, Chi [12] and Subramanian et al. [27] should be mentioned. In Subramanian et al. [27] two models are considered. In the first model, they use a queuing system formulation for the case with class independent cancellations and no-show probabilities. In their second model, they relax the class independence assumption and model a more general problem with class-dependent cancellations and no-shows. Unfortunately, in the second model, the resulting dynamic programming formulation cannot be solved efficiently due to the curse of dimensionality of the high-dimensional state space. Other examples of such models include, but are not limited to, Lee and Hersh [19], Chi [12] and Birbil et al. [9]. For a more complete overview on the recent literature with (discrete time) nonhomogeneous Bernouilli arrival processes the reader is referred to Aydin et al. [2]). In that paper, next to improved static models, an adaptation of the approach by Subramanian et al. is proposed.

In case we consider the more realistic case of a continuous time arrival and cancellation process Chatwin [10] formulates a similar problem in continuous time under the assumptions that the arrivals occur according to a homogenous Poisson process. He allows the refunds and fares to be time dependent. A recent paper by Frenk et al. [15] gives an overview over the literature with continuous time arrival and cancellation processes and analyzes in detail the most general case of nonhomogeneous Poisson arrival processes and a fare class independent Markovian cancellation process. Observe in our static setting we study a similar problem under more general conditions than in Frenk et. al. Using the techniques of dynamic optimal control theory, they identify the optimal control strategy for the above single leg revenue management problem with fare class independent refunds for cancellation and show-up probabilities. Although they also point out that a complete characterization of the optimal control policy can be given for fare class dependent show-ups and cancellations it becomes computationally impractical to compute the optimal dynamic policy due to the well-known dimensionality curse of dynamic programming. To overcome this problem, some researchers propose approximation methods to solve dynamic programming problems which are computational intractable (see Secomandi [26], Chi [12], and Chatwin [11]).

In real life using dynamic programming might be impractical and computationally infeasible. This problem might even occur for simpler static models. Hence some researchers and practitioners have developed heuristics to solve static overbooking problems. The outcomes of these static computations are then adapted to a dynamic environment involving time. Starting in the 1970s, some airlines started to offer special discounted fare classes. For example, British Airways (formally BOAC) offered early booking that charge lower fares for customers who booked at least 21 days before departure time (see McGill and Van Ryzin [21]). In 1972, Littlewood [20] proposed a control policy for such a system. He states that discounted fare class should be offered till the time that their revenue value exceed the expected revenue of future full fare booking (see Bhatia and Parekh [8] and Richter [23] for two extensions). Extending Littlewood's results to more than two fare classes, Belobaba in [4] and [5] developed the Expected Marginal Seat Revenue (EMSR) heuristic for the no overbooking problem. By using some heuristic procedure replacing the actual capacity by a larger virtual capacity Belobaba in the same paper adapted the EMSR heuristic to the problem with overbooking. Again one is referred to Aydin et al. [2] for an overview on different procedures to replace the actual capacity by the virtual capacity. Robinson [24] reports that, for general arrival processes, the EMSR method used in a nested way can produce poor results. For the detailed treatment of EMSR heuristic we refer readers to McGill and Van Ryzin [21] and its discussion in the Appendix.

In this thesis, using mathematical programming, our goal is to develop a static model for revenue management model with overbooking, cancellation, and no showups. As we will show in our computational results in Chapter 5, although the static strategy is not the optimal control policy within the class of all admissable policies, it performs extremely well in the computational experiments presented in this thesis. In these cases we compare our optimal static strategy with the optimal dynamic strategy derived in Frenk et al. [15]. It turns out that the expected revenue obtained by our static strategy is near to that of the expected revenue of the optimal dynamic strategy. We describe first in Chapter 2 our model and introduce the notations. Then, in Chapter 3 we derive the static model formulation for single and multiple fare classes. Algorithms to solve the general static model as well as some easy special cases are discussed in Chapter 4. Chapters 5 and 6 present computational results and concluding remarks respectively.

Chapter 2

Problem Description

In this section we propose a static airline revenue management model with overbooking. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space hosting the stochastic elements of our model and introduce the following notation.

- C: the total capacity of the airplane.
- T: the departure time of the airplane.
- m: the number of fare classes.
- r_j: the fare or revenue of fare class j, $j \in \{1, 2, ..., m\}$.
- κ_j: the cancelation refund of fare class j, $j \in \{1, 2, ..., m\}$
- γ: the overbooking cost per overbooked customer.
- p_j: the showup probability for fare class j at the departure time T.
- t_j: the time at which reservation for fare class j is closed , $j \in \{1, 2, ..., m\}$.

It is assumed without loss of generality that $0 < r_1 < ... < r_m$ and so fare class 1 denotes the cheapest fare class and fare class m the most expensive. By the definition of the parameters it is also clear that $r_j \geq \kappa_j$ for every $1 \leq j \leq m$. To avoid pathological cases we also assume $p_j > 0$ for every j. If $p_j = 0$ for some $j \in \{1, 2, ..., m\}$ then it is obvious using $r_j \geq \kappa_j$ that we will open fare class j up to the departure time T and so we can reduce our problem having only $m - 1$ fare classes. The class of all so-called static policies is given by the set of vectors $\mathbf{t} = (t_1, ..., t_m)$ satisfying $0 \leq t_j \leq T, 1 \leq j \leq m$. For each strategy/policy $\mathbf{t}$ an expected revenue $R(\mathbf{t})$ will be computed and we are now interested in the strategy achieving the maximum expected revenue. To compute this maximum expected revenue and show the existence of a strategy achieving this we first need to model the different random processes occurring within the model.

The random arrival time of the ith fare class j request, $1 \leq j \leq m$ is denoted by T_{ij}. The arrival processes of the different fare class requests are assumed to be independent and for each fare class j the sequence of random variables $(T_{ij})_{i \in \mathbb{N}}$ is a nonhomogeneous Poisson process (see [13]) with locally bounded Borel arrival intensity function λ_j and mean arrival function $\Lambda_j : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Lambda_j(t) := \int_0^t \lambda_j(s) ds. \tag{2.1}$$

To describe the random cancellation processes it is assumed that the cancellation behaviour of each customer is independent of the cancellation behaviour of other customers. Since customers among different fare classes might behave differently we allow the probability law describing this behaviour to be fare class dependent. In particular, the time to cancellation of a fare class j request arriving at time T_{ij} is denoted by a random variable Y_{ij} and the sequence of random variables $(Y_{ij})_{i \in \mathbb{N}}$ for fixed j are independent and identically distributed (iid) with (right continous) cdf F_j. This means that under a given static strategy $\mathbf{t} = (t_1, ..., t_m)$ a fare class j request arriving at the random time T_{ij} cancels if and only if $T_{ij} \leq t_j$ and $T_{ij} + Y_{ij} \leq T$.

Finally, to model the total number of random show ups at the departure time T, it is assumed that the show up behaviour of each customer is independent of the show up behaviour of other customers. However the probability law describing this behaviour might be fare class dependent and so within each fare class a non-cancelling fare class j customer having a reservation will show up with positive probability p_j. Hence under the previous assumption and using a static strategy $\mathbf{t}$ the random number of fare class j show ups $S_j(t_j)$ is given by

$$S_j(t_j) := \sum_{i=1}^{\infty} 1_{\{T_{ij} \leq t_j, T_{ij}+Y_{ij} > T\}} B_{ij} \tag{2.2}$$

with $(B_{ij})_{i \in \mathbb{N}}$ a sequence of independent Bernoulli random variables with success probability p_j. The different functions determining the total expected revenue under a fixed static strategy $\mathbf{t}$ are

- The fare class j expected revenue $I_j(t_j)$, $j \in \{1, 2, ..., m\}$ given by

$$I_j(t_j) := r_j \mathbb{E}\left(\sum_{i=1}^{\infty} 1_{\{T_{ij} \leq t_j\}}\right). \tag{2.3}$$

- The fare class j expected cancellation refund $C(t_j)$, $j \in \{1, 2, ..., m\}$ given by

$$K_j(t_j) := \kappa_j \mathbb{E}\left(\sum_{i=1}^{\infty} 1_{\{T_{ij} \leq t_j, T_{ij}+Y_{ij} \leq T\}}\right). \tag{2.4}$$

- The expected total overbooking cost $\Theta(\mathbf{t})$ given by

$$\Theta(\mathbf{t}) := \gamma \mathbb{E}\left(\left(\sum_{j=1}^{m} S_j(t_j) - C\right)^+\right) \tag{2.5}$$

  with $(x)^+ := \max\{x, 0\}$.

By (2.3), (2.4) and (2.5) it is clear that the total expected revenue under static strategy $\mathbf{t}$ is given by

$$R(\mathbf{t}) = \sum_{j=1}^{m} (I_j(t_j) - K_j(t_j)) - \Theta(\mathbf{t}). \tag{2.6}$$

and we need to solve the optimization problem

$$\sup_{0 \leq t_j \leq T, j \in \{1,\ldots,m\}} R(\mathbf{t}) \tag{P}$$

and show under which conditions an optimal strategy exists and compute it. Instead of using the penalty function $f(x) = (x - C)^+$ one can also apply without any additional problems a convex penalty function f satisfying $f(x) = 0$ for every $x \leq 0$ and f increasing on $\mathbb{R}_+$. In this case we obtain

$$\Theta(\mathbf{t}) = \mathbb{E}(f(\sum_{j=1}^{m} S_j(t_j) - C))$$

We will not pursue this extension in the remainder of this paper but refer the reader to Aydin et al. [2] for a discussion of the same extension of a related static problem not involving the dynamic nature of the arrival processes. To determine this expected total revenue and how to solve the optimization problem (P) we first consider in the next section the overbooking problem with a single fare class and give an easy algorithm for this case. It will turn out that this algorithm for the single fare class case is needed in our dynamic programming solution procedure solving the overbooking problem with multiple fare classes.

Chapter 3

Analysis of the static model

In this chapter, we first start with modeling the problem for a single fare class in section 3.1, and then, we extend it to the case of multiple fare classes in section 3.2.

3.1 Static Model With a Single Fare Class.

Since there is only one fare class we omit the index j representing the different fare classes. The pairs $(T_i, Y_i)_{i\in\mathbb{N}}$, of which T_i and Y_i are defined in the previous section, can be considered as the atoms of a Poisson random measure N given by

$$N(\omega, A) := \sum\nolimits_{i\in\mathbb{N}} 1_A(T_i(\omega), Y_i(\omega)),\ \forall(\omega, A) \in \Omega \times \mathbb{B}(\mathbb{R}^2_+).$$

with mean measure

$$\nu(ds\,dy) = \lambda(s)ds \cdot F(dy).$$

Also, we introduce the function $S_C : (0, \infty) \to \mathbb{R}_+$ given by

$$S_C(x) := \sum\nolimits_{j=C+1}^{\infty} (j - C)\frac{e^{-x}x^j}{j!}. \tag{3.1}$$

Using the definition of an expectation it is easy to see that $S_C(x) = \mathbb{E}(\max\{\mathbf{Z}(x) - C, 0\})$ with the random variable $\mathbf{Z}(x)$ having a Poisson distribution with parameter x. For this function one can show the following integral representation.

Lemma 1 *It follows that the function $S_C : (0,\infty) \to \mathbb{R}_+$ is continuously differentiable on $(0,\infty)$ and it has the alternative representation*

$$S_C(x) = \int_0^x \int_0^y \frac{e^{-z} z^{C-1}}{(C-1)!} dz dy. \tag{3.2}$$

Moreover, the derivative function $x \mapsto S_C'(x)$ is strictly increasing and for every $x > 0$

$$S_C'(x) = e^{-x} \sum\nolimits_{j=C}^{\infty} \frac{x^j}{j!} = \mathbb{P}(\mathbf{Y}(x) \geq C) \tag{3.3}$$

with $\mathbf{Y}(x)$ having a Poisson distribution with parameter x.

