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Bohr property of bases in the space of entire functions and its generalizations

Aydin Aytuna and Plamen Djakov

Dedicated to Tosun Terzioğlu on the occasion of his seventieth birthday

Abstract

We prove that if $(\varphi_n)_{n=0}^{\infty}$, $\varphi_0 \equiv 1$, is a basis in the space of entire functions of d complex variables, $d \geq 1$, then for every compact $K \subset \mathbb{C}^d$ there is a compact $K_1 \supset K$ such that for every entire function $f = \sum_{n=0}^{\infty} f_n \varphi_n$ we have $\sum_{n=0}^{\infty} |f_n| \sup_K |\varphi_n| \leq \sup_{K_1} |f|$. A similar assertion holds for bases in the space of global analytic functions on a Stein manifold with the Liouville Property.

1. *Introduction*

In 1914 H. Bohr [**10**] proved that if $f = \sum c_n z^n$ is a bounded analytic function on the unit disc $U \subset \mathbb{C}$, then

$$\sum_{n=0}^{\infty} |c_n| r^n \leq \sup_{z \in U} |f(z)|$$

for every $0 \leq r \leq \frac{1}{6}$. The largest $r \leq 1$ such that the above inequality holds is referred to as the *Bohr radius*, κ_1, for the unit disc. The exact value of κ_1 was computed, by M. Riesz, I. Schur and N. Wiener, to be $\frac{1}{3}$.

In 1997 H. P. Boas and D. Khavinson [**9**] showed that a similar phenomenon occurs for polydiscs in $\mathbb{C}^d$. If we let U^d denote the unit polydisc in $\mathbb{C}^d$, the largest number r such that if $|\sum_\alpha c_\alpha z^\alpha| < 1$ for all $z \in U^d$, then $\sum_\alpha |c_\alpha z^\alpha| < 1$ holds for the homothetic domain $r\,U^d$, is referred to as the *Bohr radius*, κ_d, for the unit polydisc U^d. Boas and Khavinson obtained upper and lower bounds for κ_d, in terms of d, and showed that $\kappa_d \to 0$ as $d \to \infty$. However the *exact* value of κ_d is still not known. Recently A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes and K. Seip [**14**] showed that κ_d behaves asymptotically as $\sqrt{\frac{\log d}{d}}$, modulo a factor bounded away from zero and infinity. Other multidimensional variants of Bohr's phenomenon were given by L. Aizenberg [**1**]. He proved Bohr-type theorems for bounded complete Reinhardt domains and obtained estimates for the corresponding Bohr radii.

P. G. Dixon [**16**] has used Bohr's original theorem to construct a Banach algebra which is not an operator algebra, yet satisfies the non-unital von Neumann's inequality. V. Paulsen, G. Popescu and D. Singh [**25**] have applied operator-theoretic techniques to obtain refinements and multidimensional generalizations of Bohr's inequality.

Interesting interconnections among multidimensional Bohr radii, local Banach space theory and complex analysis in infinite number of variables established in [**12**] and [**15**] triggered a further wave of investigations. For this line of research and recent related references we refer the reader to the survey [**13**].

Ramifications and extensions of Bohr-type theorems also attracted attention. Various authors studied versions of Bohr phenomena in different settings. See for example [**2**], [**11**], [**17**], [**19**].

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In [**3**] and [**4**] we (along with L. Aizenberg) took a more abstract approach to Bohr phenomena and considered general bases in the space of global analytic functions on complex manifolds rather than monomials. For a complex manifold $\mathcal{M}$, a given basis $(\varphi_n)_{n=0}^{\infty}$ in the space $H(\mathcal{M})$ of global analytic functions is said to have the *Bohr Property* (BP) if there exist an open set $U \subset \mathcal{M}$ and a compact set $K \subset \mathcal{M}$ such that, for every $f = \sum c_n \varphi_n$ in $H(\mathcal{M})$, the inequality

$$\sum |c_n| \sup_U |\varphi_n(z)| \le \sup_K |f(z)| \tag{1.1}$$

is valid. In [**4**, Theorem 3.3] it is shown that a basis $(\varphi_n)_{n=0}^{\infty}$ has BP if $\varphi_0 = 1$ and there is a point $z_0 \in \mathcal{M}$ such that $\varphi_n(z_0) = 0$, $n = 1, 2, \ldots$.

Let us note that Theorem 3.3 in [**4**] has a *local character*, namely in fact it proves that for any compact neighborhood $K \ni z_0$ there is an open set U with $z_0 \in U \subset K$ such that (1.1) holds. Moreover, its proof is based on considering sets U that "shrink" to z_0.

