Single-Leg Airline Revenue Management With Overbooking

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ABSTRACT: Airline revenue management is about identifying the maximum revenue seat allocation policies. Since a major loss in revenue results from cancellations and no-show passengers, over the years overbooking has received a significant attention in the literature. In this study, we propose new models for static and dynamic single-leg overbooking problems. In the static case, we introduce computationally tractable models that give upper and lower bounds for the optimal expected revenue. In the dynamic case, we propose a new dynamic programming model, which is based on two streams of arrivals. The first stream corresponds to the booking requests and the second stream represents the cancellations. We also conduct simulation experiments to illustrate the proposed models and the solution methods.

Keywords: Revenue management; airline; overbooking; cancellation; static model; dynamic model; dynamic programming; simulation

1. Introduction. Airline revenue management (ARM) deals with effective strategies that determine the allocation of seats in an airplane to different fare-classes over time so that the total revenue is maximized. Due to the increasing competition, ARM has become an important tool for airline companies. Capacity allocation and overbooking are the two main strategies used by revenue management specialists. While capacity allocation deals with reserving seats to fare-classes, overbooking is concerned with the number of additional booking requests (demand) to be allowed above the physical capacity. It is quite common that a certain percentage of the customers with reservations do not show-up at the departure time (no-shows). Consequently, the capacity becomes available for the overbooked passengers. On the other hand, it may also happen that some of the customers could not embark because of lack of capacity at the departure time. In such a case, the airline faces penalties like monetary compensations and loss of good will. Even though the overbooking decision involves uncertainties regarding no-shows and cancellations, booking more seats than the available capacity is still a commonly-used, profitable strategy because the revenue collected by overbooking usually exceeds the losses due to monetary/non-monetary penalties.

The overbooking limit, which is the maximum number of seats the airline company is willing to overbook, is added as a pad to the physical capacity of the plane to obtain the total booking limit. The individual allocations of the total booking limit to different fare-classes are referred to as the booking limits. Thus, to determine the booking limit for a fare-class, one needs to take into account the overbooking limit. This implies that the capacity allocation and overbooking decisions are closely related when booking requests are to be accepted or denied. Therefore, due to the correlated random demands for each fare-class as well as the uncertain no-shows and cancellations of reserved seats, the problem of determining the optimal booking limits for different fare-classes is in general difficult to solve. In the subsequent discussion, we refer to the joint capacity allocation and overbooking problem simply as the overbooking problem. Although several overbooking problems have been proposed over the years, these models generally make some simplifying assumptions. Hence, relaxing some of these assumptions may lead to more realistic models that would enrich the literature and motivate the practitioners.

In practice many airline companies are interested in managing their revenues over a network of flights. However, solving single-leg problems is still crucial because (i) the network based airline seat allocation problems are extremely difficult to solve, and hence, methods that require solving a series of single-leg problems are implemented; (ii) some small airline companies, like charter flight companies commonly seen in Europe, have special one-hub networks with single legs. Therefore, for those companies revenue management over the network boils down to solving many single-leg problems.

In this paper, we propose new mathematical programming models for static and dynamic single-leg problems that cover overbooking and cancellations. In a static model one does not consider the dynamics of the (random) processes of customer arrivals, cancellations and no-shows over time. On the other hand, a dynamic model accounts for the behavior of the system over time.

Since 1950's several approaches for both static and dynamic overbooking problems have been proposed in the literature. The first scientific work on overbooking is proposed by Beckman [2] in 1958. Beckman works on the single-leg one fare-class problem. He presents a simple static overbooking model, which determines the overbooking limit by balancing the lost revenue due to empty seats with the cost of denied bookings. A more implementable model is studied by Thompson [18]. In this work, Thompson entirely ignores the probability distribution of demand and requires only the data on cancellation proportions (rates). Given the capacities for two fare-classes, he aims at determining the overbooking amount per fare-class so that the probability of overbooking equals a prespecified value. His work has been examined and refined by Taylor [17] as well as by Rothstein and Stone [15]. Rothstein [14] also presents a study on the history of overbooking in the airline industry. A later work on the static overbooking problem is published by Bodily and Pfeifer [3]. They give optimal decision rules for overbooking in a single fare-class problem. As in the model proposed by Beckman, they discuss a trade-off between the number of empty seats and the number of denied customers.

Chi [5] formulates a multi-class static overbooking problem as a dynamic programming model. Given the flight capacity, the fares and the distributions of the demand, he finds the maximum number of seats allowed for the cheapest open fare-class (an open fare-class denotes that there is at least one seat available for this fare-class on that flight). In this model, it is assumed that the demand for the cheapest fare occurs first, and the booking for a class starts whenever all the bookings are made for the cheaper classes. As a direct consequence of this setup, the fare-classes constitute the stages of a dynamic programming model. To simplify the model, he also assumes that the reservations can be canceled without any penalty. He then proposes an approximate algorithm as a solution method. Coughlan [6] has also studied the overbooking problem in the multi-class case. In this work, the last minute passengers are considered and the seats empty at the departure time are allocated to these passengers at the same price. Such passengers, who show up without any booking at the departure time, are referred to as go-shows. Coughlan [6] presumes that the demand and the number of bookings for each fare-class are independent and also makes the simplifying assumption that both are normally distributed. Moreover, he assumes that the minimum of the demand and the number of bookings are also independently normally distributed. However, the latter assumption is mathematically incorrect and in the literature it is common to assume that the demand follows a Poisson distribution (see, for instance, [1] and [16]). Furthermore, Coughlan assumes that the number of seats allocated to go-shows in any fare-class is independent of the number of show-ups (passengers with reservations that actually embark) in that class. These assumptions may not be valid in practice, since the number of show-ups limits the number of seats allocated to go-shows. However, these assumptions enable him to derive a complicated closed form revenue function and as a solution method, he proposes two direct search methods with no optimality guarantee.

Several researchers have addressed dynamic overbooking models for single-leg problems. Generally, the dynamic overbooking problem is modeled as a Markov Decision Process (MDP). Rothstein [13], Alstrup et al.[1] and Subramanian et al.[16] are three examples that use such an approach. Rothstein [13] has formulated the single fare-class overbooking problem and constructed a general model for determining the overbooking policies. The number of reservations is the state space of the system, and the system changes the state space according to time-dependent transition probabilities. In order to simplify the model, he assumes that cancellation probabilities are independent of the number of already booked seats. On the other hand, Alstrup et al.[1] have developed a dynamic-programming approach to solve an overbooking model with two fare-classes by extending the work of Rothstein. The objective is to determine the optimal allocation of seats so that the total loss is minimized. Here the loss is defined as the maximal attainable gain for a flight minus the actual gain. Different than Rothstein, they also consider the cost of assigning the passengers requesting more expensive fare-class seats to the cheaper seats (downgrading). As a solution method, a two dimensional stochastic dynamic programming has been used. However, their dynamic programming treatment of overbooking grows exponentially in size and becomes computationally intractable for real-world problems.

Subramanian et al.[16] formulate the multi-class overbooking problem as a discrete-time MDP. Although they propose a model with class dependent cancellations and no-shows, their model can only be applied to small-sized problems. Due to this computational intractability, they also propose a model with a one dimensional state space. It is assumed that only an arrival of a booking request, a cancellation or a null event can be realized at each stage. Furthermore, cancellation, no-arrival and arrival probabilities of booking requests are assumed to depend on the number of already reserved seats. However, one may easily argue that it is more realistic when arrival and no-arrival probabilities of booking requests are independent of the number of reserved seats. Chatwin [4] formulates the problem as a birth-and-death process and proposes two models. While he ignores refunds and no-show penalties in his first model, in his second model he allows refunds and fares vary over time. He assumes that customers cancel their reservations independently according to an exponential distribution with a common rate, and the number of booking requests depends on the number of current bookings like in Subramanian et al. In a closely related study, Feng and Xiao [9] take into account fare-dependent no-show rates and refunds, but do not consider cancellations.

Karaesmen and Van Ryzin [11] examine the overbooking problem differently. Their model permits that fare-classes can substitute for one another. They formulate the overbooking problem as a two-period problem, where reservations are made in the first period based on the probabilistic information of cancellations. In the second period, after observing the cancellations and no-shows, all the remaining customers are either assigned to a reserved seat or denied by considering the substitution options. The second period allocation problem is modeled as a network flow problem. In this formulation, they assume that the service provider decides upon the allocation with the perfect knowledge of the number of show-ups in each class, and they propose a stochastic gradient algorithm.

In our study, we develop new overbooking models and associated solution methods for static and dynamic single-leg problems that incorporate no-shows and cancellations. In the proposed models, we relax some of the common simplifying assumptions in the literature that are reviewed above. As a summary, our contributions can be listed as follows:

- ◇ **Static case:** In the static case we introduce two alternative models, which require different types of demand information. Depending on the availability of the demand data, the decision maker can select one of the proposed models. Due to the (unknown) correlation between the individual fare-class demands, the revenue function in the second model is difficult to evaluate, and consequently, the corresponding optimization problem cannot be solved efficiently. As a remedy, we introduce two additional models, which are easy to solve and they yield a lower and an upper bound on the optimal expected revenue. To obtain more realistic models, the refund amounts as well as the no-show and cancellation probabilities within the models are considered to be fare-class-dependent.
- ◇ **Dynamic case:** We propose a new dynamic model based on two arrival streams; arrivals of cancellations and booking requests. In the dynamic model, the booking requests are assumed to be independent of the number of seats already booked, whereas the cancellation and no-show probabilities are assumed to depend on the total number of already booked seats.
- ◇ In both cases, the cancellation and no-show probabilities are distinguished and treated separately within the models.

The rest of the paper is organized as follows. In Section 2, we introduce our notation and present our proposed models for static and dynamic overbooking problems. We present the computational results in Section 3 and conclude the paper in Section 4.

