An exploration in school formation: Income vs. Ability

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ABSTRACT

We study stable school formation among four students that differ in ability and income. In the presence of ability complementarities and school costs to be shared, we identify the conditions under which a stable allocation is efficient, inefficient, nonexistent, and tell who become peers.

1. Introduction

We consider a *school formation* game among four students: a high-ability high-income, a high-ability low-income, a low-ability high-income, and a low-ability low-income student. A school consists of two students, and any two students can form a school provided they agree on how to share the cost. A high-ability *peer* enhances one's educational achievement. Under what conditions is there *stable* school formation? Who become peers? In this note we give an exact description that answers these questions.

There are three possible formations in this four-student game, one where peers are the same in ability, another where they are the same in income, and a third where they are opposite in both attributes. These we term the *ability assortative, income assortative*, and *cross assortative* formations respectively. We show that each of these formations may occur as a stable outcome, and that a stable outcome need not exist, depending on the direction and magnitude of *peer effects* measured against income levels and school cost. Interestingly, the ability assortative formation dominates the picture, occurring whenever peer effects are *supermodular*, and also when they are *submodular* but relatively high in magnitude. Stable ability assortative outcomes are efficient if and only if peer effects are supermodular. Income assortative and cross assortative stable formations do but coexist, exactly when peer effects are submodular and the associated outcomes are efficient. Nonexistence of stable outcomes is conditional on the size of the income gap between the rich and the poor.

Our study is an exercise in *stable matching* theory[1] in the less treated one-sided market context and with a fairly general hybrid transferable/nontransferable utility feature. In our model, each pairwise utility possibility set has unit slope at its efficient frontier, but the joint-minimum base of the efficient frontier is not at the origin and not uniform: It varies with peers' abilities. What we call peer effects is precisely a differential of this variation.

In two-sided matching, a special hybrid utility model that unites the stable "marriage"[2] and "assignment"[3] games was introduced by Sotomayor (2000) and Eriksson and Karlander (2000). The more general model of Fujishige and Tamura (2007) allows for the peer effects our study features. In one-sided matching, on the other hand, the transferable-utility "assignment" game has been considered only recently (Talman and Yang, 2011) although

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1 See the classical reference Roth and Sotomayor (1990).
2 Gale and Shapley (1962).
3 Shapley and Shubik (1972).

the nontransferable-utility "marriage/roommate" game has been studied extensively for long (e.g., Gusfield and Irving, 1989). Pycia (2012) has studied stable matching in a general one-sided environment, allowing for peer effects under unrestricted coalition size, but under specific division rules, e.g., the Nash bargaining or the equal sharing rule.

A particular aim we carry is to understand assortativeness: In two-sided one-to-one matching, when agents are described by a one-dimensional attribute, positive (negative) assortativeness of stable outcomes follows from supermodularity (submodularity) of household production functions. There has been a search for extending this celebrated result (Becker, 1973) and finding conditions that give rise to assortativeness (e.g., Eeckhout, 2000, Clark, 2006, Legros and Newman, 2007, 2010), but not explicitly in the form of partially ordered attributes as in our exercise.

In a general equilibrium framework with fee-setting schools and school-selecting students on two sides of a market, the *school competition* literature[4] treats a student population that is partially ordered in income and ability. Utility of a student – a function of own-income, own-ability and peer-ability – is generally assumed to have the positive *single-crossing-in-income* property (amounting to a positive income elasticity of demand for peer quality which in our model is zero). Under this assumption, equilibrium outcomes exhibit *stratification* in income. This literature typically assumes, in addition, that utility functions have the positive *single-crossing-in-ability* property, which is identical to peer effect *supermodularity* in our model. Then, outcomes exhibit stratification in ability as well. On the other hand, the case with the negative single-crossing-in-ability property (*submodularity* in our model) has remained untreated.