Recently P. Lassère and E. Mazzilli [**21**] (see also [**20**] and [**22**]) have studied the Bohr phenomenon for the Faber polynomial basis $(\psi_n)_{n=0}^{\infty}$ associated to a continuum in $\mathbb{C}$. By using Theorem 3.3 in [**4**] and some properties of Faber polynomials they proved that for *every* relatively compact $W \subset \mathbb{C}$ there is a compact $K \subset \mathbb{C}$ such that, for every entire function $f = \sum c_n \psi_n$,

$$\sum |c_n| \sup_W |\psi_n(z)| \le \sup_K |f(z)|.$$

Let us note that the latter assertion has a *global character*.

In this paper we give a characterization of the bases that possess *global* BP in the above sense, for a class of complex manifolds which contains $\mathbb{C}^d, 1 \le d < \infty$ and more generally parabolic Stein manifolds (see [**6**, **7**]). Our results extend and generalize the above mentioned theorem of P. Lassère and E. Mazzilli [**21**, Theorem 3.1]. See more comments on their results in Section 3, after Remark 4.

We recall some basic definitions and facts and get preliminary results in Section 2. In Section 3 we prove that the Global Bohr Property takes place for every basis $(\varphi_n)_{n=0}^{\infty}$ in the space of entire functions $H(\mathbb{C}^d)$ provided one of the functions φ_n is a constant. In Section 4 we generalize the results from Section 3 for Stein manifolds with Liouville Property, i.e., for manifolds with the property that every bounded analytic function reduces to a constant.

2. *Preliminaries*

Let $D \subset \mathbb{C}^d$ be a domain in $\mathbb{C}^d$ (or, $D \subset \mathcal{M}$, where $\mathcal{M}$ is a Stein manifold). We denote by $H(D)$ the space of analytic functions on D. For any compact subset $K \subset D$ we set

$$|f|_K := \sup_K |f(z)|, \quad f \in H(D).$$

Further we write $K \subset\subset D$ if K is a compact subset of D. The system of seminorms $|f|_K$, $K \subset\subset D$, defines the topology of uniform convergence on compact subsets of D. Regarded with it $H(D)$ is a nuclear Fréchet space (e.g. [**24**]).

Recall that a sequence $(\varphi_n)_{n=0}^{\infty}$ of analytic functions on D is called *basis* in $H(D)$ if for every function $f \in H(D)$ there exists a unique sequence of complex numbers f_n such that

$$f = \sum_{n=0}^{\infty} f_n \varphi_n,$$

where the series converges uniformly on any compact subset of D.

We will use the following well known fact.

Proposition 1. *If $(\varphi_n)_{n=0}^{\infty}$ is a basis in $H(D)$ then for every $K_1 \subset\subset D$ there exist $K_2 \subset\subset D$ and $C > 0$ such that*

$$\text{if } f = \sum f_n \varphi_n \quad \text{then} \quad \sum |f_n||\varphi_n|_{K_1} \le C|f|_{K_2}.$$

The assertion follows from the theorem of Dynin and Mityagin on absoluteness of bases in nuclear spaces ([**18**, **23**], e.g. [**24**], Theorem 28.12).

Lemma 2. *Let $(\varphi_n)_{n=0}^{\infty}$ be a basis in $H(\mathbb{C}^d)$, $d \ge 1$, such that $\varphi_0(z) \equiv 1$. Then for every compact $K \subset \mathbb{C}^d$ we have*

$$\inf_{K_1 \subset\subset \mathbb{C}^d} \sum_{n=1}^{\infty} \frac{|\varphi_n|_K}{|\varphi_n|_{K_1}} = 0. \tag{2.1}$$

Proof. Since $\varphi_0(z) \equiv 1$, we have that $\varphi_n \ne \mathit{const}$ for $n \in \mathbb{N}$. Therefore, each of the function φ_n, $n \in \mathbb{N}$ is unbounded. This implies that

$$\inf_{K_1 \subset\subset \mathbb{C}^d} \frac{|\varphi_n|_K}{|\varphi_n|_{K_1}} = 0, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

On the other hand, by Grothendieck-Pietsch criterion for nuclearity (e.g. [**24**], Theorem 28.15) we have

$$\forall\, K \subset\subset \mathbb{C}^d \quad \exists\, K_1 \subset\subset \mathbb{C}^d \ : \quad \sum_0^{\infty} |\varphi_n|_K / |\varphi_n|_{K_1} < \infty. \tag{2.3}$$

Now (2.1) follows immediately from (2.2) and (2.3).

