STOCHASTIC DISCOUNTING IN REPEATED GAMES: AWAITING THE ALMOST INEVITABLE

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Abstract

This thesis studies repeated games with pure strategies and stochastic discounting under perfect information. We consider infinite repetitions of any finite normal form game possessing at least one pure Nash action profile. We consider stochastic discounting processes satisfying Markov property, Martingale property, having bounded increments (across time) and possessing an infinite state space with a rich ergodic subset. We further require that there are states of the stochastic process with the resulting stochastic discount factor arbitrarily close to 0, and such states can be reached with positive (yet possibly arbitrarily small) probability in the long run. In this study, a player's discount factor is such a process. In this setting, we, not only establish the (subgame perfect) Folk Theorem, but also prove the main result of this study: In any equilibrium path, the occurrence of any finite number of consecutive repetitions of the period Nash action profile, must almost surely happen within a finite time window. That is, any equilibrium strategy almost surely contains arbitrary long realizations of consecutive period Nash action profiles.

SONSUZ TEKRARLI OYUNLARDA STOKASTİK İSKONTOLAMA: NEREDEYSE KAÇINILMAZI BEKLEMEK

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Özet

Bu tez tam bilgi altında sonsuz tekrar edilen ve stokastik olarak iskonto edilen oyunlar hakkındadır. Bu çalışmamızda içinde en az bir adet saf stratejilerden oluşan Nash dengesi bulunan sonsuz tekrarlı oyunları inceliyoruz. Bu oyunlarda stokastik iskonto süreçleri Markov özelliğini ve martingale özelliğini içeren, sınırlı artışları olan ve sonsuz bir durumlar uzayına, ve bu uzayın içinde zengin bir ısrarlı durum uzayına sahip olan süreçlerle ilgileniyoruz. Ayrıca bu durum uzayının 0 a çok yakın elemanları olmasını da istemekteyiz. Tüm bu şartlar sağlandığı durumda yalnızca alt-oyun yetkin Folk teoremini değil aynı zamanda bu çalışmanın ana sonucunu da elde etmekteyiz: hangi denge patikası olursa olsun, o patikanın içerisinde uzun, ardışık periyodlar süresince saf stratejilerden oluşan Nash dengesi hareketleri, neredeyse kesinlikle sonlu bir gelecek içerisinde gözlenmek zorundadır.

Contents

Chapter 1

Introduction

In this thesis our aim is to consider strategic interactions where a stage game, a normal form game in which players make simultaneous choices, is played infinitely often and the parties involved discount future returns in a *stochastic* manner.

Repeated games are standard models used in analyzing strategic interactions that occur repeatedly. Thus, they constitute the cornerstone of modeling dynamic strategic relations, hence, are essential in the theory of economics.

In fact, repeated games are a certain type of simple dynamic games in which players face the same stage game in every period. The results obtained from infinitely repeated games depend critically on the number of repetitions and change drastically from cases where the stage game is played a finite number of times.

The important feature of the repeated game structure is the ability of players to condition their actions to the past. This distinctive ability of players allows game theorists to obtain very attractive and striking results that cannot be obtained in standard one shot games, as Robert Aumann also points out in his Nobel Prize Lecture:

> The theory of repeated games is able to account for phenomena such as altruism, cooperation, trust, loyalty, revenge, threats (self destructive or otherwise), phenomena that may at first seem irrational, in terms of the "selfish" utility-maximizing paradigm of game theory and neoclassical economics. That it "accounts" for such phenomena does not mean that people deliberately choose to take revenge, or to act generously, out of consciously self-serving, rational motives. Rather, over the millennia, people have evolved norms of behavior that are by and large successful, indeed optimal. Such evolution may actually be biological, genetic. Or, it may (even) be "memetic".

Clearly, the techniques and results in repeated games are widespread not only in the economics theory, but also in the theories of biology, finance, operation research and political science. In order to discuss some of these results, we need to introduce some notions that will be employed.

1.1 Payoff Notions

In finitely repeated games, the payoffs associated with the game are usually defined as the sum of the payoffs obtained at each period. Particularly, this notion of summing through period payoffs becomes problematic in infinitely repeated games as the total payoff implies an infinite sum that might not converge to a finite value. Thus, an intuitive method of forming payoffs is to consider discounted (i.e. geometrically weighted) summation of period returns.

However, identifying payoffs in infinitely repeated games is not restricted to this particular method hinted above. In general, obtaining a payoff in an infinitely repeated game involves mapping an infinite sequence of real numbers into a single one. One may find situations in which considering simple average returns more plausible than discounted ones. Likewise, there also may be situations in which the payoff of the infinitely repeated game is given by the infimum of the infinite sequence of real numbers, each of which corresponds to some period returns. [1] Thus, one may imagine many forms of payoff notions for infinitely repeated games.

In the literature of repeated games, the following three forms of payoff notions are widely used: Payoffs' description by *limits of the means* is considered by Aumann and Shapley (1994), *overtaking criterion* is due to Rubinstein (1979) and the most common description is the *discounting* payoff structure in which players' payoffs at the end of the repeated game is the summation of discounted stage game payoffs obtained at each stage.

[1] Consider a strategic interaction where two countries decide whether or not to launch nuclear missiles toward each other at every period. In such a game it might be argued that the above given payoff notion is plausible. This is because, any one of the parties being the subject of a nuclear attack, even though once, is more than enough.

1.1.1 Limits of the Means

In the limits of the means payoff notion in an infinitely repeated game, return streams (infinite sequence of real numbers, each of which corresponds to some return obtained in some period) are evaluated with respect to the average returns associated in order to obtain payoffs. In other words, every period is equally important.

The drawback of this evaluation criterion is that anything that happens finitely often (no matter how large this finite integer may be) does not matter at all. Clearly, such a restriction imposed by this payoff notion limits the scope of applications to be considered. Consequently, one may even argue that such a payoff notion makes the game somewhat pathological. To see this, consider the following infinite sequence of real numbers: For every period up to T, the period return is 1; and thereafter it is 0. Under the notion of limits of the means, this sequence will be associated with a payoff 0, no matter how big T maybe.

On the other hand, it is not difficult to think of situations, in which, decision makers put overwhelming emphasis on the long run averages rather than the short run concerns. An extreme form of this concern is reflected with the payoff notion of the limits of the means.

1.1.2 Overtaking

The overtaking payoff notion of an infinitely repeated game is developed by Rubinstein (1979) in order to keep the advantages of the limits of the means payoff notion, while overcoming its most serious shortcoming: The return streams are evaluated with respect to the average payoffs associated, however, this notion also emphasizes single period differences between two return streams. In other words, this criterion also treats each period equally (i.e. placing the priority to the long run), while allowing a single period to affect the overall payoff structure. More details along with examples will be given in the next chapter of this thesis.

In the literature, the payoff notions of limits of the means and overtaking criterion is often referred to as the no discounting payoff notions.

1.1.3 Discounting

In the discounting payoff structure of repeated games, return streams are evaluated with respect to the discounted summation of returns. Furthermore, the discounted summations are normalized in order to associate them with overall payoff values.

This payoff notion does not treat periods equally, and puts greater emphasis on returns obtained in the short run. However, long run concerns are also present, and they can be captured under the consideration of high discount factors.

It is important to remark that discounting is the most common payoff structure in the literature. Moreover, the level of the discount factor is often referred to as the patience of a particular decision maker.

Next, we wish to introduce one of the most important results in the theory of economics. But before doing that, we need to define the notion of equilibrium.

1.2 Subgame Perfection

This concept, which is the standard notion of equilibrium in extensive form games under perfect information, is due to Reinhard Selten, initially introduced in Selten (1965) and later criticized and extended in Selten (1975). Due to his contributions, he was awarded the 1994 Nobel Memorial Prize in Economic Sciences (shared with John Harsanyi and John Nash).

The basic motivation for this equilibrium concept is that common knowledge of rationality implies that rationality should be expected in all of the states of the game. Informally, given what has happened in the past, agents look forward to do the best they can from that point on, provided that they have such a foresight in all the states of the remainder of the game.

Consequently, subgame perfection asks for each player's plan of actions, strategies, to be optimal starting at any given history of the game. That is, there should not be any histories such that a player finds it optimal to deviate from the prescribed behavior, if that behavior is subgame perfect. That is why subgame perfection treats all histories in the same fashion, unlike Nash equilibrium which discriminates between histories that will happen (under the given prescribed behavior) and histories that will never be reached (often called off the path behavior).

It needs to be pointed out that this notion of equilibrium can be used with any of the payoff notions discussed in the previous section.

1.3 Folk Theorems

An important observation emerges in the analysis of infinitely repeated games: With sufficiently patient players or under the payoff notions of no discounting, an infinitely repeated game permits players to design a joint long run behavior, supported by threats, which result in equilibria with socially optimal (in the Pareto sense) outcomes. When the equilibrium notion of subgame perfection is employed, then these threats have to be enforceable (i.e. credible). Such threats, in turn, sustain behavior described above because in an infinitely repeated game a player always has enough time to credibly retaliate, i.e. to punish a deviator in an enforceable manner.

On the other hand, the *subgame perfect Folk Theorem*[2], one of the most hated–celebrated results in repeated games, simply displays that the above given construction (with either sufficiently patient players or under the payoff notions of no discounting) can be employed to support in subgame perfection not only the Pareto optimal outcomes, but also any payoff profile that can be obtained as a result of an individually rational behavior profile in the repeated game. It should be pointed out that we say that a behavior is individually rational whenever it results in a payoff vector in which each players' payoff exceeds the least return level that he could guarantee to himself in the stage game. Thus, behavior that is not individually rational can never be sustained under subgame perfection, because then the relevant player could simply deviate and continue even with the least payoff that he can guarantee to himself

[2] This name is due to the fact that there is no well defined author of the first version of it.

in the stage game.

It should be noticed that the payoffs that one needs to concentrate on are the individually rational ones. This is because, as was displayed in the previous paragraph, payoffs that are not individually rational can never be obtained with subgame perfection.

However, the subgame perfect Folk Theorem, displays that any individually rational payoff vector can obtained under subgame perfection with sufficiently patient players. This, in turn, implies that game theoretic analysis of infinitely repeated games does not have any predictive power, because *anything goes*. Therefore, the subgame perfect Folk Theorem is a powerful and negative result. Consequently, the systematic check of whether or not the Folk Theorem holds in various settings is of great value in the theory of economics.

Subgame perfect Folk Theorems under various settings have been proven with the limit of the means and overtaking criterion payoff notions. The most significant of those are by Aumann and Shapley (1994), for the limits of the means payoff notion, and Rubinstein (1979) for the overtaking criterion notion. However, since this thesis will be concentrated on discounting, we will not put further emphasis on no discounting payoff notions.

In infinitely repeated discounted games, Folk Theorems have been proven under a variety of settings as well. The pioneering works on the subgame perfect Folk Theorem in infinitely repeated discounted games was done by Aumann and Shapley (1994) and Fudenberg and Maskin (1986), where they showed the following: Any individual rational payoff for the one-shot game can be achieved as the discounted, normalized payoff of the repeated game via the use of public randomization, a technical tool which is often interpreted as communication among the players in the stage game. Later in Fudenberg and Maskin (1991), it is shown that public randomization is inessential, hence can be dispensed with, for their subgame perfect Folk Theorem. The considerations of limited memory and bounded rationality, does not change this conclusion documented by Kalai and Stanford (1988), Sabourian (1998), Barlo, Carmona, and Sabourian (2009), Barlo, Carmona, and Sabourian (2007). Considering cases where the actions of other players are not perfectly observable, Fudenberg, Levine, and Maskin (1994), Hörner and Olszewski (2006), Mailath and Olszewski (2011) show that the Folk Theorem still holds. For the instances when there is uncertainty about the returns of the stage game, Dutta (1995), Fudenberg and Yamamato (2010), Hörner, Sugaya, Takahashi, and Vieille (2010), show that the Folk Theorem still remains.

Out of these Folk Theorems, Fudenberg and Maskin (1986) and Fudenberg and Maskin (1991) are of special interest to us. In those studies, they not only obtain the subgame perfect Folk Theorem but also dispense with the use of public randomization. They also develop techniques generating any individual rational outcome exactly as a sequence of actions while the resulting continuation values are within the neighborhood of the desired payoff level. Furthermore, in order to sustain such sequences in the presence of unobservable mixed actions, they show that a uniform level on the discount factor, strictly below one, can be identified so that the continuation values still remain the same, regardless of the realized actions as long as they are in the support of the equilibrium behavior. If any of the realized actions is not in the support of the permitted equilibrium behavior, punishments are triggered. These punishments have to be credible, thus, they are constructed so that conforming with the prescribed behavior always results in a higher (continuation) payoff than the one obtained by deviating today and being punished from thereon. Often, this construction is referred to as the enforceability of the punishments.

1.4 Our Contributions

As discussed above, the Folk Theorems of Aumann and Shapley (1994) and Fudenberg and Maskin (1986) establish that payoffs which can be approximated in equilibrium with patient players are equal the set of individually rational ones. Players' ability to coordinate their actions using past behavior allows such a large set of equilibria. In turn, this vast multiplicity of equilibrium payoffs, considerably weakens the predictive power of game theoretic analysis. [3] An important aspect of these findings is the use of constant discounting. The accepted interpretation of the use of discounting in repeated games, offered by Rubinstein (1982) and Osborne and Rubinstein (1994), is that the discount factor determines a player's perception about the probability of the game continuing into the next period. Thus, constant discounting implies that this

[3]Moreover, the consideration of limited memory and bounded rationality, lack of perfect observability of the other players' behavior and the past, and uncertainty of future payoffs do not change this conclusion, documented by Kalai and Stanford (1988), Sabourian (1998), Barlo, Carmona, and Sabourian (2009), Barlo, Carmona, and Sabourian (2007); Fudenberg, Levine, and Maskin (1994), Hörner and Olszewski (2006), Mailath and Olszewski (2011); Dutta (1995), Fudenberg and Yamamato (2010), and Hörner, Sugaya, Takahashi, and Vieille (2010).

probability is independent of the history of the game, in particular, invariant.

On the other hand, keeping the same interpretation, but allowing for the discount factor to depend on the history of the game and/or vary across time, is not extensively analyzed in the literature on repeated games. Indeed, to our knowledge, the only relevant work in the study of repeated games is Baye and Jansen (1996) which considers stochastic discounting with period discounting shocks independent from the history of the game. Related work concerning stochastic interest rates can be found in the theory of finance, see Ross (1976), Harrison and Kreps (1979), and Hansen and Richard (1987).

In this thesis, we consider a wide class of games with stochastic discounting, when the discounting process is not independent of the past and has a *rich* state space. In such a setting, we impose the restriction that players expectation of the future discount factor is equal to the current one, and only the current value is relevant when trying to make assertions about the future values of the discount factor. Under this construction, we not only prove a Folk Theorem for repeated games with stochastic discounting, but we also, show that no matter how patient players are, every subgame perfect equilibrium path must entail arbitrarily long (yet, finite) consecutive repetitions of period Nash behavior, and these consecutive periods almost surely happen in a finite time window.

In order to present these results in full detail, the next chapter presents the preliminaries for infinitely repeated games. Chapter 3, on the other hand, will introduce the notion of stochastic discounting, and we will present our contributions in chapter 4. Finally, chapter 5 concludes.