Empirical evidence on the nature of peer effects, in fact, is mixed and scarce[5] and need for further research is documented in several studies.[6] Why the "submodular" peer effects case has been neglected may have to do with the analytical difficulties this case poses relative to the "supermodular" case but is not altogether justified. Indeed, it is in this region that we find the relatively more interesting occurrences in our exercise, e.g., possible inefficiency or nonexistence of stable outcomes. Extending our query to more general student populations could contribute to a better understanding of the forces behind school or partnership formation and assortativeness or stratification.

2. Model and result

A student $s \in S$ is characterized by two endowments ($y(s)$, $b(s)$). One of these is *income*, $y(s) \in \{y_H, y_L\}$. It costs $c > 0$ to form a school. Any two students $s, s' \in S$ can form a school by making nonnegative *contributions* $p(s), p(s')$ from their incomes that satisfy $p(s) + p(s') = c$. We assume

$$c \geq y_H > y_L \geq c/2.^7$$

The other endowment is *ability*, $b(s) \in \{h, l\}$. We study the case $S = \{(H, h), (H, l), (L, h), (L, l)\}$ and call these students *Max, Rich, Abel, Minn* respectively.

Fundamental to our investigation are the complementarities in abilities: The *educational achievement* of a student s in school with

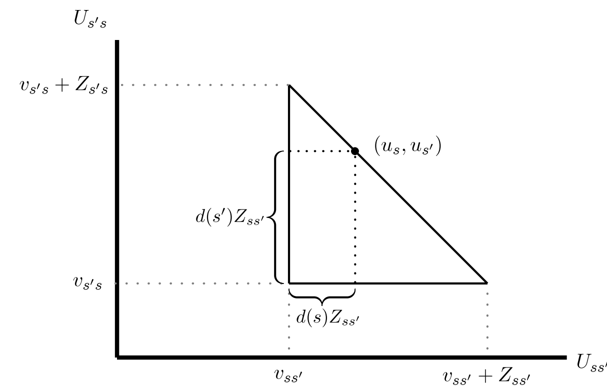

Fig. 1. Bargaining set for ss'.

peer s' depends on both their abilities $b(s), b(s')$. We assume this achievement is a positive constant and denote it $a_{b(s)b(s')}$. Utility of s in school with s' is simply the sum of educational achievement and residual income, i.e.,

$$a_{b(s)b(s')} + y(s) - p(s).$$

For short, let us denote

$$v_{ss'} = a_{b(s)b(s')}.$$

Formation of a school by s and s' requires mutual agreement on their contributions $p(s), p(s')$ bounded by their incomes $y(s), y(s')$, equivalently, agreement on $d(s), d(s') \in [0, 1]$ (satisfying $d(s) + d(s') = 1$) for sharing the *surplus*

$$Z_{ss'} = Z_{s's} = y(s) + y(s') - c,$$

and realizing the utilities

$$u_s = v_{ss'} + d(s)Z_{ss'},$$
$$u_{s'} = v_{s's} + d(s')Z_{ss'}.$$

Fig. 1 illustrates a *bargaining* set and one possible utility realization. When seeking a partner to form a school, each student considers all possible utility realizations with every potential partner.

An *allocation* is a triplet $\{\mu, d, u\}$ where μ is a partition of S into two pairs, the shares $d(s), d(s') \in [0, 1]$ satisfy $d(s) + d(s') = 1$ for each $ss' \in \mu$, and $u_s = v_{ss'} + d(s)Z_{ss'}$ is the utility of s with *peer* s'. We denote the peer s' by $\mu(s)$.

An allocation $\{\mu, d, u\}$ is *blocked* by a pair $ss' \notin \mu$ if there is a $\lambda \in [0, 1]$ such that

$$v_{ss'} + \lambda Z_{ss'} > v_{s\mu(s)} + d(s)Z_{s\mu(s)} = u_s,$$
$$v_{s's} + (1 - \lambda)Z_{ss'} > v_{s'\mu(s')} + d(s')Z_{s'\mu(s')} = u_{s'}.$$

An allocation $\{\mu, d, u\}$ is *individually rational* if $u_s \geq y(s)$ for every s, i.e., no student prefers *standing alone*.

