

# Application of the no-signaling principle to obtain quantum cloners for any allowed value of fidelity

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Special relativity forbids superluminal influences. Using only the no-signaling principle and an assumption about the form of the Schmidt decomposition, we show that for *any* allowed fidelity there is a *unique* approximate qubit cloner which can be written explicitly. We introduce the prime cloners whose fidelities have multiplicative property and show that the fidelity of the prime cloners for the infinite copy limit is 1/2.

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## I. INTRODUCTION

According to theory of special relativity, it is not possible to send instantaneous signals between two spatially separated observers. The no-signaling (NS) principle is necessary for consistency of the theory of relativity and quantum mechanics. In this work, we show that NS principle can be used to *derive* universal and symmetric 1-to- $M$  qubit cloning transformation for *any* allowed value of fidelity.

Impossibility of faster than light communication based on quantum correlations, after presentation of a signaling protocol based on perfect cloning [1], led to the discovery of the no-cloning theorem [2,3]. However, the NS principle leaves room for an approximate cloning. Imperfect or approximate 1-to-2 optimal quantum cloners have been shown to exist [4,5], and the results have been generalized to 1-to- $M$  cloning [6]. An expression for the maximum fidelity of  $N$ -to- $M$  cloning of qudits has been found [7], and corresponding cloners have been obtained [8–10]. In this work we obtain the universal symmetric quantum cloners for any allowed fidelity, including the best (optimal) one, using the NS principle. We show that fidelity determines the cloning transformation uniquely.

Gisin has analyzed universal symmetric 1-to-2 cloning under the NS principle and has shown that the optimal value is the same as that of the optimal quantum cloner [11]. It has been argued that modification of the quantum theory by introducing nonlinear time evolution for pure states [12,13] might lead to superluminal communication [14–16], and hence, the NS principle implies linearity of quantum mechanics. Inspired by the Gisin’s formalism, Simon has used the *linearity* of quantum dynamics to rederive the *optimal* 1-to- $M$  cloners [17]. In this work we apply *NS principle directly*, rather than utilizing the linearity, to 1-to- $M$  cloning, and we obtain the unique cloners for *all possible* values of fidelity. Furthermore, we construct the prime cloners which have the property that the fidelity of a 1-to- $MN$  cloner can be obtained by the successive use of 1-to- $M$  and 1-to- $N$  cloners. For a given number of copies, we obtain the *unique* prime cloner.

The article is organized as follows. We first introduce the pseudo-spin formalism for universal symmetric cloning. Next, we discuss the consequences of impossibility of instantaneous signaling. We examine the implications of the NS principle

on the cloning transformation and, hence, explicitly obtain all possible values of fidelity along with the corresponding cloners. We then derive the quantum cloners along with what we call *prime cloners*.

## II. PSEUDO-SPIN FORMALISM

Symmetry of the output state, namely, invariance of the wave function under the exchange operation, reduces the dimension of the Hilbert space from  $2^M$  to  $M + 1$ . Pseudo-spin formalism utilizes this dimensional reduction. Let  $|\hat{n}\rangle$  be the state vector of the qubit to be cloned. In the so-called pseudo-spin representation, we treat the qubit as a spin-1/2 object, and thus  $|\hat{n}\rangle$  corresponds to a spin-up state in the  $\hat{n}$  direction. Then, symmetric  $M$ -qubit states can simply be represented by the total spin states with  $j = M/2$ . Therefore, we can use the states

$$|\hat{n}; jm\rangle = \binom{2j}{j+m}^{-1/2} \mathcal{P}\{\underbrace{|\hat{n}\rangle \otimes \dots \otimes |\hat{n}\rangle}_{j+m} \otimes \underbrace{|-\hat{n}\rangle \otimes \dots \otimes |-\hat{n}\rangle}_{j-m}\} \quad (1)$$

as the basis elements in the  $M$ -qubit symmetric space. Here  $\mathcal{P}$  denotes all possible permutations of the product state in the parentheses and  $m = -j, -j + 1, \dots, j - 1, j$ . Pseudo-spin formulation allows us to solve the problem by using the techniques of rotations in quantum mechanics.

