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A class of multipartner matching markets with a strong lattice structure

Ahmet Alkan

Sabanci University, 81474 Tuzla, Istanbul, TURKEY
(e-mail: alkan@sabanciuniv.edu)

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Summary. For a two-sided multipartner matching model where agents are given by path-independent choice functions and no quota restrictions, Blair [7] had shown that stable matchings always exist and form a lattice. However, the lattice operations were not simple and not distributive. Recently Alkan [3] showed that if one introduces quotas together with a monotonicity condition then the set of stable matchings is a distributive lattice under a natural definition of supremum and infimum for matchings. In this study we show that the quota restriction can be removed and replaced by a more general condition named *cardinal monotonicity* and all the structural properties derived in [3] still hold. In particular, although there are no exogenous quotas in the model there is endogenously a sort of quota; more precisely, each agent has the same number of partners in every stable matching. Stable matchings also have the *polarity* property (supremum with respect to one side is identical to infimum with respect to the other side) and a property we call *complementarity*.

Keywords and Phrases: Stable matchings, Revealed preference, Path independent choice function, Lattice, Two-sided market.

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1 Introduction

In the original college admissions problem (Gale and Shapley [8]), it was assumed that each college had a strict ordering on the set of all of its acceptable applicants and a quota giving the maximum number it could admit. In [7], Blair considered a broad generalization of this model where there could be multiple partners on both sides, preferences were given by rather general (path-independent) choice functions that do not necessarily respect any ordering on individuals, and there were no quota restrictions. He showed that the set of stable[1] matchings is nonempty and has the structure of a lattice under the common preferences of all agents on any one side the market. However, the lattice operations were not simple and not distributive. In [3], Alkan showed that if one reintroduces quotas along with a monotonicity condition then the set of stable matchings is a distributive lattice under a natural definition of supremum and infimum for matchings.[2]

In this study we show that the quota restriction can be removed and replaced by a more general condition that we call *cardinal monotonicity* and all the structural properties derived in [3] still hold. In particular, we have the somewhat surprising result that although there are no exogenous quotas in the model there is endogenously a sort of quota; more precisely, each agent has the same number of partners in every stable matching.

The main condition that choice functions obey here in this paper, as in Alkan [3] and Blair [7], is *path-independence*, mentioned above, which requires that what is chosen from the union of any two sets T, T' is identical to what is chosen from the union of T and the choice from T'. We begin our paper by giving some properties of preferences (on partner-sets) revealed by path-independent choice. The derivation of our results on stable matchings makes use of these properties and is substantially different than that in Alkan [3].

Our results on stable matchings have the following summary: *The set of stable matchings in any two-sided market with path-independent cardinal-monotone choice functions is a distributive lattice under the common preferences of all agents on any one side of the market. The supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of associated agents. The lattice has the polarity, unicardinality and complementarity properties.*

The polarity property refers to the fact that the supremum of stable matchings with respect to one side of the market coincides with their infimum with respect to the other side. The unicardinality property is our name for the property we

[1] By stable, in this paper, we mean individually rational and pairwise stable. We are tacitly assuming that coalitions of bigger size cannot form. The stable set and core need not be the same otherwise; see Sotomayor [16].

[2] Alkan [2] and Baiou and Balinski [5] had previously noted some special properties that stable multipartner matchings have when preferences are given by an ordering on individuals. Among the related recent literature, we also cite Martinez *et al* [13] on many-to-one matching on a domain that is essentially the same as in Alkan [3] and Martinez *et al* [14] that gives an algorithm to find the set of all stable matchings on the path-independent domain.

mentioned earlier that each agent has the same number of partners in every stable matching.

