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Fuzzy Extent Analysis: Comparison of Defuzzification Methods

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***Abstract*—This paper conducts a comparison analysis of some of the most common defuzzification techniques which helps rank fuzzy weights calculated from fuzzy comparison matrices through Fuzzy Analytic Hierarchy Process. Ranking of weights calculated through FAHP algorithms is of critical importance as it directly effects the decision making process. More so ranking of fuzzy numbers is also important in data analysis, artificial intelligence, socioeconomic systems and other areas which utilize fuzzy logic. Many research articles have been published over the past five years solely focusing on this subject. This paper aims to conduct an empirical study on some of the most common defuzzification techniques and identify the best one in terms of accuracy of weights being calculated.**

Keywords—ranking fuzzy numbers; fuzzy extent analysis; fuzzy analytic hierarchy process; defuzzification

I. INTRODUCTION

Analytic Hierarchy Process (AHP) proposed by Thomas L. Saaty [1] is one of the most popular Multiple-Criteria Decision Making (MCDM) techniques which utilize pair wise comparisons in order to elicit the preferences of the decision makers. The comparison matrix is used to determine the relative *weight* of each criterion, as well as the *scores* of each alternative particularly for criteria that don't have natural scales (e.g., a qualitative criterion). Later, the overall score of the alternative is merely obtained by calculating the sum of the *weighted scores* over each criterion and the best decision is identified based on this overall score.

During the elicitation of the preferences process a 1-9 constructed scale is used where the decision maker is asked to compare the alternatives and linguistically state if s/he is indifferent, or has weak, normal, strong or extremely strong preference for one of them. The utilization of such a constructed scale and assuming in the analysis that these *vague* linguistic variables (i.e., weak, strong, etc.) can be represented with a ratio scale based crisp numbers is criticized by many researchers. Among the others, one such criticism is due to utilization of *crisp* numbers disregards the vagueness of human language which as a result may lead to wrong decisions in the decision analysis process [2].

In his seminal paper, Zadeh [3] introduced fuzzy set theory in order to address the vagueness intrinsic to various concepts (e.g., linguistic variables, human behavior, etc.) in which they are represented by fuzzy sets. As opposed to the classical dichotomous set theory where laws of *excluded middle* and *non contradiction* holds and as a result an object is either part of a set or not, fuzzy set theory represents the belongingness of an object to a set by means of membership functions that attain a value from a continuum which ranges from zero to one. Fuzzy sets enabled the researchers to incorporate the uncertainties in human reasoning in knowledge based systems and has been successfully applied in various fields such as control theory [4], health care [5], system modeling [6,7], etc.

As an extension of the fuzzy set theory, fuzzy numbers are introduced, which are basically fuzzy sets of real numbers with a convex and continuous membership function of bounded support. In the context of analytical hierarchical process, characterizing the imprecise (i.e., *vague*) linguistic labels with fuzzy numbers captures the uncertainties in human knowledge. Therefore, Fuzzy Analytical Hierarchy Process (FAHP) became popular among the researchers where the 1-9 constructed scale formed in such a way that its elements are represented with fuzzy numbers and thus fuzzy comparison matrices are produced. Over the years various FAHP algorithms [8] [9] [10] [11] have been proposed to accurately extract weights from fuzzy comparison matrices. Extensive review of literature on FAHP algorithms is provided in [12].

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Among multiple FAHP algorithms proposed over the past many years, literature shows that Fuzzy Extent Analysis (FEA) Method [13] is one of the most widely used FAHP algorithm. In this technique, concept of extent analysis combined with degree of possibility (ranking of fuzzy numbers) is utilized in order to calculate the weights from the fuzzy comparison matrices. However, this method has been criticized [14] mainly due to the way fuzzy numbers are compared using degree of possibility. Over the last five years around hundred research articles have been published on how to compare fuzzy numbers which shows that there is no general consensus on a single method to rank and order fuzzy numbers [15].

Therefore, in this research we conducted an empirical study in which we will compare three methods including Centroid Defuzzification [16], defuzzification as proposed in [17] and defuzzification by simply using the mid number of the triangular fuzzy number to evaluate the performance of each method in terms of accuracy of the weights calculated from fuzzy comparison matrices.

Rest of the paper is organized as follows. In Section II, a brief overview of fuzzy arithmetic and FEA will be introduced. In section III the set up used for the experimental analysis will be presented. Later in section IV, the results of the experimental analysis will be provided and discussed. The paper will be finalized with some concluding remarks as well as future research directions in section V.

