Alternative Characterizations of Boston Mechanism

Mustafa Oğuz Afacan*

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Faculty of Arts and Social Sciences, Sabancı University, 34956, İstanbul, Turkey.

Abstract

Kojima and Ünver (2011) are the first to characterize the class of mechanisms coinciding with the Boston mechanism for some priority order. By mildly strengthening their central axiom, we are able to pin down the Boston mechanism outcome for every priority order. Our main result shows that a mechanism is outcome equivalent to the Boston mechanism at every priority if and only if it respects both preference rankings and priorities and satisfies individual rationality for schools. In environments where each student is acceptable to every school, respecting both preference rankings and priorities is enough to characterize the Boston mechanism.

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*E-mail: mafacan@sabanciuniv.edu

1 Introduction

A recent paper by Kojima and Ünver (2011), hereafter K&Ü, offers two characterizations of the class of mechanisms each member of which coincides with the Boston mechanism (hereafter *BM*) for some priority order. In this paper, we axiomatize *BM* outcome at every priority order in the school choice environment.

K&Ü's results are remarkable in that their setting and axioms are priority-free, yet, they are able to characterize the class of mechanisms coinciding with *BM* for some priority order. The main axiom in K&Ü is "respecting preference rankings". A mechanism respects preference rankings if whenever a student i is assigned to one of his dispreferred schools to school c, then all the seats of latter are assigned to the students who rank school c at least as high as student i. A mechanism is consistent if a student is removed along with his assignment, then the assignments of all other remaining students obtained by applying the mechanism to the smaller problem are the same as those at the original problem. A mechanism is resource monotonic if each student's welfare is affected in the same way whenever the set of schools shrinks or expands. K&Ü characterize the class of mechanisms coinciding with *BM* for some priority order by respecting preference rankings, consistency, resource monotonicity, and an auxiliary invariance property. In the restricted domain where every school has only one seat, the characterization is given by respecting preference rankings, individual rationality, population monotonicity[1], and the invariance axiom.

In contrast to K&Ü, in this study, the priority order profile of schools is a primitive of the model, and we pin down *BM* outcome for every priority order. We say that a mechanism *respects both preference rankings and priorities* if whenever a student prefers a school to his assignment, then either the school finds him unacceptable (i.e., it prefers keeping seat vacant to being matched with him) or the school has no excess capacity and each assigned student to the school ranks it at least as high as him, and in the case of equal ranking, each of them has higher priority than him. A mechanism is *individually rational for schools* if no seat of any school is assigned to an unacceptable student of it. Our main result shows that a mechanism coincides with *BM* at every problem (i.e., at every preference and priority order profiles) if and only if it respects both preference rankings and priorities and satisfies individual rationality for schools. Therefore, while respecting both preference rankings and priorities is a natural extension of respecting preference rankings to the priority-based setting, it along with individual rationality for schools is able to strengthen the characterization result of K&Ü. In many school choice problems, on the other hand, each student is acceptable to

[1] A mechanism is population monotonic if students are affected in the same direction after some students leave without their assignments.

every school. In such setting, the characterization is given by respecting both preference rankings and priorities.

In order to emphasize the difference between the current paper and K&Ü, we can consider the two-agent serial dictatorship. In this well-known mechanism, agents choose their favorite object one at a time following the exogenously given ordering. It is easy to see that two-agent serial dictatorship is equivalent to BM for some priority order, yet, it is not outcome equivalent to BM at every priority order. Namely, they coincide with each other at the priority order under which the first agent in the ordering has the highest priority at every object. However, except for this particular priority order profile, equivalence no longer holds.

This paper is broadly related to the axiomatic characterization line of research. Kojima and Manea (2010) characterize the class of mechanisms coinciding with the deferred acceptance algorithm for some priority order. Then, similar to the current paper, Morrill (2011a) strengthens their characterization result through axiomatizing the deferred acceptance outcome at every priority order. Papai (2000) characterizes the class of hierarchical exchange rules, which includes the Top Trading Cycles mechanism. Recently, Morrill (2011b) and Abdulkadiroglu and Che (2010) provide alternative axiomatizations of the Top Trading Cycles mechanism in the one-to-one matching environment. Then, it is characterized in the more general many-to-one setting by Dur (2012). In the setting of house allocation with existing tenants, Sönmez and Ünver (2010) axiomatize the you request my house-I get your turn (YRMH-IGYT) mechanism. The axiomatization of the Probabilistic Serial mechanism of Bogomolnaia and Moulin (2001) has received much attention recently. Hashimoto et al. (2012) and Bogomolnaia and Heo (2012) independently offer characterizations of the Probabilistic Serial mechanism. Another well-known mechanism serial dictatorship is axiomatized by Svensson (1999).