Definition. An allocation $\{\mu, d, u\}$ is *stable* if it is individually rational and not blocked by any pair.

We call

$$\alpha_h = a_{hh} - a_{hl},$$
$$\alpha_l = a_{lh} - a_{ll},$$

the *peer effect* for a high-ability and low-ability student respectively, assume

$$\alpha_h \geq 0, \qquad \alpha_l \geq 0,$$

and say peer effects are *supermodular* if $\alpha_h \geq \alpha_l$ and *submodular* if $\alpha_h < \alpha_l$. We shall restrict our attention to stable allocations where no student stands alone and to this end assume

$$a_{hl} \geq c/2, \qquad a_{ll} \geq y_L.$$

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4 E.g., Epple and Romano (1998); Epple et al.(2003,2006); Hanushek et al. (2011); Sarpça (2010).

5 Summers and Wolfe (1977) find some support for submodularity of peer effects. The findings of Ding and Lehrer (2007) support supermodularity. Henderson et al. (1978) and Hanushek et al. (2003) find an even effect.

6 E.g., Dale and Krueger (2002), Hoxby and Weingarth (2005), Sarpça (2010).

7 It is not difficult to see that a model with $y_H > c$ is equivalent to a model with $y_H = c$. We exclude treating the case $y_L < \frac{c}{2}$ for expositional simplicity. Results are qualitatively similar.

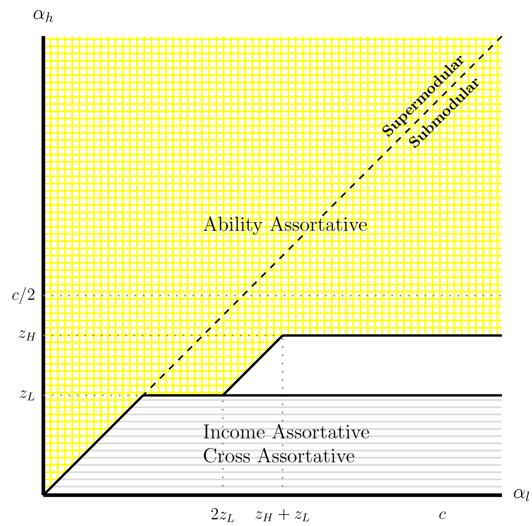

Fig. 2. Regions of existence of stable allocations.

We call two allocations $\{\mu, d, u\}$ and $\{\mu', d', u'\}$ *equivalent* if $u = u'$. We say an allocation is *ability assortative* if Max Abel are in one school and Rich Minn in the other, *income assortative* if the partition is Max Rich and Abel Minn, *cross assortative* if it is Max Minn and Rich Abel.

We shall denote

$$z_H = y_H - c/2, \qquad z_L = y_L - c/2.$$

Our result is:

Theorem. *There exists a stable* ability assortative *allocation iff*

$$\alpha_h \geq \min\{\alpha_l, z_H, \max\{z_L, \alpha_l - z_L\}\},$$

a stable cross assortative *allocation, equivalent to a stable* income assortative *allocation, iff*

$$\alpha_h \leq \min\{\alpha_l, z_L\},$$

no *stable allocation iff*

$$z_L < \alpha_h < \min\{z_H, \alpha_l - z_L\}.$$

Proof. See Appendix. $\square$

We display our result in Fig. 2: The axes stand for the peer effects α_l and α_h; the diagonal $\alpha_l = \alpha_h$ divides the supermodular and submodular regions.

We make the following observations:

(i) A stable allocation exists, except where peer effects are submodular and $z_L < \alpha_h < z_H$, a region that shrinks as income gap $y_H - y_L = z_H - z_L$ narrows.
(ii) When peer effects are supermodular, a stable allocation is ability assortative and efficient.
(iii) When peer effects are submodular and $\alpha_h \geq z_L$, a stable allocation, provided it exists, is ability assortative and inefficient.
(iv) When peer effects are submodular and $\alpha_h \leq z_L$, a stable allocation is equivalently income or cross assortative and efficient.