The property we name complementarity is associated with the fact that, for each agent, the union (intersection) of any two stable partners-sets coincides with the union (intersection) of their join and meet; in other words, the join (meet) of any two stable partner-sets is the complement of their meet (join) from their union, united with their intersection. Complementarity is an interesting property in that, given an arbitrary pair of partner-sets, their join does not necessarily contain their intersection, their meet is not necessarily contained in their union and, moreover, their join and meet together do not necessarily cover their union. It is worth mentioning that complementarity takes on a simple form when preferences are given by an ordering on individuals. On this domain, as it was shown by Alkan [2], given any two stable partner-sets, one is always their join and the other their meet, so that stable partner-sets in fact form a chain.[3] The complementarity property may be seen as the extension of this fact to the path-independent cardinal-monotone domain where stable partner-sets now form a distributive lattice for each agent.

Let us mention that it is the polarity property which is at the root of our findings. The unicardinality and complementarity properties each follow from polarity. We use polarity (but neither unicardinality nor complementariness) also in showing that stable matchings form a lattice. Our proof of distributivity makes use of complementarity and a result in abstract lattice theory (which is the only outside fact used in the paper.)

An implication of the distributivity property worth mentioning is that, for any agent, stable partner-sets partition into levels of desirability. Consequently, the notion of *sex-equal* stable matchings, that Gusfield and Irving [9] had suggested for monogamous matching, may be well-defined for multipartner matching on the path-independent cardinal-monotone domain. Relatedly, it seems that the algorithmic task of tracing the set of all stable matchings would be considerably simpler on this domain in comparison to the path-independent domain undertaken in Martinez *et al* [14].

Our results may in fact *all* fail to hold if agents' choice functions are path independent but not cardinal monotone, as one would see upon inspecting the examples in Blair [7]. We give here, in the last section of the paper, a simple example where polarity fails to hold and the supremum (infimum) of a pair of stable matchings is not stable (individually rational). We also give an example to note the extent to which our model here is broader than the one in Alkan [3].

2 Basic definitions

A *choice function* on a set U is a map $\mathscr{C} : 2^U \longrightarrow 2^U$ such that $\mathscr{C}(T) \subset T$ for all $T \subset U$. (Notation: $T \subset U$ includes the possibility that $T = U$.)

[3] This property was first shown by Roth and Sotomayor [15] for many-to-one matchings.

A *matching market* $(M, W; \mathscr{C}_M, \mathscr{C}_W)$ consists of two finite sets of agents M, W, say *men* and *women*, where each man m is described by a choice function $\mathscr{C}_m$ on W and each woman w by a choice function $\mathscr{C}_w$ on M. A *matching* is a map $\mu : M \cup W \longrightarrow 2^W \cup 2^M$ such that

$$\mu(m) \subset W, \ \ \mu(w) \subset M,$$

and $m \in \mu(w)$ if and only if $w \in \mu(m)$ for all m, w.

Notation: For any $S \subset M$ and $m \notin S$, we write $m \succ_w S$ to mean $m \in \mathscr{C}_w(S \cup m)$. We denote $\overline{S}^w$ the union of S with all m such that $m \notin \mathscr{C}_w(S \cup m)$. We call $\overline{S}^w$ the *closure* of S under $\mathscr{C}_w$.

A matching μ is *individually rational* if $\mathscr{C}_m(\mu(m)) = \mu(m), \mathscr{C}_w(\mu(w)) = \mu(w)$ for all m, w and *pairwise stable* if $w \succ_m \mu(m)$ implies $m \not\succ_w \mu(w)$, in other words, if

$$w \succ_m \mu(m) \text{ implies } m \in \overline{\mu(w)}^w.$$

(For simplicity, we shall henceforth write $\overline{\mu(w)}$ for $\overline{\mu(w)}^w$.)

We call a matching *stable* if it is individually rational and pairwise stable.

3 Preliminaries: path independent choice and the revealed preference lattice

Let $\mathscr{C}$ be a choice function on a set U. We call any subset S of U a (feasible or acceptable) *partner-set* if S is in the range of $\mathscr{C}$, namely if $S = \mathscr{C}(T)$ for some $T \subset U$. We denote by $\mathscr{A}$ the set of all partner-sets.

