A continuous model for dynamic pricing under costly price modifications

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Abstract

This paper presents a heuristic method to solve a dynamic pricing problem under costly price modifications. This is a remarkably difficult problem that is solvable only under very few special cases. The method is applied to a more general form of the problem and is numerically tested for a variety of demand functions in the literature. The results show that the method is quite accurate, approximating the optimal profit within usually much less than 1%. A more important result is that the accuracy tend to be much greater as the number of price changes increases, precisely when the underlying optimization problem becomes much harder, which makes this approach particularly desirable.

Keywords: Revenue management, dynamic pricing, continuous approximation

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1 Introduction

Dynamic pricing is one of the basic tools in a revenue management. In part due to advances in information technology, the theory and practice of dynamic pricing have witnessed a tremendous growth recently. Today, real time price changes have become a common fixture in several industries such as airline, car rental, and hospitality. However, such a practice is infeasible or too costly in certain industries due to significant amount of managerial and labor resources and other associated costs. For example, Levi et al. [10] report that a supermarket's price adjustment cost could be as high as 35% of profits while Zbaracki [15] cite that a large manufacturer spends about one million dollars a year, which is roughly one percent of the revenues or 20% of the reported profits. Continuous pricing might not also be feasible when the seller and the customers desire a certain level of price stability to facilitate their business plans.

In this paper, we present a new solution approach to a dynamic pricing problem with fixed costs under deterministic conditions. This problem has been first studied by Netessine [13]. The problem is to determine the number and time of price changes, prices to charge, and the initial inventory or capacity. The demand is assumed to be an arbitrary deterministic process that depends on the price at the time. If the firm wishes to change the current price it has to incur a fixed cost. Netessine derives various structural results on the uniqueness of the solution and monotonicity of prices during the sales period. While he is able to provide more results on the prices, his work demonstrates the difficulty of finding optimal timing of price changes. He was able to offer analytical solutions to two demand functions to derive optimal timing of price changes and prices. Subsequently, Li and Huh [11] analyze a more restricted model in more detail. They also provide exact solutions for particular linear demand functions while deriving structural properties of optimal prices and optimal timing of price changes under more general conditions. Both works, however noteworthy in their comprehensive treatments of the problem, is a testament to the difficulty of the problem, which defies any analytical or numerical approach under reasonably general conditions. Therefore, we adopt an alternative approach which relies on an approximate reformulation.

While dynamic pricing problems have received considerable attention in the literature particularly recently, the version with limited number of price changes received much less attention. As early examples, Feng and Gallego [7] study timing of a single price change and the optimal price given a discrete set of prices under uncertainty. Subsequently, in a second paper ([8]) they study the timing of exogenously given number of price changes and the prices. A more general version of the problem with uncertain demand and endogenous number of price changes is studied by Celik et al. [2], who consider a time-homogenous demand function and develop heuristics based on fluid approximation. These works consider demand uncertainty however deal with more restrictive environments such a time-homogeneity or limited number of changes. We on the other hand explicitly consider time-inhomogenous demand process with endogenously determined price changes, We however study the problem under deterministic conditions as in Netessine [13] and Li and Huh [11].

The solution approach that we adopt here is called "continuous approximation" (a.k.a. "continuum mechanics) which is introduced to the operations research literature with the seminal work of Newell [14]. Continuous approximation is a technique that is used in reformulation of finite-dimensional problems using continuous variables so that elementary calculus techniques can be utilized. Although logistics literature appears to be forerunner in utilizing this technique (see for example, Daganzo [4] and Langevin et al. [9]), recently, a variety of involved location problems have also been treated with this technique (see for example, Dasci and Laporte [5, 6], Cui et al. [3], and Li and Ouyang [12]).

The structure of the model that we study here is almost exactly the same one presented in Netessine [13] and Li and Huh [11]. However unlike these works, we focus on an approximate solution method for a more general class of demand functions rather than exact solutions for a subset of instances with more restrictive demand functions. Furthermore, cost side of our model is slightly more general than theirs. We believe that such problems under deterministic conditions are more useful at the strategic level and therefore, perhaps does not need the most accurate treatment. After all, in practice where conditions are naturally random, such operational or tactical level decisions are better made dynamically with real time demand observation and information updating. However, for example, if a firm is interested to know the value of dynamic pricing over the static pricing, one probably does not need to find the precise timings of price changes and prices optimally but a rough solution may serve a better purpose. This is precisely what we offer here: Our method, although does not attempt to find the optimal decisions, it approximates the optimal objective function value fairly accurately; within well below 1% most of the time.