□

3. *Bohr property of bases in the space of entire functions*

We say that a basis $(\varphi_n)_{n=0}^{\infty}$ in the space of entire functions $H(\mathbb{C}^d)$ has the *Global Bohr Property* (*GBP*) if for every compact $K \subset \mathbb{C}^d$ there is a compact $K_1 \supset K$ such that

$$\text{if } f = \sum_0^{\infty} f_n \varphi_n, \ \text{ then } \sum_0^{\infty} |f_n||\varphi_n|_K \le |f|_{K_1} \quad \forall f \in H(\mathbb{C}^d). \tag{3.1}$$

Theorem 3. *A basis $(\varphi_n)_{n=0}^{\infty}$ in the space of entire functions $H(\mathbb{C}^d)$ has GBP if and only if one of the functions φ_n is a constant.*

Proof. Let $(\varphi_n)_{n=0}^{\infty}$ be a basis in $H(\mathbb{C}^d)$ which has *GBP*. If $1 = \sum_{n=0}^{\infty} c_n \varphi_n(z)$ is the expansion of the constant function 1, then at least one of the coefficients c_n is nonzero, say $c_{n_0} \ne 0$. By (3.1), it follows that for every $K \subset\subset \mathbb{C}^d$ there is $K_1 \subset\subset \mathbb{C}^d$ such that

$$|c_{n_0}||\varphi_{n_0}|_K \le |1|_{K_1} = 1.$$

Hence $|\varphi_{n_0}(z)| \le 1/|c_{n_0}|$, i.e., $\varphi_{n_0}(z)$ is a bounded entire function, so it is a constant by the Liouville theorem. (The necessity assertion follows also from the argument that proves Proposition 3.1 in [**4**].)

Suppose that $(\varphi_n)_{n=0}^{\infty}$ is a basis in $H(\mathbb{C}^d)$ such that one of the functions φ_n is a constant, say

$$\varphi_0(z) \equiv 1. \tag{3.2}$$

Let $B(r) = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \sum_1^d |z_k|^2 \le r^2\}$. It is enough to prove that for every $r > 0$ there is $R > r$ such that

$$\text{if } f = \sum_0^\infty f_n \varphi_n, \text{ then } \sum_0^\infty |f_n||\varphi_n|_{B(r)} \le |f|_{B(R)} \quad \forall f \in H(\mathbb{C}^d). \tag{3.3}$$

One can easily see that the system

$$\psi_0(z) \equiv 1, \quad \psi_n(z) = \varphi_n(z) - \varphi_n(0), \quad n \in \mathbb{N}, \tag{3.4}$$

is also a basis in $H(\mathbb{C}^d)$. Moreover, if $f = \sum_{n=0}^\infty f_n \varphi_n$, then we have $f(0) = \sum_{n=0}^\infty f_n \varphi_n(0)$, which implies that $f = f(0) + \sum_{n=1}^\infty f_n \psi_n$.

First we show that for every $r > 0$ there is $R > r$ such that

$$\text{if } f = f(0) + \sum_1^\infty f_n \psi_n, \text{ then } |f(0)| + \sum_1^\infty |f_n||\psi_n|_{B(r)} \le |f|_{B(R)} \ \forall f \in H(\mathbb{C}^d). \tag{3.5}$$

Fix $r > 0$ and a function $f \in H(\mathbb{C}^d)$. We may assume without loss of generality that $f(0) \ge 0$ (otherwise one may multiply f by $|f(0)|/f(0)$). By Proposition 1, there are $C > 0$ and $r_1 > r$ (which do not depend on f) such that

$$f(0) + \sum_1^\infty |f_n||\psi_n|_{B(r)} \le f(0) + C|f - f(0)|_{B(r_1)}.$$

Now, for any $r_2 > r_1$, the Borel - Carathéodory theorem (see [**26**]) says that

$$|f - f(0)|_{B(r_1)} \le \frac{2r_1}{r_2 - r_1} \sup_{B(r_2)} (Ref(z) - f(0)).$$

Let $r_2 = (2C+1)r_1$. Then $\frac{2Cr_1}{r_2 - r_1} = 1$, so we obtain

$$f(0) + \sum_1^\infty |f_n||\psi_n|_{B(r)} \le f(0) + \frac{2Cr_1}{r_2 - r_1} \sup_{B(r_2)} (Ref(z) - f(0)) \le \sup_{B(r_2)} Ref(z),$$

i.e., (3.5) holds with $R = (2C+1)r_1$.