The *revealed preference* binary relation $\succcurlyeq$ over $\mathscr{A}$ is defined by

$$S \succcurlyeq S' \text{ if and only if } \mathscr{C}(S \cup S') = S.^4 \tag{1}$$

We note that $\succcurlyeq$ is *antisymmetric* since $\mathscr{C}(T)$ is unique for all T.

We assume throughout that $\mathscr{C}$ satisfies the axiom of *consistency*,

$$\mathscr{C}(T) \subset T' \subset T \text{ implies } \mathscr{C}(T') = \mathscr{C}(T), \tag{2}$$

and the axiom of *substitutability*,

$$a \in \mathscr{C}(T) \text{ implies } a \in \mathscr{C}(T' \cup a) \text{ for } T' \subset T. \tag{3}$$

As is well known and easily confirmed, consistency and substitutability in conjunction are equivalent to the axiom of *path independence*,[5]

$$\mathscr{C}(\mathscr{C}(T) \cup T') = \mathscr{C}(T \cup T') \text{ for all } T, T'.$$

In particular, $\mathscr{C}(\mathscr{C}(T)) = \mathscr{C}(T)$ for all T, namely, $\mathscr{C}$ is *idempotent*. Equivalently, $\mathscr{C}(S) = S$ for all S in $\mathscr{A}$, i.e., $\succcurlyeq$ is *reflexive*. One also sees easily that

[4] This relation $\succcurlyeq$ was used in Blair [7].

[5] This equivalence was first noted by Aizerman and Malishevsky [1]. Consistency and substitutability are also known as respectively the Aizerman *Outcast* Condition and *Heritage* Condition. The latter is also known as Chernoff's condition or Sen's Property α.

$\succcurlyeq$ is *transitive*.[6] Thus $\succcurlyeq$ is a partial order on $\mathscr{A}$. In fact, $\mathscr{C}(S \cup S')$ is the *least upper bound* of S, S' for all S, S' in $\mathscr{A}$.[7] In other words, $\mathscr{A}$ is a semilattice under the *join* operation $\vee$ given by

$$S \vee S' = \mathscr{C}(S \cup S'). \tag{4}$$

Since $\{\mathscr{A}, \succcurlyeq\}$ has a minimum element, namely the empty set, it follows that $\mathscr{A}$ is endowed with a *meet* (*greatest lower bound*) operation $\wedge$ as well. Thus $\{\mathscr{A}, \vee, \wedge\}$ is a lattice.

The lemma below identifies the meet operation. Take any $S, S' \in \mathscr{A}$ and consider their closures $\overline{S}, \overline{S'}$ under $\mathscr{C}$.

Lemma 1 $S \wedge S' = \mathscr{C}(\overline{S} \cap \overline{S'})$.

Proof. We show that $\mathscr{C}(\overline{S} \cap \overline{S'})$ is the greatest lower bound of S, S': By path independence $\mathscr{C}(\mathscr{C}(\overline{S} \cap \overline{S'}) \cup S) = \mathscr{C}((\overline{S} \cap \overline{S'}) \cup S) = \mathscr{C}((\overline{S} \cap S) \cup (\overline{S'} \cap S)) = \mathscr{C}(S \cup (\overline{S'} \cap S)) = \mathscr{C}(S) = S$, i.e., $S \succcurlyeq \mathscr{C}(\overline{S} \cap \overline{S'})$. Likewise $S' \succcurlyeq \mathscr{C}(\overline{S} \cap \overline{S'})$.