Appendix. Proof of theorem

We use the following lemma repeatedly in the proof of Theorem, which is given in the five lemmas that follow.

Denote the maximum utility s can realize in school with s' by $w_{ss'}$ $(= v_{ss'} + Z_{ss'})$ and the maximum total utility of ss' together in school by $W_{ss'}$ $(= W_{s's} = v_{ss'} + v_{s's} + Z_{ss'})$.

Blocking Lemma. A pair $ss' \notin \mu$ blocks an allocation $\{\mu, d, u\}$ if and only if (i) $u_s < w_{ss'}$, (ii) $u_{s'} < w_{s's}$, (iii) $u_s + u_{s'} < W_{ss'}$.

Proof. Suppose (i)–(iii) hold for a pair ss'. Let $\Delta(\lambda) = v_{ss'} + \lambda Z_{ss'} - u_s$ and $\Delta'(\lambda) = v_{s's} + (1-\lambda)Z_{ss'} - u_{s'}$. From (i) and (ii) respectively $\Delta(1) = w_{ss'} - u_s > 0$ and $\Delta'(0) = w_{s's} - u_{s'} > 0$. If either $\Delta'(1) \geq \Delta(1)$ or $\Delta(0) \geq \Delta'(0)$, then s, s' blocks $\{\mu, d, u\}$. If on the other hand, $\Delta'(1) < \Delta(1)$ and $\Delta(0) < \Delta'(0)$, since $\Delta(\lambda)$ is increasing, $\Delta'(\lambda)$ decreasing in λ, there exists a $\lambda^* \in (0, 1)$ such that $\Delta(\lambda^*) = \Delta'(\lambda^*)$ and $\Delta(\lambda^*) = \Delta'(\lambda^*) = (W_{ss'} - u_s - u_{s'})/2 > 0$ from (iii), so that again ss' blocks $\{\mu, d, u\}$. This proves the if part. The only if part is straightforward. $\square$

Given an allocation $\{\mu, d, u\}$ and a pair $ss' \notin \mu$, we will refer to the inequalities (i) and (ii) in the Blocking Lemma as the *constraint* of s and s' respectively and to the inequality (iii) as their *joint constraint*. We will say that s is *open* (*closed*) to s' if $u_s < w_{ss'}$ (if $u_s \geq w_{ss'}$) and that ss' can *negotiate* if $u_s + u_{s'} < W_{ss'}$.

Note that

$$Z_{\text{MaxAbel}} = Z_{\text{MaxMinn}} = Z_{\text{RichAbel}} = Z_{\text{RichMinn}} = z_H + z_L,$$
$$Z_{\text{MaxRich}} = 2z_H, \qquad Z_{\text{AbelMinn}} = 2z_L.$$

Ability assortative allocations

We let $AA(d_M, d_R)$ denote the ability assortative allocation under which the shares of Max and Rich are d_M and d_R (and so the shares of Abel and Minn are $1-d_M$ and $1-d_R$) respectively. We will make use of the following "blocking conditions" which are direct applications of the Blocking Lemma:

Blocking conditions

Max Rich block $AA(d_M, d_R)$ iff (i) $d_M(z_H + z_L) < 2z_H - \alpha_h$, (ii) $d_R(z_H + z_L) < 2z_H + \alpha_l$, (iii) $d_M(z_H + z_L) + d_R(z_H + z_L) < 2z_H - (\alpha_h - \alpha_l)$. In this case, *the constraint of Rich (ii) is vacuous.* In other words, Rich is open to Max for all $d_R \in [0, 1]$.