Proof. By brute force computation, one can verify that

$$S_C'(x) = e^{-x} \sum_{j=C}^{\infty} \frac{x^j}{j!} = \int_0^x \frac{e^{-z} z^{C-1}}{(C-1)!} dz, \tag{3.4}$$

which is the cdf of the Gamma distribution with shape parameter $C - 1$ and scale parameter 1. By (3.4) we obtain that the function S_C is continuously differentiable on $(0, \infty)$ and its second derivative exists and equals

$$S_C''(x) = \frac{e^{-x} x^{C-1}}{(C-1)!} > 0.$$

This shows $x \mapsto S_C'(x)$ is strictly increasing and with the boundary condition $S_C(0) = 0$, we obtain from (3.4) that

$$S_C(x) = \int_0^x S_C'(y) dy = \int_0^x \int_0^y \frac{e^{-z} z^{C-1}}{(C-1)!} dz\, dy.$$

and we have verified the result. □

Also introduce the function $h : [0,T] \mapsto \mathbb{R}$ given by

$$h(t) := \int_0^t \lambda(s)(1 - F(T-s))ds. \tag{3.5}$$

Since the function λ is a locally bounded Borel function and any cdf F is right continuous it follows that the function h is continuous on $(0,T)$ and it satisfies

$$0 = h(0) = \lim_{t \downarrow 0} h(t) =: h(0+),\ h(T) = \lim_{t \uparrow T} h(t).$$

As will be shown in the proof of the next result the value $h(t)$ represents the expected number of reservations who do not cancel before departure if we close the fare class at time $0 \leq t \leq T$. Before presenting this result we introduce the following well known definition.

Definition 2 *If $X \subseteq \mathbb{R}$ is a nonempty set and $f : X \to \mathbb{R}$ some real-valued function on X then f is called Lipschitz continuous on X with finite Lipschitz constant L_f if*

$$\mid f(x) - f(y) \mid \leq L_f \mid x - y \mid$$

for any x, y belonging to X.

Introducing for a function $f : [0,T] \to \mathbb{R}$ its supnorm $\| f\|_\infty$ given by

$$\|f\|_\infty := \sup_{0 \leq t \leq T} \mid f(t) \mid \tag{3.6}$$

one can easily show the following result.

Lemma 3 *The revenue function $R : [0,T] \to \mathbb{R}$ is Lipschitz continous on $[0,T]$ with Lipschitz constant*

$$L_R = (r + \gamma p)\|\lambda\|_\infty$$

and an optimal solution of optimization problem (P) exists. In particular, it follows that

$$R(t) = (r-\kappa)\int_0^t \lambda(s)ds + \kappa h(t) - \gamma S_C(ph(t)). \tag{3.7}$$

Proof. If we close the fare class at time $0 \leq t \leq T$, then the expected total amount $I(t)$ of fares received is given by

$$I(t) = r\int_{\mathbb{R}_+^2} 1_{\{s\leq t\}}\nu(ds\, dy) = r\int_0^t \lambda(s)ds. \tag{3.8}$$

Moreover, for the same strategy the expected total cancellation refunds $K(t)$ amounts to

$$\begin{aligned} K(t) &= \kappa \textstyle\int_{\mathbb{R}_+^2} 1_{\{s\leq t\}}1_{\{s+y\leq T\}}\nu(ds\, dy) \\ &= \kappa \textstyle\int_0^\infty 1_{\{s\leq t\}}\left[\int_0^\infty 1_{\{y\leq T-s\}}F(dy)\right]\lambda(s)ds \\ &= \kappa \textstyle\int_0^t \lambda(s)F(T-s)ds. \end{aligned} \tag{3.9}$$

Finally the expected overbooking cost has the form $\gamma\mathbb{E}\left(\sum_{i=1}^{Nf} B_i - C\right)^+$ where

$$Nf = \int_{\mathbb{R}_+^2} f(s,y)N(dsdy)$$

denotes the total number of non-canceling customers given by the integral of the function

$$f(s,y) = 1_{\{s\leq t\}}1_{\{s+y>T\}},\ s\geq 0, y\geq 0,$$

with respect to the random measure N. It is well known (see Chapter 6 of Çınlar [13]) that the random variable Nf has a Poisson distribution with mean

$$\mathbb{E}Nf = \nu f = \int_{\mathbb{R}_+^2} 1_{\{s\leq t\}}1_{\{s+y>T\}}\nu(ds\, dy) = \int_0^t \lambda(s)[1-F(T-s)]ds = h(t),$$

and this implies that the random variable $\sum_{i=1}^{Nf} B_i$ is a Poisson distributed random variable with mean $ph(t)$. Therefore the expected overbooking costs has the form

$$\theta(t) = \gamma \sum\nolimits_{j=C+1}^{\infty} (j-C) \frac{e^{-[p\,h(t)]} \cdot [p\,h(t)]^j}{j!} = S_C(ph(t)). \tag{3.10}$$

Substituting (3.8), (3.9) and (3.10) into (2.6) we obtain the revenue as given by (3.7). By Lemma 1 it follows $0 \leq S_C'(x) \leq 1$ and this yields applying the mean value theorem (See Rudin [32]) that S_C is Lipschitz continuous with Lipschitz constant 1. For λ a locally bounded Borel function one can now show by standard techniques that the function R listed in (3.7) is Lipschitz continuous on $[0,T]$ with Lipschitz constant $(r+\gamma p) \parallel \lambda \parallel_\infty$. Since any Lipschitz continuous function on $[0,T]$ is continuous and $[0,T]$ is a compact set we obtain by Weierstrass theorem (See Rudin [32]) that an optimal solution of optimization problem (P) exists. □

An immediate corollary is given by the following corollary.

Corollary 4 *If the cdf F and the function λ are continuous on $(0,\infty)$, then the revenue function R is differentiable. In particular, for every $0 < t < T$ its derivative $R'(t)$ at t is then given by*

$$R'(t) = \lambda(t)(r - \kappa + (1 - F(T-t))\varphi(t)) \tag{3.11}$$

with $\varphi : [0,T] \mapsto \mathbb{R}$ defined by

$$\varphi(t) = \kappa - p\gamma S_C'(ph(t))) = \kappa - p\gamma \mathbb{P}(\mathbf{Y}(ph(t)) \geq C) \tag{3.12}$$

Proof. Using (3.5) and (3.7) and the continuity of the arrival intensity function λ the derivative in relation (3.11) follows immediately. Also it is obvious that this derivative is a continuous function of t. □

If the cdf F is strictly increasing on $(0,\infty)$ and the the arrival intensity function λ is positive it follows by (3.5) that the function h is strictly increasing on $(0,\infty)$ and this implies by Lemma 1 that the function φ given in (3.12) is strictly decreasing on $(0,\infty)$. To analyse the objective function we need the following easy consequence of Lemma 1.

Lemma 5 *If the revenue function R is differentiable and the function λ is positive on $(0,\infty)$ and $R'(t^*) \geq (>) 0$ for some $0 < t^* \leq T$ then $R'(t) \geq (>) 0$ for every $t \leq t^*$.*

Proof. Since for $>$ a similar proof applies we only give the proof for $\geq$. It is obvious that the function φ listed in relation (3.12) is decreasing. Consider now some $t < t^*$. If $\varphi(t) > 0$ then by relation (3.11) the result follows and we consider the remaining case $\varphi(t) \leq 0$. Since φ is decreasing and $\varphi(t) \leq 0$ it follows that the function φ has non-positive values on $[t, T]$. This shows using $y \mapsto 1 - F(T-y)$ is increasing and φ decreasing and non-positive on $[t, T]$ that the function $y \mapsto r - \kappa + (1 - F(T-y))\varphi(y)$ is decreasing on $(t, T]$. Hence for $t < t^*$ we obtain

$$\begin{aligned} \tfrac{R'(t)}{\lambda(t)} &= r - \kappa + (1 - F(T-t))\varphi(t) \\ &\geq r - \kappa + (1 - F(T-t_0))\varphi(t^*) \\ &= \tfrac{R'(t^*)}{\lambda(t_*)} \end{aligned} \tag{3.13}$$

Since $R'(t^*) \geq 0$ and both $\lambda(t)$ and $\lambda(t^*)$ are positive this implies by relation (3.13) that $R'(t) \geq 0$ and we have shown the result. □

The next result is an easy consequence of Lemma 5. In this result we will show under which necessary and sufficient conditions we will never close the fare class during the booking period. Observe this condition has a clear intuitive interpretation.

Lemma 6 *If the cdf F is continuous on $(0,\infty)$ and it satisfies $F(0+) = 0$ and the function λ is positive and continuous on $(0,\infty)$, then an optimal solution of optimization problem (P) is given by T if and only if*

$$r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) \geq 0 \tag{3.14}$$

Proof. If $r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) \geq 0$ we obtain using $F(0^+) = 0$ that by relation (3.12)

$$r - \kappa + (1 - F(0+))\varphi(T) = r - \kappa + \varphi(T) = r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) \geq 0.$$

This implies using $\lambda(T) > 0$ and (3.11) that $R'(T) \geq 0$. Hence by Lemma 5 we obtain $R'(t) \geq 0$ for every $t \leq T$ and this shows T is an optimal solution. If T is an optimal solution then clearly $R'(T) \geq 0$ and this implies by (3.11), (3.12) and $F(0+) = 0$ that

$$r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) = r - \kappa + (1 - F(0+))\varphi(T) \geq 0.$$

This shows the desired result. □

Also an easy consequence of Lemma 5 is given by the following characterization of an optimal solution for all instances.

Lemma 7 *If the cdf F is continuous on $(0, \infty)$ and it satisfies $F(0+) = 0$ and the function λ is positive and continous on $(0, \infty)$, then an optimal solution t_{opt} of optimization problem (P) exists and it is given by*

$$t_{opt} = \begin{cases} T & \text{if } r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) \geq 0 \\ \max\{0 \leq t \leq T : g(t) \geq 0\} & \text{if } r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) < 0 \end{cases} \tag{3.15}$$

with $g(t) := r - \kappa + (1 - F(T - t))\varphi(t)$.

Proof. Clearly for $r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) \geq 0$ the result is shown in Lemma 6 and so we only need to consider the case that $r - p\gamma\mathbb{P}(\mathbf{Y}(ph(T)) \geq C) < 0$. This yields by Lemma 6 that $0 \leq t_{opt} < T$ and by Lemma 5 and $R'(0+) \geq 0$ that the set $\{0 \leq t \leq T : R'(t) \geq 0\}$ is compact, convex and nonempty. Since $[0, T]$ is a convex set also the complementary set $\{0 \leq t \leq T : R'(t) < 0\}$ is open, convex and nonempty and so an optimal solution t_{opt} is given by

$$t_{opt} = \max\{0 \leq t \leq T : R'(t) \geq 0\}.$$

Since λ is a positive function and applying Lemma 4 this yields the formula for t_{opt} in (3.15). □

Clearly for any cdf F and λ a positive and continuous function the left derivative

$$h'_{-}(t) = \lim_{s\downarrow 0} \frac{h(t) - h(t-s)}{s}$$

exists. It is given by $h'_{-}(t) = \lambda(t)(1-F(T-t))$,which is left continuous. This implies that also the left derivative $R'_{-}(t)$ of the objective function R exists and we obtain a similar representation for t_{opt} replacing $R'(t)$ (which might not exists!) by its left derivative $R'_{-}(t)$. Also it is easy to determine t_{opt} by applying a bisection procedure to the function $r - \kappa + (1 - F(T-t))\varphi(t)$ to locate its zero point.

