On nuclearity of Köthe spaces *

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Abstract

In this study we observe that the Köthe spaces $K^{l_p}(A)$ is nuclear when it is complementedly embedded in $K^{l_q}(B)$ for $1 \le p < q < \infty$ with $p < 2$ or $1 < q < p \le \infty$ with $p > 2$.

1. For a sequence $a = (a_i)$, $a_i > 0$, $i \in N$ we consider the weighted l_p-space as

$$l_p(a) := \{x = (\xi_i) : \|x\|_{l_p(a)} := \|(\xi_i\, a_i)\|_{l_p} < \infty\}$$

with $1 \le p < \infty$. Let $(a_{i,n})_{i,n\in\mathbb{N}}$ be a matrix of real numbers such that $0 < a_{i,n} \le a_{i,n+1}$, $i, n \in N$. The l_p-*Köthe space* $K^{l_p}(a_{i,n})$ is the space of all scalar sequences $x = (\xi_i)$ such that $(\xi_i a_{i,n}) \in l_p$ for each n, endowed with the topology of Fréchet space, determined by the canonical system of norms $\|x\|_{l_p((a_{i,n}))}$, $n \in N$.The notation $e = (e_i)_{i\in\mathbb{N}}, e_i := (\delta_{i,k})_{k\in\mathbb{N}}$, will be used for the canonical basis of $K^{l_p}(A)$, regardless of a matrix A.

It is known that, if $K^{l_p}(a_{i,n}) \simeq K^{l_q}(b_{i,n})$ with $p \ne q$, then $K^{l_p}(a_{i,n})$ is nuclear ([2], Proposition 4; see also, [3], Proposition 27.16). Here we extend this result (under some additional restriction to p and q) to the case when the first space is isomorphic to a complemented subspace of the second one.

2. First we prove the following

Lemma 1. *Let $1 \le p < q < \infty$ and $p < 2$. Suppose that $T : l_p(a) \to l_q(c)$, $S : l_q(c) \to l_p(b)$ are linear continuous operators such that $i := ST : l_p(a) \to l_p(b)$ is the identical embedding. Then*

$$\frac{b_n}{a_n} \le C\left(\frac{1}{n}\right)^r, \tag{1}$$

with $r = \frac{1}{p} - \frac{1}{s}$, $s := \min(2, q)$ and some constant $C > 0$.

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Proof. We can assume that $c_n \equiv 1$, otherwise we consider another pair of operators $\widetilde{S} = SD$ and $\widetilde{T} = D^{-1}T$, where $D : l_q \to l_q(c)$ is the diagonal isomorphism: $D((\xi_n)) := (\xi_n / c_n)$. First consider the case (i). Any linear continuous operator from l_q to l_p is compact ([1], v.I, Proposition 2.c.3), hence the operator S is compact, so the embedding $i = ST$ is compact. Therefore $\dfrac{b_n}{a_n} \to 0$ as $n \to \infty$ and without loss of generality, one can assume that the sequence $\left(\dfrac{b_n}{a_n}\right)$ is non-increasing. Then for every $n \in \mathbb{N}$ and each sequence (θ_i) with $\theta_i = \pm 1$, regarding that $STe_i = e_i$, we have

$$\begin{aligned} \frac{n^{1/p}\, b_n}{a_n} &\leq \left(\sum_{i=1}^{n} \left(\frac{b_i}{a_i} \right)^p \right)^{1/p} = \left\| S \left(\sum_{i=1}^{n} \frac{\theta_i\, Te_i}{a_i} \right) \right\|_{l_p(b)} \\ &\leq \|S\| \left\| \sum_{i=1}^{n} \frac{\theta_i\, Te_i}{a_i} \right\|_{l_q} . \end{aligned} \tag{2}$$

Since the space l_q is of the type $s := \min\{2, q\}$, there is a constant M such that for every n-tuple $(x_i)_{i=1}^n$ of elements from l_q the estimate

$$2^{-n} \left\| \sum_{\theta \in \Theta_n} \theta_i\ x_i \right\|_{l_q} \leq M \left(\sum_{i=1}^{n} \left(\|x_i\|_{l_q} \right)^s \right)^{1/s}$$

holds; here Θ_n is the set of all sequences $\theta = (\theta_i)_{i=1}^n$ with $\theta_i = \pm 1$ ([2]). Applying this to (2), we obtain, taking into account that $\|Te_i\|_{l_q} \leq \|T\|\ a_i$, that

$$\frac{n^{1/p}\, b_n}{a_n} \leq M\, \|S\| \left(\sum_{i=1}^{n} \left(\frac{\|Te_i\|_{l_q}}{a_i} \right)^s \right)^{1/s} \leq M\, \|S\|\, \|T\|\ n^{\frac{1}{s}}.$$

Thus (1) is proved with $C = M\, \|S\|\, \|T\|$. □

3. The next fact can be considered as a natural generalization of Proposition 4 from [2].

Theorem 2. *Suppose that $1 \leq p < q < \infty$ with $p < 2$. If $K^{l_p}(a_{in})$ is isomorphic to a complemented subspace of $K^{l_q}(b_{in})$, then $K^{l_p}(a_{in})$ is nuclear.*

Proof. Let $E := K^{l_p}(a_{in})$ and $F := K^{l_q}(b_{in})$ with the canonical systems of seminorms $|\cdot|_{l_p((a_{in}))}$ and $|\cdot|_{l_q((b_{in}))}$, respectively. Let $T : E \to F$ be an isomorphic embedding with the complemented image $T(E)$. If $P : F \to T(E)$ is a continuous projection then the operator $S = T^{-1}P : F \to E$ is the left inverse for T, that is, $ST = Id_E$.

Regarding the continuity of T and S, for each k, there exist $m = m(k)$, $n = n(k)$ such that $|Tx|_{l_q((b_{im}))} \leq C|x|_{l_p((a_{in}))}$ and $|Sy|_{l_p((a_{ik}))} \leq C|y|_{l_q((b_{im}))}$ with some constant $C = C(k) > 0$. Then the corresponding extensions of the operators T and S :

$$T_k : l_p((a_{in})) \to l_q((b_{im})), \ S_k : l_q((b_{im})) \to l_p((a_{ik}))$$

are continuous and their superposition S_kT_k is the identical embedding $i_k : l_p((a_{in})) \to l_p((a_{ik})),\ \ k \in \mathbb{N}$. Applying Lemma 1, we obtain that $\left(\frac{a_{ik}}{a_{in}}\right) \leq M\left(\frac{1}{n}\right)^{\frac{1}{p}-\frac{1}{s}}$ with $s = \min\{2, q\}$ and some constant $M = M(k)$. Hence $\left(\frac{a_{ik}}{a_{in}}\right) \in l_r,\ \ r > \frac{ps}{s-p}$, which implies nuclearity of the space $K^{l_p}(a_{nk})$ (see, e.g., [3, 2]). □

4. The following result can be derived from Theorem 2 by duality.

Theorem 3. *Suppose that $1 < q < p \leq \infty$ with $p > 2$. If $T : K^{l_q}(b_{in}) \to K^{l_p}(a_{in})$ is linear continuous operator onto and $kerT$ is complemented, then $K^{l_p}(a_{in})$ is nuclear.*

References

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