The reminder of the paper is follows: We present the general model in Section 2. Solution technique is given in Section 3, which is followed by a small set of computational results in Section 4. Section 5 concludes the paper with few remarks.

2 The model

Consider a firm that sells a standardized product over a finite time, normalized to unit interval $[0, 1]$. Demand is realized at the rate of $D[t, p(t)]$ for a price $p(t) \geq 0$ quoted at time $t \in [0, 1]$. We assume that revenue rate (i.e., $r[t, p(t)] = p(t)D[t, p(t)]$), is a nonnegative, twice-differentiable function in its arguments and is also bounded above and admits a unique maximizer $p^*(t) \in [0, \infty)$. Unit cost is normalized to zero without loss of generality. Hence the revenue rate function also represents the gross profit rate.

The set of demand functions that can be treated here consists of a wide range of functions that are commonly used in the pricing literature. However, similar to earlier models, we assume that the demand at a particular time is a function of the time and price at that time. Therefore, we assume that customers do not behave strategically such as to wait-and-see or to re-trade or affected by the past trajectory of the price.

We assume that the cost of a price adjustment at time $t \in [0, 1]$ is represented by a bounded continuous function $K(t)$. This is a slight generalization over Netessine [13] and, although not explicitly stated in their paper, Li and Huh [11] who consider constant price adjustment costs, i.e. $K(t) = K$

Given a demand and a price adjustment cost function, the objective is to find the number n of price changes during selling season, times $0 = t_1 < t_2 < \ldots < t_n < t_{n+1} = 1$ at which the prices are changed, and the prices p_1, p_2, ..., p_n to maximize the net profit, which is given as:

$$\max \pi = \max_{n} \left\{ \max_{t_i, p_i} \sum_{i=1}^{n} \left[\int_{t_i}^{t_{i+1}} p_i D[t, p_i] dt - K(t_i) \right] \right\}. \tag{1}$$

One possible way to attack this problem is to search over n and solve each resulting nonlinear problem in timing and pricing decisions sequentially. However, as noted by Netessine, even one of the simplest instances of the problem (such as fixed n, constant $K(t)$ and linear demand function) could be extremely hard, due particularly to multiplicity of stationary solutions in t_i's. Netessine demonstrates that two demand functional forms, $D[t,p] = a + bt - p$ and $D[t,p] = 1 - (a + bt)p$ can be solved in closed form (the latter is also demonstrated by Li and Huh) and report solutions for $a = b = 1$.

We also assume that the demand is a linear function of the price but we allow a more general form, i.e., $D[t,p] = \alpha(t) - \beta(t)p$. This form will be better suited to approximate virtually any nonlinear demand function such as Cobb-Douglas, exponential, and logit, which can then be handled with our approach.

3 Analysis

We will start the analysis by revisiting the problem with the demand function $D[t,p] = a + bt - p$, which lends itself to exact analysis under certain conditions, i.e., $K(t) = K$ for all $t \in [0,1]$. This instance will be the building block of our approach. We then extend the analysis incrementally to the general linear demand function. Once the problem is modelled via continuous approximation the optimal objective function value of the original problem can be found approximately without actually explicitly solving for any of the decision variables. However, we will also devise a heuristic solution method based on the outcome of the continuous approximation. This heuristic will also help us to measure the accuracy of the continuous approximation.

3.1 A special case

For $K(t) = K$ and $D[t, p(t)] = a + bt - p(t)$, for $t \in [0,1]$, the problem can be stated as:

$$\max \pi = \max_n \left\{ -nK + \max_{t_i, p_i} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} p_i[a + bt - p_i]dt \right\}. \tag{2}$$

Let us define $\Delta t_i = t_{i+1} - t_i$ for $i = 1, 2, \ldots, n$. The following proposition outlines the optimal solution:

Proposition 1 *Let $K(t) = K$ and $D[t, p(t)] = a + bt - p(t)$, then*

a) the optimal number of price changes n^* is either $\lfloor \sqrt[3]{\frac{b^2}{24K}} \rfloor$ or $\lceil \sqrt[3]{\frac{b^2}{24K}} \rceil$,

b) the optimal timing of price changes satisfy $\Delta t_i^* = t_{i+1}^* - t_i^* = 1/n^*$, i.e., $t_i^* = (i-1)/n^*$, and

c) the optimal the prices are $p_i^* = \frac{1}{2\Delta t_i^*} \int_{t_i^*}^{t_{i+1}^*} [a + bt]dt$