Now let S'' be any lower bound for S, S'. Then, $\mathscr{C}(S \cup S'') = S$ hence $S'' \subset S \cup S'' \subset \overline{S}$; likewise $S'' \subset \overline{S'}$; hence $S'' \subset \overline{S} \cap \overline{S'}$. By path independence, therefore, $\mathscr{C}(\mathscr{C}(\overline{S} \cap \overline{S'}) \cup S'') = \mathscr{C}((\overline{S} \cap \overline{S'}) \cup S'') = \mathscr{C}(\overline{S} \cap \overline{S'})$. Thus $\mathscr{C}(\overline{S} \cap \overline{S'}) \succcurlyeq S''$. □

Remark The properties of $\{\mathscr{A}, \succcurlyeq\}$ mentioned above are known and there is a growing literature related to path-independent choice lattices: See Alkan [4], Johnson and Dean [10], Koshevoy [11], Monjardet [12]. We remark that $\{\mathscr{A}, \succcurlyeq\}$ is isomorphic to the collection of "closed sets" $\mathscr{K} = \{\overline{S} \subset U \mid S \in \mathscr{A}\}$ partially ordered by inclusion. Given this isomorphism, Lemma 1 follows from the fact that intersection of closed sets are closed sets.

We use the following two lemmas in our study of stable matchings.

Lemma 2 $S \cap \overline{S'} \subset S \wedge S'$.

Proof. Take $a \in S \cap \overline{S'}$. By path independence $S = \mathscr{C}(\overline{S})$ so $a \in \mathscr{C}(\overline{S})$. Also, since $S \subset \overline{S}$, $a \in \overline{S}$. By substitutability, $a \in \mathscr{C}(\overline{S} \cap \overline{S'})$. Lemma 1 then implies $a \in S \wedge S'$. □

Lemma 3 $(S \vee S') \cap (S \wedge S') \subset S \cap S'$.

Proof. If $a \in S \vee S' = \mathscr{C}(S \cup S') \subset S \cup S'$ then $a \in \mathscr{C}(S' \cup a)$ by substitutability, namely, $a \notin \overline{S'} - S'$. This proves $(S \vee S') \cap (\overline{S} \cap \overline{S'}) \subset S \cap S'$. Now note that, by Lemma 1, $S \wedge S' \subset \overline{S} \cap \overline{S'}$. □

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[6] *Proof.* If $S \succcurlyeq S'$ and $S' \succcurlyeq S''$ for any $S, S', S'' \in \mathscr{A}$, then $\mathscr{C}(S \cup S'') = \mathscr{C}(\mathscr{C}(S \cup S') \cup S'') = \mathscr{C}(S \cup S' \cup S'') = \mathscr{C}(S \cup \mathscr{C}(S' \cup S'')) = \mathscr{C}(S \cup S') = S$, thus $S \succcurlyeq S''$.

[7] *Proof.* Suppose $S'' \succcurlyeq S, S'' \succcurlyeq S'$. By path independence $\mathscr{C}(\mathscr{C}(S \cup S') \cup S'') = \mathscr{C}(S \cup S' \cup S'') = \mathscr{C}(S \cup \mathscr{C}(S' \cup S'')) = \mathscr{C}(S \cup S'') = S''$ thus $S'' \succcurlyeq \mathscr{C}(S \cup S')$.

4 Main results: cardinal monotonicity and the lattice structure of stable matchings

Let $(M, W; \mathscr{C}_M, \mathscr{C}_W)$ be any matching market. Given any two matchings μ_1, μ_2, we define their male *supremum* as the matching μ^M where

$$\mu^M(m) = \mu_1(m) \vee_m \mu_2(m),$$

and their male *infimum* as the matching μ_M where

$$\mu_M(m) = \mu_1(m) \wedge_m \mu_2(m),$$

for every m. Female supremum and infimum are defined analogously.

Let μ_1, μ_2 be any two stable matchings.

Lemma 4 $\mu^M(m) \subset \mu_W(m)$ *for all* m.