Next we prove (3.3). Since $\varphi_0 \equiv 1$, we have

$$f(0) = f_0 + \sum_{n=1}^\infty f_n \varphi_n(0) \quad \Rightarrow \quad |f_0| \le |f(0)| + \sum_{n=1}^\infty |f_n||\varphi_n(0)|,$$

so it follows that

$$\sum_0^\infty |f_n||\varphi_n|_{B(r)} \le |f_0| + \sum_1^\infty |f_n||\varphi_n - \varphi_n(0)|_{B(r)} + \sum_{n=1}^\infty |f_n||\varphi_n(0)|$$

$$\le |f(0)| + \sum_1^\infty |f_n||\varphi_n - \varphi_n(0)|_{B(r)} + 2\sum_{n=1}^\infty |f_n||\varphi_n(0)|.$$

Applying the Schwartz Lemma, we obtain

$$\sum_0^\infty |f_n||\varphi_n|_{B(r)} \le |f(0)| + \frac{1}{3}\sum_1^\infty |f_n||\varphi_n - \varphi_n(0)|_{B(3r)} + 2\sum_{n=1}^\infty |f_n||\varphi_n(0)|. \tag{3.6}$$

On the other hand, by Lemma 2, there is $\tilde{r} \ge 3r$ such that

$$|\varphi_n(0)|/|\varphi_n|_{B(\tilde{r})} \le \frac{1}{4}, \quad \forall n \in \mathbb{N}.$$

Therefore, we have

$$|\varphi_n - \varphi_n(0)|_{B(\tilde{r})} \ge |\varphi_n|_{B(\tilde{r})} - |\varphi_n(0)| \ge 3|\varphi_n(0)|, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Since $\tilde{r} \geq 3r$, (3.6) and (3.7) imply that

$$\sum_0^\infty |f_n||\varphi_n|_{B(r)} \leq |f(0)| + \sum_1^\infty |f_n||\varphi_n - \varphi_n(0)|_{B(\tilde{r})}.$$

Hence, from (3.4) and (3.5) it follows that (3.3) holds. This completes the proof. □

REMARK 4. *Suppose that* $(\varphi_n)_{n=0}^\infty$, $\varphi_0 \equiv 1$, *is a basis in the space of entire functions* $H(\mathbb{C}^d)$ *and* $K \subset K_1$ *are compact sets such that for every entire function* $h = \sum_{n=0}^\infty h_n \varphi_n$ *we have*

$$\sum_{n=0}^\infty |h_n||\varphi_n|_K \leq |h|_{K_1}. \tag{3.8}$$

If a domain $G \supset K_1$ *has the property that* (φ_n) *is a basis in the space* $H(G)$ *of holomorphic functions on* G, *then for every bounded function* $f \in H(G)$ *with* $f = \sum f_n \varphi_n$ *we have*

$$\sum_{n=0}^\infty |f_n||\varphi_n|_K \leq \sup_G |f(z)|. \tag{3.9}$$

Proof. If f is a bounded holomorphic function on G, then

$$f = \sum_{n=0}^\infty f_n \varphi_n,$$

where the series converges uniformly on every compact subset of G, so in particular it converges uniformly on K_1.

Let (S_m) be the sequence of partial sums of the expansion of f. Then (3.8) holds for S_m, i.e.,

$$\sum_{n=0}^m |f_n||\varphi_n|_K \leq \sup_{K_1} |S_m(z)|,$$

and $S_m \to f$ uniformly on K_1. Therefore, it follows that

$$\sum_{n=0}^\infty |f_n||\varphi_n|_K \leq \sup_{K_1} |f(z)| \leq \sup_G |f(z)|,$$

which completes the proof. □

Some of the bases encountered for the space of entire functions are also common bases for the spaces of analytic functions corresponding to an exhaustive one parameter family of sub domains $\mathcal{D}_r$ (usually sub level domains of an associated plurisubharmonic function). So if such a basis, say (φ_n), has GBP then for every compact set $K \subset \mathbb{C}^d$, there exits a domain $\mathcal{D}_r$ such that (3.9) holds with $G = \mathcal{D}_r$, i.e.,

$$\forall K \subset\subset \mathbb{C}^d \ \ \exists r : \quad \sum_{n=0}^\infty |c_n||\varphi_n|_K \leq \sup_{D_r} |f(z)|. \tag{3.10}$$