Proof: Part (c) is shown by Netessine. Part (b) is also reported to be shown but not explicitly demonstrated. We briefly present these proofs with completeness. Let us apply a sequential optimization approach: Given n and t_i, $i = 1, 2, \ldots, n, n+1$, the objective function (2) separates in pricing decisions. For each interval $[t_i, t_{i+1}]$, the optimal price is

$$p_i^* = \frac{1}{2\Delta t_i} \int_{t_i}^{t_{i+1}} [a + bt]dt,$$

which reduces the problem to

$$\max \pi = \max_n \left\{ -nK + \max_{t_i} \sum_{i=1}^{n} \frac{1}{4\Delta t_i} \left(\int_{t_i}^{t_{i+1}} [a + bt]dt \right) \right\},$$

and after some algebra, to

$$\max \pi = \max_n \left\{ -nK + \max_{t_i} \frac{1}{4} \sum_{i=1}^{n} (a + \frac{b}{2}(t_{i+1} + t_i))^2 \right\}.$$

For fixed n, the second-order partial derivatives can be found as:

$$\frac{\partial^2 \pi_n}{\partial t_i \partial t_j} = \left(-\frac{32}{b^2} \right) \cdot \begin{cases} t_i - t_{i-1} & \text{if } j = i - 1 \\ t_{i+1} - t_{i-1} & \text{if } j = i \\ t_{i+1} - t_i & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 2, 3, \ldots, n$. The Hessian has a block-diagonal symmetric matrix structure, where the main diagonal entries dominates the off-diagonals. Therefore, it is negative semi-definite (pp. 15 in Berman and Shaked-Monderer [1]) and the objective function is jointly concave in t_i's. Hence, the following first-order conditions are sufficient for optimality, i.e.,

$$\frac{16}{b^2}(t_{i+1}^2 - 2t_i t_{i+1} + 2t_{i-1}t_i - t_{i-1}^2) = 0, \text{ for } i = 2, 3, \ldots, n.$$

From the condition for $i = 2$, we obtain $t_3^2 - 2t_2 t_3 = 0$, i.e., $t_3 = 2t_2$. If we proceed for $i = 3, 4, \ldots$ we obtain $t_{i-1} = (i-2)t_2$ and $t_i = (i-1)t_2$. We finally obtain

$$t_{i+1}^2 - 2(i-1)t_2 t_{i+1} + j(i-2)t_2 = 0.$$

Since $t_{i+1} > t_i$, $t_{i+1} = it_2$. Therefore, $t_i = (i-1)/n$. This completes part (b).

Having obtained optimal timings, we now proceed to solve for n^*. Substituting t_i with $(i-1)/n$ in the total profit, the problem can written as:

$$\max \pi = \max_n \left\{ -nK - \frac{b^2}{48n^2} + \frac{3a^2 + 3ab + b^2}{12} \right\}. \tag{3}$$

Figure 1: An example point-wise and finite pricing. $\pi^*_\infty(t)$ and $\pi^*_4(t)$ denote the gross profit integrand when point-wise and $n = 4$ prices are used, respectively.

Let us, for convenience, define $\sigma = (3a^2 + 3ab + b^2)/12 - \pi$ and re-write the problem more compactly as:

$$\min \sigma = \min_n \left\{ nK + \frac{b^2}{48n^2} \right\}. \tag{4}$$

It is easy to see that minimization (4) can be solved via first-order condition, which gives:

$$n^* = \sqrt[3]{\frac{b^2}{24K}}.$$

This establishes part (a).□

The solution conforms to several expected results of the problem. For example, it is clear to see that as $K \to 0$, the point-wise maximizer becomes the optimal solution. Similarly, for sufficiently large K single-price becomes optimal. Furthermore, as the absolute rate of demand change ($|b|$) increases, i.e., demand becomes more volatile, the frequency of price modifications increases. Finally, it is also easy to establish that the number of price modifications are robust with respect to changes in problem parameters, more so with respect to the fixed cost.