2. Proposed Mathematical Models. In this section, we introduce and analyze our proposed models for the static and dynamic overbooking problems. The overbooking problem can be defined as follows: Consider a flight with a known seat capacity C. The airline operator may overbook passengers from m different fare-classes up to a total booking limit, for which the upper bound is denoted by C'. Passengers, who already booked a seat, may cancel at any time before departure or do not show up without cancelling (no-shows). In case of cancellation, the customer receives a refund. In static models, the airline company refunds a fare-class i passenger a percentage $\alpha_i, 1 \leq i \leq m$, of the corresponding fare class i ticket price r_i. Without loss of generality it is assumed that $0 < r_1 < r_2 < \cdots < r_m$ and $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_m$. In the dynamic model, we assume that the refund amount is fixed and class independent. The objective is to determine the overbooking limit and the allocation of the resulting capacity to the different fare-classes in such a way that the expected revenue is maximized. In Section 3.1 we consider two different static models, while Section 3.2 is devoted to our dynamic model.

Here we introduce some notation. The random variables and vectors are denoted by uppercase and lowercase boldface letters, respectively. If $\mathbf{X}$ and $\mathbf{Y}$ are random variables, then $\mathbf{X} =^d \mathbf{Y}$ means that the cumulative distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ are the same.

2.1 Static Overbooking Problem. Static models focus on determining the overbooking limit at the beginning of a reservation period. In this section we first consider a basic model with only the *random total demand* for seats in an airplane and a Bernoulli selection mechanism allocating the total number of reserved seats to the different fare-classes. Then, we introduce the second static model with random demand for each fare-class. Hence, the first model determines just the total booking limit, while the second one also finds the individual booking limits for the fare-classes. The general parameters used in the static models are summarized in Table 1.

Table 1: The general parameters used for the static models

C:	Capacity
C':	Upper bound on total booking limit
m:	Number of fare-classes
n:	Total booking limit
n_i:	Booking limit for fare-class i
r_i:	Ticket price of fare-class i
s:	Unit overbooking penalty cost
β_i:	Show-up probability of fare-class i
δ_i:	Cancellation probability of fare-class i
α_i:	Refund percentage for fare-class i
p_i:	Probability that a reserved seat is a fare-class i seat

2.1.1 Total Booking Limit. In this model, we try to determine the total booking limit via finding the optimal overbooking limit. Let $\mathbf{D}$ denote the random total demand for seats in the airplane and L be the overbooking limit. Then, the total number of seats that can be booked equals $n := C + L \geq C$, which is simply the total booking limit. Thus, the (random) total number of reserved seats on the selected flight is given by $\mathbf{N}(n) := \min\{n, \mathbf{D}\}$. To model the allocation of the revenue over different fare-classes, we assume that a reserved seat is a fare-class i seat with probability p_i, $1 \leq i \leq m$. Clearly $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$. This shows that $\mathbf{B}(p_i, \mathbf{N}(n))$ is the random number of reserved fare-class i seats before the departure, and $r_i\mathbf{B}(p_i, \mathbf{N}(n))$ is the associated random revenue (consult Appendix A for an introduction of the Bernoulli selection scheme). If β_i denotes the show-up probability of each customer having a reserved fare-class i seat then this implies by relation (32) that $\mathbf{B}(\beta_i p_i, \mathbf{N}(n))$ is the total random number of fare-class i customers showing up at departure. When a customer does not show-up, there are two possible cases; either the customer cancels the reservation in advance and receives the refund based on the fare-class, or the customer does not show-up without cancelling the reservation and then no refund is given. Let us denote the probability of cancelling a reserved fare-class i seat by $\delta_i, 1 \leq i \leq m$. Thus, $(1 - \beta_i)\delta_i$ is the probability of giving a refund to a fare-class i customer and again by relation (32), the total random number of fare-class i cancellations is $\mathbf{B}((1 - \beta_i)\delta_i p_i, \mathbf{N}(n))$. Similarly, we obtain that the total random number of no-show fare-class i customers is $\mathbf{B}((1 - \beta_i)(1 - \delta_i)p_i, \mathbf{N}(n))$. Hence, the total random revenue generated by fare-class i customers at the departure time of the airplane is given by

$$r_i\mathbf{B}(p_i, \mathbf{N}(n)) - \alpha_i r_i \mathbf{B}((1 - \beta_i)\delta_i p_i, \mathbf{N}(n)).$$

Recall here that α_i denotes the fraction of the price refunded for a cancelled fare-class i ticket. Let us define

$$\tau_i = r_i(1 - \alpha_i(1 - \beta_i)\delta_i),\ 1 \leq i \leq m, \tag{1}$$

then by relation (30) this yields that $p_i\tau_i\mathbb{E}(\mathbf{N}(n))$ is the expected revenue of fare-class i customers. This shows that the expected revenue over all fare-class i customers equals

$$\sum\nolimits_{i=1}^m p_i\tau_i\mathbb{E}(\mathbf{N}(n)), \tag{2}$$

where $\theta_0 := \sum_{i=1}^m p_i\tau_i > 0$. To incorporate the penalty cost of overbooking, we first observe adding up over all fare-classes that the total number of overbooked seats equals

$$\max\left\{\sum\nolimits_{i=1}^m \mathbf{B}(\beta_i p_i, \mathbf{N}(n)) - C, 0\right\}. \tag{3}$$

Since $\sum_{i=1}^m \mathbf{B}(p_i, \mathbf{X}) =^d \mathbf{B}(\sum_{i=1}^m p_i, \mathbf{X})$, we have

$$\max\{\sum\nolimits_{i=1}^m \mathbf{B}(\beta_i p_i, \mathbf{N}(n)) - C, 0\} =^d \max\left\{\mathbf{B}\left(\sum\nolimits_{i=1}^m \beta_i p_i, \mathbf{N}(n)\right) - C, 0\right\}. \tag{4}$$

If s denotes the penalty cost of an overbooking, then by relations (3) and (4) the expected overbooking costs is given by

$$s\mathbb{E}\left(\max\left\{\mathbf{B}\left(\sum\nolimits_{i=1}^m \beta_i p_i, \mathbf{N}(n)\right) - C, 0\right\}\right). \tag{5}$$

Adding the costs in relations (2) and (5) we finally obtain that the expected revenue is given by

$$\phi(n) := \sum\nolimits_{i=1}^m p_i\tau_i\mathbb{E}(\mathbf{N}(n)) - s\mathbb{E}\left(\max\left\{\mathbf{B}\left(\sum\nolimits_{i=1}^m \beta_i p_i, \mathbf{N}(n)\right) - C, 0\right\}\right). \tag{6}$$

Hence, the optimal total booking limit is found by solving

$$\max\{\phi(n) : n \geq C, n \in \mathbb{Z}_+\}. \tag{P_T}$$

Analysis of Optimization Problem (P_T). To analyze the optimization problem (P_T), we first rewrite the objective function $\phi(\cdot)$. Since for any non-negative integer K and $0 \leq p \leq 1$

$$\max\{\mathbf{B}(p, K) - C, 0\} + \min\{\mathbf{B}(p, K) - C, 0\} = \mathbf{B}(p, K) - C, \tag{7}$$

we obtain by relation (30) that

$$-s\mathbb{E}(\max\{\mathbf{B}(p, \mathbf{N}(n)) - C, 0\}) = -sp\mathbb{E}(\mathbf{N}(n)) + sC + s\mathbb{E}(\min\{\mathbf{B}(p, \mathbf{N}(n)) - C, 0\}).$$

Introducing now

$$\theta_0 = \sum\nolimits_{i=1}^m p_i\tau_i, \theta_1 = s\sum\nolimits_{i=1}^m p_i\beta_i, \tag{8}$$

the objective function in (6) can be written as $\phi(n) = \mathbb{E}(f(\mathbf{N}(n)))$ with the function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ given by

$$f(x) := sC + (\theta_0 - \theta_1)x + s\mathbb{E}\left(\min\left\{\mathbf{B}\left(\sum\nolimits_{i=1}^m \beta_i p_i, x\right) - C, 0\right\}\right). \tag{9}$$

To analyze the global behavior of this function we consider the following two cases:

(i) $\theta_0 - \theta_1 \geq 0$. To analyze this case we first observe using $\mathbf{B}(p, n+1) \geq \mathbf{B}(p, n)$ that the function

$$x \mapsto \mathbb{E}\left(\min\left\{\mathbf{B}\left(\sum\nolimits_{i=1}^m \beta_i p_i, x\right) - C, 0\right\}\right)$$

is increasing. This shows by relation (9) that the function $f(\cdot)$ is increasing. Hence, by the monotonicity of $n \mapsto \mathbf{N}(n)$ the function $n \mapsto \mathbb{E}f(\mathbf{N}(n))$ is increasing and an optimal solution of our booking problem is to set $n = \infty$. An intuitive interpretation of this result is as follows. Since $(1-\beta_i)\delta_i$ is the probability of giving a refund to a fare-class i customer and α_i is the refund percentage, the expected net revenue per customer, given this customer belongs to fare-class i, is at most equal to $\tau_i - s\beta_i$. Hence, the expected revenue per customer is given by

$$\sum\nolimits_{i=1}^m p_i(\tau_i - s\beta_i) = \theta_0 - \theta_1.$$

This expression being non-negative shows that it is always profitable despite the overbooking costs to accept all booking requests. Thus, the overbooking limit should be set to infinity. Clearly this is a pathological case and will probably never happen in practice. A more reasonable assumption is given by the next case.