The current work characterizes an important school choice mechanism on the full priority domain. While the existing literature provides axiomatizations of school choice mechanisms, almost all of the existing papers assume acceptant priority structure under which every student is acceptable to every school (e.g., Kojima and Manea (2010), Morrill (2011b), Papai (2000), Abdulkadiroglu and Che (2010), and Dur (2012). One exception is Morrill (2011a) characterizing the deferred acceptance mechanism for general substitutable priority structures).[2] Therefore, having our characterization result on the full priority domain is important. The individual rationality for schools axiom plays a critical role in obtaining the result. As other well-known school choice mechanisms (The deferred acceptance and

[2]Apart from the theoretical restriction of acceptant priorities, it is also a restrictive assumption for practical purposes as not every student is acceptable to every school in real-life. One such student placement program example is Massachusetts school district in U.S.

top trading cycles mechanisms) also satisfy that property, it has general implications for not only BM but also other well-known school choice mechanisms on the full priority domain.

2 Model & Results

A school choice problem consists of a tuple $(S, C, P, \succ, q)$. The first two components are finite and disjoint sets of students and schools, respectively. Each student $i \in S$ has a *preference relation* P_i, which is a complete, strict, and transitive binary relation over the set of schools C and being unassigned (denoted by $\emptyset$). Let $\mathcal{P}$ be the set of all such preference relations, and the list $P = (P_i)_{i \in S}$ is the preference profile of students. We write cR_ic' whenever cP_ic' or $c = c'$. Let $P_i(c)$ be the ranking of school c at P_i, that is, if school c is the k^{th} choice of student i with respect to P_i, then $P_i(c) = k$. Each school $c \in C$ has a *priority order* $\succ_c$, which is a complete, strict, and transitive binary relation over the set of students S and keeping a seat vacant (denoted by $\emptyset$). Let $\succ = (\succ_c)_{c \in C}$ be the priority order profile of schools. As well as the priority orders, $q = (q_c)_{c \in C}$ is the *quota* profile of schools where q_c is of school c. The "null school" which denotes being unassigned is not scarce, i.e., $|q_\emptyset| = |S|$. We say that student i (school c) is *acceptable* to school c (student i) if $i \succ_c \emptyset$ ($cP_i\emptyset$). The priority order profile $\succ$ is *acceptant* if $i \succ_c \emptyset$ for any school $c \in C$ and student $i \in S$. Throughout the paper, we fix S, C and write $(P, \succ, q)$ to denote a problem.

A *matching* μ is an assignment of students to schools such that no student is assigned more than one school, and no school is assigned to more students than its quota. We write μ_k for the assignment of student (school) $k \in S \cup C$ under μ. A *mechanism* is a systematic way that assigns a matching for every problem $(P, \succ, q)$.

3 The Boston Mechanism

Below, we outline *BM*.

Step 1. Only the top choices of students are considered. Each school c admits the students who are acceptable and rank it first one at a time following priority order $\succ_c$ until either there are no seats left or there is no such student left. All admitted students are removed.

In general:

Step k. Only the k^{th} choices of the remaining students are considered. Each school c with still positive seats left admits the students who are acceptable and rank school c as their k^{th} choice one at a time following priority order $\succ_c$ until either there are no seats left or there is no such student left. All admitted students are removed.

We write $BM(P, \succ, q)$ for the Boston mechanism outcome at problem $(P, \succ, q)$.

Below we introduce the cental axiom in the paper, which incorporates the priorities into the main axiom of K&Ü (respecting preference rankings) in a natural way.

Definition 1. *Matching μ respects both preference rankings and priorities at a given problem $(P, \succ, q)$, if there exists a student-school pair (i, c) such that $cP_i\mu_i$ and $i \succ_c \emptyset$, then $|\{j \in \mu_c : P_j(c) < P_i(c)\} \cup \{j \in \mu_c : P_j(c) = P_i(c) \text{ \& } j \succ_c i\}| = q_c$.*

We say that a mechanism ψ respects both preference rankings and priorities if $\psi(P, \succ, q)$ respects both preference rankings and priorities at every problem $(P, \succ, q)$. This axiom advocates giving school seats to the ones putting them in preference rankings as high as possible, and when more students than the number of (remaining) seats apply for a school, the school wants to admit students with higher priorities. It is desirable especially in welfare terms as we will explain in detail later.