Chapter 2

Preliminaries

Let $G = (N, (A_i, u_i)_{i\in N})$ be a normal form game with $|N| \in \mathbb{N}$ and, for all $i \in N$, A_i is player i's actions with property that $|A_i| \in \mathbb{N}$; and i's payoff function denoted by $u_i : A \to \mathbb{R}$ where $A = \prod_{i\in N} A_i$ and $A_{-i} = \prod_{j\neq i} A_i$. Writing $A = \{a^1, a^2, \ldots, a^m\}$ let $w^k = u(a^k)$. Thus, $\{w^1, w^2, \ldots, w^m\}$ is the set of payoff vectors in G corresponding to pure strategies.

Let, for all $i \in N$, $S_i = \Delta(A_i)$; the set S_i is the set of mixed actions for player i. We abuse notation and let u_i, for all $i \in N$, denote the usual mixed-extension. Let $S = S_1 \times \cdots \times S_n$ and let

$$u(S) = \{(u_i)_{i\in N} \in \mathbb{R}^N : (u_i)_{i\in N} = (u_i(s))_{i\in N} \text{ for some } s \in S\}.$$

Let, for $i \in N$,

$$v_i \equiv \min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i}),$$

and let $m^i \in S$ be such that $u_i(m^i) = \max_{s_i} u_i(s_i, m^i_{-i}) = v_i$. The number

v_i denotes the **minmax payoff** of player i in G, and m^i is some action combination that is an optimal punishment of player i in G. Notice that, v_i is the least payoff level that player i can guarantee to himself. Similarly, define $\bar{u}_i \equiv \max_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$, $\bar{u}_i$ denotes the highest returns player i can get in the stage game.

The set of *individually rational payoffs* is denoted by

$$\mathcal{U} = \{u \in \mathrm{co}\,(u(A)) : u_i \geq v_i \text{ for all } i \in N\},$$

and the set of *strictly individually rational payoffs* by

$$\mathcal{U}^0 = \{u \in \mathrm{co}(u(A) : u_i > v_i \text{ for all } i \in N\}.$$

The *supergame* $\bar{G}$ consists of an infinite sequence of repetitions of G taking place in periods $t = 0, 1, 2, 3, \ldots$. Moreover, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Thus, for $k \geq 1$, a $k-$*stage history* is a $k-$length sequence $\bar{h}_k = (a_1, \ldots, a_k)$, where, for all $1 \leq t \leq k$, $a_t \in A$; the space of all $k-$stage histories is $\bar{H}_k$, i.e., $\bar{H}_k = A^k$ (the $k-$fold Cartesian product of A). We use $\bar{e}$ for the unique 0–stage history, it is a 0–length history that represents the beginning of the supergame. The set of all histories is defined by $\bar{H} = \bigcup_{n=0}^{\infty} \bar{H}_n$.

For every $\bar{h} \in \bar{H}$, define $\bar{h}^r \in A$ to be the projection of $\bar{h}$ onto its r^{th} coordinate. For every $\bar{h} \in \bar{H}$, we let $\ell(\bar{h})$ denote the *length of* $\bar{h}$. For two positive length histories, $\bar{h}$ and $\bar{h}'$ in $\bar{H}$, we define the *concatenation of* $\bar{h}$ *and* $\bar{h}'$, in that order, to be the history $(\bar{h} \cdot \bar{h}')$ of length $\ell(\bar{h}) + \ell(\bar{h}')$: $(\bar{h} \cdot \bar{h}') = (\bar{h}^1, \bar{h}^2, \ldots, \bar{h}^{\ell(\bar{h})}, \bar{h}'^1, \bar{h}'^2, \ldots, \bar{h}'^{\ell(\bar{h}')})$. We follow the convention that, $\bar{e} \cdot \bar{h} =$

$\bar{h} \cdot \bar{e} = \bar{h}$ for every $\bar{h} \in \bar{H}$.

Remember that, we assume that the game has perfect information, in other words, we assume that at stage k each player knows $\bar{h}_k$. Regarding strategies, players employ behavioral strategies, that is, in each stage k, they choose a function from $\bar{H}_{k-1}$ to A_i, denoted $\bar{f}_i^k$, for player $i \in N$. The set of player i's strategies is denoted by $\bar{F}_i$, and $\bar{F} = \prod_{i \in N} \bar{F}_i$ is the joint strategy space. Finally, a strategy vector is $\bar{f} = \left(\{\bar{f}_i^k\}_{k=1}^{\infty}\right)_{i \in N}$.

Given an individual strategy $\bar{f}_i \in \bar{F}_i$, and a history $\bar{h} \in \bar{H}$ we denote the *individual strategy induced at* $\bar{h}$ by $\bar{f}_i|\bar{h}$. This strategy is defined pointwise on $\bar{H}$: $(\bar{f}_i|\bar{h})(\bar{h}') = \bar{f}_i(\bar{h} \cdot \bar{h}')$, for every $\bar{h}' \in \bar{H}$. We will use $(\bar{f}|\bar{h})$ to denote $(\bar{f}_1|\bar{h}, \ldots, \bar{f}_n|\bar{h})$ for every $\bar{f} \in A$ and $\bar{h} \in \bar{H}$. We let $\bar{F}_i(\bar{f}_i) = \{\bar{f}_i|\bar{h} : \bar{h} \in \bar{H}\}$ and $\bar{F}(\bar{f}) = \{\bar{f}|\bar{h} : \bar{h} \in \bar{H}\}$.

Any strategy $\bar{f} \in \bar{F}$ induces an outcome $\bar{\pi}(\bar{f}) \in A^{\infty}$ as follows: $\bar{\pi}^1(\bar{f}) = \bar{f}(\bar{e})$, $\quad \bar{\pi}^k(\bar{f}) = \bar{f}(\bar{\pi}^1(\bar{f}), \ldots, \bar{\pi}^{k-1}(\bar{f}))$, for $k \in \mathbb{N}$. Letting $A^{\infty} = A \times A \times \cdots$, we have defined a function $\bar{\pi} : \bar{F} \to A^{\infty}$, which gives the outcome induced by any strategy.

2.1 Payoff Notions in Repeated Games

Suppose that, $\bar{U}_i : S^{\infty} \to \mathbb{R}$ represents the preference relation of player i on S^{∞}. We now can define the notions of Nash and subgame perfect equilibrium. Note that, when required we abuse notation letting,

$$\bar{U}_i(\bar{f}) = \bar{U}_i(\bar{\pi}(\bar{f})).$$

Definition. *A strategy vector $f \in F$ is a* ***Nash equilibrium*** *of $\bar{G}$ if for all $i \in N$, $\bar{U}_i(\bar{f}) \geq \bar{U}_i(\bar{f}'_i, \bar{f}_{-i})$ for all $\bar{f}'_i \in \bar{F}_i$. A strategy vector $\bar{f} \in \bar{F}$ is a* ***subgame perfect equilibrium*** *of $\bar{G}$ if every $\bar{f} \in \bar{F}(\bar{f})$ is a Nash equilibrium.*

We will be working with three notions of returns in the supergame $\bar{G}$. The first two are the no-discounting cases, and there the notion of limits of the means, and overtaking criterion, will be introduced. The final one is the discounting case, where all the agents discount future returns.

Definition (Limits of Means)**.** *The* ***limit of means payoff*** *in the supergame of G, $\bar{G}$ for a given $\bar{\pi} \in S^\infty$ is*

$$\bar{U}_i(\bar{\pi}) = \lim_{T \to \infty} \inf \frac{1}{T} \sum_{t=1}^{T} u_i(\bar{\pi}^t).$$

The logic behind the limits of means criterion is that, most research dealing with non-discounted supergames assume that, players try to maximize their average payoffs. More precisely, if $\bar{\pi}$ and $\bar{\pi}'$ are both outcome paths then, player i's strict preference ordering $\succ_i$ is assumed to be

$$\bar{\pi} \succ_i \bar{\pi}' \Leftrightarrow \lim_{T \to \infty} \inf \frac{1}{T} \sum_{t=1}^{T} (u_i(\bar{\pi}^t) - u_i(\bar{\pi}'^t)) > 0.$$

Limit inferior is used instead of the regular limit notion, as only a bound on the "limit" of the payoff stream is necessary. Technically, limit inferior always exists since the payoffs are real numbers, whereas, the existence of a limit is not guaranteed, the (averaged) stream itself may as well be unbounded.

The drawback of the evaluation criterion is that, anything that happens in a finite time interval does not matter at all. The sequence $(0, 0, \ldots, 0, 1, 1, 1, \ldots)$, in which M zeros are followed by a constant sequence of 1's, is preferred by the limit of means criterion to $(1, 0, 0, \ldots)$, for every value of M, no matter how large M is.

Due to the shortcomings of the limit of means criterion, we now will introduce the other no-discounting payoff, Ramsey/Weiszacker overtaking criterion. The most famous paper pioneering in the analysis of infinitely repeated games with these two criteria is Rubinstein (1979), which is based on Roth (1976).

Definition (Overtaking)**.** *The* ***overtaking criterion*** *in the supergame of G, $\bar{G}$ is a preference relation $\succ^o$ defined by: for any outcome paths $\bar{\pi}, \bar{\pi}' \in A^\infty$*

$$\bar{\pi} \succ_i^o \bar{\pi}' \Leftrightarrow \lim_{T \to \infty} \inf \sum_{t=1}^{T} (u_i(\bar{\pi}^t) - u_i(\bar{\pi}'^t)) > 0$$

.

The overtaking criterion is considered to be a stronger version of the limit of means criterion. Therefore, all the results relating to equilibria with the overtaking criterion would also hold for the limit of means criterion.

According to the overtaking criterion, the sequence $(-1, 2, 0, 0, \ldots)$ is preferred to $(0, 0, \ldots)$, but the two sequences are indifferent according to the limit of means criterion. On the other hand, the sequences $(1, -1, 0, 0, \ldots)$ and $(0, 0, \ldots)$ are indifferent according to both criteria.

The following gives a representation of preferences over a given outcome path $\pi \in A^\infty$ with discounting, using a common discount factor $\delta \in [0, 1)$.

Definition (Discounting)**.** *The* ***discounting payoff*** *in the supergame of G,*

$\bar{G}$ is, for $\delta \in [0,1)$, for a given $\bar{\pi} \in A^\infty$ is the discounted sum of stage game payoffs:

$$\bar{U}_i^\delta(\bar{\pi}) = (1-\delta)\sum_{k=1}^{\infty} \delta^{k-1} u_i(\bar{\pi}^k).$$

Clearly, defining $\bar{V}_i : A^\infty \to \mathbb{R}$ by

$$\bar{V}_i(\bar{\pi}) = (1-\delta)\sum_{k=1}^{\infty} \delta^{k-1} u_i(\bar{\pi}^k).$$

Note that, we have $\bar{U}_i = \bar{V}_i \circ \bar{\pi}$. For $\bar{\pi} \in A^\infty$, $k \in \mathbb{N}$, and $i \in N$, we let

$$\bar{V}_i^k(\pi) = (1-\delta)\sum_{t=k}^{\infty} \delta^{t-k} u^i(\pi^t),$$

be called the player i's value function in date k under $\bar{\pi}$, and it denotes the continuation payoff of player i, starting from period k, under $\bar{\pi} \in A^\infty$.

To see more about the distinction of these three concepts consider the following examples: The sequence $(1,-1,0,0,\ldots)$ is preferred for any $\delta \in [0,1)$ to the sequence $(0,0,\ldots)$. However, according to the other two criteria, the two sequences are indifferent. Finally, the sequence $(0,0,\ldots,0,1,1,1,\ldots)$, in which M zeros are followed by a constant sequence of 1's, is preferred by the limit of means criterion to $(1,0,0,\ldots)$ for every value of M. On the other hand, for every $\delta \in [0,1)$, there exists M^* large enough, so that for all $M > M^*$, the latter is preferred to the former according to the discounting criterion for the fixed value of δ.

2.2 Folk Theorem Without Public Randomization

In this section, we will focus on the Folk Theorem without Public Randomization of Fudenberg and Maskin (1991). The proof of this essential result is included in this thesis because that we will be using some of the ingredients of the proof in our constructing, hence, would like to present them in full detail

Theorem (Folk Theorem Without Public Randomization)**.** *Consider an n-player game, in which public randomization is not available and only the players' choice of actions are observable, assume that the dimension of* $\mathcal{U}^0$ *is equal to n. Then, for any* $u = (u_1, u_2, \ldots, u_n) \in \mathcal{U}^0$*, there is a* $\underline{\delta} < 1$ *such that for all* $\delta \in (\underline{\delta}, 1)$*, there is a subgame perfect equilibrium of the infinitely repeated game with discount factor* δ*, in which the discounted average payoffs are* u*.*

Proof. Consider u' in the interior of $\mathcal{U}^0$ such that $u'_i < u_i$ for all i. Take $\rho > 0$ such that for all players i the vector $u'(i) = (u'_1 + \rho \ldots u'_{i-1} + \rho, u'_i, u'_{i+1} + rho, \ldots, u'_n + \rho)$ is in $\mathcal{U}^0$. Furthermore, set $u'(0) = u$ Let $w_i^j = u_i(m^j)$ be player i's period payoff when j is being punished with m^j. Choose $\varepsilon > 0$ such that for all i and j, $\varepsilon < u'_j$ and $-w_i^j < \frac{u'_i - \varepsilon}{u'_i}(\rho - w_i^j)$. Then, by the Lemma presented below, there exists some δ_ε such that for all $\delta > \delta_\varepsilon$ and each i, there exists deterministic sequences $\{a^i(t, \delta)\}$, whose average discounted payoffs are u'_i, and whose continuation payoffs are within ε of u'_i.

Lemma. *For any* $\varepsilon > 0$*, there exists* δ_ε *such that for all* $\delta \geq \delta_\varepsilon$*, and every* $u \in \mathcal{U}^0$ *with* $u_i \geq \varepsilon$ *for all i, there is a deterministic sequence of pure strategies* *whose discounted average payoffs are u, and whose continuation payoffs at each time t are within ε of u.*

Proof. Given any u in $\mathcal{U}^0$, and $\varepsilon > 0$, let $\varepsilon' = \varepsilon/4$. Let $B(u, \varepsilon') = \{u' \in \mathcal{U}^0 : \|u' - u\| < \varepsilon'\}$ be the ball of radius ε' centered at u. Let Z be a polygon with vertices $\{z^l\}$ such that: (i) each z^l is within $2\varepsilon'$ of u, (ii) every $u' \in B(u, \varepsilon')$ can be expressed as a convex combination of $\{z^l\}$, and (iii) each z^l can be expressed as $\sum_k = 1^m \lambda^k(l) w^k$, where each weight $\lambda^k(l)$ is a rational number between zero and one, and the weights sum to 1. Since the weights are rational, one can find integers c and $\{r^k(l)\}_{k=1}^m$ such that for all l and k, $\lambda^k(l) = r^k(l)/c$. Let *cycle* l be the c-period sequence of pure strategies, in which a^1 is played for the first $r^1(l)$ periods, a^2 is played for the first $r^2(l)$ periods and so on. Let $z^l(\delta)$ be the average discounted payoff of *cycle* l. Using the algoritmh of Sorin (1986) (which is also their lemma 1) with $z^l(\delta)$, they verify that they can generate each $u' \in B(u, \varepsilon')$ by a sequence $z^l(\delta)'s$ for $\delta > 1 - 1/m$. Now, for any given u' each of these cycles are of length c, and each $z^l(\delta)$ is in $3\varepsilon'$ of u'. Then, for all $u \in \mathcal{U}^0$ and all $\varepsilon > 0$ there is a $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$, and all $u' \in B(u, \varepsilon/4)$, there is a deterministic sequence whose payoffs are equal to u' and whose continuation payoffs at each date are within ε of u and u'.