Recently, a partial case of this situation have been considered by P. Lassère and E. Mazzilli [**21**]. For a fixed continuum $K \subset \mathbb{C}$, they studied the Bohr phenomenon for Faber polynomial basis $(F_{K,n})_{n=0}^\infty$ in $H(\mathbb{C})$ (which is a common basis for $H(\mathcal{D}_r)$, where $\mathcal{D}_r$ are the sub level domains of the Green function of the complement of K with pole at infinity). They established Bohr's phenomenon in the form (3.10), i.e., there exists $r > 0$ such that, for every bounded function $f \in H(\mathcal{D}_r)$ with $f = \sum c_n F_{K,n}$,

$$\sum_{n=0}^{\infty} |c_n||F_{K,n}|_K \le \sup_{D_r} |f(z)|$$

holds. P. Lassère and E. Mazzilli called the infimum R_0 of such r *Bohr radius* associated to the family $(K, \mathcal{D}_r, (F_{K,n})_{n=0}^{\infty})$ (see also [**22**]).

The definition of P. Lassère and E. Mazzilli could be extended to the general situation mentioned above. Namely, if $(\varphi_n)_{n=0}^{\infty}$, $\varphi_0 \equiv 1$, is a basis in the space of entire functions $H(\mathbb{C}^d)$ such that it is a common basis for the spaces $H(\mathcal{D}_r)$, where $(\mathcal{D}_r)$ is an exhaustive family of domains in $\mathbb{C}^d$, then by Theorem 3 and Remark 4 it follows that (3.10) holds. So, following P. Lassère and E. Mazzilli, we may introduce *Bohr radius* associated to the family $(K, \mathcal{D}_r, (\varphi_n)_{n=0}^{\infty})$, where K is a fixed compact set, as the infimum R_0 of all r such that the inequality in (3.10) holds.

4. *Stein manifolds with Liouville Property*

In this section we generalize Theorem 3 to Stein manifolds $\mathcal{M}$ without nontrivial global bounded analytic functions.

DEFINITION 5. *We say that a Stein manifold $\mathcal{M}$ has Liouville Property if every bounded analytic function on $\mathcal{M}$ is a constant.*

Next we introduce another property that is crucial for our considerations.

DEFINITION 6. *We say that a Stein manifold $\mathcal{M}$ has Schwarz Property if for every $z_0 \in \mathcal{M}$, compact $K \ni z_0$ and $\delta > 0$ there is a compact K_1 with*

$$|f - f(z_0)|_K \le \delta\, |f - f(z_0)|_{K_1} \quad \forall f \in H(K_1), \tag{4.1}$$

where $H(K_1)$ denotes the set of functions analytic in a neighborhood of K_1.

It turns out that Schwarz Property and Liouville Property are equivalent. In order to prove this fact we introduce yet another property.

Let $\mathcal{M}$ be a Stein manifold. Fix a point $z_0 \in \mathcal{M}$, and choose an exhaustion $(D_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$ consisting of open relatively compact sets so that

$$z_0 \in D_1, \quad \overline{D_n} \subset D_{n+1} \quad \forall n \in \mathbb{N}. \tag{4.2}$$

For each n, consider the family of functions

$$\mathcal{F}_n = \{f \in H(D_n) :\ f(z_0) = 0,\ \sup_{D_n} |f(z)| \le 1\}, \tag{4.3}$$

and set

$$\gamma_n(z) = \sup\{|f(z)| :\ f \in \mathcal{F}_n\}, \quad z \in D_n, \quad n \in \mathbb{N}. \tag{4.4}$$

The functions $\gamma_n(z)$ are continuous. Indeed, fix $n \in \mathbb{N}$, $w \in D_n$ and $\varepsilon > 0$. The family of functions $\mathcal{F}_n$ is compact, and therefore, equicontinuous on D_n. Therefore, there is $\delta > 0$ such that if $z \in D_n$ and $dist(z, w) < \delta$ then

$$|\,|f(z)| - |f(w)|\,| \le |f(z) - f(w)| < \varepsilon \quad \forall f \in \mathcal{F}_n.$$

Thus we obtain, for every fixed $z \in D_n$ with $dist(z, w) < \delta$,

$$|f(z)| \leq |f(w)| + \varepsilon \quad \forall f \in \mathcal{F}_n \Rightarrow |f(z)| \leq \gamma_n(w) + \varepsilon \quad \forall f \in \mathcal{F}_n,$$

which implies that

$$\gamma_n(z) \leq \gamma_n(w) + \varepsilon.$$

On the other hand, we have also that if $dist(z, w) < \delta$ then

$$|f(w)| \leq |f(z)| + \varepsilon \quad \forall f \in \mathcal{F}_n \Rightarrow |f(w)| \leq \gamma_n(z) + \varepsilon \quad \forall f \in \mathcal{F}_n,$$

which implies that

$$\gamma_n(w) \leq \gamma_n(z) + \varepsilon.$$

Hence we obtain that $|\gamma_n(z) - \gamma_n(w)| < \varepsilon$ if $z \in D_n$ and $dist(z, w) < \delta$, i.e., γ_n is continuous at w.