(ii) $\theta_0 - \theta_1 < 0$. To analyze this case we observe by Lemma B.3 that the function

$$x \mapsto \mathbb{E}\left(\min\left\{\mathbf{B}\left(\sum_{i=1}^{m} \beta_i p_i, x\right) - C, 0\right\}\right)$$

is discrete concave. Hence, by relation (9) the function $f(\cdot)$ is a discrete concave function in x. Since $\lim_{x\uparrow\infty} f(x) = -\infty$ this shows that the optimization problem $\max\{f(n) : n \geq C, n \in \mathbb{Z}_+\}$ is easy to solve and there exist a finite optimal solution $n_{opt} \geq C$. Applying now Lemma B.4 yields that n_{opt} is also a solution of the optimization problem (P_T). A surprising consequence of this result is that the total booking limit does not depend on the distribution function of the total demand $\mathbf{D}$. To give a procedure to compute the optimal solution under the above consequence we first need to evaluate the function $n \mapsto f(n+1) - f(n)$ for every $n \geq C$ with function f given by relation (9). It follows by relation (9) and (33) that

$$f(n+1) - f(n) = \theta_0 - \theta_1 + \theta_1 \mathbb{E} f_0\left(\mathbf{B}\left(\sum_{i=1}^{m} \beta_i p_i, n\right)\right) \tag{10}$$

with

$$f_0(x) = \min\{x - C + 1, 0\} - \min\{x - C, 0\} = \begin{cases} 1, & \text{if } x \leq C - 1; \\ 0, & \text{otherwise.} \end{cases}$$

This shows for every $n \geq C$ that

$$f(n+1) - f(n) = \theta_0 - \theta_1 + \theta_1 \mathbb{P}\left(\mathbf{B}\left(\sum_{i=1}^{m} \beta_i p_i, n\right) \leq C - 1\right). \tag{11}$$

By our assumption we know that $0 < \theta_0 \theta_1^{-1} < 1$, and this implies by relation (11) that

$$f(n+1) - f(n) \leq 0 \Leftrightarrow \mathbb{P}\left(\mathbf{B}\left(\sum_{i=1}^{m} \beta_i p_i, n\right) \leq C - 1\right) \leq 1 - \theta_0 \theta_1^{-1}.$$

Using the discrete concavity of the function $f(\cdot)$, an optimal solution of our optimization problem is therefore given by

$$n_{opt} = \inf\left\{n \geq C : \mathbb{P}\left(\mathbf{B}\left(\sum_{i=1}^{m} \beta_i p_i, n\right) \leq C - 1\right) \leq 1 - \theta_0 \theta_1^{-1}\right\}. \tag{12}$$

The first static model in the airline revenue management literature was proposed by Beckman [2] and all of the other models have extended his study. Beckman proposed a cost-based static model for a single fare-class including overbooking costs and opportunity costs of empty seats. His model can be expressed more simply using our notation: Let $\mathbf{M}$ be the demand for seats of last minute customers arriving without reservations at the airport and assume that $\mathbf{M}$ is independent of $\mathbf{D}$. Then the random variable $\mathbf{M}$ is also independent of $\mathbf{B}(\beta, \mathbf{N}(n))$, where β denotes the show-up probability of each customer having reserved a seat. If there are still available seats, some of those go-show customers get an empty seat. To model the opportunity cost for empty seats, the random number of empty seats at the departure of the airplane is given by

$$\max\{C - \mathbf{B}(\beta, \mathbf{N}(n)) - \mathbf{M}, 0\}.$$

Hence, if b denotes the cost of an empty seat, the expected total cost, which consists of the expected overbooking costs and the expected opportunity costs of empty seats, is given by

$$\phi_B(n) = s\mathbb{E}(\max\{\mathbf{B}(\beta, \mathbf{N}(n)) - C, 0\}) + b\mathbb{E}(\max\{C - \mathbf{B}(\beta, \mathbf{N}(n)) - \mathbf{M}, 0\}). \tag{13}$$

Consequently, Beckman's optimization problem reduces to

$$\min\{\phi_B(n) : n \geq C, n \in \mathbb{Z}_+\}. \tag{P_B}$$

Converting this into a maximization problem we obtain

$$\min\{\phi_B(n) : n \geq C\} = -\max\{-\phi_B(n) : n \geq C\}.$$

Now by relation (13) the function $-\phi_B(n)$ can be written as

$$-\phi_B(n) = \mathbb{E}(f_B(\mathbf{N}(n)))$$

with the function $f_B : \mathbb{Z}_+ \to \mathbb{R}$ given by

$$f_B(x) := s\mathbb{E}(\min\{C - \mathbf{B}(\beta, x), 0\}) + b\mathbb{E}(\min\{-C + \mathbf{B}(\beta, x) + \mathbf{M}, 0\}).$$

To show that the function $f_B : \mathbb{Z}_+ \to \mathbb{R}$ is discrete concave, we observe using $\mathbf{M}$ being independent of $\mathbf{B}(\beta, x)$ that

$$f_B(x) = \mathbb{E}(h(\mathbf{B}(\beta, x))), \tag{14}$$

where

$$h(n) := s\min\{C - n, 0\} + b\mathbb{E}(\min\{-C + n + \mathbf{M}, 0\}). \tag{15}$$

Since for each realization $\boldsymbol{M}(\omega)$ of the random variable $\mathbf{M}$, the function

$$n \mapsto s\min\{C - n, 0\} + b\min\{-C + n + \mathbf{M}(\omega), 0\}$$

is discrete concave, we obtain that the function $h(\cdot)$ in relation (15) is discrete concave, and hence, by Lemma B.3 the function $f_B(\cdot)$ in relation (14) is discrete concave. This shows by Lemma B.4 that an optimal solution of problem (P_B) can be found by solving

$$\max\{f_B(n) : n \geq C, n \in \mathbb{Z}_+\}.$$

Thus, the random total demand $\mathbf{D}$ for seats does not play any role in the determination of the optimal overbooking limit, and consequently, neither does it in the total booking limit. We can now easily compute the differences $n \mapsto f_B(n+1) - f_B(n)$ and determine the optimal n in problem (P_B). In the next subsection, we consider a more elaborate static model, which takes into account the individual random demand for each fare-class.

2.1.2 Booking Limits for Individual Fare-Classes. In optimization problem (P_T), we only consider the total random demand for seats. Alternatively, in this section, we propose a model incorporating total random demand for each fare-class. We assume that the distributions of total demand for each fare-class is known. Let $\mathbf{D}_i$ denote the random demand for fare-class i, $1 \leq i \leq m$, and n_i be the booking limit for fare-class i such that $C \leq \sum_{i=1}^m n_i \leq C'$. Note that instead of the total booking limit as in the first model, we consider individual booking limits denoted by n_i, $1 \leq i \leq m$. The random variable $\mathbf{N}_i(n_i) = \min\{n_i, \mathbf{D}_i\}$ denotes the number of fare-class i customers having a reserved seat. Since with probability β_i a customer with a reserved fare-class i seat will show up, the random variable $\mathbf{B}(\beta_i, \mathbf{N}_i(n_i))$ denotes the number of occupied fare-class i seats at the departure time of the airplane, while the random number of no-shows within fare-class i is $\mathbf{B}((1-\beta_i)(1-\delta_i), \mathbf{N}_i(n_i))$ and the random number of cancellations is $\mathbf{B}((1-\beta_i)\delta_i, \mathbf{N}_i(n_i))$. Since the total random number of overbookings equals $\max\{\sum_{i=1}^m \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\}$, the random overbooking cost is given by $s\max\{\sum_{i=1}^m \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\}$. Hence, for any feasible booking vector $\mathbf{n} = (n_1, \cdots, n_m)$ the random revenue, denoted by $\Phi(\mathbf{n})$, becomes

$$\Phi(\mathbf{n}) = \sum_{i=1}^m r_i \mathbf{N}_i(n_i) - \sum_{i=1}^m \alpha_i r_i \mathbf{B}((1-\beta_i)\delta_i, \mathbf{N}_i(n_i)) - s\max\left\{\sum_{i=1}^m \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\right\}. \tag{16}$$

Then by (1) and (30), we obtain that the expected revenue for a given $\mathbf{n}$ as

$$\phi(\mathbf{n}) = \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(n_i)) - s\mathbb{E}\left(\max\left\{\sum_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\right\}\right). \tag{17}$$

This shows that we need to solve the optimization problem

$$\max\{\phi(\mathbf{n}) : C \leq \sum_{i=1}^{m} n_i \leq C', n_i \in \mathbb{Z}_+, 1 \leq i \leq m\}.$$

By the definition of the Bernoulli type random variables, it is immediately clear that

$$\mathbb{E}\left(\max\left\{\sum_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\right\}\right) = 0$$

for every $\sum_{i=1}^{m} n_i \leq C$ and the function

$$\mathbf{n} \mapsto \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(n_i))$$

is increasing. Therefore, the above problem reduces to

$$\max\{\phi(\mathbf{n}) : \sum_{i=1}^{m} n_i \leq C', n_i \in \mathbb{Z}_+, 1 \leq i \leq m\}. \tag{P_I}$$

In general the random variables $\mathbf{D}_i$, $1 \leq i \leq m$, and hence, the random variables $\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)), 1 \leq i \leq m$, are correlated. Therefore, it is extremely difficult to compute the expected overbooking costs

$$\mathbf{n} \mapsto s\mathbb{E}\left(\max\left\{\sum_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\right\}\right)$$

for every $\mathbf{n}$ satisfying $\sum_{i=1}^{m} n_i > C$. Besides the correlation issue, the non-separability of the above objective function makes it difficult to solve optimization problem (P_I) in an efficient way. Therefore, we consider upper and lower bounding functions on the expected overbooking costs and develop computationally efficient methods to find approximate optimal solutions for optimization problem (P_I).