Proof. Take any $w \in \mu^M(m) = \mathscr{C}_m(\mu_1(m) \cup \mu_2(m)) \subset \mu_1(m) \cup \mu_2(m)$. If w is in both $\mu_1(m)$ and $\mu_2(m)$, then m is in $\mu_1(w) \cap \mu_2(w)$, so $m \in \mu_W(w)$ by Lemma 2, hence $w \in \mu_W(m)$. Say

$$w \in \mu_2(m) - \mu_1(m).$$

But then, $w \succ_m \mu_1(m)$ by substitutability, so $m \in \mu_2(w) \cap \overline{\mu_1(w)}$ by stability, therefore $m \in \mu_1(w) \wedge_w \mu_2(w)$ by Lemma 2. Thus $m \in \mu_W(w)$ so $w \in \mu_W(m)$. □

We will assume from this point on that the choice function $\mathscr{C}$ of each agent is *cardinal monotone* in the sense that

$$|\mathscr{C}(T')| \leqslant |\mathscr{C}(T)| \text{ for all } T' \subset T.$$

Our first proposition below says that male supremum and female infimum of stable matchings are identical; so are, of course, female supremum and male infimum by symmetry. We will refer to this property as the *polarity* property.

Proposition 5 $\mu^M = \mu_W$.

Proof. By cardinal monotonicity, $|\mu_W(w)| \leqslant |\mu^W(w)|$ for all w. (To see this, let $T' = \mu_W(w), T = \mu_W(w) \cup \mu^W(w)$, then note that $T' \subset T$ and $\mathscr{C}_w(T') = \mu_W(w), \mathscr{C}_w(T) = \mu^W(w)$.) So

$$|\mu_W| = \sum_w |\mu_W(w)| \leqslant \sum_w |\mu^W(w)| = |\mu^W|. \tag{5}$$

By Lemma 4, on the other hand, $|\mu^M| \leqslant |\mu_W|, |\mu^W| \leqslant |\mu_M|$ so $|\mu^M| \leqslant |\mu_W| \leqslant |\mu^W| \leqslant |\mu_M| \leqslant |\mu^M|$ hence

$$|\mu^M| = |\mu_W| = |\mu^W| = |\mu_M|. \tag{6}$$

So, by Lemma 4, $\mu^M = \mu_W$. □

We now continue the proof of Proposition 5 above and obtain our second proposition below which we will refer to as the *unicardinality* property of stable matchings.

Proposition 6 *Each agent is matched with the same number of partners in every stable matching.*

Proof. Let w be any agent. From (5) and (6),

$$|\mu_W(w)| = |\mu^W(w)| . \tag{7}$$

Now consider the pair of matchings μ_W, μ_1. Note that μ_W is the infimum and μ_1 the supremum of this pair. By (7), then, $|\mu_W(w)| = |\mu_1(w)|$. Similarly $|\mu_2(w)| = |\mu^W(w)|$. Thus

$$|\mu_1(w)| = |\mu_W(w)| = |\mu^W(w)| = |\mu_2(w)| . \tag{8}$$

□

We next show that, on the set of stable partner-sets for each agent, join and meet are complements in the sense that

$$\mu^W(w) = (\mu_1(w) \cap \mu_2(w)) \cup ((\mu_1(w) \cup \mu_2(w)) - \mu_W(w)),$$

$$\mu_W(w) = (\mu_1(w) \cap \mu_2(w)) \cup ((\mu_1(w) \cup \mu_2(w)) - \mu^W(w)).$$

This property, which we name *complementarity,* follows directly from the proposition below.

Proposition 7 *$\mu^W(w) \cap \mu_W(w) = \mu_1(w) \cap \mu_2(w)$ and $\mu^W(w) \cup \mu_W(w) = \mu_1(w) \cup \mu_2(w)$ for any agent w.*