3.2 Static Model With Multiple Fare Classes.

In this section we consider a static model with m multiple fare classes. As for the one fare class case discussed in the previous section the pairs $(T_{ij}, Y_{ij})_{i\in\mathbb{N}}$ can be considered as atoms of a Poisson random measure N_j given by

$$N_j(\omega, A) := \sum\nolimits_{i\in\mathbb{N}} 1_A(T_{ij}(\omega), Y_{ij}(\omega)),\ \forall(\omega, A) \in \Omega \times \mathbb{B}(\mathbb{R}_+^2).$$

with mean measure

$$\nu_j(ds\,dy) = \lambda_j(s)ds \cdot F_j(dy).$$

It is assumed that $i)$ the Poisson random measures N_j, $1 \leq j \leq m$ are independent, $ii)$ the functions $\lambda_j, 1 \leq j \leq m$ are locally bounded and Borel, $iii)$ the cdf $F_j, 1 \leq j \leq m$ representing the cdf of the times to cancellation of a fare class j customer satisfy $F_j(0+) = 0$. If we apply the static strategy $\mathbf{t} = (t_1, ..., t_m)$, meaning fare class j is closed at time t_j for $j \leq m$, we obtain that the total random number of non-cancelling fare class j customers is given by

$$N_j f_j = \int_{\mathbb{R}_+^2} f_j(s, y) N_j(dsdy) \tag{3.16}$$

with $f_j : \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$f(s,y) := 1_{\{s \leq t_j\}} 1_{\{s+y>T\}}, \; \forall (s,y) \in \mathbb{R}_+^2 \tag{3.17}$$

Once again the random variable $N_j f_j$ is Poisson distributed with mean $h_j(t_j)$ with

$$h_j(t) := \int_0^t \lambda_j(s)(1 - F_j(T-s))ds. \tag{3.18}$$

Also the total number of cancelling fare class j customers is given by

$$N_j g_j = \int_{\mathbb{R}_+^2} g_j(s,y) N_j(dsdy) \tag{3.19}$$

with $g_j : \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$g_j(s,y) = 1_{\{s \leq t_j\}} 1_{\{s+y \leq T\}}, \; \forall (s,y) \in \mathbb{R}_+^2 \tag{3.20}$$

The random variable $N_j g_j$ is Poisson distributed with mean $\Lambda_j(t_j) - h_j(t_j)$ with Λ_j the maen arrival function listed in (2.1). Although not used in our analysis one can also show (see Chapter 6 of [13]) that the random variables $N_j g_j$ and $N_j f_j$ are independent. Now the total random number $S_j(t_j)$ of fare class j customers arriving at the departure time T is given by

$$S_j(t_j) = \sum_{i=1}^{N_j f_j} B_{ij} \tag{3.21}$$

and by the Bernoulli selection mechanism it is well known that the random variable $S_j(t_j)$ is Poisson distributed with mean $p_j h_j(t_j)$. Since the Poisson random measures are independent the random variables $S_j(t_j)$, $1 \leq j \leq m$, are also independent. This implies that the total random number $S(\mathbf{t})$ of customers arriving at the departure time T is given by

$$S(\mathbf{t}) = \sum_{j=1}^{m} S_j(t_j) = \sum_{j=1}^{m} \sum_{i=1}^{N_j f_j} B_{ij} \tag{3.22}$$

and this random variable is Poisson distributed with mean $\sum_{j=1}^{m} p_j h_j(t_j)$. An equivalent representation of the random arrival processes, which proves to be useful in the simulation part of the computational section, is that the overall arrival process of request is given by a nonhomogeneous Poisson process with arrival intensity $\lambda(t) = \sum_{j=1}^{m} \lambda_j(t)$ and, given an arrival occurs at time s, this arrival is a fare class j request with probability $\lambda_j(s)(\sum_{i=1}^{m} \lambda_i(s))^{-1}$. Using the above observations and (2.6) it is easy to verify that for m different fare classes the expected revenue function $R : [0,T]^m \to \mathbb{R}$ is given by

$$R(\mathbf{t}) = \sum_{j=1}^{m}((r_j - \kappa_j)\Lambda_j(t_j) + \kappa_j h_j(t_j)) - \gamma S_C(\sum_{j=1}^{m} p_j h_j(t_j). \tag{3.23}$$

Particular instances of the above problem are given by $r_j = \kappa_j$ for every $1 \leq j \leq m$ or no cancellations occur. For the first case ($r_j = \kappa_j$ for every $1 \leq j \leq m$) the expected revenue function reduces to

$$R(\mathbf{t}) = \sum_{j=1}^{m} r_j h_j(t_j) - \gamma S_C(\sum_{j=1}^{m} p_j h_j(t_j)). \tag{3.24}$$

while in the later case (no cancellations occur: that is $F_j(T) = 0$ for every $1 \leq j \leq m$) it follows that

$$R(\mathbf{t}) = \sum_{j=1}^{m} r_j \Lambda_j(t_j) - \gamma S_C(\sum_{j=1}^{m} p_j \Lambda_j(t_j)). \tag{3.25}$$

We now need to solve the optimization problem

$$v(P) = \sup\{\sum_{j=1}^{m} g_j(t_j) - \gamma S_C(\sum_{j=1}^{m} p_j h_j(t_j)) : \mathbf{t} \in [0,T]^m\}. \tag{P}$$

with $g_j : [0,T] \to \mathbb{R}$ given by

$$g_j(t) := (r_j - \kappa_j)\Lambda_j(t) + \kappa_j h_j(t).$$

We will first show some useful properties of the function R. The next definition is well-known for multivariate functions (see Definition 2 for single variate functions.)

Definition 8 *If $X \subseteq \mathbb{R}^n$ is a nonempty set and $f : X \to \mathbb{R}$ some real-valued function on X then f is called Lipschitz continuous on X with finite Lipschitz constant L_f if*

$$\mid f(\mathbf{x}) - f(\mathbf{y}) \mid \leq L_f \|\mathbf{x} - \mathbf{y}\|_M$$

for every $\mathbf{x}, \mathbf{y} \in X$ *with* $\|\mathbf{x}-\mathbf{y}\|_M := \sum_{i=1}^m | x_i - y_i |$ *the well -known Manhattan norm* .

One can now show the following generalization of Lemma 3.

Lemma 9 *The revenue function* $R : [0,T]^m \to \mathbb{R}$ *in (3.23) is Lipschitz continuous on* $[0,T]^m$ *with Lipschitz constant*

$$L_R = \sum\nolimits_{j=1}^m (r_j + \gamma p_j)\|\lambda_j\|_\infty \tag{3.26}$$

and an optimal solution of optimization problem (P) *exists.*

Proof. It follows by Lemma 1 that the function $S_C : [0,\infty) \to \mathbb{R}$ is Lipschitz continuous on $[0,\infty)$ with Lipschitz constant 1. Also the functions $\Lambda_j : [0,T] \to \mathbb{R}$ and $h_j : [0,T] \to \mathbb{R}$ are Lipschitz continuous on $[0,T]$ with Lipschitz constant $\|\lambda_j\|_\infty$, and this shows by standard techniques that the function $R : [0,T]^m \to \mathbb{R}$ listed in (3.23) is Lipschitz continuous with Lipschitz constant $\sum_{j=1}^m (r_j + \gamma p_j)\|\lambda_j\|_\infty$. Since any Lipschitz continuous function on $[0,T]^m$ is continuous and $[0,T]^m$ is a compact set the last result follows by Weierstrass theorem (See Rudin [32]) □

Under the following stronger conditions it is easy to show that the function R is differentiable. For notational convenience we introduce the constants α_j and $\beta_j, 1 \leq j \leq m$, given by

$$\alpha_j = \frac{\kappa_j}{p_j},\ \beta_j := \frac{r_j - \kappa_j}{p_j} \tag{3.27}$$

Lemma 10 *If the cdf* F_j *and the function* λ_j *are continuous on* $(0,\infty)$ *for every* $1 \leq j \leq m$*, then the revenue function* R *is differentiable. In particular, for every* $\mathbf{t} \in (0,T)^m$ *its partial derivative* $\frac{\partial R}{\partial t_k}(\mathbf{t})$*,* $1 \leq k \leq m$ *at* $\mathbf{t}$ *is given by*

$$\frac{\partial R}{\partial t_k}(\mathbf{t}) = p_k \lambda_k(t_k)(\alpha_k + (1 - F_k(T - t_k))\varphi_k(\mathbf{t})) \tag{3.28}$$

with

$$\varphi_k(\mathbf{t}) := \beta_k - \gamma S_C'(\sum\nolimits_{i=1}^m p_i h_i(t_j)). \tag{3.29}$$

Since the functions λ_j and F_j, $1 \leq j \leq m$ might have any type of behavior, it is easy to see that in general the objective function R will not be convex. This means that the optimization problem (P) belongs to the field of global optimization [17]. This means it is difficult to solve. So we will first focus on deriving necessary and sufficient conditions for a special case in which all the fare classes will be opened during the booking period. This problem was solved for the one fare class case in Lemma 6. To solve this for multiple fare classes we first observe for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^m$ that $\mathbf{x} \geq \mathbf{y}$ denotes the componentwise ordering. The next generalization of an increasing function to more dimensions is well known.

Definition 11 *A function $f : \mathbb{R}^m \to \mathbb{R}$ is called increasing if $\mathbf{x} \geq \mathbf{y}$ implies $f(\mathbf{x}) \geq f(\mathbf{y})$. The function f is decreasing if -f is decreasing.*

The next result generalizes Lemma 5 for a single fare class.

Lemma 12 *Suppose that the revenue function R is differentiable and for some k the function λ_k is positive on $(0, \infty)$. If $\frac{\partial R}{\partial t_k}(\mathbf{t}^*) \geq (>) 0$ for some $\mathbf{t}^* \in (0, T)^m$, then $\frac{\partial R}{\partial t_k}(\mathbf{t}) \geq (>) 0$ for every $\mathbf{t} \leq \mathbf{t}^*$.*

Proof. Since the proof for $>$ is similar we only give a proof for $\geq$. Consider some $\mathbf{t} \leq \mathbf{t}^*$ such that $\mathbf{t} \neq \mathbf{t}^*$. If $\phi_k(\mathbf{t}) \geq 0$ then by (3.29) and (3.28) we obtain

$$\frac{\partial R}{\partial t_k}(\mathbf{t}) \geq 0$$

and so we only need to consider the remaining case $\phi_k(\mathbf{t}) \leq 0$. Since $\mathbf{t} \leq \mathbf{t}^*$ and $\mathbf{t} \neq \mathbf{t}^*$, we know using the definition of h_j listed in (3.18) that the function

$$\alpha \mapsto \sum\nolimits_{i=1}^{m} p_i h_i(\alpha t_i^* + (1-\alpha) t_i)$$

is increasing on $[0, 1]$. This implies by Lemma 1 that the function $g_k : [0, 1] \to \mathbb{R}$ given by

$$g_k(\alpha) = \varphi_j(\alpha \mathbf{t}^* + (1-\alpha)\mathbf{t})$$

is decreasing on $[0, 1]$. Since $g_k(0) = \varphi_k(\mathbf{t}) \leq 0$ this shows that g_k has non-positive values on $[0, 1]$. Also using again $\mathbf{t} \leq \mathbf{t}^*$ we obtain that the function

$$\alpha \mapsto 1 - F_k(T - (\alpha t_k^* + (1 - \alpha)t_k))$$

is increasing and nonnegative on $[0, 1]$ and so the function $g_k : [0, 1] \to \mathbb{R}$ given by

$$g_k(\alpha) = [1 - F_k(T - (\alpha t_k^* + (1 - \alpha)t_k))]\varphi_k(\alpha\mathbf{t}^* + (1 - \alpha)\mathbf{t})$$

is decreasing on $[0, 1]$. Hence we obtain

$$\begin{aligned} \frac{\frac{\partial R}{\partial t_k}(\mathbf{t})}{p_k\lambda_k(t_k)} &= \alpha_k + (1 - F_k(T - t_k))\varphi_k(\mathbf{t}) \\ &= \alpha_k + g_k(0) \\ &\geq \alpha_k + g_k(1) \\ &= \frac{\frac{\partial R}{\partial tk}(\mathbf{t}^*)}{p_k\lambda_k(t_k^*)} \end{aligned} \tag{3.30}$$

Since $\frac{\partial R}{\partial t_k}(\mathbf{t}^*) \geq 0$ and both $\lambda_k(t_k)$ and $\lambda_k(t_k^*)$ are positive and $p_k > 0$ this implies by relation (3.30) that $\frac{\partial R}{\partial t_k}(\mathbf{t}) \geq 0$ for every $\mathbf{t} \leq \mathbf{t}^*$ and we have shown the result.. □

An easy consequence of Lemma 12 is the following necessary and sufficient condition for keeping in the optimal solution all the fare classes open during the booking period. This result generalizes Lemma 6 for the one fare class case. Observe the vector $\mathbf{e} \in \mathbb{R}^m$ denotes the vector of which all of its componets are 1.