K&Ü introduce the *respecting preference rankings* property, which turns out to be the central axiom in their priority-free setting. They call student i qualified at school c under mechanism ψ if $\psi_i(P, q) = c$ for some problem (P, q). Then, they say that a matching induced by mechanism ψ respects preference rankings for ψ at problem (P, q) if student i is qualified at school c under ψ and $cP_i\mu_i$, then $|\{j \in \mu_c : P_j(c) \leq P_i(c)\}| = q_c$.

In order to understand the logical relation between our and K&Ü's axioms, first, we should observe that, in our setting, any student is qualified at every school under any mechanism that respects both preference rankings and priorities. This is due to the fact that our domain includes priorities as a primitive. That is, we can easily construct problem instances where any student is matched with any school under any mechanism satisfying our axiom.[3] Given that, we can easily observe that our axiom implies respecting preference rankings, however, the converse is not true. To see this, consider a variant of Boston mechanism under which, at some step, schools accept lower priority students among the ones applying to them in that step. This defines a mechanism different from BM respecting preference rankings, nevertheless, it violates respecting both preference rankings and priorities.

Another interesting implication of the above axioms is related to a certain welfare maximization problem. K&Ü demonstrate that, for a given priority profile, the outcome of a Boston mechanism is equivalent to the solution of a particular welfare maximization problem.

Proposition 1 (K&Ü). *For a given problem $(P, \succ, q)$, the solution of the following linear program is equal to the outcome of the Boston mechanism:*

[3]For instance, if we want a matching where each student i is matched with school c_i (schools might be the same across all students), then we can think of the problem: $P_i : \; c_i, \; \emptyset$; and $q_{c_i} = |N|$ for every $c_i \in C$, and $\succ$ is such that every student is acceptable to every school. Then, any mechanism satisfying our axiom assigns the desired matching.

$$\max_{[z_{i,c}]} \sum_{i\in I, c\in C\cup\{\emptyset\}} z_{i,c} u_i(c).$$
$$\textit{subject to}$$
$$\emptyset \succ_c i \;\Rightarrow\; z_{i,c} = 0 \;\forall i \in I, c \in C$$
$$0 \le z_{i,c} \;\forall i \in I, c \in C \cup \{\emptyset\}$$
$$\sum_{c\in C\cup\emptyset} z_{i,c} = 1 \;\forall i \in I$$
$$\sum_{i\in I} z_{i,c} \le q_c \;\forall c \in C \cup \{\emptyset\}.$$

where $q_\emptyset = |I|$, *and* $(u_i)_{i\in I} \subseteq ((R_+^{|C|\cup\{\emptyset\}}))^{|I|}$ *are utility functions consistent with the preference profile* P[4] *satisfying*

(i) $2u_i(d) < u_j(c)$ $\forall i, j \in I$ *and* $c, d \in C \cup \{\emptyset\}$ *with* $P_j(c) < P_i(d)$

(ii) $2u_i(c) < u_j(c)$ $\forall i, j \in I$ *and* $c \in C \cup \{\emptyset\}$ *with* $P_i(c) = P_j(c)$ *and* $j \succ_c i$.

By slightly modifying the proof of above proposition (hence, omitted), we can easily show that the solution(s) of the same maximization problem but with the following condition (ii') instead of condition (ii) above gives the set of all matchings respecting preference rankings.[5]

(ii') $2u_i(c) < u_j(c)$ or $2u_j(c) < u_i(c)$ $\forall i, j \in I$ and $c \in C \cup \{\emptyset\}$ with $P_i(c) = P_j(c)$.

Hence, Conditions (ii) and (ii') show that respecting both preference rankings and priorities selects the allocation among the solutions of the problem with condition (ii') which is obtained through breaking the ties by favoring higher priority students. This way of selection can easily be argued to be fair, which also justifies our main axiom on the fairness ground.

K&Ü show that a mechanism is *BM* for some priority order if it is consistent, resource monotonic, r.r invariant, and respects preference rankings. While consistency and resource monotonicity are standard axioms in the characterization literature, the invariance property is a weaker version of Maskin monotonicity. In the rest of the paper, we will demonstrate that our natural extension of respecting preference rankings to the priority based setting along with a weaker version of the standard individual rationality enables us to pin down *BM* outcome at every priority rather than for some priority.

A mechanism ψ is *individually rational for schools* if, at every problem $(P, \succ, q)$ and for any school $c \in C$, there is no student $i \in \psi_c(P, \succ, q)$ such that $\emptyset \succ_c i$.

Theorem 1. *A mechanism* ψ *respects both preference rankings and priorities and satisfies individual rationality for schools if and only if* $\psi(P, \succ, q) = BM(P, \succ, q)$ *for every problem* $(P, \succ, q)$.