Analysis of Optimization Problem (P_I). To compute an upper bounding function on the expected overbooking costs, let $\mathbf{y} = (y_1, \cdots, y_m) \in \mathbb{Z}_+^m$ be a partition of all the available airplane seats into fare-classes. Clearly, $\sum_{i=1}^{m} y_i = C$ and by the subadditivity of the function $x \mapsto \max\{x, 0\}$, we observe that

$$\begin{aligned} \max\{\textstyle\sum_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\} &= \max\{\textstyle\sum_{i=1}^{m} (\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - y_i), 0\} \\ &\leq \textstyle\sum_{i=1}^{m} \max\{\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - y_i, 0\}. \end{aligned}$$

Thus, for any $\sum_{i=1}^{m} y_i = C$, $y_i \in \mathbb{Z}_+$ it follows that

$$\mathbb{E}\left(\max\left\{\sum_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - C, 0\right\}\right) \leq \sum_{i=1}^{m} \mathbb{E}\left(\max\left\{\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - y_i, 0\right\}\right), \tag{18}$$

and by relation (17) we obtain

$$\phi(\mathbf{n}) \geq \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(n_i)) - s\sum_{i=1}^{m} \mathbb{E}\left(\max\{\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - y_i, 0\}\right).$$

Hence, a lower bound on the optimal objective value of problem (P_I) is given by the optimal objective value ($v(P_I^{LB})$)of the problem

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(n_i)) - s\sum_{i=1}^{m} \mathbb{E}\left(\max\{\mathbf{B}(\beta_i, \mathbf{N}_i(n_i)) - y_i, 0\}\right) \\ \text{subject to} & \sum_{i=1}^{m} n_i \leq C', \\ & \sum_{i=1}^{m} y_i = C, \\ & \mathbf{n} \in \mathbb{Z}_+^m, \mathbf{y} \in \mathbb{Z}_+^m. \end{array} \tag{P_I^{LB}}$$

Since the optimization problem (P_I^{LB}) is separable, it can be solved by dynamic programming with a two-dimensional state space, where the stages correspond to the fare-classes. The associated dynamic programming recursion can be formulated as follows: Introduce the functions $\rho_i : \mathbb{Z}_+^2 \mapsto \mathbb{R}$ given by

$$\rho_i(n, y) := \tau_i \mathbb{E}(\mathbf{N}_i(n)) - s\mathbb{E}(\max\{\mathbf{B}(\beta_i, \mathbf{N}_i(n)) - y, 0\}), \tag{19}$$

and consider for $1 \leq i \leq m$, $c \in \{0, 1, \cdots, C'\}$ and $y \in \{0, 1, \cdots, C\}$ the parameterized optimization problems

$$R_p(c, d) = \max\left\{\sum\nolimits_{i=p}^{m} \rho_i(n_i, y_i) : \sum\nolimits_{i=p}^{m} n_i \leq c, \sum\nolimits_{i=p}^{m} y_i = d,\ n_i, y_i \in \mathbb{Z}_+, i = p, \cdots, m\right\}. \tag{20}$$

Clearly, $R_1(C', C)$ equals the optimal objective value of problem (P_I^{LB}) and by relation (20), the bounding condition for $c \in \{0, 1, \cdots, C'\}$ and $d \in \{0, 1, \cdots, C\}$ becomes

$$\begin{aligned} R_m(c, d) &= \max\nolimits_{n_m \in \{0,1,\cdots,c\}} \{\rho_m(n_m, d)\} \\ &= \begin{cases} \max_{n_m \in \{y,\cdots,c\}} \rho_m(n_m, d), & \text{if } c \geq d; \\ \tau_m \mathbb{E}(\mathbf{N}_m(c)), & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, by the dynamic programming optimality principle for separable programs, we obtain for every $1 \leq p \leq m-1$, $c \in \{0, 1, \cdots, C'\}$ and $y \in \{0, 1, \cdots, C\}$ that

$$R_p(c, y) = \max\{\rho_p(n_p, y_p) + R_{p+1}(c - n_p, y - y_p\} : n_p \leq c, y_p \leq y, n_p, y_p \in \mathbb{Z}_+\}, \tag{21}$$

where (c, y) belongs to the set $\{0, \cdots, C'\} \times \{0, \cdots, C\}$.

We remark that the lower bounding problem (P_I^{LB}) has a nice interpretation. The decision maker first determines an allocation of the available airplane seats into fare-classes by setting y_i, $1 \leq i \leq m$ values. Then, the risk she takes is the possibility of observing that the total number of arriving fare-class i customers exceeds the pre-allocated capacity, y_i, in which case she ends up paying penalty costs. This means a penalty is incurred even if a customer occupies a pre-allocated seat belonging to a different fare-class. With this interpretation at hand, it is clear that by solving problem (P_I^{LB}), we obtain a lower bound on the actual expected revenue that would be secured by solving problem (P_I).

To measure the quality of the optimal solution of the approximate optimization problem (P_I^{LB}) with respect to the optimization problem (P_I), we also find an upper bound on the optimal objective function of problem (P_I). By Jensen's inequality and relation (30) it follows that

$$\begin{aligned} \mathbb{E}\left(\max\left\{\sum\nolimits_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(x_i)) - C, 0\right\}\right) &\geq \max\{\mathbb{E}(\sum\nolimits_{i=1}^{m} \mathbf{B}(\beta_i, \mathbf{N}_i(x_i)) - C), 0\} \\ &= \max\left\{\sum\nolimits_{i=1}^{m} \beta_i \mathbb{E}(\mathbf{N}_i(x_i)) - C, 0\right\}. \end{aligned}$$

Therefore, an upper bound on the optimal objective value of problem (P_I) can be obtained by solving the optimization problem

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(n_i)) - s\ \max\left\{\sum_{i=1}^{m} \beta_i \mathbb{E}\left(\mathbf{N}_i(n_i)\right) - C, 0\right\} \\ \text{subject to} & \sum_{i=1}^{m} n_i \leq C', \\ & \mathbf{n} \in \mathbb{Z}_+^m. \end{array} \tag{$\tilde{P}_I^{UB}$}$$

The objective function of this integer nonlinear programming problem is not separable and so the dynamic programming is not applicable in an efficient way. However, due to the special structure of the objective function the above optimization problem ($\tilde{P}_I^{UB}$) can be reformulated as

$$\begin{array}{ll} \text{maximize} & \min\left\{\sum_{i=1}^{m} (\tau_i - s\beta_i)\mathbb{E}(\mathbf{N}_i(n_i)) + sC, \tau_i \mathbb{E}(\mathbf{N}_i(n_i))\right\} \\ \text{subject to} & \sum_{i=1}^{n} n_i \leq C', \\ & \mathbf{n} \in \mathbb{Z}_+^m. \end{array}$$

Then by introducing the set

$$S := \{\mathbf{n} \in \mathbb{Z}_+^m \mid \sum_{i=1}^{m} n_i \leq C'\},$$

the upper bounding problem ($\tilde{P}_I^{UB}$) is rewritten as

$$v(U) = \max_{\mathbf{n}\in S} \min\left\{\sum\nolimits_{i=1}^{m}(\tau_i - s\beta_i)\mathbb{E}(\mathbf{N}_i(n_i)) + sC, \sum\nolimits_{i=1}^{m} \tau\mathbb{E}(\mathbf{N}_i(n_i))\right\}.$$

By weak duality, we then obtain

$$v(U) \leq \min\left\{\max_{\mathbf{n}\in S}\sum\nolimits_{i=1}^{m}(\tau_i - s\beta_i)\mathbb{E}(\mathbf{N}_i(n_i)) + sC, \max_{\mathbf{n}\in S}\sum\nolimits_{i=1}^{m} \tau_i\mathbb{E}(\mathbf{N}_i(n_i))\right\}.$$

Thus, an upper bound on the optimal objective function value of problem (P_I) is obtained by solving

$$v(P_I^{UB}) = \min\left\{\max_{\mathbf{n}\in S}\sum\nolimits_{i=1}^{m}(\tau_i - s\beta_i)\mathbb{E}(\mathbf{N}_i(n_i)) + sC, \max_{\mathbf{n}\in S}\sum\nolimits_{i=1}^{m} \tau_i\mathbb{E}(\mathbf{N}_i(n_i))\right\}. \qquad (P_I^{UB})$$

The two inner problems in the objective function of problem (P_I^{UB}) are separable and they can be easily solved by dynamic programming (or see Birbil et al. [7] for an alternate faster procedure). Due to this computational efficiency, we consider (P_I^{UB}) as the upper bounding problem instead of problem ($\tilde{P}_I^{UB}$).

2.2 Dynamic Overbooking Problem In this section, we introduce a discrete-time dynamic model for the overbooking problem. When we have information about the behavior of the system over time, it is more natural to consider such a model, which allows us to capture the randomness of the input processes. Suppose again that there are m fare-classes with r_i being the price of a fare-class i ticket. As before $0 < r_1 < r_2 < \cdots < r_m$ and for notational convenience the no-sales class is represented by $r_0 := 0$. The reservation horizon, which is the time interval between the opening of the flight for reservations and its departure, is partitioned into T periods. At the beginning of period 1 the reservation process for the flight starts, while at the beginning of period T the flight departs. At the beginning of each period $1 \leq t \leq T-1$ we assume that two streams of independent events occur in the following order: First a possible cancellation of a reserved seat might arrive. By assumption the probability $q_t(n)$ of such an event depends on the total number n of reserved seats at the end of period $t-1$. It is also assumed that these cancellation events are independent over different time periods and that cancellations are refunded a fixed amount κ. After the occurrence of a possible cancellation, we observe the arrival of at most one booking request. Let p_{it} be the probability of a fare-class i, $1 \leq i \leq m$ booking request arriving in period t, $1 \leq t \leq T-1$. We assume that the booking requests are independent over the periods. Denoting the probability of no-arrival of a booking request in period t by p_{0t}, it is obvious that $\sum_{i=0}^{m} p_{it} = 1, p_{it} \geq 0$. Now in each period we decide to accept or reject a possible request. Finally, at the beginning of period T just before the departure of the plane we might observe no-shows. It is assumed that the show-up probability of each reserved seat does not depend on its fare-class and is given by $0 \leq \beta < 1$. The above model is related to the models discussed in [16]. However, in our model we treat the arrival processes of cancellations and booking requests consecutively. Thus, contrary to the models in [16], the probability vector in each period of the booking request arrival process is independent of the number of reserved seats. To give a more detailed mathematical description of the above model, we introduce the independent random vectors $\xi_t = (\xi_{t1}, \xi_{t2}) \in R^2$, $1 \leq t \leq T-1$. The first component, ξ_{t1} represents cancellation or no cancellation, and the second component, ξ_{t2} denotes the price of the arriving request. Then all possible outcomes for $1 \leq t \leq T-1$ and $0 \leq i \leq m$ are as follows: $\xi_t = (0, r_i)$, no cancellation followed by a fare-class i reservation request, and $\xi_t = (1, r_i)$, a cancellation followed by a fare-class i reservation request. The general parameters used in the dynamic model are summarized in Table 2.