Proof. If $m \in \mu_W(w)$ then by polarity $m \in \mu^M(w)$ so $w \in \mu^M(m) \subset \mu_1(m) \cup \mu_2(m)$ thus $m \in \mu_1(w) \cup \mu_2(w)$. Thus $\mu_W(w) \subset \mu_1(w) \cup \mu_2(w)$. Hence

$$\mu^W(w) \cup \mu_W(w) \subset \mu_1(w) \cup \mu_2(w). \tag{9}$$

So

$$|\mu^W(w)| + |\mu_W(w)| - |\mu^W(w) \cap \mu_W(w)| \leqslant |\mu_1(w)| + |\mu_2(w)| - |\mu_1(w) \cap \mu_2(w)| . \tag{10}$$

From (8) now $|\mu^W(w) \cap \mu_W(w)| \geqslant |\mu_1(w) \cap \mu_2(w)|$ and consequently from Lemma 3

$$\mu^W(w) \cap \mu_W(w) = \mu_1(w) \cap \mu_2(w). \tag{11}$$

So (10) and therefore (9) must both be equations. Thus

$$\mu^W(w) \cup \mu_W(w) = \mu_1(w) \cup \mu_2(w). \tag{12}$$

The proposition follows from (11) and (12). □

Proposition 8 *The set of stable matchings is a lattice under the supremum and infimum operations for each side of the market.*

Proof. We show that, for each side of the market, the supremum and the infimum of stable matchings are themselves stable matchings. By polarity and symmetry, it suffices to show that μ^M is stable.

By idempotency $\mathscr{C}_m(\mu^M(m)) = \mu^M(m)$ for all m and, by polarity in addition, $\mathscr{C}_w(\mu^M(w)) = \mathscr{C}_w(\mu_W(w)) = \mu_W(w) = \mu^M(w)$ for all w. Thus μ^M is individually rational.

It remains to show that μ^M is pairwise stable. To this end, take any pair mw such that $w \succ_m \mu^M(m)$. That is,

$$w \notin \mu^M(m) \tag{13}$$

and $w \in \mathscr{C}_m(\mathscr{C}_m(\mu_1(m) \cup \mu_2(m)) \cup w)$. By path independence $w \in \mathscr{C}_m(\mu_1(m) \cup \mu_2(m) \cup w)$. In particular, w is not in $\mu_1(m) \cup \mu_2(m)$ (otherwise $w \in \mu^M(m)$) so by substitutability $w \succ_m \mu_1(m)$ and $w \succ_m \mu_2(m)$. By stability, therefore, m belongs to the set

$$T = \overline{\mu_1(w)} \cap \overline{\mu_2(w)}.$$

Recall $\mathscr{C}_w(T) = \mu_W(w)$ by Lemma 1 while $m \notin \mu^M(w) = \mu_W(w)$ by (13) and polarity. By path independence and polarity, therefore, $m \notin \mathscr{C}_w(T) = \mathscr{C}_w(T \cup m) = \mathscr{C}_w(\mathscr{C}_w(T) \cup m) = \mathscr{C}_w(\mu_W(w) \cup m) = \mathscr{C}_w(\mu^M(w) \cup m)$, which says $m \in \overline{\mu^M(w)}$, proving μ^M is pairwise stable. □

Proposition 9 *The supremum and infimum operations for each side of the market are distributive on the set of stable matchings.*

Proof. We need to show that the join and meet operations are distributive on the set of stable partner-sets for each agent. Let w be any agent, μ, μ', μ'' be any three stable matchings and denote $S = \mu(w), S' = \mu'(w), S'' = \mu''(w)$. We will show that $\vee_w, \wedge_w$ are distributive on S, S', S''.