Lemma 13 *If the cdf F_j is continuous on $(0, \infty)$ and it satisfies $F_j(0+) = 0$ for every $1 \leq j \leq m$ and the functions $\lambda_j, 1 \leq j \leq m$ are positive, then an optimal solution of optimization problem (P) is given by $T\mathbf{e}$ if and only if*

$$r_j - p_j\gamma\mathbb{P}(\mathbf{Y}(\sum_{i=1}^{m} p_i h_i(T)) \geq C) \geq 0,\ 1 \leq j \leq m \tag{3.31}$$

Proof. By our assumption and relation (3.28) it follows $\nabla R(T\mathbf{e}) \geq \mathbf{0}$ with ∇ denoting the gradient operator. This shows by Lemma 12 that $\nabla R(\mathbf{t}) \geq \mathbf{0}$ for every $\mathbf{t} \in (0,T)^m$. Hence the function R is increasing on B and we obtain that $T\mathbf{e}$ is an optimal solution. If $T\mathbf{e}$ is an optimal solution then it follows for every $1 \leq j \leq m$ that

$$\frac{\partial R}{\partial t_j}(T\mathbf{e}) = \lim_{h \downarrow 0} \frac{R(T\mathbf{e} + h\mathbf{e}_j) - R(T\mathbf{e})}{h} \geq 0$$

Applying now relation (3.28) we obtain the desired result. □

As for the one dimensional case (see the remark after Lemma 7) it follows for λ_j continuous on $(0,\infty)$ for every $1 \leq j \leq m$ that the left partial derivative at $\mathbf{t}$

$$\frac{\partial R_-}{\partial t_k}(\mathbf{t}) := \lim_{s \downarrow 0} \frac{R(\mathbf{t}) - R(\mathbf{t} - \mathbf{s}\mathbf{e}_k)}{s}$$

always exists and equals the formula in relation (3.28). This means that one can derive for arbitrary cdf F_j satisfying $F_j(0+) = 0$ a similar type of result as in Lemma 12 and 13 for continuous cdfs. To identify the optimal solution the next step would be to write down for a differentiable revenue function the KKT conditions for (P). However, these conditions do not show for the general case any special properties of an optimal solution useful in the construction of a fast algorithm unless we consider the very special instances as considered in (3.24) and (3.25). For these no cancellation cases the optimization problem has the same type of structure as the optimization problem considered in Topaloglu et al. [31]. This enables us to show that finding an optimal solution reduces to solving m single fare class problems. This approach will be pursued in the last section in this paper. Due to this and the already considered global optimization structure of the general problem we will first propose in the next section a dynamic programming procedure to solve problem (P) approximately.

Chapter 4

Algorithms

In this chapter, we present the solution approaches for solving the problem have been formulated in chapter 3. A dynamic programming representation is used for solving the problem in general setting and a error bound is also given. Next, for some easy solvable subcases, we present specified algorithms.

4.1 On Algorithms Solving The Static Model

Since for the general case it is difficult to solve problem (P) we will first propose a dynamic programming formulation. To start with we discretize the set $[0, T]$. Consider some $\epsilon > 0$ such that $T\epsilon^{-1}$ belongs to $\mathbb{N}$ and let $D = \{t_0, ..., t_{n_\epsilon}\} \subseteq [0, T]$ with $t_i = i\epsilon, i \in \{0, ..., n_\epsilon\}$ with $n_\epsilon = T\epsilon^{-1}$. Consider now the optimization problem

$$v(P_D) = \max\{\sum_{j=1}^{m} g_j(t_j) - \gamma S_C(\sum_{j=1}^{m} p_j h_j(t_j)) : \mathbf{t} \in D^n\} \qquad (P_D)$$

Using Lemma 9 the following result can be shown by standard techniques.

Lemma 14 *It follows that*

$$0 \leq \nu(P) - \nu(P_D) \leq L_R \epsilon$$

with L_R listed in (3.26).

In the next part we will give a DP algortihm to solve problem (P_D). To do so we introduce the values $0 = \rho_0 < \rho_1 < \rho_2 < ... < \rho_m$ with

$$\rho_k := \sum\nolimits_{j=1}^{k} p_j h_j(T),\ 1 \leq k \leq m,$$

and let $\mathcal{F}_k$ denote the set of bounded functions $f : [0, \rho_k] \to \mathbb{R}, 0 \leq k \leq m$. If we admit all the requests for fare class 1 up to k, then ρ_k denotes the expected number of fare class 1 up to k customers showing up at the departure time. For locally bounded Borel functions $\lambda_j, 1 \leq j \leq m$ and continous cdf's $F_j, 1 \leq j \leq m$, satisfying $F_j(0+) = 0$, introduce also the bounded functions $\nu_k : [0, \rho_{k-1}] \to \mathbb{R}$, $1 \leq k \leq m$ as

$$\nu_k(y) := \max\nolimits_{t_j \in D, j \geq k} \{\sum\nolimits_{j=k}^{m} g_j(t_j) - \gamma S_C(y + \sum\nolimits_{j=k}^{m} p_j h_j(t_j))\}. \quad (4.1)$$

By Lemma 1 the functions $\nu_k, 2 \leq k \leq m$ are decreasing on $(0, \rho_{k-1})$ and $\nu_1(0)$ denotes the objective value of optimization problem (P_D). Also applying Lemma 1 it is easy to show by standard techniques the following result.

Lemma 15 *The functions $\nu_k : [0, \rho_{k-1}] \to \mathbb{R}$, $1 \leq k \leq m$ are Lipschitz continuous on $[0, \rho_{k-1}]$ with Lipschitz constant L_{ν_k} equal to 1.*

To verify the next result we introduce for every $1 \leq k \leq m$ the operators $V_k : \mathcal{F}_k \to \mathcal{F}_{k-1}$ defined by

$$V_k[f](y) := \sup\nolimits_{t \in D} \{g_k(t) + f(y + p_k h_k(t))\}. \quad (4.2)$$

Lemma 16 *If the function $f : [0, \rho_m] \to \mathbb{R}$ is given by $f(y) = -\gamma S_C(y)$ then it follows that*

$$\nu_m(y) = V_m[f](y) \quad (4.3)$$

for every $0 \leq y \leq \rho_{m-1}$ and

$$\nu_k(y) = V_k[\nu_{k+1}](y) \quad (4.4)$$

for every $1 \leq k \leq m-1$ and $0 \leq y \leq \rho_{k-1}$.

Proof. It follows by the definition of ν_m in (4.1) and (4.2) that

$$\nu_m(y) = \max_{t\in D}\{g_m(t) - \gamma S_C(y + p_m h_m(t))\} = V_m[f](y)$$

for every $0 \leq y \leq \rho_{m-1}$ and so the result is proved for $k = m$. Also for every $k \leq m-1$ it follows by (4.1) that

$$\begin{aligned}
\nu_k(y) &= \max_{t_k\in D}\sup_{t_j\in D, j\geq k+1}\{\textstyle\sum_{j=k}^m g_j(t_j) - \gamma S_C(y + \sum_{j=k}^m p_j h_j(t_j))\} \\
&= \max_{t\in D}\left\{g_k(t) + \max_{t_j\in D, j\geq k+1}\{\textstyle\sum_{j=k+1}^m g_j(t_j) - \gamma S_C(y + p_k h_k(t) + \sum_{j=k+1}^m p_j h_j(t_j)\}\right\} \\
&= \sup_{t\in D}\{g_k(t) + \nu_{k+1}(y + p_k h_k(t))\} \\
&= V_k[v_{k+1}](y)
\end{aligned} \tag{4.5}$$

for every $0 \leq y \leq \rho_{k-1}$ and we have verified (4.4). □

By the above result one can compute by means of m successive iterations the value $\nu_1(0) = v(P_D)$. To compute this value we need to compute for every $2 \leq k \leq m$ iteratively the functions $\nu_k : [0, \rho_{k-1}] \to \mathbb{R}$. Since one can only evaluate finite sequences on a computer the iterative procedure in Lemma 16 due to the continous domains $[0, \rho_k]$ needs to be adapted. This can be achieved replacing the set $[0, \rho_k], 1 \leq k \leq m$ by a finite set D_k and for each element y of this finite subset we will compute an accurate approximation of the value $\nu_k(y)$. To construct such a finite set consider some $N \in \mathbb{N}$ and introduce $D_m \subseteq [0, \rho_m]$ given by

$$D_m = \{ih_m : i = 0, ..., N\}$$

with $h_m = \rho_m N^{-1}$. Similarly, to construct a finite subset of $[0, \rho_k], 1 \leq k \leq m-1$ let

$$D_k := \{ih_k : i = 0, ..., N\}$$

with $h_k = \rho_k N^{-1}$. If the mappings $L_k : [0, \infty) \to D_k, 1 \leq k \leq m$ are given by

$$L_k(y) := \min\{h_k \lceil y h_k^{-1} \rceil, \rho_k\} \tag{4.6}$$

with $\lceil x \rceil$ denoting the smallest integer greater then or equal to x, we introduce the composite function $f \circ L_k : [0, \infty] \to \mathbb{R}$ given by

$$(f \circ L_k)(y) := f(L_k(y)).$$

Before discussing our computational algorithm we show the following result.

Lemma 17 *If the functions* $\nu_k^{(a)} : [0, \rho_{k-1}] \to \mathbb{R}$,$1 \leq k \leq m$ *are defined iteratively by*

$$\nu_m^{(a)}(y) := \nu_m(y),\ \nu_k^{(a)}(y) := V_k[\nu_{k+1}^{(a)} \circ L_k](y), 1 \leq k \leq m-1 \tag{4.7}$$

then

$$\nu_k^{(a)}(y) \leq \nu_k(y)$$

for every $y \in [0, \rho_{k-1}]$ *and* $1 \leq k \leq m$.

Proof. For every $y \in [0, \rho_{m-1}]$ it follows by definition that $\nu_m^{(a)}(y) \leq \nu_m(y)$. Assume now for some $k \leq m-2$ that

$$\nu_{k+1}^{(a)}(y) \leq \nu_{k+1}(y) \tag{4.8}$$

for every $y \in [0, \rho_k]$. To show that $\nu_k^{(a)}(y) \leq \nu_k(y)$ for every $y \in [0, \rho_{k-1}]$ consider some $y \in [0, \rho_{k-1}]$. For this selected y we obtain for every $t \leq T$ that $y + p_k h_k(t)) \leq \rho_k$ and hence by the definition of L_k given in (4.6) it follows

$$L_k(y + p_k h_k(t)) \geq \min\{y + p_k h_k(t), \rho_k\} = y + p_k h_k(t). \tag{4.9}$$

Since by (4.1) it is obvious that the function $\nu_{k+1} : [0, \rho_k] \to \mathbb{R}$ is decreasing this yields using (4.9) that

$$(\nu_{k+1} \circ L_k)(y + p_k h_k(t))) \leq \nu_{k+1}(y + p_k h_k(t)) \tag{4.10}$$

Applying now (4.8) and (4.10) it follows for every $y \in [0, \rho_{k-1}]$ that

$$\begin{aligned} \nu_k^{(a)}(y) &= \max_{t \in D}\{g_k(t) + \nu_{k+1}^{(a)}(L_k(y + p_k h_k(t)))\} \\ &\leq \max_{t \in D}\{g_k(t) + \nu_{k+1}(L_k(y + p_k h_k(t)))\} \\ &\leq \max_{t \in D}\{g_k(t) + \nu_{k+1}(y + p_k h_k(t))\} \\ &= V_k[\nu_{k+1}](y) \\ &= \nu_k(y) \end{aligned}$$

and our induction step is completed. □

Our computational algorithm now evaluates the sequences $\nu_k^{(a)}(y)$ for every $y \in D_{k-1}$ and $1 \leq k \leq m$ and is given by the following algorithm below. Observe for every $y \in D_{k-1}$ that by definition $L_k(y + p_k h_k(t))$ belongs to D_k for every $0 \leq t \leq T$. This means that the approximation algorithm can be evaluated on a computer.

Approximation algorithm.

- Let $f : [0, \rho_m] \to \mathbb{R}$ be given by $f(y) = -\gamma S_C(y)$. Evaluate for every $y \in D_{m-1}$ the finite sequence
$$\nu_m^{(a)}(y) = \nu_m(y).$$
- For $k = m - 1$ down to $k = 1$ evaluate for every $y \in D_{k-1}$ the value
$$\nu_k^{(a)}(y) := V_k[\nu_{k+1}^{(a)} \circ L_k](y)$$
- Output $\nu_1^{(a)}(0)$ and backtrack the solution achieving $\nu_1^{(a)}(0)$.

The next result is an immediate consequence of Lemma 17, and since $\nu_1(0) = v(P_D)$ it requires no proof .

Corollary 18 *We have* $\nu_1^{(a)}(0) \leq v(P_D)$.