The objective function (3) deserves a further elaboration. The constant term in the objective function is the optimal gross profit under point-wise pricing, i.e., $\pi^*_\infty = \int_0^1 \pi^*_\infty(t)dt$. The optimal point-wise price path is $p^*(t) = (a + bt)/2$, which gives $\pi^*_\infty(t) = (a + bt)^2/4$, from which the constant term in (3) easily follows. For a given number of price modifications n, the first term is the total fixed cost and the middle term is the loss due to infrequent price changes. An example for point-wise pricing and finite pricing with $n^* = 4$ are depicted in Figure 1. On the left, the optimal point-wise and finite pricing functions are given. On the right, the corresponding optimal profit functions are drawn for

Figure 2: Example...

a sample interval. The shaded area is the revenue loss due to single-pricing in a particular interval, which can easily be found. Sum of these areas produces the desired result for the total revenue loss. Hence, the profit maximizing price policy minimizes the sum of the fixed costs and the revenue loss. At the optimal solution the revenue loss and the sum of fixed costs come to:

$$\sigma * = \frac{1}{4}(3bK)^{2/3}.$$

3.2 Reformulation via continuous approximation

The previous analysis suggest that the problem can be transformed into a minimization problem that involves the total price modification costs and the revenue loss due to finite pricing. To compute the overall objective, one also needs to find the point-wise maximizer, which is rather an easy task.

Now, consider the hypothetical example given in Figure 2, which resembles a product going through a traditional product life cycle: The optimal prices are low when the demand is relatively low during product introduction and phase-out and higher in between. Therefore, the optimal prices also follow the pattern. Similar to the earlier example, the shaded area represent the revenue loss. Considering the wide range of demand functions and time-dependent variables, estimation of these shaded areas are quite complicated. Therefore, the problem seems to defy analytical approaches even under slightly more general conditions. The first part of our reformulation aims to approximate this area, which is based on a linear approximation of the demand.

Let us assume that n and t_i for $i = 1, 2, \ldots, n$ are given. Furthermore, the demand function in any interval $[t_i, t_{i+1}]$ can be approximated by a linear function $D[t, p(t)] = a_i + b(t)t - p(t)$ for $t \in [t_i, t_{i+1}]$ and the fixed cost is given in the general form. We however assume that both $b(t)$ and $K(t)$ is somewhat constant in a given interval, i.e., $b(t) \approx b_i$ and $K(t) \approx K_i$ for $t \in [t_i, t_{i+1}]$. Let us also define $\pi^*_\infty(t_i, t_{i+1})$ and $\pi^*_n(t_i, t_{i+1})$ gross profits under point-wise maximization and single pricing in the interval $[t_i, t_{i+1}]$ under these assumptions. It is easy to establish that

$$\pi^*_\infty(t_i, t_{i+1}) = \frac{1}{4} \int_{t_i}^{t_{i+1}} (a_i + b_i t)^2 dt$$

and

$$\pi^*_n(t_i, t_{i+1}) = \frac{1}{4\Delta t_i} \left(\int_{t_i}^{t_{i+1}} [a_i + b_i t] dt \right)^2 .$$

Then the revenue loss in this interval can be found approximately as

$$\pi^*_\infty(t_i, t_{i+1}) - \pi^*_n(t_i, t_{i+1}) = \frac{b_i^2 (t_{i+1} - t_i)^3}{48}.$$

Hence, the problem can approximately be reformulated as:

$$\min \sigma = \min_n \left\{ \min_{t_i} \sum_{i=1}^{n} \left[K_i + \frac{b_i^2 (t_{i+1} - t_i)^3}{48} \right] \right\}. \tag{5}$$

Above formulation does not create any simplifications under general demand and price modification cost parameters. We also like to note that b_i's also depend on the price modifications times, which makes the problem particularly harder. Therefore, we need to further simplify the problem. Now let us re-write the minimization problem as:

$$\min \sigma = \min_n \left\{ \min_{t_i} \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} \left[\frac{b_i^2 (t_{i+1} - t_i)^2}{48} + \frac{K_i}{t_{i+1} - t_i} \right] dt \right\}.$$

Now let us define a step function, $n_s(t) = 1/(t_{i+1} - t_i)$ for $t \in [t_i, t_{i+1}), i = 1, 2, \ldots, n$. This is a peculiar step function that contains all the information of the number and timing of price modifications in a compact way. At the moment it only helps us to write the problem more compactly, but its significance will be clear shortly. Similarly, let us define $b_s(t) = b_i$ and $K_s(t) = K_i$ for $t \in [t_i, t_{i+1}), i = 1, 2, \ldots, n$. Then

$$\min \sigma = \min_{n_s(t)} \int_0^1 \left[n_s(t) K_s(t) + \frac{b_s(t)^2}{48 n_s(t)^2} \right] dt. \tag{6}$$