Max Minn block $AA(d_M, d_R)$ iff (i) $d_M(z_H + z_L) < z_H + z_L - \alpha_h$, (ii) $-\alpha_l < d_R(z_H + z_L)$ (iii) $\alpha_h - \alpha_l < d_R(z_H + z_L) - d_M(z_H + z_L)$. *The constraint of Minn* (ii) *is vacuous.*

Abel Minn block $AA(d_M, d_R)$ iff (i) $z_H - z_L + \alpha_h < d_M(z_H + z_L)$, (ii) $z_H - z_L - \alpha_l < d_R(z_H + z_L)$, (iii) $2z_H + \alpha_h - \alpha_l < d_M(z_H + z_L) + d_R(z_H + z_L)$. *The constraint of Minn (ii) is implied by the joint constraint* $d_R(z_H + z_L) > \alpha_h - \alpha_l + 2z_H - d_M(z_H + z_L) \geq \alpha_h - \alpha_l + z_H - z_L > -\alpha_l + z_H - z_L$. In other words, Minn is open to Abel at all $AA(d_M, d_R)$ where Abel Minn can negotiate.

Rich Abel block $AA(d_M, d_R)$ iff (i) $d_R(z_H + z_L) < z_H + z_L + \alpha_l$, (ii) $\alpha_h < d_M(z_H + z_L)$, (iii) $\alpha_h - \alpha_l < d_M(z_H + z_L) - d_R(z_H + z_L)$. *The constraint of Rich (i) is vacuous.*

Lemma 1. *There is a stable ability assortative allocation when $\alpha_h \geq \alpha_l$ or $\alpha_h \geq \min\{z_H, \max\{z_L, \alpha_l - z_L\}\}$.*

Proof. When $\alpha_h \geq \alpha_l$, the ability assortative allocation $AA(d_M, d_R)$ where

$$d_M = d_R = z_H/(z_H + z_L)$$

cannot be blocked because no pair can negotiate. The utilities under this allocation are

$$u_{\text{Max}} = a_{hh} + z_H, \qquad u_{\text{Abel}} = a_{hh} + z_L,$$
$$u_{\text{Rich}} = a_{ll} + z_H, \qquad u_{\text{Minn}} = a_{ll} + z_L.$$

This allocation is individually rational if

$$a_{hh} \geq c/2, \qquad a_{ll} \geq c/2.$$

(Moreover, this allocation is the only stable ability assortative allocation when $(\alpha_h, \alpha_l) = (0, 0)$, so these bounds are necessary for individual rationality.)

When $z_H \le \alpha_h \le \alpha_l$ (*Case* 1) or $\max\{z_L, \alpha_l - z_L\} \le \alpha_h \le \min\{\alpha_l, z_H\}$ (*Case* 2), $AA(d_M, d_R)$ where

$$d_M = z_H/(z_H + z_L), \qquad d_R = 1$$

is not blocked: In *Case* 1, Maxx and Abel are each closed to both Rich and Minn. In *Case* 2, neither Max Rich nor Rich Abel can negotiate, and Max and Abel are each closed to Minn. The utilities under this allocation are

$$u_{\text{Max}} = a_{hh} + z_H, \qquad u_{\text{Abel}} = a_{hh} + z_L,$$
$$u_{\text{Rich}} = a_{ll} + z_H + z_L, \qquad u_{\text{Minn}} = a_{ll}.$$

This allocation is individually rational if

$$a_{hh} \ge c/2, \qquad a_{ll} \ge y_L.$$

(Moreover, this allocation is the only stable ability assortative allocation when $(\alpha_h, \alpha_l) = (z_H, z_H + z_L)$, so these bounds are necessary for individual rationality.) □

Lemma 2. *There is no stable ability assortative allocation when* $\alpha_h < \min\{\alpha_l, z_L\}$.

Proof. Suppose $AA(d_M, d_R)$ is stable.

If Rich Abel cannot negotiate, i.e., $d_M(z_H + z_L) \le d_R(z_H + z_L) + \alpha_h - \alpha_l$, then Max Minn can negotiate since $d_R(z_H + z_L) \ge d_M(z_H + z_L) - (\alpha_h - \alpha_l) > d_M(z_H + z_L) + (\alpha_h - \alpha_l)$ (because $\alpha_h < \alpha_l$). So Max must be closed to Minn, i.e., $d_M(z_H + z_L) \ge z_H + z_L - \alpha_h$. But then Abel is open to Minn because $d_M(z_H + z_L) \ge z_H + z_L - \alpha_h > z_H - z_L + \alpha_h$, and Abel Minn can negotiate since $d_M(z_H + z_L) + d_R(z_H + z_L) > 2d_M(z_H + z_L) + \alpha_h - \alpha_l \ge 2(z_H + z_L - \alpha_h) + \alpha_h - \alpha_l = 2z_H + 2(z_L - \alpha_h) + \alpha_h - \alpha_l > 2z_H + \alpha_h - \alpha_l$ (because $\alpha_h < z_L$). So Abel Minn block $AA(d_M, d_R)$. Contradiction.