Now, consider the set $\mathbf{Q} = \{u \in \mathcal{U}^0 : u_i \geq \varepsilon \text{ for all } i\}$. The collection $\mathrm{B} = \bigcup_{u \in \mathbf{Q}} B(u, \varepsilon/4)$ is an open cover of $\mathbf{Q}$ and $\mathbf{Q}$ is compact, therefore, B has a finite subcover. Using that subcover, let δ_ε be the maximum of associated $\underline{\delta}'s$. Then, for all $u \in \mathbf{Q}$, there is a deterministic sequence with properties asserted by lemma. $\square$

Choose $\underline{\delta} > \delta_\varepsilon$ such that for all $\delta > \underline{\delta}$, there exists an integer $N(\delta)$ such that for all i and j, the following holds:

$$\begin{aligned}
&(1-\delta)\bar{u}_i + \delta^{N(\delta)+1} u_i' < u_i' - \varepsilon \\
&(1-\delta)\bar{u}_i + \delta^{N(\delta)+1} u_i' < (1-\delta^{N(\delta)}) w_i^j + \delta^{N(\delta)}(u_i' + \rho) \\
&(1-\delta)\bar{u}_i + \delta^{N(\delta)+1} u_i' < (1-\delta) w_i^j + \delta(u_i' + \rho)
\end{aligned}$$

If there is more than one such integer, let $N(\delta)$ be the smallest. Now, consider the following strategy for player i:

(A) Begin by playing the sequence $\{a_i^0(t,\delta)\}$, and continue to do so as long as $\{a^0(t,\delta)\}$ was played the previous period or at least two players deviated that period.

(B^j) Play m_i^j for $N(\delta)$ periods, if player k unilaterally chooses an action outside the support of m_i^j, go to phase B^k, ignore simultaneous deviations.

(C^j) At the end of phase B^j switch to phase C^j, which requires further explanation. Observe that, in the presence of mixed minmax strategies, the payoffs of B^j will be a random variable. Let r_i^j be the player i's discounted average payoff during phase B^j. Furthermore, set

$$z_i^j = \begin{cases} r_i^j(1-\delta^{N(\delta)})/\delta^{N(\delta)} & i \neq j \\ 0 & i = j. \end{cases}$$

Let $\{a(t,\delta,\{z_i^j\})\}$ be a deterministic sequence that results in the payoffs $(u_i' + \rho - z_1^j, \ldots, u_{j-1}' + \rho - z_{j-1}^j, u_j', u_n' + \rho - z_n^j, u_n' + \rho - z_n^j)$, with the continuation values being in ε neighborhood of these values. Now, we are ready to define the strategy at (C^j).

(C^j) Play $\{a_i(t,\delta,\{z_i^j\})\}$ unless player k unilaterally deviates, in which case go to (B^k). Observe that, by the construction of (C^j), each player $i \neq j$ is indifferent among all actions during punishment phase (B^j). His continuation payoff is equal to $\delta^{N(\delta)}(u_i+\rho)$ regardless of the actions realized.

Now, due to the selection of δ_ε, deviating at phase (A) then conforming gives a continuation value strictly less than $u_i' - \varepsilon$. If a player who is being punished deviates during phase (B), he receives $\delta \cdot \delta^{N(\delta)} u_i'$, which is strictly less than $\delta^{N(\delta)} u_i'$, the punishment payoff. If a player i deviates from phase B^j, again the selection of $\delta\varepsilon$ ensures that deviating results in a strictly worse payoff. Finally, if a player i deviates at (C^k), deviation will result in a payoff strictly less than $u_i' - \varepsilon$. Therefore, no player will find it profitable to deviate at any date of any phase. Now, the only thing to show is the existence of the sequences of actions used in (C^j). Now, consider the sequence $\{(\varepsilon_n,\delta_n)\}$, where ε_n tends to 0 and δ_n tends to 1. Then, rearranging the equations used in identifying δ_ε, we reach

$$\delta^{N(\delta)+1} < (u_i' - \varepsilon - (1-\delta)\bar{u}_i)/u_i'.$$

However, when ε_n tends to 0 and δ_n tends to 1, the right-hand side tends to 1, implying $\delta_n^{N(\delta_n)} \approx 1$ for n sufficiently large similarly, $z_i^j \approx 0$ and $\rho - z_i^j > 0$. Moreover, for large n the payoffs are in the interior of $\mathcal{U}^0$, and bounded away from the axes by at least ε_n. Now, using the lemma presented earlier ascertains the existence of $\{a_i(t,\delta,\{z_i^j\})\}$, as was to be shown. $\square$

Chapter 3

Stochastic Discounting

The accepted interpretation of the use of discounting in repeated games, offered by Rubinstein (1982) and Osborne and Rubinstein (1994), is that the discount factor determines the probability of the strategic interaction surviving into the next period. Thus, constant discounting implies that this probability is independent of the history of the game, in particular, invariant. Keeping the same interpretation, but allowing for the discount factor to depend on the history of the game and/or vary across time, results in the consideration of stochastic discounting, which in fact, is not extensively analyzed in the literature on repeated games.

Particularly, a tangible set of applications of repeated games can be found in industrial organization settings. In such settings, firms can invest at the present rate of interest to obtain principal and interest tomorrow. Thus, in such settings a natural interpretation of the discount factor would be $\frac{1}{1+r_t}$, where r_t is the real interest rate between the periods t and $t+1$. Under such a construction, the use of a constant discount factor would imply the interest rates to be restricted to fixed constants. Clearly, one can easily see that a model with the interest rate varying over time has more appeal. Surprisingly, most of the results in the existing literature assume the discount factor be deterministic. In order to formally represent time preferences with discount factors (interest rates) that may vary over time, the one-shot discount factors need to be a *Stochastic Process*.

This chapter introduces and presents the specifics of the construction of the stochastic discounting that we will employ. It is appropriate to mention that, in order to render a formal treatment, we have to go over some mathematical concepts of the theory of probability.

3.1 Related Concepts in Probability Theory

Before defining how a *stochastic discounting process* is constructed, let us review a few concepts in probability theory.

Definition ($\sigma - algebra$)**.** *Given a set Ω and its power set 2^{Ω}, a set $\mathcal{F} \subseteq 2^{\Omega}$ is a $\sigma - algebra$ over Ω if (i)$\mathcal{F}$ is non-empty, (ii)for all $A \in \mathcal{F}$, $A^c \in \mathcal{F}$, and (iii) for all countable collections $\{A_1, A_2, \ldots, \}$ in $\mathcal{F}$, $A_1 \cup A_2 \cup \ldots$ is in $\mathcal{F}$. An ordered pair $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a $\sigma - algebra$ over Ω is called a measurable space.*

Definition (Measure)**.** *Given a set Ω and a $\sigma - algebra$, $\mathcal{F}$ of Ω, a function $\mathcal{P} : \mathcal{F} \to \mathbb{R}$ is called a measure if (i) $\mathcal{P}(E) \geq 0$ for all $E \in \mathcal{F}$, (ii) for all countable collections $\{E_i\}_{i \in I}$ of pairwise disjoint sets, $\mathcal{P}(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mathcal{P}(E_i)$, and* *(iii)* $\mathcal{P}(\emptyset) = 0$.

Definition (Probability Space)**.** *In probability theory, a probability space for a probabilistic experiment(random variable) is a triple* $(\Omega, \mathcal{F}, \mathcal{P})$*, where* Ω *denotes the set of outcomes of the probabilistic experiment,* $\mathcal{F}$ *denotes the* $\sigma - algebra$ *of* Ω*, the collection of all the events that are considered, and* $\mathcal{P}$ *is a function, measuring the probability of an event.*

Let us remind that $\mathcal{P} : \mathcal{F} \to [0, 1]$. An *event* is considered to have happened when the *outcome* is a member of the event. An outcome can be a member in more than one events.

Definition (Measurable Function)**.** *Given two measurable spaces,* $(\Omega, \mathcal{F})$ *and* $(\mathcal{S}, \mathcal{B})$*, a function* $X : \Omega \to \mathcal{S}$ *is called a measurable function if* $X^{-1}(E) \in \mathcal{F}$ *for every* $E \in \mathcal{B}$*.*

Definition (Random Variable)**.** *Given a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *and a measurable space* $(\mathcal{S}, \mathcal{B})$*, a random variable* $X : \Omega \to \mathcal{S}$ *is a measurable function.*

Definition (Stochastic Process)**.** *Given a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *a stochastic process* $\{X_t\}_t$ *is a collection of random variables* $\{X_t : t \in T\}$*, where the index* t *belongs to the index set* T*.*

Definition (Filtration)**.** *Given a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *and a stochastic process* $\{X_t\}_t$*, a filtration is a collection of sub-*$\sigma - algebras$ *of the* $\sigma - algebra$ $\mathcal{F}$ *such that if* $s \leq t$*,* $\mathcal{F}_s \subset \mathcal{F}_t$*, and* X_t *is* $\mathcal{F}_t$ *measurable.*

Definition (Martingale Process)**.** *A process* $\{X_t\}_t$ *with a filtration* $\{\mathcal{F}_t\}_t$ *satisfies the martingale property if* $\mathrm{E}(X_t|\mathcal{F}_s) = X_s$ *for all* $s \leq t$*.*

Definition (Markov Process)**.** *A process* $\{X_t\}_t$ *with a filtration* $\{\mathcal{F}_t\}_t$ *satisfies the Markov property if* $\mathrm{P}(X_{t+s} \in B|\mathcal{F}_t) = \mathrm{P}(X_{t+s} \in B|X_t)$ *for all* $s, t \in T$*.*

With these theoretical preliminaries, we might say more about a stochastic discounting process. Starting from the very basics, since for any kind of discounting we must have $\delta \in (0,1)$, the stochastic discounting processes we consider also must have $\Omega \subseteq (0,1)$.

The information of the players regarding the realizations of the stochastic process can be captured by the filtration construction. If a random variable is measurable in some $\sigma - algebra$, its value is known at that $\sigma - algebra$. Furthermore, since the filtration is a collection of growing sets, at any $\mathcal{F}_t$ all the values of $X_0, X_1, \ldots, X_t$ are known.

The martingale property, although not essential, is a *nice* property because if we were to consider models in industrial organization where the decision makers are firms and discount factors are inverse interest rates, the martingale property is equivalent to a *no-arbitrage condition.* [1]

The Markov property is a technically *nice* property. In repeated games, the critical point is that players face the same continuation game at every period. Without the Markov property, even in cases where the realizations in two different periods are equal, players would have to face different situations as the entire history of the process would be important. Thus, making the

[1] The usage of martingales as a notion of arbitrage free market conditions is common practice in the theory of finance, documented by Harrison and Kreps (1979)

analysis significantly difficult. Moreover, even tough the Markov property is limiting, the efficient-market hypothesis of Fama (1970) lays the theoretical framework, and provides empirical evidence supporting the use of Markovian Models in Economic Analysis.

A stochastic discounting process is a stochastic process $\{\mathbf{d}_t\}_t$, with an outcome space $\Omega \subseteq (0,1)$, and a suitable filtration $\{\mathcal{F}_t\}_t$. $\mathbf{d}_t$ denotes the discount factor from period t to period $t+1$. Similar to the convention of taking the n-th power of the discount factor for evaluating future returns, the stochastic discount factor from period t to period τ, with $\tau > t$, is defined by multiplying the respective random variables, $\prod_{s=t}^{\tau-1} \mathbf{d}_s$.

3.2 An Example: Stochastic Discounting via Polya's Urn

A well known example of a stochastic process, that is a good candidate for being a stochastic discounting process, is the normalized beta–binomial distribution with two dimensions, more commonly known as the Polya's urn scheme.

Define $\{\mathbf{d}_t\}_t$ as follows: Without loss of generality, let $\mathbf{d}_0 = \hat{\delta}$ be a rational number in $(0,1)$. Thence, $\hat{\delta} = \frac{g}{g+b}$ for some $g, b \in \mathbb{N}$, where g is interpreted as the number of "good", b as the "bad", balls in the urn. A ball is drawn randomly, and is put back into the urn along with a new ball of the same nature, and this process is repeated in each round. Thus, the support of $\mathbf{d}_1$ is $\{\frac{g+1}{g+1+b}, \frac{g}{g+1+b}\}$ where the first observation happens with probability $\mathbf{d}_0$. Inductively, for any $t > 1$ given d_{t-1} (a realization of $\mathbf{d}_{t-1}$) the support of $\mathbf{d}_t$

equals $\{\frac{g+k+1}{g+b+t}, \frac{g+k}{g+b+t}\}$ where $k \leq t$ denotes the number of good balls drawn up to period t and the first element of this support is drawn with a probability given by d_{t-1}.

Figure 3.1 illustrates how the process proceeds starting from an initial value of $\frac{6}{10}$ and displays the possible states reachable in 3 turns.

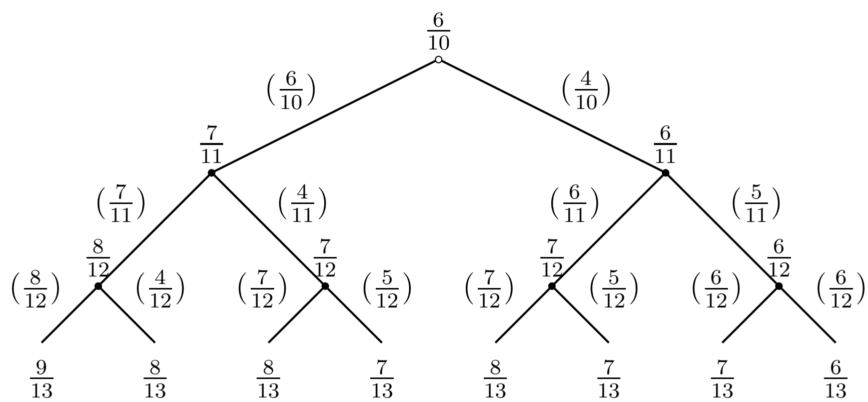

Figure 3.1: Polya Tree

In figure 3.1, the urn initially contains 6 good balls and 4 bad balls. Then, a ball randomly drawn from the urn will be a good ball with $\frac{6}{10}$ probability, and a bad ball with $\frac{4}{10}$. Suppose we draw a good ball in the first turn, and as dictated by the mechanism we put the good ball back in together with a new good ball. Then, there will be 7 good balls in the urn and 11 balls total, and the resulting ratio will be $\frac{7}{11}$. From this state, we will repeat the experiment, but this time the probability of drawing a good ball will be $\frac{7}{11}$, and the probability of drawing a bad ball will be $\frac{4}{11}$. Suppose this time we draw a bad ball, and we return the bad ball to the urn with a new bad ball. Then, there will be 7 good balls in the urn and 12 balls total. Our new state will be $\frac{7}{12}$. From this state, we will repeat the same experiment again, but this time the probability of drawing a good ball will be $\frac{7}{12}$, and the probability of drawing a bad ball will be $\frac{5}{12}$; which essentially means that in the next period, the state will be $\frac{8}{13}$ with probability $\frac{7}{12}$, and it will be $\frac{7}{13}$ with probability $\frac{5}{12}$.