To compute the error of using the above approximative algorithm we need to derive an upperbound on

$$v(P_D) - \nu_1^{(a)}(0).$$

To achieve this introduce on the set $\mathcal{F}_k$, $0 \leq k \leq m$ of functions $f : [0, \rho_k] \to \mathbb{R}$ the supnorm $\|f\|_k$ given by

$$\|f\|_k = \sup_{0 \leq y \leq \rho_k} \mid f(y) \mid . \tag{4.11}$$

It is easy to verify the following result.

Lemma 19 *It follows for every $f, g \in \mathcal{S}_k$ that*

$$\|V_k f - V_k g\|_{k-1} \leq \|f - g\|_k$$

Proof. By the definition of the operator V_k in (4.2) there exists for every $0 \leq y \leq \rho_{k-1}$ some $t_y \in D$ satisfying

$$V_k[f](y) = g_k(t_y) + f(y + t_y p_k h_k(t_y)).$$

This shows using $y + p_k h_k(t_y) \leq \rho_k$ that

$$V_k[f](y) - V_k[g](y) \leq f(y + p_k h_k(t_y)) - g(y + t_y p_k h_k(t_y)) \leq \| f - g \|_k$$

By a similar argument one can show that

$$V_k[f](y) - V_k[g](y) \geq - \| f - g \|_k$$

and so we obtain

$$\mid V_k[f](y) - V_k[g](y) \mid \leq \| f - g \|_k . \tag{4.12}$$

Since (4.12) holds for any $0 \leq y \leq \rho_{k-1}$ the desired result follows. □

Applying the above result we will now derive an upperbound on the the error caused by using the approximative algorithm to solve (P_D).

Theorem 20 *It follows that*

$$0 \leq v(P_D) - \nu_1^{(a)}(0) \leq \gamma N^{-1} \sum_{k=1}^{m-1} \rho_k$$

with $N+1$ the cardinality of the set D_m.

Proof. It follows for every $1 \leq k \leq m$ and $0 \leq y \leq [0, \rho_{k-1}]$ that

$$\nu_k(y) - \nu_k^{(a)}(y) = V_k[\nu_{k+1}](y) - V_k[\nu_{k+1} \circ L_k](y) + V_k[\nu_{k+1} \circ L_k](y) - V_k[\nu_{k+1}^{(a)} \circ L_k](y)$$

This implies by the subadditivity of a norm in (4.11) and Lemma 19 that

$$\begin{aligned} \|[\nu_k] - \nu_k^{(a)}\|_{k-1} &\leq \|V_k[\nu_{k+1}] - V_k[\nu_{k+1} \circ L_k]\|_{k-1} + \|V_k[\nu_{k+1} \circ L_k] - V_k[\nu_{k+1}^{(a)} \circ L_k]\|_{k-1} \\ &\leq \|\nu_{k+1} - \nu_{k+1} \circ L_k\|_k + \|\nu_{k+1} \circ L_k - \nu_{k+1}^{(a)} \circ L_k\|_k. \end{aligned} \tag{4.13}$$

Since $L_k([0, \rho_k]) \subseteq [0, \rho_k]$, we have

$$\|\nu_{k+1} \circ L_k - \nu_{k+1}^{(a)} \circ L_k\|_k \leq \|\nu_{k+1} - \nu_{k+1}^{(a)}\|_k$$

and we obtain by (4.13)

$$\|\nu_k - \nu_k^{(a)}\|_{k-1} \leq \quad \| \nu_{k+1} - \nu_{k+1} \circ L_k\|_k + \|\nu_{k+1} - \nu_{k+1}^{(a)}\|_k$$

Hence by iterating over k and using $\nu_m(y) = \nu_m^{(a)}(y)$ it follows that

$$\| \nu_1 - \nu_1^{(a)} \|_0 \leq \sum_{k=1}^{m-1} \| \nu_{k+1} - \nu_{k+1} \circ L_k \|_k$$

Applying now Lemma 15 we know that

$$\| \nu_{k+1} - \nu_{k+1} \circ L_k \|_k \leq h_k \gamma = \gamma \rho_k N^{-1}$$

and this shows the upperbound. The lowerbound is already verified in Corollary 18. □

The next corollary is an immediate consequence of Lemma 14 and Theorem 20 above.

Corollary 21 *It follows that*

$$0 \leq v(P) - \nu_1^{(a)}(0) \leq m\epsilon \sum_{j=1}^{m} (r_j + \gamma p_j) \parallel \lambda_j \parallel_\infty + \gamma N^{-1} \sum_{k=1}^{m-1} \rho_k.$$

Proof. It is obvious that $v(P_D) \leq v(P)$. Apply now Lemma 14 and Theorem 20. □

In the next section we will evaluate the easy subcases and show that these problems have an easy optimal solution.

4.2 On Solving the Easy Subcases

If customers do not cancel before departure it is easy to see that the considered optimization problem reduces to

$$\max\{\sum_{j=1}^{m} r_j \int_0^{t_j} \lambda_j(s)ds - \gamma S_C(\sum_{j=1}^{m} p_j \int_0^{t_j} \lambda_j(s)ds) : 0 \leq t_j \leq T, 1 \leq j \leq m\}$$

Also if the refund κ_j is the the same as the revenue r_j we obtain the optimization problem

$$\max\{\sum_{j=1}^{m} \kappa_j h_j(t_j) - \gamma S_C(\sum_{j=1}^{m} p_j h_j(t_j)) : 0 \leq t_j \leq T, 1 \leq j \leq m\}.$$

In case there is no cancellation before departure we consider the optimization problem

$$\max\{\sum_{j=1}^{m} r_j x_j - \gamma S_C(\sum_{j=1}^{m} p_j x_j) : 0 \leq x_j \leq \int_0^T \lambda_j(s)ds, 1 \leq j \leq m\}$$

Also if the refund is the same as the revenue we consider the problem

$$\max\{\sum_{j=1}^{m} \kappa_j x_j - \gamma S_C(\sum_{j=1}^{m} p_j x_j) : 0 \leq x_j \leq h_j(T), 1 \leq j \leq m\}$$

Hence in both cases we need to solve the optimization problem

$$\max\{\sum_{j=1}^{m} a_j x_j - \gamma S_C(\sum_{i=1}^{m} p_i x_i) : 0 \leq x_j \leq u_j\} \tag{Q}$$

with $a_j > 0$. Suppose now that

$$\frac{a_1}{p_1} < \frac{a_2}{p_2} < ... < \frac{a_m}{p_m}$$

Clearly the objective function of optimization problem (Q) is strictly concave and so this optimization problem has a unique optimal solution $\mathbf{x}^*$. Also it is well known ([3] or page 196 of [7]) that the KKT conditions given by

$$\begin{aligned}
&a_j - p_j \gamma S_C'(\textstyle\sum_{i=1}^{m} p_i x_i^*) = \lambda_j^* - \mu_j^* && 1 \leq j \leq m \\
&\lambda_j^*(u_j - x_j^*) = 0 && 1 \leq j \leq m \\
&\mu_j^* x_j = 0 && 1 \leq j \leq m \\
&\lambda_j^*, \mu_j^* \geq 0 && 1 \leq j \leq m
\end{aligned} \tag{4.14}$$

are necessary and sufficient. Using the necessary and sufficient KKT conditions we can now derive the following structural result for the unique optimal solution $\mathbf{x}^*$ of optimization problem (Q).

Lemma 22 *The unique optimal solution $\mathbf{x}^*$ of optimization problem (Q) is either given by $\mathbf{x}^* = (u_1, ..., u_m)$ or there exists some index $1 \leq j_0 \leq m$ such that $0 \leq x_{j_0}^* \leq u_{j_0}$, $x_j^* = 0$ for every $j \leq j_0 - 1$ and $x_j = u_j$ for every $j \geq j_0 + 1$*

Proof. If $(u_1, ..., u_m)$ is not optimal and we know that there exists an optimal solution $\mathbf{x}^*$ it must follow that or some $1 \leq j \leq m$ satisfying $x_j^* < u_j$ for some some $1 \leq j \leq m$. Introduce now $j_0 := \max\{j \leq m : x_j^* < u_j\}$. By the definition of j_0 we obtain that $x_j^* = u_j$ for every $j \geq j_0 + 1$. Since $x_{j_0}^* < u_{j_0}$ and any optimal solution x^* satisfies the

KKT conditions it must follow that $\lambda^*_{j_0} = 0$ and so

$$\frac{a_{j_0}}{p_{j_0}\gamma} - S'_C(\sum_{i=1}^{m} p_i x_i^*) = \frac{-\mu^*_{j_0}}{p_{j_0}\gamma} \leq 0$$

This shows using $\frac{a_1}{p_1} < \frac{a_2}{p_2} < ... < \frac{a_m}{p_m}$ that

$$\begin{aligned} a_j - p_j\gamma S'_C(\textstyle\sum_{i=1}^m p_i x_i^*) &= p_j\gamma(\frac{a_j}{p_j\gamma} - S'_C(\textstyle\sum_{i=1}^m p_i x_i^*)) \\ &< p_j\gamma(\left(\frac{a_{j_0}}{p_{j_0}\gamma} - S'_C(\textstyle\sum_{i=1}^m p_i x_i^*)\right) \qquad (4.15) \\ &\leq 0 \end{aligned}$$

for every $j \leq j_0 - 1$. If $x_j^* = u_j$ then again by the KKT conditions $\mu_j = 0$ and we obtain a contradiction with relation (4.15) Also for $0 < x_j^* < u_j$ it follows using the left and right partial derivative and the optimality of $\mathbf{x}^*$ that

$$a_j - p_j\gamma S'_C(\sum_{i=1}^m p_i x_i^*) = 0$$

contradicting again relation (4.15). Hence we must have $x_j^* = 0$. □

Using the above result it follows for any optimal solution that either all fare classes are open until departure or there is at most one fare class j_0 such that fare classes $j \geq j_0 + 1$ are opened and fare classes $j, j \leq j_0 - 1$ are closed until departure. Hence we need to check these m possibilities and this can be done by a one dimensional optimization with all closing times of the fare classes fixed except one.

Chapter 5

Numerical results

In this chapter, we test in the first subsection the optimal static strategy generated by the proposed static model (SM) against the optimal dynamic strategy (ODP) derived in Frenk et al. [15]. Since the latter is an optimal policy among all admissible policies and hence serves as a benchmark we can compare the performance of our static strategy against the optimal one under some restrictive conditions on the input parameters. Remember in Frenk et al. [15] we consider an overbooking model with independent nonhomogeneous Poisson arrival processes, fare class independent refunds and showup probabilities and fare class independent exponentially distributed random times to cancellation. We also compare for this instance our proposed optimal static strategy against the policy generated by the EMSR/MP heuristic proposed by Belobaba [6]. Despite its deterministic and heuristic approach to select the virtual capacity of the plane the latter policy applied in a nested way in each simulation run performs well among other policies using more sophisticated heuristics to determine the virtual capacity [2], [29]. Due to this we decided to test our optimal static policy only against this simple heuristic. For more details on this heuristic we refer the reader to Belobaba [6] and Talluri and Van Ryzin [29]. Next, relaxing our assumption on the cancellation distribution and replacing it by a hyperexponential distribution, we discuss in the second subsection the performance of our optimal static policy and benchmark it against the policy generated by the EMSR/MP heuristic.