Any quantum operation performed on qubits can be modeled as a unitary operation acting on the qubits plus an ancillary system. In the case of cloning, this system is called the cloning machine. After the cloning interaction, the  $M$  qubits will in general be entangled with the cloning machine, and the state of the whole system will be pure. This pure entangled state can be written in the Schmidt form. We assume that the Schmidt basis for  $M$  qubits consists of the states  $|\hat{n}; jm\rangle$ . Therefore, in the most general sense, the transformation for universal and symmetric pure state cloning is given by

$$|\hat{n}\rangle \otimes \underbrace{|0\rangle \otimes \dots \otimes |0\rangle}_{M-1} \otimes |R\rangle \rightarrow \sum_{m=-j}^j a_{jm} |\hat{n}; jm\rangle \otimes |R_{jm}(\hat{n})\rangle, \quad (2)$$

where  $|0\rangle$  and  $|R\rangle$  are blank copy and initial machine states, respectively. The normalization of the output state implies

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that  $\sum_m p_{jm} = 1$ , where  $p_{jm} = a_{jm}^2$ . Independence of the probabilities  $p_{jm}$  from  $\hat{n}$  is necessary for the transformation to be universal. As a result of the Schmidt decomposition, the machine states  $|R_{jm}(\hat{n})\rangle$ , are orthonormal, i.e.,  $\langle R_{jm}(\hat{n})|R_{j'm'}(\hat{n})\rangle = \delta_{mm'}$ . After tracing out the states of the machine, we can formulate the problem in terms of the original state and its copies. The reduced transformation becomes

$$T_j(|\hat{n}\rangle\langle\hat{n}|) = \sum_{m=-j}^j p_{jm}|\hat{n}; jm\rangle\langle\hat{n}; jm|. \quad (3)$$

We see that due to the orthonormality of the machine states, the output state of the cloning transformation is described by a diagonal density matrix.

Fidelity, the measure of the quality of cloning, is defined as the projection of the final single qubit state (obtained by tracing out the other  $M - 1$  qubits) onto the original state. Therefore,  $(j - m)$  combinations of  $2j - 1$  elements of the sum in (3) contribute to the fidelity expression, and the resulting value is given by

$$\begin{aligned} F_j &= \sum_{m=-j}^j \binom{2j-1}{j-m} \binom{2j}{j-m}^{-1} p_{jm} \\ &= \frac{1}{2} \left( 1 + \frac{1}{j} \sum_{m=-j}^j m p_{jm} \right). \end{aligned} \quad (4)$$

Fidelity is a linear function of the expectation value of the  $z$  component of the pseudo-angular momentum. If perfect cloning were possible, we would have  $p_{jm} = \delta_{mj}$ , which results in  $F_j = 1$ . In the next section, we shall evaluate the upper and lower limits for  $F_j$  when the NS principle is taken as a constraint.

### III. NO-SIGNALING CONSTRAINT

The impossibility of superluminal communication implies that transforms of indistinguishable mixtures are also indistinguishable. This is because two observers can share entangled states, where one of them can perform projective measurements to determine the spectral decomposition of the reduced density matrix of the other observer [18–24]. Hence, for a given transformation  $f$ , when two convex linear combinations are equal to  $\sum_i x_i |\psi_i\rangle\langle\psi_i| = \sum_j y_j |\phi_j\rangle\langle\phi_j|$ , so too must their images, i.e.,  $\sum_i x_i f(|\psi_i\rangle\langle\psi_i|) = \sum_j y_j f(|\phi_j\rangle\langle\phi_j|)$ , to prevent signaling. We note that this is a condition involving maps of pure states only, and it implies that  $\sum_i x_i f(|\psi_i\rangle\langle\psi_i|)$  should be a function of only  $\sum_i x_i |\psi_i\rangle\langle\psi_i|$ . Therefore, the NS principle requires that

$$\sum_i x_i f(|\psi_i\rangle\langle\psi_i|) = g \left( \sum_i x_i |\psi_i\rangle\langle\psi_i| \right), \quad (5)$$