Given any initial value in $\hat{\delta} \in (0,1) \cap \mathbb{Q}$, a Polya scheme has some very *nice* properties, which makes it a good candidate for a stochastic discounting process. First of all, observe that the process is defined as the number of good balls over the number of total balls. Hence, it is easy to verify that the only possible outcomes in the process are in $(0,1) \cap \mathbb{Q}$. In other words, $\Omega \subseteq (0,1) \cap \mathbb{Q}$. Furthermore, the process in the numerator, the number of good balls, obviously satisfies the Markov property, since the number of good balls can only increase by one or remain the same at any given period, regardless of the history of the process. On the other hand, the process in the denominator is a degenerate random process, it just increases by one at every period. Hence, Polya scheme is also Markovian. [2] The Polya process also satisfies the other nice property, it is a martingale process. Suppose at any time $t \in \mathbb{N}_0$ there are g good balls, and b bad balls in the urn. Then, the value of the process will be

[2]In some sources, the Polya scheme is defined directly by the rational number obtained from the ratio (of the number of good balls over the number of total balls). That is, such definitions do not distinguish between having 1 good ball among 2 and having 50 good balls among 100. Consequently the Markov property does not hold when such a definition is employed. On the other hand, the same stochastic process can be defined by the number of good balls divided by the total number of balls, where the information kept consists of the number of good balls and the number of total balls. Then, the process is a Markovian martingale. To see this, observe that a stochastic process defined by the number of good balls is clearly Markovian, and it is a martingale with respect to 1 divided by the number of total balls. For more information about martingales with respect to a specific filtration we refer the reader to Karlin and Taylor (1975)

$\frac{g}{g+b} = \mathbf{d}_t$. Hence, the expected value of the process at time $t+1$ is equal to:

$$\mathrm{E}\left(\mathbf{d}_{t+1}|\mathcal{F}_t\right) = \frac{g}{g+b}\frac{g+1}{g+b+1} + \frac{b}{g+b}\frac{g}{g+b+1} = \frac{g}{g+b}.$$

Now, since time is discrete in the Polya Scheme, just showing one period ahead is sufficient to show the martingale property. [3] Furthermore, even tough the process may seem prone to *snowballing*, it will never become a degenerate process. In fact, the probability of reaching from any rational number in $(0,1)$ to another rational number in $(0,1)$ is always positive (although it might take some time). In other words, the entire outcome space of Polya is *ergodic*. This is also easy to verify because the support of $\mathbf{d}_t$ equals $\{\frac{g+k+1}{g+b+t}, \frac{g+k}{g+b+t}\}$, where $k \leq t$ and for any $t \in \mathbb{N}$ the probability that $k = n$ for any $n \leq t$ is strictly positive. Hence, the normalized negative binomial process constitutes a *good* example of a stochastic discounting process.

[3]For more information on discrete time Martingales, we refer the reader to Williams (1991).

Chapter 4

Awaiting the Almost Inevitable

The Folk Theorems of Aumann and Shapley (1994) and Fudenberg and Maskin (1986) establish that payoffs, which can be approximated in equilibrium with patient players are equal the set of individually rational ones. The main reason for this observation is players' ability to coordinate their actions using past behavior. In turn, this vast multiplicity of equilibrium payoffs, considerably weakens the predictive power of game theoretic analysis. Moreover, the consideration of limited memory and bounded rationality, lack of perfect observability of the other players' behavior and the past, and uncertainty of future payoffs do not change this conclusion, documented by Kalai and Stanford (1988), Sabourian (1998), Barlo, Carmona, and Sabourian (2009), Barlo, Carmona, and Sabourian (2007); Fudenberg, Levine, and Maskin (1994), Hörner and Olszewski (2006), Mailath and Olszewski (2011); Dutta (1995), Fudenberg and Yamamato (2010), and Hörner, Sugaya, Takahashi, and Vieille (2010). An important aspect of all these findings is the use of constant discounting. The accepted interpretation of the use of discounting in repeated games, offered by Rubinstein (1982) and Osborne and Rubinstein (1994), is that the discount factor determines a player's probability of surviving into the next period. Thus, constant discounting implies that this probability is independent of the history of the game, in particular, invariant.

On the other hand, keeping the same interpretation, but allowing for the discount factor to depend on the history of the game and/or vary across time, is not extensively analyzed in the literature on repeated games. Indeed, to our knowledge, the only relevant work in the study of repeated games is Baye and Jansen (1996), which considers stochastic discounting with period discounting shocks independent from the history of the game. Related work concerning stochastic interest rates can be found in the theory of finance, see Ross (1976), Harrison and Kreps (1979), and Hansen and Richard (1987).

This thesis studies repeated games with pure strategies and common stochastic discounting under perfect information. We consider infinite repetitions of any finite normal form game possessing at least one pure Nash action profile. We require the stochastic discounting process to to satisfy the following: (1) Markov property, (2) Martingale property, (3)to have bounded increments (across time) and to possess a denumerable state space with a rich ergodic subset, (4) there are states of the stochastic discounting process that are arbitrarily close to 0, and such states can be reached with positive (yet possibly arbitrarily small) probability in the long run. In this setting, we, not only establish the (subgame perfect) Folk Theorem, but also prove the main result of this study: Under any subgame perfect equilibrium strategy, the occurrence of any finite number of consecutive repetitions of the period Nash action profile, must almost surely happen within a finite time window. That is, any equilibrium strategy almost surely contains arbitrary long realizations of consecutive period Nash action profiles. In other words, every equilibrium outcome path almost surely involves a stage, i.e. the stochastic process governing the one–shot discount factor possesses a *stopping time*, after which long consecutive repetitions of the period Nash action profile must be observed. Considering the repeated prisoners' dilemma with pure strategies and stochastic discounting, our results display that: (1) the subgame perfect Folk Theorem holds; and, (2) in any subgame perfect equilibrium strategy for any natural number K, the occurrence of K consecutive defection action profiles must happen almost surely within a finite time period.

The fundamental reason of our main result is captured by a significant phrase to be found on page 101 of Williams (1991): "Whatever always stands a reasonable chance of happening, will almost surely happen – sooner rather than later." Indeed, due to the restrictions on the stochastic processes we prove that for any $\varepsilon > 0$, the one–shot discount factor must fall below ε in a finite time period almost surely. Then, given any natural number K, the restriction of bounded increments enable us to identify the level of ε (via the use of K) so that: In any equilibrium path, the one–shot discount factors cannot exceed a certain threshold even when $K + 1$ consecutive "good" shocks are realized. Hence, the occurrence of K consecutive repetitions of the period Nash action profile, must almost surely happen within a finite time window under any subgame perfect strategy.

In order to see why the subgame perfect Folk Theorem holds, first, notice that due to restricting attention to perfect information and stochastic processes with the Markov property, given any history of shocks, players evaluate future payoffs with their expected discount factors and the conclusions of Abreu (1988) applies. Moreover, we show that the following observation holds regarding players' expectations for future discount factors: In any period t with any given history of shocks up to that period, each player evaluates future return streams at least as much as a player using a constant discount factor obtained from the same shocks. That is, each player's expectation of the discount factor from period t into period τ, $\tau > t$, is not less than the discount factor from t into $t+1$ raised to the power of $\tau - (t+1)$. Hence, one may approximate a given strictly individually rational payoff vector by constructing a simple strategy profile (supporting that payoff vector via period–0 expectations) and working with its extensions to our setting.

The literature on stochastic discounting in repeated games is surprisingly not very rich. A significant contribution in that field is Baye and Jansen (1996). Their study considers a form of stochastic discounting with no stringent restrictions on the values that one–shot discount factor can take, and, the distribution of one–shot discount factors may depend on the time index. However, such a distribution in a particular period is independent from the past distributions. Moreover, they identify two significant cases: The first, when the one–shot discount factor is realized before the actions in the stage game are undertaken; the second, when the actions need to be chosen before the one–shot discount factor is realized. They prove that the Folk Theorem holds with in the latter case. They also establish that in the first case, "a full folk theorem is unobtainable... since average payoffs on the efficiency frontier are unobtainable as Nash equilibrium super–game payoffs".

Our formulation involves the stochastic discount level in a period t, δ_t^{t+1}, being common knowledge among the players in period t before they choose actions. Thus, apart from the beginning of the game our formulation corresponds to the case in Baye and Jansen (1996) where "... players choose actions in each period after having observed the current discount factor". In this setting, as was mentioned above, they show that the (full) Folk Theorem "...breaks down; payoffs on the boundary of the set of individually rational payoffs are unobtainable as Nash equilibrium average payoffs to the supergame." However, it is important to emphasize the following: (1) Their stochastic discounting formulation involves a common discount factor determined by a random variable distributed independently from the history of the game; and (2) While our formulation necessitates (due to the use of stochastic processes) the period 0 discount factor to be deterministic, the failure of the Folk Theorem shown in the setting of Baye and Jansen (1996) is primarily due to the action profile chosen in period 0 being a function of the random period 0 discount factor (drawn before the period 0 action is chosen).

There is a number of notable contributions in the context of stochastic games. Indeed, recent studies by Fudenberg and Yamamato (2010) and Hörner, Sugaya, Takahashi, and Vieille (2010) generalize the Folk Theorem of Dutta (1995) for irreducible stochastic games with the requirement of a finite state space. Our setup can be expressed as an irreducible stochastic game where each players' discounting is constant, yet their payoffs are all obtained from a (stochastic) scalar, and the actions chosen have no bearing on the future payoffs. Indeed, ours is a particular irreducible stochastic game with an *infinitely rich* state space, hence these Folk Theorems do not apply.

It is important to point out that the two theorems presented in this study are two distinct observations. The first concerns the inevitable state that the stochastic discount factor must almost surely reach in far future; the reasons why the inevitable state that the stochastic discount factor must almost surely reach in far future is not reflected in date zero evaluations of future payoffs, are the martingale property and the linearity of players payoff functions. Therefore, the first result should not be interpreted as an "Anti–Folk Theorem". It displays that when players use stochastic discounting, one should not be surprised to observe long consecutive repetitions of Nash behavior in the far yet foreseeable future, no matter how patient players were in the initial stages of the repeated interaction.

On the other hand the second theorem in our study concerns state contingent plans of actions, formulated and evaluated with the information available at date zero. Therefore the small possibilities of future shocks do not impact the expected returns evaluated at the beginning. In other words our Folk Theorem says that when players are sufficiently patient at the beginning of the game (they are expected to be just as patient in the future due to martingale property) any strictly individual payoff vector can be approximately obtained (with date zero expectations) under subgame perfection. [1]

[1]This Folk Theorem is one that concerns a special class of irreducible stochastic games with infinitely many states.

The organization of this chapter is as follows: The next section will present the basic model, notation and definitions and, some preliminary yet important results. In section 2 we characterize the set of subgame perfect equilibrium payoffs and the principle of one deviation. In section 3 we will present and prove the main theorem of this thesis. Finally, in section 4, we will find an anology between regular repeated games and their stochastically discounted brethren, and present our Folk Theorem for repeated games with stochastic discounting.

4.1 Notations and Definitions

Let $G = (N, (A_i, u_i)_{i \in N})$ be a normal form game with $|N| \in \mathbb{N}$ and for all $i \in N$, A_i is player i's actions with property that $|A_i| \in \mathbb{N}$; and i's payoff function denoted by $u_i : A \to \mathbb{R}$ where $A = \prod_{i \in N} A_i$ and $A_{-i} = \prod_{j \neq i} A_i$.

For what follows, we assume some structure on the set of actions in G and also that it has a pure strategy Nash equilibrium:

Assumption 1. $G = (N, (A_i, u_i)_{i \in N})$ *is such that there exists* $a^* \in A$ *with the property that for all* $i \in N$, $u_i(a^*) \geq u_i(a_i, a^*_{-i})$ *for all* $a_i \in A_i$.

For any $i \in N$ denote respectively the (pure strategy) *minmax payoff* and a (pure strategy) *minmax profile* for player i by $v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$ and the associated action profile by $m^i \in \arg\min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$. The set of *individually rational payoffs* is denoted by

$$\mathcal{U} = \{u \in \mathrm{co}\,(u(A)) : u_i \geq v_i \text{ for all } i \in N\},$$

the set of *strictly individually rational payoffs* by

$$\mathcal{U}^0 = \{u \in \mathrm{co}(u(A)) : u_i > v_i \text{ for all } i \in N\}.$$

The *supergame of* G consists of an infinite sequence of repetitions of G taking place in periods $t = 0, 1, 2, 3, \ldots$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In every period $t \in \mathbb{N}_0$, a random variable, $\mathbf{d}_t$, is determined. The following summarizes the assumptions needed, which allows for a wide class of random variables:

Assumption 2. $\{\mathbf{d}_t\}_{t \in \mathbb{N}_0}$ *is a stochastic process satisfying the following:*

1. *Markov property;*
2. *martingale property;*
3. *the state space* Ω *of* $\mathbf{d}_t$*, is a subset of* $(0, 1)$ *with infinitely many elements;*
4. *given the state space* Ω *of* $\mathbf{d}_t$*, the set of ergodic states, denoted by* Ω^E*, is dense in* Ω*;*
5. $\mathbf{d}_t$ *is such that for any* $\varepsilon > 0$*, there exists* $\tau \geq t$ *with* $\Pr(\mathbf{d}_\tau < \varepsilon \mid \mathcal{F}_t) > 0$*;*
6. *for any given state* $\omega \in \Omega \subseteq (0, 1)$*, the set of states* $\omega' \in \Omega$ *that are reachable from* ω *in a single period and satisfying* $\omega < \omega'$*, denoted by* $R(\omega)$*, is finite. Moreover, for any* $\omega, \omega' \in \Omega$ *with* $\omega' > \omega$*,* $\sup R(\omega') \geq \sup R(\omega)$*;*
7. $\mathbf{d}_0$ *is non-stochastic.*

The first two parts of Assumption 2 imply not only that the best guess about the future depends only on the current value of the stochastic process, but also that this best guess is equal to the current value.

The third and fourth parts of Assumption 2 imply that the set of values that are reachable both in the long run and in the short run are large, but bounded. That is, the set of aperiodic and non-transient states of $\mathbf{d}_t$ must be dense in the state space, which is a subset of $(0, 1)$.

In the fifth part of Assumption 2 we require that there are states of the stochastic process arbitrarily close to 0, and such states can be reached with positive, but possibly arbitrarily small, probability in the long run. It is essential to note that when the state space of the process is finite, then the fifth part of our assumption cannot hold.

The sixth part of Assumption 2 requires that the "upward jumps" in the process cannot involve infinitely many states. This can be considered as a special form of bounded increments requirement. This is because, due to the process itself being bounded, the above requirement limits the increments to be bounded non-trivially at every state.

The final part of Assumption 2 requires that the start of the process is deterministic.

We wish to point out that the stochastic process known as the normalized beta-binomial distribution with two dimensions, a Polya's urn scheme, satisfies all the requirements of Assumption 2, where the relevant state space Ω is a subset of rational numbers in $(0, 1)$. To see this we refer the reader to Karlin and Taylor (1975).

Given a stochastic process $\{\mathbf{d}_t\}_{t\in\mathbb{N}}$, let $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ denote its natural filtration (i.e. sequence of growing σ-algebras); and for any given $t \in \mathbb{N}_0$, $\mathcal{F}_t$ is commonly interpreted as the information in period t.

Given τ, we let a particular realization of the stochastic process $\{\mathbf{d}_t\}_{t\in\mathbb{N}}$ be denoted by $d_\tau \in \mathbb{R}$.