5.1 Comparison with optimal dynamic policy and EMSR heuristic in the Markovian case

In this section, we evaluate the revenues generated by the optimal static policy and compare it with the revenues of an optimal dynamic policy and the policy generated by the EMSR/MP heuristic (sometimes shortly denoted by MP). To compare the revenue of the optimal static strategy with the revenue of an optimal dynamic policy we assume that refunds and showup probabilities are fare class independent and the cancellation cdf is give by an exponential distribution with the same parameter μ for all fare classes. The number of different fare classes is given by $m = 4$ and without loss of generality the first fare class is the cheapest and the last one the most expensive one. In our computational setup the arrival process of the requests for the different fare classes are independent non-homogenous Poisson processes with arrival intensities

$$\lambda_1(t) = \alpha\frac{T-t}{T^2}, \qquad \lambda_2(t) = 0.8\alpha\frac{T-t}{T^2}, \qquad \lambda_3(t) = 0.55\alpha\frac{t}{T^2}, \qquad \lambda_3(t) = 0.35\alpha\frac{t}{T^2},$$

As observed in reality, the arrival intensity functions are chosen in such a way that the demand for cheap tickets occurs more frequently at the beginning of the booking period, while the demand for the more expensive tickets has a tendency to arrive later in the booking period. Also the total demand for cheaper tickets is more than that for more expensive tickets. The capacity of the plane is C and to measure the demand for seats we introduce the load factor ρ given by

$$\rho = \frac{\mathbb{E}(\text{total demand})}{C} = \frac{\sum_{i=1}^{m}\int_0^T \lambda_i(s)ds}{C} \tag{5.1}$$

In our computational experiments we set $T = 200$ and the capacity of the plane equal to $C = 300$. Also the cancellation refund for all fare classes is given by $\kappa = 25$ and we selected the following parameters for the prices of the fare classes, the fare class independent showup probabilities, the fare class independent penalty overbooking cost and the cancellation parameter:

- $r = (r_1, r_2, r_3, r_4) = (50, 100, 150, 200), \gamma = 300$
- $\mu \in \{0.001, 0.0015, 0.002, 0.0025\}$
- $\rho \in \{1.2, 1.6, 2\}$
- $p \in (0.98, 0.92)$

The choice of the parameter p reflects high and low showups, while a similar distinction is made for the cancellation rate μ and the load ρ. The higher μ the more cancellations will occur and the higher the load ρ the more demand occurs for seats. In our test problems we consider all possible combinations of the parameters μ, ρ and p given by the vector (μ, ρ, p).

In Frenk et al. [15] it is shown that one needs to compute the optimal to go function in order to determine the optimal dynamic policy. This function is the unique solution of a differential equation with respect to time, and to evaluate this optimal to go function one needs to apply a discretization procedure. In our computations the discritization step size is set equal to 0.01 (see [15] for more details). Also to determine the optimal static strategy we set $N = 1000$ (see Corollary 21 for the evaluation of the error), while in the EMSR/MP heuristic we set the virtual capacity $\overline{C} = C/p$. After calculating the booking limits for EMSR/MP we generate 2000 simulation runs and apply the booking limits using theft nesting (see [29]) to evaluate in each run the realized revenue. Then, we report the sample mean as the expected revenue of EMSR/MP policy in the 4^{th} column of Table 5.1. The same procedure is also followed for our optimal static strategy and the optimal dynamic one. To compare these policies, we introduce *% difference* between two strategies **A** and **B** as

$$\%difference = \frac{\mathbf{E}(\text{revenue associated with A}) - \mathbf{E}(\text{revenue associated with B})}{\mathbf{E}(\text{revenue associated with A})}. \quad (5.2)$$

The results in Table 5.1 shows that the revenue generated by the optimal static strategy is close to the revenue generated by the optimal dynamic one. In particular, among all 24 problems, the gap between expected revenues obtained by the optimal dynamic and static

Table 5.1: The Comparison between ODP, SM, and EMSR/MP under exponential cdf

test problem	Expected Revenue			%difference		
($\mu \times 10^4$, ρ, p)	ODP	SM	MP	ODP vs. SM	ODP vs. MP	SM vs. MP
(10, 1.2, 0.92)	35987	35544	35415	1.23%	1.59%	0.36%
(10, 1.2, 0.98)	35143	34555	34403	1.67%	2.11%	0.44%
(10, 1.6, 0.92)	42428	42041	41613	0.91%	1.92%	1.02%
(10, 1.6, 0.98)	41480	41009	40694	1.13%	1.90%	0.77%
(10, 2.0, 0.92)	47644	47036	47003	1.28%	1.34%	0.07%
(10, 2.0, 0.98)	45873	45014	44994	1.87%	1.92%	0.05%
(15, 1.2, 0.92)	36154	35960	35315	0.53%	2.32%	1.79%
(15, 1.2, 0.98)	35548	35002	34274	1.54%	3.58%	2.08%
(15, 1.6, 0.92)	42801	42564	41406	0.55%	3.26%	2.72%
(15, 1.6, 0.98)	41809	41475	40466	0.80%	3.21%	2.43%
(15, 2.0, 0.92)	48422	48302	47914	0.25%	1.05%	0.80%
(15, 2.0, 0.98)	46700	46177	45776	1.12%	1.98%	0.87%
(20, 1.2, 0.92)	36094	35876	35200	0.60%	2.48%	1.88%
(20, 1.2, 0.98)	35699	35424	34203	0.77%	4.19%	3.45%
(20, 1.6, 0.92)	43214	43094	41172	0.28%	4.72%	4.46%
(20, 1.6, 0.98)	42184	41982	40188	0.48%	4.73%	4.27%
(20, 2.0, 0.92)	49379	49326	48374	0.11%	2.03%	1.93%
(20, 2.0, 0.98)	47484	47405	46246	0.17%	2.61%	2.45%
(25, 1.2, 0.92)	35876	35550	35007	0.91%	2.42%	1.53%
(25, 1.2, 0.98)	35719	35481	34024	0.67%	4.75%	4.11%
(25, 1.6, 0.92)	43670	43668	40900	0.01%	6.34%	6.34%
(25, 1.6, 0.98)	42598	42520	39867	0.18%	6.41%	6.24%
(25, 2.0, 0.92)	49903	49902	48668	0.00%	2.47%	2.47%
(25, 2.0, 0.98)	48587	48530	46594	0.12%	4.10%	3.99%

Table 5.2: The performance gap between SM and EMSR/MP averaged over all test problems in form $(\mu, \bullet, \bullet)$

$\mu \times 10^4$	ODP vs. SM	ODP vs. MP	SM vs. MP
10	1.35%	1.80%	0.45%
15	0.80%	2.6%	1.8%
6 20	0.40%	3.46%	3.07%
25	0.31%	4.42%	3.99%

strategies does not exceed 1.87%. Moreover, in 17 problems, the differences in revenue between an optimal static and dynamic strategy are less than 1%. On the other hand, the gap between revenues generated by an optimal policy and a policy generated by the EMSR/MP heuristic can be as high as 6.34%.

By increasing the cancellation rate the optimal static policy outperforms the policy generated by the EMSR/MP heuristic. As shown in Table 5.2, the average difference between the revenues generated by SM and the revenues generated by EMSR/MP is around 0.45% for a low cancellation rate $\mu = 0.0010$, while for a high cancellation rate $\mu = 0.0025$ this gap can be as large as 4.11%. For the load factor ρ, we do not observe any monotone behavior. The gaps between ODP or SM and EMSR/MP increase from $\rho = 1.2$ to $\rho = 1.6$ and then decreases from $\rho = 1.6$ to $\rho = 2.0$.

In Table 5.3 we report the optimal closing times of the static model. First, we use the *if and only if* result of Lemma 13 to determine whether all fare classes will be open till the end of the booking period. For example, for test problem $(0.0025,\ 1.2,\ 0.92)$, the vector of partial derivatives at time T is $(49, 99, 149, 199)$. Hence we can immediately conclude that $t^* = (200, 200, 200, 200)$ shown in Table 5.3. The same holds for the test problems $(0.0020,\ 1.2,\ 0.92)$ and $(0.0025, 1.2, 0.98)$ with the vector of partial derivatives at time T given by (49, 99, 149, 199) and (39, 89, 139, 189) respectively. Secondly, looking in Table 5.3 at problem set $(0.0020, 1.2, 0.98)$, we notice that the first two fare classes are closing at times 184.14 and 194.43 respectively. This shows that the structure of the optimal static strategy presented in Lemma 22 does not hold for the model with cancellations.

Table 5.3: The Static Model's Optimal Closing Time

test problem	Optimal Closing Time			
($\mu \times 10^4$, ρ, p)	t_1^*	t_2^*	t_3^*	t_4^*
(10, 1.2, 0.92)	136.58	200	200	200
(10, 1.2, 0.98)	99.03	200	200	200
(10, 1.6, 0.92)	27.91	199.99	200	200
(10, 1.6, 0.98)	12.29	200	200	200
(10, 2.0, 0.92)	0.21	129.32	200	200
(10, 2.0, 0.98)	0.21	100.97	199.47	200
(15, 1.2, 0.92)	191.86	200	200	200
(15, 1.2, 0.98)	130.31	195.94	200	200
(15, 1.6, 0.92)	40.14	199.99	200	200
(15, 1.6, 0.98)	23.84	200	200	200
(15, 2.0, 0.92)	0.22	160.36	200	200
(15, 2.0, 0.98)	0.22	121.87	200	200
(20, 1.2, 0.92)	200	200	200	200
(20, 1.2, 0.98)	184.14	194.43	200	200
(20, 1.6, 0.92)	58.77	199.99	200	200
(20, 1.6, 0.98)	37.88	200	200	200
(20, 2.0, 0.92)	0.68	193.17	200	200
(20, 2.0, 0.98)	0.22	146.54	200	200
(25, 1.2, 0.92)	200	200	200	200
(25, 1.2, 0.98)	200	200	200	200
(25, 1.6, 0.92)	76.69	199.99	200	200
(25, 1.6, 0.98)	54.44	200	200	200
(25, 2.0, 0.92)	10.12	199.99	200	200
(25, 2.0, 0.98)	0.23	194.6	200	200

5.2 Comparison with EMSR heuristic for hyperexponential distributed times to cancellation

In this subsection, we relax our class independent and exponential distributed times to cancellation. As an example we use a *hyperexponential* distribution as a cdf for times to cancellation. The hyperexponential distribution is a commonly used distribution for positive random variables with coefficient of variation $c_X := \sqrt[2]{\frac{Var(X)}{\mathbb{E}(X)}} \geq 1$ (see for example [30] Appendix B). In particular, we use a hyperexponential distribution of order two with balanced means. This means its density is given by

$$f(t) = p_1\mu_1 e^{-\mu_1 t} + p_2\mu_2 e^{-\mu_2 t}, \qquad t \geq 0 \tag{5.3}$$

with

$$p_1 = \frac{1}{2}\left(1 + \sqrt{\frac{c_X^2 - 1}{c_X^2 + 1}}\right), \qquad p_2 = 1 - p_1, \qquad \mu_k = \frac{2p_k}{E(X)}, \quad k = 1, 2, \tag{5.4}$$

For the above class of distributions the expected time to cancellation stays the same, while c_X^2 can attain any value larger than or equal to 1. Observe for $c_X^2 = 1$ we recover the exponential distribution. Second, we also extend the model by considering fare class dependent showup probabilities and cancellation refunds. The cancellation refund is a class dependent percentage of the ticket price. In particular, let $0 < \alpha_j < 1$ denote the cancellation refund percentage for fare class j and let $\alpha = (0.2, 0.3, 0.6, 0.7)$. This yields for $r = (50, 100, 150, 200)$ that

$$\kappa = (\kappa_1, ..., \kappa_4) = (10, 30, 90, 140).$$

For the showup probability we assume that the two most expensive fare classes 3 and 4 have high show-up probability 0.98 and the other two fare classes have low showup probability 0.92 resulting in $p = (p_1, p_2, p_3, p_4) = (0.92, 0.92, 0.98, 0.98)$. Letting

- $c_X^2 \in \{1.5, 2, 2.5, 3\}$

- $\mu \in \{0.0015, 0.0020, 0.0025\}$
- $\rho \in \{1.6, 2\}$

we label each set problem in Table 5.4 by triple (c_X^2, μ, ρ). Lastly, one needs to adapt the virtual capacity for EMSR/MP and set

$$\overline{C} = \frac{C \sum_{i=1}^{m} \Lambda_i(T)}{\sum_{i=1}^{m} p_i \Lambda_i(T)} \tag{5.5}$$

Note, we construct our test problems by selecting c_X^2 and $\mathbb{E}X$. In other words, for a given c_X^2 and $\mu^{-1} = \mathbb{E}X$, we calculate p_1 and p_2 according to (5.4) and set $\mu_1 = 2p_1\mu$ and $\mu_2 = 2p_2\mu$. This guarantees that for different c_X^2 values the expected time to cancellation stays μ^{-1}.

The first observation is that the revenue increases for both the optimal static strategy and the one generated by the EMSR/MP model for increasing c_X^2 (see Table 5.4). This means that the airline companies may benefit from more variation in time to cancellation, while the expected time to cancellation stays the same. Also, as observed in section 5.1, we observe the same increasing trend in the relative difference between SM and MP for increasing μ. However, for μ given, the gap decreases for c_X^2 increasing.