where the map  $g$  is not necessarily same as  $f$ . Equivalence of  $f$  and  $g$  cannot be concluded from the above form of the NS principle. However, by introducing a proper communication protocol, we can show that  $f = g$ , and thus, convex linearity is a consequence of the NS condition [25]. Now, since  $(|\hat{n}\rangle\langle\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|)/2$  is equivalent to the identity operator for any  $\hat{n}$ , all

its images, i.e.,  $M$  clones, should be invariant under changes in  $\hat{n}$  too. Since  $|-\hat{n}; jm\rangle = |\hat{n}; j, -m\rangle$ , the indistinguishability requirement states that

$$\begin{aligned} T_j(|\hat{n}\rangle\langle\hat{n}|) + T_j(|-\hat{n}\rangle\langle-\hat{n}|) \\ = \sum_{m=-j}^j (p_{jm} + p_{j,-m})|\hat{n}; jm\rangle\langle\hat{n}; jm| \end{aligned} \quad (6)$$

is rotationally invariant in the pseudo-spin space, and thus, the coefficients of expansion should be independent of  $m$ , i.e.,

$$p_{jm} + p_{j,-m} = \frac{2}{2j+1}. \quad (7)$$

Equation (7) is satisfied by any universal symmetric cloner. However, NS principle is more restrictive than the constraint given by rotational invariance of the expression given in (6). Let us consider two arbitrary qubit states  $|\hat{n}\rangle$  and  $|\hat{n}'\rangle$ , and their arbitrary convex linear combination  $\rho = r|\hat{n}\rangle\langle\hat{n}| + (1-r)|\hat{n}'\rangle\langle\hat{n}'|$  where  $0 \leq r \leq 1$ . The density matrix  $\rho$  is diagonal for some  $|\hat{m}\rangle$ , and hence, it can be written as  $\rho = s|\hat{m}\rangle\langle\hat{m}| + (1-s)|-\hat{m}\rangle\langle-\hat{m}|$  with  $0 \leq s \leq 1$ . Different convex decompositions of the density matrix  $\rho$  can be obtained by different choices of discrete measurements performed by another observer sharing an entangled state with the first observer. In order to prevent signaling, these two representations of the same density matrix must have the same images under the transformation. Therefore,

$$\begin{aligned} rT_j(|\hat{n}\rangle\langle\hat{n}|) + (1-r)T_j(|\hat{n}'\rangle\langle\hat{n}'|) \\ = sT_j(|\hat{m}\rangle\langle\hat{m}|) + (1-s)T_j(|-\hat{m}\rangle\langle-\hat{m}|). \end{aligned} \quad (8)$$

We can choose our coordinate axes so that  $\hat{m} = \hat{z}$ . Then,  $s|\hat{m}\rangle\langle\hat{m}| + (1-s)|-\hat{m}\rangle\langle-\hat{m}|$  becomes

$$\frac{1}{2} \left[ 1 + \frac{\sin(\theta + \theta')}{\sin \theta + \sin \theta'} \right] |\hat{z}\rangle\langle\hat{z}| + \frac{1}{2} \left[ 1 - \frac{\sin(\theta + \theta')}{\sin \theta + \sin \theta'} \right] |-\hat{z}\rangle\langle-\hat{z}|, \quad (9)$$

where  $\theta$  ( $\theta'$ ) is the angle between  $\hat{z}$  and  $\hat{n}$  ( $\hat{n}'$ ), and  $r = \sin \theta' / (\sin \theta + \sin \theta')$ . Therefore, the NS constraint takes the form

$$\begin{aligned} \sum_{m=-j}^j (c_+ p_{jm} + c_- p_{j,-m}) |\hat{z}; jm\rangle\langle\hat{z}; jm| \\ = \sum_{m=-j}^j p_{jm} (\sin \theta' |\hat{n}; jm\rangle\langle\hat{n}; jm| + \sin \theta |\hat{n}'; jm\rangle\langle\hat{n}'; jm|), \end{aligned} \quad (10)$$