Table 2: The general parameters used for the dynamic model

C:	Capacity
C':	Upper bound on total booking limit
T:	Number of time periods
m:	Number of fare-classes
n:	Number of reserved seats
r_i:	Ticket price of fare-class i
s:	Unit overbooking penalty cost
κ:	Fixed refund for a cancellation
β:	Show-up probability
$q_t(n)$:	Cancellation probability at period t when there are n reservations
p_{it}:	Arrival probability of fare-class i at time period t

Recall that C' is the upper bound on the total booking limit. To compute the expected revenue of the optimal policy, let $\mathbf{R}_t(n)$ be the optimal random revenue from period t up to period T if already $n \leq C'$ seats are reserved at the end of period $t-1$ just before an event occurs in period t. Also denote by $J_t(n)$ the expected optimal value function given by $J_t(n) = \mathbb{E}(\mathbf{R}_t(n))$. Clearly, for $t = T$ we obtain by the no-show assumption and our description of events happening at the beginning of period T, the bounding condition

$$J_T(n) = -s\mathbb{E}(\max\{\mathbf{B}(\beta, n) - C, 0\}). \tag{22}$$

Moreover, to relate the optimal value functions J_t and J_{t+1}, we obtain by the conditional expectation formula that

$$J_t(n) = \sum\nolimits_{i=0}^{m} \mathbb{E}(\mathbf{R}_t(n) \mid \xi_t = (0, r_i))p_{it}(1 - q_t(n)) + \sum\nolimits_{i=0}^{m} \mathbb{E}(\mathbf{R}_t(n) \mid \xi_t = (1, r_i))p_{it}q_t(n) \tag{23}$$

for $1 \leq t \leq T-1$ and $0 \leq n \leq C'$. Since clearly, $\mathbb{E}(\mathbf{R}_t(0) \mid \xi_t = (1, r_i)) = 0$, the second summation in relation (23) vanishes for $n = 0$. Introducing for every $n \in \mathbb{Z}_+ \cup \{0\}$ and $0 \leq i \leq m$

$$\Gamma_{t+1}(i, n) := \begin{cases} \max\{r_i + J_{t+1}(n+1), J_{t+1}(n)\}, & \text{for } i \in \{1, \cdots, m\}; \\ J_{t+1}(n), & \text{for } i = 0, \end{cases} \tag{24}$$

it follows by the principle of optimality in dynamic programming that

$$\mathbb{E}(\mathbf{R}_t(n) \mid \xi_t = (0, r_i)) = \Gamma_{t+1}(i, n)$$

and

$$\mathbb{E}(\mathbf{R}_t(n) \mid \xi_t = (1, r_i)) = -\kappa + \Gamma_{t+1}(i, n-1).$$

Hence, by relation (23) we obtain for every $n \in \mathbb{Z}_+$

$$J_t(n) = \sum\nolimits_{i=0}^{m} p_{it}((1 - q_t(n))\Gamma_{t+1}(i, n) + q_t(n)\Gamma_{t+1}(i, n-1)) - \kappa q_t(n), \tag{25}$$

while for $n = 0$ using $q_t(0) = 0$, it follows

$$J_t(0) = \sum\nolimits_{i=0}^{m} \Gamma_{t+1}(i, 0)p_{it}. \tag{26}$$

We shall next focus on the optimal policy obtained by the above dynamic programming model. To elaborate on the optimal booking limit policy, we need the property that the function $n \mapsto J_t(n)$ is a discrete concave function. We will show in the next lemma under which technical conditions on the cancellation probabilities $n \mapsto q_t(n)$, one can show that $n \to J_t(n)$ is a non-increasing discrete concave function on $\{0, 1, \cdots, C'\}$.

Lemma 2.1 *If in each period t the cancellation probabilities $n \mapsto q_t(n)$ are linear given by $q_t(n) = \omega_t n$ with $\omega_t C' \leq 1$, then the function $n \mapsto J_t(n)$ is non-increasing and discrete concave on $\{0, 1, \cdots, C'\}$ for every $1 \leq t \leq T$.*

Proof. We first show that the function $n \mapsto J_T(n)$ given in relation (22) is non-increasing and discrete concave on $\{0, \cdots, C'\}$. Clearly, this function is by definition non-increasing. To show that it is discrete concave, we observe by (7) and (30) that

$$-s\mathbb{E}(\max\{\mathbf{B}(\beta, n) - C, 0\}) = -s\beta n + sC + s\mathbb{E}(\min\{\mathbf{B}(\beta, n) - C, 0\}). \tag{27}$$

Since by Lemma B.3 the function

$$n \mapsto \mathbb{E}\left(\min\{\mathbf{B}\left(\beta, n\right) - C, 0\}\right)$$

is a discrete concave function, we obtain by relation (27) that the function $n \mapsto J_T(n)$ is also discrete concave on $\{0, \cdots, C'\}$. Suppose now for a given $t+1 < T$ that the function $n \mapsto J_{t+1}(n)$ is non-increasing and discrete concave on $\{0, \cdots, C'\}$. Our proof is then completed once we have shown that the function $n \mapsto J_t(n)$ is discrete concave and non-increasing on $\{0, \cdots, C'\}$. Applying our induction hypothesis and Lemma B.2 we first obtain that the function $n \mapsto \Gamma_{t+1}(i, n)$ given in relation (24) is non-increasing and discrete concave. Since the cancellation probability vector $n \mapsto q_t(n)$ is discrete linear on $\{0, \cdots, C'\}$, this implies using Lemma B.1 that

$$n \mapsto (1 - q_t(n))\Gamma_{t+1}(i, n) + q_t(n)\Gamma_{t+1}(i, n - 1) \tag{28}$$

is a non-increasing discrete concave function for every $i \in \{0, 1, \cdots, m\}$. Finally, by the linearity of the cancellation probabilities it follows that $n \mapsto \kappa q_t(n)$ is an increasing discrete linear function on $\{0, \cdots, C'\}$. Using now (25) with (28), we obtain the desired result. □

Since by definition a discrete concave function has decreasing differences, it follows by Lemma 2.1 that the following booking limit policy for fare-class i requests in period t is optimal:

$$\text{accept fare-class } i \text{ request in period } t \Leftrightarrow \text{ number of reserved seats } \leq b_{ti}$$

with

$$b_{ti} := \max\{n \in \mathbb{Z}_+ : r_i \geq J_{t+1}(n) - J_{t+1}(n + 1)\}.$$

Since $r_1 < r_2 < \cdots < r_m$ it follows immediately that

$$b_{t1} \leq b_{t2} \leq \cdots \leq b_{tm}.$$

In our subsequent discussion we refer to the proposed dynamic model, which leads to the above optimal decision policy, as (DM).

3. Computational Results. We devote this section to a computational study for discussing different aspects of the proposed models in the previous sections. Before giving an overview of our computational study, let us summarize the environment where we have conducted our experiments: We have used a personal computer with 1.6 GHz Intel Celeron M processor and 1015 MB of RAM. The codes are written in MATLAB 7.0 running under Windows Vista operating system.

We shall present our computational results in two parts. In the first part given in Section 3.1, we concentrate on the static lower and upper bounding models given by (P_I^{LB}) and (P_I^{UB}), respectively. Recall that both models are discussed because the original static model (P_I) is not separable, and hence, it does not admit a fast solution procedure. Therefore, from a practitioner's point of view, the gap between the optimal objective function values of the lower and upper bounding problems bears an important information to assess how well one can do by solving the bounding models instead of the original problem. Another interesting question one can generally raise about the merits of static models is whether these models may hedge against the stochastic nature of reality. To answer this question, we setup a simulation experiment, where we also solve the perfect information model (the details are given in the next subsection) and compare its results against the results obtained by the lower and upper bounding problems. Notice that both static bounding problems are optimizing only the expected revenues (for different objective functions). To evaluate the effectiveness of the optimal allocations proposed by the bounding models, we also conduct an additional simulation experiment, where we generate realizations for the random variables, and for each realization, we check how well the proposed optimal solutions perform in terms of revenue. In Section 3.2 we report our results about the second part of our computational study, where we focus on the dynamic model (DM). Our main purpose in this part is to measure the difference between the expected and the actual revenues. To serve this purpose, we set up a simulation experiment and test the performance of the optimal policy suggested by the proposed dynamic model (DM).

3.1 Part I - Static Models. Since both bounding problems (P_I^{LB}) and (P_I^{UB}) are separable, we have implemented dynamic programming algorithms to solve them. In these algorithms the stages correspond to the fare-classes, but the state spaces are different. For the upper bounding problem, the state space is one dimensional and it is the available capacity. For the lower bounding problem, we have a two-dimensional state space and we have implemented a dynamic programming algorithm using the recursions defined in (21).

We first elaborate on our simulation setup. We assume that the random variable $\mathbf{D}_i$, representing the total demand for fare-class $i = 1, 2, \cdots, m$ is concentrated on $\{0, 1, \cdots, K\}$, and this demand has the probability vector $p_i = (p_{i0}, p_{i1}, \cdots, p_{iK})^\intercal$. In our experiments, these probability vectors are generated by using the truncated Poisson distribution with parameters $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ and K. The general parameters used throughout our simulation experiments for the static models are summarized in Table 3. Note that the parameters λ_i are sorted in descending order so that a high demand is assigned to a low fare-class, and vice versa. An example of the probability vectors generated by the truncated Poisson distribution is illustrated in Figure 1. As given in Table 3, the refund percentages α_i, $i = 1, 2, \cdots, m$, are also sorted, but this time, in ascending order. We presume that high refunds are given to expensive (flexible) fare-class seats, and no refund is possible for the cheapest fare-class. Like Coughlan [6], we have used the weighted average fare, where the weights are the show-up probabilities, as our unit overbooking penalty cost, s. Note that as an alternative, penalty cost of an overbooking may depend on the flight properties. For example, Turkish Airlines (THY) arranges substitute transportation to get the denied passenger to the final destination and the overbooking cost depends on the flight length not the fare-class prices [19].