II. Fuzzy Extent Analysis

In this section we will briefly introduce FEA methodology as well as the fuzzy arithmetic required to understand the methodology in depth. Fuzzy numbers are helpful in capturing vagueness in human judgments because a fuzzy number represents the linguistic variables with a set of possible values each having its own membership degree. In the literature triangular, trapezoidal or bell shaped functions are commonly used as membership functions for the fuzzy numbers. A triangular fuzzy number can be represented through *[l m u]* and membership function μ_m defined as follows and graphically illustrated in Figure 1;

$$\mu_m(x) = \begin{cases} \dfrac{x}{m-l} - \dfrac{l}{m-u}, & x \in [l\ m] \\ \dfrac{x}{m-u} - \dfrac{l}{m-u}, & x \in [m\ u] \\ 0, & otherwise \end{cases}$$

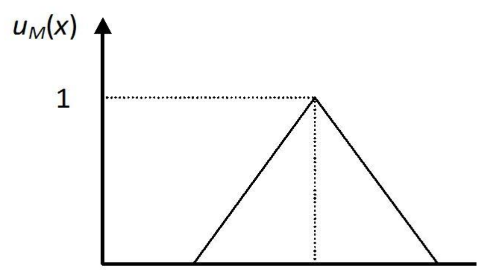

Fig. 1. Membership function of triangular fuzzy number

A. Fuzzy Arithmetic

Provided we have two fuzzy arithmetic numbers such as $(l_1\ m_1\ u_1)$ *and* $(l_2\ m_2\ u_2)$ than the basic arithmetic functions for fuzzy numbers are tabulated in Table 1;

TABLE I. FUYZZY ARITHMETIC

Description	**Fuzzy Operation**
Addition	$(l_1\ m_1\ u_1) \oplus (l_2\ m_2\ u_2) = (l_1 + l_2 m_1 + m_2\ u_1 + u_2)$
Multiplication	$(l_1\ m_1\ u_1)\ \Theta\ (l_2\ m_2\ u_2) = (l_1.l_2\ m_1.m_2\ \ u_1.u_2)$
Scalar Multiplication	$(\lambda\ \lambda\ \lambda)\ \Theta\ (l_2\ m_2\ u_2) = (\lambda.l_1\ \ \lambda.m_1\ \ \lambda.u_1)$
Inverse	$(l_1\ m_1\ u_1)^{-1} \approx (1/u_1\ 1/m_1\ 1/l_1)$

B. Fuzzy Extent Analysis Method

FEA is one of the most widely used FAHP algorithm and it utilizes the concept of extent analysis [13] in order to calculate the weights from fuzzy comparison matrices. In this method, there is an object set X = {x_1, x_2, … , x_n} and a goal set G = {g_1, g_2, … , g_m} and for each object, extent analysis for each goal is performed as follows;

$$S_i = \sum_{j=1}^{m} M_{g_i}^{j} \otimes \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^{j} \right]^{-1}$$

$$\sum_{j=1}^{m} M_{g_i}^{j} = \left(\sum_{j=1}^{m} l_j, \sum_{j=1}^{m} m_j, \sum_{j=1}^{m} u_j \right)$$

In case of conventional AHP, values of synthetic extent analysis would provide us the criteria weights. However, as mentioned earlier ordering of fuzzy numbers is not straight forward and thus in the original FEA method, degree of possibility is used to defuzzify and rank fuzzy weights. The degree of possibility of $M_1 \geq M_2$ is defined as;

$$V(M_1 \geq M_2) = \sup_{x \geq y} \left[\min(\mu_{M_1}(x), \mu_{M2}(y)) \right]$$

As M_1 and M_2 are convex fuzzy numbers therefore

$$\begin{aligned} V(M_1 \geq M_2) &= 1 \quad iff \quad m_1 \geq m_2 \\ V(M_1 \geq M_2) &= hgt(M_1 \cap M_2) \\ &= \mu_{M_1}(d) \end{aligned}$$

Where d is the ordinate of the highest intersection point as illustrated in Figure 2.

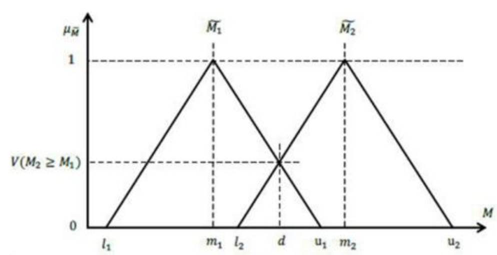

Fig. 2. Degree of possibility

If $M_1 = (l_1\ m_1\ u_1)$ and $M_2 = (l_2\ m_2\ u_2)$ than ordinate can be mathematically calculated as follows;

$$\begin{aligned} V(M_2 \geq M_1) &= hgt(M_1 \cap M_2) \\ &= \frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)} \end{aligned}$$

Finally the degree of possibility for a fuzzy number to be greater than k fuzzy numbers is given by;

$$\begin{aligned} V(M \geq M_1, M_2, \ldots, M_k) &= V\left[(M \geq M_1) \; and \; (M \geq M_2), \ldots, (M \geq M_k) \right] \\ &= \min V(M \geq M_i), \quad i = 1, 2, \ldots, k \end{aligned}$$

Weight vector is calculated by normalizing degree of possibilities. Note that this weight vector is a non-fuzzy number. In this research instead of degree of possibility, we have used previously stated three distinct techniques in order to defuzzify and rank the fuzzy numbers.