where  $2c_{\pm} = \sin \theta + \sin \theta' \pm \sin(\theta + \theta')$ . We note that  $|\langle\hat{z}; jm|\hat{n}; jm'\rangle| = |d_{mm'}^{(j)}(\theta)|$ , where  $d_{mm'}^{(j)}(\theta)$  are the elements of the reduced Wigner rotation matrix. Similarly we have  $|\langle\hat{z}; jm|\hat{n}'; jm'\rangle| = |d_{mm'}^{(j)}(\theta')|$ . Therefore, (10) can be written as

$$\begin{aligned} c_+ p_{jm} + c_- p_{j,-m} \\ = \sum_{m'=-j}^j (|d_{mm'}^{(j)}(\theta)|^2 \sin \theta' + |d_{mm'}^{(j)}(\theta')|^2 \sin \theta) p_{jm'}. \end{aligned} \quad (11)$$

Finally, using the constraint (7), the NS principle can be written as an eigenvalue equation

$$\sum_{m'=-j}^j \left[ |d_{mm'}^{(j)}(\theta)|^2 \sin \theta' + |d_{mm'}^{(j)}(\theta')|^2 \sin \theta - \frac{2c_-}{2j+1} \right] p_{jm'} = \sin(\theta + \theta') p_{jm}. \quad (12)$$

Since  $|d_{mm'}^{(j)}(\theta)| = |d_{-m,-m'}^{(j)}(\theta)|$  when  $p_{jm}$  is a solution of (12),  $p_{j,-m}$  is also a solution with the same eigenvalue. In other words, eigenvectors are (or, in case of degeneracy, can be chosen to be) either symmetric (even) or antisymmetric (odd) in  $m$ .

Let us assume that  $p_{jm}$  can be written as an analytic function  $f(m)$  of  $m$ . Since  $|d_{mm'}^{(j)}(\theta)| = |d_{m'm}^{(j)}(\theta)|$ , we have

$$\begin{aligned} & \sum_{m'=-j}^j |d_{mm'}^{(j)}(\theta)|^2 f(m') \\ &= \sum_{m'=-j}^j \langle \hat{n}; jm' | f(J_z) | \hat{z}; jm \rangle \langle \hat{z}; jm | \hat{n}; jm' \rangle = f(m \cos \theta). \end{aligned} \quad (13)$$

We see that  $\sin(\theta + \theta')$  is a twofold degenerate eigenvalue, and  $p_{jm} = 1/(2j+1)$  is the only symmetric solution, whereas  $p_{jm} = \pm m/j(2j+1)$  are the only possible antisymmetric solutions. The positivity of  $p_{jm}$  allows us to write two linearly independent solutions as  $(j+m)/j(2j+1)$  and  $(j-m)/j(2j+1)$ . Hence, the most general solution becomes

$$p_{jm}(t) = t \frac{j+m}{j(2j+1)} + (1-t) \frac{j-m}{j(2j+1)}, \quad (14)$$

where  $0 \leq t \leq 1$ . The corresponding fidelity is given by

$$F_j(t) = \frac{2j-1+2(j+1)t}{6j}, \quad (15)$$

which has its maximum value at  $(4j+1)/6j$  when  $t=1$ . This is the well-known optimal quantum cloner fidelity [6]. In this case,  $p_{jm}$  coefficients become identical to the optimal quantum machine coefficients. We observe that  $p_{j,-j}$  vanishes only for the optimal cloner. Therefore, if we exclude the worst cloning case from the set of possible output states by assuming that  $p_{j,-j} = 0$  (as has been done in Refs. [6] and [8]), we cannot find the universal cloners other than the optimal one. That

is, the optimal cloner is the only universal quantum cloning machine for which the state  $|\hat{n}; j, -j\rangle$  has zero probability.