Clearly, the problems (5) and (6) are equivalent and no simplification is achieved yet since one still needs to explicitly solve the problem for n and t_i's. This is where the continuous approximation strikes: the idea is to replace all of the step functions defined above with their smooth counterparts, which approximates the problem as follows:

$$\min \sigma \approx \min_{n(t)} \int_0^1 \left[n(t) K(t) + \frac{b(t)^2}{48 n(t)^2} \right] dt. \tag{7}$$

Here, $n(t)$ in effect represents the *density* of price modification times. As a result of this approximation the problem frees itself from finding precise values of the number and times of price changes. Above problem, which in essence is equivalent to problem (4), can easily be solved by point-wise optimization of the integrand, which results with

$$n^*(t) = \sqrt[3]{\frac{b(t)^2}{24K(t)}}$$

and

$$\sigma* \approx \frac{1}{4} \int_0^1 \left[[3b(t)K(t)]^{2/3} \right] dt.$$

The results are essentially identical to the ones presented in the previous section. Here instead of a number n^* we have a function $n^*(t)$ which gives the *density* of price changes in the planning horizon. This function does not specify the timings of the optimal price changes but it incorporates those times in its definition (approximately). At the end of this section we will see how this function can be used in a heuristic to find the precise timings. The analysis also suggests that the constant term in the demand intercept (i.e., a_i) essentially has no impact on the form of the optimal solution.

3.3 Generalized linear demand

While the form of demand function studied above is fairly general, in this section we show that a more general form of demand, i.e.,

$$D[t, p(t)] = \alpha(t) - \beta(t)p(t)$$

can be studied somewhat similarly albeit with another level of approximation.

The analysis follows similar steps outlined above and is based on the assumption that the functions $\alpha(t)$ and $\beta(t)$ vary slowly in smaller intervals. Consider any interval $[t_i, t_{i+1}]$ where the demand function can be approximated as:

$$\begin{aligned} D[t, p(t)] &= \beta(t)[\alpha(t)/\beta(t) - p(t)] \\ &\approx \beta_i[\alpha(t)/\beta(t) - p(t)] \end{aligned}$$

Applying Taylor approximation around t_i yields

$$\frac{\alpha(t)}{\beta(t)} \approx \frac{\alpha(t_i)}{\beta(t_i)} + (\frac{\alpha(t)}{\beta(t)})'_{t=t_i}(t - t_i),$$

where $(.)'_{t=t_i}$ denotes the first derivative of the expression evaluated at $t = t_i$. Let $a_i = \frac{\alpha(t_i)}{\beta(t_i)} - t_i(\frac{\alpha(t)}{\beta(t)})'_{t=t_i}$ and $b_i = (\frac{\alpha(t)}{\beta(t)})'_{t=t_i}$. Then the demand function can again be approximated as:

$$D[t, p(t)] \approx \beta_i[a_i + b_i t - p(t)].$$

Applying the results found earlier one can approximate the revenue loss in an interval $[t_i, t_{i+1}]$ as

$$\pi^*_\infty(t_i, t_{i+1}) - \pi^*_n(t_i, t_{i+1}) = \frac{\beta_i b_i^2 (t_{i+1} - t_i)^3}{48}.$$

Therefore, optimization (7) can be re-written as

$$\min \sigma \approx \min_{n(t)} \int_0^1 \left[n(t)K(t) + \frac{\beta(t)b(t)^2}{48n(t)^2} \right] dt,$$

where $b(t) = (\frac{\alpha(t)}{\beta(t)})'$. The optimal solution can then be found as

$$n^*(t) = \sqrt[3]{\frac{\beta(t)b(t)^2}{24K(t)}}$$

and

$$\sigma* \approx \frac{1}{4} \int_0^1 \left[[3\sqrt{\beta(t)}b(t)K(t)]^{2/3} \right] dt.$$

Although $\alpha(t)$ does not appear to have direct effect, it has an effect through $b(t)$.

Figure 3: Finding price change times.

3.4 A Heuristic

While the main focus of our paper is to give an approximation to the optimal objective function value, the information obtained here can also be used to devise a simple heuristic algorithm to find the prices and the timings of the changes. There are several heuristics in the literature such as those presented in Daganzo [4] and Dasci and Laporte [5]. In this paper we also devise a simple heuristic solution method based on Daganzo [4] (Chapter 3). The main purpose of the heuristic is to measure the accuracy of our continuous approximation approach.