III. EXPERIMENTAL ANALYSIS

In order to compare the performances of the three techniques, namely the centroid defuzzification, mid number of the triangular fuzzy numbers and Chen and Yang's defuzzification technique [17] we conducted an experimental analysis. As part of the experimental analysis various control parameters are considered as variables in order to evaluate the performances of the techniques in different conditions. These parameters are namely the size of the comparison matrices, the level of fuzziness and the inconsistency of the decision makers during the elicitation process. Note that even though there exist some techniques in the literature [18] which provides a methodology to generate comparison matrices with various levels of consistency levels, however those technique are limited for only conventional AHP and thus cannot be replicated for comparison matrices consisting of fuzzy numbers. Therefore, in this research a novel framework is proposed through which random fuzzy comparison matrices can be generated for various control parameters as required by the experimental set up.

This algorithm is step by step explained as follows;

Step 1: Assuming we have *n* criterion, we randomly generate crisp weights $w_1, w_2, \ldots, w_n$ and normalize them.

Step 2: Through these weights we can generate a perfectly consistent comparison matrix as follows

$$CM = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_n/w_n \\ w_2/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}$$

Step 3: Once the comparison matrix is generated, each element of the matrix is converted into a triangular fuzzy number [l` $m$` u`] with a fuzzification parameter α such that $l` = w_i/w_j - \alpha$, $m` = w_i/w_j$ and $u` = wi/wj + \alpha$

Step 4: In reality human judgments are rarely consistent and thus comparison matrices formed through these judgments are also not consistent. Therefore, we introduce different levels of inconsistency in the matrices through the inconsistency parameter β. Depending on this parameter, an interval [a b] is generated for each l` of the triangular fuzzy number such that $a = l` - l` \times \beta$ and $b = l` + l` \times \beta$. Same procedure is followed to create inconsistency intervals for $m$` and u`. Afterwards, a number is randomly selected from each one of these intervals and is correspondingly assigned as the lower, modal and upper value of the triangular fuzzy number i.e. [l m u]. However, once inconsistency parameter is increased, there is a possibility that the interval [a b] generated for each element of the triangular fuzzy number intersects and the numbers are randomly chosen in such a way that they violates the condition $l < m < u$. We address this issue as follows;

Whenever the inconsistency intervals intersect, they are shrunk in such a way that for each lower value of the triangular fuzzy number, the right endpoint of the interval is readjusted such that it is the midpoint of the right end point of the interval of lower value and the left end point of the interval generated for modular number. Similarly, both end points of the inconsistency interval of modular number are readjusted and the left endpoint of the inconsistency interval of upper number is readjusted. Numbers randomly chosen from these intervals will always satisfy the condition $l < m < u$.

This part of the algorithm is graphically explained in Figure 3 for clarity.

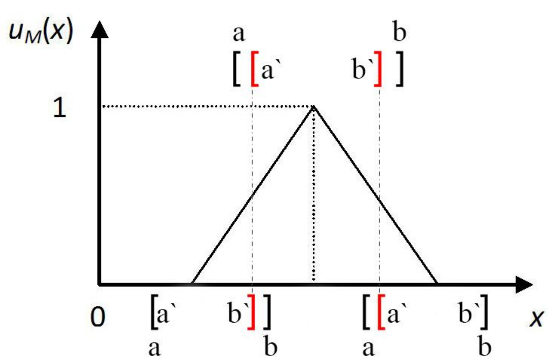

Fig. 3. Interval formation

IV. Results and Discussion

Based on the above framework, three experimental conditions are considered as part of the analysis. These are namely the namely fuzzification parameter (i.e., α), inconsistency parameter (i.e., *β*) and the size of the matrix. For the fuzzification parameter three levels are assumed as 0.05, 0.1 and 0.15. The fuzzification parameter is not inherent to the problem that the decision maker is facing but more a decision variable as part of the process. That is to say the decision analysts can set the fuzzification level and conduct FAHP accordingly. On the other hand the inconsistency parameter refers to the inconsistency of the decision maker and it is not a decision variable but depends on the decision makers. For the analysis five levels are considered for the inconsistency parameter which are 0, 0.5, 1, 1.5 and 2. Finally, four different matrix sizes are considered which are 3, 7, 11 and 15. Note that one can consider 3 as the representative of small sized problems, 7 and 11 are for medium sized problems and 15 for larger cases.