Using the general parameter values given in Table 3, if we solve both bounding problems, then the

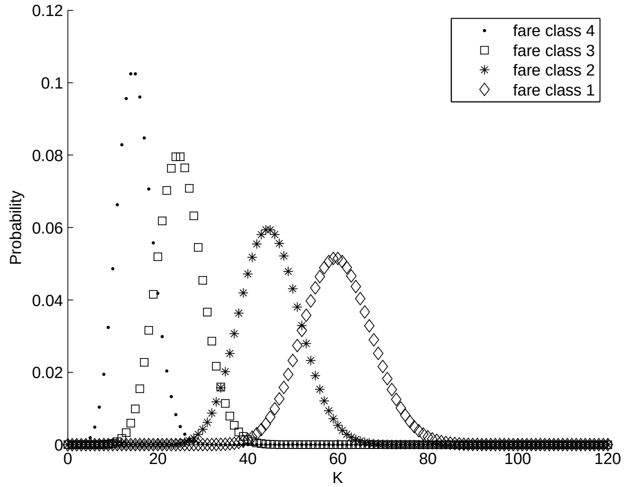


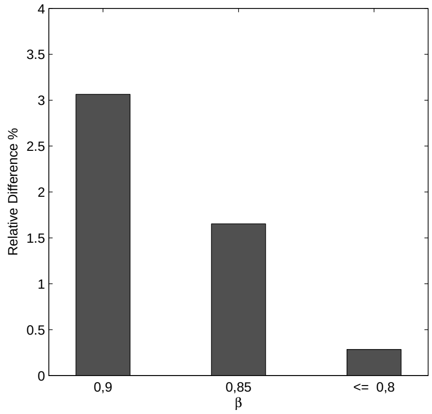
Figure 1: Truncated probability distributions for different fare-classes

Table 3: The general parameters used for the static models

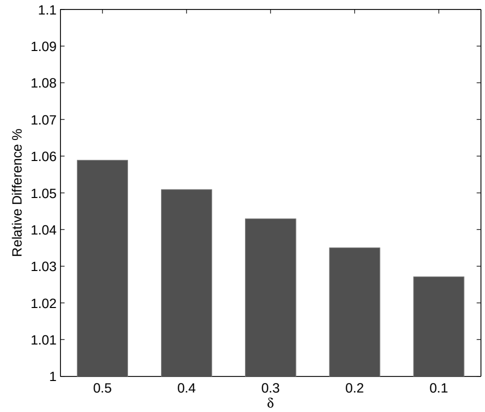
Parameters	**Values**
capacity, C; UB† on total booking limit, C'; number of fare-classes, m	100; 120; 4
ticket prices, $(r_1, \cdots, r_4)$; unit overbooking penalty cost, s	(65, 80, 95, 120); 310
truncated Poisson parameters, $(\lambda_1, \cdots, \lambda_4)$; K	(60, 45, 25, 15); 120
show-up probabilities, $(\beta_1, \cdots, \beta_4)$	(0.95, 0.90, 0.85, 0.80)
cancellation probabilities, $(\delta_1, \cdots, \delta_4)$	(0.10, 0.12, 0.15, 0.20)
refund percentages, $(\alpha_1, \cdots, \alpha_4)$	(0, 0.10, 0.25, 0.35)

†:Upper bound

relative difference between their optimal revenues $((v(P_I^{UB}) - v(P_I^{LB}))/v(P_I^{UB}))$ turns out to be 2.24%. How this difference varies according to show-up and cancellation frequencies is our next concern. Hence, we first set all the show-up probabilities to the same value, $\beta := \beta_1 = \cdots = \beta_m$ (the values of the remaining parameters are kept the same as in Table 3) and obtain the optimal objective function values of both problems. Figure 2(a) shows the relative differences between the revenues of lower and upper bounding problems for varying β values. In a similar fashion, we set all the cancellation probabilities to the same value, $\delta := \delta_1 = \cdots = \delta_m$ and solve both bounding problems (β_i values are reset to the values in Table 3). Figure 2(b) shows, in this case, the relative difference for varying δ values. As expected, Figure 2(a) shows that the difference between the objective function values of the bounding problems reduces when show-up probabilities decrease, and Figure 2(b) shows that the difference is only affected to an insignificant extent by the cancellation probabilities. Note that lower show-up probabilities imply that less people are likely to show-up and lower overbooking costs are more likely to occur. The difference between the objective function values of the bounding problems depends mainly on the overbooking cost and therefore, the objective function values almost coincide as β values decrease. The computational results show that the gap between the optimal objective function values of the proposed lower and upper bounding problems is significantly small for show-up and cancellation probabilities that are used in the



(a) Show-up probabilities



(b) Cancellation probabilities

Figure 2: The relative difference between the optimal objective function values of (P_I^{LB}) and (P_I^{UB})

literature (see, for instance, [11]).

We next consider how well the static models cope with the probabilistic information. A well-known concept in decision analysis is perfect information. Although this type of information almost never exists, it provides an upper bound on the value of real information for it pictures the *best case* scenario. In our case, having the perfect information means that there is no uncertainty with respect to the demands, cancellations and no-shows. When the total demand, the number of cancellations and the number of no-shows are fixed per fare-class, the optimal allocation is relatively straightforward: add the total number of cancellations and the total number of no shows to the capacity, then start allocating seats to the high priced customers as long as there exists available capacity. Example 1 illustrates this optimal allocation under perfect information.

Example 3.1 *Suppose that the demand realizations for fare-classes from 1 to 4 are 49, 46, 35 and 15, respectively. Moreover, the realizations for the cancellation numbers for the fare-classes are 2, 2, 2, 1, and likewise, the realizations for the no-show numbers for the fare-classes are 2,3,2,1. If we assume that the capacity is 100, then we first add the total number of cancellations and the total number of no-shows to the capacity. Then, starting from the most expensive fare-class we assign the seats, which yield the allocations 15, 35, 46, 19 to the fare-classes from 4 to 1, respectively. Notice that since we do not pay any overbooking cost, the total revenue is simply the collected fares minus the cancellation refunds. That is, the total revenue (16) is calculated as*

$$19r_1 + 46r_2 + 35r_3 + 15r_4 - 2\alpha_1 r_1 - 2\alpha_2 r_2 - 2\alpha_3 r_3 - \alpha_4 r_4.$$

The steps of our simulation experiment are given in Algorithm 1, where $\mathbf{n}^{LB} = (n_1^{LB}, \cdots, n_m^{LB})^\intercal$ and $\mathbf{n}^{UB} = (n_1^{UB}, \cdots, n_m^{UB})^\intercal$ denote the optimal solutions of problems (P_I^{LB}) and (P_I^{UB}), respectively. Since the allocations with the perfect information depends on the realizations of show-ups, cancellations and no-shows, we denote these allocations with $\mathbf{n}^{PI}(k)$, where k corresponds to the realization number.

Using Algorithm 1 we have collected statistics for 1,000 realizations. The relative differences between the static models and the perfect information for the first 50 realizations are depicted in Figure 3. Since the results are relative to the perfect information, the dashed line passing through 1.0 corresponds to the

Algorithm 1: Comparing static models and the perfect information model

1: Using Table 3 solve problems (P_I^{LB}) and (P_I^{UB}) to obtain $\mathbf{n}^{LB}$ and $\mathbf{n}^{UB}$

2: $k = 1$

3: Set β_i, δ_i, and generate realizations D_i for the random variable $\mathbf{D}_i$, $i = 1, 2, \cdots, m$

4: Calculate the realizations of the number of reserved seats for each fare-class according to the optimal solutions $\mathbf{n}^{LB}$ and $\mathbf{n}^{UB}$:

$$N_i^{LB}(n_i^{LB}) = \min\{n_i^{LB}, D_i\} \text{ and } N_i^{UB}(n_i^{UB}) = \min\{n_i^{UB}, D_i\},\ i = 1, 2, \cdots, m$$

5: Generate realizations of show-up, cancellation and no-show numbers for fare-class i using the multinomial distribution with the probabilities β_i, $(1-\beta_i)\delta_i$ and $(1-\beta_i)(1-\delta_i)$. For the lower and upper bounding models the number of trials are $N_i^{LB}(n_i^{LB})$ and $N_i^{UB}(n_i^{UB})$, respectively

6: Compute the revenue $\Phi(\mathbf{n}^{LB})$ for the lower bounding problem by plugging $\mathbf{n}^{LB}$ and the realizations generated in Step 3 and Step 5 into relation (16)

7: Compute the revenue $\Phi(\mathbf{n}^{UB})$ for the upper bounding problem by plugging $\mathbf{n}^{UB}$ and the realizations generated in Step 3 and Step 5 into relation (16)

8: Generate realizations of show-up, cancellation and no-show numbers for the perfect information case as in Step 5, where the number of trials is D_i

9: Compute $\mathbf{n}^{PI}(k)$, as illustrated in Example 3.1, for the realizations generated in Step 8 and the corresponding perfect information revenue $\Phi(\mathbf{n}^{PI}(k))$ by plugging $\mathbf{n}^{PI}(k)$ and the realizations generated in Step 8 into relation (16)

10: Calculate the relative differences with respect to the perfect information revenue

$$\frac{\Phi(\mathbf{n}^{LB})}{\Phi(\mathbf{n}^{PI}(k))} \text{ and } \frac{\Phi(\mathbf{n}^{UB})}{\Phi(\mathbf{n}^{PI}(k))}$$

11: $k \leftarrow k + 1$

12: Repeat steps 3 to 11 until the maximum number of realizations is reached and collect the statistics perfect information. This figure shows that while the relative difference between the perfect information and the upper bounding model is around 0.8492, this difference is approximately 0.7706 for the lower bounding model (the relative differences are calculated as in Step (10) of Algorithm 1). According to these results, the upper bounding problem in general provides a policy with higher realized revenues than the ones associated with the lower bounding problem. However, it is also important to note that for realizations 15, 21 and 29, the lower bounding model gives higher results than the upper bounding model. This is an expected result because both $\mathbf{n}^{LB}$ and $\mathbf{n}^{UB}$ provide bounds for the expected revenue, not for the individual realizations of the revenue. In this case, the variation around the expected revenues comes into the picture. Figure 4 gives the histograms of revenues attained by the optimal solutions of lower and upper bounding problems over 1,000 realizations. Notice also that the variance of the upper bounding model is slightly larger than the lower bounding model. The overlapping region between two histograms also confirms our observation about the possibility of obtaining larger revenues with $\mathbf{n}^{LB}$ than $\mathbf{n}^{UB}$.