Let us note that

$$\gamma_n(z) \geq \gamma_{n+1}(z) \quad \forall z \in D_n, \ \ n \in \mathbb{N}. \tag{4.5}$$

THEOREM 7. *Let $\mathcal{M}$ be a Stein manifold. The following conditions on $\mathcal{M}$ are equivalent:*
(i) $\mathcal{M}$ has Liouville Property;
(ii) $\mathcal{M}$ has Schwarz Property;
(iii) For every exhaustion (D_n) of $\mathcal{M}$ of the kind (4.2) the corresponding functions (4.4) satisfy $\lim \gamma_n(z) = 0$.

Proof. First we show that (i) implies (iii). Assume the contrary, that (i) holds but (iii) fails. Then there is an exhaustion (D_n) of $\mathcal{M}$ of the kind (4.2) and a point $w \in \mathcal{M}$ such that the corresponding functions (4.4) satisfy $\lim \gamma_n(w) = c > 0$, so $\gamma_n(w) > c/2$ for large enough n, say $n \geq n_0$. In view of the definition of $\gamma_n(w)$ (see 4.3) and (4.4)), there is a sequence of functions $(f_n)_{n \geq n_0}$ such that $f_n \in \mathcal{F}_n$ and $|f_n(w)| > c/2$. For every fixed k the sequence (f_n) is uniformly bounded by 1 on $\overline{D}_k$ for $n > k$. Now the Montel property and a standard diagonal argument show that there is a subsequence f_{n_ν} such that $|f_{n_\nu}(z)| \leq 1$ for $z \in D_{n_\nu}$, $f_{n_\nu}(z_0) = 0, \ f_{n_\nu}(w) > c/2$, and a tail of f_{n_ν} converges uniformly on every compact subset of $\mathcal{M}$. The limit function $f(z) = \lim f_{n_\nu}(z)$ will be bounded by 1 and $f(z_0) = 0, \ |f(w)| > c/2 > 0$. The existence of such a function contradicts (i). Hence $(i) \Rightarrow (iii)$.

Next we prove that $(iii) \Rightarrow (ii)$. Indeed, fix a compact K and choose $m \in \mathbb{N}$ so that $D_m \supset K$. By Dini's theorem, $(\gamma_n(z))_{n>m}$ converges uniformly to 0 on $\overline{D}_m$. Therefore, for every $\delta > 0$ there is $n_1 > m$ such that $\gamma_{n_1}(z) < \delta$ for $z \in D_m$. But this means that (4.1) holds with $K_1 = \overline{D}_{n_1}$.

Finally, we show that (ii) implies (i). Let f be a bounded analytic function on $\mathcal{M}$, say $|f(z)| \leq C$. Fix $z \in \mathcal{M}, \ z \neq z_0$. By Schwartz Property, it follows that for every $\delta > 0$ we have $|f(z) - f(z_0)| < 2C\,\delta$, so $f(z) = f(z_0)$. Hence $f = const$, which proves that (i) holds. □

Next we consider a generalization of the classical Borel-Carathéodory theorem.

THEOREM 8. *Let $\mathcal{M}$ be a Stein manifold with Liouville Property. Then for every $z_0 \in \mathcal{M}$, compact $K \ni z_0$ and $\varepsilon > 0$ there is a compact K_1 with*

$$|f - f(z_0)|_K \leq \varepsilon \sup_{K_1} Re\,(f(z) - f(z_0)) \quad \forall f \in H(K_1). \tag{4.6}$$

Proof. Fix $z_0 \in \mathcal{M}$, compact $K \ni z_0$ and $\varepsilon > 0$. Let

$$\delta = \varepsilon/(2 + \varepsilon),$$

and let $K_1 \subset \mathcal{M}$ be a compact such that (4.1) holds.