The supergame is defined for a given $d \in (0,1)$ with $d = r\mathbf{d}_0$ and $r \in (0,1]$, and is denoted by $G(\{\mathbf{d}_t\}_t)$. [2] For $k \geq 1$, a *$k-$stage history* is a k-length sequence $h_k = ((a_0, d_1), \ldots, (a_{k-1}, d_k))$, where, for all $0 \leq t \leq k-1$, $a_t \in A$; and for all $1 \leq t \leq k$, d_t is realization of $\mathbf{d}_t$; the space of all k-length histories is H_k, i.e., $H_k = (A \times \mathbb{R})^k$. We use e for the unique 0–stage history — it is a 0–length history that represents the beginning of the supergame. The set of all histories is defined by $H = \bigcup_{n=0}^{\infty} H_n$. For every $h \in H$ we let $\ell(h)$ denote the *length of h*. For $t \geq 2$, we let $d^t = (d_1, \ldots, d_t)$ denote the history of shocks up to and including period t.

We assume that players have *complete information*. That is, in period $t > 0$, knowing the history up to period t, given by h_t, the players make simultaneous moves denoted by $a_{t,i} \in A_i$. The players' choices in the unique 0–length history e are in A as well. Notice that in our setting, given t, a player not only observes all the previous action profiles, but also all the shocks including the ones realized in period t. In other words, the period–t shocks are commonly observed before making a choice in period t.

For all $i \in N$, a *strategy* for player i is a function $f_i : H \to A_i$ mapping

[2] The reason why we have chosen to formulate $d \in (0,1)$ as a multiplication of a real number r in $(0,1]$ and $\mathbf{d}_0$ is as follows: The stochastic process at hand may involve states spaces that are strict subsets of $(0,1)$. Hence, for obtaining d precisely, a multiplication with a real number in $(0,1]$ might be necessary.

histories into actions. The set of player i's strategies is denoted by F_i, and $F = \prod_{i \in N} F_i$ is the joint strategy space. Finally, a strategy vector is $f = (f_1, \ldots, f_n)$. Given an individual strategy $f_i \in F_i$ and a history $h \in H$ we denote the *individual strategy induced at* h by $f_i|h$. This strategy is defined point-wise on H: $(f_i|h)(\bar{h}) = f_i(h \cdot \bar{h})$, for every $\bar{h} \in H$. We will use $(f|h)$ to denote $(f_1|h, \ldots, f_n|h)$ for every $f \in F$ and $h \in H$. We let $F_i(f_i) = \{f_i|h : h \in H\}$ and $F(f) = \{f|h : h \in H\}$.

A strategy $f \in F$ induces an outcome $\pi(f)$ as follows: $\pi^0(f) = f(e) \in A$; and for $d_1 \in \mathbb{R}$ we have $\pi^1(f)(d^1) = f(f(e), d_1) \in A$; and, $\pi^2(f)(d^2) = f(f(e), f(f(e), d_1), d_2) \in A$, $d_1, d_2 \in \mathbb{R}$; and continuing in this fashion for all $k > 1$ and $d_1, \ldots, d_k \in \mathbb{R}$, we obtain

$$\pi^k(f)(d^k) = f\left(\pi^0(f), \pi^1(f)(d^1), \ldots, \pi^{k-1}(f)(d^{k-1}), d_k\right) \in A.$$

On the other hand, the repeated game with common and constant discounting, with a discount factor $\hat{\delta} \in (0,1)$, is denoted by $\bar{G}(\hat{\delta})$. We employ the above definitions, of course, without the parts concerning the stochastic discounting process.

Next, we wish to present the construction of expected payoffs. Due to that regard, first we will present our stochastic discounting construction, and second formulate the resulting expected utilities.

Players payoffs are evaluated with a common stochastic discount factor: The stochastic discount factor of any player i, $i \in N$, is a random variable, denoted by $\left\{\mathbf{d}_t^{t+1}\right\}_{t \in \mathbb{N}_0}$, where for any given $t \in \mathbb{N}_0$, $\mathbf{d}_t^{t+1}$ identifies the probability of the game continuing from period t to period $t+1$. Hence, the stochastic discount factor from period t to period τ, with $\tau \geq t$, given $\mathcal{F}_s$, $s \leq t$, is defined by $\mathbf{d}_t^\tau \equiv r \prod_{s=t}^{\tau-1} \mathbf{d}_s$, for some $r \in (0,1]$ with the convention that $\mathbf{d}_t^t = 1$. This trivially implies that $\mathbf{d}_t^{t+1} \equiv r\mathbf{d}_t$. We denote $\mathrm{E}(\mathbf{d}_t^{t+1} \mid \mathcal{F}_s)$ for $s \leq t-1$, by $\mathrm{E}_s\left(\mathbf{d}_t^{t+1}\right)$, which indeed is the projection of $\mathbf{d}_t^{t+1}$ on $\mathcal{F}_s$. For any $t \in \mathbb{N}_0$, we let a realization of $\mathbf{d}_t^{t+1}$ be denoted by δ_t^{t+1}, which stands for the realized probability that the game continues from period t to period $t+1$.

One thing to note is the particular *timing* and *information setting* that we employ: Given $r\mathbf{d}_0 = d$ the stochastic discount factor determining the probability that the game continues into the next period is pinned down to a constant, $\mathbf{d}_0^1 = r\mathbf{d}_0 = d$. In the next period, $t = 1$, $\mathbf{d}_1$ is realized before players decide on $a_1 \in A$. So the realization of $r\mathbf{d}_1 = \mathbf{d}_1^2$ is also known at $t = 1$. Thus, following an inductive argument in any period $t > 1$, the given d^t determines the particular level of δ_t^{t+1}, i.e. the probability that the game continues from period t into period $t+1$.

The following Lemma display that the stochastic discounting process constructed in this study involves weaker discounting than the one associated with constant discounting:

Lemma 1. *Suppose that Assumption 2 is satisfied. Then*

1. *every possible realization of* $\mathbf{d}_t^\tau$ *is in* $(0,1)$ *for every* $\tau, t \in \mathbb{N}_0$ *with* $\tau > t$,

2. $\mathrm{E}\left(\mathbf{d}_t^{t+1} | \mathcal{F}_0\right) = \delta_{(0)}$ *for some* $\delta_{(0)} \in (0,1)$ *and for all* $t \in \mathbb{N}_0$,

3. *for every given* $\hat{\delta} \in (0,1)$, *there exists* $d \in \mathbb{R}$ *such that* $\delta_{(0)} = \hat{\delta}$,

4. for every $\tau, t, s \in \mathbb{N}_0$ with $\tau > t \geq s$, given $\mathbf{d}_t^{t+1} = \delta_t^{t+1}$

$$\mathrm{E}\left(\mathbf{d}_\tau^{\tau+1} | \mathcal{F}_t\right) = \delta_t^{t+1}, \tag{4.1}$$

and

$$\mathrm{E}\left(\mathbf{d}_{t+1}^{\tau} | \mathcal{F}_s\right) \geq \left(\mathrm{E}\left(\mathbf{d}_{t+1}^{t+2} | \mathcal{F}_s\right)\right)^{\tau-(t+1)}. \tag{4.2}$$

The implications of this Lemma are essential for the proof, the interpretation and the evaluation of our results:

The first one displays that the stochastic process specified results in a well-defined construction for stochastic discounting. This is because for every $\tau, t \in \mathbb{N}_0$ with $\tau > t$, $\mathbf{d}_t^\tau = r \prod_{s=t}^{\tau-1} \mathbf{d}_s$ is in $(0, 1)$ which is due to $r \in (0, 1]$ and every possible realization of $\mathbf{d}_s$ for every $s \in \mathbb{N}_0$ being in $(0, 1)$.

The second shows that date zero expectations of future one–period discount factors are constant with respect to the time index.

And the third, displays that d can be chosen so that any given constant discount factor can be precisely obtained. In fact, we wish to point out that the reason for using a real number $r \in (0, 1]$ in the definition given by $\mathbf{d}_t^\tau \equiv r \prod_{s=t}^{\tau-1} \mathbf{d}_s$ (and not simply letting $r = 1$) is that our construction does not necessarily require the stochastic processes to have a support consisting the entirety of $(0, 1)$. Particularly, the Polya's urn scheme, employed later in the paper as an example, requires $\mathbf{d}_0$ to be a rational number in $(0, 1)$. Therefore, when dealing with stochastic processes requiring $\Omega \neq (0, 1)$, for any $\hat{\delta} \in (0, 1) \setminus \Omega$, $r \in (0, 1]$ and $\hat{\omega} \in \Omega$ can be identified such that r is sufficiently close to 1 and $\hat{\delta} = r\hat{\omega}$. Thus, without loss of generality we assume $r = 1$ in the rest of this study.

Using the first three results presented in the above Lemma, we conclude with respect to date zero expectations the repeated game at hand can be associated with one having a constant and common discount factor. Thus, our repeated game with stochastic discounting can be interpreted as a perturbation of a "standard" repeated game under perfect information with common and constant discount factor (given by $\hat{\delta}$).

Finally, the fourth implication of Lemma 1 is twofold: Given any history of shocks up to time period t, the first is that the expected level of future one–period discount factors are equal to the current one. The second shows that every player values future returns more than a player using a constant discount factor obtained from the same shocks. That is, a player discounts a return in period τ, $\tau > t$, with $\mathrm{E}_t\left(\mathbf{d}_t^{\tau}\right)$ which is greater or equal to $\left(\mathrm{E}_t\left(\mathbf{d}_t^{t+1}\right)\right)^{\tau-t}$. (Notice that given d^t, $\mathrm{E}_t\left(\mathbf{d}_t^{\tau}\right) = \delta_t^{t+1}\mathrm{E}_t\left(\mathbf{d}_{t+1}^{\tau}\right)$, because $\mathbf{d}_t^{t+1} = \delta_t^{t+1}$ is realized.) In particular, this implies d can be chosen so that

$$\mathrm{E}_0\left(\mathbf{d}_t^{\tau}\right) \geq \left(\mathrm{E}_0\left(\mathbf{d}_t^{t+1}\right)\right)^{\tau-t} = \left(\delta_{(0)}\right)^{\tau-t} = \hat{\delta}^{\tau-t},$$

and when $\tau = t+1$ then this inequality holds with an equality. Hence, these properties establish that with a date 0 point of view, our stochastic discounting construction involves weaker discounting than that associated with a constant and common discount factor.

The following Remark summarizes these observations:

Remark 1. *Given any repeated game under perfect information and a common* *and constant discount factor $\hat{\delta} \in (0,1)$, there exists a repeated game under perfect information and stochastic discounting with a process specified such that Assumption 2 holds and the perturbed game exhibits the following properties: (1) The date zero expectations of the one–shot discount factors are all equal to $\hat{\delta}$; and (2) in date 0 players employ weaker discounting than that associated with a constant and common discount factor $\hat{\delta}$; and (3) expected level of future one–period discount factors are equal to the current one.*

Proof of Lemma 1. The proofs of parts 1, 2 and 3 of the Lemma are already discussed above. The first part of the fourth result is, in fact, the martingale identity. For the second part, notice that

$$\begin{aligned}
\mathrm{E}\left(\mathbf{d}_{t+1}^{\tau}|\mathcal{F}_t\right) &= r\mathrm{E}\left(\prod_{s=t+1}^{\tau}\mathbf{d}_s|\mathcal{F}_t\right) = r\mathrm{E}\left(\mathbf{d}_{t+1}\mathrm{E}\left(\mathbf{d}_{t+2}\ldots\mathrm{E}\left(\mathbf{d}_{\tau-1}|\mathcal{F}_{\tau-1}\right)\ldots|\mathcal{F}_{t+1}\right)|\mathcal{F}_t\right) \\
&\geq r\mathrm{E}\left((\mathbf{d}_{t+1})^{(\tau-(t+1))}|\mathcal{F}_t\right) \geq \left(\mathrm{E}\left(\mathbf{d}_{t+1}^{t+2}|\mathcal{F}_t\right)\right)^{\tau-(t+1)}
\end{aligned}$$

due to the tower property (see Williams (1991)), the martingale identity and the Jensen's inequality. □

The next Assumption is about how players' employ knowledge of the past when taking expectations:

Assumption 3. *In every period $t \in \mathbb{N}_0$, each player uses the most up to date information, i.e. $\mathcal{F}_t$.*

Given a strategy profile f, because that each period's supremum return is bounded for every player, the payoff of player $i \in N$ in the supergame $G(\{\mathbf{d}_t\}_t)$

of G is, where $d = \hat{\delta} \in (0,1)$:

$$\begin{aligned} U_i(f, \{\mathbf{d}_t\}_t) &= (1-\hat{\delta})u_1\left(\pi^0(f)\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\delta_0^1 u_1\left(\pi^1(f)(d^1)\right)|\mathcal{F}_0\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\mathrm{E}\left(\delta_0^2 u_1\left(\pi^2(f)(d^2)\right)|\mathcal{F}_1\right)|\mathcal{F}_0\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\mathrm{E}\left(\mathrm{E}\left(\delta_0^3 u_2\left(\pi^3(f)(d^3)\right)|\mathcal{F}_2\right)|\mathcal{F}_1\right)|\mathcal{F}_0\right) + \dots. \end{aligned}$$

Because that $\{\mathcal{F}_s\}_{s=0,1,,2,\ldots}$ is the natural filtration, the above term reduces to

$$\begin{aligned} U_i(f, \{\mathbf{d}_t\}_t) &= (1-\hat{\delta})u_1\left(\pi^0(f)\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\delta_0^1 u_1\left(\pi^1(f)(d^1)\right)|\mathcal{F}_0\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\delta_0^2 u_1\left(\pi^2(f)(d^2)\right)|\mathcal{F}_0\right) \\ &+(1-\hat{\delta})\mathrm{E}\left(\delta_0^3 u_1\left(\pi^3(f)(d^3)\right)|\mathcal{F}_0\right) + \dots, \end{aligned}$$

i.e.

$$U_i(f, \{\mathbf{d}_t\}_t) = (1-\hat{\delta})\sum_{k=0}^{\infty} \mathrm{E}\left(\delta_0^k u_1\left(\pi^k(f)(d^k)\right)|\mathcal{F}_0\right), \tag{4.3}$$

where $\pi^0(f)(d^0) = \pi(f(e))$, and recall that $\mathrm{E}\left(\delta_t^t|\mathcal{F}_s\right) = 1$ for all $s \leq t$. Following a similar method, we can also define the continuation utility of player i as follows: Given $t \in \mathbb{N}$ and $d^t \in \mathbb{R}^t$ for $\tau \geq t$

$$V_i^{\tau,d^t}(f, \{\mathbf{d}_t\}_t) = (1-\hat{\delta})\sum_{k=\tau}^{\infty} \mathrm{E}\left(\delta_\tau^k u_i\left(\pi^k(f)(d^k)\right)|\mathcal{F}_t\right). \tag{4.4}$$

We use the convention that $V_i^{0,d^0}(f, \{\mathbf{d}_t\}_t) = U_i(f, \{\mathbf{d}_t\}_t)$.

When attention is restricted to $\bar{G}(\hat{\delta})$, i.e. the repeated game with constant discounting, the payoffs are defined as follows: For any strategy $\bar{f}$ of the repeated game $\bar{G}(\hat{\delta})$, the payoff of player i is given by $\bar{U}_i(\bar{f}, \hat{\delta}) = (1-\hat{\delta})\sum_{t=0}^{\infty}\hat{\delta}^t u_i(\bar{\pi}^t(\bar{f}))$, where $\bar{\pi}(\bar{f}) \in A^{\infty}$ is the outcome path of $\bar{G}(\hat{\delta})$ induced by $\bar{f}$. For any $\bar{\pi} \in A^{\infty}$, $t \in \mathbb{N}_0$, and $i \in N$, let $\bar{V}_i^t(\bar{\pi}, \hat{\delta}) = (1-\hat{\delta})\sum_{r=t}^{\infty}\hat{\delta}^{r-t}u^i(\bar{\pi}^r)$ be the continuation payoff of player i at date t if the outcome path $\bar{\pi}$ is played.