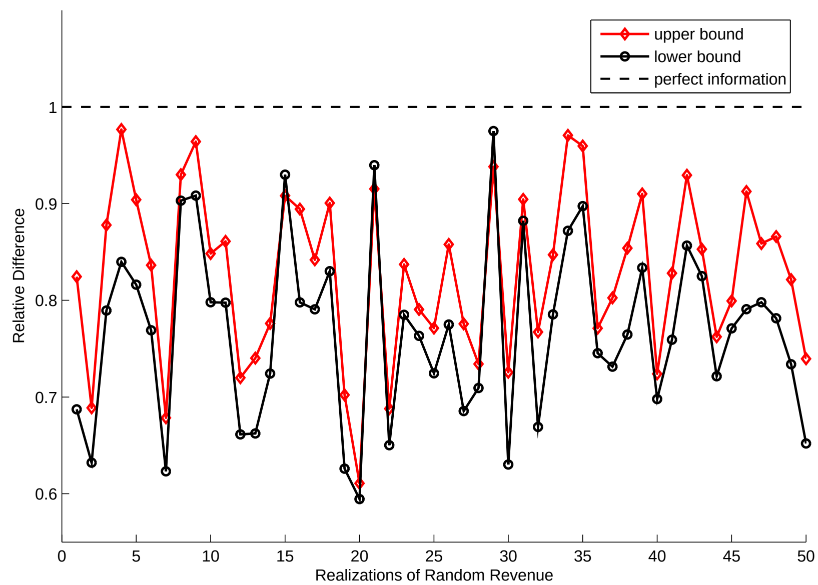


Figure 3: Relative difference between the revenue of the perfect information and the revenues of the bounding models for the first 50 realizations

3.2 Part II - Dynamic Model. To solve the dynamic model, we again implement a dynamic programming algorithm where the stages correspond to the time intervals. As given in Section 2.2, the state space is two dimensional, where the first component shows whether a cancellation occurs or not, and the second component shows the fare-class of the arriving customer (fare-class 0 denotes no-arrival).

Our simulation setup is as follows: Recall that there are two arrival processes in our dynamic model. These are arrivals of cancellations and booking requests (demand). To generate the demand arrival probability vector $p_t = (p_{0t}, p_{1t}, \cdots, p_{mt})$ at period $t = 1, 2, \cdots, T$, we use a Dirichlet distribution with parameters $\gamma_i(t)$, $i = 1, 2, \cdots, m$ (see [10]). It is reasonable to predict that as the departure time T approaches, the requests for cheaper fare-classes reduce, whereas the requests for more expensive fare-classes increase. To achieve this, we adjust the adopted Dirichlet distribution parameters monotonically. Figure 5 illustrates the change of these parameters over time. The values of Dirichlet parameters that

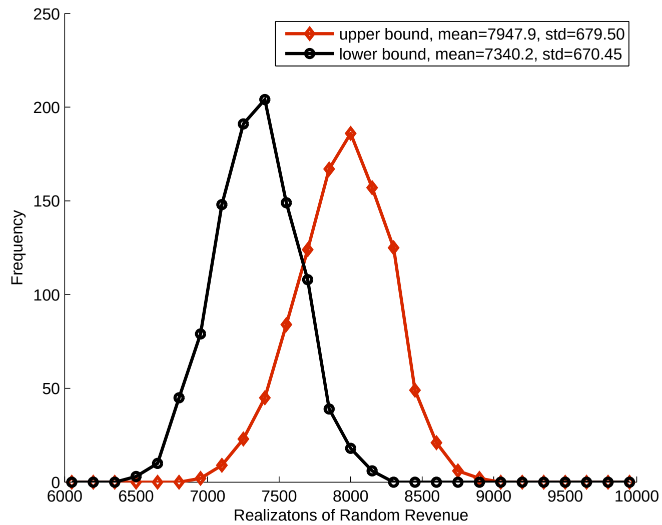


Figure 4: Histograms for the revenues obtained with the bounding models

we use in our experiments are given in line 3 of Table 4.

As we set forth in Section 2.2, in our model we work with a realistic case, where the cancellation and no-show probabilities depend on the total number of already reserved seats, n. This means that the higher the number of reserved seats, the higher the probability of cancellation $q_t(n)$, $t = 1, 2, \cdots, T$ (see also [16]). When it comes to differentiate between cancellation probabilities over time, we assume that the cancellation probabilities are slightly lower in early periods as well as in periods closer to the departure time. It is reasonable to assume that the cancellations tend to increase around the midst of the reservation horizon. To reflect this pattern, we have adjusted the coefficients w_t by using the Dirichlet distribution. The values of Dirichlet distribution parameters for generating w_t and so the cancellation probabilities ($q_t(n) = w_t n$) are given in line 4 of Table 4. The parameters v_1 and v_2 are associated with the cancellation and no-cancellation case. Figure 6 shows cancellation probabilities for different values of n over time. All parameters that we use in our experiments are given in Table 4.

Table 4: The parameters used for the dynamic model

Parameters	**Values**
capacity, C; UB† on total booking limit, C'; number of fare-classes, m	100; 120; 4
ticket prices $(r_1, \cdots, r_4)$; unit overbooking penalty cost, s; refund price, κ	(80, 90, 100, 120); 310; 30
Dirichlet parameters for demand, $(\bar{v}_0, \bar{v}, v_0, v_1, v_2, v_3, v_4)^\ddagger$	(1, 2, 3, 0.5, 1, 4, 5)
Dirichlet parameters for cancellation, $(\bar{v}_0, \bar{v}, v_0, v_1, v_2)^\ddagger$	(2, 2.87, 3, 2.5, 3)
show-up probability, β; departure time, T	0.80; 200

†:Upper bound

‡ See [7] for details

The steps of our simulation experiment is given in Algorithm 2, where k denotes a particular simulation

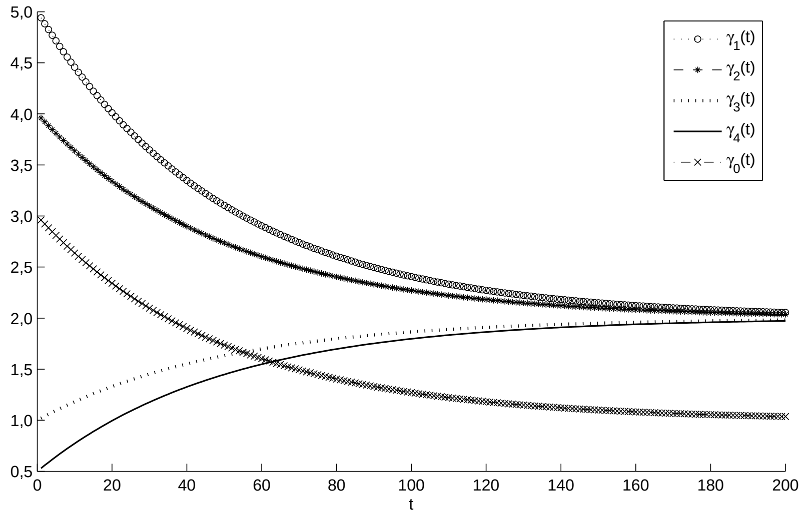


Figure 5: The change of adopted Dirichlet distribution parameters over time for generating demand arrival probabilities

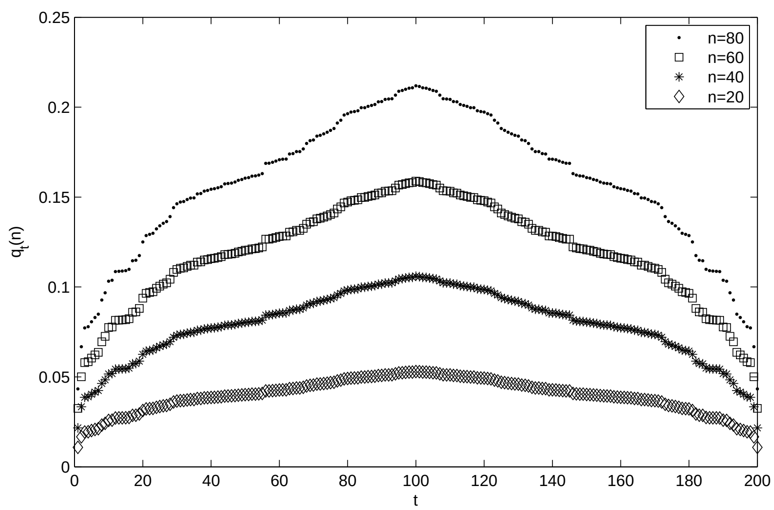


Figure 6: An example of the change of cancellation probabilities over time

run. Using this algorithm, we have collected statistics for 1,000 simulation runs. For each simulation run the revenue is calculated by using the optimal policy found by the proposed dynamic model (DM). The relative differences of those realized revenues with respect to the expected revenue $J_1(0)$ are calculated as in Step 9 of Algorithm 2 and illustrated in Figure 7. The vertical dashed line marks the expected revenue found by solving the dynamic model (DM). As seen from Figure 7, the individual realizations of the revenue are significantly close to the expected objective function value found by (DM). This shows that our proposed dynamic model performs well to hedge against the randomness inherent in the system.