By Proposition 8, S, S', S'' lie in a lattice. Therefore, using a fact from lattice theory, namely Corollary to Theorem II.13 in Birkhoff ([6])), the operations $\vee_w, \wedge_w$ are distributive on S, S', S'' if (and only if)

$$S' = S''$$

in case

$$S \vee_w S' = S \vee_w S'' \text{ and } S \wedge_w S' = S \wedge_w S''. \tag{14}$$

So suppose S, S', S'' satisfy (14). Then $(S \vee_w S') \cup (S \wedge_w S') = (S \vee_w S'') \cup (S \wedge_w S'')$ and $(S \vee_w S') \cap (S \wedge_w S') = (S \vee_w S'') \cap (S \wedge_w S'')$. Consequently $S \cup S' = S \cup S''$ by (12) and $S \cap S' = S \cap S''$ by (11). Thus $S' = S' \cup (S' \cap S) = S' \cup (S'' \cap S) = (S' \cup S'') \cap (S' \cup S) = (S' \cup S'') \cap (S'' \cup S) = S'' \cup (S' \cap S) = S'' \cup (S'' \cap S) = S''$. □

The theorem below is our summary of Propositions 5–9.

Theorem 10 *The set of stable matchings in any two-sided market with path-independent cardinal-monotone choice functions is a distributive lattice under the common preferences of all agents on one side of the market. The supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of associated agents. The lattice has the polarity, unicardinality and complementarity properties.*

5 Two examples

In this section we illustrate the difference between the class of matching markets studied here with those in Blair [7] and Alkan [3] by giving two examples respectively.

In our first example choice functions are path independent but not all cardinal monotone. We exhibit a pair of stable matchings for which polarity does not hold and the supremum (infimum) is not stable (individually rational).

Consider a market with five agents A, B, C, D, E on one side and six agents a, b, c, d, e, z on the other. A chooses a if all candidates are available and (in violation of cardinal monotonicity) he chooses cz if all but a are available; that is a and cz are A's best and second-best teams respectively. The matrix below expresses this and the best and second-best partner-sets for the others. It also says that z's third-best partner-set is E.

A	B	C	D	E	a	b	c	d	e	z
a	b	c	d	z	C	D	A	B	E	A
cz	dz	a	b	e	A	B	C	D	–	B
–	–	–	–	–	–	–	–	–	–	E

It is routine and simple to check that the two matchings where A, B, C, D, E are matched respectively with cz, b, a, d, e and a, dz, c, b, e are both stable and that their male supremum is given by a, b, c, d, e while their female infimum is given by a, bz, c, d, e. In particular, the two matchings are not identical; thus, polarity does not hold. Also, the male supremum is unstable as it is blocked by the pair Ez.[8] We further note that the male infimum of the original matchings, given by cz, dz, a, b, c, is not individually rational as z would disassociate with B.

Before giving the second example, let us mention that in Alkan [3], choice functions are assumed to be *quotafilling* in the sense that they are path independent and they choose a certain fixed number of partners whenever there are at least as many candidates. In particular, a set of partners is a (feasible) partner-set for a quotafilling agent if and only if it has cardinality less than or equal to some fixed quota. On this domain, stable partner-sets happen to be all full-quota or all identical. A partner-set that is full-quota is of course *maximal* in the sense that it is not a proper subset of any partner-set. We provide the simple example below

[8] One may check that the supremum of the two stable matchings labelled M1 and M2 in Example 5.2 given by Blair [7] is also unstable.

to show that, on the broader domain of the present paper, an agent may have several stable partner-sets none of which is maximal.

Consider a market with three agents on each side, namely A, B, C and a, b, c, where all agents but A are monogamous and have their top two choices as stated in the table below:

B	C	a	b	c
b	c	B	A	C
a	$-$	A	B	$-$

Agent A would choose the partner-sets $\{a,c\}, \{a\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}$, $\{c\}$ given the set of potential partners $\{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}$, $\{c\}$ respectively. One easily sees that this choice function is path independent and cardinal monotone but not quotafilling (as it chooses $**block\{a,c\}$ given $\{a,b,c\}$ but only $\{a\}$ given $\{a,b\}$.) One also sees easily that this market has two stable matchings in which the agents A, B, C are matched with a, b, c and b, a, c respectively. We note that the two stable partner-sets that A has, namely $\{a\}$ and $\{b\}$, are both not maximal (as they are proper subsets of the partner-sets $\{ac\}$ and $\{bc\}$ respectively.)

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