If Rich Abel can negotiate, i.e., $d_M(z_H + z_L) > d_R(z_H + z_L) + \alpha_h - \alpha_l$, then Abel must be closed to Rich, i.e., $d_M(z_H + z_L) \le \alpha_h$. But then Max is open to Rich since $d_M(z_H + z_L) \le \alpha_h < z_H + z_L - \alpha_h$ (because $\alpha_h < z_L < z_H$), and Max Rich can negotiate since $d_M(z_H + z_L) + d_R(z_H + z_L) < 2d_M(z_H + z_L) - (\alpha_h - \alpha_l) \le 2\alpha_h - (\alpha_h - \alpha_l) < 2z_H - (\alpha_h - \alpha_l)$. So Max Rich block $AA(d_M, d_R)$. Contradiction. □

Lemma 3. *There is no stable ability assortative allocation when* $\alpha_h < \min\{z_H, \alpha_l - z_L\}$.

Proof. Suppose $AA(d_M, d_R)$ is stable.

Note $\alpha_h < \alpha_l - z_L$ is equivalent to $z_H - (\alpha_h - \alpha_l) > z_H + z_L$.

If Max Rich cannot negotiate, i.e., $2z_H - (\alpha_h - \alpha_l) \le d_M(z_H + z_L) + d_R(z_H + z_L)$ then Abel is open to Rich because $d_M(z_H + z_L) \ge 2z_H - (\alpha_h - \alpha_l) - d_R(z_H + z_L) = z_H + (z_H - (\alpha_h - \alpha_l)) - d_R(z_H + z_L) > z_H + ((z_H + z_L) - d_R(z_H + z_L)) \ge z_H > \alpha_h$. Moreover Rich Abel can negotiate since $d_M(z_H + z_L) \ge 2z_H - (\alpha_h - \alpha_l) - d_R(z_H + z_L) = \alpha_h - \alpha_l + d_R(z_H + z_L) + (2z_H - 2(\alpha_h - \alpha_l)) - 2d_R(z_H + z_L) > \alpha_h - \alpha_l + d_R(z_H + z_L) + 2((z_H + z_L) - d_R(z_H + z_L)) > \alpha_h - \alpha_l + d_R(z_H + z_L)$. So Rich Abel block $AA(d_M, d_R)$. Contradiction.

If Max Rich can negotiate, i.e., $d_M(z_H + z_L) + d_R(z_H + z_L) < 2z_H - (\alpha_h - \alpha_l)$, then Max must be closed to Rich, i.e., $d_M(z_H + z_L) \ge 2z_H - \alpha_h$. But then Abel is open to Rich since $d_M(z_H + z_L) \ge 2z_H - \alpha_h > \alpha_h$ (because $\alpha_h < z_H$). Moreover, $2z_H - \alpha_h \le d_M(z_H + z_L) < 2z_H - (\alpha_h - \alpha_l) - d_R(z_H + z_L)$ so $d_R(z_H + z_L) < \alpha_l$ so $(\alpha_h - \alpha_l) + d_R(z_H + z_L) < \alpha_h < d_M(z_H + z_L)$ thus Rich Abel can negotiate and block $AA(d_M, d_R)$. Contradiction. □

Cross assortative and income assortative allocations

Denote by $CA(d_M, d_R)$ the cross assortative allocation under which the shares of Max and Rich are d_M and d_R (and so the shares of Minn and Abel are $1 - d_M$ and $1 - d_R$) respectively. From Blocking Lemma:

Blocking conditions

Max Rich block $CA(d_M, d_R)$ iff (i) $d_M(z_H + z_L) < 2z_H$, (ii) $d_R(z_H + z_L) < 2z_H$, (iii) $d_M(z_H + z_L) + d_R(z_H + z_L) < 2z_H$. The (first two) individual constraints are implied by the (third) joint constraint.