Table 5.4: The Comparison between SM and EMSR/MP under hyperexponential cdf

test problem	Expected Revenue		%difference
$(c_X^2, \mu \times 10^4, \rho)$	SM	MP	SM vs. MP
(1.5, 15, 1.6)	40961	40034	2.26%
(1.5, 15, 2.0)	45721	45639	0.18%
(1.5, 20, 1.6)	41328	39452	4.54%
(1.5, 20, 2.0)	46391	45793	1.29%
(1.5, 25, 1.6)	41762	38884	6.89%
(1.5, 25, 2.0)	46709	45779	1.99%
(2.0, 15, 1.6)	41064	40356	1.72%
(2.0, 15, 2.0)	45984	45861	0.27%
(2.0, 20, 1.6)	41509	39850	4.00%
(2.0, 20, 2.0)	46507	46123	0.82%
(2.0, 25, 1.6)	41996	39322	6.37%
(2.0, 25, 2.0)	46941	46197	1.59%
(2.5, 15, 1.6)	41150	40593	1.35%
(2.5, 15, 2.0)	46167	46000	0.36%
(2.5, 20, 1.6)	41642	40146	3.59%
(2.5, 20, 2.0)	46606	46352	0.55%
(2.5, 25, 1.6)	42172	39664	5.95%
(2.5, 25, 2.0)	47137	46507	1.34%
(3.0, 15, 1.6)	41226	40775	1.10%
(3.0, 15, 2.0)	46290	46109	0.39%
(3.0, 20, 1.6)	41758	40365	3.34%
(3.0, 20, 2.0)	46704	46515	0.40%
(3.0, 25, 1.6)	42303	39915	5.65%
(3.0, 25, 2.0)	47308	46725	1.23%

Table 5.5: The performance gap between SM and EMSR/MP averaged over all test problems in form $(C_X^2, \bullet, \bullet)$

c_X^2	SM vs. MP
1.5	2.86%
2.0	2.46%
2.5	2.19%
3.0	2.02%

Chapter 6

Concluding Remarks

In this thesis, we consider a revenue management problem with general cancellation processes, nonhomogenous Poisson arrival processes for the different fare classes and class dependent no-shows. In this model overbooking is allowed. In particular, the new static model allows for arbitrary fare class dependent distribution functions for the random time to cancellation as well as class dependent refunds and show-up probabilities. Due to its non-Markovian structure it is not possible to analyze the dynamic version of this problem. Under these general conditions it is shown by means of dynamic programming that one can relatively easy evaluate the optimal static strategy achieving a maximum expected revenue. Under the (restrictive) conditions of fare class independent refunds and show-up probabilities and a fare class independent exponential cancellation distribution we compare in the computational section the expected revenue obtained by the optimal strategy in the new static model with the expected revenue of the optimal dynamic strategy. For this particular case the relative difference of the two objective function values in all of the considered instances is at most 1.87 percent. At the same time we compare in a simulation the policy generated by our new static model with the one generated by the well-know EMSR/MP heuristic. It turns out in all most all of the simulation runs that our static model yields on average higher revenues than the those generated by the EMSR/MP heuristic.

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Appendix

A Introduction

In this note we give an overview on the well known EMSR (Expected Marginal Seat Revenue) heuristics used within airline revenue management. We start with non-overbooking and no cancellation and adapt these heuristics later to the case where overbooking is allowed and cancellation happens or does not happen. Before presenting the derivation of these heuristics we observe that the total seat capacity of the plane is given by C and the number of different fare classes are given by m with fare class 1 denoting the cheapest and fare class m the most expensive.[1] In general r_i denotes the price of fare class i and these prices satisfy

$$r_1 < r_2 < ... < r_m.$$

The random demand for fare class i is denoted by the discrete nonnegative random variable $\mathbf{D}_i$ and it is assumed that the random variables $\mathbf{D}_1, ..., \mathbf{D}_m$ are independent. Starting with $m = 2$ and $b_1 \in \{0, 1, ..., C\}$ denoting the maximum number of seats reserved for fare class 1 (also called the booking limit for fare class 1) it is easy to see that the total random revenue for this booking limit is given by

$$\mathbf{R}(b_1) = r_1 \min\{b_1, \mathbf{D}_1\} + r_2 \min\{C - \min\{b_1, \mathbf{D}_1\}, \mathbf{D}_2\}.$$

Now the total expected total revenue is given by

$$f(b_1) := \mathbb{E}\mathbf{R}(b_1) = r_1 v_{\mathbf{D}_1}(b_1) + r_2 v(b_1) \tag{A.1}$$

with

$$v_{\mathbf{D}_1}(b_1) := \mathbb{E}(\min\{b_1, \mathbf{D}_1\}), v(b_1) := \mathbb{E}(\min\{C - \min\{b_1, \mathbf{D}_1\}, \mathbf{D}_2\})$$

[1] Observe in [29] it is assumed that fare class 1 is the most expensive fare class and so in this book we have the convention $r_m < r_{m-1} < ... < r_1$.

To determine the optimal booking limit we need to solve the optimization problem

$$\max\{f(b_1) : b_1 \leq C, b_1 \in \mathbb{Z}_+\}. \tag{P}$$

For optimization problem (P) the following result holds.

Lemma 23 *The objective function f in optimization problem (P) is unimodal and an optimal solution b_{opt} of this problem is given by*

$$b_{opt} = \min\left\{b \in \mathbb{Z}_+, b \leq C : \mathbb{P}(\mathbf{D}_2 \geq C - b) > \frac{r_1}{r_2}\right\}. \tag{A.2}$$

Proof. For every b it follows

$$\begin{aligned}\min\{b+1, \mathbf{D}_1\} - \min\{b, \mathbf{D}_1\} &= (\min\{b+1, \mathbf{D}_1\} - \min\{b, \mathbf{D}_1\})\mathbf{1}_{\{\mathbf{D}_1 \geq b+1\}} \\ &= \mathbf{1}_{\{\mathbf{D}_1 \geq b+1\}}\end{aligned} \tag{A.3}$$

and this shows

$$v_{\mathbf{D}_1}(b+1) - v_{\mathbf{D}_1}(b) = \mathbb{P}(\mathbf{D}_1 \geq b+1). \tag{A.4}$$

Also for every b we obtain

$$\begin{aligned}&\min\{C - \min\{b+1, \mathbf{D}_1\}, \mathbf{D}_2\}) - \min\{C - \min\{b, \mathbf{D}_1\}, \mathbf{D}_2\} \\ = \;&\min\{C - (b+1), \mathbf{D}_2\}) - \min\{C - b, \mathbf{D}_2\})\mathbf{1}_{\{\mathbf{D}_1 \geq b+1\}}\end{aligned} \tag{A.5}$$

and this shows by the independence of the random variables $\mathbf{D}_1$ and $\mathbf{D}_2$ that

$$\begin{aligned}v(b+1) - v(b) &= \mathbb{P}(\mathbf{D}_1 \geq b+1)(\mathbb{E}(\min\{C - (b+1), \mathbf{D}_2\}) - \mathbb{E}(\min\{C - b, \mathbf{D}_2\})) \\ &= \mathbb{P}(\mathbf{D}_1 \geq b+1)(v_{\mathbf{D}_2}(C - b - 1) - v_{\mathbf{D}_2}(C - b)) \\ &= -\mathbb{P}(\mathbf{D}_1 \geq b+1)\mathbb{P}(\mathbf{D}_2 \geq C - b).\end{aligned} \tag{A.6}$$

By relations (A.1), (A.4) and (A.6) it follows for every $b \in \{0, 1, ..., C-1\}$ that

$$f(b+1) - f(b) = \mathbb{P}(\mathbf{D}_1 \geq b+1)(r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq C-b)) \tag{A.7}$$

and so

$$f(b+1) - f(b) < 0 \Leftrightarrow r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq C-b) < 0. \tag{A.8}$$

Since the function

$$b \mapsto r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq C-b)$$

is decreasing this implies by relation (A.8) that the function f is unimodal. Since for $b = C$ it is obvious that

$$r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq C-b) = r_1 - r_2 < 0$$

an optimal solution b_1^* of optimization problem (P) is then given by

$$\begin{aligned} b_1^* &= \min\{b \in \{0, ..., C\} : f(b+1) - f(b) < 0\} \\ &= \min\{b \in \{0, ..., C\} : r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq C-b) < 0\} \\ &= \min\left\{b \in \{0, ..., C\} : \mathbb{P}(\mathbf{D}_2 \geq C-b) > \tfrac{r_1}{r_2}\right\} \end{aligned} \tag{A.9}$$

and we have shown the desired result. □

In general the function f is not discrete concave and the formula in relation (A.9) is called the formula of Littlewood. Another representation of the decision variables is by introducing the protection levels $y_1 = C$ and $y_2 = C - b_1$. Clearly the protection levels satisfy $y_2 \leq y_1 = C$ and clearly

$y_2 =$ **minimum** number of seats reserved for fareclass 2

In this framework the optimal protection level for fare class 2 and higher satisfies

$$\begin{aligned} y_2^* &= \max\{y \in \{0, ..., C\} : r_1 - r_2\mathbb{P}(\mathbf{D}_2 \geq y) < 0\} \\ &= \max\left\{y \in \{0, ..., C\} : \mathbb{P}(\mathbf{D}_2 \geq y_2) > \tfrac{r_1}{r_2}\right\}. \end{aligned}$$

To generalize the above approach to more than two fare classes we define for $j = 1, ..., m$

$b_j :=$ **maximum** number of seats reserved for fare classes 1 up to j.

The decisıon variable b_j is called the **nested booking limit for class** j and **by definition** it is the maximum number of seats reserved for fare class 1 up to j. One may also introduce the **protection level** y_j of fare class $j = 1, ..., m$ and this is defined by

$y_j :=$ **minimum** number of seats reserved for fare classes j up to m

By definition of the above decision variables it follows that

$$y_j = C - b_{j-1}$$

and

$$y_m \leq y_{m-1} \leq y_{m-2} \leq ... \leq y_1 = C \text{ or } b_1 \leq b_2 \leq ... \leq b_m = C$$

In general it seems impossible to compute a closed from expression for the expected revenue of a given nested booking limit policy $\mathbf{b} = (b_1, ..., b_m)$ satisfying

$$b_1 \leq b_2 \leq ... \leq b_m = C$$

unless we assume that the demands arrive sequentially with the demand for fare class 1 arriving first and the demand for fare class m arriving last.This is related to the observation in practice that without a lot of loss requests for cheaper air fares arrrive earlier during the booking period. For a more detailed description of the different nesting policies (see [16]). It is now said that the demand arrives in stages from period 1 up to m.

A.1 EMSR-b Heuristic with no cancellations and perfect showups.