We use $D(t) = a + b(t)t - p$ as an illustration. First, define

$$H^*(t) = \frac{1}{n^*(t)} = \sqrt[3]{\frac{24K(t)}{b(t)^2}}.$$

This function, in a way, approximates the difference between the two price change times around time t. Starting at the origin (point $t_0 = 0$) draw a 45° line (Figure 3) and find a horizontal segment $t_1 - t_0$ satisfying

$$\int_{t_0}^{t_1} H(t)dt = (t_1 - t_0)^2.$$

In other words, the evaluation of t_1 should be such that the area of the leftmost square equals the area under $H^*(t)$. This locates t_1, given t_0. The construction is then repeated from t_1 to locate t_2, from t_2, to locate t_3, etc. until $t_{n+2} > 1$ since the time horizon is assumed as $[0,1]$. The last point in time t_{n+1} might be less than 1, so we allocate the residual to each time interval proportionally, i.e., the timings of price changes t'_i $(i = 0, 1, \cdots, n+1)$ are adjusted as

$$\begin{aligned} t'_0 &= 0 \\ t'_i &= t'_{i-1} + (t_i - t_{i-1}) + (1 - t_n)\frac{t_i - t_{i-1}}{t_n}, \qquad i = 1, 2, \cdots, n \\ t'_{n+1} &= 1 \end{aligned}$$

Once we obtain the times at which the prices are changed, the corresponding price and profit at each time interval can be calculated as $p_i = \frac{1}{2(t'_{i+1}-t'_i)}\int_{t'_i}^{t'_{i+1}}(a+b(t)t)dt$ and $\pi^*_i = \frac{1}{4(t'_{i+1}-t'_i)}[\int_{t'_i}^{t'_{i+1}}(a+b(t)t)dt]^2$, respectively.

4 Numerical results

In this section we present the results of a brief numerical experiment. We have tested five demand functions from Netessine [13]. For each demand function, we first found the maximum profit obtainable; i.e., point-wise maximization of the objective function by ignoring the fixed costs. We then generated instances by varying the fixed cost of price changes between 0.001% and 1% of the maximum revenue. In our experiments, we also assumed that fixed cost of price changes are constant throughout the sales period, i.e., $K(t) = K$.

Tables 1 and 2 presents the results for demand functions $D(t,p) = 1 + t - p$ and $D(t,p) = 1 - (1+t)p$, respectively. For these demand functions for a fixed number of prices n the optimal solutions are given by Netessine .These are the only two demand functions that have optimal solutions that can easily be characterized by solving the first-order conditions. We found the optimal n^* for each fixed cost by a simple search over n. The first column reports the fixed costs as a function of the maximum profit mentioned above, i.e., π^*_∞. The next three columns represent the number of prices prescribed by continuous approximation (CA), the heuristic, and the optimal solution. The number in CA is calculated by

$$\int_0^1 n^*(t)dt.$$

Keep in mind that in all these numbers as well as in the profit, the first price and its cost are also

K	Number of prices CA	Heuristic	Optimal	Profit CA	Heuristic	Optimal	Accuracy (%) CA	Heuristic
1.000%	1.9	1	2	0.56648	0.55667	0.56646	0.0042	1.7286
0.500%	2.4	2	2	0.57272	0.56938	0.57229	0.0745	0.5095
0.100%	4.1	4	4	0.57970	0.57795	0.57970	0.0009	0.3019
0.050%	5.2	5	5	0.58105	0.57987	0.58104	0.0009	0.2008
0.010%	8.9	8	9	0.58255	0.58213	0.58255	0.0001	0.0717
0.005%	11.3	11	11	0.58284	0.58255	0.58284	0.0001	0.0501
0.001%	19.3	19	19	0.58317	0.58306	0.58316	0.0000	0.0179

Table 1: $D(t,p) = 1 + t - p$, $\pi^*_\infty = 0.58333$.