As a result of this set up totally 60 (=3*5*4) different experimental conditions are constructed. For each condition 10 replications are created randomly, Hence the dataset contained a total of six hundred matrices with varying parameters of fuzzification, inconsistency and size of the matrix. After calculating the values for fuzzy synthetic extent analysis, final defuzzified weights were calculated through three different techniques. Finally the errors are calculated as the root mean squared difference between the resulting weights and the actual weights that are used in the associated experiment.

The result of the analysis is depicted in Figure 4. Note that the x-axis in Figure 4 refers to different experimental conditions (i.e., fuzzification parameter, inconsistency parameter and the size of the matrix) on the other hand the y-axis is the error.

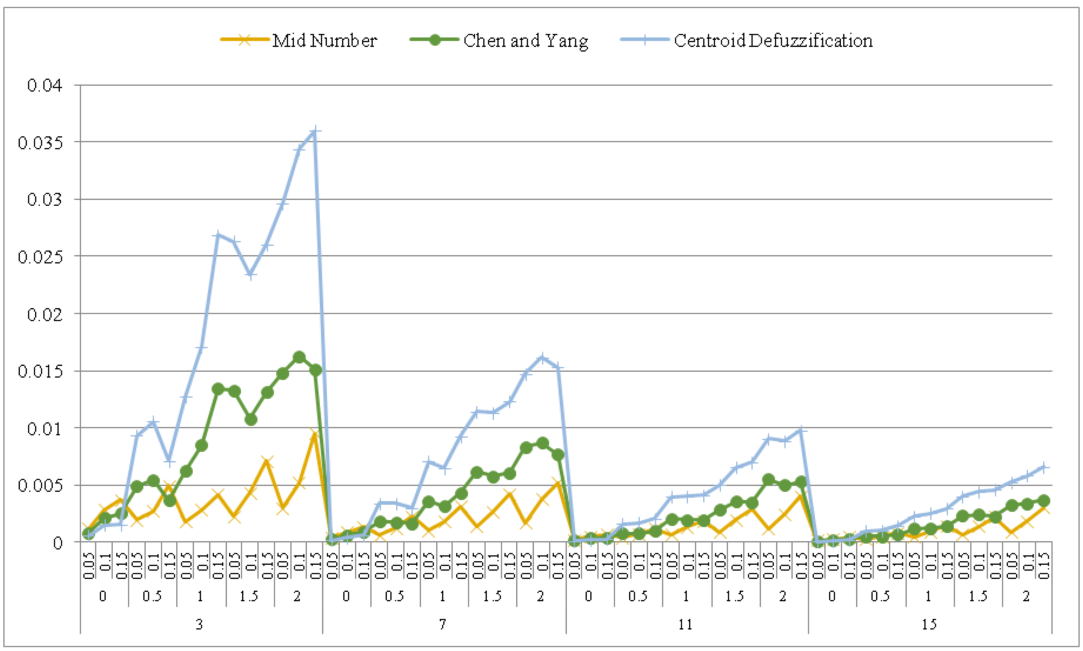

Fig. 4. The results of the three techniques for different levels of the Fuzzification Parameter (α), Inconsistency Parameter (β), Size of the Matrix (n)

Through this experimental study, we can conclude that utilizing just the mid number of a triangular fuzzy weight can give us more accurate results. While the original method of degree of possibility (not shown in this graph) is the worst performing methodology, other defuzzification techniques also performed inferior compared to using mid number of the triangular fuzzy number.

V. Future Research Directions

In this paper, a novel experimental analysis framework is introduced and the performance of a popular FAHP technique namely FEA is analyzed. In the analysis three techniques that can be utilized as part of the FEA methodology, namely the centroid defuzzification technique, mid value of the triangular fuzzy numbers and the Chen and Yang's defuzzification are also considered. The analysis revealed that the mid number technique outperformed the other two techniques in almost all experimental conditions.

Fuzzy numbers are more realistic representations of the imprecise linguistic variables that are used by the decision makers during the preference elicitation stage of the analytical hierarchy process. Therefore, FAHP really seems to serve for a cause and improve the process. As a result, abundant number of research is conducted and various FAHP techniques are proposed. However, there is virtually no throughout analysis of these techniques and measure their performance for different experimental conditions. Such a comparison would be invaluable for the researchers and the practitioners of the field since it will hint which technique might be more suitable for the problem that they are facing. In future we are going to conduct extensive experimental analysis where not only FEA but other techniques are also going to be part of. Furthermore, conventional AHP techniques will also be included in those analysis and the contribution (if any) of the FAHP will be assessed as well.

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