We will now discuss the heuristic EMSR-b. After applying this heuristic we obtain the (not necessary optimal) protection levels

$$C = y_1^{EMRS} \geq y_2^{EMRS} \geq ... \geq y_m^{EMRS}.$$

To derive the value y_m^{EMRS} we assume at the first stage m of the algorithm that we compute the protection level of fare class m against fare class $m-1$. Hence by the rule of Littlewood we obtain

$$\begin{aligned} y_m^L &= \max\{y \in \{0, ..., C\} : r_{m-1} - r_m \mathbb{P}(\mathbf{D}_m \geq y) < 0\} \\ &= \max\left\{y \in \{0, ..., C\} : \mathbb{P}(\mathbf{D}_m \geq y) > \tfrac{r_{m-1}}{r_m}\right\} \end{aligned}$$

and we set

$$y_m^{EMSR} := y_m^L. \tag{A.10}$$

If we are at stage $m-1$ of the algorithm (second iteration) we compute the protection level of the fare classes $m-1$ and m against fare class $m-2$. Now the total demand of fare classes $m-1$ and m is given by $\sum_{i=m-1}^{m} \mathbf{D}_i$ and the revenue of the aggregated fare classes $m-1$ and m is approximated by the weighted average

$$\overline{r}_{m-1} = \frac{\sum_{i=m-1}^{m} r_i \mathbb{E}(\mathbf{D}_i)}{\sum_{i=m-1}^{m} \mathbb{E}(\mathbf{D}_i)}.$$

By Littlewoods rule applied to the above parameters it follows that

$$\begin{aligned} y_{m-1}^L &= \max\{y \in \{0, ..., C\} : r_{m-2} - \overline{r}_{m-1} \mathbb{P}(\textstyle\sum_{i=m-1}^{m} \mathbf{D}_i \geq y) < 0\} \\ &= \max\left\{y \in \{0, ..., C\} : \mathbb{P}(\textstyle\sum_{i=m-1}^{m} \mathbf{D}_m \geq y) > \tfrac{r_{m-2}}{\overline{r}_{m-1}}\right\}. \end{aligned}$$

In these computations it might happen that $y_{m-1}^L > y_m^{EMSR}$ and this yields a contradiction with the interpretation of y_{m-1}^L. This happens in case $\frac{r_{m-1}}{r_m} \geq \frac{r_{m-2}}{\overline{r}_{m-1}}$ and so for this case we obtain using

$$\mathbb{P}(\sum_{i=m-1}^{m}\mathbf{D}_i \geq y_m^L) \geq \mathbb{P}(\mathbf{D}_m \geq y_m^L) > \frac{r_{m-1}}{r_m} \geq \frac{r_{m-2}}{\overline{r}_{m-1}}$$

that

$$y_{m-1}^L = \max\{y \in \{0,...,C\} : \mathbb{P}(\sum_{i=m-1}^{m}\mathbf{D}_m \geq y) > \frac{r_{m-2}}{\overline{r}_{m-1}}\} \geq y_m^{L-1}$$

However, for $\frac{r_{m-1}}{r_m} < \frac{r_{m-2}}{\overline{r}_{m-1}}$ it can happen that $y_{m-1}^L < y_m^L$ and to guarantee that the values y_{m-1}^{EMSR} and y_m^{EMSR} are nested or equivalently that $y_m^{EMSR} \leq y_{m-1}^{EMSR}$ we set

$$y_{m-1}^{EMSR} = \max\{y_{m-1}^L, y_m^{EMSR}\} \tag{A.11}$$

In general after computing the nested values

$$y_{m+1}^{EMSR} \leq .... \leq y_{j+1}^{EMSR}$$

we are at stage j of the algorithm ($(m-j+1)$th iteration) and compute the protection level of the fare classes j up to m against fare class $j-1$. Now the total demand of the $m-j$ most expensive fare classes $i=j$ up to m is given by $\sum_{i=j}^{m}\mathbf{D}_i$ and the revenue obtained from aggregating fare classes j up to m is approximated by the weighted average

$$\overline{r}_j = \frac{\sum_{i=j}^{m} r_i\mathbb{E}(\mathbf{D}_i)}{\sum_{i=j}^{m}\mathbb{E}(\mathbf{D}_i)}$$

By the rule of Littlewood and using the above parameters yields

$$\begin{aligned} y_j^L &= \max\{y \in \{0,...,C\} : r_{j-1} - \overline{r}_j\mathbb{P}(\sum_{i=j}^{m}\mathbf{D}_i \geq y) < 0\} \\ &= \max\left\{y \in \{0,...,C\} : \mathbb{P}(\sum_{i=j}^{m}\mathbf{D}_i \geq y) < \tfrac{r_{j-1}}{\overline{r}_j}\right\}. \end{aligned}$$

Again it might happen that $y_j^L < y_{j+1}^{EMSR}$ and we set

$$y_j^{EMSR} = \max\{y_j^L, y_{j+1}^{EMSR}\}.$$

Hence by construction

$$y_j^{EMSR} \geq y_{j+1}^{EMSR} \geq ... \geq y_m^{EMSR}$$

and we continue with stage $j-1$. Clearly at stage 1 we set $y_1^{EMSR} = C$ and we have computed the decreasing sequence

$$C = y_1^{EMRS} \geq y_2^{EMRS} \geq ... \geq y_m^{EMRS}.$$

If the demands $\mathbf{D}_i$ are Poisson distributed with parameter β_i and independent we obtain that the demand $\sum_{i=j}^{m} \mathbf{D}_i$ is again Poisson distributed with parameter $\sum_{i=j}^{m} \beta_i$ and so for this case it is east to compute the cdf of the sum of independent random variables. In the next subsection we adapt the EMSR-b heuristic to the case that overbooking, cancellation and no-shows occur.

A.2 EMSR-b Heuristic with cancellations and no-shows.

We will now adapt the EMSR-b heuristic in case the model includes cancellation and no-shows and so overbooking is allowed. As before there are m fare classes with independent demands $\mathbf{D}_i$. Moreover, in this case a fare class j reservation might cancel and so

$$\delta_j := \mathbb{P}(\text{reserved fare class } j \text{ customer cancels}). \tag{A.12}$$

Since we also allow showups of a customer having a fare class j reservation the conditional probability of a showup of a fare class j customer is given by

$$p_j := \mathbb{P}(\text{reserved fare class } j \text{ customer shows up} \mid \text{fare class } j \text{ reserved customer does not cancel}) \,. \tag{A.13}$$

By the definition of a conditional probability it follows that

$$\mathbb{P}(\text{reserved fare class } j \text{ customer shows up}) = p_j(1-\delta_j) \tag{A.14}$$

Without considering the penalty of overbooking we can now adapt the EMSR-b heuristic by selecting in the following way the capacity of the airplane (see Talluri and Van Ryzin [29] or Phillips [22]).

EMSR-MP.
Replace the capacity P by the virtual capacity P_v given by

$$P_v = \frac{P \sum_{i=1}^{m} \mathbb{E}\mathbf{D}_i}{\sum_{i=1}^{m} p_i \mathbb{E}(\mathbf{D}_i)} \geq P \tag{A.15}$$

and apply the EMSR-b heuristic to an airplane having capacity P_v.

Another way of selecting the virtual capacity is also to include the cancellation behaviour occurring before the departure.

EMSR-MP1
Replace the capacity P by the virtual capacity P_v^* given by

$$P_v^* = \frac{P \sum_{i=1}^{m} \mathbb{E}\mathbf{D}_i}{\sum_{i=1}^{m} p_i (1-\delta_i) \mathbb{E}(\mathbf{D}_i)} \geq P_v \geq P \tag{A.16}$$

and apply the EMSR-b heuristic to an airplane having capacity P_v^*.

Since we assume that the arrival process of fare class j requests is given by a nonhomogeneous Poisson process with intensity function λ_j we need to compute the parameter δ_j. Since this probability is a given characteristic of every fare class j customer (with a rejected or accepted reservation) we assume that a good approximation of this probability can be determined by assuming every fare class j request in the booking period $[0, T]$ will receive a reservation. Under this assumption one can now show the following result.

Lemma 24 *If the arriving requests are given by a nonhomogeneous Poisson process with a locally bounded Borel arrival intensity function λ and the random time to cancellation has cdf F and every arriving request is accepted, then the probability δ of cancellation is* *given by*

$$\delta = \frac{\int_0^T F(T-s)\lambda(s)ds}{\int_0^T \lambda(s)ds}. \tag{A.17}$$

Proof. In this proof we give two ways to evaluate the expression for δ. It follows by (A.12) with T_i denoting the arrival time of the ith arriving request and Y_i its time to cancellation and $N_\Lambda(t)$ the total number of arriving customers up to time t that

$$\begin{aligned}\delta &= \frac{\mathbb{E}(\sum_{i=1}^\infty 1_{\{Ti \leq T, Ti+Yi \leq T\}})}{\mathbb{E}(\sum_{i=1}^\infty 1_{\{T_i \leq T\}})} \\ &= \frac{\mathbb{E}(\sum_{i=1}^\infty 1_{\{Ti \leq T, Ti+Yi \leq T\}})}{\mathbb{E}(N_\Lambda(t))} \\ &= \frac{\mathbb{E}(\sum_{i=1}^\infty 1_{\{T_i \leq T, T_i+Y_i \leq T\}})}{\int_0^T \lambda(s)ds}\end{aligned}$$

By the monotone convergence theorem and T_i has cdf G_i and the random variables T_i and Y_i are independent we obtain

$$\begin{aligned}\mathbb{E}(\sum_{i=1}^\infty 1_{\{T_i \leq T, T_i+Y_i \leq T\}}) &= \sum_{i=1}^\infty \mathbb{E}(1_{\{T_i \leq T, T_i+Y_i \leq T\}}) \\ &= \sum_{i=1}^\infty \int_0^T F(T-s)dG_i(s) \\ &= \int_0^T F(T-s)d\mathbb{E}N_\Lambda(s) \\ &= \int_0^T F(T-s)\lambda(s)ds\end{aligned}$$

This shows the expression for δ in (A.17). A different and more natural approach fitting the static nature is to count the number of point falling in the specified region A using the Poisson random measure approach (see next figure). Introducing

$$\mathbf{N}_C = \text{number of points falling into region C}$$

we need to compute

$$\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A} \mid \mathbf{N}_A+\mathbf{N}_B \geq 1\right) = \frac{\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A}\mathbf{1}_{\{\mathbf{N}_B+\mathbf{N}_A\geq 1\}}\right)}{\mathbb{P}(\mathbf{N}_A+\mathbf{N}_B \geq 1)}$$

It follows that $\mathbf{N}_B + \mathbf{N}_A$ has the same distribution as $\mathbf{N}_\Lambda(T)$ and as already observed in Section 3.2 it is known (see Chapter 6 of [13]) the random variables $\mathbf{N}_A$ and $\mathbf{N}_B$ are Poisson distributed and independent. Also the probability p that a point falling into the region $A \cup B$ will fall into the region A can be easily computed, i.e

$$p = \frac{\int_0^T F(T-s)\lambda(s)ds}{\int_0^T \lambda(s)ds}.$$

Hence conditional on $\mathbf{N}_B + \mathbf{N}_A \geq 1$ we obtain

$$\mathbf{N}_A \stackrel{d}{=} \mathbf{B}(p, \mathbf{N}_B + \mathbf{N}_A).$$

This shows by conditioning on $\mathbf{N}_B + \mathbf{N}_A$ that

$$\begin{aligned}\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A}\mathbf{1}_{\{\mathbf{N}_B+\mathbf{N}_A\geq 1\}}\right) &= \sum_{k=1}^{\infty} \frac{1}{k}\mathbb{E}(\mathbf{B}(p,k))\mathbb{P}(\mathbf{N}_B+\mathbf{N}_A=k) \\ &= p\mathbb{P}(\mathbf{N}_A+\mathbf{N}_B \geq 1)\end{aligned}$$

and so we obtain

$$\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A} \mid \mathbf{N}_A+\mathbf{N}_B \geq 1\right) = p = \frac{\int_0^T F(T-s)d\Lambda(s)}{\Lambda(T)}$$

showing the result. □

Applying Lemma 24 and (A.16) it follows that the EMSR MP1 heuristic applied to the framework of our static model has virtual capacity

$$P_v^* = \frac{P\sum_{j=1}^m \Lambda_j(T)}{\sum_{j=1}^m p_j \int_0^T (1-F_j(T-s))\lambda_j(s)ds}$$

Applying the above heuristic policy in a simulation we should apply it in a nested way. For different ways of nesting the reader is referred to Haerian and Campbell [16] or Talluri and Van Ryzin [28].

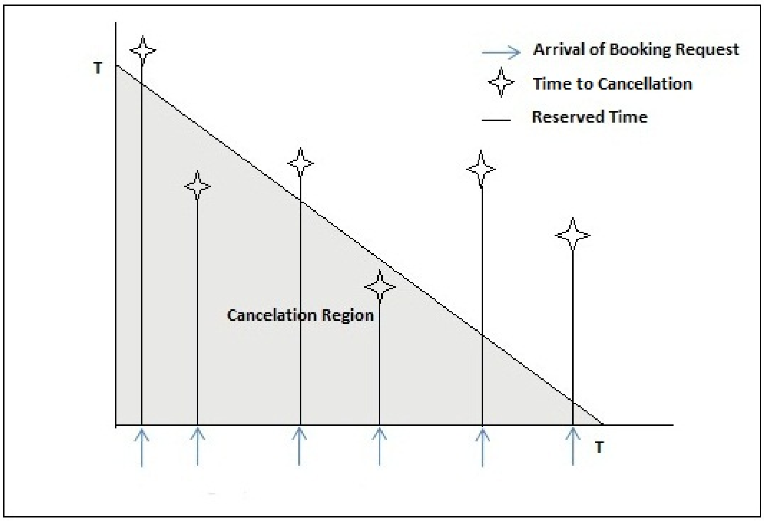

Figure 6.1: Arrivals, cancelations, and the cancellation region