Fix a non-constant function $f \in H(K_1)$ such that $f(z_0) = 0$, and set

$$A = \sup_{K_1} Re\, f(z). \tag{4.7}$$

Set

$$g(z) = f(z)/(2A - f(z));$$

then $g \in H(K_1),\ g(z_0) = 0$, and in view of (4.7) it follows that

$$|g(z)| \le 1 \quad \text{for} \quad z \in K_1.$$

Therefore, by (4.1),

$$|g(z)| = \left| \frac{f(z)}{2A - f(z)} \right| \le \delta \quad \text{if } \ z \in K,$$

so $|f(z)| \le \delta |2A - f(z)| \le 2\delta A + \delta |f(z|$, and it follows that

$$|f(z)| \le \frac{2\delta}{1-\delta} A = \varepsilon A, \quad z \in K.$$

Hence, we obtain

$$|f(z)| \le \varepsilon \sup_{K_1} Re\,(f(z)) \quad \forall z \in K,$$

which completes the proof. □

THEOREM 9. *Let $\mathcal{M}$ be a Stein manifold with Liouville Property. Suppose $(\varphi_n)_{n=0}^\infty$ is a basis in $H(\mathcal{M})$ such that*

$$\varphi_0(z) \equiv 1.$$

Then for every compact $K \subset \mathcal{M}$ there is a compact K_1 such that

$$\textit{if } \ f = \sum_0^\infty f_n \varphi_n, \ \ \textit{then } \ \sum_0^\infty |f_n||\varphi_n|_K \ \le \ |f|_{K_1} \quad \forall f \in H(\mathcal{M}). \tag{4.8}$$

Proof. Fix a compact K and a point $z_0 \in K$. The system

$$\psi_0(z) \equiv 1, \quad \psi_n(z) = \varphi_n(z) - \varphi_n(z_0), \quad n \in \mathbb{N}. \tag{4.9}$$

is also a basis in $H(\mathcal{M})$. Moreover, if $f = f_0 + \sum_{n=1}^\infty f_n \varphi_n$, then we have $f(z_0) = f_0 + \sum_{n=1}^\infty f_n \varphi_n(z_0)$, so it follows that

$$f = f(z_0) + \sum_1^\infty f_n \psi_n.$$

First we are going to show that there is a compact K_1 such that

$$\text{if } \ f = f(z_0) + \sum_1^\infty f_n \psi_n, \ \ \text{then } \ |f(z_0)| + \sum_1^\infty |f_n||\psi_n|_K \ \le \ |f|_{K_1} \quad \forall f \in H(\mathcal{M}). \tag{4.10}$$

Fix a function $f \in H(\mathcal{M})$. We may assume without loss of generality that $f(z_0) \ge 0$ (otherwise one may multiply f by $|f(z_0)|/f(z_0)$). By Proposition 1, there are $C \ge 1$ and a compact $\tilde{K} \supset K$ (which do not depend on f) such that

$$\sum_1^\infty |f_n||\psi_n|_K \ \le \ C|f - f(z_0)|_{\tilde{K}}.$$

By Theorem 8, with $\varepsilon = 1/C$, there is a compact $K_1 \supset \tilde{K}$ such that (4.6) holds with $K = \tilde{K}$. Since $\sup_{K_1} Re\,(f(z) - f(z_0)) \le |f|_{K_1} - f(z_0)$, we obtain

$$f(z_0) + \sum_1^\infty |f_n||\psi_n|_K \ \le\ f(z_0) + C\varepsilon\,(|f|_{K_1} - f(z_0)) = |f|_{K_1},$$

i.e., (4.10) holds.

Lemma 2 can be readily generalized to the case of Stein manifolds with Liouville Property, so the same argument as in the proof of Theorem 3 shows that (4.10) implies (4.8). This completes the proof. □

Corollary 10. *Let $\mathcal{M}$ be a Stein manifold with Liouville Property. A basis $(\varphi_n(z))_{n=0}^\infty$ in $H(\mathcal{M})$ has GBP if and only if one of the functions φ_n is a constant.*

Indeed, the necessity part follows from Proposition 3.1 in [**4**], or as in Theorem 3.

Obviously, the class of manifolds with Liouville Property include $\mathbb{C}^d,\ d = 1, 2, \ldots$. In order to give more examples let us recall that a complex manifold is called *parabolic* if every bounded from above plurisubharmonic function reduces to a constant. In view of the fact that the modulus of analytic functions are plurisubharmonic, parabolic manifolds possess Liouville Property. Affine algebraic manifolds, $\mathbb{C}^d \setminus Z(F)$ with $Z(F)$ being the set of zeros of an entire function F, parabolic Riemann surfaces [**5**], are parabolic. For more examples see [**7**]. However not every complex manifold with Liouville Property is parabolic – see [**7**] for a simple example.