4.2 Subgame Perfect Equilibria

A strategy vector $f \in F$ is a *Nash equilibrium* of $G(\{\mathbf{d}_t\}_t)$ if for all $i \in N$, $U_i(f, \{\mathbf{d}_t\}_t) \geq U_i((\hat{f}_i, f_{-i}), \{\mathbf{d}_t\}_t)$ for all $\hat{f}_i \in F_i$. A strategy vector $f \in F$ is a *subgame perfect equilibrium* of the supergame $G(\{\mathbf{d}_t\}_t)$ if every $f' \in F(f)$ is a Nash equilibrium. We denote the set of subgame perfect equilibrium strategies of $G(\{\mathbf{d}_t\}_t)$ by $SPE(G(\{\mathbf{d}_t\}_t))$. Let $\mathcal{V}(\{\mathbf{d}_t\}_t)$ be the subgame perfect equilibrium payoffs of $G(\{\mathbf{d}_t\}_t)$. We will abuse notation and will let $\mathcal{V}(\{\mathbf{d}_t\}_t)$ denoted by $\mathcal{V}(d)$ where $d = \mathbf{d}_0$. Moreover, $\mathcal{V}(\{\mathbf{d}_t\}_t, \tau)$ are the subgame perfect equilibrium continuation payoffs (in period τ terms), when d^{τ} is realized. In fact, abusing notation we let $\mathcal{V}(\{\mathbf{d}_t\}_t, \tau) = \mathcal{V}(\delta_{\tau}^{\tau+1})$.

Moreover, when attention is restricted to the repeated game with constant discounting, $\bar{G}(\hat{\delta})$, subgame perfection can easily be defined by excluding stochastic parts of the above definitions. We denote the set of subgame perfect strategies in $\bar{G}(\hat{\delta})$ by $SPE(\bar{G}(\hat{\delta}))$. Let $\bar{\mathcal{V}}(\hat{\delta})$ be the set of subgame perfect equilibrium payoffs in the repeated game with constant discount factor $\hat{\delta}$.

Letting $d = \mathbf{d}_0 = \hat{\delta}$ below we will show that for every t and d^t, $\mathcal{V}(\delta_t^{t+1})$ is compact, hence obtain the following characterization analogous to Abreu

(1988): A strategy f is subgame perfect if and only if for all $i \in N$ and for all $t \in \mathbb{N}_0$ and for all $d^t \in \mathbb{R}^t$, we have

$$V_i^{t,d^t}(f, \{\mathbf{d}_t\}_t) \geq (1-\hat{\delta}) \max_{a_i \in A_i} u_i(a_i, \pi_{-i}^t(f)(d^t)) + \delta_t^{t+1} \mathrm{E}\left(v_i(d^{t+1}) | \mathcal{F}_t\right), \quad (4.5)$$

where $\delta_t^{t+1} = \mathbf{d}_t^{t+1}$ (i.e. given d^t, the realization of $\mathbf{d}_t^{t+1}$ is equal to δ_t^{t+1}), and for every $i \in N$

$$v_i(d^{t+1}) = \min\left\{u_i : u_i \in \mathcal{V}(\delta_{t+1}^{t+2})\right\}. \quad (4.6)$$

Before the justification of these, we wish to describe the resulting construction briefly. Notice that, when player i decides whether or not to follow the equilibrium behavior in period t given the history of process d^t, it must be that: The player i's expected continuation payoff associated with the equilibrium behavior must be as high as player i deviating singly and optimally today, and being punished tomorrow. An important issue to notice is that, tomorrow players will know d_{t+1} (thus, δ_{t+1}^{t+2}) before deciding on their actions. Thus, players will be punishing player i, the deviator, with the most severe and credible punishment with the information they have in period $t+1$. Thus, the punishment payoff to player i with the information that players have in period $t+1$, i.e. d^{t+1}, is $v_i(d^{t+1})$. Player i forecasts these in period t, and hence, forms an expectation regarding his punishment payoff (starting from period $t+1$ onwards) with the information that he has in period t, namely d^t.

In order to show that for every t and d^t, $\mathcal{V}(\delta_t^{t+1})$ is compact, we will be employing the construction of Abreu, Pearce, and Stachetti (1990), and it is important to point out that their assumptions, 1 – 5 are all satisfied in our framework: A2, A3, and A4 are trivially satisfied as the period payoffs are deterministic, and we also impose A1 and A5.

Following their construction, given d^t for any $W \subset \mathbb{R}^N$ and the resulting level of $\delta_t^{t+1} \in (0,1)$, let $g(\delta_t^{t+1})$ be the expected discounted continuation utility vector (not including today's payoff levels and using the normalization via $\hat{\delta} \in (0,1)$) for an arbitrary strategy profile. Furthermore, for that given level of δ_t^{t+1}, consider the pair, $(g(\delta_t^{t+1}), a)$ and define $\mathrm{E}(g(\delta_t^{t+1}), a) = \delta_t^{t+1}\left((1-\hat{\delta})u(a) + g(\delta_{t+1}^{t+2})\right)$. A pair $(g(\delta_t^{t+1}), a)$ is called *admissible with respect to* W if $\mathrm{E}(g_i(\delta_t^{t+1}), a) \geq \mathrm{E}\left(g_i(\delta_t^{t+1}), (\gamma_i, a_{-i})\right)$ for all $\gamma_i \in A_i$ and for all $i \in N$. Moreover, for each set W, define $B^{d^t}(W)$ as follows $B^{d^t}(W) = \{\mathrm{E}(g(\delta_t^{t+1}), a) | (g(\delta_t^{t+1}), a)$ is admissible w.r.t $W\}$. Any set that satisfies $W \subset B^{d^t}(W)$ is called *self-generating* at d^t. At this point it is useful to recall that $\mathcal{V}(\delta_t^{t+1}) = \{V^{t,d^t}(f, \{\mathbf{d}_t\}_t) |\ f$ is a subgame perfect equilibrium strategy profile$\}$.

Notice that

$$g(\delta_t^{t+1}) = (1-\hat{\delta}) \sum_{k=t+1}^{\infty} \mathrm{E}\left(\delta_t^k u\left(\pi^k(f)(d^k)\right) | \mathcal{F}_t\right),$$

for some strategy profile f. Furthermore since, δ_t^{t+1} is actually realized before the actions are taken and the multiplicative nature of our discount factor, the above equation becomes

$$g(\delta_t^{t+1}) = (1-\hat{\delta})\delta_t^{t+1}\left[u\left(\pi^{t+1}(f)(d^t)\right) + \sum_{k=t+2}^{\infty} \mathrm{E}\left(\delta_t^k u\left(\pi^k(f)(d^k)\right) | \mathcal{F}_t\right)\right]$$

which is equal to

$$g(\delta_t^{t+1}) = \delta_t^{t+1}\left[(1-\hat{\delta})u\left(\pi^{t+1}(f)(d^t)\right) + g(\delta_{t+1}^{t+2})\right]. \tag{4.7}$$

Now, it is easy to see that $\mathcal{V}(\delta_t^{t+1})$ is self-generating, as the pair, $(g(\delta_t^{t+1}),$ $\pi^{t+1}(f)(d^t))$ is admissible with respect to $\mathcal{V}(\delta_t^{t+1})$ whenever f is a subgame perfect equilibrium strategy profile with

$$V^{t,d^t}(f,\{\mathbf{d}_t\}_t) = (1-\hat{\delta})u(a_t) + g(\delta_t^{t+1}), \tag{4.8}$$

where $a_t = \pi^t(f)(d^{t-1})$. The two further points to notice is that, due to lemma 1 of Abreu, Pearce, and Stachetti (1990), $B^{d^t}(W)$ is compact whenever W is compact, and the operator B^{d^t} is monotone. Furthermore, since $\mathcal{V}(\delta_t^{t+1})$ is bounded (by $(1/1-\delta_t^{t+1})\mathcal{U}$), closure of $\mathcal{V}(\delta_t^{t+1})$, denoted by $\mathrm{cl}(\mathcal{V}(\delta_t^{t+1}))$ is compact. Hence, $B^{d^t}(\mathrm{cl}(\mathcal{V}(\delta_t^{t+1})))$ is compact, and due to $\mathrm{cl}(\mathcal{V}(\delta_t^{t+1})) \subset B^{d^t}(\mathrm{cl}(\mathcal{V}(\delta_t^{t+1})))$ and self-generation $\mathrm{cl}(\mathcal{V}(\delta_t^{t+1})) \subset \mathcal{V}(\delta_t^{t+1})$. Thus, by Theorem 2 of Abreu, Pearce, and Stachetti (1990), $B^{d^t}(\mathcal{V}(\delta_t^{t+1})) = \mathcal{V}(\delta_t^{t+1})$, thus $\mathcal{V}(\delta_t^{t+1})$ is compact.

4.3 Inevitability of Nash behavior

In this section, we wish to present the main result of this study:

Theorem 1. *Suppose Assumptions 1, 2, 3 hold. Then, for every $K \in \mathbb{N}$, for every $\hat{\delta} \in (0,1)$, for every stochastic discounting process $\{\mathbf{d}_t\}_t$ with $\mathbf{d}_0 = \hat{\delta}$, and for every subgame perfect strategy profile f of the repeated game with stochastic* *discounting; there exists T which is almost surely in $\mathbb{N}_0$, and the probability of $\pi^\tau(f)$ being a Nash equilibrium action profile of the stage game conditional on the information available at s, equals 1, for all $s = T, \ldots, T+K$ and for all $\tau = s, \ldots, T+K$.*

The above theorem establishes that when Assumptions 1, 2 and 3 hold, then arbitrary long (yet, finite) consecutive repetitions of the period Nash action profile must almost surely happen in a finite time window no matter which subgame perfect equilibrium strategy is considered and no matter how high the initial discount factor is. That is, any equilibrium strategy almost surely entails arbitrary long consecutive observations of the period Nash action profile.

Showing this result involves 2 steps: The first displays that every subgame perfect strategy must involve the prescription of the Nash behavior whenever the current discount factor is sufficiently small. The second displays that for any given level of the initial discount factor and any given natural number K, the stochastic process governing the one–shot discount factors possesses a *stopping time*, after which the return to some sufficiently high level of one–shot discount rates within a K–period time window, has zero probability with the evaluation being made in any period within that time window.

The Proof of Theorem 1. The result follows from Lemmas 2 and 3.

Lemma 2. *Suppose Assumptions 1, 2, 3 hold. Then, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta_t^{t+1} \leq \underline{\delta}$, $t \in \mathbb{N}_0$, every subgame perfect strategy profile f of $G(\{\mathbf{d}_t\})$ must be such that $f(d^t, a^t) \in A$ is a Nash equilibrium of G.*

Proof. Without loss of generality, assume that the subgame perfect strategy f is such that $\left(\max_{a_i \in A_i} u_i(a_i, \pi^t_{-i}(f)(d^t)) - u_i(\pi^t(f)(d^t))\right) > 0$ for some $t \in \mathbb{N}_0$ and for some d^t and for some $i \in N$. Because otherwise, the strategy is resulting in a repetition of period Nash behavior. Then, by equation 4.5, for any such subgame perfect strategy f and i and t and d^t

$$\delta_t^{t+1}\left(V_i^{t+1,d^t} - \mathrm{E}\left(v_i(d^{t+1})|\mathcal{F}_t\right)\right) \geq (1-\hat{\delta})\left(\max_{a_i \in A_i} u_i(a_i, \pi^t_{-i}(f)(d^t)) - u_i(\pi^t(f)(d^t))\right).$$

Both the left and the right hand sides of this inequality are strictly positive. Yet, when the prescribed action is not a Nash equilibrium of G, then the left hand side converges to 0 when δ_t^{t+1} tends to 0, but the right hand side is constant. □

Lemma 3. *Suppose Assumptions 1, 2, 3 hold. Then, for every $\underline{\delta} \in (0,1)$, for every $K \in \mathbb{N}$, for every $\hat{\delta} \in (0,1)$, and for every stochastic discounting process $\{\mathbf{d}_t\}_t$ with $\hat{\delta} = \mathbf{d_0}$; there exists T which is almost surely in $\mathbb{N}_0$ and* $\Pr\left[\mathbf{d}_\tau^{\tau+1} < \underline{\delta} \mid \mathcal{F}_s\right] = 1$*, for every $s = T, \ldots, T+K$ and $\tau = s, \ldots, T+K$.*

Proof. Let $\underline{\delta} \in (0,1)$ and $K \in \mathbb{N}$ and $\hat{\delta} \in (0,1)$ with $\hat{\delta} = \mathbf{d_0}$. Let $\omega_0 \in \{\omega \in \Omega^E : \omega < \underline{\delta}\} \neq \emptyset$ due to part (4) and (5) of Assumption 2. Consider $\omega_1 \in \Omega^E$ with $\omega_0 \geq \max R(\omega_1)$, and such an ω_1 exists due to part (4), (5) and (6) of Assumption 2. [3] Now, define $\omega_2 \in \Omega^E$ that satisfies $\omega_1 \geq \max R(\omega_2)$. Inductively, for a given $\omega_{K-1} \in \Omega^E$ define $\omega_K \in \Omega^E$ likewise. Again notice that due to Assumption 2 such an ω_K exists.

[3] Recall that for any given state $\omega \in \Omega \subseteq (0,1)$, the set of states $\omega' \in \Omega$ that are reachable from ω in a single period and satisfying $\omega < \omega'$, denoted by $R(\omega)$.

Following Karlin and Taylor (1975), define the following event

$$\zeta \equiv \min\{\tau \in \mathbb{N}_0 : \mathbf{d}_\tau \leq \omega_K\}.$$

Then by construction, it must be that $\Pr[\mathbf{d}_{s+k} \geq \underline{\delta} \mid \mathcal{F}_s] = 0$, for all $s = \zeta, \ldots, \zeta + K$ and $k = 0, \ldots, K - s$. Finally, due to the ergodicity of ω_K, ζ is a stopping time, that is it will almost surely happen in a finite time period, i.e. $\Pr[\zeta < \infty] = 1$. Hence, ζ is almost surely in $\mathbb{N}_0$ with $\Pr[\mathbf{d}_{s+k} < \underline{\delta} \mid \mathcal{F}_s] = 1$ for all $s = \zeta, \ldots, \zeta + K$ and $k = 0, \ldots, K - s$. Indeed, this also implies that ζ is almost surely in $\mathbb{N}_0$ with $\mathrm{E}_\tau\left(\delta_\tau^{\tau+1}\right) < \underline{\delta}$ for every $\tau = \zeta, \ldots, \zeta + K$. □

□

4.4 The Subgame Perfect Folk Theorem

In this section we prove the following subgame perfect Folk Theorem for repeated games with stochastic discounting.

Theorem 2. *Suppose Assumptions 1, 2, 3 hold, and either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$. *Then, for all* $\varepsilon > 0$, *there exists* $\underline{\delta} \in (0, 1)$ *such that for all* $u \in \mathcal{U}^0$ *and for all stochastic discounting processes* $\{\mathbf{d}_t\}_{t \in \mathbb{N}_0}$ *with* $\hat{\delta} = \mathbf{d}_0 \geq \underline{\delta}$, *there exists a subgame perfect strategy* f *of* $G(\{\mathbf{d}_t\}_t)$ *such that* $\|U(f, \{\mathbf{d_t}\}_t) - u\| < \varepsilon$.