[4]That is, for any $i \in I$ and $c, d \in C \cup \{\emptyset\}$, $u_i(c) > u_i(d) \Leftrightarrow cP_id$.

[5]I am grateful to the anonymous referee for suggesting me to look at the welfare maximization problem.

Proof. See Appendix. □

Remark 1. In this remark, we show that our axioms are independent. The "null" mechanism which leaves each student unassigned satisfies individual rationality for schools, yet, it does not respect both preference rankings and priorities. Let us consider a mechanism which works as the same as *BM* except it treats every student as acceptable at every school regardless of the actual priority orders. It respects both preference rankings and priorities, yet, it is not individually rational for schools.

Remark 2. After I wrote this paper, I figured out that Chen (2011) independently characterizes *BM* in the same setting as the current paper. Yet, our work is a clear improvement over Chen (2011) for two important reasons. First, the combination of the axioms in the main theorem (Theorem 1) of Chen (2011) is equivalent to respecting both preference rankings and priorities. While she claims that those axioms characterize *BM* in Theorem 1, it is not true as respecting both preference rankings and priorities is not enough as we explained above. Yet, as her characterizations are valid in the restricted acceptant priorities domain in which any student is acceptable to every school, Chen (2011) can be assumed in that restricted domain, while, the current paper provides characterization in the full strict priority domain. Another advantage of the current paper is that the proofs are much simpler and shorter than those in Chen (2011).

In some school districts, students might not be acceptable due to the living outside of the districts or discipline problems.[6] Yet, students are often acceptable to schools in school-choice problems. In the acceptant priorities domain, we do not need individual rationality for schools.

Proposition 2. *In the acceptant priorities domain, a mechanism* ψ *respects both preference rankings and priorities if and only if* $\psi(P, \succ, q) = BM(P, \succ, q)$ *at every problem* $(P, \succ, q)$.

The proof of the above result is a straightforward modification of the proof of Theorem 1, hence, it is omitted.

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[6] For instance, not all school districts in Massachusetts accept students from outside of their districts.

Appendix

The proof of Theorem 1. **"If" part:** It is straightforward to verify that BM respects both preference rankings and priorities and satisfies individual rationality for schools.

"Only If" part:[7] Let ψ be mechanism which is individually rational for schools and respecting preference rankings and priorities. Assume for a contradiction that, for some problem $(P, q, \succ)$, $\psi(P, q, \succ) \neq BM(P, q, \succ)$. Hereafter, for ease of notation, we write μ and π for the outcomes, respectively. Let $\tilde{k}$ be the earliest step of Boston algorithm in which student s is assigned and $\mu_s \neq \pi_s$. If we write $T(k)$ for the assigned students at step k in the course of Boston algorithm, then, our above construction implies that $\mu_s = \pi_s$ for all $s \in \bigcup_{k<\tilde{k}} T(k)$ (if $\tilde{k} = 1$, then $\bigcup_{k<\tilde{k}} T(k) = \emptyset$).

Now, we have $s \in T(\tilde{k})$ and $\mu_s \neq \pi_s$. We first claim that $\pi_s P_s \mu_s$. By the definition of Boston algorithm, at step $\tilde{k}$, student s applies to his favorite school among the ones having left seat. This fact and $\mu_s = \pi_s$ for all $s \in \bigcup_{k<\tilde{k}} T(k)$ imply that all schools student s preferring to school π_s are filled up by other students at matching μ, thereby, $\pi_s R_s \mu_s$. This along with $\mu_s \neq \pi_s$ implies that $\pi_s P_s \mu_s$. On the other hand, as μ respects both preference rankings and priorities, we have $|\mu_{\pi_s}| = q_{\pi_s}$ (note that student s is acceptable to both schools π_s and μ_s as he is matched with those schools under mechanisms BM and ψ which are individually rational for schools). This implies that there exists a student s' such that $\mu_{s'} = \pi_s \neq \pi_{s'}$. As μ respects both preference rankings and priorities, we have either $P_{\pi_s}(s') < P_{\pi_s}(s)$ or $P_{\pi_s}(s') = P_{\pi_s}(s)$ and $s' \succ_{\pi_s} s$. This along with π_s being the $\tilde{k}^{th}$ choice of student s shows that student s' matches with $\pi_{s'}$ in the course of Boston algorithm in a step earlier than $\tilde{k}$. That is, $s' \in \bigcup_{k<\tilde{k}} T(k)$. On the other hand, by our construction, we have $\mu_s = \pi_s$ for all $s \in \bigcup_{k<\tilde{k}} T(k)$. This, yet, contradicts $\mu_{s'} \neq \pi_{s'}$.

□

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