Finally, we present CPU times to demonstrate the computational performance of the proposed solution methods. It required 18.56 and 0.09 seconds to solve the lower and the upper bounding problem, respectively. Moreover, computing the optimal policy corresponding to the dynamic model (DM) took

Algorithm 2: Step of the simulation experiment conducted for the dynamic model

1: Using Table 4, find the optimal policy for the dynamic model and obtain the expected revenue $J_1(0)$

2: $k = 1$

3: $t = 1$

4: Generate a realization for demand and cancellation pair, ξ_t:

- ⋄ Generate a realization for demand from a multinomial distribution with a single trial, $m+1$ classes and the associated probabilities $p_{0t}, p_{2t}, \ldots, p_{mt}$
- ⋄ Obtain a realization for cancellation by generating a Bernoulli random variable with success probability of $q_t(n)$

5: Using the optimal policy, accept or reject the arriving demand; mark the cancellation; adjust the remaining capacity

6: $t \leftarrow t + 1$

7: Repeat steps 4 to 6 until $t = T$

8: Generate a realization of the number of no-shows: generate a Bernoulli trial for each reserved seat with the probability of success being equal to $(1 - \beta)$ and count the number of successes

9: Compute the total revenue (TR) associated with the generated realizations and calculate the relative difference with respect to the expected revenue $J_1(0)$

$$\frac{\mathrm{TR} - J_1(0)}{J_1(0)}$$

10: $k \leftarrow k + 1$

11: Repeat steps 3 to 10 the maximum number of simulation runs is reached and collect the statistics

0.31 seconds. In static (dynamic) case a simulation run took on average 23.35 (1.28) seconds.

4. Conclusion In this study, we have developed new optimization models for static and dynamic single-leg problems that consider overbooking, no-shows and cancellations. In the proposed models, we relax some of the common simplifying assumptions of the existing models in the literature. In particular, cancellation and no-show probabilities are distinguished in all the models, and in the dynamic model, while cancellation probabilities depend on the number of already reserved seats, arrival and no-arrival probabilities of booking requests are independent of the current bookings. In addition, the proposed models are computationally tractable and they allow us to handle large size problems compared to the existing relevant studies.

The computational results for the static case show that the gap between the optimal objective function values of the proposed lower and upper bounding problems is significantly small for show-up and cancellation probabilities that are common in practice. Thus, solving the lower or upper bounding problems instead of the computationally challenging original problem, provides us with reasonable booking policies. Numerical results show that the upper bounding problem in general yields better solutions in terms of the revenue. Depending on the availability of the data, a decision maker may use one of the proposed

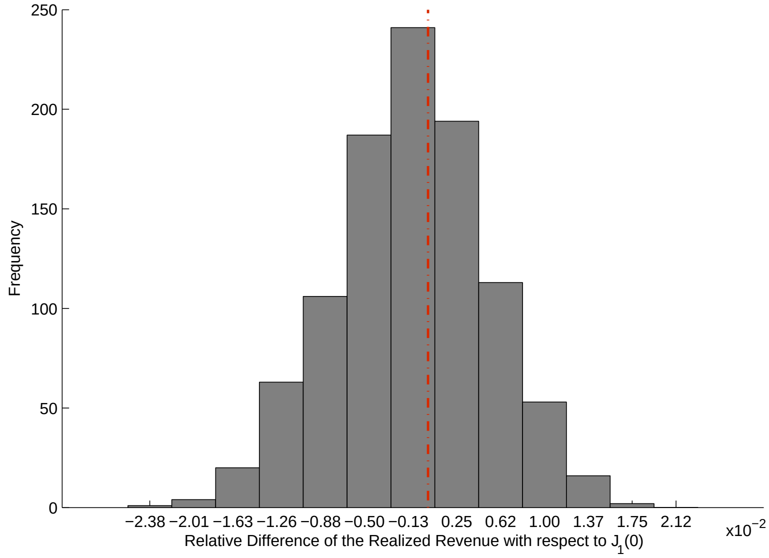


Figure 7: Histogram for the relative differences with respect to the expected revenue obtained by (DM)

static models or the dynamic model. According to the performed simulation studies both type of models perform reasonably well in capturing the randomness inherent in the system. As a future work we are planning to study the extensions of the models in the network environment and in the robust framework.

Appendix A. Review on Bernoulli Selection Scheme. In this appendix we first define a Bernoulli selection type random variable to model the demand arrivals. If $\mathbf{X}$ denotes the non-negative integer random size of a population, then the random variable $\mathbf{B}(p, \mathbf{X})$ denotes the total number within the population of size $\mathbf{X}$ having a certain property under the condition that each member in the population has this property with probability p independent of each other. Hence, the random variable $\mathbf{B}(p, \mathbf{X})$ is given by

$$\mathbf{B}(p, \mathbf{X}) := \begin{cases} \sum_{i=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{U}_i \leq p\}}, & \text{if } \mathbf{X} \geq 1; s \\ 0, & \text{if } \mathbf{X} = 0, \end{cases} \tag{29}$$

where $\mathbf{U}_n, n \in \mathbb{N}$ is a sequence of independent standard uniformly distributed random variables, and the random variable $\mathbf{X}$ is independent of the sequence $\mathbf{U}_n,\ n \in \mathbb{N}$. By relation (29), we obtain

$$\mathbb{E}(\mathbf{B}(p, \mathbf{X})) = p\mathbb{E}(\mathbf{X}). \tag{30}$$

Furthermore, it is well-known that the generating function of the random variable $\mathbf{B}(p, \mathbf{X})$ is given by

$$\mathbb{E}(z^{\mathbf{B}(p,\mathbf{X})}) = \mathbb{E}\left((1 - p + pz)^{\mathbf{X}}\right) \tag{31}$$

and

$$\mathbf{B}(q, \mathbf{B}(p, \mathbf{X})) =^d \mathbf{B}(pq, \mathbf{X}) \tag{32}$$

for any $0 \leq p, q \leq 1$ [8].

Appendix B. Results on Discrete Concave Functions. In this appendix we shall mention some results related to the discrete concavity that are used in our analysis of the proposed models. We start with a definition.

Definition B.1 *A function $f : \mathbb{Z}_+ \mapsto \mathbb{Z}$ is discrete concave if and only if the differences $n \mapsto f(n+1) - f(n)$ are decreasing.*

The proofs for the following two lemmas are given in [12].

Lemma B.1 *If the function $\mu : \mathbb{Z}_+ \mapsto [0, 1]$ is a discrete concave function and the function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ is a non-increasing discrete concave function, then the function $g : \mathbb{Z}_+ \mapsto \mathbb{R}$ given by*

$$g(n) = \mu(n)f(n-1) + (1 - \mu(n))f(n)$$

is a non-increasing discrete concave function.

Lemma B.2 *If the function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ is discrete concave, then the function $h : \mathbb{Z}_+ \mapsto \mathbb{R}$ given by*

$$h(n) = \begin{cases} \max\{r + f(n+1), f(n)\} & \text{if } n \in \mathbb{N} \\ f(0) & \text{if } n = 0 \end{cases}$$

is also discrete concave.

In the next lemma we will derive an important property of expectations of discrete concave functions of the random variable $\mathbf{B}(p, n)$.

Lemma B.3 *If the function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ is discrete concave, then the function $n \mapsto \mathbb{E}f(\mathbf{B}(p, n))$ is also discrete concave.*

Proof. We need to show that $n \mapsto \mathbb{E}f(\mathbf{B}(p, n+1)) - \mathbb{E}f(\mathbf{B}(p,n))$ is decreasing. By the definition of $\mathbf{B}(p, n+1)$ in relation (29) and the conditional expectation formula we obtain that

$$\begin{aligned} \mathbb{E}f(\mathbf{B}(p,n+1)) - \mathbb{E}f(\mathbf{B}(p,n)) &= p\mathbb{E}(f(\mathbf{B}(p,n+1)) - f(\mathbf{B}(p,n))|\mathbf{U}_{n+1} \leq p) \\ &= p(\mathbb{E}(f(1+\mathbf{B}(p,n)) - f(\mathbf{B}(p,n))|\mathbf{U}_{n+1} \leq p) \\ &= p\mathbb{E}(f(1+\mathbf{B}(p,n)) - f(\mathbf{B}(p,n))). \end{aligned} \tag{33}$$

Since $\mathbf{B}(p, n+1) \geq \mathbf{B}(p,n)$ and f is discrete concave we obtain that $n \mapsto f(1+\mathbf{B}(p,n)) - f(\mathbf{B}(p,n))$ is decreasing and by relation (33) the result follows. □

For any non-negative random variable $\mathbf{D}$, we define the random variable $\mathbf{N}(n) = \min\{n, \mathbf{D}\}$.

Lemma B.4 *If* $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ *is a discrete concave function and the optimization problem* $\max\{f(n) : n \geq C\}$ *has a finite optimal solution* n_{opt}*, then this is also an optimal solution of the problem* $\max\{\mathbb{E}f(\mathbf{N}(n)) : n \geq C\}$.

Proof. By the discrete concavity of f implying discrete unimodality, we obtain for every $n \geq n_{opt}$ that

$$f(n+1) \leq f(n) \tag{34}$$

and for every $n < n_{opt}$

$$f(n+1) \geq f(n). \tag{35}$$

This shows by relations (34) and (35) that $\mathbb{E}(f(\mathbf{N}(n+1)) \leq \mathbb{E}f(\mathbf{N}(n))$ for every $n \geq n_{opt}$, and that $\mathbb{E}(f(\mathbf{N}(n+1)) \geq \mathbb{E}(f(\mathbf{N}(n))$ for every $n < n_{opt}$. Hence, n_{opt} is also an optimal solution of the optimization problem $\max\{\mathbb{E}f(\mathbf{N}(n)) : n \geq C\}$. □

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