In order to establish this result, an analogy between repeated games with stochastic discounting and those with constant discounting is constructed as follows: Given any repeated game with stochastic discounting, we consider the repeated game with a constant discount factor that equals the initial level of the stochastic discounting process. Particularly, in the repeated game with constant discounting we concentrate on strictly enforceable strategies, those to which players strictly prefer to conform, in each date and state including equilibrium and punishment phases. It is useful to remind the reader that due to the monotonicity result of Abreu, Pearce, and Stachetti (1990), Theorem 6, such strategies are strictly enforceable for higher discount factors as well.

Formulating an extension of such a strategy and requiring it to be subgame perfect in the repeated game with stochastic discounting, turns out to be an arduous, yet feasible (as proven in Lemma 4), endeavor whenever the initial level of the stochastic discounting is sufficiently high.

To see the difficulties involved, consider a repeated prisoners' dilemma, with actions $A_i = \{c, d\}$ and $u_1(d, c) = u_2(c, d) > u_i(c, c) > \frac{1}{2}(u_1(c, d) + u_2(c, d)) > u_i(d, d) > u_1(c, d) = u_2(d, c)$, $i = 1, 2$. Clearly, there exists $\underline{\delta} \in (0, 1)$ such that the cooperative payoff (hence, path given by $((c, c); \infty)$) is sustained with a strictly enforceable strategy profile for all $\delta > \underline{\delta}$. Now, consider any stochastic discounting process satisfying our restrictions and possessing a sufficiently high initial level, and any strategy such that its utility (evaluated at the beginning of the game) is arbitrarily close to $(u_i(c, c))_{i=1,2}$. Due to Lemma 2, for any realizations of the one-shot discount factor that is strictly below $\underline{\delta}$, any such strategy must dictate the play of (d, d), if it were to be subgame perfect. Thus, any subgame perfect strategy in the repeated game with stochastic discounting sustaining the cooperative payoff must be contingent on the realizations of the one-shot discount factors. A simple formulation is one where this contingency is represented by a date and state independent threshold, $\delta_* \geq \underline{\delta}$, so that: the play continues on the cooperative path as long as every past realization of the one-shot discount factors is above δ_*; and otherwise, the play switches to the defection phase. Then, the verification of subgame perfection in the repeated game with stochastic discounting calls for checking every subgame, in particular, those with the current one-shot discount factor arbitrarily close, yet, strictly exceeding δ_*. In such a subgame where additionally there have not been any single player deviations in the past and all past one-shot discount factors have been above δ_*, this strategy should call for the play of (c, c). However, it is not subgame perfect whenever the following holds: The stochastic discounting process is one where the probability of the next period's one-shot discount factor being strictly less than δ_*, is high enough such that any one of the players finds it profitable to deviate in the current period.

Therefore, given a strictly enforceable strategy in the repeated game with constant discounting, the extended strategy we employ is contingent on the stochastic discounting process in the following manner: It will prescribe the play to continue on the paths dictated by its counterpart in the repeated game with constant discounting, whenever each of the past realizations of the one-shot discount factors exceeds a date and state specific threshold. Otherwise, our strategy will recommend the play to consist of the repetitions of a Nash action profile of the stage game thereafter. The initial level of the stochastic discounting process can be chosen sufficiently high so that we can construct date and state specific thresholds such that, given a date and state, the probability (evaluated in that date and state) of the one-shot discount factor in the next period falling below its associated threshold, is sufficiently low. This and strict enforceability, in turn, imply that the relevant incentive conditions hold for any date and state. Meanwhile, choosing the initial level of the stochastic discounting process to be sufficiently high, also results in the utility (evaluated at the beginning of the game) of this strategy profile in the repeated game with stochastic discounting to be arbitrarily close to the utility of its counterpart in the game with the constant discount factor given by that initial level.

Finally, our result is obtained by combining the above construction and the observation that when restricted to pure actions, the strategy profile in the proof of the subgame perfect Folk Theorem of Fudenberg and Maskin (1991) is, in fact, strictly enforceable.

The rest of this section presents the details about the proof of Theorem 2.

Suppose Assumptions 1, 2, and 3 hold. Then, for any $\hat{\delta} \in (0,1)$, consider the repeated game with stochastic discounting $G(\{\mathbf{d_t}\}_t)$ with $\mathbf{d_0} = \hat{\delta}$; and, the repeated game with constant discounting $\bar{G}(\hat{\delta})$. For any $k-$stage history $h_k = ((a_0, d_1), \ldots, (a_{k-1}, d_k))$ of $G(\{\mathbf{d_t}\}_t)$ where for all $0 \leq t \leq k-1$, $a_t \in A$, and for all $1 \leq t \leq k$, d_t is realization of $\mathbf{d}_t$, define its deterministic counterpart, a $k-$stage history, in $\bar{G}(\hat{\delta})$ by $\bar{h}_k = (a_0, \ldots, a_{k-1})$.

Following Abreu (1988), it is well known that one may restrict attention to simple strategies in the analysis of subgame perfection in repeated games with constant discounting: $\bar{f}$ in $\bar{G}(\delta)$, $\delta \in (0,1)$ is *a simple strategy profile represented by* $n+1$ *paths* $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ if $\bar{f}$ specifies: (i) play $\bar{\pi}^{(0)}$ until some player deviates singly from $\bar{\pi}^{(0)}$; (ii) for any $j \in N$, play $\bar{\pi}^{(j)}$ if the jth player deviates singly from $\bar{\pi}^{(i)}$, $i = 0, 1, \ldots, n$, where $\bar{\pi}^{(i)}$ is the ongoing previously specified path; (iii) continue with the ongoing specified path $\bar{\pi}^{(i)}$, $i = 0, 1, \ldots, n$, if no deviations occur or if two or more players deviate simultaneously. These strategies are simple because the play of the game is always in only $(n + 1)$ *states*, namely, in state $j \in \{0, \ldots, n\}$ where $\bar{\pi}^{(j),t}$ is played, for some $t \in \mathbb{N}_0$. In this case, we say that the play is in *phase t of state j*. A profile $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ of $n+1$ outcome paths is *subgame perfect* if the simple strategy represented by it is a subgame perfect equilibrium. Moreover, following Barlo, Carmona, and Sabourian (2009), we say that a simple strategy $\bar{f}$ in $\bar{G}(\delta)$, $\delta \in [0, 1)$, is *weakly enforceable* if for all $i \in N$ and for all $j \in \{0, 1, .., n\}$ and for all $t \in \mathbb{N}_0$

$$\bar{V}_i^t(\bar{\pi}^{(j)}, \delta) \geq (1 - \delta) \max_{a_i \in A_i} u_i(a_i, \bar{\pi}_{-i}^{(j),t}) + \delta \bar{V}_i(\bar{\pi}^{(i)}, \delta), \tag{4.9}$$

where $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ is the simple strategy associated with $\bar{f}$. Due to Abreu (1988), we know that a simple strategy $\bar{f} \in SPE(\bar{G}(\delta))$ if and only if $\bar{f}$ in $\bar{G}(\delta)$ is weakly enforceable. Moreover, we say that a simple strategy $\bar{f}$ in $\bar{G}(\delta)$ with associated outcome paths $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)} \ldots, \bar{\pi}^{(n)})$ is *strictly enforceable* if for all $i \in N$ and for all $j \in \{0, 1, .., n\}$ and for all $t \in \mathbb{N}_0$

$$\inf_{i,j,t} \left(\bar{V}_i^t(\bar{\pi}^{(j)}, \delta) - \left((1 - \delta) \max_{a_i \in A_i} u_i(a_i, \bar{\pi}_{-i}^{(j),t}) + \delta \bar{V}_i(\bar{\pi}^{(i)}, \delta) \right) \right) > 0 \tag{4.10}$$

Let $\bar{f}$ be a strictly enforceable simple strategy in $\bar{G}(\delta)$, $\delta \in (0, 1)$, and the profile of outcome paths associated be $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$. Now, let us formulate an analogy between $G(\{\mathbf{d_t}\}_t)$ with $\mathbf{d}_0 = \hat{\delta}$ and $\bar{G}(\hat{\delta})$ for $\hat{\delta} \geq \delta$.

Lemma 4. *Suppose that Assumptions 1, 2, 3 hold, and $\bar{f}$ of $\bar{G}(\delta)$, where*

$\delta \in (0,1)$ and the associated outcome paths are $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$, is a strictly enforceable simple strategy. Then, for all $\eta > 0$ there exists $\delta^* \in (\delta, 1)$ such that for all $\hat{\delta} > \delta^*$ there is f in $G(\{\mathbf{d}_t\}_t)$ with $\mathbf{d}_0 = \hat{\delta}$, with the properties that f is subgame perfect in $G(\{\mathbf{d}_t\}_t)$ and

$$\left\| U\left(f, \{\mathbf{d}_t\}_t\right) - \bar{U}\left(\bar{f}, \hat{\delta}\right) \right\| < \eta.$$

Proof. Let ν^* be defined by

$$\nu^* \equiv \inf_{i,j,t} \left(\bar{V}_i^t(\bar{\pi}^{(j)}, \delta) - \left((1-\delta) \max_{a_i \in A_i} u_i(a_i, \bar{\pi}_{-i}^{(j),t}) + \delta \bar{V}_i(\bar{\pi}^{(i)}, \delta) \right) \right) > 0,$$

and consider $\nu > 0$ with $\nu < \min\{\nu^*, \eta\}$. Then, there exists $\delta_\nu > \delta$ and $p_\nu \in (0,1)$ sufficiently close to 0 such that the following conditions hold: For all $i \in N$, $j \in N \cup \{0\}$, $t \in \mathbb{N}_0$

$$\left\| \delta_\nu \bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_\nu) - \left(\delta_\nu ((1-p_\nu) \bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_\nu) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_\nu)) \right) \right\| < \frac{\nu}{6} \tag{4.11}$$

$$\begin{aligned} &\inf_{i,j,t} \Big[(1-\delta_\nu) u_i(\bar{\pi}^{(j),t}) + \delta_\nu ((1-p_\nu) \bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_\nu) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_\nu)) \\ &\quad - ((1-\delta_\nu) \max_{a_i \in A_i} u_i(a_i, \bar{\pi}_{-i}^{(j),t}) + \delta_\nu \left((1-p_\nu) \bar{V}_i^{t+1}(\bar{\pi}^{(i)}, \delta_\nu) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_\nu) \right) \Big] \\ &\quad > \nu \end{aligned} \tag{4.12}$$

$$\left\| \sum_{t=0}^{\infty} \mathrm{E}_0\left(\delta_0^t\right) \left(u_i\left(\bar{\pi}^{(j),t}(\bar{f}) \right) \right) - \sum_{t=0}^{\infty} \delta_\nu^t \left(u_i\left(\bar{\pi}^{(j),t}(\bar{f}) \right) \right) \right\| < \frac{\nu}{6}. \tag{4.13}$$

$$\left\| \sum_{t=0}^{\infty} \mathrm{E}_0\left(\delta_0^t\right) \left(u_i\left(a^* \right) \right) - \sum_{t=0}^{\infty} \delta_\nu^t \left(u_i\left(a^* \right) \right) \right\| < \frac{\nu}{6}. \tag{4.14}$$

Condition 4.11 holds trivially. On the other hand, condition 4.12 holds because $\bar{f}$ is strict enforceable at δ, and the monotonicity result, Theorem 6 of Abreu, Pearce, and Stachetti (1990), implies that $\bar{f} \in SPE(\bar{G}(\delta'))$ for any $\delta' \geq \delta$. Indeed, it can easily be verified (by using the same techniques of the proof of this result) that $\bar{f}$ is also strictly enforceable at $\delta' \geq \delta$. Furthermore, since p_ν can be selected arbitrarily close to 0, the associated slack (the left hand side of condition 4.12, which converges to ν^* when p_ν tends to 0) can be chosen to strictly exceed ν. Moreover, conditions 4.13 and 4.14 are due to the following: Observe that for any process satisfying Assumption 2 with $\mathbf{d}_0 = \delta_\nu$, the fourth part of Lemma 1 and the Sandwich Lemma directly imply that as δ_ν tends to 1, $\mathrm{E}\left(\delta_t^\tau | \mathcal{F}_0\right)$ tends to $\left(\delta_\nu\right)^{\tau - t}$ for all $\tau, t \in \mathbb{N}_0$ with $\tau \geq t$. It is important to point out that because p_ν can be selected arbitrarily small, all these conditions, 4.11 – 4.14, keep holding when they are evaluated at p_ν and $\hat{\delta} > \delta_\nu$ and $\mathbf{d}_0 = \hat{\delta}$.

Furthermore, observe that since $\{\mathbf{d}_t\}_t$ is a non–negative bounded martingale, $\{\mathbf{e}_t\}_t$ defined by $\mathbf{e}_t \equiv (1 - \mathbf{d}_t)$ for all $t \in \mathbb{N}_0$ is also a non-negative, bounded martingale. Using Doob's Maximal Inequality (we refer the reader to Doob (1984)) [4] for this martingale we obtain for any $t < T$ and $\bar{\delta} \in (0,1)$

$$
\begin{aligned}
(1-\bar{\delta})\Pr\left[\sup_{t\le s\le T}(1-\delta_s^{s+1}) \ge (1-\bar{\delta}) \,\middle|\, \mathcal{F}_t\right] &\le \mathrm{E}\left((1-\delta_T^{T+1})|\mathcal{F}_t\right)\\
(1-\bar{\delta})\Pr\left[\inf_{t\le s\le T}\delta_s^{s+1} \le \bar{\delta} \,\middle|\, \mathcal{F}_t\right] &\le \mathrm{E}\left((1-\delta_T^{T+1})|\mathcal{F}_t\right)\\
\Pr\left[\inf_{t\le s\le T}\delta_s^{s+1} \le \bar{\delta} \,\middle|\, \mathcal{F}_t\right] &\le \frac{\mathrm{E}\left((1-\delta_T^{T+1})|\mathcal{F}_t\right)}{(1-\bar{\delta})}.
\end{aligned}
$$

Moreover, since $\{\mathbf{d}_t\}_t$ is a martingale the right hand side of the above condition is constant for all $T \in \mathbb{N}_0$ and $t < T$, i.e.

$$\Pr\left[\inf_{t\le s\le T}\delta_s^{s+1} \le \bar{\delta} \,\middle|\, \mathcal{F}_t\right] \le \frac{1-\delta_t^{t+1}}{1-\bar{\delta}}. \tag{4.15}$$

In the following we will inductively construct the set of states in which the strategy that we will employ in the game with stochastic discounting, would prescribe the play to continue following $\bar{f}$. Consider $\mathbf{d}_0 > \delta_\nu$, and recall that it is deterministic. Then, let $\bar{\delta}_{(1)}$ be such that $\bar{\delta}_{(1)} \ge \delta_\nu$ and

$$\frac{1-\mathbf{d}_0}{1-\bar{\delta}_{(1)}} \le p_\nu,$$

and define

$$\Omega_{(1)}^{\nu} = \left\{\delta \in \Omega : \delta > \delta_\nu \text{ and } \frac{1-\delta}{1-\bar{\delta}_{(1)}} \le p_\nu\right\}.$$

Now, given $\bar{\delta}_{(t-1)}$ and $\Omega_{(t-1)}^{\nu}$, define $\bar{\delta}_{(t)}$ such that $\bar{\delta}_{(t)} \ge \delta_\nu$ and for any $\delta \in$

[4]Doob's Maximal Inequality for nonnegative submartingales is as follows: Let $\{\mathbf{X}_t\}_{t\in\mathbb{N}_0}$ be a nonnegative submartingale with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ and $\ell > 0$. Then for any T for any $s < T$, $\ell\Pr\left[\sup_{s\le t\le T} X_t \ge \ell \middle| \mathcal{F}_s\right] \le \mathrm{E}\left(X_T|\mathcal{F}_s\right)$.