K	Number of prices CA	Heuristic	Optimal	Profit CA	Heuristic	Optimal	Accuracy (%) CA	Heuristic
1.000%	2.0	1	2	0.16809	0.16493	0.16811	0.0126	1.8875
0.500%	2.5	2	3	0.17001	0.16897	0.16992	0.0531	0.5612
0.100%	4.3	4	4	0.17217	0.17164	0.17216	0.0027	0.3023
0.050%	5.4	5	5	0.17258	0.17223	0.17258	0.0026	0.2010
0.010%	9.3	9	9	0.17305	0.17291	0.17305	0.0001	0.0799
0.005%	11.7	11	12	0.17314	0.17305	0.17313	0.0002	0.0502
0.001%	20.0	20	20	0.17324	0.17320	0.17323	0.0001	0.0187

Table 2: $D(t,p) = 1 - (1+t)p$, $\pi^*_\infty = 0.17329$.

accounted. Note that both the heuristic and continuous approximation give very close results to the optimal value. The rest of the tables give the objective function values and the accuracies of continuous approximation and the heuristic solutions, which are calculated as $100 * \frac{|\text{Optimal}-\text{CA}|}{\text{Optimal}}$ and $100 * \frac{(\text{Optimal}-\text{Heuristic})}{\text{Optimal}}$, respectively. A quick glance at the numbers show that both the heuristic and the continuous approximation perform remarkably well, particularly when the optimal number of price changes are more than two. This result is somehow expected, because as the number of price changes increases, the intervals gets smaller in which problem parameters vary slowly, or stay somewhat constant, which improves the accuracy of the continuous approximation. Also note that continuous approximation gives neither an upper-bound nor a lower-bound but an approximation of the optimal value.

Tables 3, 4, and 5 present the result for demand functions $D(t,p) = 1 + t^2 - p$, $D(t,p) = \sqrt{t} - p$, and $D(t,p) = 1 - (1+t^2)p$, respectively. Unfortunately, we could not find the optimal solutions in these cases, since the first order conditions were much too involved. In lieu of the optimal objective however,

K	Number of prices CA	Heuristic	Profit CA	Heuristic	UB	Accuracy (%) CA	Heuristic
1.000%	1.2	1	0.45795	0.43978	0.46200	4.1329	4.8100
0.500%	1.6	1	0.46118	0.44211	0.46433	4.3126	4.7858
0.100%	2.7	1	0.46479	0.44398	0.46620	4.6875	4.7666
0.050%	3.4	2	0.46548	0.45930	0.46643	1.3472	1.5302
0.010%	5.8	4	0.46626	0.46470	0.46662	0.3367	0.4119
0.005%	7.3	6	0.46641	0.46572	0.46664	0.1488	0.1982
0.001%	12.4	11	0.46658	0.46638	0.46666	0.0425	0.0601

Table 3: $D(t,p) = 1 + t^2 - p$, $\pi^*_\infty = 0.46667$.

K	Number of prices CA	Heuristic	Profit CA	Heuristic	UB	Accuracy (%) CA	Heuristic
1.000%	4.8	4	0.11595	0.11522	0.12375	6.3038	6.8913
0.500%	6.1	5	0.11930	0.11871	0.12438	4.0820	4.5556
0.100%	10.4	10	0.12305	0.12245	0.12488	1.4615	1.9387
0.050%	13.1	12	0.12377	0.12344	0.12494	0.9329	1.1962
0.010%	22.4	22	0.12458	0.12443	0.12499	0.3260	0.4484
0.005%	28.2	28	0.12474	0.12463	0.12499	0.2070	0.2878
0.001%	48.3	48	0.12491	0.12487	0.12500	0.0710	0.0998

Table 4: $D(t,p) = \sqrt{t} - p$, $\pi^*_\infty = 0.12500$.

we used a very crude upper-bound, which is the maximum profit less one price setting cost, i.e., $\text{UB} = \pi^*_\infty - K$. The accuracy of the heuristic is calculated in a similar way, i.e., $100 * \frac{(\text{UB}-\text{Heuristic})}{\text{UB}}$. However, since continuous approximation does not necessarily give an upper- or a lower-bound, we have used both the upper-bound and the heuristic (as a lower-bound) to compute the accuracy of continuous approximation. That is we have taken the maximum of $100 * \frac{(\text{UB}-\text{CA})}{\text{UB}}$ and $100 * \frac{(\text{Heuristic}-\text{CA})}{\text{Heuristic}}$, which overestimates the continuous approximation's error.

The last two columns show that while continuous approximation and the heuristic appear to be less accurate, they are still reasonable and tend the decrease rapidly with the number of price changes. Nevertheless, based on the results of the first two demand cases, we also have a reason to believe that the actual accuracies of the continuous approximation and the heuristic are more likely to be much better than what we report here.