Equations (14) and (15) can be used to find the quantum cloner for a given fidelity  $F_j$  in the allowed interval  $[1 - (F_j)_{\max}, (F_j)_{\max}]$ . They can also be used to construct a quantum cloner satisfying some specific property. For example, let us consider the cloners where successive use of 1-to- $M$  and 1-to- $N$  cloners gives the same fidelity as a single 1-to- $MN$  cloner. We call such a cloner as *prime cloner* since it is enough to have 1-to- $p$  cloners, where  $p$  is a prime number, to construct any 1-to- $M$  cloner. The fidelities of prime cloners  $F^P$  should satisfy

$$F_{M/2}^P F_{N/2}^P + (1 - F_{M/2}^P)(1 - F_{N/2}^P) = F_{MN/2}^P. \quad (16)$$

Substituting the fidelity expressions given by (15), we find the coefficients of expansion as

$$p_{jm} = \frac{1}{2j+1} \left[ 1 + \frac{3m}{2j(j+1)} \right], \quad (17)$$

which corresponds to  $t = (2j+5)/4(j+1)$ . We note that fidelity  $F_j^P = (2j+1)/4j$  tends to  $F_\infty^P = 1/2$ , which is just at the center of the allowed fidelity interval at infinite copy limit.

#### IV. CONCLUSION

We presented a method for constructing universal symmetric 1-to- $M$  qubit cloners. In particular, we systematically derived the properties of universal symmetric quantum cloning machines instead of postulating them first and proving them afterwards. Direct use of NS principle allowed us to find the best (optimal) and the worst cloners along with all other cloners having a fidelity between the maximum and the minimum values. For a given fidelity, cloning transformation is unique. We introduced the prime cloners whose fidelities have multiplicative property, and we found the corresponding machines.

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- [1] N. Herbert, *Found. Phys.* **12**, 1171 (1982).  
 [2] W. K. Wootters and W. H. Zurek, *Nature (London)* **299**, 802 (1982).  
 [3] D. Dieks, *Phys. Lett. A* **92**, 271 (1982).  
 [4] V. Bužek and M. Hillery, *Phys. Rev. A* **54**, 1844 (1996).  
 [5] D. Bruß, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. A. Smolin, *Phys. Rev. A* **57**, 2368 (1998).  
 [6] N. Gisin and S. Massar, *Phys. Rev. Lett.* **79**, 2153 (1997).  
 [7] R. F. Werner, *Phys. Rev. A* **58**, 1827 (1998).  
 [8] H. Fan, K. Matsumoto, and M. Wadati, *Phys. Rev. A* **64**, 064301 (2001).

- [9] V. Scarani, S. Iblisdir, and N. Gisin, *Rev. Mod. Phys.* **77**, 1225 (2005).  
 [10] H. Fan, Y.-N. Wang, L. Jing, J.-D. Yue, H.-D. Shi, Y.-L. Zhang, and L.-Z. Mu, *arXiv:1301.2956* [quant-ph], *Phys. Rep.* (to be published).  
 [11] N. Gisin, *Phys. Lett. A* **242**, 1 (1998).  
 [12] S. Weinberg, *Ann. Phys. (NY)* **194**, 336 (1989).  
 [13] S. Weinberg, *Phys. Rev. Lett.* **62**, 485 (1989).  
 [14] N. Gisin, *Phys. Lett. A* **143**, 1 (1990).  
 [15] M. Czachor, *Found. Phys. Lett.* **4**, 351 (1991).  
 [16] J. Polchinski, *Phys. Rev. Lett.* **66**, 397 (1991).  
 [17] C. Simon, *arXiv:quant-ph/0103057v1*.

- [18] E. Schrödinger, *Proc. Camb. Phil. Soc.* **32**, 446 (1936).
- [19] E. T. Jaynes, *Phys. Rev.* **108**, 171 (1957).
- [20] N. Hadjisavvas, *Lett. Math. Phys.* **5**, 327 (1981).
- [21] N. Gisin, *Helv. Phys. Acta* **62**, 363 (1989).
- [22] L. P. Hughston, R. Jozsa, and W. K. Wootters, *Phys. Lett. A* **183**, 14 (1993).
- [23] N. D. Mermin, *Found. Phys.* **29**, 571 (1999).
- [24] K. A. Kirkpatrick, *Found. Phys. Lett.* **19**, 95 (2006).
- [25] Z. Gedik (unpublished).