Abel Minn block $CA(d_M, d_R)$ iff (i) $2z_H < d_R(z_H + z_L)$, (ii) $2z_H < d_M(z_H + z_L)$, (iii) $2z_H < d_R(z_H + z_L) + d_M(z_H + z_L)$. Again the individual constraints are implied by the joint constraint.

Rich Minn block $CA(d_M, d_R)$ iff (i) $d_R(z_H + z_L) < z_H + z_L - \alpha_l$, (ii) $\alpha_l < d_M(z_H + z_L)$, (iii) $2\alpha_l < d_M(z_H + z_L) - d_R(z_H + z_L)$. Again the individual constraints are implied by the joint constraint.

Max Abel block $CA(d_M, d_R)$ iff (i) $d_M(z_H + z_L) < \alpha_h + z_H + z_L$, (ii) $-d_R(z_H + z_L) < \alpha_h$, (iii) $d_M(z_H + z_L) - d_R(z_H + z_L) < 2\alpha_h$. The individual constraints are vacuous.

Thus a cross assortative allocation $CA(d_M, d_R)$ is stable iff no pair can negotiate.

Lemma 4. *There exists a stable cross assortative allocation if and only if* $\alpha_h \le \min\{\alpha_l, z_L\}$.

Proof. If $\alpha_h \le \min\{\alpha_l, z_L\}$ then $CA(d_M, d_R)$ with

$$d_M = (z_H + \alpha_h)/(z_H + z_L), \qquad d_R = (z_H - \alpha_h)/(z_H + z_L)$$

is not blocked because no pair can negotiate. The utilities under this allocation are

$$u_{\text{Max}} = a_{hl} + z_H + \alpha_h, \qquad u_{\text{Rich}} = a_{lh} + z_H - \alpha_h,$$
$$u_{\text{Abel}} = a_{hl} + z_L + \alpha_h, \qquad u_{\text{Minn}} = a_{lh} + z_L - \alpha_h.$$

This allocation is individually rational if

$$a_{hl} \ge c/2, \qquad a_{lh} \ge y_L.$$

(Moreover, this allocation is the only stable cross assortative allocation when $(\alpha_h, \alpha_l) = (0, 0)$, so these bounds are necessary for individual rationality.)

In the other direction, from the blocking conditions above, a cross assortative allocation $CA(d_M, d_R)$ is blocked by neither Max Rich nor Abel Minn iff $d_M(z_H + z_L) + d_R(z_H + z_L) = 2z_H$, and it is blocked by neither Rich Minn nor Max Abel iff $2\alpha_h \le d_M(z_H + z_L) - d_R(z_H + z_L) \le 2\alpha_l$. Thus $CA(d_M, d_R)$ is blocked by no pair if and only if $\alpha_h \le \alpha_l$ and $\alpha_h \le d_M(z_H + z_L) - z_H \le z_L$. □

Lemma 5. *If* $\alpha_h \le \min\{\alpha_l, z_L\}$, *a stable cross assortative allocation is equivalent to a stable income assortative allocation.*

Proof. Take a stable $CA(d_M, d_R)$. Observe that Max's contribution $p(\text{Max}) = y_H - d_M(z_H + z_L)$ is equal to Abel's contribution $p(\text{Abel}) = y_L - (1 - d_R)(z_H + z_L)$ (because $2z_H = d_M(z_H + z_L) + d_R(z_H + z_L)$). Therefore there is an income assortative allocation equivalent to $CA(d_M, d_R)$. This income assortative allocation is stable: For otherwise it is blocked by either Max Abel or Rich Minn; but then $CA(d_M, d_R)$ is blocked by the same pair; contradiction. □

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