	Number of prices		Profit			Accuracy (%)	
K	CA	Heuristic	CA	Heuristic	UB	CA	Heuristic
1.000%	1.9	1	0.19078	0.18554	0.19439	2.8261	4.5523
0.500%	2.4	1	0.19284	0.18652	0.19537	3.3899	4.5294
0.100%	4.1	3	0.19515	0.19425	0.19615	0.5112	0.9675
0.050%	5.1	4	0.19559	0.19505	0.19625	0.3347	0.6107
0.010%	8.8	8	0.19609	0.19592	0.19633	0.1214	0.2085
0.005%	11.1	10	0.19619	0.19608	0.19634	0.0775	0.1326
0.001%	18.9	18	0.19629	0.19626	0.19635	0.0270	0.0460

Table 5: $D(t,p) = 1 - (1+t^2)p$, $\pi^*_\infty = 0.19635$.

5 Concluding Remarks

We have presented an alternative method for a dynamic pricing problem under costly price modifications. This is an extremely difficult problem that defies treatment except for few special cases. The continuous approximation approach that we provide has a number of desirable features. First of all, it can be implemented under variety of demand and parameter functions. Second, it provides closed form or simple ways to calculate approximation to the optimal objective function. Finally, the accuracy of this approximation increases when the optimal number of price changes increases, precisely when discrete approaches would have failed.

We should note that our method would be more useful to managers who like to have an approximation of the benefits of dynamic pricing to make strategic pricing decisions. The approach would be of limited use in operational decision making environments which needs to consider uncertainties and take demand learning into account. Although, the generalized linear demand case we have solved allows us to better approximate different forms of demand functions, such as exponential or Cobb-Douglas types, in general, continuous approximation approach could be extended to those and some other type of demand, such as logic or probit. Finally, we believe that continuous approximation has much to offer other dynamic pricing and revenue management settings, especially at the strategic decision making level.

References

[1] A. Berman and N. Shaked-Monderer. *Completely Positive Matrices.* World Scientific Publishing Co. Pte. Ltd., Singapore, 2003.

[2] S. Celik, A. Muharremoglu, and S. Savin. Revenue management with costly price adjustments. *Operations Research*, 57(5):1206–1219, 2009.

[3] T. Cui, Y. Ouyang, and Z.J.M. Shen. Reliable facility location design under the risk of disruptions. *Operations Research*, 58(4):998–1011, 2010.

[4] C.F. Daganzo. *Logistics Systems Analysis.* Springer-Verlag, Berlin, Heidelberg, 1991.

[5] A. Dasci and G. Laporte. Location and pricing decisions of a multi-store monopoly in a spatial market. *Journal of Regional Science*, 44(3):489–515, 2004.

[6] A. Dasci and G. Laporte. A continuous model for multistore competitive location. *Operations Research*, 53(2):263–280, 2005.

[7] Y. Feng and G. Gallego. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science*, 41(8):1371–1391, 1995.

[8] Y. Feng and G. Gallego. Perishable asset revenue management with markovian time dependent demand intensities. *Management Science*, 46(7):941–956, 2000.

[9] A. Langevin, P. Mbaraga, and J.F. Campbell. Continuous approximation models in freight distribution: An overview. *Transportation Research B*, 30:163–188, 1996.

[10] D. Levy, M. Bergen, S. Dutta, and R. Venable. The magnitude of menu costs: Direct evidence from large u. s. supermarket chains. *The Quarterly Journal of Economics*, 112(3):791–824, 1997.

[11] H. Li and W.Y. Huh. Optimal pricing for a short life-cycle product when customer price-sensitivity varies over time. *Naval Research Logistics*, 59(7):552–576, 2012.

[12] X. Li and Y. Ouyang. A continuum approximation approach to reliable facility location design under correlated probabilistic disruptions. *Transportation Research Part B: Methodological*, 44(4):535–548, 2010.

[13] S. Netessine. Dynamic pricing of inventory/capacity with infrequent price changes. *European Journal of Operational Research*, 174(1):553–580, 2006.

[14] G.F. Newell. Scheduling, location, transportation, and continuum mechanics: Some simple approximations to optimization problems. *SIAM Journal on Applied Mathematics*, 25:346–360, 1973.

[15] M.J. Zbaracki, M. Ritson, D. Levy, S. Dutta, and M. Bergen. Managerial and customer costs of price adjustment: Direct evidence from industrial markets. *The Review of Economics and Statistics*, 86(2):514–533, 2004.