$\Omega^{\nu}_{(t-1)}$,

$$\frac{1-\delta}{1-\bar{\delta}_{(t)}} \leq p_{\nu},$$

and let

$$\Omega^{\nu}_{(t)} = \left\{ \delta \in \Omega : \delta > \delta_{\nu} \text{ and } \frac{1-\delta}{1-\bar{\delta}_{(t)}} \leq p_{\nu} \right\}.$$

Notice that for any history h, with $\delta_t^{t+1} \in \Omega^{\nu}_{(t)}$, it must be that δ_t^{t+1} is not only strictly above δ_{ν}, but also, the probability of any one of the future one-shot discount factors being less than or equal to $\bar{\delta}_{(t)}$ is less than or equal to p_{ν}. An important observation is that when $\mathbf{d}_0$ is chosen sufficiently high, then $\Omega^{\nu}_{(t)} \neq \emptyset$ for all $t \in \mathbb{N}$. This follows from the denseness of Ω^E (the ergodic set of states) in Ω following the fourth part of Assumption 2.

The strategy we use as follows: For any history $h = (\bar{h}, d^t)$ for some $t \in \mathbb{N}_0$ with $\ell(h) = \ell(\bar{h}) = t$

$$f(h) = \begin{cases} \bar{f}(\bar{h}) & \text{if } \delta_s^{s+1} \in \Omega^{\nu}_{(s)} \text{ for all } s \leq t \\ a^* & \text{otherwise.} \end{cases} \tag{4.16}$$

In words, this strategy prescribes the continuation along the simple strategy $\bar{f}$ whenever the history is one in which the following hold: In any period t, all realizations of one-shot discount factors up to period t, δ_s^{s+1} with $s \leq t$, have been such that (1) each one of them is strictly above δ_{ν}, and (2) the probability evaluated with date s information of any one of the future one-shot discount factors, δ_k^{k+1} with $k \geq s$, being less than or equal to $\bar{\delta}_{(s)}$ is less than or equal to p_{ν}, $s \leq t$. For all other cases, the strategy prescribes the repetitions of the Nash action profile of the stage game. An interesting observation about this strategy f is that it induces the play to be in only $(n+2)$ states, namely, $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)}, \pi^*)$, where $\pi^{*,t} = a^*$ for all $t \in \mathbb{N}_0$.

Clearly, this strategy is well defined.

Consider any history h. Below, we will prove that when $\mathbf{d}_0 = \hat{\delta}$ is chosen sufficiently high, f is Nash in the subgame starting at h, hence, subgame perfect.

If $\delta_s^{s+1} \notin \Omega_{(s)}^{\nu}$ for some $s \leq t$, f recommends the repetition of a^* thereafter. Hence, is clearly Nash in such subgames.

If $\delta_s^{s+1} \in \Omega_{(s)}^{\nu}$ for all $s \leq t$, f recommends the continuation of the simple strategy given by $\bar{f}$, which is associated with $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$. Then, the continuation utility, equation 4.4, can be written as follows:

$$V_i^{t,d^t}(f, \{\mathbf{d}_t\}_t) = (1-\hat{\delta}) \sum_{k=t}^{\infty} \mathrm{E}_t\left(\delta_t^k\right) \left(u_i\left(\pi^k(f)(d^k)\right)\left(1-\rho_k^{(t)}\right) + u_i\left(a^*\right)\rho_k^{(t)}\right), \tag{4.17}$$

where $\mathbf{d}_0 = \hat{\delta}$ and for any $k \geq t$,

$$\rho_k^{(t)} = 1 - \Pr\left[\delta_s^{s+1} \in \Omega_{(s)}^{\nu}, \text{ for all } s \text{ with } t \leq s \leq k \,\middle|\, \mathcal{F}_t\right] \leq p_\nu.$$

Notice that, given h, hence a^t, $(\pi^k(f)(d^k))$ is equal to some $(\bar{\pi}^{(j),\kappa})$ for some $j \in N \cup \{0\}$ and κ, whenever $\delta_s^{s+1} \in \Omega_{(s)}^{\nu}$ for all s with $t \leq s \leq k$, an event which happens with probability $1 - \rho_k^{(t)}$. That is, in such cases the play must be in some phase of $\bar{\pi}^{(j)}$ for some $j \in N \cup \{0\}$.

Observe that for any process satisfying Assumption 2 (specifically the Markov property) with $\mathbf{d}_0 = \hat{\delta} > \delta_\nu$, condition 4.13 directly implies (recall that the history is such that $\delta_t^{t+1} > \delta_\nu$ for all t, and $\mathrm{E}_t\left(\delta_k^{k+1}\right) = \delta_t^{t+1}$, $k \geq t$)

$$\left\|\sum_{k=t}^{\infty} \mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(\bar{\pi}^k(\bar{f})\right)\right) - \sum_{k=t}^{\infty}\left(\delta_t^{t+1}\right)^k\left(u_i\left(\bar{\pi}^k(\bar{f})\right)\right)\right\| < \frac{\nu}{6}. \tag{4.18}$$

Similarly due to the same reasons, condition 4.14 implies that

$$\left\|\sum_{k=t}^{\infty} \mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(a^*\right)\right) - \sum_{k=t}^{\infty}\left(\delta_t^{t+1}\right)^k\left(u_i\left(a^*\right)\right)\right\| < \frac{\nu}{6}. \tag{4.19}$$

Conditions 4.13 and 4.14 together with the fact that $\delta_t^{t+1} > \delta_\nu$ bring about

$$\begin{aligned}\frac{\nu}{3} \quad > \quad & \Bigg\| \delta_t^{t+1}((1-p_\nu)\bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_t^{t+1}) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_t^{t+1})) \\ & - \frac{\delta_t^{t+1}}{1-\hat{\delta}}\left((1-p_\nu)\sum_{k=t}^{\infty}\mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(\bar{\pi}^{(j),k}(\bar{f})\right)\right) + p_\nu \sum_{k=t}^{\infty}\mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(a^*\right)\right)\right)\Bigg\|.\end{aligned}$$

Now, using condition 4.11 we obtain

$$\begin{aligned}\frac{\nu}{2} \quad > \quad & \Bigg\| \delta_t^{t+1}((1-p_\nu)\bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_t^{t+1}) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_t^{t+1})) \\ & - \frac{\delta_t^{t+1}}{1-\hat{\delta}}\left((1-\rho_{t+1}^{(t)})\sum_{k=t}^{\infty}\mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(\bar{\pi}^{(j),k}(\bar{f})\right)\right) + \rho_{t+1}^{(t)} \sum_{k=t}^{\infty}\mathrm{E}_t\left(\delta_t^k\right)\left(u_i\left(a^*\right)\right)\right)\Bigg\| \\ = \quad & \Bigg\| \delta_t^{t+1}((1-p_\nu)\bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_t^{t+1}) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_t^{t+1})) \\ & - \delta_t^{t+1}\left((1-\rho_{t+1}^{(t)})V_i^{t+1,d^{t+1}}(f, \{\mathbf{d}_t\}_t) + \rho_{t+1}^{(t)} V_i^{t+1,d^{t+1}}(a^*, \{\mathbf{d}_t\}_t)\right)\Bigg\|\end{aligned}$$

delivering

$$\begin{aligned}\frac{\nu}{2} \quad > \quad & \Big\|(1-\hat{\delta})u_i(\bar{\pi}^{(j),t}) + \delta_t^{t+1}((1-p_\nu)\bar{V}_i^{t+1}(\bar{\pi}^{(j)}, \delta_t^{t+1}) + p_\nu \bar{V}_i^{t+1}(a^*, \delta_t^{t+1})) \\ & - V_i^{t,d^t}(f, \{\mathbf{d}_t\}_t)\Big\|,\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\frac{\nu}{2} \quad > \quad & \Bigg\|(1-\hat{\delta})\max_{a_i\in A_i} u_i(a_i,\bar{\pi}_{-i}^{(j),t}) + \delta_t^{t+1}((1-p_\nu)\bar{V}_i^{t+1}(\bar{\pi}^{(i)},\delta_t^{t+1}) + p_\nu \bar{V}_i^{t+1}(a^*,\delta_t^{t+1})) \\ & -\Bigg((1-\hat{\delta})\max_{a_i\in A_i} u_i(a_i,\bar{\pi}_{-i}^{(j),t}) \\ & +\delta_t^{t+1}\left((1-\rho_{t+1}^{(t)})V_i^{t+1,d^{t+1}}(f,\{\mathbf{d}_t\}_t) + \rho_{t+1}^{(t)}V_i^{t+1,d^{t+1}}(a^*,\{\mathbf{d}_t\}_t)\right)\Bigg)\Bigg\|, \quad (4.21)\end{aligned}$$

where $V_i^{t+1,d^{t+1}}(f,\{\mathbf{d}_t\}_t)$ in condition 4.21, is the continuation payoff of player i's punishment path in the stochastic game when $\delta_{t+1}^{t+2} \in \Omega_{(t+1)}^{\nu}$ (otherwise, player i's deviation is followed by the repetitions of the Nash action).

Conditions 4.12, and conditions 4.20 and 4.21 together imply that

$$\begin{aligned}& V_i^{t,d^t}(f,\{\mathbf{d}_t\}_t) - \Bigg((1-\hat{\delta})\max_{a_i\in A_i} u_i(a_i,\bar{\pi}_{-i}^{(j),t}) \\ & +\delta_t^{t+1}\left((1-\rho_{t+1}^{(t)})V_i^{t+1,d^{t+1}}(f,\{\mathbf{d}_t\}_t) + \rho_{t+1}^{(t)}V_i^{t+1,d^{t+1}}(a^*,\{\mathbf{d}_t\}_t)\right)\Bigg) \\ & > \nu - \frac{\nu}{2} - \frac{\nu}{2} = 0,\end{aligned}$$

showing that f is Nash in every subgame that starts with h such that $\delta_s^{s+1} \in \Omega_{(s)}^{\nu}$ for all $s \le t$ with $\ell(h) = t$.

Thus, f is subgame perfect.

Choose $\mathbf{d}_0 = \hat{\delta} > \delta_\nu$ such that $\hat{\delta} \in \Omega_{(1)}^{\nu}$. Then, conditions 4.11, and 4.13 and 4.14 imply

$$\left\| V_i^{0,d^0}(f,\{\mathbf{d}_t\}_t) - (1-\hat{\delta})\sum_{t=0}^{\infty} \mathrm{E}_0\left(\delta_0^t\right)\left(u_i\left(\bar{\pi}^t(\bar{f})\right)\right)\right\| < \frac{\nu}{2},$$

and

$$\left\|(1-\hat{\delta})\sum_{t=0}^{\infty}\mathrm{E}_0\left(\delta_0^t\right)\left(u_i\left(\bar{\pi}^t(\bar{f})\right)\right)-(1-\hat{\delta})\sum_{t=0}^{\infty}\hat{\delta}^t\left(u_i\left(\bar{\pi}^t(\bar{f})\right)\right)\right\|<\frac{\nu}{6}.$$

These, in turn, finishes the proof because of the following conclusion

$$\left\|U\left(f,\{\mathbf{d_t}\}_t\right)-\bar{U}\left(\bar{f},\hat{\delta}\right)\right\|=\left\|V_i^{0,d^0}(f,\{\mathbf{d}_t\}_t)-\bar{U}\left(\bar{f},\hat{\delta}\right)\right\|<\frac{4}{6}\nu<\nu<\eta.$$

□

Now, we are ready to present the proof of our subgame perfect Folk Theorem for repeated games with stochastic discounting:

Proof of Theorem 2. The proof of the Folk Theorem of Fudenberg and Maskin (1991) shows that for any $u \in \mathcal{U}^0$, there exists some $\bar{\delta} \in (0,1)$ and a strictly enforceable simple strategy $\bar{f}$ in $\bar{G}(\bar{\delta})$ such that for all $\delta \in (\bar{\delta}, 1)$, $\bar{U}(\bar{f}, \delta) = u$.

This follows from considering conditions $2-4$ and 8 in their proof, which guarantee that each phase of play (which they denote A for the equilibrium, B_j for the minmax and C_j for the reward phases of $j \in N$) they consider satisfies incentive conditions strictly. Additionally, their strategy becomes simple and strictly enforceable when attention is restricted to obtaining individually rational payoffs constructed with the pure strategy minmax.

Hence, Lemma 4 applies and delivers the conclusion that for all $\eta > 0$ there exists $\delta^* \in (\bar{\delta}, 1)$ such that for all $\hat{\delta} > \delta^*$ there is f in $G(\{\mathbf{d}_t\}_t)$ with $\mathbf{d}_0 = \hat{\delta}$, such that f is subgame perfect in $G(\{\mathbf{d}_t\}_t)$ and $\left\|U\left(f,\{\mathbf{d}_t\}_t\right)-\bar{U}\left(\bar{f},\hat{\delta}\right)\right\|<\eta$. Thus, letting $\eta \leq \varepsilon$, renders the desired conclusion. □

Chapter 5

Conclusions

In this thesis, we consider a wide class of games with stochastic discounting with rich state spaces, when the discounting process is not independent of the past. In such a setting, we imposed the restriction that players expectation of the future discount factor is equal to the current one, and only the current value is relevant when trying to make assertions about the future values of the discount factor. Under this construction, we not only proved a Folk Theorem for repeated games with stochastic discounting but we also, showed that no matter how patient players are every subgame perfect equilibrium path must entail arbitrarily (yet finite) consecutive repetitions of period Nash behaviour, and these consecutive periods almost surely happen in a finite time window.

The reason why these seemingly contradictory results appear together is about the timing. The Folk Theorem involves complete contingent plans of actions, drafted at the beginning of the game, and the expected results from such plans again calculated at the beginning of the game. The inevitability result on the other hand, concerns observations about the behavior of the stochastic discounting process in the far, yet almost surely finite future.

It is well worth to investigate this construction with weaker assumptions on the stochastic process, namely reducing the martingale requirement to a submartingale requirement. Under such a construction, the evaluation of the continuation values becomes more problematic as the compactness of the continuation values are hard to ascertain. Moreover, the inevitability result becomes even more complicated, because we have to consider same stopping time in the martingale case as a moving boundary in the submartingale case. Our initial findings suggest that when the submartingale is generated from a markovian martingale via a convex transformation, the same stopping time in the martingale case can be found as a moving boundary however, the almost sureness can not be guaranteed and is dependent on both the initial martingale, and the transformation used.

Relating our work to the existing literature is also essential in understanding our contribution. There is a number of notable contributions in the context of stochastic games. Indeed, recent studies by Fudenberg and Yamamato (2010) and Hörner, Sugaya, Takahashi, and Vieille (2010) generalize the Folk Theorem of Dutta (1995) for irreducible stochastic games with the requirement of a finite state space. Nevertheless, the setting we employ can be perceived as a specific form of an irreducible stochastic game with perfect information, with an *infinite* state space. Even tough Folk Theorems for irreducible stochastic games can be found in the literature, to our knowledge, ours is the only one with an infinitely rich state space. [1]

As a further research question one can look at the case when every player has an identically and independently(with other players) distributed stochastic discounting process. It is interesting to try to find analogies between deterministically discounted repeated games and stochastically discounted repeated games, and even just characterizing the set of subgame perfect strategies in such a setting constitutes an avenue for future research.

[1] It is essential to note that the studies mentioned have less stringent requirements than ours beside the richness of the state space.

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