Let us note that the class of parabolic manifolds is remarkable by the fact that the space of global analytic functions on a parabolic manifold admits a basis (see [**6**]). For a general complex manifold $\mathcal{M}$ it is an open question as to whether or not $H(\mathcal{M})$ possesses a basis.

References

1. L. Aizenberg, Multidimensional analogues of Bohr's theorem on power series, *Proc. Amer. Math. Soc.* 128 (2000), 1147–1155.

2. L. Aizenberg, Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series. *Selected topics in complex analysis, Oper. Theory Adv. Appl.* 158, Birkhäuser, Basel, (2005), 87–94.

3. L. Aizenberg, A. Aytuna and P. Djakov, An abstract approach to Bohr's phenomenon. *Proc. Amer. Math. Soc.* 128 (2000), no. 9, 2611–2619.

4. L. Aizenberg, A. Aytuna and P. Djakov, Generalization of a theorem of Bohr for bases in spaces of holomorphic functions of several complex variables. *J. Math. Anal. Appl.* 258 (2001), no. 2, 429–447.

5. L. Ahlfors and L. Sario, *Riemann surfaces*, Princeton University Press, Princeton, New Jersey, 1960.

6. A. Aytuna, J. Krone, and T. Terzioglu, Complemented infinite type power series subspaces of nuclear Fréchet spaces. *Math. Ann.* **283** (1989), 193–202.

7. A. Aytuna and A. Sadullaev, Parabolic Stein manifolds. Preprint arXiv:1112.1626v1 [math.CV] 2011.

8. C. Beneteau, A. Dahlner and D. Khavinson, Remarks on the Bohr phenomenon, *Comput. Methods Funct. Theory* **4** no.1 (2004), 1–19.

9. H. P. Boas and D. Khavinson, Bohr's power series theorem in several variables, *Proc. Amer. Math. Soc.* **125** (1997), 2975–2979.

10. H. Bohr, A theorem concerning power series, *Proc. London Math. Soc.* (2) **13** (1914), 1–5.

11. E. Bombieri and J. Bourgain, A remark on Bohr's inequality, *Int. Math. Res. Not.* **80** (2004), 4307–4330.

12. A. Defant, D. García and M. Maestre, Bohr's power series theorem and local Banach space theory. *J. Reine Angew. Math.* 557 (2003), 173–197.

13. A. Defant and C. Prengel, Christopher Harald Bohr meets Stefan *Banach. Methods in Banach space theory, 317–339, London Math. Soc. Lecture Note Ser.* 337, Cambridge Univ. Press, Cambridge, 2006.

14. A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes and K. Seip, The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive. *Annals of Mathematics* **174** (2011), Issue 1, 485–497.

15. S. Dineen and R. M. Timoney, Absolute bases, tensor products and a theorem of Bohr, *Studia Math.* 94 (1989), 227–234.

16. P. G. Dixon, Banach algebra satisfying the non-unital von Neumann inequality. *Bull. London Math. Soc.* 27 1995, 359–362.

17. P. Djakov and M. Ramanujan, A remark on Bohr's theorem and its generalizations. *J. Anal.* 8 (2000), 65–77.
18. A. Dynin and B. Mityagin, Criterion for nuclearity in terms of approximative dimension, *Bull. Pol. Acad. Sci., Math.*, 8 (1960), 535–540.
19. G. Roos, H. Bohr's theorem for bounded symmetric domains, arXiv:0812.4815.
20. H. T. Kaptanoglu and N. Sadik, Bohr Radii of elliptic functions. *Russian Journal of Mathematical Physics*, 12-3 (2005), 365–368.
21. P. Lassère and E. Mazzilli, Bohr's phenomenon on a regular condensator in the complex plane, *Comput. Methods Funct. Theory* 12, no.1 (2012), 31–43.
22. P. Lassere and E. Mazzilli, The Bohr Radius for an Elliptic Condenser, arXiv:1109.4511.
23. B. S. Mityagin, Approximative dimension and bases in nuclear spaces, *Uspekhi Mat. Nauk* 16, (1961), 63–132, (Russian), English Transl.-*Russian Math. Surveys* 16 (1961), 59–127.
24. R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.
25. V. Paulsen, G. Popescu and D. Singh, On Bohr's inequality, *Proc. London Math. Soc.* 85 (2002), no.2, 493–512.
26. E. C. Titchmarsh, *The theory of functions*, Oxford University Press, 1939.

A. Aytuna and P. Djakov
Sabanci University, Orhanli, 34956 Tuzla,
Istanbul, Turkey

aytuna@sabanciuniv.edu
djakov@sabanciuniv.edu